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ON THE STRUCTURE OF A LOG SMOOTH PAIR IN THE EQUALITY CASE OF THE BOGOMOLOV–GIESEKER INEQUALITY

by Masataka IWAI (*)

Abstract. — We study the structure of a log smooth pair when the equality holds in the Bogomolov–Gieseker inequality for the logarithmic tangent bundle and this bundle is semistable with respect to some ample divisor. We also study the case of the canonical extension sheaf.

Résumé. — Nous étudions la structure d'une paire lisse logarithmique lorsque l'égalité tient dans l'inégalité de Bogomolov–Gieseker pour le faisceau tangent logarithmique et que ce faisceau est semistable par rapport à un certain diviseur ample. Nous étudions également le cas du sheaf d'extension canonique.

1. Introduction

Let E be a vector bundle on a smooth projective variety X. If E is semistable with respect to some ample divisor H, then the Bogomolov–Gieseker inequality holds:

$$\left(c_2(E) - \frac{r-1}{2r} c_1(E)^2\right) H^{n-2} \geqslant 0.$$

Keywords: Bogomolov–Gieseker inequality, Logarithmic tangent bundle, Log smooth pair, Projectively flat, Numerically projectively flat, Stability, Uniformization, Rational curve, MRC fibration, Rationally connected.
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If the equality holds, then E is projectively flat. Therefore, in the equality case of the Bogomolov–Gieseker inequality, the structure of a vector bundle is restricted. Moreover, in [17] and [16], we already know the structure of X when the equality holds in the Bogomolov–Gieseker inequality for the tangent bundle T_X or the canonical extension sheaf $\mathcal{E}$ (see Definition 1.3 below) under the some assumptions.

Theorem 1.1 ([17, Theorem 1.3]). — *Let X be a projective klt variety. Assume that $-K_X$ is nef. Then the following are equivalent.*

(1) *There exists an ample Cartier divisor H on X such that the canonical extension sheaf $\mathcal{E}$ is H-semistable and the equality holds in the Bogomolov–Gieseker inequality for $\mathcal{E}$:*

$$\left(\widehat{c_2}(\mathcal{E})-\frac{n}{2(n+1)}\widehat{c_1}(\mathcal{E})^2\right)[H]^{n-2}=\left(\widehat{c_2}(\Omega_X^{[1]})-\frac{n}{2(n+1)}\widehat{c_1}(\Omega_X^{[1]})^2\right)[H]^{n-2}$$
$$=0.$$

(2) *X is a quotient of a projective space or an Abelian variety by the action of a finite group of automorphisms without fixed points in codimension one.*

Theorem 1.2 ([16, Theorem 1.2]). — *Let X be a projective klt variety of dimension $n \geqslant 2$ and H be an ample divisor on X. If $\Omega_X^{[1]}$ is H-semistable and*

$$\left(\widehat{c_2}(\Omega_X^{[1]})-\frac{n-1}{2n}\widehat{c_1}(\Omega_X^{[1]})^2\right)[H]^{n-2}=0,$$

then X is a quasi-Abelian variety, that is, there exists a quasi-étale cover $\widetilde{X}\to X$ from an Abelian variety $\widetilde{X}$ to X.

We point out that $c_1(\Omega_X^1)^2=c_1(T_X)^2=c_1(\mathcal{E})^2$ and $c_2(\Omega_X^1)=c_2(T_X)=c_2(\mathcal{E})$ for any smooth projective variety X and the canonical extension sheaf $\mathcal{E}$. By Theorem 1.1 and 1.2, we have the structure theorem of a klt variety X when the equality holds in the Bogomolov–Gieseker inequality for the tangent sheaf T_X (or the canonical extension sheaf $\mathcal{E}$) and this sheaf is H-semistable.

In this paper, we study a generalization of Theorem 1.1 and 1.2 to a log smooth pair (X,D). Before the main theorems, we recall the definition of the canonical extension sheaf.

Definition 1.3 ([27, Proposition 2.10], [17, Chapter 4]). — *Let X be a smooth projective variety, D be a simple normal crossing divisor on X, and L be a line bundle on X. By the natural homomorphism of cohomology* *groups*

$$H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^1(X,\Omega_X^1) \xrightarrow{\Phi} H^1(X,\Omega_X^1(\log D)) = \operatorname{Ext}^1(\mathcal{O}_X,\Omega_X^1(\log D)),$$

there exist a vector bundle W_L induced by $\Phi(c_1(L))$ and the following exact sequence

$$0 \to \Omega_X^1(\log D) \to W_L \to \mathcal{O}_X \to 0.$$

Let $\mathcal{E}_L$ be a dual bundle of W_L. Then we have

$$0 \to \mathcal{O}_X \to \mathcal{E}_L \to T_X(-\log D) \to 0. \tag{1.1}$$

$\mathcal{E}_L$ is called the extension sheaf of $T_X(-\log D)$ by $\mathcal{O}_X$ with the extension class $c_1(L)$. In particular, $\mathcal{E}_{\mathcal{O}_X(-(K_X+D))}$ is called the canonical extension sheaf of $T_X(-\log D)$ by $\mathcal{O}_X$.

By [31, Theorem 0.1], if $D=0$ and a Fano variety X has a Kähler–Einstein metric, then the canonical extension sheaf $\mathcal{E}_{\mathcal{O}_X(-K_X)}$ is $-K_X$-semistable, thus the Miyaoka–Yau inequality holds by the Bogomolov–Gieseker inequality for $\mathcal{E}_{\mathcal{O}_X(-K_X)}$. In the case of a log smooth pair, by [27, Theorem 1.4], if (X,D) is a log smooth log-Calabi–Yau pair, then the extension sheaf $\mathcal{E}_H$ is H-semistable for any ample line bundle H, thus $c_2\big(T_X(-\log D)\big)H^{n-2} \geqslant 0$ holds by the Bogomolov–Gieseker inequality for $\mathcal{E}_H$. It is easily seen that $c_1(\mathcal{E}_L) = c_1\big(T_X(-\log D)\big)$ and $c_2(\mathcal{E}_L) = c_2\big(T_X(-\log D)\big)$.

