Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

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Article à paraître, mis en ligne le 3 juillet 2024, 21 p.

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
Article à paraître
Mis en ligne le 3 juillet 2024.

INDUCED DIFFERENTIAL CHARACTERS ON NONLINEAR GRASSMANNIANS

by Tobias DIEZ, Bas JANSSENS, Karl-Hermann NEEB & Cornelia VIZMAN (*)

ABSTRACT. — Using a nonlinear version of the tautological bundle over Graßmannians, we construct a transgression map for differential characters from M to the nonlinear Graßmannian $\mathrm{Gr}^S(M)$ of submanifolds of M of a fixed type S. In particular, we obtain prequantum circle bundles of the nonlinear Graßmannian endowed with the Marsden–Weinstein symplectic form. The associated Kostant–Souriau prequantum extension yields central Lie group extensions of a group of volume-preserving diffeomorphisms integrating Lichnerowicz cocycles.

RÉSUMÉ. — En utilisant une version non-linéaire du fibré tautologique sur les Graßmanniennes, nous construisons une application de transgression pour les caractères différentiels de M à la Graßmannienne non-linéaire $\mathrm{Gr}^S(M)$ des sous-variétés de M d'un type fixé S. En particulier, nous obtenons des fibrés en cercles préquantiques au dessus de la Graßmannienne non-linéaire doté de la forme symplectique de Marsden–Weinstein. L'extension préquantique de Kostant–Souriau associée donne des extensions centrales de groupes de Lie d'un groupe de difféomorphismes préservant le volume et intégrant les cocycles de Lichnerowicz.

1. Introduction

The orbit method provides a powerful framework to construct irreducible unitary representations of an arbitrary Lie group. In its simplest form, the method proceeds in two stages: first, construct an equivariant prequantum line bundle over certain coadjoint orbits and, second, pass to the

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Keywords: Volume-preserving diffeomorphism, Nonlinear Graßmannian, Tautological bundle, Differential character.
2020 *Mathematics Subject Classification:* 58D05, 58D10, 53D50, 53C08, 37K65.
(*) C. V. was supported by a grant of the Romanian Ministry of Education and Research, CNCS-UEFISCDI, project number PN-III-P4-ID-PCE-2020-2888, within PNCDI III. B. J. and T. D. were supported by the NWO grant 639.032.734 "Cohomology and representation theory of infinite dimensional Lie groups". K.-H. N. acknowledges support by DFG-grant NE 413/10-1.

space of sections that are covariantly constant relative to a chosen polarization. Although originally developed in a finite-dimensional setting, the orbit method also has been successfully applied to infinite-dimensional Lie groups [12, 20, 23]. In this paper, we are concerned with the construction of prequantum bundles on a certain class of coadjoint orbits of the infinite-dimensional Lie group of volume-preserving diffeomorphisms.

Let (M, μ) be a compact manifold of dimension $n \geqslant 2$ endowed with a volume form μ. In [9, 11], certain coadjoint orbits of the group of volume-preserving diffeomorphisms $\mathrm{Diff}(M, \mu)$ were described in terms of the nonlinear Graßmannian $\mathrm{Gr}^S(M)$ of all oriented submanifolds of M of type S, where S is a compact manifold of dimension $n-2$. The tautological bundle over this nonlinear Graßmannian is a nonlinear version of the tautological vector bundle over the ordinary linear Graßmannian. It is defined by

$$\mathcal{T} := \big\{(N, x) \in \mathrm{Gr}^S(M) \times M \ : \ x \in N\big\},$$

with bundle projection $q_1 : \mathcal{T} \to \mathrm{Gr}^S(M)$, $q_1(N, x) = N$. We use the tautological bundle to define a transgression of a differential character $h \in \widehat{H}^{n-1}(M, \mathbb{T})$ to a differential character $\widetilde{h} = (q_1)_!(q_2^* h) \in \widehat{H}^1(\mathrm{Gr}^S(M), \mathbb{T})$ on the nonlinear Graßmannian, where $(q_1)_!$ denotes integration along the fibers of the tautological bundle and $q_2 : \mathcal{T} \to M$ is defined by $q_2(N, x) = x$.

Differential characters were introduced by Cheeger and Simons [5], see also [2] for a systematic exposition. Differential characters of degree one classify principal circle bundles with connections through their holonomy maps. The map that associates to a differential character its curvature form is a surjective group homomorphism $\mathrm{curv} : \widehat{H}^k(M, \mathbb{T}) \to \Omega^{k+1}_{\mathbb{Z}}(M)$ onto the group of differential forms with integral periods. In this way, starting with a volume form $\mu \in \Omega^n(M)$ that has integral periods, we get via transgression a differential character of degree one on the nonlinear Graßmannian $\mathrm{Gr}^S(M)$. This yields an isomorphism class of principal circle bundles $\mathcal{P} \to \mathrm{Gr}^S(M)$ equipped with a connection 1-form $\Theta_{\mathcal{P}} \in \Omega^1(\mathcal{P})$ whose curvature is the Marsden–Weinstein symplectic form $\widetilde{\mu}$ induced by μ [19]. That is, $(\mathcal{P}, \Theta_{\mathcal{P}})$ is a prequantization of $(\mathrm{Gr}^S(M), \widetilde{\mu})$.

Theorem A. — *Let M be a compact manifold of dimension n endowed with a volume form μ having integral periods, and let S be a closed, oriented manifold of dimension $n-2$. For every choice of a differential character $h \in \widehat{H}^{n-1}(M, \mathbb{T})$ with curvature μ, the transgression $\widetilde{h} \in \widehat{H}^1(\mathrm{Gr}^S(M), \mathbb{T})$ of h yields an isomorphism class of prequantum bundles of the nonlinear Graßmannian $\mathrm{Gr}^S(M)$ endowed with the Marsden–Weinstein symplectic form $\widetilde{\mu}$.*

Note that a prequantum bundle over the nonlinear Graßmannian with curvature $\widetilde{\mu}$ has been previously constructed in [3] and [9]. These constructions yield a description of the prequantum bundle with connection in terms of local data. An advantage of our approach is the additional control over the holonomy; for a fixed volume form μ, the set of prequantum line bundles obtained with our construction is naturally a torsor over $H^{n-1}(M,\mathbb{T})$.

Associated to the volume form μ, there is a flux homomorphism $\mathrm{Flux}_\mu : \mathrm{Diff}(M,\mu)_0 \to J^{n-1}(M)$ taking values in the Jacobian torus. The kernel $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ of Flux_μ acts on the nonlinear Graßmannian while preserving its connected components. We restrict the above constructed prequantum bundle $\mathcal{P} \to \mathrm{Gr}^S(M)$ to the connected component $\mathrm{Gr}^S_N(M)$ of $N \in \mathrm{Gr}^S(M)$. Following ideas of Ismagilov [11], the pull-back of the prequantum extension by the action of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ yields a central Lie group extension of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ that integrates the Lichnerowicz cocycle $\psi_N(X,Y) = \int_N i_X i_Y \mu$ on the Lie algebra $\mathfrak{X}_{\mathrm{ex}}(M,\mu)$ of exact divergence free vector fields.

Theorem B. — *Let M be a compact manifold of dimension n endowed with a volume form μ having integral periods. For every closed, oriented manifold S of dimension $n-2$ and for every differential character $h \in \widehat{H}^{n-1}(M,\mathbb{T})$ with curvature μ, the 1-dimensional central extension $\widehat{\mathrm{Diff}}_{\mathrm{ex}}(M,\mu)$ of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ obtained by pull-back of the prequantum extension (4.4) is a Fréchet–Lie group that integrates the Lichnerowicz cocycle ψ_N.*

Since $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ acts transitively on connected components of $\mathrm{Gr}^S(M)$ according to [9, Prop. 2], this shows that $\mathrm{Gr}^S_N(M)$ is a coadjoint orbit of $\widehat{\mathrm{Diff}}_{\mathrm{ex}}(M,\mu)$. A similar result for the identity component of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ is obtained in [9, Thm. 2].

In [6] we have used transgression of differential characters from S and M to get differential characters of degree one on the mapping space $C^\infty(S,M)$. The associated central extension of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ integrates the Lichnerowicz cocycle as well. In Section 4.3 we show that the two central extensions of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ constructed using transgression on the connected component of f of the embedding space $\mathrm{Emb}(S,M)$ on the one hand and of the connected component of $f(S)$ of the nonlinear Graßmannian $\mathrm{Gr}^S(M)$ on the other hand are isomorphic as central extensions of Lie groups.

Notation. — We write $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ for the circle group which we identify with $\mathbb{R}/\mathbb{Z}$. Accordingly, we write $\exp_{\mathbb{T}}(t) = e^{2\pi i t}$ for its exponential function.

