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BRAIDS, INNER AUTOMORPHISMS AND THE ANDREADAKIS PROBLEM

by Jacques DARNÉ (*)

ABSTRACT. — In this paper, we generalize the tools that were introduced in [13] in order to study the Andreadakis problem for subgroups of IA_n . In particular, we study the behaviour of the Andreadakis problem when we add inner automorphisms to a subgroup of IA_n . We notably use this to show that the Andreadakis equality holds for the pure braid group on n strands modulo its center acting on the free group F_{n-1} , that is, for the (pure, based) mapping class group of the n -punctured sphere acting on its fundamental group.

RÉSUMÉ. — Nous généralisons les outils introduits dans [13] pour étudier le problème d'Andreadakis pour les sous-groupes de IA_n . En particulier, nous étudions comment la réponse au problème d'Andreadakis varie lorsque les automorphismes intérieurs sont ajoutés à un sous-groupe donné. Nous utilisons les résultats obtenus pour montrer notamment que l'égalité d'Andreadakis est vraie pour le groupe de tresses pures à n brins modulo son centre agissant sur le groupe libre F_{n-1} . Cette action est celle du groupe de difféotopie (pur, pointé) de la sphère avec n points marqués sur le groupe fondamental de la sphère privée de n points.

Introduction

In his 1962 PhD. thesis [2], Andreadakis studied two filtrations on the group of automorphisms of the free group F_n . More precisely, they were filtrations on the subgroup IA_n of $\text{Aut}(F_n)$ consisting of automorphisms acting trivially on the abelianization $F_n^{ab} \cong \mathbb{Z}^n$. These filtrations were:

Keywords: Lower central series, Central filtrations, Lie algebras, Automorphisms of free groups, Braid groups.

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- The *lower central series* $IA_n = \Gamma_1(IA_n) \supseteq \Gamma_2(IA_n) \supseteq \cdots$.
- The filtration $IA_n = \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \cdots$ now known as the *Andreadakis filtration*.

He showed that there are inclusions $\mathcal{A}_i \supseteq \Gamma_i(IA_n)$, and he conjectured that these were equalities. This question became known as the *Andreadakis conjecture*, and it turned out to be a very difficult one, which is still nowadays shrouded with mystery. The interest of this question (and its difficulty) lies notably in the fact that the definitions of these filtrations are very different in nature, and thus what we understand about them is too. For instance, it is very easy to test whether a given element lies in some \mathcal{A}_j , but there is no known efficient procedure for testing whether the same element belongs to Γ_j or not. On the other hand, producing elements of Γ_j is not difficult, but we do not know any good recipe for producing elements of \mathcal{A}_j (apart, of course, from the one producing elements of Γ_j). So far, these difficulties have been overcome only for very small values of $n \geq 3$ and very small degrees to show, using computer calculations, that the conjecture is in fact false [6].

The same question can be asked for any subgroup G of IA_n . Namely, if G is such a subgroup, then we can ask whether the inclusions $\Gamma_i(G) \subseteq G \cap \mathcal{A}_i$ are equalities. The answer is obviously negative for some subgroups which are embedded in IA_n in a wrong way (take for instance a cyclic subgroup of $\Gamma_2(IA_n)$). But if G is nicely embedded in IA_n , we can hope that these filtrations on G are equal, in which case we say that the subgroup G of IA_n *satisfies the Andreadakis equality*.

Interesting examples of such subgroups include the pure braid group, embedded in IA_n via Artin's action on the free group, for which the Andreadakis equality was shown in [13]. Another example is the pure welded braid group, also acting on the free group via Artin's action. This subgroup is the group of (pure) basis-conjugating automorphisms of F_n , also known as the McCool group $P\Sigma_n$. We still do not know whether the Andreadakis equality holds for this one. A version of the latter problem up to homotopy has been considered in [11]. In all these cases, the Andreadakis equality can be seen as a comparison statement between different kinds of invariants of elements of the group (see [11]).

Automorphisms of free groups

Consider the 2-sphere \mathbb{S}^2 with n marked points, and let us choose a basepoint different from the marked points. The group of isotopy classes of

orientation-preserving self-homeomorphisms of \mathbb{S}^2 fixing the base point and each marked point is isomorphic to the quotient P_n^* of the pure braid group on n strands by its center. This group acts canonically on the fundamental group of the sphere with the n marked points removed, which is free on $n-1$ generators. In [21], this action is shown to be faithful (see also Appendix B in the present paper). Moreover, the induced action on the homology of the punctured sphere, which is the abelianization of its fundamental group, is trivial. As a consequence, the mapping class group P_n^* identifies with a subgroup of IA_{n-1} . Our main goal is the following theorem :

THEOREM 5.6. — *The subgroup P_{n+1}^* of IA_n satisfies the Andreadakis equality.*

It turns out that the subgroup P_{n+1}^* of IA_n is generated by pure braids, together with inner automorphisms of F_n . Our strategy of proof relies on this fact. Indeed, our key result (Theorem 4.4 and Corollary 5.1) is a theorem allowing us to decide, for a given subgroup K of IA_n , whether $K \cdot \text{Inn}(F_n)$ satisfies the Andreadakis equality when K does. The reader should note that such a result allowing us to pass from a subgroup to another one is quite exceptional: the Andreadakis problem (and, more generally, the lower central series) does not usually behave well when passing to smaller or bigger groups. In this regard, although P_{n+1}^* strictly contains P_n and is contained in $P\Sigma_n$, we can hardly see our present result as a step further in the study of the Andreadakis problem for $P\Sigma_n$. It has to be considered as a new interesting example in itself, and also as a good pretext to develop new tools in the study of the Andreadakis problem.

We give two other applications of our key result (Theorem 4.4 and Corollary 5.1). By applying it to the subgroup $K = IA_n^+$ of triangular automorphisms, we get the Andreadakis equality for a somewhat bigger subgroup (Theorem 5.3). Moreover, by applying it to the subgroup $K = P\Sigma_n^+$ of triangular basis-conjugating automorphisms, we recover the result obtained in [16] showing that the Andreadakis equality holds for the group of *partial inner automorphisms* defined and studied in [5].

Inner automorphisms

The question of whether or not the Andreadakis equality holds can be asked, more generally, for subgroups of IA_G , where G is any group, and IA_G is the group of automorphisms of G acting trivially on its abelianization G^{ab} . In order to get our results about subgroups of IA_n , we need

to show that the Andreadakis equality holds for the subgroup $\text{Inn}(F_n)$ of inner automorphisms of F_n . Although this result is fairly easy to get (see Section 2.2.2), we choose to develop the general theory of the Andreadakis problem for inner automorphisms, and we show, in particular, the following result (where $\mathcal{L}(G)$ denotes the graded Lie ring obtained from the lower central series of G):

COROLLARY 2.2. — *Let G be a group. The Andreadakis equality holds for $\text{Inn}(G)$ if and only if every central element of $\mathcal{L}(G)$ is the class of some central element of G .*

We give several examples and counter-example, the main one being the case of the group $G = P_n$ of pure braids. In order to study it, we give a calculation of the center of P_n which can readily be adapted to a calculation of the center of its Lie ring, and we prove:

THEOREM 3.10. — *The subgroup $\text{Inn}(P_n)$ of $IA(P_n)$ ($\subset \text{Aut}(P_n)$) satisfies the Andreadakis equality.*

Outline of the paper

The first section is devoted to recalling the needed definitions and results from the theory of group filtrations, in particular filtrations on automorphism groups and on braid groups. In Section 2, we study the Andreadakis problem for inner automorphisms, which turns out to be very much related to calculations of centers of groups and of their Lie rings. We then turn to the calculation of the center of the pure braid group, of which we give a version that generalizes easily to a calculation of the center of the associated Lie ring; this allows us to solve the Andreadakis problem for inner automorphisms of the pure braid group (Section 3). Then, in Section 4, we prove our key result (Theorem 4.4), giving a criterion for deducing the Andreadakis equality for a product of subgroups from the Andreadakis equality for these subgroups. Finally, the last section is devoted to applications to subgroups of automorphisms of free groups, namely triangular automorphisms, triangular basis-conjugating automorphism, and the pure braid group on n strands modulo its center acting on F_{n-1} .

In addition to our main results, we put in an appendix a comparison between the Drinfeld–Kohno Lie ring and the Lie ring of so-called *special derivations* of the free Lie ring (which we call *braid-like*), boiling down to some rank calculations. In a second appendix, we write down a new proof

of the faithfulness of the action of the braid group on n strands modulo its center on F_{n-1} , which involves less calculations than the ones in the literature, and we gather some useful group-theoretic results.

1. Reminders

We recall here some of the basics of the general theory of (strongly central) group filtrations and the Andreadakis problem. Details may be found in [12, 13].

1.1. Filtrations on groups

Since the only filtrations we consider in the present paper are *strongly central* ones (in the sense of [12]), we adopt Serre's convention [25] and we simply call them *filtrations*. The systematic study of such filtrations was initiated by Lazard [20], who called them *N-series*.

Notation 1.1. — Let G be a group. If $x, y \in G$, we denote by $[x, y]$ their commutator $xyx^{-1}y^{-1}$, and we use the usual exponential notations $x^y = y^{-1}xy$ and ${}^yx = yxy^{-1}$ for conjugation in G . If $A, B \subseteq G$ are subsets of G , we denote by $[A, B]$ the subgroup generated by commutators $[a, b]$ with $a \in A$ and $b \in B$.

DEFINITION 1.2. — A filtration G_* on a group G is a sequence of nested subgroups $G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots$ satisfying:

$$\forall i, j \geq 1, \quad [G_i, G_j] \subseteq G_{i+j}.$$

If G_* and H_* are two filtrations on the same group G , we write $G_* \subseteq H_*$ if $G_i \subseteq H_i$ for all i . The minimal filtration (for the inclusion relation) on a given group G is its lower central series $\Gamma_*(G)$, defined as usual by $\Gamma_1(G) = G$ and $\Gamma_{i+1}(G) = [G, \Gamma_i(G)]$ when $i \geq 1$. Recall that G is called *nilpotent* (resp. *residually nilpotent*) if $\Gamma_i(G) = \{1\}$ for some i (resp. if $\bigcap \Gamma_i(G) = \{1\}$). Since the lower central series is the minimal filtration on G , if $G_i = \{1\}$ for some i (resp. if $\bigcap G_i = \{1\}$) for any filtration G_* on $G = G_1$, then G is nilpotent (resp. residually nilpotent).

CONVENTION 1.3. — Let G be a group endowed with a filtration G_* . Let g be an element of G . If there is an integer d such that $g \in G_d - G_{d+1}$, it is obviously unique. We then call d the *degree of g with respect to G_** .

The notation \bar{g} will denote the class of g in some quotient G_i/G_{i+1} ; if the integer i is not specified, it will be assumed that $i = d$, which means that \bar{g} denotes the only non-trivial class induced by g in some G_i/G_{i+1} . If such a d does not exist (that is, if $g \in \bigcap G_i$), we say that g has degree ∞ and we put $\bar{g} = 0$.

Recall that to a filtration G_* we can associate a graded Lie ring (that is, a graded Lie algebra over \mathbb{Z}):

PROPOSITION-DEFINITION 1.4. — *If G_* is a group filtration, the graded abelian group $\mathcal{L}(G_*) := \bigoplus G_i/G_{i+1}$ becomes a graded Lie ring when endowed with the Lie bracket $[-, -]$ induced by commutators in G . Precisely, with the above convention, this bracket is defined by:*

$$\forall x \in G_i, \forall y \in G_j, [\bar{x}, \bar{y}] := \overline{[x, y]} \in \mathcal{L}_{i+j}(G_*),$$

where $\mathcal{L}_k(G_*) = G_k/G_{k+1}$ denotes the set of homogeneous elements of degree k in $\mathcal{L}(G_*)$.

CONVENTION 1.5. — *When no filtration is specified on a group G , it is implied that G is endowed with its lower central series $\Gamma_*(G)$. In particular, we denote $\mathcal{L}(\Gamma_*(G))$ simply by $\mathcal{L}(G)$.*

Since products of commutators become sums of brackets inside the Lie algebra, the following fundamental property follows easily from the definition of the lower central series:

PROPOSITION 1.6. — *The Lie ring $\mathcal{L}(G)$ is generated in degree 1. Precisely, it is generated (as a Lie ring) by $\mathcal{L}_1(G) = G^{ab}$. As a consequence, if G is finitely generated, then each $\mathcal{L}_n(G)$ is too.*

Let us recall that the construction of the associated Lie ring from a filtration is a functor from the category of filtrations and filtration-preversing group morphisms to the category of graded Lie rings. This functor $\mathcal{L} : G_* \mapsto \mathcal{L}(G_*)$ will be referred to as *the Lie functor*. We will use the fact that it is exact. Precisely, if G_* , H_* and K_* are group filtrations, a *short exact sequence of filtrations* is a short exact sequence $H_1 \hookrightarrow G_1 \twoheadrightarrow K_1$ of groups, such that the morphisms are filtration-preserving, and such that they induce a short exact sequence of groups $H_i \hookrightarrow G_i \twoheadrightarrow K_i$ for all integer $i \geq 1$. As a consequence of the Nine Lemma in the category of groups, the Lie functor sends a short exact sequence of filtrations to a short exact sequence of graded Lie rings [12, Prop. 1.24].

1.2. Actions and Johnson morphisms

Before introducing the Andreadakis problem, we introduce one of our main tools in its study, which is the *Johnson morphism* associated to an *action* of a filtration on another one. Recall that the categorical notion of an *action* of an object on another one in a protomodular category leads to the following definition:

DEFINITION 1.7. — Let K_* and H_* be filtrations on groups $K = K_1$ and $H = H_1$. An action of K_* on H_* is a group action of K on H by automorphisms such that:

$$\forall i, j \geq 1, [K_i, H_j] \subseteq H_{i+j},$$

where commutators are computed in $H \rtimes K$. Precisely:

$$\forall k \in K, \forall h \in H, [k, h] = (k \cdot h)h^{-1},$$

where $k \cdot h$ denotes the image of h by the action of k .