Now we state the main results.

Theorem 1.4. — *Let X be a smooth projective variety of dimension $n \geqslant 2$, D be a simple normal crossing divisor on X, and H be an ample divisor on X. Assume that $-(K_X+D)$ is nef.*

If the extension sheaf $\mathcal{E}_L$ is H-semistable for some line bundle L and

$$\left(c_2\big(T_X(-\log D)\big) - \frac{n}{2(n+1)}c_1\big(T_X(-\log D)\big)^2\right)H^{n-2} = 0, \tag{1.2}$$

then one of the following statements holds.

1. *(X,D) is a toric fiber bundle over a finite étale quotient of an Abelian variety. Strictly speaking, there exists a smooth morphism $f: X \to Y$ such that Y is a finite étale quotient of an Abelian variety (i.e. there exists a finite étale cover $A \to Y$ from an Abelian variety A to Y), $f:(X,D) \to Y$ is locally trivial for the analytic topology, and any fiber F of f is a smooth toric variety with a boundary divisor $D|_F$.*
2. *(X,D) is isomorphic to $(\mathbb{P}^n,0)$.*

Theorem 1.5. — *Let X be a smooth projective variety of dimension $n \geqslant 2$, D be a simple normal crossing divisor on X, and H be an ample divisor on X. Assume that $-(K_X + D)$ is nef.*

If $T_X(-\log D)$ is H-semistable and

$$\left(c_2\big(T_X(-\log D)\big) - \frac{n-1}{2n} c_1\big(T_X(-\log D)\big)^2\right) H^{n-2} = 0, \tag{1.3}$$

then one of the following statements holds.

(1) *(X, D) is a toric fiber bundle over a finite étale quotient of an Abelian variety.*
(2) *X is rationally connected, $K_X + D \not\equiv 0$, and there exists a Cartier divisor B on X such that $T_X(-\log D) \cong \mathcal{O}_X(B)^{\oplus n}$.*

Moreover, if (2) holds and (X, D) is a Mori fiber space, then (X, D) is isomorphic to $(\mathbb{P}^n, H_{\mathbb{P}^n})$, where $H_{\mathbb{P}^n}$ is a hyperplane of $\mathbb{P}^n$.

By Theorem 1.4 and [27, Theorem 1.4], we obtain the following corollary.

Corollary 1.6 (A characterization of a toric fiber bundle). — *Let (X, D) and H be as in Theorem 1.4. If*

$$c_1\big(T_X(-\log D)\big) = 0 \text{ and } c_2\big(T_X(-\log D)\big) H^{n-2} = 0,$$

then (X, D) is a toric fiber bundle over a finite étale quotient of an Abelian variety.

We emphasize that Corollary 1.6 is also an easy consequence of [11, Corollary 1.7] and [27, Theorem 1.4]. In Remark 3.2, we give an another short proof of [11, Corollary 1.7].

As a difference from Theorem 1.2, even if $T_X(-\log D)$ is H-semistable and Equality (1.3) holds, (X, D) is not necessarily a toric fiber bundle over a finite étale quotient of an Abelian variety. In fact, we obtain the following examples which are different from toric fiber bundles.

Proposition 1.7 (= Subsection 4.1 and Subsection 4.2).

(1) *Let $H_{\mathbb{P}^n}$ be a hyperplane of $\mathbb{P}^n$. Then $-(K_{\mathbb{P}^n} + H_{\mathbb{P}^n})$ is nef, $T_{\mathbb{P}^n}(-\log H_{\mathbb{P}^n})$ is $H_{\mathbb{P}^n}$-semistable, and Equality (1.3) holds*
(2) *Let m be a positive integer and $\mathbb{F}_m := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-m))$ be the m-th Hirzebruch surface. Then there exists a simple normal crossing divisor D on $\mathbb{F}_m$ such that $-(K_{\mathbb{F}_m} + D)$ is nef, $T_{\mathbb{F}_m}(-\log D)$ is H-semistable, Equality (1.3) holds, and $(\mathbb{F}_m, D)$ is not a Mori fiber space. Moreover, if $m \geqslant 2$, then a minimal model of $(\mathbb{F}_m, D)$ is not isomorphic to $(\mathbb{P}^2, H_{\mathbb{P}^2})$.*

We recall some earlier works related to the structure theorem of a log smooth pair (X, D). In [33, Theorem 1] and [32, Theorem 3.1], under the assumption that $K_X + D$ is nef, big, and ample modulo D, if the equality holds in the Miyaoka–Yau inequality, then the universal cover of $X \setminus D$ is a unit ball in $\mathbb{C}^n$. In [9, Theorem A], if the natural log Higgs bundle $(\Omega^1_X(\log D) \oplus \mathcal{O}_X, \theta)$ is H-polystable and the equality holds in the Bogomolov–Gieseker inequality, then $X \setminus D \cong \mathbb{B}^n/\Gamma$, where $\mathbb{B}^n$ is a unit ball in $\mathbb{C}^n$ and Γ is a lattice of $PU(n, 1)$. In the above works, they studied the structure of (X, D) when $\Omega^1_X(\log D)$ is positive. In [11, Corollary 1.7], if $T_X(-\log D)$ is numerically flat, then (X, D) is a toric fiber bundle over a finite étale quotient of an Abelian variety. In this work, they studied the structure of (X, D) when $T_X(-\log D)$ is flat.

Under the assumptions in Theorem 1.4 or 1.5, we know that $T_X(-\log D)$ is nef. Therefore, in this paper, we study the structure of (X, D) when $T_X(-\log D)$ is (semi)positive.

Remark 1.8. — After the author submitted this paper, Druel established the structure theorem of a reduced log smooth pair (X, D) such that the logarithmic tangent bundle $T_X(-\log D)$ is H-semistable and

$$\left(c_2\big(T_X(-\log D)\big) - \frac{n-1}{2n} c_1\big(T_X(-\log D)\big)^2\right) H^{n-2} = 0$$

for some ample line bundle H. For more details, we refer the reader to [10].

