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THE SAITO MODULE AND THE MODULI OF A GERM OF CURVE IN $(\mathbb{C}^2, 0)$.

by Yohann GENZMER

Abstract. — This article proposes to study the moduli space of a germ of curve S in the complex plane, that is to say the equisingularity class of S up to analytical equivalence relation. The first part is devoted to proving that this last quotient can be endowed with a reasonable, yet not canonical, complex structure. The second part deals with the computation of its generic dimension in terms of topological invariants of S. It can be obtained from the study of the valuations of the Saito module of S, $\mathrm{Der}(\log S)$, i.e. the module of vector fields tangent to S.

Résumé. — Cet article est une étude des espaces de module d'une courbe S dans le plan complexe, c'est-à-dire, de la classe d'équisingularité de S modulo la relation d'équivalence analytique. La première partie établit l'existence d'une structure non canonique de variété complexe sur ce quotient. La seconde partie se consacre au calcul de sa dimension générique à partir de la donnée d'invariants primitifs topologiques de S. Ce calcul est le fruit de l'étude des valuations des champs de vecteurs tangents à S.

Introduction

The number of moduli of a germ of curve S in $(\mathbb{C}^2, 0)$ is basically the number of parameters on which depends a topologically miniversal family for S. It is also the *generic dimension* of the quotient of the topological class of S up to analytical equivalence relation, provided that this quotient admits a structure from which a notion of dimension can be derived. Indeed, this moduli space defined by the quotient of the topological class of S

$$\{S' \mid S' \sim_{\text{top}} S\}$$

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by the following action of $\mathrm{Diff}(\mathbb{C}^2,0)$

$$\phi \cdot S' = \phi(S'), \quad \phi \in \mathrm{Diff}(\mathbb{C}^2,0)$$

a priori has no particular structure beyond being a set.

The first determination of such a number of moduli goes back to the work of Sherwood Ebey in 1965 [6] who dealt with the irreducible curves, those having only one irreducible component. Ebey proved that the moduli space of S carries a complex structure compatible with a non separated topology and computed the number of moduli for a particular topological class of curve, namely, that given by the equation $y^5 = x^9$. In 1973, in [26], Oscar Zariski proposed various approaches to get the number of moduli for irreducible curves beyond the case treated previously by Ebey. He introduced most of the concepts on which future work will be based. In 1978, Delorme [4] studied extensively the case of an irreducible curve with one Puiseux pair. In 1979, Granger [13] and later, in 1988, Briançon, Granger and Maisonobe [1] produced an algorithm to compute the number of the moduli of a non irreducible quasi-homogeneous curve. In 1988, Laudal, Martin and Pfister in [19], improved the work of Delorme and gave an explicit description of a miniversal family. From 2009, in a series of papers [14, 15, 16], Abramo Hefez and Marcelo Hernandes greatly improved the previous studies and achieved the analytical classification of irreducible curves. Their algorithmic approach provided in particular the number of moduli.

In 2010 and 2011, in [10, 11], Emmanuel Paul and the author described the moduli space of a topologically quasi-homogeneous curve S as the spaces of leaves of an algebraic foliation defined on the moduli space of a foliation whose analytic invariant curve is precisely S. This work initiated an approach based on the theory of foliations. In 2019, in [7], the author gave an explicit formula for the number of moduli for irreducible curves S, *generic* in its topological class : this formula involves only very elementary topological invariants of S, such as, the topological class of its desingularization.

The aim of this article is to investigate the full general case, that is the number of moduli of a germ of curve in the complex plane. We emphasize that our objective is far from being as ambitious as a complete analytical classification, which would require at least some deep algorithmic procedures, but is rather to obtain a *geometric* interpretation of these moduli and a procedure to calculate their number from primitive topological invariants.

This work follows the ideas introduced in [7] and illustrated in [9], which focuses on the irreducible case.

Section 1 establishes an extension of the result of Ebey [6, Theorem 4] to the non irreducible curves : it concerns the structure of the moduli space. As noticed by Ebey himself at the end of its article, its *machinery* derives from the theory of algebraic groups and depends on the groups being solvable and *connected*. Therefore, it cannot be directly carried over to several component curves. Here, we overcome this issue by considering not only curves but curves enriched with a *marking* which allows us to recover the necessary connexity. As Ebey, we use an adapted complete topological invariant, the semi ring of values, introduced by R. Waldi [25] and some of its properties identified by M. Hernandes and E. de Carvalho in [17]. Finally, we obtain the following result

THEOREM. — *The marked moduli space* $\mathbb{M}^\bullet(S)$ *of a germ of curve* S *in* $\mathbb{C}^2$*, that is its marked topological class up to analytical marked equivalence relation, can be identified with the quotient of a complex constructible set by an action of a connected solvable algebraic group. In particular, it is endowed with a non separated complex structure.*

Notice that passing from the moduli space to the *marking* moduli space has no effect on the generic dimension.

Section 1 can be read independently from the rest of the article.

Sections 2 and 3 aim to develop the study of the module $\operatorname{Der}(\log S)$ of vector fields tangent to S, on which depends the computation of the number of moduli of S. The starting point is a remark of K. Saito in [23], that, highlighted the freeness of this module, which is specific to the curves embedded in the complex plane. An immediate consequence of the work of Saito is that, the smallest valuation of the vector fields in $\operatorname{Der}(\log S)$ cannot be too big compared to the valuation of S, namely, the following upper bound holds

$$\min_{X\in\operatorname{Der}(\log S)} \nu(X) \leqslant \frac{\nu(S)}{2}.$$

Our purpose is to prove that, *generically,* this bound is essentially reached. In Section 2, the existence of a *flat* basis of $\operatorname{Der}(\log S)$ is shown in the generic situation, that is, a basis admitting an analytic extension as a basis for the modules $\operatorname{Der}(\log C)$ where C are in a *neighborhood* of S in $\mathbb{M}^\bullet(S)$. As a consequence, using the theory of infinitesimal deformations of foliations of X. Gómez-Mont [12], we obtain the following theorem:

Theorem. — *For S generic in its moduli space $\mathbb{M}^\bullet(S)$, one has*

$$\min_{X\in\mathrm{Der}(\log S)} \nu(X) \geqslant \begin{cases} \left\lfloor \frac{\nu(S)}{2} \right\rfloor & \text{if } S \text{ is not of radial type,}\\ \left\lceil \frac{\nu(S)}{2} \right\rceil - 1 & \text{else.}\end{cases}$$

The definition of S being of *radial type* will be given in the article. Note that if S is not generic in its moduli space, the above lower bound is false, as it will be illustrated by some examples in the article. Moreover, in Section 3, we proceed with the precise description of the various possibilities for the flat basis of $\mathrm{Der}(\log S)$.

Finally, Section 4 illustrates our approach for the computation of the generic dimension of $\mathbb{M}^\bullet(S)$. As a consequence of Sections 2 and 3, we recover the classical dimension of the moduli space of the singularity $x^n + y^n = 0$ with $n \geqslant 1$.

Corollary 1 ([13]). — *The generic dimension of* $\mathbb{M}^\bullet(S)$ *where*

$$S = \{x^n + y^n = 0\}$$

is equal to

$$\begin{cases} \frac{(n-2)^2}{4} & \text{if } n \text{ is even,}\\ \frac{(n-1)(n-3)}{4} & \text{if } n \text{ is odd.}\end{cases}$$

In an upcoming article, we will build an algorithm based upon the results presented here, that computes the generic dimension of the moduli space for more general curves, namely, curves with many but smooth irreducible components. We implemented, among other procedures, this algorithm on Sage 9.*. See the routine *Courbes.Planes* following the link

`https://perso.math.univ-toulouse.fr/genzmer/`

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1. Moduli space of marked curve

Throughout this article, S stands for a germ of singular curve in the complex plane $(\mathbb{C}^2, 0)$. In particular, its algebraic valuation is at least 2.

From now on, we fix a decomposition of S in irreducible components

$$S = S^1 \cup \cdots \cup S^r$$

where r is the number of irreducible components. Here and subsequently, $\mathrm{Comp}(S)$ stands for the set of irreducible components of S.

Let C be a germ of curve topologically equivalent to S by a germ of homeomorphism of the ambient space $(\mathbb{C}^2, 0)$ denoted by h and such that

$$h(S) = C.$$

The application h induces a bijective map

$$\sigma_h : \mathrm{Comp}(S) \longrightarrow \mathrm{Comp}(C).$$

Two such homeomorphisms h and h' are said to be *equivalent* if and only if

$$\sigma_h = \sigma_{h'}. \tag{1.1}$$

Definition 1.1. — *A curve* marked by S *is a couple* $(C, \overline{h})$ *where* C *is curve topologically equivalent to* S *and* $\overline{h}$ *a class of homeomorphism between* C *and* S *for the equivalence relation defined above. We will denote by* $\mathrm{Top}^\bullet(S)$ *the set of curves marked by* S.

The group $\mathrm{Diff}(\mathbb{C}^2, 0)$ of germs of automorphisms of the ambient space $(\mathbb{C}^2, 0)$ acts on the set $\mathrm{Top}^\bullet(S)$ by

$$\phi \cdot (C, \overline{h}) = \left(\phi(C), \overline{\phi \circ h}\right).$$

In what follows, the quotient of $\mathrm{Top}^\bullet(S)$ by $\mathrm{Diff}(\mathbb{C}^2, 0)$ will be denoted by

$$\mathbb{M}^\bullet(S)$$

and will be refered to as *the marked moduli space of S*. Although $\mathbb{M}^\bullet(S)$ cannot be endowed with a complex structure by some general statements about group actions, the result below provides such a structure. Indeed, generalizing a result of Ebey [6], we obtain the

Theorem 1.2. — *The quotient* $\mathbb{M}^\bullet(S)$ *can be identified with the quotient of a complex constructible set by an action of a connected solvable algebraic group.*

This result still holds if we drop the assumption of S being a *plane* curve, once we replace the topological equivalence by the equisingularity which corresponds to the equality of the semirings of valuations as defined in [17]. Since the general proof consists at most in increasing the complexity of the notations, we state Theorem 1.2 and prove it only for a curve embedded in the complex plane. We follow Theorem 5 in [6] observing that a connected solvable algebraic action on a complex constructible set admits a complete transversal, that is a constructible subset in correspondance one to one with the orbits of the action. Thus, from Theorem 1.2, $\mathbb{M}^{\bullet}(S)$ inherits of the complex structure of this transversal. Its compatible topology is just the quotient topology : in most case, it is not separated (see for instance [14, 15]).

The goal of the current section is to prove Theorem 1.2.

1.1. The ring of functions of $(C, \overline{h})$

Let $(C, \overline{h})$ be in $\mathrm{Top}^{\bullet}(S)$ and

$$\gamma_C = \left\{\gamma_c : t \in (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^2, 0)\right\}_{c \in \mathrm{Comp}(C)}$$

be any system of parametrizations of the irreducible components of C. We denote by C_i the component of C defined by the marking $\overline{h}$

$$C_i = \sigma_h(S^i).$$

From the marking $\overline{h}$ of C, we derived a morphism of rings defined by

$$\begin{cases} \mathbb{C}[\![x, y]\!] \to (\mathbb{C}[\![t]\!])^r \\ u \mapsto \left(\gamma_{C_i}^{\star} u\right)_{i=1,\ldots,r} \end{cases}.$$

which factorizes in an monomorphism

$$\mathfrak{E}_{(C,\overline{h})} : \widehat{\mathcal{O}}_C = \frac{\mathbb{C}[\![x, y]\!]}{(f)} \hookrightarrow (\mathbb{C}[\![t]\!])^r \tag{1.2}$$

where f is any reduced equation of C and $\widehat{\mathcal{O}}_C$ is the completion of $\mathcal{O}_C = \frac{\mathbb{C}\{x,y\}}{(f)}$.

The following result is classic, see [6] for the irreducible case.

Lemma 1.3. — *Let $(C, \overline{h})$ and $(C', \overline{h'})$ be two marked curves in $\mathrm{Top}^{\bullet}(S)$. The following propertie are equivalent*

1. *The curves $(C, \overline{h})$ and $(C', \overline{h'})$ are analytically equivalent by a conjugacy preserving the markings.*
2. *The images of the monomorphisms* (1.2) *associated to both curves are conjugated by a diagonal formal automorphism of $(\mathbb{C}[\![t]\!])^r$.*

1.2. The tropical semiring of values of $(C, \overline{h})$

Following [17], we consider Γ_C the set defined by

$$\Gamma_C = \{\nu(G) \,|\, G \in \mathcal{O}_C\} \subset \left(\overline{\mathbb{N}}\right)^r$$

where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. The valuation ν is defined by

$$\nu(G) = \left(\nu_0\left(\gamma_{C_i}^{\star} G\right)\right)_{i=1,\dots,r}$$

where ν_0 is the standard valuation $\mathbb{C}\{t\}$. Notice that this set depends not only on the curve C but also on its marking.

The set Γ_C inherits of a semiring structure defined by

$$\alpha \oplus \beta = (\min\{\alpha_i, \beta_i\})_{i=1\dots r} \qquad \alpha \odot \beta = (\alpha_i + \beta_i)_{i=1\dots r}$$

where we set $k + \infty = \infty$. Γ_C is also partially ordered by the product order $\leqslant$. The quadruplet $(\Gamma_C, \oplus, \odot, \leqslant)$ is called *the tropical semiring of values of* $(C, \overline{h})$.

Definition 1.4. — *An element $\alpha \in \Gamma_C$ is said* irreducible *if and only if*

$$(\alpha = a + b \text{ with } a, b \in \Gamma_C) \Longrightarrow \alpha = a \text{ or } \alpha = b.$$

It is said to be absolute *if for any non empty proper subset J of the set*

$$\mathcal{I}_\alpha = \{i \in \{1, \dots r\} \mid \alpha_i \neq \infty\}, \tag{1.3}$$

the following set

$$F_J(\alpha) = \{a \in \Gamma_C \mid \forall i \in \mathcal{I}_\alpha \setminus J, a_i > \alpha_i \text{ and } \forall i \notin \mathcal{I}_\alpha \setminus J, a_i = \alpha_i\} \tag{1.4}$$

is empty.

The following result gathers some known properties of the semiring of values.

Theorem 1.5 ([17, 20, 25]). — *Two germs of plane curves are topologically equivalent if and only if they share the same semiring of values* [25]. *More precisely $C_1 \cup C_2 \cup \cdots \cup C_r$ and $C'_1 \cup C'_2 \cup \cdots \cup C'_r$ are two curves with same semiring if and only if there exists an homeomorphism ϕ of the ambient space $(\mathbb{C}^2, 0)$ such that for any i*

$$\phi(C_i) = C'_i.$$

Moreover,

1. *Γ_C has a conductor, i.e, there exists a minimal $\sigma \in \Gamma_C$ such that $\sigma + \overline{\mathbb{N}}^r \subset \Gamma_C$* [20].
2. *The set g of irreducible absolute points of Γ_C is finite and minimaly generates Γ_C as semiring* [17].
3. *Any family G of $\mathcal{O}_C$ such that $\nu(G) = g$ is a minimal standard basis of $\mathcal{O}_C$ as defined in* [17].

As a consequence, for any element $C \in \mathrm{Top}^\bullet(S)$, one has

$$\Gamma_C = \Gamma_S.$$

For now on, we will denote the mutual semiring for curves in $\mathrm{Top}^\bullet(S)$ simply by Γ.

1.3. Trunctation and conductor

The following lemma allows us to truncate elements in the ring $\mathcal{O}_C$ (resp. $\widehat{\mathcal{O}}_C$).

Lemma 1.6. — *Suppose that $G = (\sum_{k=0}^{\infty} a_{lk}t^k)_{l=1,\ldots,r}$ is an element of $\mathcal{O}_C$ (resp. of its completion $\widehat{\mathcal{O}}_C$). Then, for any $p = (p_1, \ldots, p_r) \in \mathbb{N}^r$ with $p_l \geqslant \sigma_l - 1$ for $l = 1, \ldots, r$, one has*

$$\left(\sum_{k=0}^{p_l} a_{lk}t^k\right)_{l=1,\ldots,r} \in \mathcal{O}_C,\ \text{(resp. } \widehat{\mathcal{O}}_C\text{)}.$$

Proof. — By definition of σ, for any $l = 1, \ldots, r$ and for any $k \geqslant p_l + 1 \geqslant \sigma_l$, the r-uple

$$\left(\infty, \ldots, \infty, \underbrace{k}_{l^{th}}, \infty, \ldots, \infty\right)$$

belongs to Γ. Thus, an inductive argument on the rank $k \geqslant p_l + 1$ shows that there exists a formal series $\widehat{F}_l \in \mathbb{C}[\![x, y]\!]$ such that

$$\gamma^\star \widehat{F}_l = \left(0, \ldots, 0, \sum_{k=p_l+1}^{\infty} a_{lk}t^k, 0, \ldots, 0\right).$$

Now, following [22, Theorem 1, p. 493], if G is convergent, so is $\widehat{F}_l$ and, in any case, evaluating

$$G - \gamma^\star\left(\sum_{l=1}^{r} \widehat{F}_l\right)$$

yields the lemma. □

1.4. Γ-reduction

The notion of Γ-*reduction* will allow us to construct normal forms for systems of generators of $\widehat{\mathcal{O}}_C$.

Let $\underline{P} = (P_i)_{i=1,\ldots,r}$ be a family of r finite subsets of $\overline{\mathbb{N}}$ such that for any i, $\infty \in P_i$.