Acknowledgments

We would like to thank I. Mărcuţ for pointing out the use of tautological bundles over Graßmannians, which is one of the main pillars of this paper.

2. Differential characters

In order to keep the paper self-contained, we give a brief introduction to differential characters, following [5] and [2]. The material in this section is essentially an abridged version of [6, §2 and §3].

2.1. Basics on differential characters

In this section, M denotes a locally convex smooth manifold for which the de Rham isomorphism holds[(1)]. Let $C_k(M)$ be the group of smooth singular k-chains, and let $Z_k(M)$ and $B_k(M)$ denote the subgroups of k-cycles and k-boundaries, so that $H_k(M) := Z_k(M)/B_k(M)$ is the k-th smooth singular homology group.

A *differential character (Cheeger–Simons character) of degree* k is a group homomorphism $h : Z_k(M) \to \mathbb{T}$ for which there exists a differential form $\omega \in \Omega^{k+1}(M)$ such that

$$h(\partial z) = \exp_{\mathbb{T}}\left(\int_z \omega\right)$$

for all $z \in C_{k+1}(M)$. Then ω is uniquely determined by h and is called the *curvature of* h, denoted by $\operatorname{curv}(h)$. We write

$$\widehat{H}^k(M,\mathbb{T}) \subseteq \operatorname{Hom}(Z_k(M),\mathbb{T})$$

for the group of differential characters[(2)] of degree k. The curvature $\omega = \operatorname{curv}(h)$ satisfies $1 = h(\partial z) = \exp_{\mathbb{T}}(\int_z \omega)$ for all $z \in Z_{k+1}(M)$, so it belongs to the abelian group of forms with integral periods

$$\Omega^{k+1}_{\mathbb{Z}}(M) := \left\{\omega \in \Omega^{k+1}(M) \,:\, \int_{Z_{k+1}(M)} \omega \subseteq \mathbb{Z}\right\}.$$

(1) See [14, Thm. 34.7] for a de Rham Theorem in this context and sufficient criteria for it to hold.
(2) In [2] this group is denoted $\widehat{H}^{k+1}(M,\mathbb{Z})$. In this sense our notations are compatible, although the degree is shifted by 1. Our convention follows the original one introduced by Cheeger and Simons in [5].

On the other hand, the identification $H^k(M,\mathbb{T}) \cong \mathrm{Hom}(H_k(M),\mathbb{T})$ yields a natural inclusion $j : H^k(M,\mathbb{T}) \to \widehat{H}^k(M,\mathbb{T})$, whose image is the subgroup of differential characters with zero curvature. We get an exact sequence

$$0 \longrightarrow H^k(M,\mathbb{T}) \xrightarrow{j} \widehat{H}^k(M,\mathbb{T}) \xrightarrow{\mathrm{curv}} \Omega^{k+1}_{\mathbb{Z}}(M) \longrightarrow 0. \tag{2.1}$$

Remark 2.1. — The holonomy map $h_{(P,\theta)}$ of a principal circle bundle $P \to M$ with connection form $\theta \in \Omega^1(P)$ assigns to each piecewise smooth 1-cycle $c \in Z_1(M)$ an element $h_{(P,\theta)}(c)$ in $\mathbb{T}$. If the principal connection has curvature $\omega \in \Omega^2(M)$, then $h_{(P,\theta)}(\partial z) = \exp_{\mathbb{T}}(\int_z \omega)$ for all $z \in C_2(M)$, so that $h_{(P,\theta)} \in \widehat{H}^1(M,\mathbb{T})$ is a differential character with $\mathrm{curv}(h_{(P,\theta)}) = \omega$. The assignment $(P,\theta) \mapsto h_{(P,\theta)}$ defines an isomorphism between the group of isomorphism classes of pairs (P,θ) and the group $\widehat{H}^1(M,\mathbb{T})$ of differential characters of degree 1. To see this, note that principal circle bundles with connection are classified by Deligne cohomology [3, Thm. 2.2.12], which is an alternative model for differential cohomology, cf. [2, Sec. 5.2]. For a direct proof, see also [6, App. B].

2.2. Stabilizer groups and Lie algebras

For any manifold M, the action of the diffeomorphism group $\mathrm{Diff}(M)$ from the right on $\widehat{H}^k(M,\mathbb{T})$ by pull-back [2, Rk. 15],

$$(\varphi^* h)(c) := h(\varphi \circ c) \quad \text{for } c \in Z_k(M), \tag{2.2}$$

extends to the exact sequence (2.1) of abelian groups:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^k(M,\mathbb{T}) & \xrightarrow{j} & \widehat{H}^k(M,\mathbb{T}) & \xrightarrow{\mathrm{curv}} & \Omega^{k+1}_{\mathbb{Z}}(M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \varphi^*} & & \\ 0 & \longrightarrow & H^k(M,\mathbb{T}) & \xrightarrow{j} & \widehat{H}^k(M,\mathbb{T}) & \xrightarrow{\mathrm{curv}} & \Omega^{k+1}_{\mathbb{Z}}(M) & \longrightarrow & 0. \end{array} \tag{2.3}$$

We denote the stabilizer group of the curvature form $\omega \in \Omega^{k+1}_{\mathbb{Z}}(M)$ by

$$\mathrm{Diff}(M,\omega) := \{\varphi \in \mathrm{Diff}(M) \,:\, \varphi^*\omega = \omega\}.$$

The stabilizer group of a differential character $h \in \widehat{H}^k(M,\mathbb{T})$,

$$\mathrm{Diff}(M,h) := \{\varphi \in \mathrm{Diff}(M) \,:\, \varphi^* h = h\},$$

is a subgroup of $\mathrm{Diff}(M,\omega)$ for $\omega = \mathrm{curv}(h)$, by (2.3). If $H_k(M) = \{0\}$, then $H^k(M,\mathbb{T})$ is trivial, and thus $\mathrm{Diff}(M,h) = \mathrm{Diff}(M,\omega)$.

Remark 2.2. — Let $h_{(P,\theta)} \in \widehat{H}^1(M,\mathbb{T})$ be the differential character defined by the holonomy of the principal $\mathbb{T}$-bundle $P \to M$ with connection θ as in Remark 2.1. Then $\varphi \in \mathrm{Diff}(M, h_{(P,\theta)})$ if and only if, for every smooth loop c in M, the holonomy of c coincides with the holonomy of $\varphi \circ c$. Since this is equivalent to the existence of a lift to a connection-preserving automorphism $\widetilde{\varphi} \in \mathrm{Aut}(P,\theta)$ by [22, Thm. 2.7], one can view $\mathrm{Diff}(M, h_{(P,\theta)})$ as the group of *liftable* diffeomorphisms, cf. [13, 26].

Although $\mathrm{Diff}(M,\omega)$ need not be a locally convex Lie group, we can still define its Lie algebra as follows.

DEFINITION 2.3. — *We call a curve* $(\varphi_t)_{t\in[0,1]}$ *in* $\mathrm{Diff}(M)$ smooth *if the map* $[0,1] \times M \to M \times M \colon (t,x) \mapsto (\varphi_t(x), \varphi_t^{-1}(x))$ *is smooth. For a subgroup* $G \subseteq \mathrm{Diff}(M)$*, we denote by* G_0 *the group of diffeomorphisms that are connected to the identity by a piecewise smooth path in* G*. We denote by* $\delta^l\varphi$ *the left logarithmic derivative*

$$\delta^l\varphi_t(x) := \frac{\mathrm{d}}{\mathrm{d}\tau}\Big|_t \varphi_t^{-1}\big(\varphi_\tau(x)\big), \tag{2.4}$$

yielding a curve of vector fields on M*. Then a Lie subalgebra* $\mathfrak{g} \subseteq \mathfrak{X}(M)$ *is the Lie algebra of* $G \subseteq \mathrm{Diff}(M)$ *if for every smooth curve* $(\varphi_t)_{t\in[0,1]}$ *in* $\mathrm{Diff}(M)$ *with* $\varphi_0 = \mathrm{id}_M$*, the curve* $(\varphi_t)_{t\in[0,1]}$ *is contained in* G *if and only if its logarithmic derivative* $(\delta^l\varphi_t)_{t\in[0,1]}$ *is a curve in* $\mathfrak{g}$.

