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# COHOMOLOGICAL SUPPORT LOCI AND PLURICANONICAL SYSTEMS ON IRREGULAR VARIETIES

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ABSTRACT. — Given an irregular variety  $X$  of general type, we show that if a general fiber  $F$  of the Albanese morphism of  $X$  satisfies certain Hodge theoretic conditions, the 0-th cohomological support locus of  $K_X$  generates  $\mathrm{Pic}^0(X)$ . We then show that the condition that the 0-th cohomological support locus of  $K_X$  generates  $\mathrm{Pic}^0(X)$  can often be applied to prove the birationality of certain pluricanonical maps of  $X$ .

RÉSUMÉ. — Étant donné une variété irrégulière  $X$  de type général, nous montrons que si une fibre générale du morphisme d'Albanese de  $X$  satisfait certaines conditions de la théorie de Hodge, le lieu  $V^0(K_X)$  engendre  $\mathrm{Pic}^0(X)$ . Nous montrons ensuite que la condition que  $V^0(K_X)$  engendre  $\mathrm{Pic}^0(X)$  peut souvent être appliquée pour prouver la birationalité de certaines applications pluricanoniques de  $X$ .

## 1. Introduction

We work over complex number field  $\mathbb{C}$  throughout this paper.

We study the pluricanonical systems of irregular varieties. This topic was initiated in a series of article of Jungkai Chen and Hacon [5, 6, 7] and has been studied by many other authors (see for instance [2, 4, 13, 24]).

Let  $X$  be a smooth projective variety of general type with  $q(X) = h^1(X, \mathcal{O}_X) > 0$ . Fix the Albanese morphism  $a_X : X \rightarrow A_X$  from  $X$  to

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its Albanese variety  $A_X$ . We denote by  $F$  a connected component of a general fiber of  $a_X$ . We say that  $X$  is of Albanese fiber dimension  $m$  if  $\dim F = m$ . When  $X$  is of Albanese fiber dimension 0, we say that  $X$  is of maximal Albanese dimension. The properties of the pluricanonical maps of  $X$  are closely related to the properties of the pluricanonical systems of  $F$ . When  $F$  or  $X$  is of low dimensions, the general picture is well-understood.

The following theorem was due to Chen and Hacon [5] and Lahoz, Tirabassi and the author [13].

**THEOREM 1.1.** — *Let  $X$  be a smooth projective variety of general type, of maximal Albanese dimension. The linear system  $|mK_X + P|$  induces a birational map of  $X$  for any integer  $m \geq 3$  and  $P \in \text{Pic}^0(X)$ .*

This is an optimal result, which can be easily seen from the properties of pluricanonical systems of smooth projective curves of genus 2. A similar result was proved by the author and Hao Sun in [14] when  $F$  is a curve.

**THEOREM 1.2.** — *Let  $X$  be a smooth projective variety of general type, of Albanese fiber dimension 1. The linear system  $|mK_X + P|$  induces a birational map of  $X$  for any integer  $m \geq 4$  and  $P \in \text{Pic}^0(X)$ .*

The structure of

$$V^0(K_X) := \{P \in \text{Pic}^0(X) \mid H^0(X, K_X \otimes P) \neq 0\}$$

plays an important role in the proof of Theorem 1.1 and Theorem 1.2. We know from generic vanishing theory (see [10] and [23]) that

$$V^j(K_X) := \{P \in \text{Pic}^0(X) \mid H^j(X, K_X \otimes P) \neq 0\},$$

for  $j \geq 0$ , is a union of torsion translates of abelian subvarieties of  $\text{Pic}^0(X)$ . We say that  $V^j(K_X)$  generates  $\text{Pic}^0(X)$  if the irreducible components of  $V^j(K_X)$  generate  $\text{Pic}^0(X)$  as a group. One of the main points of the proof of Theorem 1.1 and 1.2 is the following (see [5] and [14]).

**THEOREM 1.3.** — *Let  $X$  be a smooth projective variety of general type. Assume that  $X$  is of maximal Albanese dimension or is of Albanese fiber dimension 1,  $V^0(K_X)$  generates  $\text{Pic}^0(X)$ .*

The first main result of this paper is a general criterion for the property that  $V^0(K_X)$  generates  $\text{Pic}^0(X)$ .

We say a smooth projective variety  $V$  satisfies the infinitesimal Torelli condition if the natural cup product

$$H^1(V, T_V) \longrightarrow \text{Hom}(H^0(V, K_V), H^1(V, \Omega_V^{\dim V - 1}))$$

is injective. The infinitesimal Torelli condition is of course closely related with the variation of Hodge structures of  $V$ .

Given a smooth projective variety of general type  $V$ , the birational transformation group  $\text{Bir}(V)$  is a finite group which acts naturally on  $H^0(V, K_V)$ .

**THEOREM 1.4.** — *Assume that  $X$  is of general type and  $F$  satisfies the following conditions:*

(C1) *the canonical model of  $F$  is a smooth projective variety  $V$  which satisfies the infinitesimal Torelli condition;*

(C2)  *$\text{Bir}(F)$  acts faithfully on  $H^0(F, K_F)$ ,*

*$V^0(K_X)$  generates  $\text{Pic}^0(X)$ .*

This is a generalization of Theorem 1.3. Indeed, let  $C$  be a non-hyperelliptic smooth projective curve of genus  $\geq 3$ . We know that  $K_C$  is very ample, i.e. the canonical map  $\varphi_{K_C} : C \rightarrow \mathbb{P}(H^0(K_C))$  is an embedding. Since  $C$  is of dimension 1,  $\text{Bir}(C) = \text{Aut}(C)$ . It is also clear that the canonical map  $\varphi_{K_C}$  is  $\text{Aut}(C)$ -equivariant. Thus  $\text{Aut}(C)$  acts faithfully on  $H^0(C, K_C)$ . Moreover, by taking Serre duality, it is easy to see that the infinitesimal Torelli condition for  $C$  is equivalent to the statement that the natural product  $H^0(C, K_C) \otimes H^0(C, K_C) \rightarrow H^0(C, 2K_C)$  is surjective. This holds when  $C$  is non-hyperelliptic by a classical theorem of Max Noether (see [1, p. 117]). Thus Theorem 1.4 implies that if  $X$  is a smooth projective variety of general type and a connected component of a general fiber of  $a_X$  is a non-hyperelliptic curve,  $V^0(K_X)$  generates  $\text{Pic}^0(X)$ .

The proof of Theorem 1.4 is quite different from the proof of Theorem 1.3. One of the main ingredients of the proof of Theorem 1.4 is a decomposition theorem of Hodge modules on abelian varieties due to Pareschi, Popa, and Schnell in [20]. Our proof also works for higher cohomological support locus  $V^0(R^j f_* K_X)$  (see Theorem 4.1).

We also explore the relation between the structure of  $V^0(K_X)$  and the properties of the pluricanonical systems of  $X$ .

**THEOREM 1.5.** — *Assume that  $V^0(K_X)$  generates  $\text{Pic}^0(X)$  and  $|K_F|$  induces a birational map of  $F$ . The linear system  $|3K_X + P|$  induces a birational map of  $X$  for  $P \in \text{Pic}^0(X)$  general.*

## 2. Notation and preliminaries

A variety is a separated integral scheme of finite type over  $\mathbb{C}$ . Let  $X$  be a smooth projective variety. We denote by  $K_X$  the canonical bundle of  $X$  and

we write  $\omega_X = \mathcal{O}_X(K_X)$  the canonical sheaf of  $X$ . We denote by  $p_g(X) := \dim H^0(X, K_X)$  the *geometric genus* of  $X$ ,  $p_m(X) := \dim H^0(X, mK_X)$  the  $m$ -th *plurigenus* of  $X$  for  $m \geq 2$ , and  $q(X) = \dim H^1(X, \mathcal{O}_X)$  the *irregularity* of  $X$ .