Definition 1.7. — *The family* $\underline{P}$ *is* said to be Γ-reduced *if and only if*

$$\Gamma \cap \prod_{i=1,\ldots,r} P_i = \{\underline{\infty}\}$$

where $\underline{\infty} = (\infty, \infty, \ldots, \infty)$

A Γ-*reduction of* $\underline{P}$ is an elementary transformation of $\underline{P}$ of the following form : suppose that there exists $\underline{n} = (n_1, \ldots, n_r)$ such that

$$\underline{n} \in \left(\Gamma \cap \prod_{i=1,\ldots,r} P_i\right) \setminus \{\underline{\infty}\}.$$

Consider an integer i such that $n_i \neq \infty$. Then the family $\underline{P}^{(1)} = (P_i^{(1)})_{i=1,\ldots,r}$ defined by

$$\begin{cases} P_j^{(1)} = P_j & \text{for } j \neq i \\ P_i^{(1)} = P_i \setminus \{n_i\} \end{cases}$$

is called a Γ-reduction of $\underline{P}$. To keep track of a Γ-reduction, we denote it by

$$\underline{P} = \underline{P}^{(0)} \xrightarrow{\underline{n},i} \underline{P}^{(1)}.$$

The following lemma is obvious

Lemma 1.8. — *For any* $\underline{P}$*, there exists a finite sequence of* Γ*-reductions*

$$\underline{P} = \underline{P}^{(0)} \xrightarrow{\underline{n}_0,i_0} \underline{P}^{(1)} \xrightarrow{\underline{n}_1,i_1} \cdots \xrightarrow{\underline{n}_{q-1},i_{q-1}} \underline{P}^{(q)}$$

such that $\underline{P}^{(q)}$ *is* Γ*-reduced.*

Notice that this sequence is not unique.

1.5. Parametrization of the set $\mathrm{Top}^\bullet(S)$

Let $g = \{g^1, \ldots, g^q\}$ be the set of irreducible absolute points of Γ and $G = \{G^1, \ldots, G^q\} \subset \mathcal{O}_C$ such that for all i,

$$\nu\left(G^i\right) = g^i.$$

Lemma 1.9. — *Among the family* G *and in the identification*

$$\mathcal{O}_C = \frac{\mathbb{C}[\![x,y]\!]}{(f)},$$

there are two components G^i *whose linear parts are independent.*

Proof. — Assume that C contains an irreducible singular component, say C_1, and consider some coordinates (x,y) such that it is parametrized by

$$t \longrightarrow (t^n, t^m + \cdots), \quad n \nmid m.$$

Evaluating the valuation of the coordinate functions x and y, we obtain that Γ contains two elements of the form

$$\nu(x) = (n, \ldots) \in \Gamma \text{ and } \nu(y) = (m, \ldots) \in \Gamma. \tag{1.5}$$

If the linear parts of the functions G^i are dependent two by two, then the set of valuations of the complete ring generated by the family G can contains either $(n,\ldots)$ or $(m,\ldots)$ or none of them, but certainly not both. However, according to Theorem 1.5, the complete ring generated by G is the whole ring $\widehat{\mathcal{O}}_C$, which contradicts (1.5). If C contains two smooth components, transversal or not, a contradiction can be obtained much the same way by considering coordinates in which these components are written

$$t \longrightarrow ((t,0),(0,t),\ldots) \text{ or } t \longrightarrow ((t,0),(t,t^n),\ldots), \; n \geqslant 2. \qquad \square$$

Changing the numbering of the elements in g, we may assume that the two elements identified by the above lemma are G^1 and G^2 with $g^1 < g^2$ minimal for the lexicographic order among those satisfying the property of Lemma 1.9. Let us denote G^i, $i = 1, 2$

$$G^i = \left(\sum_{k=g_l^i}^{\infty} a_{lk}^i t^k \right)_{l=1,\ldots,r}. \tag{1.6}$$

Notice that in the above expression, g_l^i may be equal to ∞ and the corresponding component $(G^i)_l$ be equal to 0 . However, one has the following

Lemma 1.10. — *Assume that C is not the union of two smooth curves. If $g_l^i \neq \infty$, then $g_l^i \leqslant \sigma_l - 1$.*

Proof. — The proof is by contradiction. Suppose that for some l, $g_l^i \neq \infty$ and $g_l^i \geqslant \sigma_l$. Applying Lemma 1.6 to G^i with

$$(p_i)_{i=1,\ldots,r} = (\infty, \ldots, \infty, \sigma_l, \infty, \ldots, \infty)$$

yields an element $\bar{g} \in \Gamma$ such that $\bar{g}_l = \infty$ and $\bar{g}_k = g_k^i$ for $k \neq l$. Consider the proper subset of $\mathcal{I}_{g^i}$ defined by

$$J = \mathcal{I}_{g^i} \setminus \{l\},$$

and suppose it is non empty. Definition 1.4 of absolute point ensures that $F_J(g^i)$ is empty. However, by construction, $\bar{g}$ belongs to $F_J(g^i)$ which is a contradiction. Thus, J is empty and $\mathcal{I}_{g^i} = \{l\}$. Therefore, g^i is written

$$g^i = \left(\infty, \ldots, \infty, g^i_l, \infty, \ldots, \infty\right).$$

- If $r \geqslant 3$, we are lead to a contradiction noticing that G^i would be a function with non trivial linear part vanishing along two distinct components of C.
- Assume $r = 2$. Since G^i is a regular function and $g^i = (\infty, g^i_2)$ or (g^i_1, ∞), one of the component of C, say C_1, is smooth. One can choose some coordinates (x, y) such that

$$C_1 = \{\alpha y + \beta x = 0\}, \ \alpha, \beta \in \mathbb{C}$$
$$C_2 = \{y^p + x^q + \cdots = 0\}$$

with $p < q$. The hypothesis of the lemma ensures that the case $p = 1$ is excluded. According to [17], the conductor σ of C is written

$$\sigma = (0, c_2) + \begin{cases} (p, p) & \text{if } \beta \neq 0 \\ (q, q) & \text{if } \beta = 0 \end{cases}.$$

where $c_2 \geqslant 1$ is the conductor of the component C_2. By construction, the function G^1 is equal to $\alpha y + \beta x$. Therefore,

$$g^1 = \left(\infty, \begin{cases} p & \text{if } \beta \neq 0 \\ q & \text{if } \beta = 0 \end{cases}\right),$$

thus $g^1_2 < \sigma_2$. □

If C is a union of two smooth curves then one has

$$\sigma = (n, n), \ g^1 = (\infty, n) \text{ and } g^2 = (n, \infty)$$

where n is the order of tangency between C_1 and C_2. Actually, C is analytically equivalent to the curve

$$y\,(y + x^n) = 0.$$

Thus the moduli space of C reduces to a point and the problem of the analytic classification is trivial. For now on, we will assume that C is not a union of two smooth curves.

Lemma 1.6 yields truncations of G^i, $i = 1, 2$ that we keep on denoting by

$$G^i = \left(\sum_{k=g^i_l}^{\max\left(\sigma_l - 1, g^i_l\right)} a^i_{lk} t^k \right)_{l=1,\ldots,r}. \tag{1.7}$$

Notice that some components of (1.7), but not all, may vanish.

We are going to normalize the expressions of G^i in order to make it unique and depending only on the marked curve $(C,\bar{h})$. The first normalization consists in the following: for $i = 1,\ 2$ let us consider the smallest l_i such that $g^i_{l_i} \neq \infty$; we impose that

$$a^i_{l_i g^i_{l_i}} = 1.$$

To go further in the normalization, we will use Γ-reductions. For $i = 1, 2$ let us consider $\underline{P}^i = (P^i_1, \dots, P^i_r)$ defined by

$$\begin{cases} P^i_l = \left[g^i_l, \max\left(\sigma_l - 1, g^i_l\right)\right] \cap \mathbb{N} \cup \{\infty\} & \text{if } l \neq l_i \\ P^i_{l_i} = \left[g^i_{l_i} + 1, \max\left(\sigma_{l_i} - 1, g^i_{l_i}\right)\right] \cap \mathbb{N} \cup \{\infty\}. & \end{cases}$$

Notice that if $g^i_l = \infty$ then $P^i_l = \{\infty\}$. In the same way, if $g^i_{l_i} = \sigma_{l_i} - 1$ then $P^i_{l_i} = \{\infty\}$.

For any $\underline{n} \in \overline{\mathbb{N}}^r$, we denote by $\operatorname{Init}^i(\underline{n})$ the integer defined by

$$\min\left\{k \,\middle|\, (\underline{n})_k \neq \infty \text{ and } (\underline{n})_k \neq g^i_k\right\}.$$

If $\underline{n} \neq \underline{\infty}$ and $\underline{n} \in \Gamma \cap \prod_{l=1,\dots,r} P^i_l$ then $\operatorname{Init}^i(\underline{n})$ is well defined since the set of which it is the minimum is non-empty : indeed, if for any l, one has $(\underline{n})_l = g^i_l$ or ∞, then in particular, $(\underline{n})_{l_i} = \infty$. Moreover, $\underline{n}$ belongs to $F_J(g^i)$ where J is defined by

$$J = \{k \,|\, (\underline{n})_k \neq \infty\}.$$

The set J is non-empty since $\underline{n} \neq \underline{\infty}$ and is proper since $l_i \notin J$. That is impossible because by definition of absolute point, $F_J(g^i)$ is empty.

We choose a sequence of Γ-reductions of $\underline{P}^i$

$$\underline{P}^i = \underline{P}^{i,(0)} \xrightarrow{\underline{n}_0, k_0} \underline{P}^{i,(1)} \xrightarrow{\underline{n}_1, k_1} \cdots \underline{P}^{i,(q_i-1)} \xrightarrow{\underline{n}_{q_i-1}, k_{q_i-1}} \underline{P}^{i,(q_i)}$$

such that for any $t \in \{0, \dots, q_i - 1\}$,

(P_1) $k_t = \operatorname{Init}^i(\underline{n}_t) = \min\{\operatorname{Init}^i(\underline{n}) \mid \underline{n} \neq \underline{\infty},\ \underline{n} \in \Gamma \cap \prod_{l=1,\dots,r} P^{i,(t)}_l\}$ and

(P_2) among the $\underline{n}_t$'s that satisfy the previous equality, we choose the one for which the integer

$$(\underline{n}_t)_{\operatorname{Init}^i(\underline{n}_t)}$$

is the smallest possible.

The element $(\underline{n}_t)$ might not be unique which is why we keep track of the choice in the Γ-reduction. As one can see, the sequence of Γ-reductions is constructed by induction on the integer $(\underline{n}_t)_{\operatorname{Init}^i(\underline{n}_t)}$. Finally, notice that $\underline{P}^i$ does not depend on G^i but only on Γ.

Let us show how the Γ-reduction

$$\underline{P}^{i,(t)} \xrightarrow{\underline{n}_t, k_t} \underline{P}^{i,(t+1)}$$

allows us to normalize G^i. The r-uple $\underline{n}_t$ being an element of Γ, by definition, there exists a sum of the form

$$W^{i,(t)} = \sum_{\beta\in\mathbb{N}^2} w_{\beta}^{i,(t)} \left(G^1\right)^{\beta_1} \left(G^2\right)^{\beta_2}$$

such that $\nu(W^{i,(t)}) = \underline{n}_t$ and the coefficient of $t^{(\underline{n}_t)_{k_t}}$ in the k_t^{th} component is equal to 1. The difference

$$G^i - a^i_{k_t\left(\underline{n}_t\right)_{k_t}} W^{i,(t)} \tag{1.8}$$

belongs to $\widehat{\mathcal{O}}_C$ and the coefficient of $t^{(\underline{n}_t)_{k_t}}$ in the k_t^{th} component vanishes. By construction, after a Γ-reduction, the new couple of functions defined by (1.8) still generates $\widehat{\mathcal{O}}_C$. Doing the whole process of Γ-reductions for both $G^i,\ i = 1,2$ and a final truncation at σ, we obtain *a normalized family of generators that we denote* $(\mathfrak{N}^i(G^i))_{i=1,2}$. By construction, following the properties (P_1) and (P_2), a normalized family of generators $(\mathfrak{N}^i(G^i))_{i=1,2}$ is written

$$\mathfrak{N}^i(G^i) = \left(\sum_{k\in P_1^{i,(q_i)}} a^i_{1k}t^k, \dots, t^{g^i_{l_i}} + \sum_{k\in P_{l_i}^{i,(q_i)}} a^i_{l_ik}t^k, \dots, \sum_{k\in P_r^{i,(q_i)}} a^i_{rk}t^k\right). \tag{1.9}$$

The main characteristic of this normalized basis is that its parameters are unique: indeed, G and G' being two couples of normalized generators as in (1.9), we consider the valuation

$$\gamma = \nu\left(G^i - (G')^i\right).$$

By definition, γ is an element of Γ. By construction of the normalized family, it is also an element of $\prod_{l=1,\dots,r} P_l^{i,(q_i)}$. Since $\underline{P}^{i,(q_i)}$ is Γ-reduced, γ is equal to $\underline{\infty}$ and G^i and $(G')^i$ are equal. Therefore the normalized basis is unique and we can consider the following *well defined* map

$$\mathbb{M}_S : \begin{cases} \mathrm{Top}^\bullet(S) \longrightarrow \prod_{l,i} \mathbb{C}^{P_l^{i,(q_i)}} \\ (C, \overline{h}) \longmapsto \left(a^i_{lk}\right) \end{cases}$$

that associates to a marked curve in $\mathrm{Top}^\bullet(S)$, the ordered coefficients of a normalized family of generators of $\mathcal{O}_C$.

1.6. $\mathrm{Top}^{\bullet}(S)$ as a constructible set

In this section, we are going to prove the

Proposition 1.11. — *The image of $\mathbb{M}_S$ is a constructible algebraic set, i.e, a finite union of finite intersections of algebraic subsets and complements of algebraic subsets of the affine set $\prod_{l,i}\mathbb{C}^{P_l^{i,(q_i)}}$.*

Proof. — Consider an element of $\prod_{l,i}\mathbb{C}^{P_l^{i,(q_i)}}$ and the associated couple (G^1,G^2) as in (1.9). The complete ring generated by G is the completion of the ring of a plane curve C with r components $C^1,\ldots,C^r$ given by the coordinates of G. Fix some i in $\{1,\ldots,r\}$. We begin by proving that the condition

$$g^i\in\Gamma_C$$

is a constructible condition. Choose any reduced equation $h_i(x,y)$ of the curve

$$\bigcup_{j\notin\mathcal{I}_{g^i}} C^j.$$

If the complement of $\mathcal{I}_{g^i}$ is empty, choose simply $h_i=1$. Consider $\mathcal{N}$ the finite set of couples of integers $(u,v)\in\mathbb{N}^2$ such that

$$\nu\left(h_i\left(G^1,G^2\right)\left(G^1\right)^u\left(G^2\right)^v\right)\ngeq\sigma.$$

and the set of expressions of the form

$$h_i\left(G^1,G^2\right)\times\sum_{(u,v)\in\mathcal{N}}\beta_{uv}\left(G^1\right)^u\left(G^2\right)^v,\qquad\beta_{uv}\in\mathbb{C}\tag{1.10}$$

where the β_{uv}'s are coefficients. It follows that $g^i\in\Gamma_C$ is equivalent to the existence of a family $\{\beta_{uv}\}_{uv}$ so that the expression (1.10) has a valuation equal to g^i. Let

$$L^i_{l,k}$$

be the coefficient of t^k in the l^{th} component of (1.10). The functions $L^i_{l,k}$ are linear forms in the variables β_{uv} whose coefficients are algebraic expressions in the coefficients of the generators G^i. The condition $g^i\in\Gamma_C$ is equivalent to require that for each $l=1,\ldots,r$, the linear form L^i_{l,g^i_l} is linearly independent of the linear forms $L^i_{l,k}$ for $l=1,\ldots,r$ and $k<g^i_l$. The latter condition is a constructible one in the coefficients of the generators G^i since it can be expressed using the ranks of the minors of the matrix of these linear forms. It follows that $g^i\subset\Gamma_C$ and thus

$$\Gamma\subset\Gamma_C$$

is a constructible condition. We can now proceed analogously to prove that $\Gamma = \Gamma_C$ is also a constructible condition : indeed, according to [17], provided that $\Gamma \subset \Gamma_C$, the equality $\Gamma = \Gamma_C$ is equivalent to the equality

$$\Gamma \cap \prod_{i=1}^{r} [0, \sigma_l] = \Gamma_C \cap \prod_{i=1}^{r} [0, \sigma_l]$$

which induces a finite number of conditions, that can be proven to be constructible with similar arguments. □

1.7. Action on $\mathrm{Top}^{\bullet}(S)$

The group $(\mathrm{Diff}\,(\mathbb{C}, 0))^r$ acts on the image of $\mathbb{M}_S$ the following way : given a point A in the image, consider its corresponding couple of generators $\left(G^1, G^2\right)$. Take an element $\phi \in (\mathrm{Diff}\,(\mathbb{C}, 0))^r$ and right compose G^i, $i = 1, 2$, by ϕ ; apply the process of normalization following a sequence of Γ-reductions initially fixed and truncate the final expressions. In the end, the coefficients of the new normalized couple of generators

$$\left(\mathfrak{N}^i\left(G^i \circ \phi\right)\right)_{i=1,2}$$

corresponds to some expressions $\phi \cdot A$ which depend on A and ϕ.