In this sense, the Lie algebra of $\mathrm{Diff}(M,\omega)$ is the stabilizer Lie algebra

$$\mathfrak{X}(M,\omega) := \big\{X \in \mathfrak{X}(M) \ : \ L_X\omega = 0\big\}.$$

2.3. Flux homomorphism

The isotropy group $\mathrm{Diff}(M,h)$ is the kernel of the flux cocycle

$$\mathrm{Flux}_h : \mathrm{Diff}(M,\omega) \to H^k(M,\mathbb{T}), \quad \mathrm{Flux}_h(\varphi) = \varphi^* h - h. \tag{2.5}$$

The restriction of Flux_h to the identity component $\mathrm{Diff}(M,\omega)_0$ takes values in the Jacobian torus $J^k(M) \cong \mathrm{Hom}(H_k(M),\mathbb{R})/\mathrm{Hom}(H_k(M),\mathbb{Z})$:

$$\begin{tikzcd} \mathrm{Diff}(M,\omega)_0 \arrow[r, "\mathrm{Flux}_\omega"] \arrow[d, "\iota"] & J^k(M) \arrow[d, "\exp_{\mathbb{T}}"] \\ \mathrm{Diff}(M,\omega) \arrow[r, "\mathrm{Flux}_h"] & H^k(M,\mathbb{T}). \end{tikzcd} \tag{2.6}$$

We denote this restriction by Flux_ω, since it depends only on $\omega = \mathrm{curv}(h)$. Indeed, we can express Flux_ω as

$$\mathrm{Flux}_\omega(\varphi) = \left[\int_0^1 i_{\delta^l \varphi_t} \omega \, \mathrm{d}t \right], \tag{2.7}$$

where $(\varphi_t)_{t\in[0,1]}$ is any smooth curve in $\mathrm{Diff}(M,\omega)$ with $\varphi_0 = \mathrm{id}_M$ and $\varphi_1 = \varphi$ [1, 4]. To see that the expression in (2.7) is indeed the restriction of Flux_h, note that for all $c \in Z_k(M)$,

$$\begin{aligned} \exp_{\mathbb{T}} \mathrm{Flux}_\omega(\varphi)(c) &= \exp_{\mathbb{T}} \left(\int_c \int_0^1 i_{\delta^l \varphi_t} \omega \, \mathrm{d}t \right) = \exp_{\mathbb{T}} \left(\int_\sigma \omega \right) = h(\partial\sigma) \\ &= h(\varphi \circ c) - h(c) = (\varphi^* h - h)(c) = \mathrm{Flux}_h(\varphi)(c), \end{aligned}$$

where σ is the $(k+1)$-chain swept out by the k-cycle c under the path of diffeomorphisms $\{\varphi_t\}$. The kernel of Flux_ω is the group

$$\mathrm{Diff}_{\mathrm{ex}}(M,\omega) := \mathrm{Diff}(M,h) \cap \mathrm{Diff}(M,\omega)_0, \tag{2.8}$$

which is independent of the choice of h with $\mathrm{curv}(h) = \omega$. The groups $\mathrm{Diff}(M,h)$ and $\mathrm{Diff}_{\mathrm{ex}}(M,\omega)$ have the same Lie algebra

$$\mathfrak{X}_{\mathrm{ex}}(M,\omega) := \left\{ X \in \mathfrak{X}(M,\omega) \ : \ i_X\omega \text{ is exact} \right\}.$$

Example 2.4. — If M is compact and $\omega \in \Omega^{k+1}_{\mathbb{Z}}(M)$, then the following special cases are of particular importance:

(1) For $k = 1$ and ω a symplectic form, $\mathfrak{X}_{\mathrm{ex}}(M,\omega)$ is the Lie algebra $\mathfrak{X}_{\mathrm{ham}}(M,\omega)$ of *Hamiltonian vector fields* and $\mathrm{Diff}_{\mathrm{ex}}(M,\omega)_0$ is the group $\mathrm{Diff}_{\mathrm{ham}}(M,\omega)$ of *Hamiltonian diffeomorphisms*.
(2) For $k = n-1$ and $\omega = \mu$ a volume form, we get the Lie algebra $\mathfrak{X}_{\mathrm{ex}}(M,\mu)$ of *exact divergence free vector fields* and the group $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)_0$ of *exact volume-preserving diffeomorphisms*.

The corresponding groups $\mathrm{Diff}_{\mathrm{ham}}(M,\omega)$ and $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)_0$ are Fréchet–Lie groups [14, Thm. 43.7, 43.12]. In both cases mentioned above, the same holds for the possibly non-connected groups $\mathrm{Diff}(M,h)$ and $\mathrm{Diff}_{\mathrm{ex}}(M,\omega)$ with $\mathrm{curv}(h) = \omega$ for $h \in \widehat{H}^k(M,\mathbb{T})$, cf. [6, Prop. 3.8].

3. Tautological bundle over nonlinear Graßmannians

3.1. Transgression of differential forms

Let M be a finite dimensional manifold, and let S be a compact oriented k–dimensional manifold. The nonlinear Graßmannian $\mathrm{Gr}^S(M)$ of all compact, oriented, k–dimensional submanifolds of M of type S is a Fréchet manifold, cf. [14, Thm. 44.1]. The tangent space of $\mathrm{Gr}^S(M)$ at a submanifold N can be identified with the space of smooth sections of the normal bundle $TN^\perp = (TM|_N)/TN$. The natural surjection

$$\pi : \mathrm{Emb}(S, M) \to \mathrm{Gr}^S(M), \quad \pi(f) = f(S), \tag{3.1}$$

where the orientation on the submanifold $f(S)$ is chosen such that the diffeomorphism $f : S \to f(S)$ is orientation-preserving, defines a principal bundle $\mathrm{Emb}(S, M) \to \mathrm{Gr}^S(M)$ with structure group $\mathrm{Diff}_+(S)$, the group of orientation-preserving diffeomorphisms of S, cf. [14, Thm. 44.1].

The *transgression*, or *tilda map*, [9] associates to any n–form ω on M an $(n-k)$–form $\widetilde{\omega}$ on $\mathrm{Gr}^S(M)$ by

$$\widetilde{\omega}_N(\widetilde{Y}_1, \dots, \widetilde{Y}_{n-k}) := \int_N \iota_N^*(i_{Y_{n-k}} \cdots i_{Y_1}\omega), \tag{3.2}$$

where $\iota_N : N \hookrightarrow M$ is the inclusion. Here $\widetilde{Y}_j$ are tangent vectors at $N \in \mathrm{Gr}^S(M)$, i.e. sections of $TN^\perp$, represented by sections Y_j of $TM|_N$. Moreover, $\iota_N^*(i_{Y_{n-k}} \cdots i_{Y_1}\omega) \in \Omega^k(N)$ is defined by

$$\begin{aligned} &\big(\iota_N^*(i_{Y_{n-k}} \cdots i_{Y_1}\omega)\big)_x(X_1, \dots, X_k) \\ &\qquad = \omega_{\iota_N(x)}\big(Y_1(x), \dots, Y_{n-k}(x), \mathrm{T}_x\iota_N(X_1), \dots, \mathrm{T}_x\iota_N(X_k)\big) \end{aligned} \tag{3.3}$$

for $x \in N$ and $X_i \in T_xN$, so it does not depend on the representatives Y_j of $\widetilde{Y}_j$. Finally, integration in (3.2) is well defined since $N \in \mathrm{Gr}^S(M)$ comes with an orientation.

The natural action of the group $\mathrm{Diff}(M)$ on the nonlinear Graßmannian $\mathrm{Gr}^S(M)$ is given by $\varphi \cdot N = \varphi(N)$. With the notations $\widetilde{\varphi}$ for the diffeomorphism of $\mathrm{Gr}^S(M)$ induced by the action of $\varphi \in \mathrm{Diff}(M)$ on $\mathrm{Gr}^S(M)$, and $\widetilde{X}$ for the infinitesimal action of $X \in \mathfrak{X}(M)$, the following functorial identities hold:

$$\widetilde{\varphi}^*\widetilde{\omega} = \widetilde{\varphi^*\omega}, \qquad L_{\widetilde{X}}\widetilde{\omega} = \widetilde{L_X\omega}, \qquad i_{\widetilde{X}}\widetilde{\omega} = \widetilde{i_X\omega}, \qquad \mathtt{d}\widetilde{\omega} = \widetilde{\mathtt{d}\omega}. \tag{3.4}$$

Similarly, S being oriented, the *hat map* [29] associates to any form $\omega \in \Omega^n(M)$ the form $\widehat{\omega} \in \Omega^{n-k}(\mathrm{Emb}(S, M))$ defined by

$$\widehat{\omega}(Z_1, \dots, Z_{n-k}) := \int_S f^*(i_{Z_{n-k}} \dots i_{Z_1}\omega), \tag{3.5}$$

with $Z_j \in T_f \mathrm{Emb}(S, M) = \Gamma(f^*TM)$. It is easy to check that the hat map on $\mathrm{Emb}(S, M)$ and the tilda map on $\mathrm{Gr}^S(M)$ are related by

$$\widehat{\omega} = \pi^*\widetilde{\omega} \tag{3.6}$$

for every $\omega \in \Omega^n(M)$.