Let  $f : X \rightarrow A$  be a morphism from  $X$  to an abelian variety  $A$ . We say that  $f$  is *primitive* if  $f^* : \text{Pic}^0(A) \rightarrow \text{Pic}^0(X)$  is injective. Note that  $f$  is primitive iff  $f$  does not factor through a non-trivial étale cover of  $A$ . We say that  $A$  is *simple* if there does not exist a positive dimensional proper abelian subvariety of  $A$ .

Let  $\mathcal{F}$  be a coherent sheaf on  $X$ . The  $i$ -th *cohomological support locus* of  $\mathcal{F}$  with respect to  $f$  is the closed subset

$$V^i(\mathcal{F}, f) := \{P \in \text{Pic}^0(A) \mid H^i(X, \mathcal{F} \otimes f^*P) \neq 0\} \subset \text{Pic}^0(A).$$

Similarly, we write

$$V^i(\mathcal{F}, f)_m := \{P \in \text{Pic}^0(A) \mid \dim H^i(X, \mathcal{F} \otimes f^*P) \geq m\} \subset \text{Pic}^0(A),$$

for each  $m \geq 2$ . When  $f$  is the Albanese morphism of  $X$ , we often write  $V^i(\mathcal{F})$  (resp.  $V^i(\mathcal{F})_m$ ) instead of  $V^i(\mathcal{F}, f)$  (resp.  $V^i(\mathcal{F}, f)_m$ ). Given a morphism  $f : X \rightarrow A$  from a smooth projective variety to an abelian variety, Green and Lazarsfeld proved in [10] that  $V^i(K_X, f)$  or  $V^i(K_X, f)_m$  is a union of translates of abelian subvarieties of  $\text{Pic}^0(A)$  for each  $i \geq 0$ . Simpson later showed in [23] that each component of  $V^i(K_X, f)_m$  is a torsion translate of an abelian subvariety of  $\text{Pic}^0(A)$ .

Let  $\mathcal{F}$  be a coherent sheaf on an abelian variety  $A$ . We say that  $\mathcal{F}$  is an *IT sheaf of index 0* if  $V^i(\mathcal{F}) = \emptyset$  for each  $i > 0$ . We say that  $\mathcal{F}$  is *M-regular* if  $\text{codim}_{\text{Pic}^0(A)} V^i(\mathcal{F}) > i$  for each  $i > 0$  and we say that  $\mathcal{F}$  is *GV* if  $\text{codim}_{\text{Pic}^0(A)} V^i(\mathcal{F}) \geq i$  for each  $i > 0$ .

GV sheaves and M-regular sheaves were introduced respectively in [11] and [19]. The following properties of GV and M-regular sheaves are very useful and will be applied frequently in this paper. If  $\mathcal{F}$  is GV,  $V^0(\mathcal{F}) \neq \emptyset$  (see [11]) and if  $\mathcal{F}$  is M-regular,  $\mathcal{F}$  is *continuously globally generated*, i.e. for any Zariski open subset  $U \subset \text{Pic}^0(A)$ , the natural evaluation map

$$\bigoplus_{P \in U} H^0(A, \mathcal{F} \otimes P^{-1}) \otimes P \longrightarrow \mathcal{F}$$

is surjective, and in particular,  $V^0(\mathcal{F}) = \text{Pic}^0(A)$  (see [19]). We then conclude that if  $\mathcal{F}$  is M-regular, its holomorphic Euler characteristic  $\chi(A, \mathcal{F})$  is positive. Indeed, let  $P \in \text{Pic}^0(A)$  be general,  $\chi(A, \mathcal{F}) = \chi(A, \mathcal{F} \otimes P) = h^0(A, \mathcal{F} \otimes P) > 0$ . Similarly, if  $\mathcal{F}$  is GV,  $\chi(A, \mathcal{F}) \geq 0$  and equality holds

iff  $V^0(\mathcal{F})$  is a proper subset of  $\mathrm{Pic}^0(A)$ . Moreover, we know that M-regular sheaves are ample (see [9]).

Given a morphism  $f : X \rightarrow A$  from a smooth projective variety to an abelian variety, Hacon proved in [11] that  $R^j f_* \omega_X$  is GV for each  $j \geq 0$ . This result was strengthened by Chen and the author when  $f$  is generically finite onto its image (see [8]) and by Pareschi, Popa, and Schnell in general (see [20]). More precisely, let  $f : X \rightarrow A$  be a morphism from a smooth projective variety to an abelian variety, for each  $j \geq 0$ , there exist surjective morphisms between abelian varieties  $p_i : A \rightarrow A_i$  with connected fibers, M-regular sheaves  $\mathcal{F}_i$  on  $A_i$ , and torsion line bundles  $P_i$  on  $X$  such that

$$R^j f_* \omega_X = \bigoplus_i p_i^* \mathcal{F}_i \otimes P_i.$$

This result is usually called *the Chen-Jiang decomposition*. If a coherent sheaf  $\mathcal{F}$  on  $A$  satisfies this decomposition form, we shall say that  $\mathcal{F}$  has *the Chen-Jiang decomposition property*. If  $\mathcal{F}$  has the Chen-Jiang decomposition property and there exist two coherent sheaves  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  on  $A$  such that  $\mathcal{F} = \mathcal{Q}_1 \oplus \mathcal{Q}_2$ ,  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  also have the Chen-Jiang decomposition property (see [18, Proposition 4.6]). If  $X$  is of general type and  $f_* \omega_X^{\otimes m}$  is non-zero for some integer  $m \geq 2$ , Lombardi, Popa, and Schnell proved that  $f_* \omega_X^{\otimes m}$  is an IT sheaf of index 0 in [18]. Thus,  $p_m(X) = h^0(X, mK_X + f^*P)$  for any  $P \in \mathrm{Pic}^0(A)$ .

### 3. Hodge modules

The proof of Theorem 1.4 relies heavily on the language of Hodge modules. We recall some results about Hodge modules on abelian varieties which will be applied later. All results can be found in [20], where Pareschi, Popa, and Schnell applied the machinery of Hodge modules to prove the Chen-Jiang decomposition. A decomposition theorem of polarizable Hodge modules on abelian varieties proved by Pareschi, Popa, and Schnell is crucial for our purpose.

Let  $X$  be a complex manifold. We denote  $\mathbf{HM}_{\mathbb{R}}(X, w)$  be the category of real Hodge modules on  $X$  of weight  $w$ , which is a semi-simple  $\mathbb{R}$ -linear abelian category, endowed with a faithful functor to the category of real perverse sheaves. An object  $M$  of  $\mathbf{HM}_{\mathbb{R}}(X, w)$  consists of a regular holonomic left  $\mathcal{D}$ -module  $\mathcal{M}$ , a good filtration  $F.M$  by coherent  $\mathcal{O}_X$ -modules, a perverse sheaf  $M_{\mathbb{R}}$  with coefficients in  $\mathbb{R}$  and an isomorphism  $M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{DR}(\mathcal{M})$ . The support of  $M$  is defined to be the support of the

perverse sheaf  $M_{\mathbb{R}}$ . One can also define polarizable Hodge modules. One can check [20] or [22] for more details about this category.