Given a group action of K on H , the above conditions are exactly the ones required for the sequence of subgroups $(H_i \rtimes K_i)_{i \geq 1}$ to be a filtration on $H \rtimes K$, denoted by $H_* \rtimes K_*$. Then $\mathcal{L}(H_* \rtimes K_*)$ is a semi-direct product of $\mathcal{L}(H_*)$ and $\mathcal{L}(K_*)$, encoding an action of $\mathcal{L}(K_*)$ on $\mathcal{L}(H_*)$ by derivations, described explicitly by the formula:

$$\forall k \in K, \forall h \in H, \bar{k} \cdot \bar{h} = \overline{(k \cdot h)h^{-1}}.$$

This action can also be seen as a morphism from $\mathcal{L}(K_*)$ to the Lie ring $\text{Der}(\mathcal{L}(H_*))$ of derivations of $\mathcal{L}(H_*)$, called the *Johnson morphism* associated to the action of K_* on H_* :

$$\tau : \begin{cases} \mathcal{L}(K_*) & \longrightarrow & \text{Der}(\mathcal{L}(H_*)) \\ \bar{k} & \longmapsto & (\bar{h} \mapsto \overline{(k \cdot h)h^{-1}}). \end{cases}$$

Actions of filtrations can be obtained from group actions via the following:

PROPOSITION-DEFINITION 1.8. — Let K be a group acting on another group H by automorphisms, and let H_* be a filtration on $H = H_1$. Then, there is a greatest one among filtrations K_* on a subgroup K_1 of K such that the action of K on H induces an action of K_* on H_* . This filtration is denoted by $\mathcal{A}_*(K, H_*)$ and is defined by:

$$\mathcal{A}_j(K, H_*) = \{k \in K \mid \forall i \geq 1, [k, H_i] \subseteq H_{i+j}\}.$$

The filtration $\mathcal{A}_*(\text{Aut}(H), H_*)$, denoted simply by $\mathcal{A}_*(H_*)$, is called the Andreadakis filtration associated to H_* . When furthermore $H_* = \Gamma_* H$, we denote it by $\mathcal{A}_*(H)$, and we call it the Andreadakis filtration associated to H . The filtration $\mathcal{A}_*(H)$ is a filtration on $\mathcal{A}_1(H) =: IA_H$, which is the group of automorphisms of H acting trivially on H^{ab} .

The following fact was the initial motivation for introducing such filtrations [18]:

FACT 1.9. — *In the above setting, if K acts faithfully on H , and if $H_i = \{1\}$ for some i (resp. if $\bigcap H_i = \{1\}$), then we have $\mathcal{A}_{i-1}(K, H_*) = \{1\}$ (resp. $\bigcap \mathcal{A}_i(K, H_*) = \{1\}$). In particular, $\mathcal{A}_1(K, H_*)$ must then be nilpotent (resp. residually nilpotent).*

Remark that if a morphism $a : K \rightarrow \text{Aut}(H)$ represents a group action of K on H , then $\mathcal{A}_*(K, H_*) = a^{-1}(\mathcal{A}_*(H_*))$.

1.3. Lower central series of semi-direct products

We now briefly recall the description of lower central series of semi-direct product, which will be an important ingredient in the proof of our key theorem (Theorem 4.4). This description is based on the following general construction:

PROPOSITION-DEFINITION 1.10 ([13, Def. 3.3]). — *Let G be a group, and let H be a normal subgroup of G . We define a filtration $\Gamma_*^G(H)$ on H by:*

$$\begin{cases} \Gamma_1^G(H) := H, \\ \Gamma_{k+1}^G(H) := [G, \Gamma_k^G(H)]. \end{cases}$$

If a group K acts on another group H by automorphisms, then we denote $\Gamma_*^{H \rtimes K}(H)$ only by $\Gamma_*^K(H)$. It is the minimal filtration on H which is acted upon by $\Gamma_*(K)$ via the action of K on H .

PROPOSITION 1.11 ([13, Prop. 3.4]). — *Let K be a group acting on another group H by automorphisms. Then:*

$$\forall i \geq 1, \Gamma_i(H \rtimes K) = \Gamma_i^K(H) \rtimes \Gamma_i(K).$$

Moreover, under the right conditions, $\Gamma_*^K(H)$ is in fact equal to $\Gamma_*(H)$:

PROPOSITION 1.12 ([13, Prop. 3.5]). — *Let K be a group acting on another group H by automorphisms. The following conditions are equivalent:*

- *The action of K on H^{ab} is trivial.*
- *$[K, H] \subseteq [H, H] = \Gamma_2(H)$.*
- *The action of K on H induces an action of $\Gamma_* K$ on $\Gamma_* H$.*
- *$\Gamma_*^K(H) = \Gamma_*(H)$.*
- *$\forall i \geq 1, \Gamma_i(H \rtimes K) = \Gamma_i(H) \rtimes \Gamma_i(K)$.*
- *$\mathcal{L}(H \rtimes K) \cong \mathcal{L}(H) \rtimes \mathcal{L}(K)$.*
- *$(H \rtimes K)^{ab} \cong H^{ab} \times K^{ab}$.*

When these conditions are satisfied, we say that the semi-direct product $H \rtimes K$ is an almost-direct one.

1.4. The Andreadakis problem

Let G_* be a group filtration. The Andreadakis problem is concerned with the comparison of two filtrations on $\mathcal{A}_1(G_*)$, namely the Andreadakis filtration $\mathcal{A}_*(G_*)$ and the filtration $\Gamma_*(\mathcal{A}_1(G_*))$. The latter is the minimal filtration on the group $\mathcal{A}_1(G_*)$, hence $\mathcal{A}_*(G_*)$ contains $\Gamma_*(\mathcal{A}_1(G_*))$. Now, if K is a subgroup of $\mathcal{A}_1(G_*)$ (that is, K acts on G by automorphisms preserving the G_i , and the induced action on $\mathcal{L}(G_*)$ is trivial), then we can restrict these filtrations to K , and the filtrations so obtained must contain $\Gamma_*(K)$, which is the minimal filtration on K .

DEFINITION 1.13. — *Let G_* be a group filtration and K be a subgroup of $\mathcal{A}_1(G_*)$. We say that K satisfies the Andreadakis equality with respect to G_* if the following inclusions are equalities:*

$$\Gamma_*(K) \subseteq K \cap \Gamma_*(\mathcal{A}_1(G_*)) \subseteq K \cap \mathcal{A}_*(G_*).$$

If K is a subgroup of IA_G , we simply say that K satisfies the Andreadakis equality when it does with respect to $\Gamma_*(G)$.

Remark 1.14. — This definition can be generalized to groups acting on G (in a possibly non-faithful way). Precisely, let K act on a group G via a morphism $a : K \rightarrow \text{Aut}(G)$, and let G be endowed with a filtration G_* . Suppose that $a(K) \subseteq \mathcal{A}_1(G_*)$ (that is, K acts by automorphisms preserving the filtration, and the induced action on $\mathcal{L}(G_*)$ is trivial). Then we get inclusions of filtrations on K :

$$\Gamma_*(K) \subseteq a^{-1}(\Gamma_*(\mathcal{A}_1(G_*))) \subseteq a^{-1}(\mathcal{A}_*(G_*)) = \mathcal{A}_*(K, G_*).$$

However, since $\Gamma_*(a(K)) = a(\Gamma_*(K))$, these filtrations are equal if and only if $a(K)$ satisfies the Andreadakis equality with respect to G_* , so we can (and will) focus on subgroups of $\text{Aut}(G)$ when studying the difference between such filtrations.

The following result is deduced easily from the definitions, and will be our main tool in proving the Andreadakis equality for subgroups of IA_G :

PROPOSITION 1.15 ([12, Lem. 1.28]). — *Let K_* and H_* be group filtrations. The Johnson morphism associated to a given action of K_* on H_* is injective if and only if $K_* = \mathcal{A}_*(K_1, H_*)$.*

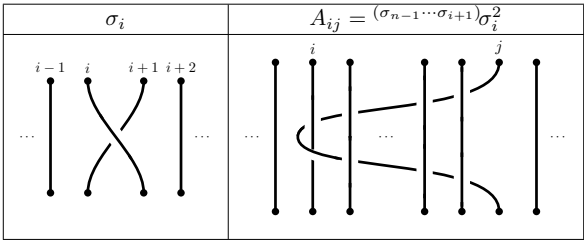
The Andreadakis problem for automorphisms of free groups. — The classical setting is the one when $G = F_n$ is the free group on n generators, and G_* is its lower central series. Then IA_G , denoted by IA_n , is the subgroup of $\text{Aut}(F_n)$ made of automorphisms acting trivially on $F_n^{ab} \cong \mathbb{Z}^n$. The Andreadakis filtration associated to F_n is a filtration on IA_n simply denoted by \mathcal{A}_* and referred to as *the Andreadakis filtration*. Recall that F_n is residually nilpotent, which implies that $\bigcap \mathcal{A}_i = \{1\}$, thus IA_n is residually nilpotent (Fact 1.9). Since the Lie algebra of F_n is the free Lie ring \mathfrak{L}_n on n generators, the Johnson morphism associated to the action of \mathcal{A}_* on $\Gamma_*(F_n)$ is an injection (Proposition 1.15 above) $\tau : \mathcal{L}(\mathcal{A}_*) \hookrightarrow \text{Der}(\mathfrak{L}_n)$.

1.5. Braids

We gather here the results we need about Artin’s braid groups and filtrations on them. Our main reference here is Birman’s book [8]. The reader can also consult the original papers of Artin [3, 4].

1.5.1. Generalities

We denote Artin’s braid group by B_n and the subgroup of pure braids by P_n . Recall that B_n is generated by $\sigma_1, \dots, \sigma_{n-1}$ and P_n is generated by the A_{ij} (for $i < j$), which are drawn as follows:



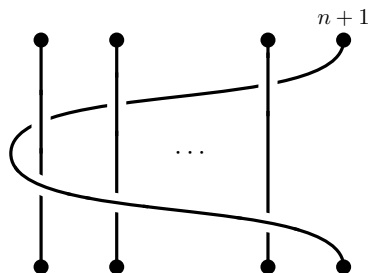
CONVENTION 1.16. — *It is often convenient to have A_{ij} defined for all i, j (not only for $i < j$), using the formulas $A_{ji} = A_{ij}$ and $A_{ii} = 1$.*

CONVENTION 1.17. — *Drawing braids from top to bottom corresponds to taking products from left to right. Otherwise said, the product $\alpha\beta$ denotes the braid obtained by putting α on top of β .*

Forgetting the $(n+1)$ -th strand induces a projection $P_{n+1} \rightarrow P_n$ which is split (a section is given by adding a strand away from the other ones). The kernel of this projection identifies with the fundamental group of the plane with n punctures, which is the free group F_n . We thus get a decomposition into a semi-direct product:

$$P_{n+1} \cong F_n \rtimes P_n,$$

which encodes an action of P_n on F_n via automorphisms. A basis of F_n is given by the elements $x_i := A_{i,n+1}$. A classical result of Artin [8, Cor. 1.8.3 and Thm. 1.9] says that this action is faithful, and its image is exactly the group $\text{Aut}_C^\partial(F_n)$ of automorphisms preserving the conjugacy class of each generator and fixing the following *boundary element*:



$$\partial_n := x_1 \cdots x_n = A_{1,n+1} \cdots A_{n,n+1}.$$

Since $P_n \cong \text{Aut}_C^\partial(F_n)$, such automorphisms are also called *braid automorphisms*. We will often identify P_n with $\text{Aut}_C^\partial(F_n)$ in the sequel.

Notice that there are many possible choices of generators of P_n , but the choices that we have made are coherent : they allow us to interpret the above split projection $P_{n+1} \rightarrow P_n$ (defined by forgetting the $(n+1)$ -th strand) as the projection from $\text{Aut}_C^\partial(F_{n+1})$ onto $\text{Aut}_C^\partial(F_n)$ induced by $x_{n+1} \mapsto 1$ (precisely, any braid automorphism β sends x_{n+1} to one of its conjugates, thus preserves the normal closure of x_{n+1} and induces an automorphism of $F_{n+1}/x_{n+1} \cong F_n$). The section of this projection defined by adding a strand corresponds to extending canonically automorphisms of F_n to automorphisms of F_{n+1} fixing x_{n+1} .

Remark 1.18. — The free subgroup of B_{n+1} generated by the $A_{i,n+1}$ is in fact normalized not only by P_n but also by B_n . The corresponding action of B_n on F_n (induced by conjugation in B_{n+1}) is faithful, and its image is the group of automorphisms of F_n permuting the conjugacy classes of the x_i and fixing the boundary element.

Recall that B_n and P_n can be seen as fundamental groups of configuration spaces. Namely, if \mathbb{D} is the closed disc and $F_n(\mathbb{D}^\circ) := \{(x_1, \dots, x_n) \in \mathbb{D}^\circ \mid \forall i \neq j, x_i \neq x_j\}$ is the usual configuration space, P_n (resp. B_n) is the fundamental group of $F_n(\mathbb{D}^\circ)$ (resp. of its quotient $F_n(\mathbb{D}^\circ)/\mathfrak{S}_n$ by the symmetric group).

They can also be interpreted as Mapping Class Groups. Indeed, if we fix a base configuration in the open disk, evaluation at this base configuration gives a continuous map from the space $\text{Homeo}_\partial(\mathbb{D})$ of self-homeomorphisms of the disk fixing the boundary pointwise to $F_n(\mathbb{D}^\circ)$. This map is a locally trivial fibration, whose fiber is the subspace $\text{PHomeo}_\partial(\mathbb{D}, n)$ of homeomorphisms fixing n points of \mathbb{D}° . Using Alexander's trick, one can use the long exact sequence in homotopy to get an isomorphism:

$$P_n \cong \pi_0(\text{PHomeo}_\partial(\mathbb{D}, n)).$$

This extends to an isomorphism $B_n \cong \pi_0(\text{Homeo}_\partial(\mathbb{D}, n))$ identifying B_n with isotopy classes of self-homeomorphisms of the disk fixing the boundary pointwise and permuting n given points of \mathbb{D}° .

With this point of view, the Artin action on the free group is induced by the canonical action of homeomorphisms permuting n points on the fundamental group of the disk with these n points removed, which is free on n generators.

1.5.2. The Drinfeld–Kohno Lie ring

A braid acts on $F_n^{ab} \cong \mathbb{Z}^n$ via the associated permutation of the basis. As a consequence, the pure braid group P_n acts trivially on F_n^{ab} , hence it is a subgroup of IA_n . Since IA_n is residually nilpotent, so is P_n . The Lie ring associated to the lower central series of P_n was first determined rationally in [19], and it was shown not to have torsion in [14], where its ranks were computed. Details about the complete description over \mathbb{Z} that we now recall may be found in the appendix of [13].