3.2. Tautological bundle

A nonlinear version of the tautological bundle over the usual Graßmannian is the associated bundle $\mathcal{T} = \mathrm{Emb}(S, M) \times_{\mathrm{Diff}_+(S)} S$ over the nonlinear Graßmannian $\mathrm{Gr}^S(M)$, a smooth bundle with typical fiber S. The tautological bundle can also be expressed as

$$\mathcal{T} = \big\{(N, x) \in \mathrm{Gr}^S(M) \times M \,:\, x \in N\big\},$$

with bundle projection $q_1 : \mathcal{T} \to \mathrm{Gr}^S(M)$, $q_1(N, x) = N$. From this point of view, the quotient map $\Pi : \mathrm{Emb}(S, M) \times S \to \mathcal{T}$, an S-bundle morphism over $\pi : \mathrm{Emb}(S, M) \to \mathrm{Gr}^S(M)$, becomes $\Pi(f, s) = \big(f(S), f(s)\big)$, and the projection $q_2 : \mathcal{T} \to M$ defined by $q_2(N, x) = x$ satisfies $q_2 \circ \Pi = \mathrm{ev}$. Thus, the following diagrams commute:

$$\begin{tikzcd} \mathrm{Emb}(S,M) \times S \arrow[r, "\Pi"] \arrow[d, "p_1"] & \mathcal{T} \arrow[d, "q_1"] \\ \mathrm{Emb}(S,M) \arrow[r, "\pi"] & \mathrm{Gr}^S(M), \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Emb}(S,M) \times S \arrow[r, "\Pi"] \arrow[dr, "\mathrm{ev}"'] & \mathcal{T} \arrow[d, "q_2"] \\ & M. \end{tikzcd} \tag{3.7}$$

The transgression (3.2) of a differential form $\omega \in \Omega^n(M)$ to the nonlinear Graßmannian $\mathrm{Gr}^S(M)$ can be expressed with the help of the tautological bundle over the nonlinear Graßmannian as

$$\widetilde{\omega} = (q_1)_!(q_2^*\omega) \in \Omega^{n-k}(\mathrm{Gr}^S(M)), \tag{3.8}$$

where $(q_1)_!$ denotes integration along the fibers of the tautological bundle $q_1 : \mathcal{T} \to \mathrm{Gr}^S(M)$. Indeed, since π is a submersion, this relation follows from

$$\pi^*\big((q_1)_!(q_2^*\omega)\big) = (p_1)_!(\Pi^* q_2^*\omega) \overset{(3.7)}{=} (p_1)_!(\mathrm{ev}^*\,\omega) = \widehat{\omega} \overset{(3.6)}{=} \pi^*\widetilde{\omega},$$

using that fiber integration commutes with pull-back [8].

In a similar spirit, tautological bundles over manifolds of nonlinear flags in M, i.e. nested sets of submanifolds of M, have been used in [10] to handle the transgression of differential forms on M to differential forms on the manifold of nonlinear flags.

3.3. Transgression of differential characters

The pull-back and the fiber integration make sense also for differential characters [2, Ch. 7]. This allows us to define the *transgression of a differential character* $h \in \widehat{H}^{n-1}(M, \mathbb{T})$ to the nonlinear Graßmannian $\mathrm{Gr}^S(M)$ with the help of the tautological bundle $\mathcal{T}$, in the same way as in formula (3.8):

$$\widetilde{h} = (q_1)_!(q_2^*h) \in \widehat{H}^{n-k-1}(\mathrm{Gr}^S(M), \mathbb{T}). \tag{3.9}$$

The transgression map for differential characters

$$\widehat{H}^{n-1}(M, \mathbb{T}) \longrightarrow \widehat{H}^{n-k-1}(\mathrm{Gr}^S(M), \mathbb{T}) \tag{3.10}$$

has functorial properties that we describe below.

Proposition 3.1. — *The transgression map for differential characters in* (3.10) *makes the following diagram commutative:*

$$\begin{tikzcd}
H^{n-1}(M,\mathbb{T}) \arrow[r, "j"] \arrow[d, "\sim"] & \widehat{H}^{n-1}(M,\mathbb{T}) \arrow[r, "\mathrm{curv}"] \arrow[d, "\sim"] & \Omega^n_{\mathbb{Z}}(M) \arrow[d, "\sim"] \\
H^{n-k-1}(\mathrm{Gr}^S(M),\mathbb{T}) \arrow[r, "j"] & \widehat{H}^{n-k-1}(\mathrm{Gr}^S(M),\mathbb{T}) \arrow[r, "\mathrm{curv}"] & \Omega^{n-k}_{\mathbb{Z}}(\mathrm{Gr}^S(M)),
\end{tikzcd}$$

where the transgression $\widetilde{a} \in H^{n-k-1}(\mathrm{Gr}^S(M), \mathbb{T})$ *of* $a \in H^{n-1}(M, \mathbb{T})$ *is defined by* $\widetilde{a} = (q_1)_!(q_2^*a)$*. In particular,* $\mathrm{curv}(\widetilde{h}) = \widetilde{\mathrm{curv}(h)}$*.*

Proof. — Using the compatibility of both the pull-back and the fiber integration of differential characters with the curvature explained in [2, Rk. 15, Def. 38], we compute

$$\mathrm{curv}(\widetilde{h}) = \mathrm{curv}\big((q_1)_!(q_2^*h)\big) = (q_1)_!\,\mathrm{curv}(q_2^*h) = (q_1)_!q_2^*\,\mathrm{curv}(h) = \widetilde{\mathrm{curv}(h)}.$$

This shows the commutativity of the right-hand side of the diagram.

To prove the commutativity of the left-hand side, we use [2, Prop. 48]:

$$\widetilde{j(a)} = (q_1)_!\big(q_2^*j(a)\big) = (q_1)_!\big(j(q_2^*a)\big) = j\big((q_1)_!(q_2^*a)\big) = j(\widetilde{a}),$$

for all $a \in H^{n-1}(M, \mathbb{T})$. □

Corollary 3.2. — *The transgression map* $\Omega^n(M) \to \Omega^{n-k}(\mathrm{Gr}^S(M))$*,* $\omega \mapsto \widetilde{\omega}$*, preserves the integrality of differential forms.*

Proof. — Every integral form $\omega \in \Omega^n_{\mathbb{Z}}(M)$ is the curvature of a character $h \in \widehat{H}^{n-1}(M, \mathbb{T})$. By Proposition 3.1, its transgression $\widetilde{\omega} \in \Omega^{n-k}(\mathrm{Gr}^S(M))$ is the curvature of the transgressed character $\widetilde{h} \in \widehat{H}^{n-k-1}(\mathrm{Gr}^S(M), \mathbb{T})$, hence an integral form. □

Consider the natural action of $\mathrm{Diff}(M)$ on $\mathcal{T}$ defined by assigning to every diffeomorphism $\varphi \in \mathrm{Diff}(M)$ the diffeomorphism $\varphi_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ given by $\varphi_{\mathcal{T}}(N,x) = \big(\varphi(N),\varphi(x)\big)$. The following diagram commutes:

$$
\begin{tikzcd}
\mathrm{Gr}^S(M) \arrow[d, "\widetilde{\varphi}"] & \mathcal{T} \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, "\varphi_{\mathcal{T}}"] & M \arrow[d, "\varphi"] \\
\mathrm{Gr}^S(M) & \mathcal{T} \arrow[l, "q_1"'] \arrow[r, "q_2"] & M.
\end{tikzcd}
\tag{3.11}
$$

Proposition 3.3. — *The transgression map for differential characters defined in* (3.10) *is compatible with the action of* $\mathrm{Diff}(M)$, *that is,* $\widetilde{\varphi^* h} = \widetilde{\varphi}^*\widetilde{h}$ *holds for every* $h \in \widehat{H}^{n-1}(M,\mathbb{T})$ *and* $\varphi \in \mathrm{Diff}(M)$.

Proof. — The claim follows from the direct calculation

$$\widetilde{\varphi^* h} = (q_1)_!(q_2^*\varphi^* h) = (q_1)_!(\varphi_{\mathcal{T}}^* q_2^* h) = \widetilde{\varphi}^*\big((q_1)_!(q_2^* h)\big) = \widetilde{\varphi}^*\widetilde{h},$$

by [2, Def. 38] and (3.11). □

The case of a volume form $\omega \in \Omega^n(M)$ and $k = n-2$, with $\widetilde{\omega} \in \Omega^2(\mathrm{Gr}^S(M))$, has been considered in [9, Thm. 1], where a principal circle bundle $(\mathcal{P},\theta)$ over $\mathrm{Gr}^S(M)$ with curvature $\widetilde{\omega}$ has been constructed through its Čech 1-cocycle. In our setting, we get such a prequantum bundle over $\mathrm{Gr}^S(M)$ using the transgression $\widetilde{h} \in \widehat{H}^1(\mathrm{Gr}^S(M),\mathbb{T})$ of a differential character $h \in \widehat{H}^{n-1}(M,\mathbb{T})$ with curvature ω(3). This is described in the next theorem, a direct consequence of Proposition 3.1 and Remark 2.1.