- (A) Every object  $M \in \mathbf{HM}_{\mathbb{R}}(X, w)$  admits a locally finite decomposition by strict support:

$$M \simeq \bigoplus_{i=1}^n M_i,$$

where each  $M_i \in \mathbf{HM}_{\mathbb{R}}(X, w)$  has strict support equal to an irreducible analytic subvariety  $Z_i \subset X$ .

- (B) The category of polarizable real Hodge modules of weight  $w$  with strict support  $Z \subset X$  is equivalent to the category of generically defined polarizable real variations of Hodge structure of weight  $w - \dim Z$  on  $Z$ .
- (C) Denote by  $\mathbb{R}_X[\dim X] \in \mathbf{HM}_{\mathbb{R}}(X, \dim X)$  the polarizable real Hodge module corresponding to the constant real variation of Hodge structure on  $X$ . For any morphism  $f : X \rightarrow A$  from  $X$  to an abelian variety and an integer  $j \geq 0$ ,  $\mathcal{H}^j f_* \mathbb{R}_X[\dim X]$  is a polarizable real Hodge module of weight  $j + \dim X$  on  $A$ . Let  $M = (\mathcal{M}, F\mathcal{M}, M_{\mathbb{R}})$  be the direct summand of  $\mathcal{H}^j f_* \mathbb{R}_X[\dim X]$  with strict support  $f(X)$ . Then the first non-trivial piece of the Hodge filtration of  $\mathcal{M}$  is  $R^j f_* \omega_X$ .

- (D) Let  $M$  be as in (C) above. Then we associate it with a complex polarizable Hodge module  $(M \oplus M, J_M)$  as in [20], whose underlying perverse sheaf is simply  $M \otimes_{\mathbb{R}} \mathbb{C}$ . Then by [20, Theorem 7.1 and Corollary 7.3], we know that

$$(M \oplus M, J_M) \simeq \bigoplus_{i=1}^n p_i^{-1}(N_i, J_i) \otimes_{\mathbb{C}} \mathbb{C}_{\rho_i},$$

where  $p_i : A \rightarrow A_i$  is a surjective morphism with connected fibers between abelian varieties,  $\rho_i \in \text{Char}(A)$  is a unitary character, and  $(N_i, J_i) \in \mathbf{HM}_{\mathbb{C}}(A_i, \dim X - \dim p_i)$  is a simple polarizable complex Hodge module with  $\chi(A_i, N_i, J_i) > 0$ .

- (E) Under the assumption of (D), by [20, Theorem D], for any  $k \in \mathbb{Z}$ ,

$$\text{gr}_k^F \mathcal{M} \simeq \bigoplus_{i=1}^n p_i^* \mathcal{F}_i \otimes P_i,$$

where  $\mathcal{F}_i$  is an M-regular sheaf on  $A_i$  and  $P_i$  is a torsion line bundle on  $A$ . Pareschi, Popa, and Schnell indeed proved the Chen-Jiang decomposition for each graded quotient of a polarizable real Hodge module.

#### 4. The proof of Theorem 1.4

We prove a more general version of Theorem 1.4.

**THEOREM 4.1.** — Assume that  $X$  is a smooth projective variety of general type. Let  $f : X \rightarrow A$  be a morphism from  $X$  to an abelian variety and let  $F$  be a connected component of a general fiber of  $f$ . Fix a positive integer  $j \geq 0$ . Assume that  $F$  satisfies the following conditions:

(C1') the canonical model of  $F$  is a smooth projective variety  $V$  satisfying the following infinitesimal Torelli condition: the cup product

$$H^1(V, T_V) \longrightarrow \mathrm{Hom}(H^j(V, K_V), H^{j+1}(V, \Omega_V^{\dim V - 1}))$$

is injective;

(C2')  $\mathrm{Bir}(F)$  acts faithfully on  $H^j(F, K_F)$ ,

then  $V^0(R^j f_* \omega_X)$  generates  $\mathrm{Pic}^0(X)$ .

*Proof.* — We argue by contradiction. The proof consists of several steps.

*Step 1.* We may assume that  $\dim V^0(R^j f_* \omega_X) = 0$ . — By the assumption (C2'),  $H^j(F, K_F) \neq 0$ . Thus,  $R^j f_* \omega_X$  is non-zero. By Kollár's theorem (see [16]),  $R^j f_* \omega_X$  is a torsion-free sheaf supported on  $f(X)$ . Assume that  $V^0(R^j f_* \omega_X)$  generates a proper abelian subvariety  $\widehat{B}$  of  $\mathrm{Pic}^0(A)$ . Consider the dual  $B$  of  $\widehat{B}$ , let  $q : A \rightarrow B$  be the natural quotient with connected fibers and denote  $g = p \circ f$ . Let  $K$  be the fiber of  $q : A \rightarrow B$  over a general point  $b \in g(X)$  and  $X_b$  the corresponding fiber of  $g$ . Let  $f_b : X_b \rightarrow K$  be the restriction of  $f$  on  $X_b$ . Note that  $X_b$  may not be connected. We write  $X_b = \bigsqcup_{k=1}^m Y_k$ , where each  $Y_k$  is a connected component of  $X_b$  and let  $f_k : Y_k \rightarrow K$  be the restriction of  $f$  on  $Y_k$ . We have the following commutative diagram:

$$(4.1) \quad \begin{array}{ccccc} & & X & \longleftrightarrow & Y_1 \\ & \nearrow f & \downarrow & & \downarrow f_1 \\ f(X) & \hookrightarrow & A & \longleftrightarrow & K \\ & \downarrow & \downarrow q & & \\ & g(X) & \hookrightarrow & B. \end{array}$$

Since  $b \in g(X)$  is general,  $R^j f_* \omega_X|_K \simeq R^j f_{b*} \omega_{X_b} = \bigoplus_{k=1}^m R^j f_{k*} \omega_{Y_k}$  (see for instance [18, Proposition 5.1]). Note that  $R^j f_{k*} \omega_{Y_k}$  are also non-zero GV sheaves on  $K$ . Hence  $V^0(R^j f_{k*} \omega_{Y_k}) \neq \emptyset$  for each  $1 \leq k \leq m$ . We then consider the natural restriction  $\pi : \mathrm{Pic}^0(A) \rightarrow \mathrm{Pic}^0(K)$ . We claim that

$$(4.2) \quad \dim V^0(R^j f_{1*} \omega_{Y_1}) = 0.$$



Indeed, for  $Q \in V^0(R^j f_{1*} \omega_{Y_1})$ , we choose  $P \in \text{Pic}^0(A)$  such that  $P|_K = Q$ . Then

$$q_*(R^j f_* \omega_X \otimes P)$$

is non-trivial, since its stalk at  $b$  is exactly  $H^0(K, R^j f_{b*} \omega_{X_b} \otimes Q) \neq 0$ . On the other hand, it is known that  $q_*(R^j f_* \omega_X \otimes P)$  is GV by Hacon’s theorem (see [11]). Thus there exists  $P' \in \text{Pic}^0(B)$  such that

$$H^0(A, R^j f_* \omega_X \otimes P \otimes q^* P') = H^0(B, q_*(R^j f_* \omega_X \otimes P) \otimes P') \neq 0.$$

Hence

$$P \otimes q^* P' \in V^0(R^j f_* \omega_X)$$

and  $(P \otimes q^* P')|_K = Q$ . This shows that  $V^0(R^j f_{1*} \omega_{Y_1}) \subset \pi(V^0(R^j f_* \omega_X))$ . Since  $V^0(R^j f_* \omega_X)$  consists of a finite union of torsion translates of abelian subvarieties contained in  $\widehat{B}$ , we conclude that  $V^0(R^j f_{1*} \omega_{Y_1})$  consists of finitely many torsion points.