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2. Preliminaries

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers. We denote by $\mathbb{N}_{>0}$ the set of positive integers and denote $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ by $\mathcal{F}^*$ for any torsion-free coherent sheaf $\mathcal{F}$ on any variety X. A pair (X, D)

is *log smooth* if X is a smooth projective variety and D is a simple normal crossing divisor on X.

First, we recall some notions of algebraic positivities of vector bundles and torsion-free coherent sheaves.

Definition 2.1. — *Let X be a smooth projective variety.*

1. [8, Definition 1.9] *A vector bundle E is* nef *if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on $\mathbb{P}(E)$.*
2. [8, Definition 1.17] *A vector bundle E is* numerically flat *if E is nef and $c_1(E) = 0$.*
3. [25, Chapter 1. Proposition 4.22] *A vector bundle E is projectively Hermitian flat if E admits a smooth Hermitian metric h such that the Chern curvature tensor $\Theta_{E,h}$ satisfies $\Theta_{E,h} = \alpha \mathrm{Id}_E$ for some 2-form α;*
4. [25, Chapter 1. Corollary 2.7] *A vector bundle E is projectively flat if E admits a connection ∇ such that $\nabla^2 = \alpha \mathrm{Id}_E$ for some 2-form α, equivalently, there exists a representation*
$$\rho : \pi_1(X) \to \mathbb{P}GL(r, \mathbb{C})$$
such that $\mathbb{P}(E) \cong X_{\mathrm{univ}} \times_\rho \mathbb{P}^{r-1}$, where X_{univ} is the universal cover of X.
5. [30, Definition 3.20] [1, Definition 7.1] [13, Definition 3.1.1] *A torsion-free coherent sheaf $\mathcal{E}$ is* pseudo-effective *(*weakly positive in the sense of Nakayama*) if for any $a \in \mathbb{N}_{>0}$ and for any ample line bundle A on X, there exists $b \in \mathbb{N}_{>0}$ such that $\mathrm{Sym}^{ab}(\mathcal{E})^{**} \otimes A^b$ is generically generated by global sections.*

Our definition of pseudo-effective vector bundles is stronger than this definition as in [26, Example 6.1.23]. In fact, our definition requires that the image of the non-nef locus of $\mathcal{O}_{\mathbb{P}(E)}(1)$ is properly contained in X in addition to this condition (cf. [20, Proposition 2.2]).

Second, we recall the definition of numerically projectively flat.

Theorem-Definition 2.2 ([30, Chapter 4. Theorem 4.1] [28, Definition 4.1]). — *Let X be a smooth projective variety of dimension $n \geqslant 2$ and $\mathcal{E}$ be a rank r reflexive coherent sheaf on X. $\mathcal{E}$ is said to be numerically projectively flat if it satisfies one of the equivalent following conditions:*

1. *$\mathcal{E}$ is locally free and the $\mathbb{Q}$-twisted vector bundle $\mathcal{E}\langle \frac{\det \mathcal{E}^*}{r} \rangle$ is nef (for the definition of a $\mathbb{Q}$-twisted coherent sheaf, see* [26, Section 6.2.A]*).*

(2) *$\mathcal{E}$ is H-semistable and*

$$\left(c_2(\mathcal{E}) - \frac{r-1}{2r}c_1(\mathcal{E})^2\right) H^{n-2} = 0$$

holds for some ample line bundle H.

(3) *$\mathcal{E}$ is locally free and there exists a filtration of subbundles:*

$$0 =: E_0 \subset E_1 \subset \cdots \subset E_l := \mathcal{E}$$

such that $G_i := E_i/E_{i-1}$ is a projectively Hermitian flat vector bundle and $c_1(G_i)/\mathrm{rank}G_i = c_1(E)/r \in H^{1,1}(X,\mathbb{R})$ holds for any $i = 1, \ldots, l$.

By [28, Lemma 4.3], if $\mathcal{E}$ is numerically projectively flat, then $\mathcal{E}^*$ is so. We use the following lemma in the proof of Theorem 1.4.

Lemma 2.3. — *If X is simply connected and E is numerically projectively flat, then there exists a line bundle L such that $E \cong L^{\oplus r}$, where r is a rank of E.*

Proof. — By [28, Theorem 1.7], E is projectively flat, and thus $\mathbb{P}(E) \cong X \times \mathbb{P}^{r-1}$ from [25, Chapter 1. Corollary 2.7] and simply connectedness of X, which completes the proof. □

3. Proofs

Proof of Theorem 1.4. — By Theorem-Definition 2.2, $\mathcal{E}_L\langle\frac{\det \mathcal{E}_L^*}{n+1}\rangle$ is nef. Hence $T_X(-\log D)\langle\frac{K_X+D}{n+1}\rangle$ is also nef by (1.1). Since $-(K_X+D)$ is nef, $T_X(-\log D)$ is nef by [28, Lemma 4.3]. Since the inclusion map $T_X(-\log D) \to T_X$ is generically surjective, by [13, Lemma 3.1.12 (ii)], T_X is pseudo-effective. By [20, Theorem 1.1], there exists a smooth morphism $f : X \to Y$ such that Y is a finite étale quotient of an Abelian variety and any fiber F of f is rationally connected.

Claim 3.1. — *$f : X \to Y$, as well as the restriction of f to D, is isomorphic to a projection from a product space, i.e. for any $x \in X$, there exist an open neighborhood U of x and an isomorphism $\phi : U \to V \times W$, where $V := f(U)$ and $W := U \cap f^{-1}(f(x))$, such that the following diagram*

$$\begin{tikzcd} U \arrow[rr, "\phi"] \arrow[dr, "f"'] & & V \times W \arrow[dl, "\mathit{projection}"] \\ & V & \end{tikzcd}$$

is commutative and $\phi(U\cap D) = V\times(W\cap D)$*. In particular,* $f:(X,D)\to Y$ *is a logarithmic deformation in the sense of* [23, Definition 3].