Theorem C. — *Let M be a compact manifold of dimension n endowed with a volume form μ having integral periods, and let S be a closed, oriented manifold of dimension $n-2$. For every choice of a differential character $h \in \widehat{H}^{n-1}(M,\mathbb{T})$ with curvature μ, the transgression $\widetilde{h} \in \widehat{H}^1(\mathrm{Gr}^S(M),\mathbb{T})$ of h yields an isomorphism class of prequantum bundles of the nonlinear Graßmannian $\mathrm{Gr}^S(M)$ endowed with the Marsden–Weinstein symplectic form $\widetilde{\mu}$.*

A *hat product* of differential characters has been introduced in [6, §4] yielding the transgression of a pair of differential characters from S and from M to a differential character on $C^\infty(S,M)$. We specialize it here to a *hat map* that assigns to every $h \in \widehat{H}^{n-1}(M,\mathbb{T})$ the differential character

$$\widehat{h} := (p_1)_!(\mathrm{ev}^*\, h) \in \widehat{H}^{n-k-1}(\mathrm{Emb}(S,M),\mathbb{T}), \tag{3.12}$$

where $p_1 : \mathrm{Emb}(S,M)\times S \to \mathrm{Emb}(S,M)$ is the natural projection.

(3) We do not know whether the holonomy $h_{(\mathcal{P},\theta)} \in \widehat{H}^1(\mathrm{Gr}^S(M),\mathbb{T})$ of the principal bundle constructed in [9] coincides with the transgression of a differential character $h \in \widehat{H}^{n-1}(M,\mathbb{T})$. The differential character $h_{(\mathcal{P},\theta)}$ may differ from $\widetilde{h} \in \widehat{H}^1(\mathrm{Gr}^S(M),\mathbb{T})$ by an element in $H^1(\mathrm{Gr}^S(M),\mathbb{T})$.

Proposition 3.4. — *Given a differential character h on M, the differential character $\widehat{h}$ on $\operatorname{Emb}(S,M)$ is $\operatorname{Diff}_+(S)$ invariant and coincides with the pull-back $\pi^*\widetilde{h}$ of the differential character $\widetilde{h}$ on $\operatorname{Gr}^S(M)$ under the map $\pi : \operatorname{Emb}(S,M) \to \operatorname{Gr}^S(M)$ from (3.1).*

Proof. — By (3.7), the transgression diagrams for $\operatorname{Emb}(S,M)$ and for $\operatorname{Gr}^S(M)$ are connected by the projection $\pi\colon \operatorname{Emb}(S,M) \to \operatorname{Gr}^S(M)$, yielding the commutative diagram

$$\begin{tikzcd}
\operatorname{Emb}(S,M)\times S \arrow[rr, two heads, "\Pi"] \arrow[dr, "\mathrm{ev}"] \arrow[dd, "p_1"'] & & \mathcal{T};\text{"}b\text{"};\text{"}d\text{"} \arrow[d, "q_1"'] \arrow[dr, "q_2"] & \\
& M \arrow[r, equal] & {} \arrow[r, equal] & M \\
\operatorname{Emb}(S,M) \arrow[rr, two heads, "\pi"] & & \operatorname{Gr}^S(M). &
\end{tikzcd}$$

Accordingly, we have

$$\pi^*\widetilde{h} = \pi^*(q_1)_!(q_2^*h) = (p_1)_!(\Pi^* q_2^* h) = (p_1)_!(\mathrm{ev}^*\, h) = \widehat{h}, \tag{3.13}$$

where we used the naturality of fiber integration [2, Def. 38]. □

Note that the $\operatorname{Diff}_+(S)$-invariance of the differential character $\widehat{h}$ is not enough to conclude that $\widehat{h}$ descends to a differential character $\widetilde{h}$ on $\operatorname{Gr}^S(M)$. In fact, we are not aware of a direct proof that $\widehat{h}$ descends without using transgression to $\operatorname{Gr}^S(M)$ as defined in (3.10). Even for differential forms, invariance is not enough to conclude that they descend to the base (they have to be basic!).

4. Integration of Lichnerowicz cocycles

Let M be a closed, connected manifold of dimension $n \geqslant 2$, and let $\mu \in \Omega^n_{\mathbb{Z}}(M)$ be an integral volume form. Then each oriented codimension two submanifold $N \subset M$ determines a 2-cocycle on the Lie algebra $\mathfrak{X}_{\mathrm{ex}}(M,\mu)$ of exact divergence free vector fields by

$$\psi_N(X,Y) := \int_N i_X i_Y \mu. \tag{4.1}$$

If $[\eta] \in H^2_{\mathrm{dR}}(M)$ is Poincaré dual to $[N] \in H_{n-2}(M)$, then (4.1) is cohomologous to the *Lichnerowicz cocycle* [18] $\psi_\eta(X,Y) := \int_M \eta(X,Y)\,\mu$, cf. [28].

In [6], we used transgression of differential characters over mapping spaces to integrate these Lie algebra cocycles to smooth central extensions of the Lie group $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$. In this section, we apply ideas from [9, 11] to show that the same can be done using transgression over nonlinear Graßmannians, and we indicate the relation between these two methods.

4.1. Construction using the nonlinear Graßmannian $\mathrm{Gr}^S(M)$

First we turn to the use of transgression over nonlinear Graßmannians. Here S is a closed, oriented manifold of dimension $n-2$. Let $h \in \widehat{H}^{n-1}(M,\mathbb{T})$ be a differential character on M with curvature the integral volume form μ, i.e. a group homomorphism $h : Z_{n-1}(M) \to \mathbb{T}$ that assigns to each boundary its enclosed volume modulo $\mathbb{Z}$. Using the transgression diagram

$$\begin{tikzcd} & \mathcal{T} \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ \mathrm{Gr}^S(M) & & M, \end{tikzcd} \tag{4.2}$$

we saw that h yields $\widetilde{h} := (q_1)_!(q_2^* h)$ in $\widehat{H}^1(\mathrm{Gr}^S(M),\mathbb{T})$, a differential character with curvature $\widetilde{\mu} := (q_1)_!(q_2^*\mu)$ in $\Omega^2_{\mathbb{Z}}(\mathrm{Gr}^S(M))$. Since differential characters of degree 1 correspond to isomorphism classes of principal circle bundles with connection (see Remark 2.1), this yields an isomorphism class of principal circle bundles $\mathcal{P} \to \mathrm{Gr}^S(M)$ equipped with a connection 1-form $\Theta_{\mathcal{P}} \in \Omega^1(\mathcal{P})$ whose curvature is $\widetilde{\mu}$. In fact, the closed 2-form $\widetilde{\mu}$ is symplectic [9, 11, 19], so that $\mathcal{P} \to \mathrm{Gr}^S(M)$ is a prequantum circle bundle. As any two differential characters h and h' with curvature μ differ by an element of $H^{n-1}(M,\mathbb{T})$, we obtain a distinguished class of "transgressed" prequantum bundles of $(\mathrm{Gr}^S(M),\widetilde{\mu})$, forming a torsor over $H^{n-1}(M,\mathbb{T})$.

Let $\mathrm{Gr}^S_N(M)$ be the connected component of $N \in \mathrm{Gr}^S(M)$ and let $\mathcal{P}_N$ denote the restriction of $\mathcal{P}$ to $\mathrm{Gr}^S_N(M)$. The quantomorphism group $\mathrm{Aut}\big(\mathcal{P}_N,\Theta_{\mathcal{P}}\big)$ is then a central extension

$$\mathbb{T} \to \mathrm{Aut}\big(\mathcal{P}_N,\Theta_{\mathcal{P}}\big) \to \mathrm{Diff}\big(\mathrm{Gr}^S_N(M),\widetilde{h}\big) \tag{4.3}$$

of the group $\mathrm{Diff}\big(\mathrm{Gr}^S_N(M),\widetilde{h}\big)$ of holonomy-preserving diffeomorphisms by the circle group $\mathbb{T}$, cf. [13, 26], and [22] for the infinite-dimensional case.