Lemma 1.12. — *The application $(\phi, A) \to \phi \cdot A$ is an action. More precisely, for any ϕ, ψ in $(\mathrm{Diff}(\mathbb{C}, 0))^r$*

$$\phi \cdot (\psi \cdot A) = (\psi \circ \phi) \cdot A.$$

Proof. — For $i = 1, 2$, consider a normalized basis (G^1, G^2) and the two following normalizations

$$\left(\mathfrak{N}^i\left(G^i \circ \psi \circ \phi\right)\right)_{i=1,2} \text{ and } \left(\mathfrak{N}^i\left(\mathfrak{N}^i\left(G^i \circ \psi\right) \circ \phi\right)\right)_{i=1,2}.$$

Both are normalized bases of the ring

$$(\psi \circ \phi)^{\star}\, \mathcal{O}_C = \{\gamma \circ \psi \circ \phi \,|\, \gamma \in \mathcal{O}_C\}.$$

The rings $\mathcal{O}_C$ and $(\psi \circ \phi)^{\star}\mathcal{O}_C$ share the same semiring of valuations Γ. Thus, for $i = 1, 2$, the valuation

$$\nu\left(\mathfrak{N}^i\left(G^i \circ \psi \circ \phi\right) - \mathfrak{N}^i\left(\mathfrak{N}^i\left(G^i \circ \psi\right) \circ \phi\right)\right)$$

is an element of $\Gamma \cap \prod_{l,i} P_l^{i,(q_i)}$. Since $\underline{P}^{i,(q_i)}$ is Γ-reduced, this valuation is $\underline{\infty}$ and one has

$$\mathfrak{N}^i\left(G^i \circ \psi \circ \phi\right) = \mathfrak{N}^i\left(\mathfrak{N}^i\left(G^i \circ \psi\right) \circ \phi\right),$$

which is the lemma. □

Let us denote by $\text{Diff}^c(\mathbb{C},0)$ the quotient of $\text{Diff}\,(\mathbb{C},0)$ by the normal subgroup of elements of the form

$$t \longrightarrow t + ut^c + \cdots .$$

The truncation at σ being part of the normalization process, it follows that the previous action factorizes through

$$\prod_{i=1}^{r} \text{Diff}^{\sigma_i}(\mathbb{C},0) . \tag{1.11}$$

Since the group (1.11) is a connected solvable algebraic group, Theorem 1.2 follows from Lemma 1.3 and the previous constructions.

1.8. An example

Let S be the curve $y\left(y^2 - x^3\right) = 0$. Figure 1.1 shows the semiring Γ_S. In this rather simple situation, it can computed *by hand.* In the general case, there exist algorithms to compute the semiring of a curve with several components, see [2].

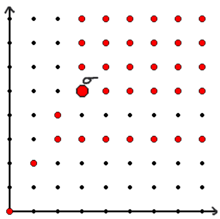

Figure 1.1. Semiring of values the curve $\left\{y\left(y^2 - x^3\right) = 0\right\}$.

Let $(C, \overline{h})$ be in $\text{Top}^{\bullet}(S)$. The conductor σ of Γ is $(3,5)$. The set of irreducible absolute points of Γ is

$$\{(1,2)\,,(2,4)\,,(3,\infty)\,,(\infty,3)\} .$$

Since $(2,4) = 2 \times (1,2)$, following [17], the set that minimaly generates Γ as semiring is

$$g = \{(1,2)\,,(3,\infty)\,,(\infty,3)\} .$$

Since $\nu(x) = (1,2)$ and $\nu(y) = (\infty,3)$, applying Lemma 1.6 leads to a couple of generators that are written

$$\left\{G^1 = \left(a^1_{11}t + a^1_{12}t^2, a^1_{22}t^2 + a^1_{23}t^3 + a^1_{24}t^4\right),\ G^2 = \left(0, a^2_{13}t^3 + a^2_{14}t^4\right)\right\}$$

with $a^1_{11} \neq 0,\ a^1_{22} \neq 0,\ a^2_{13} \neq 0$. Normalizing some initial non vanishing coefficients provides the following couple of generators

$$\left\{G^1 = \left(t + a^1_{12}t^2, a^1_{22}t^2 + a^1_{23}t^3 + a^1_{24}t^4\right),\ G^2 = \left(0, t^3 + a^2_{14}t^4\right)\right\}.$$

To reduce G^1 we consider the following data

$$P^{1,(0)}_1 = \{\infty, 2\} \qquad P^{1,(0)}_2 = \{\infty, 2, 3, 4\},$$

and the two successive Γ_S-reductions defined as follows

$$\begin{array}{lcl} \underline{P}^{1,(0)} & \xrightarrow{(2,3),1} & \underline{P}^{1,(1)} = (\{\infty\}, \{\infty, 2, 3, 4\}). \\ \underline{P}^{1,(1)} & \xrightarrow{(\infty,3),2} & \underline{P}^{1,(2)} = (\{\infty\}, \{\infty, 2, 4\}). \end{array}$$

Observe $\underline{P}^{1,(2)}$ is Γ_S-reduced. Since $\nu\left(\left(G^1\right)^2 + G^2\right) = (2,3)$, the transformation associated to the first Γ_S-reduction as in (1.8) is written

$$G^1 - a^1_{12}\left(\left(G^1\right)^2 + G^2\right) = \left(t + t^3(\cdots), (\star)\, t^2 + (\star)\, t^3 + (\star)\, t^4 + t^5(\cdots)\right),$$

which leads to a new generator that we still denote by G^1. Noticing that $\nu(G^2) = (\infty, 3)$ yields the transformation

$$G^1 - (\star)\, G^2.$$

The truncation at $\sigma = (3,5)$ finishes the normalization of G^1. The generator G^2 is already normalized since $(\infty, 4) \notin \Gamma$.

Therefore, the normalized family G has for final form

$$\left\{G^1 = \left(t, at^2 + bt^4\right), G^2 = \left(0, t^3 + ct^4\right)\right\} \tag{1.12}$$

and its elements depend only on $\mathfrak{E}_{(C,\overline{h})}$. The map $\mathbb{M}_S$ is defined by

$$\mathbb{M}_S : \begin{cases} \mathrm{Top}^\bullet(S) \longrightarrow \mathbb{C}^3 \\ (C, \overline{h}) \longmapsto (a, b, c) \end{cases}$$

following (1.12). By construction if $a \neq 0$, any curve C associated to a ring generated by such a family admits a semiring of values Γ_C that contains $(1,2)$ and $(\infty, 3)$. It can be checked that $a \neq 0$ is the sole condition to ensure that actually, $\Gamma_C = \Gamma_S$. Thus the image of $\mathbb{M}_S$ is the constructible set $\mathbb{C} \setminus \{0\} \times \mathbb{C}^2$.

Let us compute the action of $\phi \in \mathrm{Diff}^3(\mathbb{C}, 0) \times \mathrm{Diff}^5(\mathbb{C}, 0)$ on $A \in \mathbb{C} \setminus \{0\} \times \mathbb{C}^2$ induced by the present construction where

$$\phi = \left(ut + vt^2, \alpha t + \beta t^2 + \gamma t^3 + \delta t^4\right) \text{ and } A = (a, b, c).$$

The action of ϕ on A is written

$$\phi \cdot A = \left(\frac{a\alpha^2}{u}, \frac{-a^2\alpha^4 v - 2a\alpha^2\beta c u^2 + \alpha^4 b u^2 + 2a\alpha\gamma u^2 - 5a\beta^2 u^2}{u^3}, \alpha c + 3\frac{\beta}{\alpha}\right)$$

and the quotient reduces to the class of the point $(1,0,0)$. As a matter of fact, the curve S has no moduli [11].

2. Optimal vector field for a germ of curve S

The space $\mathbb{M}^{\bullet}(S)$ is now endowed with a complex structure. The remainder of the article is interested in the generic dimension of $\mathbb{M}^{\bullet}(S)$. In order to reach this purpose, subsequently, we proceed to the study of the module of vector fields tangent to S.

Let S be a germ of curve in $(\mathbb{C}^2,0)$ and f a reduced equation of S. Throughout this article, $\mathrm{Der}(\log S)$ will stand for be the $\mathcal{O}_{(\mathbb{C}^2,0)}$- module of vector fields tangent to S, that is such that the set of vector fields X such that

$$X \cdot f \in (f)\,.$$

It will be called *the Saito module of S* in reference to [23]. Associated to the latter, we consider the following analytical invariant

Definition 2.1. — *The* Saito number *of S is the integer*

$$\mathfrak{s}(S) = \min_{X \in \mathrm{Der}(\log S)} \nu(X),$$

where ν is the valuation defined by

$$\nu\left(a\partial_x + b\partial_y\right) = \min\left(\nu\left(a\right), \nu\left(b\right)\right).$$

According to [23], the Saito module of S is a free $\mathcal{O}_{(\mathbb{C}^2,0)}$- module of rank 2. If $\{X_1, X_2\}$ is one of its basis, said to be *a Saito basis for S*, it is easily seen that the number of Saito of S satisfies

$$\mathfrak{s}(S) = \min\left(\nu(X_1), \nu(X_2)\right).$$

Following again [23], $\{X_1, X_2\}$ is a Saito basis for S if and only if the following property holds.

Criterion (Criterion of Saito). — *$\{X_1, X_2\}$ is a Saito basis for S if and only if there exists a germ of unit u such that*

$$X_1 \wedge X_2 = uf, \tag{2.1}$$

where $\cdot\wedge\cdot$ stands for the determinant of the vector fields in any coordinates.

The property (2.1) will be referred to as *the criterion of Saito.* Evaluating the valuation of (2.1) gives the inequality

$$\nu(X_1) + \nu(X_2) \leqslant \nu\left(X_1 \wedge X_2\right) = \nu\left(f\right) = \nu(S). \tag{2.2}$$

In particular, one has

$$\mathfrak{s}(S) \leqslant \frac{\nu(S)}{2}. \tag{2.3}$$

Definition 2.2. — *A vector field $X \in \mathrm{Der}(\log S)$ is said to be* optimal *for S if $\nu(X) = \mathfrak{s}(S)$.*

Example 2.3. — Let S be the double cusp given by

$$S = \left\{ \left(x^2 - y^3\right)\left(y^2 - x^3\right) = 0 \right\}.$$

Then an optimal vector field can be given by

$$X = \left(2x^2 + \frac{5}{2}y^3 - \frac{9}{2}x^3 y\right)\partial_x + \left(3xy - 3x^2y^2\right)\partial_y.$$

In particular

$$\mathfrak{s}(S) = 2.$$

Proposition 2.4. — *If X is optimal for S, then there exists a vector field Y such that $\{X, Y\}$ is a Saito basis for S.*

Proof. — Let $\{X_1, X_2\}$ be any Saito basis for S. According to the criterion of Saito, there exists a unit u such that

$$X_1 \wedge X_2 = uf. \tag{2.4}$$

Since $\{X_1, X_2\}$ is a basis, there exist functions $u_i,\ i = 1, 2$ such that

$$X = u_1 X_1 + u_2 X_2.$$

Since $\nu(X) = \mathfrak{s}(S) = \min(\nu(X_1), \nu(X_2))$, for some i, say $i = 1$, u_i is a unit. Then, using (2.4) yields

$$X \wedge X_2 = u_1 u f.$$

and thus, $\{X, X_2\}$ is a Saito basis for S. □

2.1. Curve of radial type

Let E be the single blowing-up at 0. The total space of the blowing-up will be denoted by $\mathcal{M}$,

$$E : (\mathcal{M}, D) \longrightarrow (\mathbb{C}^2, 0).$$

For any curve S, S^E will stand for the strict transform of S by E, that is the closure in $\mathcal{M}$ of $E^{-1}(S \setminus \{0\})$. Moreover, for any vector field Y, Y^E will be the blown-up vector field $E^\star Y$ divided by the maximal power of a local equation of D.

Definition 2.5. — *Let* Y *be a germ of vector field in* $(\mathbb{C}^2,0)$*. It is said* dicritical *if* Y^E *is generically transverse to the exceptional divisor* D*.*

Being dicritical is a property that can be read on the homogeneous component of smallest degree. Indeed,

Proposition 2.6. — Y *is dicritical if and only if its homogeneous component of smallest degree* $Y^{\nu(Y)}$ *is tangent to the radial vector field, i.e, there exist an homogeneous polynomial function* R *such that*

$$Y^{\nu(Y)} = R(x,y)\left(x\partial_x + y\partial_y\right).$$

Suppose that X is dicritical and optimal for S and let Y be such that $\{X,Y\}$ is a basis of $\operatorname{Der}(\log S)$. Writing

$$X^{\nu(X)} = R(x,y)\left(x\partial_x + y\partial_y\right),$$

we have that for any couple of non-vanishing functions (a,b), the initial part of $aX+bY$ is equal to

$$a(0)R(x,y)\left(x\partial_x + y\partial_y\right) + b(0)Y^{(\mathfrak{s}(S))}.$$

where $Y^{(\star)}$ stands for the homogeneous part of degree $\star$ of Y. If Y is optimal and not dicritical then for a and b generic, $aX+bY$ is not dicritical. Which is why, we consider the following definition.

Definition 2.7. — S *is said to be* of radial type *if all optimal vector fields for* S *are dicritical.*

2.2. Flat Saito basis

In this section, we are going to identify an open dense set $U \subset \mathbb{M}^\bullet(S)$ for which, the Saito basis of $C \in U$, can be extended locally around C in $\mathbb{M}^\bullet(S)$ into a family of Saito bases. Further on, an example will illustrate that this property holds only generically.

Theorem 2.8. — *There exist an open dense set* $U \subset \mathbb{M}^\bullet(S)$ *on which the Saito number is constant. More precisely, for any* $(C,\overline{h}) \in U$*, there exist two germs of analytical families of vector fields*

$$c \in \left(\mathbb{M}^\bullet(S),(C,\overline{h})\right) \longmapsto X_i\left(c\right),\ i=1,2$$

such that for any c*,* $\{X_1(c),X_2(c)\}$ *is a Saito basis for which the multiplicity*

$$\nu\left(X_1\left(c\right)\right) = \mathfrak{s}\left(c\right)$$

is a constant depending only on S*.*

Proof. — Let $(C,\overline{h}) \in \mathbb{M}^{\bullet}(S)$ be a regular point for the complex structure of $\mathbb{M}^{\bullet}(S)$. Consider a miniversal deformation of C

$$(\Sigma, C) \subset \left(\mathbb{C}^{2+N}, \mathbb{C}^{2} \times \{0\}\right) \xrightarrow{\pi} \left(\mathbb{C}^{N}, 0\right), \qquad \pi(x,t) = t \in \mathbb{C}^{N} \tag{2.5}$$

versal for topologically trivial deformations of C and for which the singular locus of Σ is $\{0\} \times \mathbb{C}^{N}$: it is enough to consider the miniversal deformation of any reduced equation of C and to restrict it to the associated smooth μ-constant stratum. We fix an open neighborhood $\mathbb{C}^{2+N} \supset \mathcal{U} \ni 0$ on which Σ and C are well defined. By shrinking $\mathcal{U}$ if necessary, we can also suppose that, out of its singular locus, Σ is transverse to the fiber of π, that is for any $p \in \mathcal{U} \setminus \{0\} \times \mathbb{C}^{N}$,

$$\pi^{-1}(\pi(p)) \not\subset T_p\Sigma. \tag{2.6}$$

The deformation (2.5) is topologically trivial : more precisely, there exists an homeomorphism $\mathcal{H}{:}(\mathbb{C}^{2+N}, 0) \to (\mathbb{C}^{2+N}, 0)$ such that

(1) $\pi\mathcal{H} = \pi$
(2) $\mathcal{H}|_{\pi^{-1}(0)} = h$
(3) The following diagram commutes

$$\begin{array}{ccc} \left(S \times \left(\mathbb{C}^{N}, 0\right), S\right) & \xrightarrow{\mathcal{H}} & (\Sigma, C) \\ & \searrow^{\pi} & \downarrow{\pi} \\ & & \left(\mathbb{C}^{N}, 0\right) \end{array}$$

By construction, the map $\mathfrak{C}$ defined by

$$t \in \left(\mathbb{C}^{N}, 0\right) \overset{\mathfrak{C}}{\mapsto} \left(\Sigma|_{\pi^{-1}(t)}, \overline{\mathcal{H}}|_{\pi^{-1}(t)}\right) \in \mathbb{M}^{\bullet}(S) \tag{2.7}$$

is a local diffeomorphism.

For technical reason, we add to Σ an hyperplane H not contained in Σ and transverse to π. Consider $\Sigma^{\circ} = \Sigma \cup H$. In what follows, $f_{\Sigma^{\circ}}$ stands for a reduced equation of Σ. The kernel of the evaluation map

$$\mathrm{Der}\left(\log \Sigma^{\circ}\right) \xrightarrow{d\pi(\cdot)} \left(\mathcal{O}_{N+2}\right)^{N}$$

is the sheaf $\mathrm{Der}^{\uparrow}(\log \Sigma^{\circ})$ of *vertical* vector fields tangent to Σ°. In the initial coordinates (x,y,t) a section of $\mathrm{Der}^{\uparrow}(\log \Sigma^{\circ})$ is written

$$a(x,y,t)\,\partial_x + b(x,y,t)\,\partial_y$$

where a and b are analytic functions. The sheaf $\mathrm{Der}\left(\log \Sigma^{\circ}\right)$ is coherent, so is $\mathrm{Der}^{\uparrow}\left(\log \Sigma^{\circ}\right)$. Note that if X is a section of $\mathrm{Der}^{\uparrow}\left(\log \Sigma^{\circ}\right)$, then for any $t \in \pi(\mathcal{U})$, $X|_{\pi^{-1}(t)}$ is tangent to $\Sigma^{\circ}|_{\pi^{-1}(t)}$. Fix a system of generators

$$\{X_1, \ldots, X_n\} \tag{2.8}$$

of $\mathrm{Der}^{\uparrow}(\log\Sigma^{\circ})(\mathcal{U})$. We are going to use the following remarks which are consequences of the coherence property : for any open set $\mathcal{V}\subset\mathcal{U}$, the vector fields $X_1|_{\mathcal{V}},\ldots,X_n|_{\mathcal{V}}$ generate $\mathrm{Der}^{\uparrow}(\log\Sigma^{\circ})(\mathcal{V})$. Moreover,

(1) if $\mathcal{V}$ does not meet Σ° then $\mathrm{Der}^{\uparrow}(\log\Sigma^{\circ})(\mathcal{V})$ is the set of all holomorphic vertical vector fields on $\mathcal{V}$.