Note that  $Y_1$  is a smooth projective variety of general type and a general fiber of  $f_1$  is a also a general fiber of  $f$ . Thus after replacing  $f : X \rightarrow A$  by  $f_1 : Y_1 \rightarrow K$ , we may assume that  $\dim V^0(R^j f_* \omega_X) = 0$ .

*Step 2. We may assume that  $\dim V^0(R^j f_* \omega_X) = 0$  and  $A$  is simple.* — We repeat the same argument in Step 1. If  $A$  is not simple, we may take a quotient  $q : A \rightarrow B$  such that its kernel  $K$  is simple. We then go back to the setting in the commutative diagram (4.1). Since  $\dim V^0(R^j f_* \omega_X) = 0$ , we again have  $\dim V^0(R^j f_{1*} \omega_{Y_1}) = 0$  by the same proof of Claim (4.2). After replacing  $f : X \rightarrow A$  by  $f_1 : Y_1 \rightarrow K$ , we may assume that  $\dim V^0(R^j f_* \omega_X) = 0$  and  $A$  is simple.

*Step 3.  $f : X \rightarrow A$  is birationally isotrivial.* — Let  $m = \dim X$ . Since  $\dim V^0(R^j f_* \omega_X) = 0$ , we conclude by the Chen-Jiang decomposition and the assumption that  $A$  is simple that  $R^j f_* \omega_X$  is a direct sum of torsion line bundles on  $A$ . In particular,  $f : X \rightarrow A$  is surjective.

Let  $M$  be the direct summand of  $\mathcal{H}^j f_* \mathbb{R}[m]$  with strict support  $f(X)$ . Note that the first non-trivial piece of the Hodge filtration of  $\mathcal{M}$  of  $M$  is  $R^j f_* \omega_X$ .

Since  $A$  is simple, by Subsection 3(D), we know that

$$\begin{aligned} (M \oplus M, J_M) &= \mathcal{H}' \bigoplus \mathcal{H}'' \\ &= \left( \bigoplus_j q^{-1} V_j \otimes \mathbb{C}_{\rho_j} \right) \bigoplus \mathcal{H}'', \end{aligned}$$

where  $q : A \rightarrow \text{Spec } \mathbb{C}$  is the constant morphism,  $V_j$  are Hodge structures of weights  $\dim F + j$ ,  $\rho_j \in \text{Char}(K)$  are unitary characters, and  $\mathcal{H}''$  is a direct

sum of simple complex polarizable Hodge modules with positive holomorphic Euler characteristic. The fact that  $\dim V^0(R^j f_* \omega_X) = 0$  implies that  $R^j f_* \omega_X \oplus R^j f_* \omega_X$  is indeed the first non-trivial piece of the underlying  $\mathcal{D}$ -module of  $\mathcal{H}'$ . Indeed, since the holomorphic Euler characteristic of  $\mathcal{H}''$  is positive, each graded quotient  $\mathcal{Q}$  of  $\mathcal{H}''$  has positive holomorphic Euler characteristic (see [20, Lemma 15.1]) and thus by the Chen-Jiang decomposition for  $\mathcal{Q}$  (see Section 3(B)),  $V^0(\mathcal{Q}) = \text{Pic}^0(A)$ .

By Section 3(B),  $\mathcal{H}^j f_* \mathbb{R}[m]$  corresponds to the variation of Hodge structure of  $H^{j+\dim F}(X_t)$ , where  $t$  belongs to the smooth locus  $U \subset A$  of  $f$  and  $X_t$  is the corresponding fiber of  $f$ . Thus,  $\mathcal{H}'|_U$  is a polarizable variation of complex Hodge structure, whose fiber over  $t$  is a sub-Hodge structure of  $H^{j+\dim F}(X_t, \mathbb{C})$  containing  $H^j(X_t, K_{X_t})$ . Note that  $\mathcal{H}'$  is trivial up to an étale cover of  $A$ .

Since  $F$  is a connected component of  $X_t$  and  $\mathcal{H}'$  is flat, the composition of the Kodaira-Spencer map with the cup product of variation of Hodge structures:

$$T_{A,t} \longrightarrow H^1(F, T_F) \longrightarrow \text{Hom}(H^j(F, K_F), H^{j+1}(F, \Omega_F^{\dim F-1}))$$

is zero.

We now consider the relative canonical model of  $f$ :

$$\tilde{f}: \tilde{X} \longrightarrow A.$$

A connected component of the fiber  $\tilde{f}$  over  $t$  is the canonical model  $V$  of  $F$ . By assumption,  $V$  is a smooth projective variety which satisfies the infinitesimal Torelli condition (C1'). Note that  $H^j(F, K_F) = H^j(V, K_V)$ , and  $H^{j+\dim V}(V, \mathbb{C}) \subset H^{j+\dim F}(F, \mathbb{C})$  is a direct summand, we have the commutative diagram:

$$\begin{array}{ccccc} T_{A,t} & \longrightarrow & H^1(F, T_F) & \longrightarrow & \text{Hom}(H^j(F, K_F), H^{j+1}(F, \Omega_F^{\dim F-1})) \\ \parallel & & \downarrow & & \downarrow \\ T_{A,t} & \longrightarrow & H^1(V, T_V) & \longrightarrow & \text{Hom}(H^j(V, K_V), H^{j+1}(V, \Omega_V^{\dim V-1})). \end{array}$$

Thus the composition of maps

$$T_{A,t} \longrightarrow H^1(V, T_V) \longrightarrow \text{Hom}(H^j(V, K_V), H^{j+1}(V, \Omega_V^{\dim V-1}))$$

is also zero.

By the assumption that  $V$  satisfies the infinitesimal Torelli condition, the Kodaira-Spencer map  $T_{A,t} \rightarrow H^1(V, T_V)$  induced by the family  $\tilde{f}$  is thus zero. Hence  $\tilde{f}$  is locally isotrivial around  $t \in A$  and thus the morphism  $f: X \rightarrow A$  is birationally isotrivial.

*Step 4. The contradiction.* — In this step, we show that given a birationally isotrivial morphism  $f : X \rightarrow A$  whose general fiber satisfies condition (C2'), it is absurd that  $\dim V^0(R^j f_* \omega_X) = 0$ .

We assume without loss of generalities that  $f : X \rightarrow A$  is primitive, i.e.  $f$  does not factor through a non-trivial étale cover of  $A$ . We consider the Stein factorization

$$f : X \xrightarrow{h_1} N \xrightarrow{h_2} A$$

of  $f$ . After birational modifications, we may and will assume that  $N$  is smooth,  $h_1$  is a fibration and  $h_2$  is generically finite and surjective.

Note that  $h_1$  is a birationally isotrivial fibration. Since  $F$  is of general type, its birational automorphism group is finite, there exists a Galois cover  $\rho : M \rightarrow N$  with Galois group  $G$  such that the flat base change  $M \times_N X$  is birational to  $M \times F$ . In other words,  $X$  is birational to the diagonal quotient  $(M \times F)/G$ , where  $G$  acts rationally on  $F$  via a homomorphism  $G \rightarrow \text{Bir}(F)$ . We may take  $G$  minimal so that  $G \rightarrow \text{Bir}(F)$  is injective. After birational modifications of  $F$ , we may assume that  $G$  acts regularly on  $F$  and the corresponding homomorphism  $G \rightarrow \text{Aut}(F)$  is injective.