The group $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ in Example 2.4 (with identity component the group of exact volume-preserving diffeomorphisms) might be non-connected. Being a subgroup of $\mathrm{Diff}(M,\mu)_0$, by continuity, the natural action σ of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu) \subseteq \mathrm{Diff}(M)_0$ leaves the connected component $\mathrm{Gr}^S_N(M)$ invariant. Moreover, Proposition 3.3 implies that $\mathrm{Diff}_{\mathrm{ex}}(M,\mu) \subseteq \mathrm{Diff}(M,h)$

preserves $\widetilde{h}$. Thus, a central group extension of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ can be obtained by pull-back of the prequantization central extension (4.3) by the action σ,

$$
\begin{tikzcd}
1 \arrow[r] & \mathbb{T} \arrow[r] & \mathrm{Aut}\big(\mathcal{P}_N,\Theta_{\mathcal{P}}\big) \arrow[r] & \mathrm{Diff}\big(\mathrm{Gr}_N^S(M),\widetilde{h}\big) \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{T} \arrow[r] \arrow[u] & \widehat{\mathrm{Diff}}_{\mathrm{ex}}(M,\mu) \arrow[r] \arrow[u] & \mathrm{Diff}_{\mathrm{ex}}(M,\mu) \arrow[r] \arrow[u, "\sigma"] & 1.
\end{tikzcd}
\tag{4.4}
$$

The group $\mathrm{Aut}\big(\mathcal{P}_N,\Theta_{\mathcal{P}}\big)$ need not be a locally convex Lie group. But it follows from the generalization of [22, Thm. 3.4] to non-connected Lie groups given in [6, Thm. A.1] that the pull-back $\widehat{\mathrm{Diff}}_{\mathrm{ex}}(M,\mu)$ is a Lie group, with the manifold structure coming from the pull-back of $\mathcal{P}_N \to \mathrm{Gr}_N^S(M)$ along the orbit map $\mathrm{Diff}_{\mathrm{ex}}(M,\mu) \to \mathrm{Gr}_N^S(M)$, cf. [9, Rk. 4].

Although (4.3) is not a central extension of locally convex Lie groups, its Lie algebra extension in the sense of Definition 2.3 is the one with the Kostant–Souriau cocycle

$$\psi_{KS}(\widetilde{Y}_1,\widetilde{Y}_2) = \widetilde{\mu}_N(\widetilde{Y}_1,\widetilde{Y}_2), \quad \widetilde{Y}_1,\widetilde{Y}_2 \in T_N\,\mathrm{Gr}_N^S(M) = \Gamma(TN^{\perp}).$$

The central Lie algebra extension corresponding to the Lie group extension $\widehat{\mathrm{Diff}}_{\mathrm{ex}}(M,\mu) \to \mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ in (4.4) is the pull-back along the infinitesimal action $\sigma_*\colon \mathfrak{X}_{\mathrm{ex}}(M,\mu) \to \mathfrak{X}(\mathrm{Gr}_N^S(M),\widetilde{\mu})$, $X \mapsto \widetilde{X}$ of the Kostant–Souriau cocycle,

$$\psi_{KS}(\sigma_*X,\sigma_*Y) = \widetilde{\mu}_N(\widetilde{X},\widetilde{Y}) \overset{(3.4)}{=} \widetilde{i_Y i_X\mu}(N) \overset{(3.2)}{=} \int_N i_Y i_X \mu \overset{(4.1)}{=} \psi_N(X,Y).$$

It follows that $\mathbb{T} \to \widehat{\mathrm{Diff}}_{\mathrm{ex}}(M,\mu) \to \mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ is a central extension of Fréchet–Lie groups integrating the Lichnerowicz cocycle ψ_N defined in (4.1). We have thus proven the following result:

Theorem D. — *Let M be a compact manifold of dimension n endowed with a volume form μ having integral periods. For every closed, oriented manifold S of dimension $n-2$ and for every differential character $h \in \widehat{H}^{n-1}(M,\mathbb{T})$ with curvature μ, the 1-dimensional central extension $\widehat{\mathrm{Diff}}_{\mathrm{ex}}(M,\mu)$ of* $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ *obtained by pull-back of the prequantum extension* (4.4) *is a Fréchet–Lie group that integrates the Lichnerowicz cocycle ψ_N.*

Remark 4.1. — In [9, 11] another approach to integrate the Lichnerowicz cocycle ψ_N to smooth central extensions of the group of exact volume-preserving diffeomorphisms, i.e. the identity component of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$, is presented. We do not know whether these extensions coincide with the extensions given in Theorem B. The construction in [9] also uses a prequantum bundle over $\mathrm{Gr}^S(M)$, which is constructed by hand in a rather complex process through its Čech 1-cocycle. The main novelty of the present work is the use of differential characters to obtain the prequantum bundle over $\mathrm{Gr}^S(M)$ that is needed for the construction of the smooth central extension $\widehat{\mathrm{Diff}}_{\mathrm{ex}}(M,\mu)$.

4.2. Construction using the embedding space $\mathrm{Emb}(S,M)$

In [6], we constructed a central extension of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ using transgression to the mapping space $C^\infty(S,M)$, where S is a closed, oriented manifold of dimension $n-2$. We now briefly recall the construction in [6] and adapt it to the case of embeddings. Using the transgression diagram

$$\begin{tikzcd} & \mathrm{Emb}(S,M)\times S \arrow[dl, "p_1"'] \arrow[dr, "\mathrm{ev}"] & \\ \mathrm{Emb}(S,M) & & M, \end{tikzcd} \tag{4.5}$$

a differential character $h \in \widehat{H}^{k-1}(M,\mathbb{T})$ with curvature μ transgresses to the differential character $\widehat{h} := (p_1)_!(\mathrm{ev}^* h)$ in $\widehat{H}^1(\mathrm{Emb}(S,M),\mathbb{T})$, with curvature $\widehat{\mu} := (p_1)_!(\mathrm{ev}^*\mu)$ in $\Omega^2_{\mathbb{Z}}(\mathrm{Emb}(S,M))$, see Section 3.3. This yields a principal $\mathbb{T}$-bundle $\mathcal{Q} \to \mathrm{Emb}(S,M)$ with connection $\Theta_{\mathcal{Q}}$ and curvature $\widehat{\mu}$.

For a smooth embedding $f\colon S \to M$, let $\mathrm{Emb}_f(S,M)$ be the connected component of f in $\mathrm{Emb}(S,M)$ and let $\mathcal{Q}_f$ denote the restriction of $\mathcal{Q}$ to $\mathrm{Emb}_f(S,M)$. The group of connection-preserving automorphisms of $\mathcal{Q}_f \to \mathrm{Emb}_f(S,M)$ is a central extension of the group of $\widehat{h}$-preserving diffeomorphisms of $\mathrm{Emb}_f(S,M)$. If we pull this back along the action $\sigma\colon \mathrm{Diff}_{\mathrm{ex}}(M,\mu) \to \mathrm{Diff}(\mathrm{Emb}_f(S,M),\widehat{h})$ of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ on $\mathrm{Emb}_f(S,M)$, we obtain a central extension $\widehat{\mathrm{Diff}}'_{\mathrm{ex}}(M,\mu)$ of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ by $\mathbb{T}$,

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{T} \arrow[r] & \mathrm{Aut}(\mathcal{Q}_f,\Theta_{\mathcal{Q}}) \arrow[r] & \mathrm{Diff}(\mathrm{Emb}(S,M),\widehat{h}) \arrow[r] & 1 \\ 1 \arrow[r] & \mathbb{T} \arrow[r] \arrow[u] & \widehat{\mathrm{Diff}}'_{\mathrm{ex}}(M,\mu) \arrow[r] \arrow[u] & \mathrm{Diff}_{\mathrm{ex}}(M,\mu) \arrow[r] \arrow[u, "\sigma"'] & 1. \end{tikzcd}$$

The group $\widehat{\mathrm{Diff}}'_{\mathrm{ex}}(M,\mu)$ is a Fréchet–Lie group, and the Lie algebra 2-cocycle corresponding to this central extension is the Lichnerowicz cocycle (4.1) with $N = f(S)$ by [6, Thm. 5.4].

4.3. Comparison

Since the corresponding Lie algebra cocycles coincide, the two central extensions $\widehat{\mathrm{Diff}}_{\mathrm{ex}}(M,\mu)$ and $\widehat{\mathrm{Diff}}'_{\mathrm{ex}}(M,\mu)$ constructed using transgression, respectively, on the nonlinear Graßmannian $\mathrm{Gr}^S_N(M)$ and on the embedding space $\mathrm{Emb}_f(S,M)$, are isomorphic on the infinitesimal level if $f(S)=N$. To show that they are isomorphic also as central extensions of Lie groups, we will use the following general result.

Lemma 4.2. — *Assume that a Lie group G acts on the connected manifolds M and N. Let $P\to M$ be a principal $\mathbb{T}$-bundle with connection θ whose holonomy is G-invariant. For a G-equivariant map $\psi: N\to M$, let $\widehat{G}$ be the central extension of G obtained from the bundle (P,θ) and let $\widehat{G}_\psi$ the one obtained from the pull-back bundle $(\psi^*P,\psi^*\theta)$. For every $\phi\in\mathrm{Aut}(P,\theta)$ covering the action of some $g\in G$, the map*

$$\overline{\phi}:\psi^*P\to\psi^*P,\qquad (n,p)\mapsto\big(g\cdot n,\phi(p)\big) \tag{4.6}$$

is a bundle automorphism of ψ^*P which preserves the pull-back connection $\psi^*\theta$ and covers the action of $g\in G$ on N. The resulting Lie group homomorphism $\widehat{G}\ni\phi\mapsto\overline{\phi}\in\widehat{G}_\psi$ yields a smooth isomorphism of central extensions.