DEFINITION 1.19. — *The Drinfeld–Kohno Lie ring is $DK_n := \mathcal{L}(P_n)$.*

Since P_n acts trivially on F_n^{ab} , the semi-direct product $P_{n+1} \cong F_n \rtimes P_n$ is an almost-direct one, thus it induces a decomposition of the associated Lie rings:

$$\mathcal{L}(P_{n+1}) \cong \mathfrak{L}_n \rtimes \mathcal{L}(P_n),$$

where $\mathfrak{L}_n = \mathcal{L}(F_n)$ is the free Lie ring on n generators $X_i = t_{i,n+1} := \bar{A}_{i,n+1}$ (for $i \geq n$). Thus, $\mathcal{L}(P_n)$ decomposes as an iterated semi-direct product of free Lie rings. From that, it is not difficult to get a presentation of this Lie ring:

PROPOSITION 1.20 ([13, Prop A.3]). — *The Drinfeld–Kohno Lie ring $DK_n = \mathcal{L}(P_n)$ is generated by the $t_{ij} = \bar{A}_{ij}$ ($1 \leq i, j \leq n$), under the relations:*

$$\begin{cases} t_{ij} = t_{ji}, t_{ii} = 0 & \forall i, j, \\ [t_{ij}, t_{ik} + t_{kj}] = 0 & \forall i, j, k, \\ [t_{ij}, t_{kl}] = 0 & \text{if } \{i, j\} \cap \{k, l\} = \emptyset. \end{cases}$$

Recall that the decomposition $DK_{n+1} \cong \mathfrak{L}_n \rtimes DK_n$ is encoded in the corresponding Johnson morphism $\tau : DK_n \rightarrow \text{Der}(\mathfrak{L}_n)$. The following result allows us to identify DK_n with a Lie subring of the Lie ring of derivations of the free Lie ring:

THEOREM 1.21 ([13, Thm. 6.2]). — *The subgroup P_n of IA_n satisfies the Andreadakis equality. This means exactly that the corresponding Johnson morphism $\tau : DK_n \rightarrow \text{Der}(\mathfrak{L}_n)$ is injective.*

Remark 1.22. — This statement can be seen as a statement about Milnor’s μ -invariants. Namely, up to a slight change of viewpoint (corresponding to taking Magnus expansions), $\tau(\bar{\beta})$ corresponds exactly to the set of non-vanishing Milnor invariants of the (pure) braid β of minimal degree. In particular, $\tau(\bar{\beta})$ is of degree at least d if and only if the Milnor invariants of degree at most $d - 1$ of β vanish. With this point of view, the injectivity of τ means exactly that Milnor invariants of degree at most $d - 1$ distinguish (pure) braids up to elements of $\Gamma_d(P_n)$ (see also [15, 22]). Moreover, since P_n (which is an almost-direct product of free groups) is residually nilpotent, this also means that Milnor invariants distinguish braids.

In the sequel, we identify DK_n with $\tau(DK_n)$. It is not difficult to see that DK_n is in fact a Lie subring of the Lie ring $\text{Der}_t^\partial(\mathfrak{L}_n)$ of *tangential derivations* (derivations sending each X_i to $[X_i, w_i]$ for some $w_i \in \mathfrak{L}_n$) vanishing on the *boundary element*, defined by:

$$\bar{\partial}_n = X_1 + \cdots + X_n = t_{1,n+1} + \cdots + t_{n,n+1} \in \mathfrak{L}_n \subset DK_{n+1}.$$

DEFINITION 1.23. — A derivation of the free Lie ring \mathfrak{L}_n is called braid-like (or special) if it is an element of $\text{Der}_t^\partial(\mathfrak{L}_n)$. It is called a braid derivation if it stands inside DK_n .

Remark 1.24 (On terminology). — Ihara [17] called *special* the derivation we call *tangential*. The ones we call *braid-like*, he called *normalized special*. More recently, authors working on the Kashiwara–Vergne problem (see for instance [1]), who were using the word “tangential” for the former, kept the word “special” only for Ihara’s *normalized special* derivations (dropping the adjective “normalized”). In the present paper, we use the word “tangential”, but we prefer “braid-like” over “special”, because we feel that it conveys more meaning.

Not all braid-like derivations are braid derivations. The difference between the two Lie subrings of $\text{Der}(\mathfrak{L}_n)$ is investigated in Appendix A.

Remark that the projection of DK_n onto DK_{n-1} giving the above decomposition $DK_n \cong \mathfrak{L}_{n-1} \rtimes DK_{n-1}$ can be seen as the restriction of the projection of $\text{Der}_t^\partial(\mathfrak{L}_n)$ onto $\text{Der}_t^\partial(\mathfrak{L}_{n-1})$ induced by $X_n \mapsto 0$.

2. Inner automorphisms

This section is devoted to the study of the Andreadakis problem for inner automorphisms of a group G , with respect to any filtration G_* on the group. We prove a general criterion (Theorem 2.1), involving a comparison between the center of G and the center of the Lie ring associated to G_* . Since most of our applications will be to the case when G_* is the lower central series of G , we spell out the application of our general criterion to this case in Corollary 2.2. We then turn to examples, including the easy case of the free group. Our most prominent application will be to the pure braid group, which will be the goal of the next section.

2.1. A general criterion

Recall that for any element g of a group G , the inner automorphism c_g associated to g is defined by $c_g(x) = {}^gx (= gxg^{-1})$ for all $x \in G$. The map $c : g \mapsto c_g$ is a group morphism whose image is the normal subgroup $\text{Inn}(G)$ of $\text{Aut}(G)$. The kernel of this morphism is the set of elements $g \in G$ such that for all $x \in G$, $gxg^{-1} = x$, which is the center $\mathcal{Z}(G)$ of G . As a consequence, $\text{Inn}(G) \cong G/\mathcal{Z}(G)$.

The same story can be told for Lie algebras: a Lie algebra \mathfrak{g} acts on itself via the adjoint action, the image of this action $\text{ad} : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$ is the Lie algebra $\text{ad}(\mathfrak{g})$ of inner derivations, and the kernel of ad is the center $\mathfrak{z}(\mathfrak{g})$ of \mathfrak{g} , so that $\text{ad}(\mathfrak{g}) \cong \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$.

We now explain how these two stories are related to the Andreadakis problem. Let G_* be a filtration on $G = G_1$, and let $\mathcal{A}_*(G_*)$ be the associated Andreadakis filtration. The filtration G_* acts on itself via the adjoint action, which is induced by the action of G on itself by inner automorphisms. The latter is represented by the morphism $c : G \rightarrow \text{Aut}(G)$ described above, and the fact that it induces an action of G_* on itself is reflected in the fact that c sends G_* to $\mathcal{A}_*(G_*)$. Thus, if we consider the image of G_* under the corestriction $\pi : G \rightarrow \text{Inn}(G)$ of c , we get an inclusion of filtration $\pi(G_*) \subseteq \mathcal{A}_*(G_*)$. The following theorem gives a criterion for the inclusion $\pi(G_*) \subseteq \text{Inn}(G) \cap \mathcal{A}_*(G_*)$ to be an equality:

THEOREM 2.1. — *Let G_* be a filtration on $G = G_1$. Its image in $\text{Inn}(G)$ coincides with $\text{Inn}(G) \cap \mathcal{A}_*(G_*)$ if and only if the inclusion $\mathfrak{z}(\mathcal{L}(G_*)) \supseteq \mathcal{L}(G_* \cap \mathcal{Z}(G))$ is an equality, that is, exactly when every central element of $\mathcal{L}(G_*)$ is the class of some central element of G .*

If G_* is the lower central series of G , whose image in $\text{Inn}(G)$ is the lower central series of $\text{Inn}(G)$, then $\mathcal{A}_*(\Gamma_*G)$ is the usual Andreadakis filtration on $IA(G)$.

COROLLARY 2.2. — *For a group G , the following conditions are equivalent to each other:*

- *The Andreadakis equality holds for $\text{Inn}(G)$.*
- *Every central element of the Lie ring $\mathcal{L}(G)$ is the class of some central element of the group G .*
- *The canonical projection $\mathcal{L}(G/\mathcal{Z}(G)) \rightarrow \mathcal{L}(G)/\mathfrak{z}(\mathcal{L}(G)) = \text{ad}(\mathcal{L}(G))$ is an isomorphism.*

This introduces a motivation for solving the Andreadakis problem for inner automorphisms of a group : it allows one to compute the Lie algebra of the quotient $G/\mathcal{Z}(G)$. We will apply this to the pure braid group P_n later (Section 3.4).

Proof of Corollary 2.2. — As a direct application of Theorem 2.1, the first assertion is equivalent to the inclusion $\mathfrak{z}(\mathcal{L}(G)) \supseteq \mathcal{L}(\Gamma_*(G) \cap \mathcal{Z}(G))$ being an equality, which is clearly equivalent to the second assertion. In order to see that it is also equivalent to the third condition, let us consider

the short exact sequence of filtrations :

$$\Gamma_*(G) \cap \mathcal{Z}(G) \hookrightarrow \Gamma_*(G) \twoheadrightarrow \Gamma_*(G/\mathcal{Z}(G)).$$

By applying the Lie functor, we get an isomorphism:

$$\mathcal{L}(G/\mathcal{Z}(G)) \cong \mathcal{L}(G)/\mathcal{L}(\Gamma_*(G) \cap \mathcal{Z}(G)).$$

As a consequence, the inclusion $\mathfrak{z}(\mathcal{L}(G)) \supseteq \mathcal{L}(\Gamma_*(G) \cap \mathcal{Z}(G))$ induces a canonical projection $\mathcal{L}(G/\mathcal{Z}(G)) \rightarrow \mathcal{L}(G)/\mathfrak{z}(\mathcal{L}(G))$. The latter is an isomorphism if and only if the former is an equality. \square

Proof of Theorem 2.1. — Let us denote by $\pi(G_*)$ the image of G_* under $\pi : G \twoheadrightarrow \text{Inn}(G)$, and the inclusion $\pi(G_*) \subseteq \text{Inn}(G) \cap \mathcal{A}_*(G_*)$ by i . The latter induces a morphism $i_* : \mathcal{L}(\pi(G_*)) \rightarrow \mathcal{L}(\text{Inn}(G) \cap \mathcal{A}_*(G_*))$, whose injectivity is equivalent to the two filtrations on $\text{Inn}(G)$ being equal. By definition of π and i , we have a commutative square:

$$\begin{array}{ccc} \pi(G_*) & \xleftarrow{\pi} & G_* \\ \downarrow i & & \downarrow c \\ \text{Inn}(G) \cap \mathcal{A}_*(G_*) & \hookrightarrow & \mathcal{A}_*(G_*). \end{array}$$

By taking the associated graded, we get the left square in:

$$\begin{array}{ccccc} \mathcal{L}(\pi(G_*)) & \xleftarrow{\pi_{\#}} & \mathcal{L}(G_*) & & \\ \downarrow i_{\#} & & \downarrow c_{\#} & \searrow \text{ad} & \\ \mathcal{L}(\text{Inn}(G) \cap \mathcal{A}_*(G_*)) & \hookrightarrow & \mathcal{L}(\mathcal{A}_*(G_*)) & \xrightarrow{\tau} & \text{Der}(\mathcal{L}(G_*)). \end{array}$$

The fact that the triangle on the right commutes can be seen *via* an abstract argument (c represents the adjoint action of G_* , hence $c_{\#}$ represents the adjoint action of $\mathcal{L}(G_*)$), or can be obtained *via* a direct calculation using the usual explicit description of the Johnson morphism :

$$(2.1) \quad \tau(\overline{c_g}) : \bar{x} \mapsto \overline{c_g(x)x^{-1}} = \overline{[g, x]} = [\bar{g}, \bar{x}] = \text{ad}_{\bar{g}}(\bar{x}).$$

From this fact and the injectivity of the Johnson morphism, we deduce that the kernel of $c_{\#}$ is $\mathfrak{z}(\mathcal{L}(G_*))$. The Lie subring $\mathcal{L}(G_* \cap \mathcal{Z}(G))$ of $\mathcal{L}(G_*)$, on the other hand, appears as the kernel of $\pi_{\#}$ by applying the Lie functor to the following short exact sequence of filtrations:

$$G_* \cap \mathcal{Z}(G) \hookrightarrow G_* \xrightarrow{\pi} \pi(G_*).$$

Now, $\pi_{\#}$ induces a surjection from $\ker(c_{\#}) = \ker(i_{\#}\pi_{\#})$ onto $\ker(i_{\#})$, whose kernel is exactly $\ker(\pi_{\#})$ (this can be seen as an application of the

usual exact sequence $0 \rightarrow \ker(v) \rightarrow \ker(uv) \rightarrow \ker(u) \rightarrow \operatorname{coker}(v) \rightarrow u = i_{\#}$ and $v = \pi_{\#}$). Thus we have:

$$(2.2) \quad \ker(i_{\#}) \cong \frac{\mathfrak{z}(\mathcal{L}(G_*))}{\mathcal{L}(G_* \cap \mathcal{Z}(G))},$$

whence the conclusion. \square

Remark 2.3. — The isomorphism (2.2) gives more information than the statement of Theorem 2.1, which is the case when this kernel is trivial. However, we will mainly use the latter case in the sequel.

2.2. Examples

We now apply Corollary 2.2 in order to give examples of groups whose group of inner automorphism satisfies (or not) the Andreadakis equality.

2.2.1. Counter-examples

The symmetric group. — The first counter-examples we can give are groups G with no center and a non-trivial abelian Lie ring (that is, a Lie ring reduced to $G^{ab} \neq 0$), such as the symmetric group Σ_n ($n \geq 3$), whose abelianization is $\Sigma_n/A_n \cong \mathbb{Z}/2$.

The braid group. — Another slightly more interesting counter-example in the braid group B_n , whose classical generators $\sigma_1, \dots, \sigma_{n-1}$ are all conjugate. Its Lie algebra is reduced to its abelianization $B_n^{ab} \cong \mathbb{Z}$, generated by the common class $\bar{\sigma}$ of all σ_i . If $n \geq 3$, the center of B_n is cyclic, generated by $\xi_n = (\sigma_1 \cdots \sigma_{n-1})^n$. Thus its image in $B_n^{ab} \cong \mathbb{Z}$ is not equal to all of B_n^{ab} , but to $n(n-1)\mathbb{Z} \subset \mathbb{Z}$.

Such examples are not very interesting to us, since filtrations on them do not contain a lot of information (they contain only information about the abelianization). We now describe a way of obtaining nilpotent (or residually nilpotent) counter-examples.

Constructing counter-examples as semi-direct products. — Let G be a group, and α be an automorphism of G . Consider the semi-direct product $G \rtimes \mathbb{Z}$ encoding the action of \mathbb{Z} on G through powers of α . This semi-direct product is an almost-direct one if and only if $\alpha \in IA(G)$. Under this condition, $\Gamma_*(G \rtimes \mathbb{Z}) = \Gamma_*(G) \rtimes \Gamma_*(\mathbb{Z})$, so that $G \rtimes \mathbb{Z}$ is nilpotent (resp. residually nilpotent) whenever G is.