Proof of Claim 3.1. — From the differential map $f^*\Omega^1_Y\to\Omega^1_X$ of f, we obtain an injective morphism $s: f^*\Omega^1_Y\to\Omega^1_X(\log D)$. Hence we obtain a morphism

$$\wedge^{\dim Y}s : f^*\det\Omega^1_Y \to \wedge^{\dim Y}\Omega^1_X(\log D).$$

Since $\wedge^{\dim Y}s$ belongs to $H^0(X,\wedge^{\dim Y}\Omega^1_X(\log D)\otimes f^*\det T_Y)$ and $\wedge^{\dim Y}T_X(-\log D)\otimes f^*\det\Omega^1_Y$ is nef, $\wedge^{\dim Y}s$ has no zero point on X by [5, Proposition 1.2 (12)].

Fix $x\in X$. Let U be a neighborhood of x, $(z_1,\ldots,z_n)$ be a local coordinate on U, and Δ be a unit disk of $\mathbb{C}$. We may regard U as Δ^n and regard $x\in U$ as an origin. We may assume that $D\cap U=\{z_{n-r+1}\cdots z_n=0\}$. Set $m\coloneqq\dim Y$ and $V\coloneqq f(U)$. Let $(w_1,\ldots,w_m)$ be a local coordinate on V. Then f is written in U as follows:

$$\begin{array}{cccc} f: & U & \to & V \\ & (z_1,\ldots,z_n) & \mapsto & (f_1(z),\ldots,f_m(z)), \end{array}$$

where $f_1(z),\ldots,f_m(z)$ are holomorphic functions on U. Now we define the $m\times n$ matrix J of holomorphic functions as follows:

$$J\coloneqq\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_{n-r}} & \frac{\partial f_1}{\partial z_{n-r+1}}z_{n-r+1} & \cdots & \frac{\partial f_1}{\partial z_n}z_n \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial z_1} & \cdots & \frac{\partial f_m}{\partial z_{n-r}} & \frac{\partial f_m}{\partial z_{n-r+1}}z_{n-r+1} & \cdots & \frac{\partial f_m}{\partial z_n}z_n \end{pmatrix}.$$

Set $\mathcal{I}\coloneqq\{(i_1,i_2,\ldots,i_m)\in\mathbb{N}^m \mid 1\leqslant i_1<i_2<\cdots<i_m\leqslant n\}$. For any $I=(i_1,\ldots,i_m)\in\mathcal{I}$, we define $J_I\coloneqq(J_{k,i_l})_{1\leqslant k,l\leqslant m}$, where J_I is an $m\times m$ matrix of holomorphic functions. Then we have

$$f^*(\mathrm{d}w_1\wedge\cdots\wedge\mathrm{d}w_m)=\sum_{I=(i_1,\ldots,i_m)\in\mathcal{I}}\det(J_I)\,\delta_{i_1}\wedge\cdots\wedge\delta_{i_m}$$

on U, where δ_i is defined by

$$\delta_i\coloneqq\begin{cases}\mathrm{d}z_i & (1\leqslant i\leqslant n-r)\\ \frac{\mathrm{d}z_i}{z_i} & (n-r+1\leqslant i\leqslant n).\end{cases}$$

Since $\wedge^{\dim Y}s$ has no zero point, $f^*(\mathrm{d}w_1\wedge\cdots\wedge\mathrm{d}w_m)$ also has no zero point at $x=(0,\ldots,0)$. Therefore $m\leqslant n-r$ and there exists $I_0=(i_1,i_2,\ldots,i_m)\in\mathcal{I}$ such that $J_{I_0}(x)\neq 0$ and $1\leqslant i_1<i_2<\cdots<i_m\leqslant n-r$. We may assume that $I_0=(1,2,\ldots,m)$ and J_{I_0} has no zero point on U. Hence we define a

morphism ϕ as follows:

$$\begin{array}{cccc} \phi: & U & \to & V \times \Delta^{n-m} \\ & (z_1,\dots,z_n) & \mapsto & (f_1(z),\dots,f_m(z),z_{m+1},\dots,z_n), \end{array}$$

then ϕ is isomorphism and $\phi(U\cap D)=V\times(\Delta^{n-m}\cap D)$. □

Hence any fiber F of f intersects D transversally. Set $D_F := D|_F$, then $(K_{X/Y}+D)|_F = K_F + D_F$. By the argument of [24, Lemma 2.13 (2.13.1)] (cf. [12, Properties 2.3 (a)]),

$$0 \to T_F(-\log D_F) \to T_X(-\log D)|_F \to N_{F/X} \cong \mathcal{O}_F^{\oplus \dim Y} \to 0. \tag{3.1}$$

In particular, $T_F(-\log D_F)$ is nef by [5, Proposition 1.2 (8)].

First, we consider the case where $K_F+D_F\equiv 0$ for any fiber F of f. In this case, $T_F(-\log D_F)$ is numerically flat. Since F is rationally connected, F is simply connected by [7, Corollary 4.18 (c)]. Hence $T_F(-\log D_F)$ is trivial. Hence F is a smooth toric variety with a boundary divisor $D|_F$ by [34, Corollary 1] and [11, Chapter 1] (cf. [2, Theorem 1.2] and [29, Theorem 4.5]). From $H^1(F, T_F(-\log D_F)) = 0$ by [7, Chapter 4. Corollary 4.18], $f : (X, D) \to Y$ is locally trivial for the analytic topology by [23, Corollary 2].

Second, we consider the case where there exists a fiber F of f with $K_F+D_F \not\equiv 0$. Since $T_X(-\log D)\langle \frac{K_X+D}{n+1}\rangle$ is nef, $T_X(-\log D)|_F\langle \frac{K_F+D_F}{n+1}\rangle$ is also nef by [26, Theorem 6.2.12. (i)]. If $\dim Y \neq 0$, then $\mathcal{O}_F^{\oplus \dim Y}\langle \frac{K_F+D_F}{n+1}\rangle$ is nef by (3.1) and [26, Theorem 6.2.12. (i)], hence $K_F + D_F \equiv 0$ since $T_F(-\log D_F)$ is nef, which is contrary to the assumption. Therefore $\dim Y = 0$ and X is rationally connected. By Lemma 2.3, there exists a Cartier divisor B on X such that $\mathcal{E}_L \cong \mathcal{O}_X(B)^{\oplus(n+1)}$. Thus $-(K_X+D) \sim (n+1)B$. Since $K_X + D$ is not nef, by [14, Theorem 2.1], $X \cong \mathbb{P}^n$ and $D = 0$. □

Proof of Theorem 1.5. — By the proof of Theorem 1.4, if $T_X(-\log D)$ is numerically projectively flat, then (1) or (2) of Theorem 1.5 holds. We show that, if (2) of Theorem 1.5 holds and (X, D) is a Mori fiber space, then (X, D) is isomorphic to $(\mathbb{P}^n, H_{\mathbb{P}^n})$.