Proof. — Let $\phi\in\mathrm{Aut}(P,\theta)$ covering the action of $g\in G$. By the G-equivariance of ψ, the prescription (4.6) indeed defines a smooth bundle map $\overline{\phi}:\psi^*P\to\psi^*P$ that covers the action of g on N by construction. It is straightforward to see that $\overline{\phi}$ is a bundle automorphism preserving the connection $\psi^*\theta$, as $\phi\in\mathrm{Aut}(P,\theta)$. Clearly, $\widehat{G}\ni\phi\mapsto\overline{\phi}\in\widehat{G}_\psi$ is a group homomorphism fitting into the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{T} \arrow[r] & \widehat{G}_\psi \arrow[r] & G \arrow[r] & 1 \\ 1 \arrow[r] & \mathbb{T} \arrow[r] \arrow[u, "\mathrm{id}"'] & \widehat{G} \arrow[r] \arrow[u] & G \arrow[r] \arrow[u, "\mathrm{id}"'] & 1 \end{tikzcd} \tag{4.7}$$

and it thus yields an isomorphism of central extensions [21, Def. V.1.1]. ☐

Corollary 4.3. — *Assume that a Lie group G acts on the connected manifolds M and N. Let $h_M\in\widehat{H}^1(M,\mathbb{T})$ and $h_N\in\widehat{H}^1(N,\mathbb{T})$ be G-invariant differential characters. If there exists a G-equivariant map $\psi: N\to M$ such that $\psi^*h_M=h_N$, then the central extensions of G obtained from h_M and from h_N are isomorphic.*

Let us return to the main objective of comparing the central extensions $\widehat{\mathrm{Diff}}_{\mathrm{ex}}(M,\mu)$ and $\widehat{\mathrm{Diff}}'_{\mathrm{ex}}(M,\mu)$ constructed using transgression on the nonlinear Graßmannian $\mathrm{Gr}^S_N(M)$ and on the embedding space $\mathrm{Emb}_f(S,M)$. Here and in the following, f and N are related via $f(S)=N$. For $h\in \widehat{H}^{n-1}(M,\mathbb{T})$, the transgressed differential characters $\widehat{h}\in\widehat{H}^1(\mathrm{Emb}(S,M),\mathbb{T})$ and $\widetilde{h}\in\widehat{H}^1(\mathrm{Gr}^S(M),\mathbb{T})$ are related by $\widehat{h}=\pi^*\widetilde{h}$ according to Proposition 3.4. Since π is $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$-equivariant, Corollary 4.3 yields the following comparison result.

Proposition 4.4. — *Let M be a compact manifold of dimension n endowed with a differential character $h\in\widehat{H}^{n-1}(M,\mathbb{T})$ with curvature μ. Let S be a closed, oriented manifold of dimension $n-2$. For every embedding $f:S\to M$, the central extensions $\widehat{\mathrm{Diff}}_{\mathrm{ex}}(M,\mu)$ and $\widehat{\mathrm{Diff}}'_{\mathrm{ex}}(M,\mu)$ constructed using transgression to the nonlinear Graßmannian $\mathrm{Gr}^S_{f(S)}(M)$ and to the embedding space $\mathrm{Emb}_f(S,M)$ are smoothly isomorphic.*

In particular, it follows that the extensions obtained from $\mathcal{P}\to\mathrm{Gr}^S(M)$ over different connected components $\mathrm{Gr}^S_N(M)$ and $\mathrm{Gr}^S_{N'}(M)$ are isomorphic as soon as N and N' are the images of homotopic embeddings. Furthermore, if a submanifold $N\subseteq M$ is the image of an embedding $f\colon S\to M$ that is homotopic to a "thin" map $g\colon S\to M$ (meaning that $\int_S g^*\alpha=0$ for all $\alpha\in\Omega^{n-2}(M)$), then the corresponding central extension $\widehat{\mathrm{Diff}}_{\mathrm{ex}}(M,\mu)$ is trivial at the Lie algebra level.

Remark 4.5 (Comparison of the constructions). — As $\pi:\mathrm{Emb}(S,M)\to\mathrm{Gr}^S(M)$ is surjective, every extension of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ that can be obtained using the nonlinear Graßmannian can also be obtained starting from the embedding space, showing that the two approaches are essentially equivalent. An advantage of the construction using Graßmannians is that the trangressed 2-form $\widetilde{\mu}$ on $\mathrm{Gr}^S(M)$ is not only closed, but also nondegenerate; in contrast to the principal circle bundle over $\mathrm{Emb}(S,M)$, the circle bundle $\mathcal{P}_N$ (with connection $\Theta_{\mathcal{P}}$) over $\mathrm{Gr}^S_N(M)$ (with symplectic form $\widetilde{\mu}$) is thus a prequantum bundle. Under certain conditions, the Graßmannian $\mathrm{Gr}^S_N(M)^+$ of oriented embedded submanifolds admits a formally integrable almost complex structure [3, 7, 16, 17, 27]. In future work, we hope to use this almost complex structure in order to define a suitable polarization on the associated prequantum line bundle $\mathbb{L}=\mathcal{P}_N\times_{\mathbb{T}}\mathbb{C}$, making the step from prequantization to quantization.

Remark 4.6 (Construction based on $C^\infty(S,M)$). — In [6], we constructed a central extension of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ using transgression to the mapping space

$C^\infty(S,M)$ instead of the embedding space $\mathrm{Emb}(S,M)$ as described in Section 4.2. Clearly, the inclusion $\iota : \mathrm{Emb}(S,M) \to C^\infty(S,M)$ is $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$-equivariant and relates the transgressed characters $\widehat{h} \in \widehat{H}^1(\mathrm{Emb}(S,M),\mathbb{T})$ and $\overline{h} \in \widehat{H}^1(C^\infty(S,M),\mathbb{T})$ by $\widehat{h} = \iota^*\overline{h}$. Thus, Corollary 4.3 implies that the central extensions of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ obtained from $(\mathrm{Emb}_f(S,M),\widehat{h})$ and from $(C^\infty_f(S,M),\overline{h})$ are smoothly isomorphic for every embedding $f \in \mathrm{Emb}(S,M)$. However, there could be connected components of $C^\infty(S,M)$ that do not contain an embedding[(4)], and such connected components might give rise to central extensions of $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)$ which are not accessible via embedding spaces and nonlinear Graßmannians.

4.4. The special case $H^1(M,\mathbb{Z}) = \{0\}$

In this section we treat the special case when $H^1(M,\mathbb{Z}) = \{0\}$. The group $\mathrm{Diff}_{\mathrm{ex}}(M,\mu)_0$ of exact volume-preserving diffeomorphisms then coincides with the smooth arc-component of the identity of the group $\mathrm{Diff}(M,\mu)$ of volume-preserving diffeomorphisms. Indeed, $H^{n-1}_{\mathrm{dR}}(M) = \{0\}$, since by Poincaré duality $H_{n-1}(M,\mathbb{Z}) \cong H^1(M,\mathbb{Z}) = \{0\}$. Since $H^{n-1}(M,\mathbb{T}) = \{0\}$, it now follows from the exact sequence (2.1) that $\mathrm{curv} : \widehat{H}^{n-1}(M,\mathbb{T}) \to \Omega^n_{\mathbb{Z}}(M)$ is an isomorphism. In particular, to any volume form μ with integral periods one can assign in a natural way a uniquely determined differential character $h_\mu \in \widehat{H}^{n-1}(M,\mathbb{T})$ with curvature μ. It is defined as $h_\mu(c) := \exp_{\mathbb{T}}(\int_D \mu)$ for any $c \in Z_{n-1}(M)$ and $D \in C_n(M)$ with $\partial D = c$. Such smooth singular chains D exist because the smooth singular homology group $H_{n-1}(M) = 0$, and the result is independent of the choice of D, because μ is integral.