(2) if $\mathcal{V}$ meets the smooth part of Σ°, then $\mathrm{Der}^{\uparrow}(\log\Sigma^{\circ})(\mathcal{V})$ is locally freely generated on $\mathcal{V}$ by the vertical vector fields $u\partial_u$ and ∂_v where (u,v,t) is a local system of coordinates preserving the fibration π for which $u=0$ is an equation of the trace of Σ° on $\mathcal{V}$: such a local system of coordinates exists under the transversality property (2.6). In particular, the product

$$u\partial_u\wedge\partial_v$$

vanishes at order 1 along Σ°.

All the $X_i's$ cannot vanish identically on a given component of Σ° because for instance the section of $\mathrm{Der}^{\uparrow}(\log\Sigma^{\circ})$ defined by

$$\partial_x\left(f_{\Sigma^{\circ}}\right)\partial_y-\partial_y\left(f_{\Sigma^{\circ}}\right)\partial_x$$

does not vanish on any component of Σ°. Considering if necessary a combination of the $X_i's$, we can suppose that X_1 does not vanish identically on any component of Σ°. We can also suppose that X_1 is singular in codimension 2 : indeed, if not, there exists $\widetilde{X}_1$ such that $X_1=h\widetilde{X}_1$ where h is an holomorphic map with $h(0)=0$. Since h cannot vanish identically on any component of Σ°, $\widetilde{X}_1$ is tangent to Σ° and the family

$$\left\{\widetilde{X}_1,\ldots,X_n\right\}$$

still generates the sheaf $\mathrm{Der}^{\uparrow}(\log\Sigma^{\circ})$. Now, if there exists $j\neq 1$ such that

$$X_1\wedge X_j\equiv 0$$

then, by division, there exists ϕ such that $X_j=\phi X_1$, which contradicts the minimality of the system of generators (2.8). Thus, for any $j\neq 1$, there exists a function $g_j\not\equiv 0$ such that

$$X_1\wedge X_j=f_{\Sigma^{\circ}}g_j=xf_{\Sigma}g_j \tag{2.9}$$

where the system of coordinates (x,y,t) is chosen so that x is an equation of the added hyperplan H. Consider a point p in the zero set $Z(g_2,\ldots,g_n)$ of the ideal $(g_2,\ldots,g_n)$. If p is not in Σ° then all the generators of $\mathrm{Der}^{\uparrow}(\log\Sigma^{\circ})$ are tangent two by two at p, which is impossible in view of the above remark (1). Therefore, one has

$$Z\left(g_2,\ldots,g_n\right)\subset\Sigma^{\circ}.$$

We are going to improve the above inclusion, showing that one can suppose that

$$Z\left(g_2,\ldots,g_n\right) \subset \{0\} \times \mathbb{C}^N.$$

Consider the following set

$$\Delta = \left\{ t \in \left(\mathbb{C}^N, 0\right) \,\middle|\, \pi^{-1}\big|_{\Sigma_0}(t) \subset Z\left(g_2,\ldots,g_n\right) \right\}.$$

It is a closed analytic subset of $(\mathbb{C}^N, 0)$ and we remove $\pi^{-1}(\Delta)$ of $\mathcal{U}$. Now, fixed some t and denote by denote by I the canonical injection $I : (\mathbb{C}^2, 0) \to \pi^{-1}(t)$, $I(x) = (x,t)$. If the intersection

$$\Delta_t = \Sigma^\circ\big|_{\pi^{-1}(t)} \cap Z\left(g_2,\ldots,g_n\right)$$

contains $0 \times \{t\}$ has a non isolated point of Δ_t, it contains also an analytic curve which is a component of $I^\star f_{\Sigma^\circ} = 0$. Therefore there is a factor h of $I^\star f_{\Sigma^\circ}$ that divides $I^\star g_i$ for any i. Thus, for $i \geqslant 2$, one has

$$X_1\big|_{\pi^{-1}(t)} \wedge X_i\big|_{\pi^{-1}(t)} = h^2(\cdots).$$

Since the vector fields $X_i\big|_{\pi^{-1}(t)}$ are tangent to $h=0$, any couple of elements in $\operatorname{Der}^\uparrow(\log \Sigma^\circ)\Big|_{\pi^{-1}(t)}$ has a contact of order 2 locally around the zero locus of h, which is impossible according to the above remark (2). As a consequence, for any $t \in (\mathbb{C}^N, 0)$, if Δ_t contains $0 \times \{t\}$, it is as an isolated point in Δ_t. So, Δ_t is a finite set.

Lemma 2.9. — *Let $W \subset (\mathbb{C}^{2+N}, 0)$ be an analytic set such that for any t, $W \cap \pi^{-1}(t)$ is finite. Then*

$$\{0\} \times \mathbb{C}^N \not\subset \overline{W \setminus \{0\} \times \mathbb{C}^{\mathbb{N}}}.$$

Proof. — The hypothesis ensures that $\operatorname{codim} W \geqslant 2$. If $\operatorname{codim} W \geqslant 3$, the lemma is clear since $\operatorname{codim}\Big(\underbrace{\{0\} \times \mathbb{C}^N}_{V}\Big) = 2$. Suppose $\operatorname{codim} W = 2$. Let us write

$$V = (W \cap V) \cup \left(\overline{V \setminus W}\right).$$

Since V is irreducible, either $W \cap V = V$, and V is an irreducible component of W, or $\overline{V \setminus W} = V$, and $V \cap W$ is an analytic subset of V of codimension at least 1 in V, and thus of codimension at least 3 in $\mathbb{C}^{2+N}$. In any case, the lemma is proved. □

Following the lemma, the analytic set K defined by

$$K = \{0\} \times \mathbb{C}^N \cap \overline{\Sigma^\circ \cap Z\left(g_2,\ldots,g_n\right) \setminus \{0\} \times \mathbb{C}^{\mathbb{N}}}$$

is a strict analytic subset of $\{0\} \times \mathbb{C}^N$ admitting a neighborhood on which

$$Z(g_2, \ldots, g_n) \subset \{0\} \times \mathbb{C}^N$$

At the level of the ideals, the inclusion above ensures that there exists $M \in \mathbb{N}$ such that

$$(x, y)^M \subset (g_2, \ldots, g_n)$$

where (x, y) are local coordinates for which x is a local equation of the hyperplane H. As a consequence, there exists a relation of the following form

$$x^M = \sum_{i=2}^{n} h_i g_i$$

and considering $Y = \sum_{i=2}^{n} h_i X_i$ and the relation (2.9) yields a vector field Y in $\mathrm{Der}^{\uparrow}(\log \Sigma^{\circ})(\mathcal{U})$ such that

$$X_1 \wedge Y = f_{\Sigma} x^{M+1}.$$

Notice that X_1 and Y are both tangent to $H = \{x = 0\}$. Let us write in coordinates

$$X_1 = xa^1(x, y, t) \frac{\partial}{\partial x} + \left(b_0^1(y, t) + xb_1^1(x, y, t)\right) \frac{\partial}{\partial y}$$
$$Y = xa^2(x, y, t) \frac{\partial}{\partial x} + \left(b_0^2(y, t) + xb_1^2(x, y, t)\right) \frac{\partial}{\partial y}.$$

Replacing if necessary X_1 by $X_1 + Y$, we can suppose that

$$\nu_y\left(b_0^1\right) \leqslant \nu_y\left(b_0^2\right)$$

where ν_y is the valuation in the ring $\mathbb{C}\{t\}\{y\}$. Consider the vertical vector field

$$\widetilde{Y} = \frac{1}{x}\left(Y - \frac{b_0^2}{b_0^1} X_1\right).$$

It is holomorphic removing if necessary, some fibers $\pi^{-1}(t)$ for t in some closed analytic set of $\mathbb{C}^N$ related to the zeros of $b_0^1(0, t)$. Moreover, one has

$$X_1 \wedge \widetilde{Y} = f_{\Sigma} x^M.$$

Since X_1 is tangent to $x = 0$ and since its singular locus has codimension 2, $\widetilde{Y}$ is also tangent to $x = 0$. The process can be repeated and finally, one obtains two vertical vector fields X_1 and X_2 tangent to Σ such that

$$X_1 \wedge X_2 = f_{\Sigma}.$$

The functions

$$t \in \pi(\mathcal{U}) \longmapsto \nu\left(X_i|_{\pi^{-1}(t)}\right),\ i = 1, 2$$

are lower semi-continuous. Replacing if necesssary X_1 by $X_1 + X_2$, we consider an open set $\mathcal{U}' \subset (\mathbb{C}^N, 0)$, whose closure is a neighborhood of 0, on which

$$\forall t \in \mathcal{U}',\ \nu\left(X_1|_{\pi^{-1}(t)}\right) \leqslant \nu\left(X_2|_{\pi^{-1}(t)}\right).$$

According to the criterion of Saito, for any t,

$$\left\{X_1|_{\pi^{-1}(t)}, X_2|_{\pi^{-1}(t)}\right\}$$

consists in basis of Saito for the curve $\Sigma|_{\pi^{-1}(t)}$. Therefore, for any $t \in \mathcal{U}'$, one has

$$\nu\left(X_1|_{\pi^{-1}(t)}\right) = \mathfrak{s}\left(\Sigma|_{\pi^{-1}(t)}\right).$$

From (2.7), one can consider the open set $\mathfrak{C}(\mathcal{U}') \subset \mathbb{M}^\bullet(S)$ and the union of such open sets while the above construction is done in the neighborhood of any regular point $(C, \overline{h})$ in $\mathbb{M}^\bullet(S)$. By construction, the resulting open set has the desired properties. □

From now on, a curve C in $\mathbb{M}^\bullet(S)$ will be said *generic* if it belongs to the open set identified in the theorem above : in that sense, for a generic curve C in its moduli space, we will be allowed to consider an analytical family of Saito bases following any topologically trivial deformation of C.

Example 2.10. — Consider the union of four regular transversal curves. Up to some change of coordinates, it can be written

$$S = \{xy\,(y + x)\,(y + t_1 x) = 0\}$$

where $t_1 \in \mathbb{M}^\bullet(S) = \mathbb{C}\backslash\{0, 1\}$. It can be seen [10] that it admits a miniversal deformation for the topologically trivial deformations of the form

$$\Sigma = \{F\,(x, y, t) = xy\,(y + x)\,(y + tx) = 0\} \in \left(\mathbb{C}^2 \times \mathbb{C}, (0, 0, t_1)\right).$$

The basis highlighted in Theorem 2.8 can be explicited in the above coordinates as

$$X_1 = x\partial x + y\partial y, \qquad X_2 = \partial_x F \partial_y - \partial_y F \partial_x.$$

In this case, X_1 and X_2 is a Saito basis in a whole neighborhood of $t_1 \in \mathbb{M}^\bullet(S)$. In general, the situtation is not so favourable.

Example 2.11. — Consider for instance the union of five regular transversal curves, which is written

$$S = \{xy\,(y + x)\,(y + \alpha x)\,(y + \beta x) = 0\}.$$

with $\alpha \neq 0,1$ and $\beta \neq 0,1,\alpha$. A miniversal deformation of S is written

$$\Sigma = \left\{F = xy\,(y+x)\,(y+t_1x)\left(y+t_2x+t_3x^2\right) = 0\right\} \\ \in \left(\mathbb{C}^2 \times \mathbb{C}^3, (0,0,\alpha,\beta,0)\right).$$

For the curve S, which corresponds to the parameter $(\alpha,\beta,0)$, a basis of Saito is given by

$$X_1 = x\partial x + y\partial y, \qquad X_2 = \partial_x F\partial_y - \partial_y F\partial_x.$$

However, this basis cannot be *extended,* in a whole neighborhood of $(\alpha,\beta,0)$. Since X_1 has a valuation equal to one, the Saito number of S is equal to 1. It can be seen that for any $t_3 \neq 0$, the number of Saito of $\Sigma|_{t=(\alpha,\beta,t_3)}$ is bigger than 2. Indeed, consider a vector field X tangent to $\Sigma|_{t=(\alpha,\beta,t_3)}$. If its valuation is smaller than 1, then it is dicritical. Thus it is written

$$X = k\,(x\partial_x + y\partial y) + (\cdots)$$

where k is a non vanishing constant. Following [5], X is linearizable and in some coordinates in which X is linear, the curve $\Sigma|_{t=(\alpha,\beta,t_3)}$ becomes exactly the union of five germs of straight lines, which is impossible if $t_3 \neq 0$. Finally, it can be seen that if $t_3 \neq 0$ then

$$\mathfrak{s}\left(\Sigma|_{t=(\alpha,\beta,t_3)}\right) = 2$$

and an optimal vector field for $\Sigma|_{t=(\alpha,\beta,v)}$ is written

$$X = (x+\epsilon y)\,(x\partial_x + y\partial y) + (\cdots)$$

where $\epsilon \neq 0,1$.

2.3. Saito basis for S and $S \cup l$

The process described below allows us to obtain a Saito basis for S from a Saito basis for $S \cup l$ where the curve l is a regular curve. This trick has been already introduced in the proof of Theorem 2.8. Throughout this article, it will be often a key argument.

Let S be a germ of curve and l be a germ of smooth curve that is not a component of S. Let $\{X_1, X_2\}$ be a Saito basis for $S \cup l$. The Saito criterion is written

$$X_1 \wedge X_2 = ufL \tag{2.10}$$

where u is a unity, f a reduced equation of S and L a reduced equation of l. Let us consider a local system of coordinates (x, y) in which $L = x$. Then, for $i = 1, 2$, the vector fields X_i can be written

$$X_i = xa_i\partial_x + \left(b_i^0 + xb_i^1\right)\partial_y, \quad a_i,\ b_i^1 \in \mathbb{C}\{x, y\},\ b_i^0 \in \mathbb{C}\{y\}.$$

Considering if necessary a generic change of basis

$$\{\alpha X_1 + \beta X_2, uX_1 + vX_2\}$$

where $\left|\begin{smallmatrix} \alpha & \beta \\ u & v \end{smallmatrix}\right| \neq 0$, one can suppose that

$$\nu(X_i) = \mathfrak{s}(S \cup l) \text{ and } \nu_y\left(b_1^0(y)\right) = \nu_y\left(b_2^0(y)\right)$$

where ν_y is the valuation in the ring $\mathbb{C}\{y\}$. In particular, the quotient $\frac{b_1^0}{b_2^0}$ extends holomorphically at $(x, y) = (0, 0)$ as a unit. The relation 2.10 leads to

$$\underbrace{\frac{\left(X_1 - \frac{b_1^0}{b_2^0}X_2\right)}{x}}_{X_1'} \wedge X_2 = uf, \tag{2.11}$$

where X_1' extends holomorphically at $(0, 0)$. Since $L = 0$ is not a component of S, the vector field X_1' leaves invariant S. The Saito criterion ensures that $\{X_1', X_2\}$ is a Saito basis for S.

Now, it is clear that

$$\nu(X_1') \geqslant \nu(X_1) - 1 = \mathfrak{s}(S \cup l) - 1.$$

Since, $\nu(X_2) = \mathfrak{s}(S \cup l)$, one has

$$\mathfrak{s}(S) = \mathfrak{s}(S \cup l) - 1 \text{ or } \mathfrak{s}(S \cup l).$$

Assume moreover, that S is not of radial type but $S \cup l$ is. By definition, X_1 and X_2 are dicritical. Thus, the homogeneous part of degree $\mathfrak{s}(S \cup l)$ of X_i is written

$$X_i^{(\mathfrak{s}(S \cup l))} = R_i\left(x\partial_x + y\partial_y\right).$$

Therefore the homogeneous part of degree $\mathfrak{s}(S \cup l) - 1$ of X_1' is

$$\frac{1}{x}\left(R_1 - \frac{b_1^0}{b_2^0}(0)R_2\right)\left(x\partial_x + y\partial_y\right). \tag{2.12}$$

If the above expression does not identically vanish, then X_1' would be dicritical. Since X_2 is dicritical too, S would be of radial type, which is impossible. Thus, the expression (2.12) vanishes and $\nu(X_1') \geqslant \mathfrak{s}(S \cup l)$. Since $\nu(X_2) = \mathfrak{s}(S \cup l)$ one has finally

$$\mathfrak{s}(S) = \mathfrak{s}(S \cup l).$$

Gathering the remarks above, we obtain the

Proposition 2.12. — *Let l be a germ of smooth curve that is not a component of S. Then*

(1) *In any case,* $\mathfrak{s}(S) = \mathfrak{s}(S \cup l) - 1$ or $\mathfrak{s}(S \cup l)$.
(2) *If S is not of radial type but $S \cup l$ is then*

$$\mathfrak{s}(S) = \mathfrak{s}(S \cup l).$$

The process described above can be reversed. Consider a Saito basis $\{X_1, X_2\}$ for S. Changing of basis if necessary, one can consider that

$$\nu(X_1) = \nu(X_2).$$

Let l be a *generic* smooth curve and L a reduced equation of l. Fix some coordinates (x, y) in which l has a parametrization of the form

$$\gamma(t) = (t, \epsilon(t)), \ t \in (\mathbb{C}, 0).$$

The product

$$X_1(\gamma) \wedge \gamma' \in \mathbb{C}\{t\}$$

has a valuation in $\mathbb{C}\{t\}$ equal to $\nu(X_1)$ or $\nu(X_1) + 1$ depending on whether X_1 is dicritical or not. Therefore, the quotient

$$\frac{X_1(\gamma) \wedge \gamma'}{X_2(\gamma) \wedge \gamma'}$$

extends holomorphically at $t = 0$ as a unit $\phi(t)$. By construction, the vector field

$$X_1 - \phi(x) X_2$$

is tangent to the curve l. Finally, in the coordinates (x, y), according to the criterion of Saito, the family

$$\{X_1 - \phi(x) X_2, LX_2\} \tag{2.13}$$

is a Saito basis for $S \cup l$.