If  $\rho$  factors through an abelian étale cover of  $N$  induced by an isogeny  $\tilde{A} \rightarrow A$ , we replace  $f : X \rightarrow A$  by  $\tilde{f} : \tilde{X} := X \times_A \tilde{A} \rightarrow \tilde{A}$ . It is easy to verify that  $V^0(\tilde{f}_* \omega_{\tilde{X}})$  still consists of finitely many points. Therefore, without loss of generalities, we may assume that  $\rho : M \rightarrow N$  does not factor through abelian étale covers of  $N$  induced by isogenies of  $A$ .

Taking an  $G$ -equivariant resolution of singularities, we may assume that  $M$  is smooth. Note that  $G$  acts naturally on  $\rho_* \omega_M$  and over a general point  $x \in N$ , the representation of  $G$  on  $(\rho_* \omega_M)_x$  is isomorphic to the regular representation of  $G$ . We then consider the canonical decomposition

$$\rho_* \omega_M = \bigoplus_{\chi \in \text{Irr}(G)} \mathcal{V}_\chi$$

with respect to the  $G$ -action, where  $\text{Irr}(G)$  denotes the set of characters of irreducible representations of  $G$  and  $\mathcal{V}_\chi$  is the image of  $\pi_\chi := \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} g_* : \rho_* \omega_X \rightarrow \rho_* \omega_X$ . Note that  $\mathcal{V}_{1_G} = (\rho_* \omega_M)^G = \omega_N$  and  $\mathcal{V}_\chi$  is non-zero for each  $\chi \in \text{Irr}(G)$ .

We claim that  $h_{2*} \mathcal{V}_\chi$  is  $M$ -regular on  $A$  for any  $1_G \neq \chi \in \text{Irr}(G)$ . If not, by the Chen-Jiang decomposition theorem,  $h_{2*} \mathcal{V}_\chi$  has a numerically trivial line bundle  $P$  as a direct summand. Since  $h^{\dim M}(A, h_{2*} \rho_* \omega_M) = h^{\dim M}(A, h_{2*} \omega_N) = 1$ ,  $P$  is a non-trivial torsion line bundle on  $A$ . Since  $f$  is primitive,  $h_2^* P$  is also a non-trivial torsion line bundle on  $N$ . But  $h^{\dim M}(M, \omega_M \otimes \rho^* h_2^* P^{-1}) \neq 0$ , thus  $\rho^* h_2^* P = \mathcal{O}_M$ . This implies that  $\rho$

factors through an abelian étale cover of  $N$  induced by an isogeny of  $A$ , which is a contradiction.

We are now ready to deduce the final contradiction. After birational modifications of  $X$ , we have a birational morphism  $\tau : X \rightarrow (M \times F)/G$ . Note that  $(M \times F)/G$  has quotient singularities and hence,  $(M \times F)/G$  has rational singularities or log terminal singularities (see for instance [17, Proposition 5.15 and Proposition 5.20]). Therefore,  $\mathbf{R}\tau_*\omega_X = \tau_*\omega_X = \omega_{(M \times F)/G}$ . Moreover, the fibers of  $\tau$  are rationally chain connected (see [12]) and thus  $f : X \rightarrow A$  factors as  $f : X \xrightarrow{\tau} (M \times F)/G \xrightarrow{h} A$ . We also have  $R^j f_*\omega_X = R^j h_*\tau_*\omega_X = R^j h_*\omega_{(M \times F)/G}$ . We then identify  $X = (M \times F)/G$  in the following.

Since  $G$  acts naturally on the vector space  $H^j(F, K_F)$ , we decompose the  $G$ -representation to direct sum of irreducible representations:

$$H^j(F, K_F) = \bigoplus_{\chi \in \text{Irr}(G)} V_{\chi}^{\oplus m_{\chi}},$$

where  $V_{\chi}$  is the irreducible  $G$ -representation whose character is  $\chi$  and  $m_{\chi} \geq 0$  are non-negative integers. By the condition (C2'),  $G \subset \text{Bir}(F)$  acts faithfully on  $H^j(F, K_F)$ . Hence, for some  $\chi_0 \neq 1_G$ ,  $m_{\chi_0} > 0$ .

Let  $\gamma : M \times F \rightarrow (M \times F)/G = X$  be the diagonal quotient and let  $p : M \times F \rightarrow M$  be the first projection. Note that  $\omega_{(M \times F)/G} = (\gamma_*\omega_{M \times F})^G$  and since  $\gamma$  is flat,  $R^j(h_1 \circ \gamma)_*\omega_{M \times F} = R^j h_{1*}(\gamma_*\omega_{M \times F})$ . From the commutative diagram:

$$\begin{array}{ccccc} M \times F & \xrightarrow{\gamma} & (M \times F)/G & & \\ \downarrow p & & \downarrow h_1 & \searrow f & \\ M & \xrightarrow{\rho} & M/G = N & \xrightarrow{h_2} & A, \end{array}$$

we see that

$$\begin{aligned} R^j f_*\omega_X &= h_{2*}R^j h_{1*}\omega_{(M \times F)/G} \\ &= h_{2*}R^j h_{1*}(\gamma_*\omega_{M \times F})^G \\ &= h_{2*}(R^j(h_1 \circ \gamma)_*\omega_{M \times F})^G \\ &= h_{2*}(R^j(\rho \circ p)_*\omega_{M \times F})^G \\ &= h_{2*}(\rho_*\omega_M \otimes H^j(F, K_F))^G \\ &= \bigoplus_{\chi, \chi' \in \text{Irr}(G)} h_{2*}(\mathcal{V}_{\chi} \otimes_{\mathbb{C}} V_{\chi'}^{\oplus m_{\chi'}})^G. \end{aligned}$$

By the character theory of representation of finite groups, for  $\chi_0 \neq 1_G$ , there exists  $1_G \neq \chi_1 \in \text{Irr}(G)$  such that  $(\mathcal{V}_{\chi_1} \otimes_{\mathbb{C}} V_{\chi_0}^{\oplus m_{\chi_0}})^G$  is a non-trivial direct summand of  $\mathcal{V}_{\chi_1} \otimes_{\mathbb{C}} V_{\chi_0}^{\oplus m_{\chi_0}}$ . Since  $h_{2*}\mathcal{V}_{\chi_1}$  is M-regular, so is  $h_{2*}(\mathcal{V}_{\chi_1} \otimes_{\mathbb{C}} V_{\chi_0}^{\oplus m_{\chi_0}})$ . Thus,  $h_{2*}(\mathcal{V}_{\chi_1} \otimes_{\mathbb{C}} V_{\chi_0}^{\oplus m_{\chi_0}})^G$  is also M-regular. Thus  $R^j f_* \omega_X$  has an M-regular sheaf as a direct summand. This contradicts the assumption that  $\dim V^0(R^j f_* \omega_X) = 0$ .  $\square$

*Remark 4.2.* — Perhaps the most useful case of Theorem 4.1 is when  $j = 0$ , because the conditions (C1) and (C2) have been extensively studied in the literature.

All smooth complete intersections of general type in projective spaces satisfy (C1) (see [21] or [25]). Moreover, a fairly general criterion for varieties of general type satisfying (C1) was proved in [15]: assume that  $K_F = L^{\otimes m}$  for some  $m \geq 1$ , the base locus of  $|L|$  is of codimension  $\geq 2$ , and  $h^0(F, \Omega_F^{\dim F-1} \otimes L) \leq h^0(L) - 2$ , then

$$H^1(F, T_F) \longrightarrow \text{Hom}(H^0(F, K_F), H^1(F, \Omega_F^{\dim F-1}))$$

is injective.