If $H^1(M,\mathbb{Z}) = \{0\}$, the diffeomorphism groups that preserve the volume form μ and the holonomy h_μ coincide: $\mathrm{Diff}(M,\mu) = \mathrm{Diff}(M,h_\mu)$ by the right commutative diagram in (2.3). We thus have

$$\mathrm{Diff}_{\mathrm{ex}}(M,\mu) \overset{(2.8)}{=} \mathrm{Diff}(M,\mu)_0 = \mathrm{Diff}_{\mathrm{ex}}(M,\mu)_0 = \mathrm{Diff}(M,h_\mu)_0.$$

We can therefore canonically associate to every homotopy class $[f]$ of an embedding $f\colon S \to M$ the isomorphism class of a central Lie group extension

$$\mathbb{T} \to \widehat{\mathrm{Diff}}(M,\mu)_0 \to \mathrm{Diff}(M,\mu)_0$$

with cocycle $\psi(X,Y) = \int_S f^*(i_X i_Y \mu)$.

(4) For example, [15, Cor. 1.3] yields immersions of a closed surface into a closed 4-manifold that are not homotopic to an embedding.

Example 4.7. — When M is a surface with $H_1(M) = \{0\}$ and $S = \{*\}$ one point, then the nonlinear Graßmannian is $\mathrm{Gr}^S(M) = M$ and the transgression map is the identity. Thus the action σ is the identity, and the two rows in (4.4) coincide.

For a 2-sphere $M = S^2$ the group of volume-preserving diffeomorphisms is connected: it has the rotation group SO(3) as a deformation retract by [25]. Thus $\mathrm{Diff}_{\mathrm{ex}}(S^2,\mu)_0 = \mathrm{Diff}(S^2,\mu)$ for any volume form μ on the 2-sphere. Since $H^1(S^2,\mathbb{Z}) = \{0\}$, each $\mu \in \Omega^2_{\mathbb{Z}}(S^2)$ determines uniquely a differential character $h_\mu \in \widehat{H}^1(S^2,\mathbb{T})$. It is the holonomy of a suitable "Hopf fibration" $(S^3,\theta) \to (S^2,\mu)$, a principal circle bundle with connection θ and curvature μ.

The next example shows that every knot on the 3-sphere yields a central extension of the identity component of the volume-preserving diffeomorphism group.

Example 4.8. — On the 3-sphere $S^3 \simeq \mathrm{SU}(2)$ we consider the bi-invariant volume form

$$\mu = \kappa(\theta \wedge [\theta \wedge \theta]),$$

where θ denotes the Maurer-Cartan 1-form on SU(2). We scale the Killing form κ such that the cohomology class $[\mu]$ of the volume form generates the image of $H^3(\mathrm{SU}(2),\mathbb{Z}) \simeq \mathbb{Z}$ in $H^3(\mathrm{SU}(2),\mathbb{R})$. Let h_μ be the unique differential character with curvature μ. Each embedded knot in S^3 yields a central Lie group extension of $\mathrm{Diff}(S^3,\mu)_0$. Since $H^2(S^3) = 0$, the corresponding extension of the Lie algebra of divergence free vector fields is trivial [24].

In the next example, the second cohomology of the ambient space is non-trivial.

Example 4.9. — Let $M = S^2 \times S^2$ be endowed with the volume form $\mu = p_1^*\nu \wedge p_2^*\nu$, with ν an area form on S^2 of total area 1. Again we have a unique differential character h_μ on $S^2 \times S^2$ with the volume form μ with integral periods as curvature. The isomorphism classes of central extensions of the Lie algebra of divergence free vector fields on $S^2 \times S^2$ are classified by $H^2(S^2 \times S^2) \simeq \mathbb{R}^2$ [24].

For each embedded S^2 in $S^2 \times S^2$ one gets in a canonical way a central extension of the identity component of the group of volume-preserving diffeomorphisms of $S^2 \times S^2$. For instance, the diagonal embedding yields a Lie group extension that integrates a non-trivial Lie algebra extension: the Lichnerowicz 2-cocycle $\psi(X,Y) = \int_{\Delta_{S^2}} i_X i_Y \mu$ has non-trivial cohomology class.

BIBLIOGRAPHY

[1] A. Banyaga, "Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique", *Comment. Math. Helv.* **53** (1978), p. 174-227.

[2] C. Bär & C. Becker, *Differential Characters*, Lecture Notes in Mathematics, vol. 2112, Springer, 2014.

[3] J.-L. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization*, Modern Birkhäuser Classics, Birkhäuser, 2007.

[4] E. Calabi, "On the group of automorphisms of a symplectic manifold", in *Problems in Analysis: A Symposium in Honor of Salomon Bochner*, Princeton Mathematical Series, vol. 31, Princeton University Press, 1970.

[5] J. Cheeger & J. Simons, "Differential characters and geometric invariants", in *Geometry and Topology (College Park, Md., 1983/84)*, Lecture Notes in Mathematics, vol. 1167, Springer, 1983, p. 50-80.

[6] T. Diez, B. Janssens, K.-H. Neeb & C. Vizman, "Central extensions of Lie groups preserving a differential form", *Int. Math. Res. Not.* **2021** (2021), no. 5, p. 3794-3821.

[7] D. Fiorenza & H. V. Lê, "Formally integrable complex structures on higher dimensional knot spaces", *J. Symplectic Geom.* **19** (2021), no. 3, p. 507-529.

[8] W. Greub, S. Halperin & R. Vanstone, *Connections, curvature, and cohomology. Vol. I: De Rham cohomology of manifolds and vector bundles*, Pure and Applied Mathematics, vol. 47, Academic Press Inc., 1972.

[9] S. Haller & C. Vizman, "Nonlinear Graßmannians as coadjoint orbits", *Math. Ann.* **329** (2004), no. 4, p. 771-785.

[10] ———, "Nonlinear flag manifolds as coadjoint orbits", *Ann. Global Anal. Geom.* **58** (2020), no. 4, p. 385-413.

[11] R. S. Ismagilov, *Representations of infinite-dimensional groups*, Translations of Mathematical Monographs, vol. 152, American Mathematical Society, 1996.

[12] A. A. Kirillov & D. V. Yur′ev, "Representations of the Virasoro algebra by the orbit method", *J. Geom. Phys.* **5** (1988), no. 3, p. 351-363.

[13] B. Kostant, "Quantization and unitary representations. I: Prequantization", in *Lectures in modern analysis and applications. III*, Lecture Notes in Mathematics, vol. 170, Springer, 1970, p. 87-208.

[14] A. Kriegl & P. W. Michor, *The Convenient Setting of Global Analysis*, Mathematical Surveys and Monographs, vol. 53, American Mathematical Society, 1997.

[15] P. B. Kronheimer, "An obstruction to removing intersection points in immersed surfaces", *Topology* **36** (1997), no. 4, p. 931-962.

[16] C. LeBrun, "A Kähler structure on the space of string worldsheets", *Class. Quant. Grav.* **10** (1993), no. 9, p. L141-L148.

[17] L. Lempert, "Loop spaces as complex manifolds", *J. Differ. Geom.* **38** (1993), no. 3, p. 519-543.

[18] A. Lichnerowicz, "Algèbre de Lie des automorphismes infinitésimaux d'une structure unimodulaire", *Ann. Inst. Fourier* **24** (1974), no. 3, p. 219-266.

[19] J. Marsden & A. Weinstein, "Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids", *Physica D* **7** (1983), p. 305-323.

[20] K.-H. Neeb, "Infinite-dimensional groups and their representations", in *Lie theory. Lie algebras and representations*, Progress in Mathematics, vol. 228, Birkhäuser, 2004, p. 213-328.

[21] ———, "Towards a Lie theory of locally convex groups", *Jpn. J. Math.* **1** (2006), no. 2, p. 291-468.

[22] K.-H. Neeb & C. Vizman, "Flux homomorphisms and principal bundles over infinite dimensional manifolds", *Monatsh. Math.* **139** (2003), no. 4, p. 309-333.

[23] A. Pressley & G. Segal, *Loop Groups*, Oxford University Press, 1986.

[24] C. Roger, "Extensions centrales d'algèbres et de groupes de Lie de dimension infinie,algèbre de Virasoro et généralisations", *Rep. Math. Phys.* **35** (1995), no. 2-3, p. 225-266.

[25] S. Smale, "Diffeomorphisms of the 2-sphere", *Proc. Am. Math. Soc.* **10** (1959), p. 621-626.

[26] J.-M. Souriau, *Structure des systèmes dynamiques. Maitrises de mathématique*, Dunod, 1970.

[27] M. Verbitsky, "A formally Kähler structure on a knot space of a G_2-manifold", *Sel. Math., New Ser.* **18** (2012), no. 3, p. 539-555.

[28] C. Vizman, "Lichnerowicz cocycles and central Lie group extensions", *An. Univ. Vest Timiș., Ser. Mat.-Inform.* **48** (2010), no. 1-2, p. 285-297.

[29] ———, "Induced differential forms on manifolds of functions", *Arch. Math., Brno* **47** (2011), no. 3, p. 201-215.

Manuscrit reçu le 5 septembre 2020,
révisé le 3 août 2022,
accepté le 17 novembre 2022.

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