3. Generic element in $\mathrm{Der}(\log S)$ and adapted Saito Bases

In (2.3), we remark that

$$\mathfrak{s}(S) \leqslant \frac{\nu(S)}{2}.$$

In this section, we will prove that for a curve S generic in its moduli space the latter inequality is essentially reached, as it will be stated in Theorem 3.4.

3.1. Generic value of $\mathfrak{s}(S)$

Let S be a curve generic in its topological class and $\{X_1, X_2\}$ be a Saito basis for S.

The exceptional divisor D of the blowing-up $E : (\mathcal{M}, D) \to (\mathbb{C}^2, 0)$ can be covered by two open sets U_1 and U_2 and two charts (x_1, y_1) and (x_2, y_2) defined respectively in some neighborhoods of U_1 and U_2 such that

$$E\left(x_1, y_1\right) = \left(x_1, y_1 x_1\right) \qquad \text{and} \qquad E\left(x_2, y_2\right) = \left(x_2 y_2, y_2\right).$$

Let Θ_S be the sheaf on $\mathcal{M}$ of vector fields tangent to $E^{(-1)}(S) = S^E \cup D$. Let ω be a 1-form with an isolated singularity tangent to the vector field X_1 : if X_1 is written

$$X_1 = a\partial_x + b\partial_y,$$

one can choose

$$\omega = a\,\mathrm{d}y - b\,\mathrm{d}x.$$

Let us consider the global 1-form on $\mathcal{M}$ defined by the pull-back

$$\Omega = E^\star\omega.$$

We denote by $\mathfrak{B}$ the *basic operator* : this is a morphism of sheaves

$$\mathfrak{B} : \Theta_S \longrightarrow \Omega^2\left(\mathcal{M}\right)$$

that is written

$$\mathfrak{B}(T) = L_T\Omega \wedge \Omega = d(\Omega(T)) \wedge \Omega - \Omega(T)d\Omega.$$

Here, $\Omega^2(\mathcal{M})$ is the sheaf on $\mathcal{M}$ of holomorphic 2-forms and L_T is the Lie deriviative with respect to the vector field T. Following [3], the kernel of $\mathfrak{B}$ consists in the infinitesimal generators of the sheaf of automorphisms of the foliation induced by X_1^E, that is,

$$L_T\Omega \wedge \Omega \equiv 0 \Longrightarrow \forall t \in (\mathbb{C}, 0),\ \left(\left(e^{tT}\right)^\star X_1^E\right) \wedge X_1^E \equiv 0.$$

The lemma below describes partially the image of $\mathfrak{B}$.

Lemma 3.1. — *$\mathfrak{B}(\Theta_S) \subset \Omega^2(-\overline{n}D - S^E)$ where*

- $\overline{n} = 2\nu(X_1) + \begin{cases} 2 & \text{if } X_1 \text{ is dicritical,} \\ 1 & \text{if not.} \end{cases}$
- *$\Omega^2(-\overline{n}D - S^E)$ is the sheaf of 2-forms that vanish along D and S^E with at least respective orders $\overline{n}$ and 1.*

Proof. — It is a computation which can be performed in local coordinates. If X_1 is dicritical, then out of $\mathrm{Sing}(X_1^E)$, one can write

$$\Omega = u x_1^{\nu(X_1)+1} \mathrm{d}y_1,$$

where u is a local unit. A section T of Θ_S is written

$$T = \alpha x_1 \partial_{x_1} + \beta \partial_{y_1}, \quad \alpha, \beta \in \mathbb{C}\{x_1, y_1\}.$$

Thus, applying the morphism $\mathfrak{B}$ yields

$$\mathfrak{B}(T) = -\left(u^2 x_1^{2\nu(X_1)+2} \partial_{x_1} \beta\right) \mathrm{d}x_1 \wedge \mathrm{d}y_1.$$

If X_1 is not dicritical, then out of the locus of tangency between X_1^E and D, one can write in some coordinate

$$\Omega = u x_1^{\nu(X_1)} \mathrm{d}x_1,$$

and

$$\mathfrak{B}(T) = -\left(u^2 x_1^{2\nu(X_1)+1} \partial_{y_1} \alpha\right) \mathrm{d}x_1 \wedge \mathrm{d}y_1.$$

Finally, along a regular point of S^E, one can write

$$\Omega = u\, \mathrm{d}y_1,$$

where $y_1 = 0$ is a local equation of S^E. A local section of T of Θ_S is written

$$T = \alpha \partial_{x_1} + \beta y_1 \partial_{y_1}, \quad \alpha, \beta \in \mathbb{C}\{x_1, y_1\}$$

and

$$\mathfrak{B}(T) = -\left(u^2 y_1 \partial_{x_1} \beta\right) \mathrm{d}x_1 \wedge \mathrm{d}y_1. \qquad \square$$

Notice that if c is not a tangency point between X_1^E and D, then at the level of the stack, one has

$$\left(\mathfrak{B}\left(\Theta_S\right)\right)_c = \left(\Omega^2\left(-\overline{n}D - S^E\right)\right)_c,$$

thus the two sheaves $\mathfrak{B}(\Theta_S)$ and $\Omega^2(-\overline{n}D - S^E)$ are essentially equal.

The proof of the next lemma is a corollary of an adaptation of the theory of infinitesimal deformations of foliations developped in [12] by Gómez-Mont.

Lemma 3.2. — *The map in cohomology induced by the inclusion of Lemma 3.1*

$$H^1\left(\mathcal{M}, \Theta_S\right) \xrightarrow{\overline{\mathfrak{B}}} H^1\left(\mathcal{M}, \Omega^2\left(-\overline{n}D - S^E\right)\right)$$

is the zero map.

Proof. — Let us denote by Θ_{X_1} the sheaf of tangent vector fields to the foliation induced on $\mathcal{M}$ by X_1^E. Let us consider the morphism of sheaves

$$\Theta_S \xrightarrow{\mathfrak{D}} \operatorname{Hom}\left(\Theta_{X_1}, \Theta_S/\Theta_{X_1}\right) \tag{3.1}$$

defined by $\mathfrak{D}(T) = (X \mapsto \pi[X,T])$ where $[\,\cdot\,]$ stands for the Lie bracket and π the quotient map $\pi : \Theta_S \to \Theta_S/\Theta_{X_1}$. Following [12, Theorem 1.6], one has the following exact sequence

$$\mathbb{H}^1\left(\mathcal{M}, \Theta_{X_1}\right) \longrightarrow H^1\left(\mathcal{M}, \Theta_S\right) \xrightarrow{\overline{\mathfrak{D}}} H^1\left(\mathcal{M}, \operatorname{Hom}\left(\Theta_{X_1}, \Theta_S/\Theta_{X_1}\right)\right). \tag{3.2}$$

In this sequence, $\mathbb{H}^1(\mathcal{M}, \Theta_{X_1})$ is the first hypercohomology group of the leaf complex associated to the morphism (3.1) as defined in [12]. It is identified with the space of infinitesimal deformations of the foliation induced by X_1^E. The cohomological group $H^1(\mathcal{M}, \bullet)$ is the standard Cěch cohomology of sheaves. The first cohomology group $H^1(\mathcal{M}, \Theta_S)$ is identified with the space of infinitesimal deformations of S^E. Finally, $\overline{\mathfrak{D}}$ is the map induced in cohomology by $\mathfrak{D}$.

Assume that S is generic in its moduli space $\mathbb{M}^\bullet(S)$. Theorem 2.8 ensures that any small deformation of S^E can be followed by a deformation of X_1^E. As a consequence, any infinitesimal deformation of S^E can be followed by an infinitesimal deformation of X_1^E. In other words, in (3.2) the map

$$\mathbb{H}^1\left(\mathcal{M}, \Theta_{X_1}\right) \longrightarrow H^1\left(\mathcal{M}, \Theta_S\right)$$

is onto. Since the sequence (3.2) is exact, the map $\overline{\mathfrak{D}}$ is the zero map.

Now, let us consider a covering $\{U_i\}_{i\in I}$ of $\mathcal{M}$ and a cocycle $\{T_{ij}\}_{ij}$

$$\{T_{ij}\}_{ij} \in Z^1\left(\mathcal{M}, \{U_i\}_{i\in I}, \Theta_S\right).$$

The map $\overline{\mathfrak{D}}$ being the zero map, the cocycle $\overline{\mathfrak{D}}(\{T_{ij}\}_{ij})$ is trivial, that is,

$$\overline{\mathfrak{D}}\left(\{T_{ij}\}_{ij}\right) \equiv 0 \text{ in } H^1\left(\mathcal{M}, \operatorname{Hom}\left(\Theta_{X_1}, \Theta_S/\Theta_{X_1}\right)\right).$$

By definition, there exists $\{\mathcal{T}_i\}_i \in Z^0(\mathcal{M}, \{U_i\}_{i\in I}, \operatorname{Hom}(\Theta_{X_1}, \Theta_S/\Theta_{X_1}))$ such that

$$[T_{ij}, \cdot\,] = \mathcal{T}_j - \mathcal{T}_i.$$

At the level of the stack, the map $\mathfrak{D}$ is onto at any regular point for X_1^E. Thus we can consider a covering of $U_i \setminus \operatorname{Sing}(X_1^E) = \bigcup_{k\in K} U_{ik}$ by open sets U_{ik} such that $\mathfrak{D}$ is onto on U_{ik}. By construction, on any U_{ik} there exists a section τ_{ik} of Θ_S such that

$$\mathcal{T}_i = [\tau_{ik}, \cdot\,].$$

Therefore, on $U_{ik} \cap U_{ik'}$, $[\tau_{ik}, \cdot\,] = [\tau_{ik'}, \cdot\,]$. Thus, appyling $\mathfrak{B}$ yields

$$L_{\tau_{ik}}\Omega \wedge \Omega = \mathfrak{B}\left(\tau_{ik}\right) = \mathfrak{B}\left(\tau_{ik'}\right) = L_{\tau_{ik'}}\Omega \wedge \Omega.$$

Therefore, the 2-forms $\{L_{T_{ik}}\Omega \wedge \Omega\}_{k\in K}$ paste in a global 2-form Ω_i defined on $U_i \setminus \mathrm{Sing}(X_1^E)$ which can be extended to U_i since $\mathrm{Sing}(X_1^E)$ is of codimension 2. By construction,

$$\overline{\mathfrak{B}}\left(\{T_{ij}\}\right) \equiv \{\Omega_j - \Omega_i\},$$

which is the lemma. □

The open sets U_1 and U_2 defined at the beginning of this section are Stein as open sets in $\mathbb{C}$. Thus, following [24], they admit a system of Stein neighborhoods. Since $\Omega^2(-\overline{n}D - S^E)$ is coherent, we deduce that there is a covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of $\mathcal{M}$ that is acyclic for $\Omega^2(-\overline{n}D - S^E)$. Therefore, one can compute the cohomology using this covering and thus

$$H^1\left(\mathcal{M}, \Omega^2\left(-\overline{n}D - S^E\right)\right) = H^1\left(\{\mathcal{U}_1, \mathcal{U}_2\}, \Omega^2\left(-\overline{n}D - S^E\right)\right)$$

which is the quotient

$$\frac{H^0\left(\mathcal{U}_1 \cap \mathcal{U}_2, \Omega^2\left(-\overline{n}D - S^E\right)\right)}{H^0\left(\mathcal{U}_1, \Omega^2\left(-\overline{n}D - S^E\right)\right) \oplus H^0\left(\mathcal{U}_2, \Omega^2\left(-\overline{n}D - S^E\right)\right)}. \tag{3.3}$$

The lemma below is the key to get a lower bound for the Saito number $\mathfrak{s}(S)$ of the curve S.

Lemma 3.3. — *Let f_1 be the quotient $\frac{f\circ E}{x_1^{\nu(S)}}$ where f is a reduced equation of S. If there exists a Laurent series $A = \sum a_{i,j} x_1^i y_1^j$ holomorphic on $\mathcal{U}_1 \cap \mathcal{U}_2$ with a non vanishing residu $a_{0,-1}$, such that, in the identification* (3.3), *one has*

$$\left[A \cdot f_1 x_1^k \mathrm{d}x_1 \wedge \mathrm{d}y_1\right] \equiv 0 \in H^1\left(\mathcal{M}, \Omega^2\left(-kD - S^E\right)\right)$$

then

$$k \geqslant \nu(S).$$

Proof. — The global sections of $\Omega^2(-kD - S^E)$ on each open sets $\mathcal{U}_1, \mathcal{U}_2$ and their intersection are written

$$\Omega^2\left(-kD - S^E\right)(\mathcal{U}_1) = \left\{f(x_1, y_1) f_1 x_1^k \,\mathrm{d}x_1 \wedge \mathrm{d}y_1 \;\middle|\; f \in \mathcal{O}(\mathcal{U}_1)\right\}$$
$$\Omega^2\left(-kD - S^E\right)(\mathcal{U}_2) = \left\{g(x_2, y_2) f_2 y_2^k \,\mathrm{d}x_2 \wedge \mathrm{d}y_2 \;\middle|\; g \in \mathcal{O}(\mathcal{U}_2)\right\}$$
$$\Omega^2\left(-kD - S^E\right)(\mathcal{U}_1 \cap \mathcal{U}_2) = \left\{h(x_1, y_1) f_1 x_1^k \,\mathrm{d}x_1 \wedge \mathrm{d}y_1 \;\middle|\; h \in \mathcal{O}(\mathcal{U}_1 \cap \mathcal{U}_2)\right\}$$

where $f_2 = \frac{f\circ E}{y_2^{\nu(S)}}$. Therefore, the cohomological equation induced by the equality (3.3) is written

$$h(x_1, y_1) f_1 x_1^k \,\mathrm{d}x_1 \wedge \mathrm{d}y_1 = g(x_2, y_2) f_2 y_2^k \,\mathrm{d}x_2 \wedge \mathrm{d}y_2 - f(x_1, y_1) f_1 x_1^k \,\mathrm{d}x_1 \wedge \mathrm{d}y_1$$

which is equivalent to

$$h(x_1,y_1) = y_1^{k-\nu(S)-1} g\left(\frac{1}{y_1}, y_1 x_1\right) - f(x_1,y_1). \tag{3.4}$$

The hypothesis of Lemma 3.3 induces that if we set h to be the series $\sum a_{i,j} x_1^i y_1^j$ then the equation above has a solution. In particular, the monomial $\frac{a_{0,-1}}{y_1}$ has to appear in the Laurent expansion of one of the two terms of the expression at the right of (3.4). This is equivalent to require that the following system

$$\begin{cases} 0 = j \\ -1 = j - i + k - \nu(S) - 1 \end{cases} \iff \begin{cases} j = 0 \\ i = k - \nu(S). \end{cases}$$

has a solution in $\mathbb{N}^2$. Thus, $k \geqslant \nu(S)$. □

Theorem 3.4. — *For S generic in its moduli space $\mathbb{M}^\bullet(S)$, one has*

$$\mathfrak{s}(S) \geqslant \begin{cases} \left\lfloor \frac{\nu(S)}{2} \right\rfloor & \text{if } S \text{ is not of radial type,} \\ \left\lceil \frac{\nu(S)}{2} \right\rceil - 1 & \text{else,} \end{cases}$$

where $\lfloor\star\rfloor$ and $\lceil\star\rceil$ stands respectively for the integer part and the least integer of $\star$.

In the moduli space $\mathbb{M}^\bullet(S)$, the lower bound above holds only for the generic point. For instance, the Saito number of a union of any number of germs of straight lines is 1, since the radial vector field $x\partial x + y\partial y$ is in the Saito module, whereas the algebraic multiplicity $\nu(S)$ goes to infinity with the number of components. Even if the curve S is irreducible, one cannot drop the assumption of S being generic in its moduli space, as it can be seen in the following example due to M. Hernandes known as *deformation by socle*: let S be the irreducible curve

$$\left\{y^p - x^q + x^{q-2} y^{p-2} = 0\right\}$$

with $p \wedge q = 1$ and $4 = p < q$. Its algebraic multiplicity is equal to p whereas its Saito number $\mathfrak{s}(S)$ is equal to 2 regardless the value of p. Indeed, the vector field X_1 written

$$X_1 = \left(y + \frac{(p-2)(q-2)}{pq} x^{q-4} y^{p-3}\right)(px\partial_x + qy\partial_y) + \frac{(p-2)q - 2p}{q} x^{q-2}\partial_y - (p-2)\frac{(p-2)q-2p}{pq} x^{p-3} y^{q-3}\partial_x$$

is optimal for S.

Proof of Theorem 3.4. — Let X_1 be a generic optimal vector field for S. Since we assume S generic in its moduli space, the operator $\overline{\mathfrak{B}}$ associated to X_1 and defined in Lemma 3.2 is trivial.