For (C2), assuming that a finite subgroup  $G \subset \text{Bir}(F)$  acting trivially on  $H^0(F, K_F)$ , the canonical map of  $F$  factors through the quotient map by  $G$ . This is of course impossible if the canonical system  $|K_F|$  defines a birational map of  $F$ . Moreover, if  $|K_F|$  defines a generically finite map of  $F$ , then the condition that the canonical map of  $F$  factors through the quotient map by  $G$  implies that  $p_g(F) = p_g(F/G)$  (see [3, Théorème 3.1]), which is a very restrictive condition.

In conclusion, the assumptions (C1) and (C2) are satisfied by a large number of varieties of general type, including all smooth complete intersections of general type in projective spaces.

The proof of Theorem 4.1 provides more information.

**COROLLARY 4.3.** — *Under the assumption of Theorem 4.1. Assume the sub-Hodge structure of  $H^{j+\dim F}(F, \mathbb{C})$  containing  $H^j(F, K_F)$ , i.e. the transcendental part of  $H^{j+\dim F}(F, \mathbb{C})$ , is simple and  $f$  is a fibration onto its image,  $R^j f_* \omega_X$  is M-regular.*

*Proof.* — By Theorem 4.1,  $V^0(R^j f_* \omega_X)$  generates  $\text{Pic}^0(A)$ . If  $R^j f_* \omega_X$  is not M-regular, the Chen-Jiang decomposition for  $R^j f_* \omega_X$  has at least 2 direct summands, whose 0-th cohomological support loci are translates of different positive dimensional abelian subvarieties of  $\text{Pic}^0(A)$ . Let  $M$  be the direct summand of  $\mathcal{H}^j f_* \mathbb{R}_X[\dim X]$  with strict support  $f(X)$  as in Section 3(C) and let  $M'$  be the direct summand of  $M$  corresponding to the

transcendental part of the Hodge structure  $H^{j+\dim F}(F)$ . Then the first non-trivial piece of the Hodge filtration of  $M'$  is  $R^j f_* \omega_X$ . Moreover, by Section 3(D), we have the decomposition for the complex Hodge module

$$(4.3) \quad (M' \oplus M', J_M) \simeq \bigoplus_{i=1}^n p_i^{-1}(N_i, J_i) \otimes_{\mathbb{C}} \mathbb{C}_{\rho_i},$$

where  $p_i : A \rightarrow A_i$  is a surjective morphism with connected fibers between abelian varieties,  $\rho_i \in \text{Char}(A)$  is a unitary character, and  $(N_i, J_i) \in \mathbf{HM}_{\mathbb{C}}(A_i, \dim X - \dim p_i)$  is a simple polarizable complex Hodge module with  $\chi(A_i, N_i, J_i) > 0$ . Since  $V^0(R^j f_* \omega_X)$  has at least 2 irreducible components, these components generate  $\text{Pic}^0(A)$ , and each component is exactly a translate of  $\text{Pic}^0(A_i)$ , we have  $n \geq 2$ . Restricting (4.3) to a general fiber of  $f$ , we see that the sub-Hodge structure of  $H^{j+\dim F}(F, \mathbb{C})$  containing  $H^j(F, K_F)$  is not simple, which is a contradiction.  $\square$

## 5. Tricanonical maps of $X$ with $K_F$ sufficiently positive

**THEOREM 5.1.** — *Let  $f : X \rightarrow A$  be a morphism from a smooth projective variety to an abelian variety. Let  $F$  be a connected component of a general fiber of  $f$ . Assume that  $V^0(f_* \omega_X)$  generates  $\text{Pic}^0(A)$ , and  $|K_F|$  induces a birational map of  $F$ ,*

- (1) *the linear system  $|3K_X + f^*P|$  induces a birational map of  $X$  for  $P \in \text{Pic}^0(A)$  general;*
- (2) *the linear system  $|mK_X + f^*P|$  induces a birational map for  $m \geq 4$  and all  $P \in \text{Pic}^0(A)$ .*

*Proof.* — By the Chen-Jiang decomposition, we write

$$(5.1) \quad f_* \omega_X = \bigoplus_{i=1}^n p_i^* \mathcal{F}_i \otimes P_i^{-1},$$

where  $p_i : A \rightarrow B_i$  are surjective morphisms with connected fibers between abelian varieties,  $\mathcal{F}_i$  are  $M$ -regular sheaves on  $B_i$ ,  $P_i$  are torsion line bundles. Then

$$V^0(f_* \omega_X) = \bigcup_{i=1}^n (P_i + \text{Pic}^0(B_i)).$$

Since  $M$ -regular sheaves are continuously globally generated, the natural evaluation map

$$(5.2) \quad \bigoplus_{U_i} \bigoplus_{Q \in U_i} H^0(A, f_* \omega_X \otimes Q) \otimes Q^{-1} \longrightarrow f_* \omega_X$$

is surjective, where  $U_i \subset P_i + \text{Pic}^0(B_i)$  are non-empty open subsets.

Let  $a \in f(X)$  be general and let  $X_a$  be the corresponding fiber of  $f$ . By assumption, the canonical map of a connected component of  $X_a$  is birational. Thus  $|kK_{X_a}|$  also induces a birational map of  $X_a$ , for each  $k \geq 1$ .

We claim that  $X$  is of general type. By the Chen-Jiang decomposition and the fact that M-regular sheaves are ample, the Iitaka model of  $X$  dominates  $B_i$  for each  $i$ . In other words, for some ample divisor  $H_i$  on  $B_i$ ,  $K_X - tf^*p_i^*H$  is  $\mathbb{Q}$ -equivalent to an effective  $\mathbb{Q}$ -divisor for  $0 < t \ll 1$ . Since  $\bigcup_{i=1}^n \text{Pic}^0(B_i)$  generates  $\text{Pic}^0(A)$ , the Iitaka model of  $X$  also dominates  $A$ . On the other hand, the connected components of  $f$  are of general type. Thus  $X$  is of general type.

Since  $V^0(f_*\omega_X)$  contains positive dimensional components and  $f_*\omega_X^{\otimes k}$  is an IT sheaf of index 0 for  $k \geq 2$ ,  $h^0(X, kK_X + f^*Q) = p_k(X) \geq 2$  for any  $Q \in \text{Pic}^0(A)$ .

We first remark that (2) is easy. By the assumption that  $V^0(f_*\omega_X)$  generates  $\text{Pic}^0(A)$  and Tirabassi's argument in [24], we know that  $f_*(\omega_X^{\otimes 2} \otimes \mathcal{I}_x)$  is M-regular for a general point  $x \in X$ . For readers' convenience, we recall Tirabassi's argument. Consider the short exact sequence

$$0 \longrightarrow f_*(\omega_X^{\otimes 2} \otimes \mathcal{I}_x) \longrightarrow f_*\omega_X^{\otimes 2} \longrightarrow \mathbb{C}_{f(x)} \longrightarrow 0,$$

where  $\mathbb{C}_{f(x)}$  is the skyscraper sheaf at  $f(x)$ . Since  $f_*\omega_X^{\otimes 2}$  is an IT sheaf of index 0,  $V^i(f_*(\omega_X^{\otimes 2} \otimes \mathcal{I}_x)) = \emptyset$  for  $i \geq 2$ . Hence it suffices to show that

$$\text{codim}_{\text{Pic}^0(A)} V^1(f_*(\omega_X^{\otimes 2} \otimes \mathcal{I}_x)) \geq 2.$$

We also deduce from the above short exact sequence that

$$V^1(f_*(\omega_X^{\otimes 2} \otimes \mathcal{I}_x)) = \{P \in \text{Pic}^0(A) \mid x \in \text{Bs}(|2K_X + P|)\}.$$