We take (X, D) and B as in (2) of Theorem 1.5 and we assume that $\phi : (X, D) \to Z$ is a Mori fiber space. We show that $\dim Z = 0$. To obtain a contradiction, assume that $\dim Z \neq 0$. Let F be a general fiber of ϕ. Notice that $\dim F \leqslant n-1$. Set $D_F := D|_F$, then

$$-(K_F+D_F) = -(K_{X/Z}+D)|_F \sim nB|_F.$$

Since $B|_F$ is ample, we obtain $F \cong \mathbb{P}^{n-1}$, $\mathcal{O}_F(B|_F) = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$, and $D_F = 0$ by [14, Theorem 2.1]. By the argument of [24, Lemma 2.13 (2.13.1)] (cf. [12, Properties 2.3 (a)]), we obtain the following exact sequence:

$$0 \to \mathcal{O}_F^{\oplus \dim Z} \to \Omega_X^1(\log D)|_F \to \Omega_F^1(\log D_F) = \Omega_F^1 \to 0.$$

Hence we obtain a group homomorphism

$$H^1(F, \Omega_X^1(\log D)|_F) \to H^1(F, \Omega_F^1) \to H^2(F, \mathcal{O}_F)^{\oplus \dim Z}.$$

From $n \geqslant 2$,

$$\begin{aligned} H^1(F, \Omega_X^1(\log D)|_F) &= H^1(F, \mathcal{O}_X(-B)^{\oplus n}|_F) \\ &= H^1(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(-1))^{\oplus n} = 0 \end{aligned}$$

and $H^2(F, \mathcal{O}_F)^{\oplus \dim Z} = H^2(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}})^{\oplus \dim Z} = 0$. Hence $H^1(F, \Omega_F^1) = 0$, which is impossible since the Picard number of F is one.

Hence (X, D) is log Fano. If $D = 0$, then we obtain

$$H^1(X, \Omega_X^1) = H^1(X, \mathcal{O}_X(-B))^{\oplus n} = H^1(X, \mathcal{O}_X(K_X + (n-1)B))^{\oplus n} = 0,$$

which is impossible since the Picard number of X is non zero. Hence $D \neq 0$. Since the log Fano index of (X, D) is more than or equal to n, by [15, Proposition 4.1], $(X, D) \cong (\mathbb{P}^n, H_{\mathbb{P}^n})$. $\square$

Remark 3.2. — We give an another proof of [11, Corollary 1.7]. If $T_X(-\log D)$ is numerically flat, then by Claim 3.1, (X, D) is a logarithmic deformation over Y such that Y is a finite étale quotient of an Abelian variety. By (3.1), $T_F(-\log D_F)$ is numerically flat for any fiber F. Therefore, by the same argument of Theorem 1.4, (F, D_F) is a toric pair and (X, D) is a toric fiber bundle over Y.

4. Examples

We recall the first Chern class and the second Chern class of a logarithmic tangent bundle. Let X be a smooth variety and $D = \sum_{i=1}^{l} D_i$ be a simple normal crossing divisor on X. By [18, Example 3.5], we have $c_1\big(T_X(-\log D)\big) = -(K_X + D)$ and

$$c_2\big(T_X(-\log D)\big) = c_2(T_X) + K_X D + D^2 - \sum_{i<j} D_i D_j.$$

4.1. Projective spaces

In Subsection 4.1, let n be a positive integer with $n \geqslant 2$ and H be a hyperplane of $\mathbb{P}^n$.

Lemma 4.1. — *$-(K_{\mathbb{P}^n}+H)$ is nef and the equality holds in the Bogomolov–Gieseker inequality for $T_{\mathbb{P}^n}(-\log H)$:*

$$\left(c_2\big(T_{\mathbb{P}^n}(-\log H)\big)-\frac{n-1}{2n}c_1\big(T_{\mathbb{P}^n}(-\log H)\big)^2\right)H^{n-2}=0. \tag{4.1}$$

Proof. — From $c_1(T_{\mathbb{P}^n})=(n+1)H$ and $c_2(T_{\mathbb{P}^n})=\frac{n(n+1)}{2}H^2$, we have $c_1\big(T_{\mathbb{P}^n}(-\log H)\big)=nH$ and

$$c_2\big(T_{\mathbb{P}^n}(-\log H)\big)=\left(\frac{n(n+1)}{2}-(n+1)+1\right)H^2=\frac{n(n-1)}{2}H^2.$$

Hence $-(K_{\mathbb{P}^n}+H)$ is nef and Equality (4.1) holds. □

Proposition 4.2.

(1) *For any $1\leqslant r\leqslant n$, if $-r<x$, then $H^0(\mathbb{P}^n,\Omega^r_{\mathbb{P}^n}(\log H)\otimes\mathcal{O}_{\mathbb{P}^n}(-x))=0$.*
(2) *$T_{\mathbb{P}^n}(-\log H)$ is numerically projectively flat. In particular, $T_{\mathbb{P}^n}(-\log H)\cong\mathcal{O}_{\mathbb{P}^n}(H)^{\oplus n}$ holds.*

Proof. — The proof is the same as [6, Lemma 2.1 and Proposition 5.2].