Suppose, first that X_1 is dicritical. Let us suppose that in the coordinates (x_1, y_1), the vector field X_1^E is transverse to D at $(0,0)$ and that $f_1 = \frac{f\circ E}{x_1^{\nu(S)}}$ does not vanish at $(0,0)$. We can suppose that, in these coordinates, Ω is written

$$\Omega = u x_1^{\nu(X_1)+1}\,\mathrm{d}y_1, \quad u(0) \neq 0.$$

The image of the vector field

$$T = \frac{x_1}{y_1}\partial_{y_1}$$

by $\frac{1}{f_1}\mathfrak{B}$ is written

$$\frac{1}{f_1}\mathfrak{B}(T) = \frac{1}{f_1} L_T\Omega \wedge \Omega = \frac{u^2\,(0,0)}{f_1\,(0,0)} x_1^{\overline{n}} \frac{1}{y_1}\,\mathrm{d}x_1 \wedge \mathrm{d}y_1 + x_1^{\overline{n}+1}(\cdots).$$

This meromorphic 2-form considered as a cocycle in

$$Z^1\left(\{\mathcal{U}_1, \mathcal{U}_2\}, \Omega^2\left(-\overline{n}D - S^E\right)\right)$$

has to be trivial in cohomology according to Lemma 3.2. Thus, Lemma 3.3 ensures that $\overline{n} = 2\nu(X_1) + 2 \geqslant \nu(S)$, which is also written

$$\mathfrak{s}(S) = \nu(X_1) \geqslant \frac{\nu(S)}{2} - 1.$$

Therefore, if X_1 is dicritical the theorem is proved.

Suppose now that X_1 is not dicritical. Let us suppose that $(0,0)$ is a singular point of X_1^E. Locally around $(0,0)$, Ω can be written

$$\Omega = x_1^{\nu(X_1)} y_1^a\,\mathrm{d}x_1 + x_1^{\nu(X_1)+1}(\cdots)$$

where a is some positive integer. Let us write

$$f_1 = y_1^b v\,(y_1) + x_1(\cdots), \quad v(0) \neq 0$$

where b is some positive integer. Considering the meromorphic vector field

$$T = \frac{x_1}{y_1^{2a-b}}\partial_{x_1},$$

we apply the operator $\frac{1}{f_1}\mathfrak{B}$ and obtain

$$\frac{1}{f_1}\mathfrak{B}(T) = \frac{(2a-b)}{v(0)}\frac{x_1^{\overline{n}}}{y_1}\,\mathrm{d}x_1 \wedge \mathrm{d}y_1 + x_1^{\overline{n}+1}(\cdots).$$

Suppose that there exists a singular point of X_1^E such that $2a \neq b$. Then, Lemma 3.3 ensures that $\overline{n} = 2\nu(X_1) + 1 \geqslant \nu(S)$, which is written

$$\mathfrak{s}(S) = \nu(X_1) \geqslant \frac{\nu(S) - 1}{2}. \tag{3.5}$$

If the equality $2a = b$ is true for any singular points, then $\nu(S)$ is even. Thus, the theorem is proved when

- $\nu(S)$ is odd
- or $\nu(S)$ is even and for some singular points of X_1^E, one has $b \neq 2a$.
- or if S is radial.

Finally, suppose that $\nu(S)$ is even and S is not radial. Consider a Saito basis $\{X_1, X_2\}$ for S with $\nu(X_1) = \nu(X_2)$. If $\nu(X_1) = \frac{\nu(S)}{2}$ then the property is proved. Therefore, assume that $\nu(X_1) \leqslant \frac{\nu(S)}{2} - 1$. Let l_1 be a generic smooth curve. Using the construction introduced at (2.13), we obtain a Saito basis for $S \cup l_1$ of the form

$$\{X_1 + \phi_1 X_2, L_1 X_2\}, \ \phi_1(0) \neq 0 \text{ and } l_1 = \{L_1 = 0\}.$$

If $\nu\left(X_1 + \phi_1 X_2\right) = \nu(X_1)$ then

$$\nu\left(X_1 + \phi_1 X_2\right) \leqslant \frac{\nu(S)}{2} - 1 < \frac{\nu\left(S \cup l_1\right) - 1}{2}$$

which contradicts Theorem 3.4 applied to $S \cup l_1$, the valuation $\nu(S \cup l_1)$ being odd. Therefore, $\nu(X_1 + \phi_1 X_2) \geqslant \frac{\nu(S)}{2}$ and since $\nu(L_1 X_2) \geqslant \frac{\nu(S)}{2}$ and $\nu(X_2) \leqslant \frac{\nu(s)}{2} - 1$, considering if necessary $X_1 + \phi_1 X_2 + L_1 X_2$, we obtain a basis of Saito for $S \cup l_1$ written

$$\{X_1 + \phi_1 X_2, L_1 X_2\}, \qquad \phi_1(0) \neq 0 \tag{3.6}$$

both of these vector fields being non dicritical and of multiplicity $\frac{\nu(S)}{2}$. Using again the construction (2.13), we add one more generic curve l_2 and obtain a basis of Saito of the form

$$\Bigg\{\underbrace{L_2\left(X_1 + \phi_1 X_2\right)}_{Y_1}, \underbrace{L_1 X_2 + \phi_2\left(X_1 + \phi_1 X_2\right)}_{Y_2}\Bigg\}, \ \phi_2(0) \neq 0 \text{ and } l_2 = \{L_2 = 0\}.$$

We can apply Theorem 3.4 to $S \cup l_1 \cup l_2$ since the latter curve has a smooth component for which $b = 1$ is not even. Therefore, the two above vector fields are of multiplicity $\frac{\nu}{2} + 1$ and not dicritical. According to the Saito criterion applied to (3.6), one has

$$(X_1 + \phi_1 X_2) \wedge L_1 X_2 = u L_1 f, \ u(0) \neq 0.$$

Therefore, L_1 cannot divide $X_1 + \phi_1 X_2$. Now consider any couple of non vanishing functions α and β. Writing

$$\alpha Y_1 + \beta Y_2 = (\beta\phi_2 + \alpha L_2)(X_1 + \phi_1 X_2) + \beta L_1 X_2 \tag{3.7}$$

ensures that $\alpha Y_1 + \beta Y_2$ cannot be divided by $L_1 L_2$. Fix some coordinates (x, y) such that $L_1 = x$ and $L_2 = y$. Taking a suitable linear combination of Y_1 and Y_2 we can suppose that they are written

$$\begin{aligned} Y_1 &= a(x)\, x^p \partial_x + b(y) y^q \partial_y + xy(\cdots) \\ Y_2 &= c(x)\, x^p \partial_x + d(y) y^q \partial_y + xy(\cdots) \end{aligned}$$

where a, b, c and d are non-vanishing germs of functions and p and q some integers bigger than $\frac{\nu}{2} + 1$. Dividing Y_1 and Y_2 respectively by b and d, and making a suitable change of coordinates of the form $(x, y) \mapsto (u(x), y)$, we can suppose that Y_1 and Y_2 are written

$$\begin{aligned} Y_1 &= a x^p \partial_x + b y^q \partial_y + xy(\cdots) \\ Y_2 &= c(x)\, x^p \partial_x + d y^q \partial_y + xy(\cdots) \end{aligned}$$

where a, b and d belongs to $\mathbb{C} \setminus \{0\}$. Finally, considering the vector field

$$Y_2 - \frac{(c(x) - c(0))}{a} Y_1,$$

we can write

$$\begin{aligned} Y_1 &= a x^p \partial_x + b y^q \partial_y + xy(\cdots) \\ Y_2 &= c x^p \partial_x + d y^q \partial_y + xy(\cdots) \end{aligned}$$

where a, b, c and d are non vanishing complex numbers. Now, the Saito criterion written

$$Y_1 \wedge Y_2 = uxyf$$

where u is a unit ensures that

$$(cY_1 - aY_2) \wedge (dY_1 - bY_2) = (ad - bc)\, Y_1 \wedge Y_2 = u\,(ad - bc)\, xyf.$$

If $ad - bc = 0$ then considering $\left(\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right)$ in the kernel of the matrix $\left(\begin{smallmatrix}a & c \\ b & d\end{smallmatrix}\right)$ yields a linear combination written

$$\alpha Y_1 + \beta Y_2 = xy(\cdots). \tag{3.8}$$

Since neither Y_1 nor Y_2 can be divided by xy, one has $\alpha \neq 0$ and $\beta \neq 0$. According to (3.7), $\alpha Y_1 + \beta Y_2$ cannot be divided by xy too. That is a contradiction with (3.8). Therefore, $ad - bc \neq 0$ and the expression

$$\frac{(cY_1 - aY_2)}{y} \wedge \frac{(dY_1 - bY_2)}{x} = u\,(ad - bc)\, f$$

is the Saito criterion for the curve S. However, both vector fields in the product above have mutiplicities bigger than $\frac{\nu(S)}{2}$ which is a contradiction with the initial assumption. □

3.2. Generic Saito basis

The generic lower bound of Theorem 3.4 induces some properties for a Saito basis of a generic curve. In this section, we explore some of them.

To do so, we are going to use frequently the following lemma that is a direct consequence of the criterion of Saito.

Lemma 3.5. — *Let $\{X_1, X_2\}$ be a Saito basis S. Then*

1. *if $\nu(X_1) + \nu(X_2) < \nu(S)$ and X_1 is dicritical then X_2 is dicritical.*
2. *if $\nu(X_1) + \nu(X_2) = \nu(S)$ and X_1 is dicritical then X_2 is not dicritical.*
3. *If S is generic in its moduli space, then one can suppose that*

$$\nu(S) - 1 \leqslant \nu(X_1) + \nu(X_2) \leqslant \nu(S).$$

Proof. — Properties (1) and (2) are consequences of the following remark

$$X_1^{(\nu(X_1))} \wedge X_2^{(\nu(X_2))} \equiv 0 \iff \nu(X_1) + \nu(X_2) < \nu(S).$$

For (3), first, we recall that the sum $\nu(X_1) + \nu(X_2)$ cannot exceed $\nu(S)$ as noticed in (2.2). If $\nu(S)$ is odd, Theorem 3.4 gives the inequalities

$$\nu(X_i) \geqslant \frac{\nu(S) - 1}{2}, \ i = 1, 2$$

and thus $\nu(X_1) + \nu(X_2) \geqslant \nu(S) - 1$, which is the lemma.

If $\nu(S)$ is even, adding a generic line l to S yields a Saito basis of $S \cup l$ for which, in view of the previous arguments – $\nu(S \cup l)$ is odd –, one has

$$\nu(X_1) + \nu(X_2) \geqslant \nu(S \cup l) - 1 = \nu(S).$$

By the process described in Proposition 2.12, the induced Saito basis $\{X_1', X_2\}$ of S satisfies

$$\nu(X_1') + \nu(X_2) \geqslant \nu(X_1) + \nu(X_2) - 1 \geqslant \nu(S) - 1,$$

which ends the proof of the lemma. □

The next lemma ensures somehow that both inequalities identified in Theorem 3.4 cannot be reached at the same time.

Lemma 3.6. — *Let S be a generic curve of radial type. Then there is no non dicritical vector field X in* $\mathrm{Der}(\log S)$ *with* $\nu(X) = \lfloor \frac{\nu(S)}{2} \rfloor$.

Proof. — Consider an optimal dicritical vector field X_1 and X_2 a vector field such that $\{X_1, X_2\}$ is a Saito basis of S. If $\nu(S)$ is even, then Theorem 3.4 ensures that $\nu(X_1) \geqslant \frac{\nu(S)}{2} - 1$. If $\nu(X_1) = \frac{\nu(S)}{2}$ then the lemma follows from the definition of S being radial. If $\nu(X_1) = \frac{\nu(S)}{2} - 1$ then either $X_1^{(\nu(X_1))} \wedge X_2^{(\nu(X_2))} = 0$ or $\nu(X_2) \geqslant \frac{\nu(S)}{2} + 1$. In any case, the lemma follows. Finally, if $\nu(S)$ is odd and S radial, by definition, every vector field of multiplicity $\lfloor \frac{\nu(S)}{2} \rfloor = \frac{\nu(S)-1}{2}$ is dicritical. □

In the proposition below, we are going to identify precisely the type of Saito basis that may occur for a generic curve. In the statement of the theorem, we introduce some notations for the identified classes.

THEOREM 3.7. — *Let S be a curve generic in its moduli space. Then there exists a Saito basis $\{X_1, X_2\}$ for S with one of the following forms*

- *if $\nu(S)$ is even*
  $(\mathfrak{E})$: $\nu(X_1) = \nu(X_2) = \frac{\nu(S)}{2}$, *$X_1$ and X_2 are non dicritical.*
  $(\mathfrak{E}_d)$: $\nu(X_1) = \nu(X_2) - 1 = \frac{\nu(S)}{2} - 1$, *$X_1$ and X_2 are dicritical.*
  $(\mathfrak{E}'_d)$: $\nu(X_1) = \nu(X_2) - 2 = \frac{\nu(S)}{2} - 1$, *$X_1$ is dicritical but not X_2;*
- *if $\nu(S)$ is odd*
  $(\mathfrak{O})$: $\nu(X_1) = \nu(X_2) - 1 = \frac{\nu-1}{2}$, *$X_1$ and X_2 are non dicritical.*
  $(\mathfrak{O}_d)$: $\nu(X_1) = \nu(X_2) = \frac{\nu-1}{2}$, *$X_1$ and X_2 are dicritical.*
  $(\mathfrak{O}'_d)$: $\nu(X_1) = \nu(X_2) - 1 = \frac{\nu-1}{2}$, *$X_1$ is dicritical but not X_2.*

Moreover, if $\{X_1, X_2\}$ is a generic Saito basis for S then there exists an holomorphic function h such that

$$\{X_1, X_2 - hX_1\}$$

has one of the above type.

If the Saito basis of S has one of the forms given by Theorem 3.7, we will say that the basis is *adapted*.

Remark 3.8. — Notice that if the Saito basis $\{X_1, X_2\}$ of S is of type $(\mathfrak{E}'_d)$ or $(\mathfrak{O}'_d)$ then for any function L with a non trivial linear part, the family

$$\{X_1, X_2 + LX_1\}, \quad \begin{cases} L(0) \neq 0 & \text{if } S \text{ is of type } (\mathfrak{O}'_d) \\ L(0) = 0 & \text{if } S \text{ is of type } (\mathfrak{E}'_d) \end{cases}$$

is a Saito basis for S of type $(\mathfrak{E}_d)$ or $(\mathfrak{O}_d)$. In some sense, the bases of type $(\mathfrak{E}'_d)$ or $(\mathfrak{O}'_d)$ are exceptional among the one of type $(\mathfrak{E}_d)$ or $(\mathfrak{O}_d)$.

Remark 3.9. — The curves of type $(\mathfrak{E}'_d)$ are the only curves for which there exists a Saito basis $\{X_1, X_2\}$ with

$$|\nu(X_1) - \nu(X_2)| \geqslant 2.$$

Remark 3.10. — One of the interest of adapted Saito bases is their behaviour with respect to the blowing-up. For instance, suppose that S has an adapted Saito basis $\{X_1, X_2\}$ of type $(\mathfrak{E}_d)$. Then, blowing-up the Saito criterion (2.1) yields the relation

$$X_1^E \wedge X_2^E = u \circ E \frac{f \circ E}{x_1^{\nu(S)}}.$$

Therefore, according to the Saito criterion, the family $\{(X_1^E)_c, (X_2^E)_c\}$ is a Saito basis for $\left(S^E\right)_c$ for any $c \in D$ - but not necessarely *adapted*. It is a simple matter to check that the latter proprety holds for any above type of Saito bases.

Proof of Theorem 3.7. — Let us consider a Saito basis $\{X_1, X_2\}$ of S and suppose that $\nu(X_1) \leqslant \nu(X_2)$.

Case 1. — Suppose first $\nu(S)$ even. If X_1 is not dicritical then according to Theorem 3.4 and (2.2), $\nu(X_1) = \nu(X_2) = \frac{\nu(S)}{2}$. Considering if necessary $X_2 + \alpha X_1$ for a generic value $\alpha \in \mathbb{C}$, one has

$$(\mathfrak{E}) \qquad \nu(X_1) = \nu(X_2) = \frac{\nu(S)}{2}$$
$$X_1 \text{ and } X_2 \text{ are non-dicritical.}$$

Assume X_1 is dicritical. If $\nu(X_1) = \frac{\nu(S)}{2}$ then Lemma 3.5 ensures that X_2 is not dicritical and the Saito basis $\{X_1 + X_2, X_2\}$ is of type $(\mathfrak{E})$. If $\nu(X_1) = \frac{\nu(S)}{2} - 1$, following Lemma 3.5, one can suppose that

$$\nu(X_2) = \frac{\nu(S)}{2} \text{ or } \frac{\nu(S)}{2} + 1.$$

If $\nu(X_2) = \frac{\nu(S)}{2} + 1$ then X_2 is not dicritical. Thus, one has a basis of the form

$$(\mathfrak{E}'_d) \qquad \nu(X_1) = \nu(X_2) - 2 = \frac{\nu(S)}{2} - 1$$
$$X_1 \text{ is dicritical but not } X_2.$$

If $\nu(X_2) = \frac{\nu(S)}{2}$, then X_2 is dicritical, and thus

$$(\mathfrak{E}_d) \qquad \nu(X_1) = \nu(X_2) - 1 = \frac{\nu(S)}{2} - 1$$
$$X_1 \text{ and } X_2 \text{ are both dicritical.}$$

Case 2. — Suppose now $\nu(S)$ odd. In any case, $\nu(X_1)=\frac{\nu(S)-1}{2}$. Suppose X_1 dicritical. If $\nu(X_2)=\frac{\nu(S)-1}{2}$ then X_2 is dicritical, and thus

$$(\mathfrak{D}_d)\qquad \begin{gathered}\nu(X_1)=\nu(X_2)=\frac{\nu(S)-1}{2}\\ X_1 \text{ and } X_2 \text{ are dicritical.}\end{gathered}$$

If $\nu(X_2)=\frac{\nu(S)+1}{2}$ then X_2 is not dicritical, and therefore the basis satifies

$$(\mathfrak{D}'_d)\qquad \begin{gathered}\nu(X_1)=\nu(X_2)-1=\frac{\nu(S)-1}{2}\\ X_1 \text{ is dicritical and } X_2 \text{ is non-dicritical.}\end{gathered}$$

Finally, suppose that X_1 is not dicritical. If $\nu(X_2)=\frac{\nu(S)+1}{2}$ then the basis satifies

$$(\mathfrak{D})\qquad \begin{gathered}\nu(X_1)=\nu(X_2)-1=\frac{\nu(S)-1}{2}\\ X_1 \text{ and } X_2 \text{ are non-dicritical.}\end{gathered}$$

It remains to study the case in which X_1 is not dicritical and $\nu(X_2)=\frac{\nu(S)-1}{2}$. To do so, consider a generic line l. The multiplicity of $S\cup l$ is even, thus we can apply the results above to reach the description of the possible bases for S.