Since  $x \in X$  is general,  $x \notin \text{Bs}|K_X + P|$  for  $P \in P_i + \text{Pic}^0(B_i)$  general. Thus  $x \notin \text{Bs}|2K_X + Q|$  for any  $Q \in 2P_i + \text{Pic}^0(B_i)$ , because

$$|2K_X + Q| \supset |K_X + P| + |K_X + Q - P|$$

for  $P \in P_i + \text{Pic}^0(B_i)$  general. Therefore

$$V^1(f_*(\omega_X^{\otimes 2} \otimes \mathcal{I}_x)) \cap (2P_i + \text{Pic}^0(B_i)) = \emptyset$$

for  $1 \leq i \leq n$ . Since  $\bigcup_{1 \leq i \leq n} \text{Pic}^0(B_i)$  generates  $\text{Pic}^0(A)$ , the intersection of an effective divisor of  $\text{Pic}^0(A)$  with  $\bigcup_{1 \leq i \leq n} (2P_i + \text{Pic}^0(B_i))$  is always non-empty. We conclude that

$$\text{codim}_{\text{Pic}^0(A)} V^1(f_*(\omega_X^{\otimes 2} \otimes \mathcal{I}_x)) \geq 2.$$

Since  $f_*(\omega_X^{\otimes 2} \otimes \mathcal{I}_x)$  is M-regular, for any non-empty open subset  $U \subset \text{Pic}^0(A)$ , the evaluation map

$$\bigoplus_{P \in U} H^0(A, f_*(\omega_X^{\otimes 2} \otimes \mathcal{I}_x) \otimes P) \otimes P^{-1} \longrightarrow f_*(\omega_X^{\otimes 2} \otimes \mathcal{I}_x)$$

is surjective. Since  $|2K_{X_a}|$  induces a birational map of  $X_a$ , for any general point  $y \neq x$  and for  $Q \in U$  general, there exists a section  $s \in H^0(A, f_*(\omega_X^{\otimes 2} \otimes \mathcal{I}_x) \otimes Q)$  not vanishing at  $y$ . Let  $\tilde{s}$  be the corresponding section in  $H^0(X, \mathcal{I}_x(2K_X + f^*Q)) \subset H^0(X, 2K_X + f^*Q)$ . Then,  $\tilde{s}$  separates  $x$  and  $y$ . Note that when  $m \geq 4$ ,  $p_{m-2}(X) \geq 2$ . Let  $t \in H^0(X, (m-2)K_X + f^*P - f^*Q)$  be general such that  $t$  does not vanish at both  $x$  and  $y$ . Then,  $\tilde{s} \cdot t \in H^0(X, mK_X + f^*P)$  separates  $x$  and  $y$ . Hence,  $|mK_X + f^*P|$  induces a birational map of  $X$  for  $m \geq 4$ .

The proof of (1) consists of several steps. Without loss of generalities, we may assume that  $f$  is primitive and  $f(X)$  generates  $A$ .

Let  $p := \prod_{i=1}^n p_i : A \rightarrow \prod_{i=1}^n B_i$ . By the assumption that  $V^0(f_*\omega_X)$  generates  $\text{Pic}^0(A)$ , we know that  $p$  is finite onto its image. Let  $K$  be the image of  $p$ . Then  $p : A \rightarrow K \hookrightarrow \prod_{i=1}^n B_i$  is an isogeny between  $A$  and  $K$ .

*Step 1.*  $|3K_X + f^*P|$  for any  $P \in \text{Pic}^0(A)$ , separates two general points on the same general fiber of  $f$ . — Fix  $P \in \text{Pic}^0(X)$ . Let  $x, y \in X$  be two general points in the same general fiber  $X_a$  of  $f$  for  $a \in f(X) \subset A$ .

If for some  $B_i$  and  $P_i$  in (5.1) and  $Q \in P_i + \widehat{B}_i$  general, the linear system

$$|2K_X + f^*P - f^*Q|$$

separates  $x$  and  $y$ , then there exists a divisor  $D_1 \in |2K_X + f^*P - f^*Q|$  such that  $x \in D_1$  and  $y \notin D_1$ . Since  $Q \in P_i + \widehat{B}_i$  general, there exists  $D_2 \in |K_X + f^*Q|$  containing neither  $x$  nor  $y$ . We see that  $D_1 + D_2 \in |3K_X + f^*P|$  separates  $x$  and  $y$ .

We then assume that for all  $B_i$  and  $P_i$  and  $Q \in P_i + \text{Pic}^0(B_i)$ , the corresponding linear system

$$|2K_X + f^*P - f^*Q|$$

cannot separate  $x$  and  $y$ . Since  $|K_{X_a}|$  induces a birational map of  $X_a$ , we conclude from (5.2) that there exist positive integers  $N$ ,  $Q_{ik} \in P_i + \text{Pic}^0(B_i)$  general for  $1 \leq k \leq N$ , sections  $s_{ik} \in H^0(X, K_X + f^*Q_{ik})$  such that the section

$$s := \sum_{i=1}^n \sum_{k=1}^N s_{ik} |_{X_a} \in H^0(X_a, K_{X_a})$$

separates  $x$  and  $y$ , i.e.  $x \in D_s$  and  $y \notin D_s$ , where  $D_s$  is the corresponding divisor of  $s$  on  $X_a$ . We choose  $s'_{ik} \in H^0(X, 2K_X + f^*P - f^*Q_{ik})$  which take



the same nonzero value at both  $x$  and  $y$  in an affine charts of  $X_a$  containing both  $x$  and  $y$ . Note that  $s'_{ik}$  exists since  $\varphi_{ik}(x) = \varphi_{ik}(y)$ , where  $\varphi_{ik}$  is the rational map induced by  $|2K_X + f^*P - f^*Q_{ik}|$ .

Then

$$\sum_{i=1}^n \sum_{k=1}^N s_{ik} \cdot s'_{ik} \in H^0(X, 3K_X + f^*P)$$

separates  $x$  and  $y$ . Hence, the twisted tricanonical map induced by  $|3K_X + f^*P|$  separates two general points on the same general fiber of  $f$ .

*Step 2.*  $|3K_X + f^*P|$  induces a generically finite map from  $X$  onto its image for all  $P \in \text{Pic}^0(A)$ . — Assume the contrary, there exists a curve  $C$  through a general point of  $X$  contracted by the map  $|3K_X + f^*P|$ . Then the rank of the restriction map

$$H^0(X, 3K_X + f^*P) \longrightarrow H^0(C, (3K_X + f^*P)|_C)$$

is 1. By Step 1,  $f|_C : C \rightarrow A$  is generically finite onto its image. By the assumption, there exists a quotient  $p_B : A \rightarrow B$  appearing in the decomposition formula (5.1) such that  $p_B \circ f|_C : C \rightarrow B$  is generically finite onto its image. Let  $T = P + \text{Pic}^0(B)$  be the corresponding subset of  $V^0(f_*\omega_X)$ . For  $Q \in T$  general, the image  $V_Q$  of the restriction map

$$H^0(X, K_X + f^*Q) \longrightarrow H^0(C, (K_X + f^*Q)|_C)$$

is not zero. We denote by  $V'_Q$  the non-zero image of the restriction map

$$H^0(X, 2K_X + f^*P - f^*Q) \longrightarrow H^0(C, (2K_X + f^*P - f^*Q)|_C).$$

Observe that the image of the natural pull-back  $\text{Pic}^0(B) \rightarrow \text{Pic}^0(C)$  is positive dimensional, since  $p_B \circ f|_C$  is generically finite from  $C$  onto its image.