(1). — Fix $1\leqslant r\leqslant n$. For any torsion-free coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, the slope $\mu_H(\mathcal{F})$ with respect to H is defined by $\mu_H(\mathcal{F}):=\frac{c_1(\mathcal{F})H^{n-1}}{\mathrm{rank}\mathcal{F}}$. By [12, Properties 2.3 (b)],

$$0\to\Omega^r_{\mathbb{P}^n}\to\Omega^r_{\mathbb{P}^n}(\log H)\to\Omega^{r-1}_H\to 0. \tag{4.2}$$

Since $\Omega^r_{\mathbb{P}^n}$ is H-semistable and $\mu_H(\Omega^r_{\mathbb{P}^n})=\frac{-r(n+1)}{n}$, if $\frac{-r(n+1)}{n}<x$, then $H^0(\mathbb{P}^n,\Omega^r_{\mathbb{P}^n}\otimes\mathcal{O}_{\mathbb{P}^n}(-x))=0$. By the same argument, if $\frac{-(r-1)n}{n-1}<x$, then $H^0(H,\Omega^{r-1}_H\otimes\mathcal{O}_H(-x))=0$. Therefore by (4.2), if $-r<x$, then $H^0(\mathbb{P}^n,\Omega^r_{\mathbb{P}^n}(\log H)\otimes\mathcal{O}_{\mathbb{P}^n}(-x))=0$.

(2). — By Theorem-Definition 2.2 and Lemma 4.1, it is enough to show that $\Omega^1_{\mathbb{P}^n}(\log H)$ is H-semistable. To obtain a contradiction, assume that there exists a rank r torsion-free coherent sheaf $\mathcal{F}\subset\Omega^1_{\mathbb{P}^n}(\log H)$ with $\mu_H(\mathcal{F})>\mu_H(\Omega^1_{\mathbb{P}^n}(\log H))=-1$. Let x be a real number with $\det\mathcal{F}\cong\mathcal{O}_{\mathbb{P}^n}(x)$. By the assumption, $\mu_H(\mathcal{F})=\frac{x}{r}>-1$. From $\det\mathcal{F}\subset\Omega^r_{\mathbb{P}^n}(\log H)$, we have $H^0(\mathbb{P}^n,\Omega^r_{\mathbb{P}^n}(\log H)\otimes\mathcal{O}_{\mathbb{P}^n}(-x))\neq 0$, contrary to (1). □

4.2. Hirzebruch surfaces

In Subsection 4.2, let m be a positive integer, $\mathbb{F}_m:=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-m))$ be the m-th Hirzebruch surface, and $\sigma:\mathbb{F}_m\to\mathbb{P}^1$ be the ruling of $\mathbb{F}_m$. By [19, Chapter V. Proposition 2.8], there exists a section C_0 with

$\mathcal{O}_{\mathbb{F}_m}(C_0) \cong \mathcal{O}_{\mathbb{F}_m}(1)$. Let f be a fiber of σ. By [19, Chapter V. Theorem 2.17], there exists a section C_∞ with $C_\infty \sim C_0+mf$. Set $D \coloneqq C_0+C_\infty$. Notice that D is a simple normal crossing divisor on $\mathbb{F}_m$.

Lemma 4.3. — *$-(K_{\mathbb{F}_m}+D)$ is nef and the equality holds in the Bogomolov–Gieseker inequality for $T_{\mathbb{F}_m}(-\log D)$:*

$$c_2\big(T_{\mathbb{F}_m}(-\log D)\big) - \frac{1}{4}c_1\big(T_{\mathbb{F}_m}(-\log D)\big)^2 = 0. \tag{4.3}$$

Proof. — We have $(C_0)^2=-m$, $C_0f=1$, and $f^2=0$. From

$$-(K_{\mathbb{F}_m}+D) \sim (2C_0+(m+2)f) - (2C_0+mf) = 2f, \tag{4.4}$$

$-(K_{\mathbb{F}_m}+D)$ is nef and $c_1\big(T_{\mathbb{F}_m}(-\log D)\big)^2 = (2f)^2 = 0$. From

$$c_2(T_{\mathbb{F}_m}) = c_1(T_{\mathbb{F}_m/\mathbb{P}^1})c_1(\sigma^*T_{\mathbb{P}^1}) = (2C_0+mf)2f = 4,$$

we obtain

$$\begin{aligned} c_2\big(T_{\mathbb{F}_m}(-\log D)\big) &= c_2(T_{\mathbb{F}_m}) + K_{\mathbb{F}_m}D + D^2 - C_0C_\infty \\ &= 4 - (2C_0+(m+2)f)(2C_0+mf) \\ &\qquad + (2C_0+mf)^2 - C_0(C_0+mf) \\ &= 4-4+0-0 = 0. \end{aligned}$$

Therefore Equality (4.3) holds. □

Proposition 4.4.

$$\Omega^1_{\mathbb{F}_m}(\log D)\otimes\sigma^*(\mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathcal{O}_{\mathbb{F}_m}^{\oplus 2}$$

holds. In particular, $T_{\mathbb{F}_m}(-\log D)$ is numerically projectively flat.

Proof. — From $\mathbb{F}_m = \{([x_1:x_2],[y_0:y_1:y_2]) \in \mathbb{P}^1\times\mathbb{P}^2 \mid y_1x_2^m = y_2x_1^m\}$, we obtain

$$C_0 = \{([x_1:x_2],[1:0:0]) \in \mathbb{F}_m \mid [x_1:x_2]\in\mathbb{P}^1\}$$

and

$$C_\infty = \{([x_1:x_2],[0:x_1^m:x_2^m]) \in \mathbb{F}_m \mid [x_1:x_2]\in\mathbb{P}^1\}.$$

We define the Zariski open sets $W_k \cong \mathbb{C}^2$ in $\mathbb{F}_m$ for $k=1,2,3,4$ as follows:

$$\begin{array}{llcl} \tau_1: & W_1 & \to & \mathbb{F}_m \\ & (x,y) & \mapsto & ([1:x],[1:y:x^my]) \\ \tau_3: & W_3 & \to & \mathbb{F}_m \\ & (\xi,\eta) & \mapsto & ([\xi:1],[1:\xi^m\eta:\eta]) \end{array} \qquad \begin{array}{llcl} \tau_2: & W_2 & \to & \mathbb{F}_m \\ & (u,v) & \mapsto & ([1:u],[v:1:u^m]) \\ \tau_4: & W_4 & \to & \mathbb{F}_m \\ & (z,w) & \mapsto & ([z:1],[w:z^m:1]). \end{array}$$