(1) Suppose first that the Saito basis $\left\{X_1^l,X_2^l\right\}$ of $S\cup l$ has the form $(\mathfrak{C})$;

$$\nu\left(X_1^l\right)=\nu\left(X_2^l\right)=\frac{\nu(S)+1}{2},$$

none of these vector fields being dicritical. Let us consider some coordinates in which $l=\{x=0\}$ and let us write

$$X_i^l=xA_i\partial_x+\left(y^{\alpha_i}b_i(y)+xB_i\right)\partial_y$$

with $b_i(0)\neq 0$. By symmetry, one can suppose $\alpha_1\leqslant\alpha_2$. Thus, the family

$$\left\{X_1^l,\overline{X}_2^l=\frac{1}{x}\left(X_2-y^{\alpha_2-\alpha_1}\frac{b_2}{b_1}X_1\right)\right\}$$

is a Saito basis for S such that

$$\nu\left(X_1^l\right)=\frac{\nu(S)+1}{2}\text{ and }\nu\left(\overline{X_2}^l\right)=\frac{\nu(S)-1}{2}.$$

$\overline{X_2}^l$ has to be not dicritical since X_1 is not dicritical. Therefore, S admits a basis of the form $(\mathfrak{D})$.

(2) Suppose that the Saito basis $\{X_1^l, X_2^l\}$ of $S \cup l$ has the form $(\mathfrak{E}_d)$

$$\nu\left(X_1^l\right) = \nu\left(X_2^l\right) - 1 = \frac{\nu(S)+1}{2} - 1$$

both vector fields being dicritical. As before, let us consider some coordinates in which $l = \{x = 0\}$ and let us write

$$X_i^l = xA_i\partial_x + \left(y^{\alpha_i}b_i(y) + xB_i\right)\partial_y$$

with $b_i(0) \neq 0$. If $\alpha_1 \leqslant \alpha_2$ then the induced Saito basis $\left\{X_1^l, \overline{X}_2^l\right\}$ for S satisfies $\nu\left(X_1^l\right) = \frac{\nu(S)-1}{2}$. Therefore,

$$\nu\left(\overline{X_2}^l\right) = \frac{\nu(S)-1}{2} \text{ or } \frac{\nu(S)+1}{2}.$$

In any case of the alternative above, there is no non dicritical vector fields of multiplicity $\frac{\nu(S)-1}{2}$ in the Saito module of S, which is a contradiction with the property of X_1. Therefore, $\alpha_1 > \alpha_2$. In the induced basis $\{\overline{X}_1^l, X_2^l\}$, the vector field $\overline{X}_1^l$ is written

$$\overline{X}_1^l = \frac{1}{x}\left(X_1^l - y^{\alpha_1-\alpha_2}\frac{b_1(y)}{b_2(y)}X_2^l\right).$$

Therefore, $\overline{X}_1^l$ is dicritical since $\nu(X_2^l) > \nu(X_1^l)$. But since X_1 is not dicritical, it is a contradiction.

(3) Finally, suppose that the Saito basis of $S \cup l$ has the form $(\mathfrak{E}'_d)$

$$\nu\left(X_1^l\right) = \nu\left(X_2^l\right) - 2 = \frac{\nu(S)+1}{2} - 1$$

with X_1^l dicritical and X_2^l not dicritical. Then for any linear function L the Saito basis for $S \cup l$

$$\left\{X_1^l, X_2^l + LX_1^l\right\}$$

is of type $(\mathfrak{E}_d)$ which brings us back to the previous case. □

Table 3.1. Examples of different types of Saito bases.

S	$f = x$	$f = xy$	$f = xy(x+y)$	$f = xy\left(x^2 - y^2\right)$
$\nu(S)$	1	2	3	4
$X_1,\ X_2$	$\partial_x,\ x\partial_y$	$x\partial_x,\ y\partial_y$	$x\partial_x + y\partial_y,\ \sharp f$	$x\partial_x + y\partial_y,\ \sharp f$
$\nu(X_1),\ \nu(X_2)$	$0,\ 1$	$1,\ 1$	$1,\ 2$	$1,\ 3$
Type	$(\mathfrak{D})$	$(\mathfrak{E})$	$(\mathfrak{D}'_d)$	$(\mathfrak{E}'_d)$

S	$f = xy\left(x^3 - y^3 + \cdots\right)$	$f = xy\left(x^2 - y^2\right)(x + 2y + \cdots)(x + 3y + \cdots)$
$\nu(S)$	5	6
$X_1,\ X_2$	$x\left(x\partial_x + y\partial_y\right) + \cdots$ $y\left(x\partial_x + y\partial_y\right) + \cdots$	$\left(x + \frac{29}{15}y\right)\left(x\partial_x + y\partial_y\right) + \cdots$ $x^2\left(x\partial_x + y\partial_y\right) + \cdots$
$\nu(X_1),\ \nu(X_2)$	$2,\ 2$	$2,\ 3$
Type	$(\mathfrak{D}_d)$ - 1 free point	$(\mathfrak{E}_d)$ - 1 free point

3.3. Base of type $(\mathfrak{E}'_d)$ and $(\mathfrak{D}'_d)$

Beyond Example 2.11, the curve S defined by

$$S = \left\{y^5 + x^5 + x^6 = 0\right\}$$

belongs to the generic component of the moduli space of five smooth and transversal curves. An optimal vector field X_1 for S can be written

$$X_1 = \left(\frac{1}{5}xy - \frac{1}{25}x^2y + \frac{6}{125}x^3y + \frac{36}{125}x^4y\right)\partial_x + \left(\frac{1}{5}y^2 + \frac{216}{625}x^3y^2\right)\partial_y$$

whose initial part is

$$\frac{y}{5}\left(x\partial_x + y\partial_y\right). \tag{3.9}$$

Thus X_1 is dicritical and of multiplicity 2. However, after one blowing-up, X_1^E is not transverse to D at every point: indeed, following (3.9), it is tangent to D at the point corresponding to the direction $y = 0$. To formalize these remarks, let us recall the following definition

Definition 3.11. — *Let D be a divisor and X a vector field defined in a neighborhood of D that does not leave invariant D. The locus of tangency between X and D is the common zeros of F and $X \cdot F$ where F is any local equation of D. It is denoted by*

$$\mathrm{Tan}(X, D).$$

By definition, the locus of tangency between D and X contains the singular points of X which are on D. In the example (3.9), we have

$$\mathrm{Tan}\left(X_1^E, D\right) = \{(x_1 = 0, y_1 = 0)\} \neq \mathrm{Tan}(S^E, D) = \emptyset.$$

This leads us to introduce the following notion.

Definition 3.12. — *A curve S of radial type is said to be of* pure radial type *if for any optimal vector field X_1 the following equality holds*

$$\mathrm{Tan}\left(X_1^E, D\right) = \mathrm{Tan}(S^E, D).$$

If S is not pure radial, then the non empty set

$$\mathrm{Tan}\left(X_1^E, D\right) \setminus \mathrm{Tan}(S^E, D)$$

is called the set of free points of X_1

Notice that by construction of X_1, in any case, the inclusion

$$\mathrm{Tan}(S^E, D) \subset \mathrm{Tan}\left(X_1^E, D\right)$$

holds. The main feature of this definition relies on the fact that it allows to state a characterization of the curves admitting a basis of type $(\mathfrak{E}'_d)$ or $(\mathfrak{D}'_d)$.

Theorem 3.13. — *The following properties are equivalent :*

(1) *S is of pure radial type.*
(2) *S admits a Saito basis of type $(\mathfrak{E}'_d)$ or $(\mathfrak{D}'_d)$.*

Proof. — We begin by proving (2) $\implies$ (1). Assume that S admits an adapted Saito basis of type $(\mathfrak{E}'_d)$ or $(\mathfrak{D}'_d)$. According to Remark 3.10, for any point $c \in D$, the family

$$\left\{(X_1^E)_c, \left(X_2^E\right)_c\right\}$$

is a Saito basis of the germ of curve $(S^E)_c$. Let $c \in D \setminus \text{Tan}(S^E, D)$. Suppose first that $c \notin \text{Sing}(E^{-1}(S))$. Then following Remark 3.10, the product $X_1^E \wedge X_2^E$ is a unity at c. Now, X_1 is dicritical and X_2 is not. Thus in local coordinates (x, y) at c in which $x = 0$ is local equation of D, we can write

$$\begin{aligned} X_1^E \wedge X_2^E &= (u\partial x + v\partial y) \wedge (ax\partial x + b\partial y), \qquad u, v, a, b \in \mathbb{C}\{x, y\} \\ &= avx - bu. \end{aligned}$$

Therefore u is a unity and X_1^E is transverse to D. Suppose now that $c \in \text{Sing}(E^{-1}(S))$. Since $c \in D \setminus \text{Tan}(S^E, D)$ then S^E is regular and transverse to D. Now, considering local coordinates (x, y) in which $xy = 0$ is a local equation of $E^{-1}(S)$, we obtain

$$\begin{aligned} X_1^E \wedge X_2^E &= (u\partial x + vy\partial y) \wedge (ax\partial x + by\partial y) \qquad u, v, a, b \in \mathbb{C}\{x, y\} \\ &= avxy - buy \end{aligned}$$

which has to be of the form (unity) $\times\, y$ according to the criterion of Saito. Therefore, u is a unity and X_1^E is still transverse to D, which completes the proof of the equality

$$\text{Tan}\left(X_1^E, D\right) = \text{Tan}(S^E, D).$$

We now proceed to the proof of (1) $\implies$ (2). Let $\{X_1, X_2\}$ be an adapted Saito basis for S. The curve S being radial, let us write

$$X_1 = h_1 (x\partial x + y\partial y) + \cdots . \tag{3.10}$$

The hypothesis is equivalent to assume that the tangent cone of h_1 coincide with the locus of tangency $\text{Tan}(S^E, D)$ for any optimal vector field X_1.

Assume first that $\nu(S)$ is odd. According to Proposition (3.7), the valuation of X_1 is

$$\nu(X_1) = \frac{\nu - 1}{2}.$$

If X_2 is not dicritical, then $\nu(X_2) = \frac{\nu+1}{2}$. Therefore, the basis $\{X_1, X_2\}$ is of type $(\mathfrak{O}'_d)$ and the proposition is proved. Assume X_2 is dicritical and $\nu(X_2) = \frac{\nu-1}{2}$. As in (3.10), we write

$$X_2 = h_2\,(x\partial x + y\partial y) + \cdots$$

and

$$h_2 = q_2 \cdot \overline{h_2}$$

where the tangent cone of q_2 does not meet $\mathrm{Tan}(S^E, D)$. For any value of α and β, the initial part of $\alpha X_1 + \beta X_2$ is written

$$\left(\alpha h_1 + \beta q_2 \overline{h_2}\right)(x\partial_x + y\partial_y)\,.$$

The hypothesis ensures that the tangent cone of $\alpha h_1 + \beta q_2 \overline{h_2}$ is in $\mathrm{Tan}(S^E, D)$. Since the tangent cone of h_1 is in $\mathrm{Tan}(S^E, D)$, it can be seen that the function q_2 is constant and that there exists a constant u such that

$$h_2 = uh_1$$

Then the basis $\{X_1, X_2 - uX_1\}$ is of type $(\mathfrak{O}'_d)$.

Assume finally that $\nu(S)$ is even and consider a smooth curve l which is attached to a point in $\mathrm{Tan}(S^E, D)$ after on blowing-up. Let $\left\{X_1^l, X_2^l\right\}$ be an adapted Saito basis for $S \cup l$. Consider some coordinates in which $l = \{x = 0\}$ and write

$$X_i^l = xa_i\partial_x + \left(y^{\alpha_i} b_i^0(y) + xb_i^1\right)\partial_y$$

with $b_i(0) \neq 0$. Since $\nu(S \cup l)$ is odd, a few cases may occur :

(1) Assume the basis is of type $(\mathfrak{O})$ Then, we can suppose that $\nu\left(X_1^l\right) = \nu\left(X_2^l\right) = \frac{\nu(S)}{2}$ and $\alpha_1 = \alpha_2$. The family

$$\left\{X_1^l, \overline{X}_2^l = \frac{1}{x}\left(X_2^l - \frac{b_2^0}{b_1^0} X_1^l\right)\right\}$$

is a Saito basis for S with

$$\nu\left(X_1^l\right) = \frac{\nu}{2} \qquad \text{and} \qquad \nu(\overline{X}_2^l) \geqslant \frac{\nu}{2} - 1.$$

If $\nu(\overline{X}_2^l) \geqslant \frac{\nu}{2}$ then $\mathfrak{s}(S) \geqslant \frac{\nu(S)}{2}$ which is impossible. Thus $\nu(\overline{X}_2^l) = \frac{\nu}{2} - 1$. But then, following Lemma 3.5 $\overline{X}_2^l$ cannot be dicritical which contradicts the radiality of S. Finally, the Saito basis of $S \cup l$ cannot be of type $(\mathfrak{O})$.

(2) Assume it is of type $(\mathfrak{D}_d)$ but not of type $(\mathfrak{D}'_d)$. Applying Theorem 3.13 to this case for which $\nu(S\cup l)$ is odd ensures that $S\cup l$ is not pure radial. Thus up to some changes of basis, $\mathrm{Tan}((X_1^l)^E, S\cup l)$ and $\mathrm{Tan}((X_2^l)^E, S\cup l)$ contains some points out of $\mathrm{Tan}((S\cup l)^E, D) = \mathrm{Tan}(S^E, D)$. Therefore, we obtain a Saito basis for S of the form
$$\{X_1^l, \overline{X}_2^l\}$$
where the tangent cone of $\overline{X}_2^l$ is not contained in $\mathrm{Tan}(S^E, D)$, which contradicts the assumption of S being pure radial. Therefore, the Saito basis of $S\cup l$ cannot be of type $(\mathfrak{D}_d)$ but not of type $(\mathfrak{D}'_d)$.

(3) Finally, $S\cup l$ admits a Saito basis $\{X_1^l, X_2^l\}$ of type $(\mathfrak{D}'_d)$ with $\nu(X_1^l) = \nu(X_2^l) - 1 = \frac{\nu}{2}$. If $\alpha_2 > \alpha_1$ then
$$\left\{X_1^l, \overline{X}_2^l = \frac{1}{x}\left(X_2^l - y^{\alpha_2-\alpha_1}\frac{b_2^0}{b_1^0}X_1^l\right)\right\}$$
is a Saito basis for S with
$$\nu\left(X_1^l\right) = \frac{\nu}{2} \qquad \text{and} \qquad \nu(\overline{X}_2^l) \geqslant \frac{\nu}{2}$$
which is impossible. Thus $\alpha_2 \leqslant \alpha_1$ and
$$\left\{\overline{X}_1^l = \frac{1}{x}\left(X_1^l - y^{\alpha_1-\alpha_2}\frac{b_1^0}{b_2^0}X_2^l\right), X_2^l\right\}$$
is a Saito basis for S of type $(\mathfrak{E}'_d)$. □

Finally, from the proof above we deduce the following

Lemma 3.14. — *If S is of type $(\mathfrak{D})$ then $S\cup l$ is of type $(\mathfrak{E})$. If S is of type $(\mathfrak{E}_d)$ or $(\mathfrak{E}'_d)$ then $S\cup l$ is of type $(\mathfrak{D}_d)$ or $(\mathfrak{D}'_d)$.*

3.4. Cohomology of Θ_S

As we will explain in the next section, the cohomology of the sheaf Θ_S computes the generic dimension of $\mathbb{M}^\bullet(S)$. The associated formula depends on the type of Saito basis of S.

Proposition 3.15. — *The dimension of the cohomology group $H^1(D,\Theta_S)$ can be obtained from the multiplicities of an adapted Saito basis of S the following way*

(1) *If $\nu(X_1) + \nu(X_2) = \nu(S)$ then*
$$\dim H^1(D,\Theta_S) = \frac{(\nu_1-1)(\nu_1-2)}{2} + \frac{(\nu_2-1)(\nu_2-2)}{2}.$$

(2) *If* $\nu(X_1)+\nu(X_2)=\nu(S)-1$ *then*

$$\dim H^1(D,\Theta_S)=\frac{(\nu_1-1)(\nu_1-2)}{2}+\frac{(\nu_2-1)(\nu_2-2)}{2}+\nu(S)-2-\nu_0$$

where $\nu_i=\nu(X_i),\ i=1,2$ *and* $\nu_0=\nu(\gcd(X_1^{(\nu(X_1))},X_2^{(\nu(X_2))}))$.

Proof. — The proof of the first equality is in [7]. Below, we only give a proof of the second equality. Let us consider the standard system of coordinates defined in a neighborhood of D and introduced in Section 3.1.