Let us denote by t the generator of \mathbb{Z} acting via α on G , and consider its class \bar{t} in $\mathcal{L}(G \rtimes \mathbb{Z})$. This element is central in $\mathcal{L}(G \rtimes \mathbb{Z}) = \mathcal{L}(G) \rtimes \mathbb{Z}$ if and only if its action on $\mathcal{L}(G)$ is trivial. This happens exactly when $\alpha \in \mathcal{A}_2(G)$. Indeed, for any class $\bar{g} \in \mathcal{L}_i(G)$, we have $[\bar{t}, \bar{g}] = \overline{[t, g]} = \overline{[\alpha, g]}$, and this is trivial in $\mathcal{L}_{i+1}(G)$ if and only if $[\alpha, g] \in \Gamma_{i+2}G$. Thus, $[\bar{t}, \bar{g}] = 0$ for all \bar{g} if and only if for all $i \geq 1$, $[\alpha, \Gamma_i G] \subseteq \Gamma_{i+2}G$.

When is \bar{t} the class of a central element in $G \rtimes \mathbb{Z}$? Lifts of \bar{t} to $G \rtimes \mathbb{Z}$ are elements of the form gt with $g \in \Gamma_2 G$. Using Proposition 3.1 below, we see that such an element is central in $G \rtimes \mathbb{Z}$ if and only if t acts on G via $c_{g^{-1}}$. Thus, \bar{t} is the class of a central element of $G \rtimes \mathbb{Z}$ if and only if α is an inner automorphism c_x ($x \in \Gamma_2 G$), in which case (x^{-1}, t) is the only central element whose class in $(G \rtimes \mathbb{Z})$ is \bar{t} .

We conclude that this construction gives counter-examples whenever there exist automorphisms α in \mathcal{A}_2 which are not inner. Such automorphisms exist for the free group, or for the free nilpotent group of any class at least 3, giving counter-examples of any nilpotency class greater than 3.

2.2.2. The free group

Let $G = F_n$ be the free group on n generators, with $n \geq 2$. It is centerless, so that $\text{Inn}(F_n) \cong F_n$. We can apply the above machinery to the lower central series $\Gamma_*(F_n)$. The associated graded ring $\mathcal{L}(F_n)$ is free, whence centerless. As a consequence, we deduce directly from Corollary 2.2 the following :

COROLLARY 2.4. — *The subgroup $\text{Inn}(F_n)$ of IA_n satisfies the Andreadakis equality.*

We do not need the full strength of the above theory of filtrations on inner automorphisms in order to get this result. In fact, it may be enlightening to write down a direct proof of this corollary.

Direct proof of Corollary 2.4. — If $w \in F_n$, suppose that $c_w \in \mathcal{A}_j$, and let us show that $w \in \Gamma_j(F_n)$. The hypothesis means that:

$$\forall x \in F_n, \quad c_w(x) \equiv x \pmod{\Gamma_{j+1}(F_n)}.$$

But $c_w(x)x^{-1} = [w, x]$, and $[w, x_i] \in \Gamma_{j+1}(F_n)$ implies that the class \bar{w} of w in $\mathcal{L}(F_n)$ either is of degree at least j or commutes with the generator \bar{x}_i . The latter is possible only if $\bar{w} \in \mathbb{Z}\bar{x}_i$ (see the classical Lemma 3.6 recalled below), which can be true only for one value of i . Thus \bar{w} must be of degree at least j , which means that $w \in \Gamma_j(F_n)$. \square

We can give yet another proof, which looks more like a simple version of the above theory in this particular case, and enhances the fact (needed later) that the Lie ring of $\text{Inn}(F_n)$ identifies with the ring of inner derivations of \mathfrak{L}_n .

Yet another proof of Corollary 2.4. — The Andreadakis equality is equivalent to the associated Johnson morphism $\tau : \mathcal{L}(F_n) \rightarrow \text{Der}(\mathfrak{L}_n)$ being injective (Proposition 1.15). Because of the direct calculation (2.1), this morphism can be identified with $\text{ad} : \mathfrak{L}_n \rightarrow \text{Der}(\mathfrak{L}_n)$, whose kernel is the center of \mathfrak{L}_n , which is trivial. Hence the result. \square

3. Centers of semi-direct products

We now turn to showing the Andreadakis equality for inner automorphisms of the pure braid group, a goal achieved in Theorem 3.10. In order to do this, we recover the classical computation of the center of P_n in a way that can be adapted easily to a computation of the center of its Lie ring DK_n .

3.1. General theory

The following easy result, which we have already used once in Section 2.2.1, will be our main tool in computing the center of P_n , and that of DK_n :

PROPOSITION 3.1. — *Let K be a group acting on another group H by automorphisms. Then the center of $H \rtimes K$ consists of elements hk (with $h \in H$ and $k \in K$) such that:*

- k is central in K ,
- h is a fixed point of the action of K on H (that is, $K \cdot h = \{h\}$),
- k acts on H via c_h^{-1} .

Similarly, let \mathfrak{k} be a Lie algebra acting on another Lie algebra \mathfrak{h} by derivations. Then the center of $\mathfrak{h} \rtimes \mathfrak{k}$ consists of elements $x + y$ (with $x \in \mathfrak{h}$ and $y \in \mathfrak{k}$) satisfying:

- y is central in \mathfrak{k} ,
- x is a fixed point of the action of \mathfrak{k} on \mathfrak{h} (that is, $\mathfrak{k} \cdot x = \{0\}$),
- y acts on \mathfrak{h} via $-\text{ad}(x)$.

Proof of Proposition 3.2. — The result is clear for $n \leq 2$. We thus suppose that $n \geq 3$.

Lemma 3.5 below implies that $\mathcal{Z}(P_n)$ is not trivial: it contains the braid automorphism c_{∂_n} . We use the decomposition $P_n \cong F_{n-1} \rtimes P_{n-1}$ to show that this element generates $\mathcal{Z}(P_n)$. Let $w\beta \in \mathcal{Z}(F_{n-1} \rtimes P_{n-1})$. Proposition 3.1 implies that β should act on F_{n-1} via c_w^{-1} . From Lemma 3.5, we deduce that $w \in \langle \partial_{n-1} \rangle$. As a consequence:

$$(3.1) \quad \mathcal{Z}(P_n) \subseteq \langle \partial_{n-1} c_{\partial_{n-1}}^{-1} \rangle.$$

Then, the central element c_{∂_n} must be equal to $\partial_{n-1}^k c_{\partial_{n-1}}^{-k}$ for some k . But since the projection from P_n onto P_{n-1} is induced by $x_n \mapsto 1$, we see that it sends c_{∂_n} to $c_{\partial_{n-1}}$, whence $k = 1$, and (3.1) is an equality. Moreover, we have obtained the induction relation $c_{\partial_n} = \partial_{n-1}^{-1} c_{\partial_{n-1}}$, which implies the relation we wanted for ξ_n , if we define ξ_n to be $c_{\partial_n}^{-1}$.

Finally, we remark that for $n \geq 3$, the center of B_n has to be a subgroup of P_n (whence of $\mathcal{Z}(P_n)$), since its image in Σ_n must be in the center of Σ_n , which is trivial. The other inclusion can be obtained by showing directly that ξ_n commutes with the classical generators of B_n (which is obvious from the geometric picture). \square

An element of the free group is called *primitive* when it is part of a basis of the free group. Recall the following easy result:

LEMMA 3.4. — *In the free group F_n , the centralizer of any primitive element w is the cyclic group generated by w .*

Proof. — Suppose that $(x_1 = w, x_2, \dots, x_n)$ is a basis of F_n . If $g \in F_n$, the relation $x_1 g x_1^{-1} g^{-1} = 1$ cannot hold in the free group if any letter different from x_1 appears in the reduced expression of g in the letters x_i , whence our claim. \square

LEMMA 3.5. — *In $\text{Aut}(F_n)$, the intersection between P_n and $\text{Inn}(F_n)$ is cyclic, generated by c_{∂_n} , which is a central element of P_n .*

Proof. — Let $\beta \in P_n$ be a braid automorphism which is also an inner automorphism c_w for some $w \in F_n$. Then c_w must fix ∂_n , that is, w must commute with ∂_n . However, ∂_n is a primitive element of F_n . Indeed, $(\partial_n, x_2, \dots, x_n)$ is a basis of F_n , the change of basis being given by $\partial_n = x_1 \cdots x_n$ and $x_1 = \partial_n x_n^{-1} \cdots x_2^{-1}$. As a consequence, $w \in \langle \partial_n \rangle$. Thus $P_n \cap \text{Inn}(F_n) \subseteq \langle c_{\partial_n} \rangle$. This inclusion is in fact an equality, since c_{∂_n} is a braid automorphism. Moreover, for any braid automorphism $\beta \in P_n$, we have $\beta c_{\partial_n} \beta^{-1} = c_{\beta(\partial_n)} = c_{\partial_n}$, hence c_{∂_n} is central in P_n . \square

3.3. The center of the Drinfeld–Kohno Lie ring

The above computation of $\mathcal{Z}(P_n)$ can readily be adapted to compute the center of the Lie ring of P_n , which is the Drinfeld–Kohno Lie ring DK_n . We use the decomposition $DK_n \cong \mathfrak{L}_n \rtimes DK_{n-1}$, induced by the decomposition $P_n \cong F_{n-1} \rtimes P_{n-1}$ (see Section 1.5.2). The analogue of Lemma 3.4 holds in the free Lie ring:

LEMMA 3.6. — *In the free Lie ring \mathfrak{L}_n , the centralizer of an element $x \in \mathbb{Z}^n$ (of degree 1) is $\mathbb{Z} \cdot \frac{x}{d}$, where d is the gcd of the coefficients of x .*

Proof. — Consider a basis $(x_1 = x/d, x_2, \dots, x_n)$ of \mathbb{Z}^n . In the tensor algebra $T(\mathbb{Z}^n)$, which is the enveloping ring of \mathfrak{L}_n , the centralizer of dx_1 consists of all polynomials in x_1 only. Among these, the only ones belonging to \mathfrak{L}_n (the only *primitive* ones, in the sense of Hopf algebras) are the linear ones. \square

We also have an analogue of Lemma 3.5 in this context (see Definition 1.23 for the definition of braid-like derivation):

LEMMA 3.7. — *In $\text{Der}(\mathfrak{L}_n)$, the intersection $\text{Der}_t^\partial(\mathfrak{L}_n) \cap \text{Inn}(\mathfrak{L}_n)$ is cyclic, generated by $\text{ad}_{\bar{\partial}_n}$, which is a central element of $\text{Der}_t^\partial(\mathfrak{L}_n)$.*

Proof. — Let $X \in \mathfrak{L}_n$ such that the inner derivation ad_X is braid-like. Then $\text{ad}_X(\bar{\partial}_n) = 0$, that is, X must be in the centralizer of $\bar{\partial}_n$. We deduce from Lemma 3.6 that $X \in \mathbb{Z} \cdot \bar{\partial}_n$. Thus $\text{Der}_t^\partial(\mathfrak{L}_n) \cap \text{ad}(\mathfrak{L}_n) \subseteq \mathbb{Z} \cdot \bar{\partial}_n$. This inclusion is in fact an equality, since $\text{ad}_{\bar{\partial}_n}$ is a braid-like derivation. Moreover, for any braid-like derivation d , we have:

$$[d, \text{ad}_{\bar{\partial}_n}] = d([\bar{\partial}_n, -]) - [\bar{\partial}_n, d(-)] = [d(\bar{\partial}_n), -] = 0,$$

hence $\text{ad}_{\bar{\partial}_n}$ is central in $\text{Der}_t^\partial(\mathfrak{L}_n)$. \square

Finally, we can use Lemmas 3.6 and 3.7 to compute the center of DK_n :

PROPOSITION 3.8. — *The center of DK_n is cyclic, generated by the element $\bar{\xi}_n$ defined by $\bar{\xi}_1 = 0$ and $\bar{\xi}_{n+1} = \bar{\partial}_n + \bar{\xi}_n$. As a braid derivation, $\bar{\xi}_n = -\text{ad}_{\bar{\partial}_n}$.*

Remark 3.9. — Out of the relation $\bar{\xi}_{n+1} = \bar{\partial}_n + \bar{\xi}_n$, we get a formula for this central element:

$$\bar{\xi}_n = \sum_{i < j} t_{ij} \in DK_n.$$

If we compare this formula with that of Remark 3.3, we can see directly that $\bar{\xi}_n$ is the class of the central element ξ_n of P_n .

Proof of Proposition 3.2. — The result is trivial for $n \leq 2$. We thus suppose that $n \geq 3$.

We first remark that the braid-like derivation $\text{ad}_{\bar{\partial}_n}$ of Lemma 3.7 is in fact a braid derivation. Indeed, DK_n contains all braid-like derivations of degree 1 (see Proposition A.1). Alternatively, we can use formula (2.1) to see that $\text{ad}_{\bar{\partial}_n}$ identifies with the class of the braid automorphism c_{∂_n} via the Johnson morphism $\tau : DK_n \hookrightarrow \text{Der}_t^{\partial}(\mathfrak{L}_n)$.

Since DK_n identifies with a Lie subring of $\text{Der}_t^{\partial}(\mathfrak{L}_n)$, Lemma 3.7 implies that $\mathfrak{z}(DK_n)$ is not trivial: it contains the braid derivation $\text{ad}_{\bar{\partial}_n}$. We use the decomposition $DK_n \cong \mathfrak{L}_{n-1} \rtimes DK_{n-1}$ to show that this element generates $\mathfrak{z}(DK_n)$. Let $X + d \in \mathfrak{L}_{n-1} \rtimes DK_{n-1}$. Proposition 3.1 implies that d should act on \mathfrak{L}_{n-1} via $-\text{ad}_X$. From Lemma 3.7, we deduce that $X \in \mathbb{Z} \cdot \bar{\partial}_{n-1}$. As a consequence:

$$(3.2) \quad \mathfrak{z}(DK_n) \subseteq \mathbb{Z} \cdot (\bar{\partial}_{n-1} - \text{ad}_{\bar{\partial}_{n-1}}) \subset \mathfrak{L}_{n-1} \rtimes DK_{n-1}.$$

The central element $\text{ad}_{\bar{\partial}_n}$ must then be equal to $k \cdot (\bar{\partial}_{n-1} - \text{ad}_{\bar{\partial}_{n-1}})$ for some integer k . But since the projection from DK_n onto DK_{n-1} is induced by $X_n \mapsto 0$, we see that it sends $\text{ad}_{\bar{\partial}_n}$ to $\text{ad}_{\bar{\partial}_{n-1}}$, whence $k = -1$, and (3.2) is an equality. Moreover, we have obtained the induction relation $\text{ad}_{\bar{\partial}_n} = -\bar{\partial}_{n-1} + \text{ad}_{\bar{\partial}_{n-1}}$, which implies the relation we wanted for $\bar{\xi}_n$, if we define $\bar{\xi}_n$ to be $-\text{ad}_{\bar{\partial}_n}$. \square

3.4. Inner automorphisms of the pure braid group

Let us denote, as usual, the quotient $P_n/\mathcal{Z}(P_n)$ by P_n^* . In a similar fashion, we denote by DK_n^* the quotient of the Drinfeld–Kohno Lie ring DK_n by its center.