The image of the restriction map

$$H^0(X, 3K_X + f^*P) \longrightarrow H^0(C, (3K_X + f^*P)|_C)$$

contains the map of natural multiplication map

$$\bigcup_{Q \in T} V_Q \otimes V'_{-Q} \longrightarrow H^0(C, (3K_X + f^*P)|_C).$$

Thus the rank of the restriction map

$$H^0(X, 3K_X + f^*P) \longrightarrow H^0(C, (3K_X + f^*P)|_C)$$

is  $> 1$ , which is a contradiction.

*Step 3.* For  $P \in \text{Pic}^0(A)$  general, let  $x, y \in X$  be two general points such that  $p \circ f(x) \neq p \circ f(y) \in K$ . The linear system  $|3K_X + f^*P|$  separates  $x$  and  $y$ . — For a general point  $x \in X$ , for each component  $P_i + \widehat{B}_i$  appearing in (5.1),  $x \notin \text{Bs}(|K_X + f^*Q|)$  for  $Q \in P_i + \widehat{B}_i$  general. Then  $x \notin \text{Bs}(|2K_X + f^*Q|)$  for all  $Q \in \widehat{T}_i := 2P_i + \widehat{B}_i$ . Considering  $g_i := p_i \circ f : X \rightarrow B_i$ , we know that  $g_{i*}(\omega_X^{\otimes 2} \otimes f^*Q)$  is an IT sheaf of index 0 for all  $Q \in \text{Pic}^0(A)$ . From the short exact sequence

$$0 \longrightarrow g_{i*}(\omega_X^{\otimes 2} \otimes f^*Q \otimes \mathcal{I}_x) \longrightarrow g_{i*}(\omega_X^{\otimes 2} \otimes f^*Q) \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

we deduce that  $g_{i*}(\omega_X^{\otimes 2} \otimes f^*Q \otimes \mathcal{I}_x)$  is also an IT sheaf of index 0 on  $B_i$ , for all  $Q \in \widehat{T}_i$ . There exists an open subset  $\mathcal{U}_i \subset X \times \text{Pic}^0(X)$  parametrizing a flat family of sheaves  $g_{i*}(\omega_X^{\otimes 2} \otimes f^*Q \otimes \mathcal{I}_x)$  on  $B_i$  for  $(x, Q) \in \mathcal{U}$ . Being an IT sheaf of index 0 is an open condition. Hence, there exists a non empty open subset  $\mathcal{U}'_i$  of  $\mathcal{U}_i$  such that for  $(x, Q) \in \mathcal{U}'_i$ ,  $g_{i*}(\omega_X^{\otimes 2} \otimes f^*Q \otimes \mathcal{I}_x)$  is an IT sheaf of index 0.

Let  $\mathcal{V}$  be the intersection of all  $\mathcal{U}'_i$  for  $1 \leq i \leq n$ . Then  $\mathcal{V}$  is again a non-empty open subset of  $X \times \text{Pic}^0(A)$ . For a general  $P \in \text{Pic}^0(A)$ , there exists an open subset  $W$  of  $X$  such that  $g_{i*}(\omega_X^{\otimes 2} \otimes f^*P \otimes \mathcal{I}_x)$  is an IT sheaf of index 0 for  $1 \leq i \leq n$  and all  $x \in W$ . Hence, for  $P \in \text{Pic}^0(A)$  general, there exists an open subset  $W'$  of  $X$  such that

$$g_{i*}(\omega_X^{\otimes 2} \otimes f^*P \otimes f^*P_i^{-1} \otimes \mathcal{I}_x)$$

is an IT sheaf of index 0 for  $1 \leq i \leq n$  and all  $x \in W'$ .

We now take  $W''$  to be the open subset of  $W'$  such that for any  $x \in W''$ ,  $x \notin \text{Bs}(|K_X + f^*P_i + f^*Q|)$  for each  $1 \leq i \leq n$  and  $Q \in \widehat{B}_i$  general. Then for  $x, y \in W''$  and  $p(x) \neq p(y)$ , there exists  $p_i : A \rightarrow B_i$  such that  $g_i(x) \neq g_i(y)$ . Since  $g_{i*}(\omega_X^{\otimes 2} \otimes f^*P \otimes f^*P_i^{-1} \otimes \mathcal{I}_x)$  is an IT sheaf of index 0, it is continuously globally generated. Thus, there exists a divisor  $D \in |2K_X + f^*P - f^*P_i - f^*Q|$  separating  $x$  and  $y$  for  $Q \in \text{Pic}^0(B_i)$  general and as before, the linear system  $|3K_X + f^*P|$  also separates  $x$  and  $y$ .

*Step 4. Conclusion.* — We now argue that for  $P \in \text{Pic}^0(A)$  general,  $|3K_X + f^*P|$  induces a birational map of  $X$ . Assume the contrary. Let  $\varphi : X \dashrightarrow Y \subset \mathbb{P}^N$  be the map induced by  $|3K_X + f^*P|$ , where  $Y$  is the image of  $\varphi$ . We claim that there exists a map  $f_Y : Y \dashrightarrow K$  such that we have the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow f & & \downarrow f_Y \\ A & \xrightarrow{p} & K. \end{array}$$

Indeed, for  $y \in Y$  general,  $\varphi^{-1}(y)$  consists of finitely many points by Step 2 and by Step 3,  $p \circ f(\varphi^{-1}(y))$  is a single point of  $K$ . Hence we can define the map  $Y \dashrightarrow K$  by  $y \rightarrow p \circ f(\varphi^{-1}(y))$ .

After birational modifications of  $X$  and  $Y$ , we may assume that  $Y$  is smooth,  $\varphi$  is an morphism and  $Y \rightarrow K$  is also a morphism. After replacing  $K$  by a suitable étale cover of it, we may also assume that  $Y \rightarrow K$  is primitive. Since  $\varphi$  maps a general fiber of  $f$  birationally onto its image by Step 1, we conclude that  $\varphi : X \rightarrow Y$  is birationally equivalent to the étale morphism  $Y \times_K A \rightarrow Y$  induced by base change of  $p : A \rightarrow K$ . But then

$$\varphi_* \omega_X^{\otimes 3} = \bigoplus_{Q \in \ker p^*} \omega_Y^{\otimes 3} \otimes Q.$$

We then take  $P' \in \text{Pic}^0(K)$  such that  $p^*P' = P$ . Then

$$H^0(X, 3K_X + f^*P) = \bigoplus_{Q \in \ker p^*} H^0(Y, 3K_Y + f_Y^*P' + f_Y^*Q).$$

Note that  $H^0(Y, 3K_Y + f_Y^*P' + f_Y^*Q) \neq 0$  for all  $Q$ . Hence it is absurd that  $\varphi$  is birationally equivalent to the base change of  $p$ .  $\square$

Combining Theorem 1.4 and Theorem 5.1, we have

**COROLLARY 5.2.** — *Let  $f : X \rightarrow A$  be a morphism from a smooth projective variety to an abelian variety. Let  $F$  be a connected component of a general fiber of  $f$ . If  $F$  satisfies the birationally infinitesimal Torelli condition (C1) and  $|K_F|$  induces a birational map of  $F$ , then*

- (1) *the linear system  $|3K_X + f^*P|$  induces a birational map of  $X$  for  $P \in \text{Pic}^0(A)$  general;*
- (2) *the linear system  $|mK_X + f^*P|$  induces a birational map for  $m \geq 4$  and all  $P \in \text{Pic}^0(A)$ .*

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