One can compute the cohomology using the associated covering and thus

$$H^1(D,\Theta_S)=H^1(\{U_1,U_2\},\Theta_S)=\frac{H^0(U_1\cap U_2,\Theta_S)}{H^0(U_1,\Theta_S)\oplus H^0(U_2,\Theta_S)}. \tag{3.11}$$

The task is now to describe in coordinates each H^0 involved in the quotient above. To deal with $H^0(U_1,\Theta_S)$, we start with the criterion of Saito

$$X_1\wedge X_2=uf. \tag{3.12}$$

As $\nu_1+\nu_2=\nu(S)-1$, blowing-up the criterion of Saito in the first chart (x_1,y_1) yields

$$\underbrace{\frac{E^\star X_1}{x_1^{\nu_1-1}}}_{X_1^1}\wedge\underbrace{\frac{E^\star X_2}{x_1^{\nu_2-1}}}_{X_2^1}=u\circ E\cdot x_1^2\frac{f\circ E}{x_1^{\nu(S)}},$$

Let Y be a section of Θ_S on U_1. By definition, there exists $g_1\in\mathcal{O}(U_1)$ such that

$$Y\wedge X_1^1=g_1x_1\frac{f\circ E}{x_1^{\nu(S)}}.$$

Hence, one has

$$\left(x_1Y-g_1\frac{1}{u\circ E}X_2^1\right)\wedge X_1^1=0.$$

Assume X_1 is not dicritical. Then, X_1^1 has only isolated singularities and there exists $h_1\in\mathcal{O}(U_1)$ such that

$$x_1Y=g_1\frac{1}{u\circ E}X_2^1+h_1X_1^1.$$

If now X_1 is dicritical, then $\frac{X_1^1}{x_1}$ extends analytically along D and has only isolated singularities. Therefore, there still exists $h_1\in\mathcal{O}(U_1)$ such that

$$x_1Y=k_1\frac{1}{u\circ E}X_2^1+\frac{h_1}{x_1}X_1^1.$$

Since, $x_1 Y$ and X_2^1 are tangent to D, x_1 divides h_1. Thus we get

$$H^0(U_1, \Theta_S) = \left\{ Y = \frac{1}{x_1}\left(\phi_1^1 X_1^1 + \phi_2^1 X_2^1\right) \;\middle|\; \begin{array}{l} (1)\ \phi_i^1 \in \mathcal{O}(U_1) \\ (2)\ Y \text{ extends analytically} \\ \quad\ \text{along } D \end{array} \right\}.$$

We now proceed to analyse the second condition highlighted above : let us write the expansion of X_i in homogeneous components

$$X_i = X_i^{(\nu_i)} + X_i^{(\nu_i+1)} + \cdots .$$

The relation $\nu_1 + \nu_2 = \nu - 1$ implies that

$$X_1^{(\nu_1)} \wedge X_2^{(\nu_2)} = 0$$

and we can write

$$X_i^{(\nu_i)} = \delta_i X_0$$

where $X_0 = \gcd(X_1^{(\nu_1)}, X_2^{(\nu_2)})$ and $\delta_i, i = 1, 2$ are homogeneous functions such that

$$\delta_1 \wedge \delta_2 = 1.$$

The expression of Y can be expanded with respect to x_1 in

$$Y = \frac{1}{x_1} \sum_{i=1,2} \underbrace{\left(\phi_i^{1,0}(y_1) + x_1(\cdots)\right)}_{\phi_i^1} \underbrace{\left(\delta_i^1 X_0^1 + x_1(\ldots)\right)}_{X_i^1}, \qquad \delta_i^1 = \frac{\delta_i \circ E}{x_1^{\nu(\delta_i)}}.$$

Thus the condition Y being extendable along D reduces to

$$\sum_{i=1,2} \phi_i^{1,0} \delta_i^1 = 0.$$

We proceed analogously for the open sets U_2 and $U_1 \cap U_2$ and obtain the following description where the exponent 2 refers to the second chart (x_2, y_2)

$$H^0(U_1, \Theta_S) = \left\{ Y^1 = \frac{1}{x_1} \sum_{i=1,2} \phi_i^1 X_i^1 \;\middle|\; \begin{array}{l} \phi_i^1 \in \mathcal{O}(U_1) \\ \sum_{i=1,2} \phi_i^{1,0} \delta_i^1 = 0 \end{array} \right\},$$

$$(3.13) \qquad H^0(U_2, \Theta_S) = \left\{ Y^2 = \frac{1}{y_2} \sum_{i=1,2} \phi_i^2 X_i^2 \;\middle|\; \begin{array}{l} \phi_i^2 \in \mathcal{O}(U_2) \\ \sum_{i=1,2} \phi_i^{2,0} \delta_i^2 = 0 \end{array} \right\},$$

$$H^0(U_1 \cap U_2, \Theta_S) = \left\{ Y^{12} = \frac{1}{x_1} \sum_{i=1,2} \phi_i^{12} X_i^1 \;\middle|\; \begin{array}{l} \phi_i^{12} \in \mathcal{O}(U_1 \cap U_2) \\ \sum_{i=1,2} \phi_i^{12,0} \delta_i^1 = 0 \end{array} \right\}.$$

We may now compute the number of obstructions involved in the cohomological equation describing the quotient (3.11), namely,

$$Y^{12} = Y^2 - Y^1.$$

In view of the description above, the cohomological equation splits into the system

$$\phi_i^{12} = \frac{\phi_i^2}{y_1^{\nu_i}} - \phi_i^1,\ i = 1, 2$$

which we filter with respect to x_1 obtaining

$$\phi_i^{12,0} = \frac{\phi_i^{2,0}}{y_1^{\nu_i}} - \phi_i^{1,0},\ i = 1, 2 \tag{3.14}$$

$$\phi_i^{12,1} = \frac{\phi_i^{2,1}}{y_1^{\nu_i}} - \phi_i^{1,1},\ i = 1, 2 \tag{3.15}$$

where

$$\phi_i^\star = \phi_i^{\star,0} + x_1\phi_i^{\star,1},$$

with $\star = 1, 2, 12$. Let us analyse the system (3.14). Since the functions δ_i are relatively prime, the conditions involved in the description of the cohomological spaces (3.13) ensures that there exist analytical functions $\dot{\phi}^{\star,0}$ such that

$$\phi_1^{\star,0} = \dot{\phi}^{\star,0}\delta_2^\star \text{ and } \phi_2^{\star,0} = -\dot{\phi}^{\star,0}\delta_1^\star$$

for $\star = 1, 2, 12$. Thus, the system (3.14) reduces to the sole equation

$$\dot{\phi}^{12,0} = \frac{\dot{\phi}^{2,0}}{y_1^{\nu_1+\nu(\delta_2)}} - \dot{\phi}^{1,0}.$$

Writing the Laurent expansions of the above functions yields the relation

$$\sum_{k\in\mathbb{Z}} \dot{\phi}_k^{12} y_1^k = \sum_{k\in\mathbb{N}} \dot{\phi}_k^2 y_1^{-k-\nu_1-\nu(\delta_2)} - \sum_{k\in\mathbb{N}} \dot{\phi}_k^1 y_1^k$$

which implies $\dot{\phi}_k^{12} = 0$ for $-\nu_1 - \nu(\delta_2) + 1 \leqslant k \leqslant -1$. These conditions provide the sole $-\nu_1 - \nu(\delta_2) + 1$ obstructions to the cohomological equation (3.11). The system (3.15) involves two independant cohomological equations. We can proceed analogously to identify $\frac{(\nu_i-1)(\nu_i-2)}{2}$ obstructions $i = 1, 2$, respectively, for the equation $i = 1, 2$. Finally, the formula (2) of the proposition follows from the relation

$$\nu(\delta_2) = \nu_2 - \nu_0. \qquad \square$$

Notice that the second case of Proposition 2.12 may occur when S is of type $(\mathfrak{D}_d)$ or $(\mathfrak{E}_d)$. In that case, it can be seen that $\nu(X_1) - \nu_0$ is the number of free points of X_1.

4. Dimension of the moduli space of a singular regular point

In this section, we intend to apply the previous results to compute the generic dimension of $\mathbb{M}^\bullet(S)$ for

$$S = \{x^n + y^n = 0\},$$

and thus, to recover a classical result due to Granger [13].

To achieve this, we have to identify precisely the topology of the generic optimal vector field for S. First, we are going to improve somehow the optimality of the generic optimal vector field studying when this optimality is preserved after one blowing-up.

In [8], we successfully apply these techniques in a slightly more general case : the curves with many but smooth components. The full general case is still open but might be a consequence of the mentioned work : indeed, up to ramification, any curve is an union of several smooth curves.

4.1. Optimality after one blowin-up

Proposition 4.1. — *Let S be a generic curve in its moduli space. Let $c \in D$ a point in the exceptional divisor of the single blowing-up E. Assume that*

($\star$) *there exists a germ of regular curve l such that $(S^E)_c \cup (l^E)_c$ has no Saito basis of type $(\mathfrak{E}'_d)$.*

Then there exists a vector field X optimal for S such that $(X^E)_c$ is optimal for $(S^E)_c$.

Proof. — Let $\{X_1, X_2\}$ be an adapted Saito basis for S. If $\nu(X_1) = \nu(X_2)$ which is satisfy when the basis is of type $(\mathfrak{E})$, $(\mathfrak{O}_d)$ then for α and β generic one has

$$\alpha X_1^E + \beta X_2^E = (\alpha X_1 + \beta X_2)^E .$$

According to Remark (3.10), $\{(X_1^E)_c, (X_2^E)_c\}$ is a Saito basis for $(S^E)_c$, therefore at c one has

$$\nu_c\left((\alpha X_1 + \beta X_2)^E\right) = \mathfrak{s}\left(\left(S^E\right)_c\right)$$

Thus, in that case, choosing $X = \alpha Y_1 + \beta Y_2$ yields the lemma.

Now, assume that $\nu(X_1) < \nu(X_2)$. Suppose first that $\nu(S)$ is odd then S is of type $(\mathfrak{O})$. Let us consider a curve l satisfying the hypothesis of the lemma. According to Lemma 3.14, an adapated Saito basis $\{X_1^l, X_2^l\}$ is of type $(\mathfrak{E})$ with

$$\nu\left(X_1^l\right) = \nu\left(X_2^l\right) = \frac{\nu(S)+1}{2}.$$

Applying the process of division, we are lead to an adapted Saito basis $\{\overline{X}_1^l, X_2^l\}$ for S with

$$\nu\left(\overline{X}_1^l\right) = \frac{\nu(S)-1}{2} < \nu\left(X_2^l\right) = \frac{\nu(S)+1}{2}. \tag{4.1}$$

The blow-up family $\{(\overline{X}_1^l)_c^E, (X_2^l)_c^E\}$ is a basis for $(S^E)_c$. Now, suppose that

$$\nu_c\left(\left(\overline{X}_1^l\right)^E\right) \geqslant \nu_c\left(\left(X_2^l\right)^E\right) + 1$$

therefore,

$$\nu_c\left(L\left(\overline{X}_1^l\right)^E\right) \geqslant \nu_c\left(\left(X_2^l\right)^E\right) + 2$$

where L is a local equation of l^E. The family $\{L(\overline{X}_1^l)^E, (X_2^l)^E\}$ is a Saito basis for $S^E \cup l^E$ at c and following Remark 3.9 it is of type $(\mathfrak{E}'_d)$. That is impossible. Hence,

$$\nu_c\left(\left(\overline{X}_1^l\right)^E\right) \leqslant \nu_c\left(\left(X_2^l\right)^E\right) \tag{4.2}$$

and, according to (4.1) and (4.2), $X = \overline{X}_1^l$ satisfies the conclusion of the lemma. Finally, if $\nu(S)$ is even then S is of type $(\mathfrak{E}_d)$. Therefore, $S \cup l$ is of type $(\mathfrak{O}_d)$ and the arguments are similar. □

Corollary 4.2. — *If any component of S^E satisfies the hypothesis $(\star)$ of Proposition 4.1, then there exists a vector field X optimal for S such that, for any c, $(X^E)_c$ is optimal for $(S^E)_c$.*

Proof. — Indeed, for any point c in the tangent cone of S, consider X_c given by Proposition 4.1 for the curve $(S^E)_c$. Then for a generic family of complex numbers $\{\alpha_c\}$, the vector field

$$X = \sum \alpha_c X_c$$

satisfies the property. □

4.2. Dimension of $\mathbb{M}^\bullet(S)$ where $S = \{x^n + y^n = 0\}$

The curve S is desingularized by a single blowing-up. From [21], the generic dimension of $\mathbb{M}^\bullet(S)$ is equal to

$$\dim H^1(D, \Theta_S).$$

Following Proposition 3.15, it can be computed from some topological data associated to an adapted basis of Saito for S. Below, we are going to describe these bases according to the value of n.

If $n = 3$ then there are coordinates (x, y) in which

$$S = \{f = xy(x + y) = 0\}.$$

The family

$$\{X_1 = x\partial_x + y\partial_y,\ X_2 = \sharp df = \partial_x f \partial_y - \partial_y f \partial_x\}$$

is a Saito basis for S. Since $\nu(X_1) = \nu(X_2) - 1 = 1$, X_1 is dicritical but not X_2, S is of type $(\mathfrak{O}'_d)$. If $n = 4$ then there are coordinates (x, y) in which $S = \{f = xy(x + y)(x + ay) = 0\}$ for some $a \notin \{0, 1\}$. Hence, the family

$$\{X_1 = x\partial_x + y\partial_y,\ X_2 = \sharp df\}$$

is a Saito basis for S. Since $\nu(X_1) = \nu(X_2) - 2 = 1$, X_1 is dicritical but not X_2, S is of type $(\mathfrak{E}'_d)$. In the latter case, the dimension of $\mathbb{M}^\bullet(S)$ is 1.

Now suppose $n \geqslant 5$.

Proposition 4.3. — *The curve S is of type $(\mathfrak{O}_d)$ or $(\mathfrak{E}_d)$. Moreover, the generic optimal vector X_1 is completely regular after a single blowing-up and has $\lceil \frac{n}{2} \rceil - 2$ free points.*

Proof. — Following notations introduced in [18, p. 657] for a germ at p of vector field X and a germ of curve S given in coordinates by

$$X = a(x, y)\partial_x + b(x, y)\partial_y \qquad S = \{x = 0\}$$

we recall the following definitions :

(1) if S is invariant by X, the integer $\nu_y(b(0, y))$ is called *the index* of X at p with respect to S and it is denoted by

$$\operatorname{ind}(X, S, p).$$

(2) if S is not invariant by X, the integer $\nu_y(a(0, y))$ is called *the tangency order* of X at p with respect to S and it is denoted by

$$\tan(X, S, p).$$

Suppose X_1 non dicritical, then according to [18, Lemma 1], one has

$$\nu(X_1) + 1 = \sum_{c \in D} \operatorname{ind}\left(X_1^E, D, c\right). \tag{4.3}$$

For any point c in the tangent cone of S, the curve $(S^E \cup D)_c$ is a union of two transversal smooth curves. Therefore, the index $\operatorname{ind}(X_1^E, D, c)$ is at least 1 since $(X_1^E)_c$ is singular. Therefore, one has

$$\sum_{c\in D} \operatorname{ind}\left(X_1^E, D, c\right) \geqslant \sharp\text{tangent cone} = n. \tag{4.4}$$

On the other hand, the optimality of X_1 ensures that

$$\nu(X_1) \leqslant \frac{n}{2}. \tag{4.5}$$

The equality 4.3 and the inequalities (4.4) and (4.5) are incompatible with $n \geqslant 5$, and thus X_1 is dicritical. Any component of S^E is a regular curve. Since the union of two curves is not of type $(\mathfrak{E}'_d)$, any component of S^E satisfies the hypothesis $(\star)$ of Propostion 4.1. As a consequence, we can consider X_1 to be not only optimal for S but also optimal after one blowing-up. Since any component of S^E is a regular curve, whose Saito number is equal to 0, the vector field X_1^E is regular at the tangent cone of S. Moreover, there exists Y such that $\{X_1, Y\}$ is an adapted Saito basis. Thus, after one blowing-up, one can write

$$X_1^E \wedge Y^E = u \circ E \frac{f \circ E}{x_1^n}$$

where $f = x^n + y^n$. Since out of the tangent cone of S, the function $\frac{f\circ E}{x_1^n}$ is a unit, X_1^E is finally regular at any point of D.

Following again [18, Lemma 1], one has

$$\nu(X_1) + 1 = \left\lceil \frac{n}{2} \right\rceil = 2 + \sum_{c\in D} \tan\left(X_1^E, D, c\right).$$

The above relation concludes the proof of the proposition : $\operatorname{Tan}(S^E, D)$ being empty, any tangency point between X_1^E and D is a free point. □

As a consequence of Proposition 4.3, we recover a classical result of Granger concerning the generic dimension of the moduli space of S [13]. According to Theorem 4.3, the Saito basis of S satisfies

$$\nu(X_1) = \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{else,} \end{cases} \quad \text{and } \nu(X_2) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{else.} \end{cases}$$

Moreover, by construction, the integer ν_0 identified in Proposition 3.15 satisfies

$$\begin{aligned} \nu_0 &= \nu(X_1) - (\text{number of free points}) \\ &= \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{else} \end{cases} - \left(\left\lceil \frac{n}{2} \right\rceil - 2\right) = 1. \end{aligned}$$

Now, following Propostion 3.15, the dimension of $\mathbb{M}^{\bullet}(S)$ is equal to

$$\begin{cases} \frac{1}{2}\left(\frac{n}{2}-2\right)\left(\frac{n}{2}-3\right)+\frac{1}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)+n-3=\frac{(n-2)^2}{4} & \text{if } n \text{ is even} \\ \left(\frac{n-1}{2}-1\right)\left(\frac{n-1}{2}-2\right)+n-3=\frac{(n-1)(n-3)}{4} & \text{if } n \text{ is odd} \end{cases}$$

which coincides with the results in [13].

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