By computations, we have

$$x = u = \frac{1}{\xi} = \frac{1}{z} \text{ and } y = \frac{1}{v} = \xi^m\eta = \frac{z^m}{w}.$$

	on W_1	on W_2	on W_3	on W_4
local basis of $\Omega^1_{\mathbb{F}_m}(\log D)$	$\mathrm{d}x, \frac{\mathrm{d}y}{y}$	$\mathrm{d}u, \frac{\mathrm{d}v}{v}$	$\mathrm{d}\xi, \frac{\mathrm{d}\eta}{\eta}$	$\mathrm{d}z, \frac{\mathrm{d}w}{w}$

Hence the local basis of $\Omega^1_{\mathbb{F}_m}(\log D)$ are as shown in the following table:

Set

$$h_{W_1W_2} := 1, h_{W_1W_3} := x, h_{W_1W_4} := x, h_{W_2W_3} := u, h_{W_2W_4} := u, h_{W_3W_4} := 1,$$

and $h_{W_jW_i} := h^{-1}_{W_iW_j}$ for any $i, j \in \mathbb{N}$ with $1 \leqslant i < j \leqslant 4$. Then $\{h_{W_iW_j}\}_{1\leqslant i,j\leqslant 4}$ are transition functions of $\sigma^*(\mathcal{O}_{\mathbb{P}^1}(1))$.

We would like to find two nowhere vanishing global sections in $\Omega^1_{\mathbb{F}_m}(\log D) \otimes \sigma^*(\mathcal{O}_{\mathbb{P}^1}(1))$. To find a global section, it is enough to find a 4-tuple (t_1, t_2, t_3, t_4) of local holomorphic logarithmic differential forms such that $t_i \in H^0(W_i, \Omega^1_{\mathbb{F}_m}(\log D))$ and $t_i = h_{W_iW_j} t_j$ for any $1 \leqslant i, j \leqslant 4$.

The first section S_1 is given by

$$S_1 := \left(\frac{\mathrm{d}y}{y}, -\frac{\mathrm{d}v}{v}, m\,\mathrm{d}\xi + \frac{\xi \mathrm{d}\eta}{\eta}, m\,\mathrm{d}z - \frac{z\mathrm{d}w}{w}\right),$$

and the second section S_2 is given by

$$S_2 := \left(m\,\mathrm{d}x + \frac{x\,\mathrm{d}y}{y}, m\,\mathrm{d}u - \frac{u\,\mathrm{d}v}{v}, \frac{\mathrm{d}\eta}{\eta}, -\frac{\mathrm{d}w}{w}\right).$$

S_1 and S_2 are nowhere vanishing global sections in $\Omega^1_{\mathbb{F}_m}(\log D) \otimes \sigma^*(\mathcal{O}_{\mathbb{P}^1}(1))$. Moreover, S_1 and S_2 are linearly independent. Hence

$$\Omega^1_{\mathbb{F}_m}(\log D) \otimes \sigma^*(\mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathcal{O}^{\oplus 2}_{\mathbb{F}_m}.$$

From $\Omega^1_{\mathbb{F}_m}(\log D) \cong \sigma^*(\mathcal{O}_{\mathbb{P}^1}(-1))^{\oplus 2}$, $\Omega^1_{\mathbb{F}_m}(\log D)$ is semistable with respect to some ample divisor. By Theorem-Definition 2.2 and Lemma 4.3, $T_{\mathbb{F}_m}(-\log D)$ is numerically projectively flat. □

By (4.4), we have $(K_{\mathbb{F}_m} + D)C_0 = -2 < 0$ and $(K_{\mathbb{F}_m} + D)f = 0$. From $\overline{NE}(\mathbb{F}_m) = \mathbb{R}_+[f] + \mathbb{R}_+[C_0]$, only $\mathbb{R}_+[C_0]$ is a $(K_{\mathbb{F}_m} + D)$-negative extremal ray. If $m = 1$, then a blow-down $(\mathbb{F}_1, D) \to (\mathbb{P}^2, H_{\mathbb{P}^2})$ along C_0 is a $(K_{\mathbb{F}_1} + D)$-negative extremal contraction induced by $\mathbb{R}_+[C_0]$. Hence $(\mathbb{F}_1, D)$ is not a Mori fiber space.

We consider the case of $m \geqslant 2$. Let R be the image of the m-th Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^m$ and Y be the projective cone over R. By [19, Chapter V. Example 2.11.4], there exists a $(K_{\mathbb{F}_m} + D)$-negative extremal contraction $\tau : (\mathbb{F}_m, D) \to (Y, R)$ induced by $\mathbb{R}_+[C_0]$ such that τ contracts C_0 to the vertex of Y. Hence $(\mathbb{F}_m, D)$ is not a Mori fiber space and a minimal model of $(\mathbb{F}_m, D)$ is not isomorphic to $(\mathbb{P}^2, H_{\mathbb{P}^2})$.

4.3. On slope rationally connected varieties

This study is motivated by the following conjecture.

Conjecture 4.5 ([4, Conjecture 1.5]). — *Let X be a smooth projective variety and D be an effective $\mathbb{Q}$-divisor on X. Assume that (X, D) is klt and $-(K_X + D)$ is nef. Then there exists an orbifold morphism $\rho : (X, D) \to (R, D_R)$ with the following properties:*

(1) *(R, D_R) is a klt pair and $c_1(K_R + D_R) = 0$.*
(2) *For general point $r \in R$, the general fiber (X_r, D_r) is slope rationally connected (for the definition of slope rationally connectedness, see* [3, Definition 1.2]*).*
(3) *ρ is locally trivial with respect to pairs.*

An orbifold morphism $\rho : (X, D) \dashrightarrow (R, D_R)$ is called *slope rationally connected quotient* (in short *sRC-quotient*) if a general fiber (X_r, D_r) is slope rationally connected and $K_R + D_R$ is pseudo-effective. By [3, Theorem 1.5], an sRC-quotient exists and is unique up to orbifold birational equivalence. Notice that an sRC-quotient is a generalization of an MRC-fibration to an orbifold pair. From Conjecture 4.5, it is expected that we can take an sRC-quotient as a smooth morphism for any klt pair (X, D) such that $-(K_X + D)$ is nef. If X is a smooth surface, then Conjecture 4.5 holds by [4, Theorem 1.6].