THEOREM 3.10. — *The subgroup $\text{Inn}(P_n)$ of $IA(P_n)$ ($\subset \text{Aut}(P_n)$) satisfies the Andreadakis equality. Equivalently:*

$$\mathcal{L}(P_n^*) \cong DK_n^*.$$

Proof. — The only central elements of DK_n are multiples of $\bar{\xi}_n$, which is the class of the central element $\xi_n \in P_n$ (Proposition 3.8). Thus, our claim is obtained as a direct application of Corollary 2.2 to $G = P_n$. \square

COROLLARY 3.11. — *The Lie algebra DK_n^* , and hence the group P_n^* , are centerless. As a consequence, the subgroup $\text{Inn}(P_n^*)$ of $IA(P_n^*)$ also satisfies the Andreadakis equality.*

Proof. — We use the fact that the only non-trivial central elements of DK_n are in degree one. Let $\bar{x} \in \mathfrak{z}(DK_n^*)$ be the class of $x \in DK_n$. Let y be any element of DK_n . Then the bracket $[x, y]$ must be in $\mathfrak{z}(DK_n)$ (since its class in DK_n^* is trivial). But since its degree is at least 2, it must be trivial. Hence $x \in \mathfrak{z}(DK_n)$, whence $\bar{x} = 0$.

In order to deduce that P_n^* is centerless, we can use the fact that it is residually nilpotent, since P_n is (Lemma B.9), and thus any non-trivial element in its center would give a non-trivial class in its Lie algebra, which must be central too.

The last statement is then a direct application of Corollary 2.2 to $G = P_n^*$. \square

Remark 3.12. — Part of the results of Theorem 3.10 and its corollary can be deduced from the (classical) calculation of the center of P_n and from the classical (non-canonical) splitting $P_n \cong P_n^* \times \mathcal{Z}(P_n)$ recalled in Appendix B (Corollary B.11). In fact, if we apply the Lie functor to this direct product, we get:

$$DK_n = \mathcal{L}(P_n) \cong \mathcal{L}(P_n^*) \times \mathcal{L}(\mathcal{Z}(P_n)) \cong \mathcal{L}(P_n^*) \times \mathbb{Z} \cdot \bar{\xi}_n,$$

where the right factor is the abelian Lie subring generated by the class $\bar{\xi}_n = \sum t_{ij}$ of the generator ξ_n of $\mathcal{Z}(P_n)$. Thus we get the computation $\mathcal{L}(P_n^*) \cong DK_n / \bar{\xi}_n$. Moreover, we see easily that P_n^* is centerless (Proposition B.10). However, neither does this imply that $\bar{\xi}_n$ generates the center of DK_n , nor do we get the statements about Andreadakis equalities without computing this center.

4. Products of subgroups of IA_n

Here we turn to the proof of our key result (Theorem 4.4), which generalises [13, Thm. 4.2]. In order to do this, we study filtrations on products HK , where H and K are subgroups of a given group G , such that K normalizes H . Namely, we investigate the behaviour of the lower central series of HK , and of a filtration $G_* \cap (HK)$ induced by a filtration G_* of G , with respect to the product decomposition of HK .

4.1. Lower central series of products of subgroups

Let G be a group, and let H and K be subgroups of G , such that K normalizes H . Then K acts on H by conjugation in G , we can form the

corresponding semi-direct product $H \rtimes K$ and we get a well-defined morphism $H \rtimes K \rightarrow G$ given by $(h, k) \mapsto hk$. The image of this morphism is the subgroup HK of G . Its kernel, given by the elements $(h, k) \in H \rtimes K$ such that $hk = 1$, is isomorphic to $H \cap K$, via $k \mapsto (k^{-1}, k)$. Thus we get a short exact sequence of groups:

$$H \cap K \hookrightarrow H \rtimes K \twoheadrightarrow HK.$$

The surjection on the right induces a surjection $\Gamma_i(H \rtimes K) \twoheadrightarrow \Gamma_i(HK)$, for all $i \geq 1$. Since $\Gamma_i(H \rtimes K) = \Gamma_i^K(H) \rtimes \Gamma_i(K)$ (see Proposition 1.11), this implies:

PROPOSITION 4.1. — *Let K and H be subgroups of a group G , such that K normalizes H . Then:*

$$\forall i \geq 1, \Gamma_i(HK) = \Gamma_i^K(H) \Gamma_i(K).$$

Moreover, the kernel of $\Gamma_i(H \rtimes K) \twoheadrightarrow \Gamma_i(HK)$ consists of elements $k \in H \cap K$ such that $(k^{-1}, k) \in \Gamma_i(H \rtimes K)$: it is $\Gamma_i^K(H) \cap \Gamma_i(K) \subseteq H \cap K$. Thus we get a short exact sequence of filtrations:

$$\Gamma_*^K(H) \cap \Gamma_*(K) \hookrightarrow \Gamma_*^K(H) \rtimes \Gamma_*(K) \twoheadrightarrow \Gamma_*(HK).$$

4.2. Decomposition of an induced filtration

Let G_* be a filtration, and let H and K be subgroups of $G = G_1$ such that K normalizes H . Then we can consider two filtrations on HK : the induced filtration $G_* \cap (HK)$ and the product of induced filtrations $(G_* \cap H)(G_* \cap K)$. The former obviously contains the latter. We now describe a criterion for this inclusion to be an equality.

PROPOSITION 4.2. — *In the above setting, the following assertions are equivalent to each other:*

- (i) $G_* \cap (HK) = (G_* \cap H)(G_* \cap K)$,
- (ii) Inside $\mathcal{L}(G_*)$, $\mathcal{L}(G_* \cap H) \cap \mathcal{L}(G_* \cap K) = \mathcal{L}(G_* \cap (H \cap K))$.

Remark 4.3. — The case considered in [13, Prop. 4.1] was exactly the case when $H \cap K = 1$. In this context, H and K were called G_* -disjoint when they satisfied the equivalent conditions of the proposition.

Proof of Proposition 4.2. — Suppose that (ii) does not hold. Then there exists an element $x \in \mathcal{L}_i(G_* \cap H) \cap \mathcal{L}_i(G_* \cap K) - \mathcal{L}_i(G_* \cap (H \cap K))$ for some $i \geq 1$. Then $x = \bar{h} = \bar{k}$ for some $h \in G_i \cap H$ and some $k \in G_i \cap K$. Since $\bar{h} = \bar{k}$ in $\mathcal{L}_i(G_*)$, the element $g = h^{-1}k$ is in G_{i+1} . It is also obviously

in HK . However, we claim that $g \notin (G_{i+1} \cap H)(G_{i+1} \cap K)$. Indeed, if we could write g as a product $h'k'$ with $h' \in G_{i+1} \cap H$ and $k' \in G_{i+1} \cap K$, then we would have $h^{-1}k = h'k'$, whence $kk'^{-1} = hh' \in H \cap K$. And by construction, $\overline{hh'} = \bar{h} = x$ in G_i/G_{i+1} . But this would imply that $x \in \mathcal{L}(G_* \cap (H \cap K))$, a contradiction. Thus g must be a counter-example to (i).

Conversely, suppose (i) false. Then there exists $g \in G_j \cap (HK)$, such that for all $(h, k) \in H \rtimes K$ satisfying $g = hk$, neither h nor k belongs to G_j (if h or k belongs to G_j , so does the other one, since their product g does). Then $h \equiv k^{-1} \not\equiv 1 \pmod{G_j}$. For all such (h, k) , there exists $i < j$ such that $h, k \in G_i - G_{i+1}$. Let us take (h, k) such that this index i is maximal. We show that the element $\bar{h} = -\bar{k} \in \mathcal{L}_i(G_*)$ gives a counter-example to the equality (ii). Indeed, it is clear that $\bar{h} = -\bar{k}$ belongs to $\mathcal{L}_i(G_* \cap H) \cap \mathcal{L}_i(G_* \cap K)$. Suppose now that this element would belong to $\mathcal{L}_i(G_* \cap (H \cap K))$. Then there would exist $x \in G_i \cap H \cap K$ such that $\bar{h} = -\bar{k} = \bar{x}$. This means that $\bar{h} - \bar{x} = -(\bar{x} + \bar{k}) = 0$ or, equivalently: $\overline{hx^{-1}} = -\overline{xk} = 0$. But then $g = hk = (hx^{-1})(xk)$, with hx^{-1} and xk in G_{i+1} , contradicting the maximality of i , or the definition of g if $i = j - 1$. Thus \bar{h} cannot belong to $\mathcal{L}_i(G_* \cap (H \cap K))$, whence our claim. \square

4.3. Application to the Andreadakis problem

We are now able to state our key theorem, which is an improvement on [13, Thm. 4.2]. We will apply it to the case when $G = IA_n$ and $G_* = \mathcal{A}_*$ is the Andreadakis filtration, but we still give a general statement.

THEOREM 4.4. — *Let G_* be a filtration, and let H and K be subgroups of $G = G_1$ such that $[K, H] \subseteq [H, H]$. Suppose that in the Lie ring $\mathcal{L}(G_*)$, the intersection of the Lie subrings $\mathcal{L}(G_* \cap H)$ and $\mathcal{L}(G_* \cap K)$ is $\mathcal{L}(G_* \cap (H \cap K))$. Then:*

$$\begin{cases} G_* \cap K = \Gamma_*(K) \\ G_* \cap H = \Gamma_*(H) \end{cases} \quad \Rightarrow \quad G_* \cap (HK) = \Gamma_*(HK).$$

Recall that in the case when $G = IA_n$ and $G_* = \mathcal{A}_*$ is the Andreadakis filtration, the Lie ring $\mathcal{L}(\mathcal{A}_*)$ embeds into the Lie ring $\text{Der}(\mathfrak{L}_n)$ of derivations of the free Lie ring, via the Johnson morphism τ . As a consequence, the hypothesis about Lie subrings of $\mathcal{L}(\mathcal{A}_*)$ can be checked there.

Proof of Theorem 4.4. — Remark that $[K, H] \subseteq [H, H]$ implies in particular that K normalizes H . Thus, we can apply the results of Section 4.1.

Since $[K, H] \subseteq [H, H]$ is exactly the condition needed for $\Gamma_*^K(H)$ to be equal to $\Gamma_*(H)$ (Proposition 1.12), we get:

$$\Gamma_*(HK) = \Gamma_*(H)\Gamma_*(K).$$

Then we use Proposition 4.2 to get:

$$G_* \cap (HK) = (G_* \cap H)(G_* \cap K),$$

whence the result. \square

5. Applications

The subgroup $H = \text{Inn}(F_n) \cong F_n$ of IA_n is an ideal candidate for applying our key result (Theorem 4.4). We begin by spelling out the consequences of Theorem 4.4 in this particular case (Corollary 5.1). It turns out that for each of the subgroups K which were shown to satisfy the Andreadakis equality in [13], we can apply Corollary 5.1 to show that HK does too. Moreover, two of the subgroups so obtained had already been considered in the literature. These are the group of *partial inner automorphisms* of [5], and the subgroup P_{n+1}^* of IA_n presented in the introduction, which was first studied in [21].

5.1. Adding inner automorphisms

We are going to apply Theorem 4.4 with $H = \text{Inn}(F_n)$ and $G = IA_n$ endowed with the Andreadakis filtration $G_* = \mathcal{A}_*$, for three different subgroups K of IA_n . We record in Corollary 5.1 below the consequences of Theorem 4.4 in this context.

COROLLARY 5.1 (to Theorem 4.4). — *Let K be a subgroup of IA_n , and $\tau : \mathcal{L}(K) \rightarrow \text{Der}(\mathfrak{L}_n)$ be the corresponding Johnson morphism. Suppose that K satisfies the Andreadakis equality, which means that τ is injective. Suppose, moreover, that every element of $\tau(\mathcal{L}(K)) \cap \text{ad}(\mathfrak{L}_n)$ comes from an element of $K \cap \text{Inn}(F_n)$ (precisely, it equals $\tau(\bar{x})$, for some $x \in K \cap \text{Inn}(F_n)$). Then $\text{Inn}(F_n)K$ satisfies the Andreadakis equality.*

Proof. — We apply Theorem 4.4 to the subgroups $H = \text{Inn}(F_n)$ and K of $G = IA_n$ endowed with the Andreadakis filtration $G_* = \mathcal{A}_*$.

- The hypothesis $[K, H] \subseteq [H, H]$ comes from the fact that IA_n (whence K) normalizes $\text{Inn}(F_n)$ and acts trivially on F_n^{ab} .

- Corollary 2.4 says that $H = \text{Inn}(F_n)$ satisfies the Andreadakis equality.
- By hypothesis, K does too.

Thus, we are left with proving that the following inclusion (which is true in general) is in fact an equality in $\mathcal{L}(\mathcal{A}_*)$:

$$\mathcal{L}(\mathcal{A}_* \cap \text{Inn}(F_n)) \cap \mathcal{L}(\mathcal{A}_* \cap K) \supseteq \mathcal{L}(\mathcal{A}_* \cap (\text{Inn}(F_n) \cap K)).$$

If we embed $\mathcal{L}(\mathcal{A}_*)$ into $\text{Der}(\mathfrak{L}_n)$ via the Johnson morphism, the Lie ring $\mathcal{L}(\mathcal{A}_* \cap \text{Inn}(F_n))$ identifies with the Lie subring of inner derivations $\text{ad}(\mathfrak{L}_n)$ (see Section 2.2.2). Moreover, since K satisfies the Andreadakis equality, the Lie ring $\mathcal{L}(\mathcal{A}_* \cap K)$ identifies with $\tau(\mathcal{L}(K))$. As for $\mathcal{L}(\mathcal{A}_* \cap (\text{Inn}(F_n) \cap K))$, it is exactly the set of $\tau(\bar{x})$, for $x \in K \cap \text{Inn}(F_n)$, whence the result. \square

5.2. Triangular automorphisms

We recall the definition of the subgroup IA_n^+ of triangular automorphisms [13, Def. 5.1]:

DEFINITION 5.2. — *Fix (x_1, \dots, x_n) an ordered basis of F_n . The subgroup IA_n^+ of IA_n consists of triangular automorphisms, i.e. automorphisms φ acting as:*

$$\varphi : x_i \longmapsto (x_i^{w_i})\gamma_i,$$

where $w_i \in \langle x_j \rangle_{j < i} \cong F_{i-1}$ et $\gamma_i \in \Gamma_2(F_{i-1})$.

Recall that IA_n^+ satisfies the Andreadakis equality. This statement first appeared as [24, Thm. 1], and a shorter proof was given in [13].

THEOREM 5.3. — *The subgroup $\text{Inn}(F_n)IA_n^+$ of IA_n satisfies the Andreadakis equality.*

Proof. — Remark that a triangular automorphism φ has to fix x_1 . As a consequence, the derivation $\tau(\bar{\varphi})$ sends x_1 to $\overline{\varphi(x_1)x_1^{-1}} = 0$. The only inner derivations vanishing on x_1 are the multiples of $\text{ad}(x_1)$ (Lemma 3.6). However, $\text{ad}(x_1)$ is the image by τ of $\overline{c_{x_1}}$, and c_{x_1} is a triangular automorphism. Thus, the hypotheses of Corollary 5.1 are satisfied, whence the desired conclusion. \square

5.3. The triangular McCool group

The triangular McCool group $P\Sigma_n^+$ was first considered in [10], where its Lie algebra was computed. It was shown to satisfy the Andreadakis equality in [13].

The subgroup $\text{Inn}(F_n)P\Sigma_n^+$ of IA_n (or, more precisely, $\text{Inn}(F_n)P\Sigma_n^-$, which is obtained from the definition of $\text{Inn}(F_n)P\Sigma_n^+$ with the opposite order on the generators) is exactly the *partial inner automorphism group* I_n defined and studied in [5]. Indeed, I_n is defined as the subgroup generated by the automorphism c_{ki} for $k \geq i$, defined by:

$$c_{ki} : x_j \mapsto \begin{cases} x_j^{x_i} & \text{if } j \leq k, \\ x_j & \text{else.} \end{cases}$$

Equivalently, it is generated by the $c_{ni} = c_{x_i^{-1}}$ (which generate $\text{Inn}(F_n)$), together with the $c_{ki}c_{k-1,i}^{-1} = \chi_{ki}$ for $k > i$ (which generate $P\Sigma_n^-$).

THEOREM 5.4. — *The subgroup $I_n = \text{Inn}(F_n)P\Sigma_n^+$ of IA_n satisfies the Andreadakis equality.*

Proof. — The proof is exactly the same as the proof of Theorem 5.3, using the fact that $P\Sigma_n^+$ satisfies the Andreadakis equality [13, Cor. 5.5], and remarking that c_{x_1} is not only in IA_n^+ , but in $P\Sigma_n^+$. \square

5.4. The pure mapping class group of the punctured sphere

The Artin action of P_n on F_n is by group automorphisms fixing $\partial_n = x_1 \cdots x_n$. As a consequence, it induces an action of P_n on the quotient $F_n/\partial_n \cong F_{n-1}$ (which is free on x_1, \dots, x_{n-1} since $(x_1, \dots, x_{n-1}, \partial_n)$ is a basis of F_n). This action is not faithful anymore, since the generator c_{∂_n} of $\mathcal{Z}(P_n)$ (see Proposition 3.2) acts trivially modulo ∂_n . However, by a classical theorem of [21], of which we give a simple proof in Appendix B (Section B.1), the kernel is not bigger than $\mathcal{Z}(P_n)$: the above induces a faithful action of $P_n/\mathcal{Z}(P_n) =: P_n^*$ on F_{n-1} .

The latter is exactly the action on the free group of the (pure, based) mapping class group of the punctured sphere presented in the introduction, described in a purely algebraic fashion. Indeed, P_n identifies with the boundary-fixing mapping class group of the n -punctured disc, and the Artin action corresponds to the canonical action on the fundamental group of this space. Then, collapsing the boundary to a point (which we choose

as basepoint) to get an action on the n -punctured sphere corresponds exactly to taking the quotient by $\partial_n = x_1 \cdots x_n$, at the level of fundamental groups.

From the description of Artin's action, we see that automorphisms of F_{n-1} obtained from the above action must send each generator x_i to one of its conjugates and, since $x_n^{-1} \equiv x_1 \cdots x_{n-1} \pmod{\partial_n}$, they must also send the boundary element $\partial_{n-1} = x_1 \cdots x_{n-1}$ to one of its conjugates. This means exactly that this action is *via* elements of $\text{Inn}(F_{n-1})P_{n-1}$:

LEMMA 5.5. — *The subgroup $\text{Inn}(F_n)P_n$ of IA_n is the subgroup of all automorphisms of F_n sending each generator x_i to one of its conjugates, and sending the boundary element $\partial_n = x_1 \cdots x_n$ to one of its conjugates.*

Proof. — The conditions of the lemma obviously describe a subgroup G of $\text{Aut}(F_n)$ (it is an intersection of stabilizers for the action of $\text{Aut}(F_n)$ on the set of conjugacy classes of F_n), and it contains P_n and $\text{Inn}(F_n)$. Now let $\sigma \in G$. Then $\sigma(\partial_n) = \partial_n^w$ for some w , hence $c_w \circ \sigma$ fixes ∂_n (where $c_w : x \mapsto {}^wx$ is the inner automorphism associated to w). As a consequence, $c_w \sigma \in P_n$, whence $\sigma \in \text{Inn}(F_n)P_n$, and our claim. \square

In fact, the proof of Magnus' theorem (Theorem B.1) that we give consists in showing that the above action of P_n on F_{n-1} induces an isomorphism between P_n^* and $\text{Inn}(F_{n-1})P_{n-1}$.

We now give the main application of our key result (Theorem 4.4):

THEOREM 5.6. — *The subgroup $P_{n+1}^* \cong \text{Inn}(F_n)P_n$ of IA_n satisfies the Andreadakis equality.*

Proof. — We apply Corollary 5.1 to the subgroup $K = P_n$ of IA_n . The Andreadakis equality holds for P_n : this is [13, Thm. 6.2], recalled as Theorem 1.21 above. We need to check the other hypothesis. Recall that $\mathcal{L}(P_n)$, which is the Drinfeld–Kohno Lie ring DK_n , is identified to a Lie subring of $\text{Der}_t^\partial(\mathcal{L}_n)$ as in Section 3.3. Moreover, the intersection of $\text{Der}_t^\partial(\mathcal{L}_n)$ with $\text{ad}(\mathcal{L}_n)$ is (linearly) generated by $\text{ad}_{\bar{\partial}_n}$ (Lemma 3.7). Now, $\text{ad}_{\bar{\partial}_n}$ is the image by τ of $\overline{c_{\partial_n}}$, which is a braid automorphism, whence our conclusion. \square

Remark 5.7. — This does not depend of an identification of P_{n+1}^* with $\text{Inn}(F_n)P_n$: one can choose the more algebraic isomorphism of Lemma B.3, or the more geometric (and arguably more interesting) isomorphism given by Magnus' theorem (Theorem B.1). The latter allows us to reformulate our result in more geometric terms: the Johnson kernels associated with the action of P_n^* on the fundamental group of the n -punctured sphere are exactly the terms of the lower central series of P_n^* . In other words, the

Milnor invariants associated with the Magnus action of B_{n+1}^* on F_n distinguish elements of B_{n+1}^* up to elements of the lower central series of P_{n+1}^* , exactly like in the classical case (see Remark 1.22).

Appendix A. Braid and braid-like derivations

Recall (from Section 1.5.2) that the Artin action of P_n on F_n induces an action of $\Gamma_*(P_n)$ on $\Gamma_*(F_n)$ and thus an action of the Drinfeld–Kohno Lie ring $\mathcal{L}(P_n) = DK_n$ on the free Lie ring $\mathcal{L}(F_n) = \mathfrak{L}_n$. Moreover, the Johnson morphism encoding the latter is in fact an injection:

$$\tau : DK_n \hookrightarrow \mathrm{Der}(\mathfrak{L}_n).$$

We identify DK_n with its image in $\mathrm{Der}(\mathfrak{L}_n)$, and its elements are called *braid derivations*. It is easy to see that this image is contained in the Lie subring $\mathrm{Der}_t^\partial(\mathfrak{L}_n)$ of *braid-like derivations* (see Definition 1.23). The goal of the present section is to compare these two Lie subrings of $\mathrm{Der}(\mathfrak{L}_n)$.

Most of the results in this appendix are not new, and are even well-known in the theory of Milnor invariants, with a different point of view ($\mathrm{Der}_t^\partial(\mathfrak{L}_n)$ corresponding to Milnor invariants of string links, whereas DK_n corresponds to Milnor invariants of braids). The only notable exception is Proposition A.4, which estimates the ranks of the graded abelian group $\mathrm{Der}_t^\partial(\mathfrak{L}_n)/DK_n$.

A.1. A calculation of ranks

As graded abelian groups, both DK_n and $\mathrm{Der}_t^\partial(\mathfrak{L}_n)$ are fairly well understood. Since the free Lie ring \mathfrak{L}_n does not have torsion, neither do they. Moreover, we can compute explicitly their ranks in each degree k , from the ranks of the free Lie algebra, denoted by $d(n, k) := \mathrm{rk}_k(\mathfrak{L}_n)$, which are in turn given by Witt’s formula [25, I.4]:

$$(A.1) \quad d(n, k) = \frac{1}{k} \sum_{st=k} \mu(s) n^t,$$

where μ is the usual Möbius function.

In order to compute the ranks of DK_n , consider the decomposition:

$$DK_n \cong \mathfrak{L}_{n-1} \rtimes (\mathfrak{L}_{n-2} \rtimes (\cdots \rtimes \mathfrak{L}_1) \cdots).$$

As an immediate consequence, we get:

$$(A.2) \quad rk_k(DK_n) = \sum_{l=1}^{n-1} d(l, k).$$

We now compute the ranks of $\text{Der}_t^\partial(\mathfrak{L}_n)$. We first recall that given any choice of n elements δ_i of \mathfrak{L}_n , there exists a unique derivation δ of \mathfrak{L}_n sending each generator X_i to δ_i . Moreover, for each X_i , the only elements Y of \mathfrak{L}_n such that $[Y, X_i] = 0$ are integral multiples of X_i (see Lemma 3.6 below). As a consequence, the map sending $(t_i) \in (\mathfrak{L}_n)^n$ to the derivation $\delta : X_i \mapsto [X_i, t_i]$ is well-defined, and its kernel is $\mathbb{Z} \cdot X_1 \times \cdots \times \mathbb{Z} \cdot X_n \cong \mathbb{Z}^n$ (concentrated in degree 1). Moreover, by definition of tangential derivations, its image is $\text{Der}_t(\mathfrak{L}_n)$, whence:

$$rk_k(\text{Der}_t(\mathfrak{L}_n)) = \begin{cases} n(n-1) & \text{if } k = 1, \\ n \cdot d(n, k) & \text{if } k \geq 2. \end{cases}$$

Now, $\text{Der}_t^\partial(\mathfrak{L}_n)$ is the kernel of the linear map (raising the degree by 1) $ev_\partial : \delta \mapsto \delta(\bar{\partial}_n) = \delta(X_1) + \cdots + \delta(X_n)$ from $\text{Der}_t(\mathfrak{L}_n)$ to \mathfrak{L}_n . This map is surjective onto $(\mathfrak{L}_n)_{\geq 2}$ (elements of \mathfrak{L}_n of degree at least 2). Indeed, \mathfrak{L}_n is generated by so-called *linear monomials* (this is easy to prove – see for instance the appendix of [11]), which are of the form $[X_i, t]$ ($t \in \mathfrak{L}_n$), and $[X_i, t]$ is the image by ev_∂ of the derivation δ sending X_i to $[X_i, t]$ and all other X_j to 0. From this, we recover the classical formula (compare [23, Thm. 15]):

$$(A.3) \quad \begin{aligned} rk_k(\text{Der}_t^\partial(\mathfrak{L}_n)) &= rk_k(\text{Der}_t(\mathfrak{L}_n)) - rk_{k+1}(\mathfrak{L}_n) \\ &= \begin{cases} n(n-1) - \frac{n(n-1)}{2} = \frac{n(n-1)}{2} & \text{if } k = 1, \\ n \cdot d(n, k) - d(n, k+1) & \text{if } k \geq 2. \end{cases} \end{aligned}$$

In order to understand the image of DK_n by the Johnson morphism $DK_n \hookrightarrow \text{Der}_t^\partial(\mathfrak{L}_n)$, we now compare formulas (A.2) and (A.3).

A.2. Comparison in degree 1

In degree one, (A.2) and (A.3) give:

$$rk_1(DK_n) = \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2} = rk_1(\text{Der}_t^\partial(\mathfrak{L}_n)).$$

In fact, the Johnson morphism in degree one is given by the explicit morphism:

$$\tau_1 : t_{ij} \mapsto \tau(t_{ij}) : X_l \mapsto \begin{cases} [X_i, X_j] & \text{if } l = i, \\ [X_j, X_i] & \text{if } l = j, \\ 0 & \text{else.} \end{cases}$$

This formula is easily deduced from the fact that the action of DK_n on \mathfrak{L}_n is encoded in $\mathfrak{L}_n \rtimes DK_n \cong DK_{n+1}$ and from the following relations in DK_{n+1} (see Proposition 1.20):

$$[t_{ij}, t_{l,n+1}] = \begin{cases} [t_{i,n+1}, t_{j,n+1}] & \text{if } l = i, \\ [t_{j,n+1}, t_{i,n+1}] & \text{if } l = j, \\ 0 & \text{else.} \end{cases}$$

The morphism τ_1 , which is known to be injective, is easily seen to be surjective. Indeed, if $\delta \in \text{Der}_t^\partial(\mathfrak{L}_n)$ is homogeneous of degree 1, then the equality $\delta(X_1 + \cdots + X_n) = 0$ implies that the coefficient λ_{ij} of $[X_i, X_j]$ in $\delta(X_i)$ equals the coefficient of $[X_j, X_i]$ in $\delta(X_j)$, from which we deduce that $\delta = \sum \lambda_{ij} \tau(t_{ij}) = \tau(\sum \lambda_{ij} t_{ij})$, the sum being taken on (i, j) with $i < j$.

Since DK_n is generated in degree one (Proposition 1.6), we deduce:

PROPOSITION A.1. — *The Drinfeld–Kohno Lie ring DK_n identifies, via the Johnson morphism, with the Lie subring of $\text{Der}_t^\partial(\mathfrak{L}_n)$ generated in degree one.*

A.3. Comparison in degree 2

In degree 2, using Witt's formula (A.1) for $k = 2$ and $k = 3$ (which give $d(n, 2) = n(n-1)/2$ and $d(n, 3) = (n^3 - n)/3$), we find:

$$\text{rk}_2(DK_n) = \frac{n(n-1)(n-2)}{6} = \text{rk}_2\left(\text{Der}_t^\partial(\mathfrak{L}_n)\right).$$

We can in fact write an explicit formula for τ_2 , and show directly that it is an isomorphism:

$$\tau_2 : [t_{ik}, t_{jk}] \mapsto \delta_{ijk} := [\tau(t_{ik}), \tau(t_{jk})] : X_l \mapsto \begin{cases} [X_i, [X_j, X_k]] & \text{if } l = i, \\ [X_j, [X_k, X_i]] & \text{if } l = j, \\ [X_k, [X_i, X_j]] & \text{if } l = k, \\ 0 & \text{else.} \end{cases}$$

Remark that the fact that $\delta_{ijk}(X_1 + \cdots + X_n) = 0$ is exactly the Jacobi identity for X_i , X_j and X_k .