At least in the special case of nef logarithmic tangent bundle, one might be tempted to propose the following conjecture.

Conjecture 4.6. — *Let (X, D) be a log smooth pair. If the logarithmic tangent bundle $T_X(-\log D)$ is nef, then we can take a smooth sRC-quotient.*

By [5] and [8] (or by the argument of [20, Theorem 1.1]), if T_X is nef, then we can take a smooth MRC fibration $f : X \to Y$ such that Y is a finite étale quotient of an Abelian variety. Since any fiber F of f is rationally connected and T_F is nef, F is Fano by [8, Proposition 3.10]. Hence if T_X is nef, then X consists of an Abelian variety and a Fano variety, up to a finite étale cover. Therefore to study Conjecture 4.6 is to study the structure of a log smooth pair with a nef logarithmic tangent bundle, such as [5] and [8].

However, there exists a counter-example of Conjecture 4.6.

Proposition 4.7. — *Let $[z_0 : z_1 : z_2]$ be a coordinate of $\mathbb{P}^2$. Set $H_1 := \{z_1 = 0\}$ and $H_2 := \{z_2 = 0\}$. Then $T_{\mathbb{P}^2}(-\log(H_1 + H_2))$ is nef but $(\mathbb{P}^2, H_1 + H_2)$ is not slope rationally connected. In this example, we can not take an sRC-quotient as a smooth morphism.*

By [3, Example 10.2], we already know that $(\mathbb{P}^2, H_1 + H_2)$ is not slope rationally connected. We give a proof of this fact for the reader's convenience.

Proof. — Let $\pi : X \to \mathbb{P}^2$ be a blow-up of $\mathbb{P}^2$ at $[1:0:0]$, E be an exceptional divisor of π, and $\widetilde{H_1}$ (resp. $\widetilde{H_2}$) be a strict transform of H_1 (resp. H_2) by π. Set $D := \widetilde{H_1} + \widetilde{H_2} + E$. From $K_X + D = \pi^*(K_{\mathbb{P}^2} + H_1 + H_2)$ and $\pi^{-1}\big(\mathrm{Supp}(H_1 + H_2)\big) = \mathrm{Supp}(D)$, we obtain

$$T_X(-\log D) \cong \pi^* T_{\mathbb{P}^2}(-\log(H_1 + H_2))$$

by [21, Chapter 11]. It is enough to show that $T_X(-\log D)$ is nef.

Notice that $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. Let $\sigma : X \to \mathbb{P}^1$ be the ruling of X and $[z_1 : z_2]$ be a coordinate of $\mathbb{P}^1$. Set $[0] := \{z_1 = 0\} \subset \mathbb{P}^1$ and $[\infty] := \{z_2 = 0\} \subset \mathbb{P}^1$. Then we have $\sigma_* \widetilde{H_1} = [0]$ and $\sigma_* \widetilde{H_2} = [\infty]$. Since $\sigma : (X, D) \to (\mathbb{P}^1, [0] + [\infty])$ is a log smooth morphism in the sense of [22, Chapter 3], there exists a line bundle F on X such that

$$0 \to F \to T_X(-\log D) \to \sigma^* T_{\mathbb{P}^1}(-\log([0] + [\infty])) \to 0$$

by [22, Proposition 3.12]. From $F \cong \mathcal{O}_X(-K_X - D) \cong \pi^* \mathcal{O}_{\mathbb{P}^2}(H_1)$, F is nef. Since $\sigma^* T_{\mathbb{P}^1}(-\log([0] + [\infty]))$ is nef, $T_X(-\log D)$ is also nef.

A general fiber of $\sigma : (X, D) \to (\mathbb{P}^1, [0] + [\infty])$ is isomorphic to $(\mathbb{P}^1, [p])$ for some $p \in \mathbb{P}^1$. Hence a general fiber of σ is slope rationally connected. Thus $\sigma \circ \pi^{-1} : (\mathbb{P}^2, H_1 + H_2) \dashrightarrow (\mathbb{P}^1, [0] + [\infty])$ is an sRC-quotient, and finally $(\mathbb{P}^2, H_1 + H_2)$ is not slope rationally connected. Since the Picard number of $\mathbb{P}^2$ is one, we can not take an sRC-quotient as a (smooth) morphism. $\square$

4.4. On the assumption of a semistability condition

Without some assumptions such as semistability in Theorem 1.4 or 1.5, it is difficult to study the structure of a log smooth pair when the equality holds in the Bogomolov–Gieseker inequality. In fact, there exist many examples of log smooth pairs such that Equality (1.2) or (1.3) holds. We give a few examples. In this subsection, Let X be a smooth projective variety and $D = \sum_{i=1}^{l} D_i$ be a simple normal crossing divisor on X.

First, we consider the case of $X = \mathbb{P}^n$. Let H be a hyperplane of $\mathbb{P}^n$ and d_i be a positive integer with $D_i \sim d_i H$ for any $1 \leqslant i \leqslant l$. Then we have at least 18 examples such that Equality (1.2) or (1.3) holds, $D \neq 0$, and $D \neq H$ by computations using a computer. For example, if $n = 7$, $l = 3$, and $(d_1, d_2, d_3) = (2, 1, 1)$, then Equality (1.2) holds, and if $n = 8$, $l = 4$, and $(d_1, d_2, d_3, d_4) = (2, 1, 1, 1)$, then Equality (1.3) holds.

Second, we consider the case where X is a degree q hypersurface of $\mathbb{P}^{n+1}$. We assume that $q \geqslant 2$ and any degree of D_i is one. Then we have at least 90 examples such that Equality (1.2) or (1.3) holds by computations using a computer. For example, if $(n, q, l) = (7, 2, 3)$, then Equality (1.2) holds, and if $(n, q, l) = (8, 2, 4)$, then Equality (1.3) holds.

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