PROPOSITION A.2. — *The Johnson morphism $\tau : DK_n \rightarrow \text{Der}_t^\partial(\mathfrak{L}_n)$ is an isomorphism in degree 2.*

Proof. — We already know that τ_2 is injective : we only need to show that it is surjective. Let $\delta \in \text{Der}_t^\partial(\mathfrak{L}_n)$ be homogeneous of degree 2. Recall that \mathfrak{L}_n is \mathbb{N}^n -graded, since the antisymmetry and Jacobi relation only relate parenthesized monomials with the same letters. We denote by $\deg : \mathfrak{L}_n \rightarrow \mathbb{N}^n$ the corresponding degree, sending X_i to e_i . Since δ is tangential, for all i , $\delta(X_i)$ can be written as a sum of $\lambda_{ij}[X_i, [X_i, X_j]]$ and of $\lambda_{ijk}[X_i, [X_j, X_k]]$ (with $j < k$), for some integral coefficients $\lambda_{ij}, \lambda_{ijk}$. Then, in the decomposition of $\delta(X_1) + \cdots + \delta(X_n)$ into \mathbb{N}^n -homogeneous components, the components are:

$$\begin{cases} \lambda_{ij}[X_i, [X_i, X_j]] & \text{for } i \neq j, \\ \lambda_{ijk}[X_i, [X_j, X_k]] + \lambda_{jik}[X_j, [X_i, X_k]] + \lambda_{kij}[X_k, [X_i, X_j]] & \text{for } i < j < k, \end{cases}$$

respectively of degree $2e_i + e_j$ and $e_i + e_j + e_k$. The first ones are trivial if and only if $\lambda_{ij} = 0$ for all i, j . The second ones are trivial if and only if they are multiples of Jacobi identities, that is, if $\lambda_{ijk} = -\lambda_{jik} = \lambda_{kij}$. Indeed, the component of degree $e_i + e_j + e_k$ of \mathfrak{L}_n is the quotient of the free abelian group generated by $[X_i, [X_j, X_k]]$, $[X_j, [X_i, X_k]]$ and $[X_k, [X_i, X_j]]$ by the Jacobi identity. When these conditions hold, we conclude that $\delta = \sum \lambda_{ijk} \delta_{ijk} \in \text{Im}(\tau_2)$. \square

A.4. Comparison in higher degree

In order to compute ranks in degree 3, we use Witt's formula in degree 4, which gives $d(n, 4) = (n^4 - n^2)/4$. We find:

$$\text{rk}_3(DK_n) = \frac{(n-2)(n-1)n(n+1)}{12},$$

$$\text{rk}_3(\text{Der}_t^\partial(\mathfrak{L}_n)) = \frac{(n-1)n^2(n+1)}{12}.$$

Hence the following:

PROPOSITION A.3. — *The inclusion $DK_n \subset \text{Der}_t^\partial(\mathfrak{L}_n)$ is a strict one: not all braid-like derivations come from braids.*

We can in fact say more:

PROPOSITION A.4. — *The cokernel of the inclusion of graded abelian groups $DK_n \subset \text{Der}_t^\partial(\mathfrak{L}_n)$ is a graded abelian group whose rank in degree $k \geq 3$ is given by a polynomial function of n , whose leading term is $\frac{n^k}{2k}$.*

Proof. — Recall that in degree k , the rank of $\text{Der}_t^\partial(\mathfrak{L}_n)$ is $n \cdot d(n, k) - d(n, k+1)$ (formula (A.3)). Witt's formula (A.1) implies that $d(n, k)$ is a polynomial function of n , whose leading term is n^k/k (since $\mu(1) = 1$). Moreover, the second non-trivial term is in degree k/p , where p is the least prime factor of k , hence its degree is at most $k/2$. As a consequence, $n \cdot d(n, k) - d(n, k+1)$ is a polynomial function of n , with leading term

$$n \cdot \frac{n^k}{k} - \frac{n^{k+1}}{k+1} = \frac{n^{k+1}}{k(k+1)},$$

and no term of degree k .

Now, recall that in degree k , the rank of DK_n is given by formula (A.2):

$$\begin{aligned} \text{rk}_k(DK_n) &= \sum_{l=1}^{n-1} d(l, k) = \sum_{l=1}^{n-1} \frac{1}{k} (l^k - l^{k/p} + \dots) \\ &= \frac{1}{k} \left(\sum_{l=1}^{n-1} l^k - \sum_{l=1}^{n-1} l^{k/p} + \dots \right). \end{aligned}$$

From Faulhaber's formula for sums of powers (recalled below), we know that $S_\alpha(n) = 1^\alpha + 2^\alpha + \dots + n^\alpha$ is a polynomial function of n , whose leading terms are given by:

$$S_\alpha(n) = \frac{n^{\alpha+1}}{\alpha+1} + \frac{n^\alpha}{2} + \dots$$

Thus, the first two terms of $\text{rk}_k(DK_n)$ are those of $\frac{1}{k} S_k(n-1)$, that is:

$$\begin{aligned} \frac{1}{k} S_k(n-1) &= \frac{1}{k} \left(\frac{(n-1)^{k+1}}{k+1} + \frac{(n-1)^k}{2} + \dots \right) \\ &= \frac{n^{k+1}}{k(k+1)} - \frac{n^k}{2k} + \dots \end{aligned}$$

We find, as announced, that the difference $\text{rk}_k(\text{Der}_t^\partial(\mathfrak{L}_n)) - \text{rk}_k(DK_n)$ is a polynomial function of n , whose leading term is $\frac{n^k}{2k}$. \square

We have used in the proof the usual formula (known as Faulhaber's formula) for sums of powers. For any integer $\alpha \geq 0$:

$$S_\alpha(n) := \sum_{l=0}^n l^\alpha = \frac{1}{\alpha+1} \sum_{j=0}^{\alpha} \binom{\alpha+1}{j} B_j \cdot n^{\alpha+1-j},$$

where the B_j are the Bernoulli numbers, defined by:

$$\frac{ze^z}{e^z - 1} = \sum_{j=0}^{\infty} B_j \frac{z^j}{j!}.$$

Recall that the above formula can be obtained by a quite straightforward calculation of the exponential generating series:

$$\sum_{\alpha=0}^{\infty} S_{\alpha}(n) \frac{z^{\alpha}}{\alpha!} = \sum_{l=1}^n e^{lz} = e^z \cdot \frac{e^{nz} - 1}{e^z - 1}.$$

Appendix B. Studying P_n^*

We gather here some classical results about $P_n^* = P_n/\mathcal{Z}(P_n)$, with some new proofs. Our first goal is a classical theorem of Magnus [21] about the faithfulness of its action on the fundamental group of the punctured sphere. We give a complete proof which avoids any difficult calculation with group presentations. This proof is partially built on the sketch of proof of [8, Lem. 3.17.2]. The proof uses the Hopf property in a crucial way, which leads us to recall some basic facts about Hopfian groups. Finally, we describe the splitting $P_n \cong P_n^* \times \mathcal{Z}(P_n)$, which can be used to give an alternative proof of part of the results of Section 3.4 (see in particular Remark 3.12).

B.1. A faithful action of P_n^* on F_{n-1}

Recall that the pure braid group P_n acts faithfully on F_n via the Artin action. Since this action is by group automorphisms fixing $\partial_n = x_1 \cdots x_n$, there is an induced action of P_n on the quotient $F_n/\partial_n \cong F_{n-1}$ (see Section 5.4). We now prove the classical:

THEOREM B.1 ([21, Formula (23)]). — *The Artin action induces a faithful action of $P_n^* = P_n/\mathcal{Z}(P_n)$ on $F_n/\partial_n \cong F_{n-1}$. Moreover, this induces an isomorphism between P_n^* and the subgroup $\text{Inn}(F_{n-1})P_{n-1}$ of $\text{Aut}(F_{n-1})$.*

Remark B.2. — The restriction to pure braids is not an important one: we can deduce readily that the result holds for the action of B_n^* on F_{n-1} , by showing that the kernel of this action is contained in P_n . Indeed, let $\beta \in B_n$ and σ_{β} its image in Σ_n . If β acts trivially on $F_n/\partial_n \cong F_{n-1}$, then it acts trivially on $F_n^{ab}/\overline{\partial_n}$. But the action of B_n on $F_n^{ab} \cong \mathbb{Z}^n$ is via the canonical action of Σ_n on \mathbb{Z}^n , and $\mathbb{Z}^n/(X_1 + \cdots + X_n)$ is a faithful representation of Σ_n , hence $\sigma_{\beta} = 1$.

Our proof will use the following lemma, giving an isomorphism between P_n^* and $\text{Inn}(F_{n-1})P_{n-1}$, which is different from the one of the theorem:

LEMMA B.3. — *The kernel of the morphism $(w, \beta) \mapsto c_w \beta$ from the semi-direct product $F_n \rtimes P_n \cong P_{n+1}$ onto $\text{Inn}(F_n)P_n \subset \text{Aut}(F_n)$ is exactly the center of P_{n+1} . Thus, $\text{Inn}(F_n)P_n \cong P_{n+1}^*$.*

Proof. — The formula does defines a morphism, because of the usual formula $\beta c_w \beta^{-1} = c_{\beta(w)}$. The center of P_{n+1} is generated by $(\partial_{n+1}, c_{\partial_{n+1}}^{-1})$ (Proposition 3.2), which is obviously sent to the identity. Conversely, if (w, β) is such that $c_w = \beta^{-1}$, then $c_w \in P_n \cap \text{Inn}(F_n)$. Lemma 3.5 implies that w must be a power of ∂_{n+1} , whence the result. \square

Proof of Theorem B.1. — Let us denote by $\pi : P_n \twoheadrightarrow P_n^*$ the canonical projection and by $\chi : P_n^* \rightarrow \text{Aut}(F_{n-1})$ the action under scrutiny.

We first show that $\text{Im}(\chi) = \text{Inn}(F_{n-1})P_{n-1}$. From the definition of the action, the characterization of braid automorphisms of F_n and the description of $\text{Inn}(F_{n-1})P_{n-1}$ given in Lemma 5.5, it follows directly that χ takes values in $\text{Inn}(F_{n-1})P_{n-1}$. The map $s : P_{n-1} \hookrightarrow P_n$ extending braid automorphisms by $x_n \mapsto x_n$ is easily seen to satisfy $\chi \pi s(\beta) = \beta$ for all $\beta \in P_{n-1}$, hence $P_{n-1} \subset \text{Im}(\chi)$. Moreover, for all $j \leq n-1$, the inner automorphism $c_{x_1 \cdots x_j}$ is the image by χ of the braid automorphism:

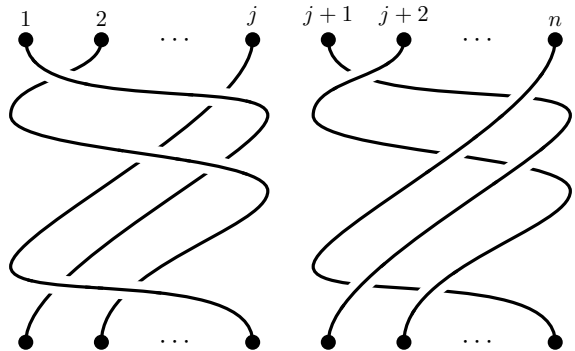
$$C_j : x_t \mapsto \begin{cases} (x_1 \cdots x_j) x_t & \text{if } t \leq j, \\ (x_{j+1} \cdots x_n)^{-1} x_t & \text{if } t > j. \end{cases}$$

Since the $x_1 \cdots x_j$ form a basis of F_{n-1} , we have $\text{Inn}(F_{n-1}) \subset \text{Im}(\chi)$.

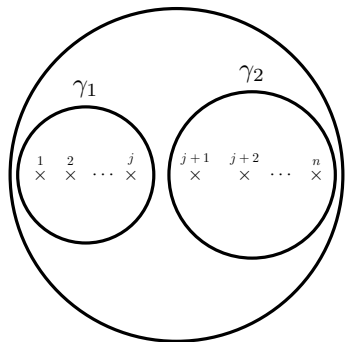
We now show that the action χ is faithful. Lemma B.3 implies that $\text{Inn}(F_{n-1})P_{n-1}$ is isomorphic to P_n^* . Thus χ restricts to a morphism from P_n^* to P_n^* . We have just showed that this endomorphism is surjective. We claim that its surjectivity implies that it is an automorphism of P_n^* . Indeed, since P_n embeds in IA_n (via the Artin action), it is residually nilpotent. Hence, by Lemma B.9, P_n^* is too. It is also finitely generated (by the usual pure braid generators A_{ij}). Thus, Proposition B.7 implies that it is *Hopfian*, which means exactly that surjective endomorphisms of P_n^* are automorphisms. \square

Remark B.4. — The braid automorphism C_j from the proof of Theorem B.1 is easy to understand as a geometric braid. Here is a drawing of

this braid:



Moreover, if we interpret the braid group as the mapping class group of the punctured disc [8, Thm. 1.10], then C_j identifies with the (commutative) product of Dehn twists along γ_1 and γ_2 (left-handed along γ_1 and right-handed along γ_2):



Remark B.5. — Instead of using the Hopf property, one could define explicitly the inverse of $\chi : P_n^* \rightarrow P_n^*$ using the explicit lifts of the generators given in the proof. However, showing that this inverse is well-defined directly from the presentation of P_n^* does involve quite a bit of calculation, which one would need to do in a clever way in order not to get lost. This method would be closer to the original proof of Magnus [21, Formula (23)]. Our method is closer to the sketch of proof of [8, Lem. 3.17.2], but the latter seems to miss the fact that the endomorphism of P_n^* at the end of our proof is a non-trivial one, whose injectivity is not obvious.

Remark B.6 (Braids on the cylinder). — The action of P_n^* on F_n described algebraically in Lemma B.3 also has a geometric interpretation. In fact, it extends to an action of $F_n \rtimes B_n$, which is the group $B_n(\mathbb{D} - pt)$ of

braids with n strands on the punctured disk (or on the cylinder), on F_n , which is the fundamental group of the disk minus n points. Precisely, let $C_n(X)$ denote the unordered configuration space of n points on X , and $\text{Homeo}_*(\mathbb{D})$ denote the group of self-homeomorphisms of \mathbb{D} fixing the base-point pt . The choice of a base configuration $\mathbf{c} = (c_i)$ on the punctured disc $\mathbb{D} - pt$ induces an evaluation map $\text{Homeo}_*(\mathbb{D}) \rightarrow C_n(\mathbb{D} - pt)$. The latter is a locally trivial fibration whose fiber is the subgroup $\text{Homeo}_*(\mathbb{D}, \mathbf{c})$ of homeomorphisms permuting the c_i . Then, using the well-known fact that $SO_2(\mathbb{R})$ is a retract of $\text{Homeo}_*(\mathbb{D})$, and the fact that $\pi_2(C_n(\mathbb{D} - pt)) = 0$ [8, Prop. 1.3] the long exact sequence in homotopy gives the short exact sequence: $0 \rightarrow \mathbb{Z} \rightarrow B_n(\mathbb{D} - pt) \rightarrow \pi_0 \text{Homeo}_*(\mathbb{D}, \mathbf{c}) \rightarrow 0$. The mapping class group $\pi_0 \text{Homeo}_*(\mathbb{D}, \mathbf{c})$ acts transparently on the fundamental group of the disk minus n points, which is F_n . It is not difficult to see that the corresponding action of $B_n(\mathbb{D} - pt) \cong F_n \rtimes B_n$ is the one from Lemma B.3, and that the generator of $\pi_1(SO_2) = \mathbb{Z}$ is sent to the central element of $B_n(\mathbb{D} - pt)$. This action was also considered for instance in [7], and with this interpretation, our Lemma B.3 is equivalent to their Theorem 2.1.

B.2. Residually nilpotent groups

Recall that a group G is called *Hopfian* if every surjective endomorphism of G is an automorphism. The relevance of this property in our context relies on the following:

PROPOSITION B.7. — *Finitely generated residually nilpotent groups are Hopfian.*

Proof. — Let G be a finitely generated residually nilpotent group and let $\pi : G \rightarrow G$ be a surjective endomorphism. The Lie ring $\mathcal{L}(G)$ is generated in degree one. This implies on the one hand, that every $\mathcal{L}_k(G)$ is finitely generated, and on the other hand, that $\mathcal{L}(\pi)$ is surjective ($\mathcal{L}_1(\pi)$ is, being the induced endomorphism of G^{ab}). Because of Lemma B.8, this implies that all the $\mathcal{L}_k(\pi)$ are isomorphisms.

Suppose now that there exists $x \in \ker(\pi)$ such that $x \neq 1$. Since G is residually nilpotent, there exists $k \geq 1$ such that $x \in \Gamma_k G - \Gamma_{k+1} G$. Then \bar{x} is a non-trivial element of $\ker(\mathcal{L}_k(\pi))$, which is impossible. \square

LEMMA B.8. — *Finitely generated abelian groups are Hopfian.*

Proof. — Let A be a finitely generated abelian group and $\pi : A \rightarrow A$ be a surjective endomorphism. If A has no torsion, then the rank of $\ker(\pi)$ must be trivial, hence $\ker(\pi) = 0$, whence the result in this case.

In general, let $\text{Tors}(A)$ denotes the torsion subgroup of A . Then π induces a commutative diagram whose rows are short exact sequences:

$$\begin{array}{ccccc} \text{Tors}(A) & \hookrightarrow & A & \twoheadrightarrow & A/\text{Tors}(A) \\ \downarrow \pi_{\#} & & \downarrow \pi & & \downarrow \bar{\pi} \\ \text{Tors}(A) & \hookrightarrow & A & \twoheadrightarrow & A/\text{Tors}(A). \end{array}$$

Since π is surjective, the induced endomorphism $\bar{\pi}$ of $A/\text{Tors}(A)$ has to be too. But $A/\text{Tors}(A)$ is a finitely generated free abelian group. As a consequence of the first part of the proof, $\bar{\pi}$ has to be an isomorphism. Then, the Snake Lemma implies that the induced endomorphism $\pi_{\#}$ of $\text{Tors}(A)$ is surjective. But $\text{Tors}(A)$ is finite, so $\pi_{\#}$ must be an isomorphism too. We can then apply the Snake Lemma again to conclude that π is injective. \square

LEMMA B.9. — *If G is a residually nilpotent group, then so is $G/\mathcal{Z}(G)$.*

Proof. — Let $x \in G$ such that $\bar{x} \in \Gamma_n(G/\mathcal{Z}(G))$. Then $x \in \mathcal{Z}(G)\Gamma_n(G)$, whence $[G, x] \subseteq \Gamma_{n+1}(G)$. Thus, if $\bar{x} \in \bigcap \Gamma_n(G/\mathcal{Z}(G))$, then $[G, x] \subseteq \bigcap \Gamma_{n+1}(G) = \{1\}$, which means that $x \in \mathcal{Z}(G)$, and $\bar{x} = 1$. \square

B.3. Splitting by the center

We now show that there is a (non-canonical) splitting $P_n \cong P_n^* \times \mathcal{Z}(P_n)$ (Corollary B.11). We recover this classical result as part of the following general:

PROPOSITION B.10. — *For a group G , the following conditions are equivalent to each other:*

- (i) $G/\mathcal{Z}G$ is centerless, and $G \cong \mathcal{Z}G \times (G/\mathcal{Z}G)$ (non-canonically).
- (ii) The canonical projection $p : G \twoheadrightarrow G/\mathcal{Z}G$ splits.
- (iii) The canonical map $\mathcal{Z}G \hookrightarrow G \twoheadrightarrow G^{ab}$ is injective and its image is a direct factor of G^{ab} .

When G is finitely generated, or Hopfian, these are equivalent to:

- (i') There exists an isomorphism $G \cong \mathcal{Z}G \times (G/\mathcal{Z}G)$.

Proof.

(iii) \Rightarrow (ii). — Suppose that the third condition is satisfied. We denote by $\pi : G \twoheadrightarrow G^{ab}$ the canonical projection. Let us identify $\mathcal{Z}G$ with its image in G^{ab} , and let W be a direct complement of $\mathcal{Z}G$ in G^{ab} . Then we claim

that G decomposes as the direct product of its subgroups $\mathcal{Z}G$ and $\pi^{-1}(W)$. Indeed, if $g \in G$, then its class $\pi(g)$ decomposes as $z + w$ with $z \in \mathcal{Z}G$ and $w \in W$. Hence $gz^{-1} \in \pi^{-1}(W)$, which implies that $g \in \mathcal{Z}G \cdot \pi^{-1}(W)$. Moreover, the intersection of the two subgroups is trivial by definition of W , and elements of $\mathcal{Z}G$ commute with elements of $\pi^{-1}(W)$, whence our claim. Then the canonical projection $p : G \twoheadrightarrow G/\mathcal{Z}G$ has to induce an isomorphism $\pi^{-1}(W) \cong G/\mathcal{Z}G$, whose inverse is a section of p .

(ii) \Rightarrow (i). — If p has a section s , then G decomposes as a semi-direct product $\mathcal{Z}G \rtimes (G/\mathcal{Z}G)$, which has to be a direct product, since the conjugation action of G on $\mathcal{Z}G$ is trivial. Moreover, if $z \in \mathcal{Z}(G/\mathcal{Z}G)$, then $s(z)$ is central in $\mathcal{Z}G \times (G/\mathcal{Z}G) \cong G$, thus $z = ps(z)$ is trivial, by definition of p .

(i) \Rightarrow (iii). — Now suppose that $G/\mathcal{Z}G$ is centerless. Then we have that $\mathcal{Z}(\mathcal{Z}G \times (G/\mathcal{Z}G)) = \mathcal{Z}G \times 1$. Recall that the canonical map $\mathcal{Z}G \hookrightarrow G \rightarrow G^{ab}$ is functorial in G . As a consequence, an isomorphism between G and $\mathcal{Z}G \times (G/\mathcal{Z}G)$ has to induce a commutative square:

$$\begin{array}{ccc} \mathcal{Z}G & \longrightarrow & G^{ab} \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{Z}G \times 1 & \hookrightarrow & \mathcal{Z}G \times (G/\mathcal{Z}G)^{ab}, \end{array}$$

whence the third condition.

If G is Hopfian, then any direct factor of G is Hopfian: if $G \cong H \times K$ and u is a surjective endomorphism of H , then $u \times 1$ is a surjective endomorphism of $H \times K \cong G$, thus is an automorphism, and u must be injective too. Thus, if $G \cong \mathcal{Z}G \times (G/\mathcal{Z}G)$, then $\mathcal{Z}G$ is Hopfian. But we also have an induced isomorphism $\mathcal{Z}G \cong \mathcal{Z}G \times \mathcal{Z}(G/\mathcal{Z}G)$. If we compose this isomorphism with the first projection, we get a surjective endomorphism of $\mathcal{Z}G$, which has to be injective, forcing $\mathcal{Z}(G/\mathcal{Z}G)$ to be trivial, whence the result.

If G is finitely generated and $G \cong \mathcal{Z}G \times (G/\mathcal{Z}G)$, then $\mathcal{Z}G$ is a quotient of G , thus it is a finitely generated abelian group, hence Hopfian, and the same reasoning as in the case when G is Hopfian leads to the desired conclusion. \square

COROLLARY B.11. — *There exists an isomorphism $P_n \cong \mathcal{Z}(P_n) \times P_n^*$.*

Proof. — The center $\mathcal{Z}(P_n)$ injects into $P_n^{ab} \cong \mathbb{Z}\{t_{ij}\}_{i < j}$, and its image is $\mathbb{Z} \cdot \sum_{i < j} t_{ij}$, which is a direct factor of P_n^{ab} . \square

Remark B.12. — There is no canonical choice of splitting of $P_n \twoheadrightarrow P_n^*$: the splitting depends on a choice of direct complement W of $\mathcal{Z}(P_n)$ inside

P_n^{ab} . For instance, we can choose $W = W_{kl}$, generated by all the t_{ij} for $(i, j) \neq (k, l)$, so that the corresponding section s_{kl} sends the class of A_{ij} to A_{ij} if $(i, j) \neq (k, l)$. In the literature, authors often choose $s_{1,2}$. Another natural choice is the section corresponding to $W = \{\sum \lambda_{ij} t_{ij} \mid \sum \lambda_{ij} = 0\}$, which sends the class of $A_{ij} A_{kl}^{-1}$ to $A_{ij} A_{kl}^{-1}$, for all i, j, k, l .

BIBLIOGRAPHY

- [1] A. ALEKSEEV & C. TOROSSIAN, “The Kashiwara-Vergne conjecture and Drinfeld’s associators”, *Ann. Math.* **175** (2008), p. 415-463.
- [2] S. ANDREADAKIS, “On the automorphisms of free groups and free nilpotent groups”, *Proc. Lond. Math. Soc.* **15** (1965), p. 239-268.
- [3] E. ARTIN, “Theorie der Zöpfe”, *Abh. Math. Semin. Univ. Hamb.* **4** (1925), no. 1, p. 47-72.
- [4] ———, “Theory of braids”, *Ann. Math.* **48** (1947), p. 101-126.
- [5] V. G. BARDAKOV & M. V. NESHCHADIM, “Subgroups, automorphisms, and Lie algebras related to the basis-conjugating automorphism group”, *Algebra Logika* **55** (2016), no. 6, p. 670-703.
- [6] L. BARTHOLDI, “Automorphisms of free groups. I—erratum [MR3084710]”, *New York J. Math.* **22** (2016), p. 1135-1137.
- [7] P. BELLINGERI & A. BODIN, “The braid group of a necklace”, *Math. Z.* **283** (2016), no. 3-4, p. 995-1010.
- [8] J. S. BIRMAN, *Braids, links, and mapping class groups*, Annals of Mathematics Studies, vol. 82, Princeton University Press; University of Tokyo Press, 1974, ix+228 pages.
- [9] W.-L. CHOW, “On the algebraical braid group”, *Ann. Math.* **49** (1948), p. 654-658.
- [10] F. R. COHEN, J. N. PAKIANATHAN, V. V. VERSHININ & J. WU, “Basis-conjugating automorphisms of a free group and associated Lie algebras”, in *Groups, homotopy and configuration spaces*, Geometry and Topology Monographs, vol. 13, Geometry and Topology Publications, 2008, p. 147-168.
- [11] J. DARNÉ, “Milnor invariants of braids and welded braids up to homotopy”, <https://arxiv.org/abs/1904.10677>, 2019.
- [12] ———, “On the stable Andreadakis problem”, *J. Pure Appl. Algebra* **223** (2019), no. 12, p. 5484-5525.
- [13] ———, “On the Andreadakis problem for subgroups of IA_n ”, *Int. Math. Res. Not.* (2021), no. 19, p. 14720-14742.
- [14] M. FALK & R. RANDELL, “The lower central series of generalized pure braid groups”, in *Geometry and topology (Athens, Ga., 1985)*, Lecture Notes in Pure and Applied Mathematics, vol. 105, Marcel Dekker, 1987, p. 103-108.
- [15] N. HABEGGER & G. MASBAUM, “The Kontsevich integral and Milnor’s invariants”, *Topology* **39** (2000), no. 6, p. 1253-1289.
- [16] A. IBRAHIM, “On a subgroup of McCool group $P\Sigma_n$ ”, <https://arxiv.org/abs/1902.10033>, 2019.
- [17] Y. IHARA, “The Galois representation arising from $\mathbb{P}^1 - \{0, 1, \infty\}$ and Tate twists of even degree”, in *Galois groups over \mathbb{Q}* , 1989.
- [18] L. KALOIJNINE, “Sur quelques propriétés des groupes d’automorphismes d’un groupe abstrait. (Généralisation d’un théorème de M. Ph. Hall)”, *C. R. Acad. Sci. Paris* **231** (1950), p. 400-402.

- [19] T. KOHNO, “Série de Poincaré-Koszul associée aux groupes de tresses pures”, *Invent. Math.* **82** (1985), no. 1, p. 57-75.
- [20] M. LAZARD, “Sur les groupes nilpotents et les anneaux de Lie”, *Ann. Sci. Éc. Norm. Supér.* **71** (1954), p. 101-190.
- [21] W. MAGNUS, “Über Automorphismen von Fundamentalgruppen berandeter Flächen”, *Math. Ann.* **109** (1934), no. 1, p. 617-646.
- [22] J. MOSTOVOY & S. WILLERTON, “Free groups and finite-type invariants of pure braids”, *Math. Proc. Camb. Philos. Soc.* **132** (2002), no. 1, p. 117-130.
- [23] K. E. ORR, “Homotopy invariants of links”, *Invent. Math.* **95** (1989), no. 2, p. 379-394.
- [24] T. SATOH, “On the Andreadakis conjecture restricted to the “lower-triangular” automorphism groups of free groups”, *J. Algebra Appl.* **16** (2017), no. 5, article no. 1750099 (31 pages).
- [25] J.-P. SERRE, *Lie algebras and Lie groups*, Lecture Notes in Mathematics, vol. 1500, Springer, 2006, 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition, viii+168 pages.

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