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A MORSE-BOTT TYPE COMPLEX AND THE BISMUT–ZHANG TORSION FOR INTERSECTION COHOMOLOGY

by Ursula LUDWIG (*)

Abstract. — In the first part of this article we establish, for a compact pseudomanifold and a given perversity in the sense of Goresky and MacPherson a Morse theoretical cochain complex, which computes the intersection cohomology of the space. In the second part we use this cochain complex as well as the model Witten Laplacian to define the Bismut–Zhang torsion of a pseudomanifold. Conjecturally the Bismut–Zhang torsion will serve as the "topological" side in a Cheeger–Müller theorem for spaces with iterated conical singularities.

Résumé. — Dans la première partie de cet article on construit, en utilisant la théorie de Morse, pour une pseudovariété stratifiée et une perversité au sens de la théorie de Goresky et MacPherson un complexe cohomologique. Ce complexe calcule la cohomologie d'intersection de la pseudovariété. Dans la deuxième partie on utilise ce complexe ainsi que le Laplacien de Witten pour définir la torsion de Bismut–Zhang, qui, conjecturellement, va servir dans un théorème de Cheeger–Müller pour des pseudovariétés à singularités coniques.

1. Introduction

The famous Morse inequalities establish a relation between the singular cohomology of a smooth compact manifold M and the number of critical points of a smooth Morse function $f : M \to \mathbb{R}$. A way of proving the Morse inequalities is to show the existence of a combinatorial complex, which is generated by the critical points of f and whose cohomology is isomorphic to the singular cohomology of M. Such a complex has first been established

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by Thom [79] and Smale [78], using the unstable cell decomposition of M for the negative gradient flow of f with respect to a generic Riemannian metric. An analytic counterpart of the Morse–Thom–Smale complex has been suggested in [84]. Rigorous proofs of Witten's ideas have been given by Helffer and Sjöstrand [38] using semi-classical analysis. The Morse–Thom–Smale complex has seen a revival in the 1990s, starting with the work of Floer [22], where the idea of counting trajectories between critical points, has been exploited in an infinite dimensional setting.

An important generalisation of the Morse–Thom–Smale complex is to smooth Morse–Bott functions on a smooth compact manifold. Several approaches to the Morse–Bott complex can be found in the literature, most of them having Floer theoretical applications in mind [6, 7, 24, 29, 39, 41] (see [7, Section 1.2] for a comparison of the different approaches). In this paper we use the Morse–Bott cochain complex of Austin and Braam [6], defined in terms of the de Rham complexes of the critical submanifolds of the Morse–Bott function and the trajectory spaces between them.

The *first aim of this article* is to generalise the Morse–Thom–Smale complex to compact stratified pseudomanifolds and their intersection cohomology. First we introduce the notion of a *stratified anti-radial gradient-like vector field* on a compact abstract stratified space X. An important property of a stratified anti-radial gradient-like vector field ξ is that it can be lifted to a smooth vector field $\widetilde{\xi}$ on the smooth unfolding $\widetilde{X}$ of X. Singularities of a stratified anti-radial gradient-like vector field ξ are isolated, but the smooth vector field $\widetilde{\xi}$ on $\widetilde{X}$ has Morse–Bott singularities. This will allow us to adapt the construction in [6] and to construct, for any given perversity $\bar{p}$ in the sense of Goresky and MacPherson [30, 31], a Morse–Bott type cochain complex $(C^\bullet_{\bar{p}}(X,\xi), \partial^\bullet)$ associated to the stratified anti-radial gradient-like vector field ξ on X, and which computes the intersection cohomology of X with perversity $\bar{p}$. The main idea of the construction is to replace the de Rham complex of critical submanifolds of $\widetilde{\xi}$ in the construction of the smooth Morse–Bott cochain complex of Austin and Braam [6] with the truncated complex of liftable intersection differential forms. The latter has been introduced by Brasselet, Hector and Saralegi in [11] and is a subcomplex of the more notorious complex of intersection differential forms of Brylinski, Goresky and MacPherson [14]. The main result regarding the Morse–Bott type complex, constructed in the first part of this article, is the following

Main Theorem 1. — *The Morse–Bott type complex* $(C^\bullet_{\bar{p}}(X,\xi),\partial^\bullet)$ *computes the intersection cohomology of* X *with perversity* $\bar{p}$*,*

$$H^\bullet(C^\bullet_{\bar{p}}(X,\xi),\partial^\bullet) \simeq IH^\bullet_{\bar{p}}(X).$$

The Morse–Bott type complex, constructed in this article, generalises the constructions in [48] and in [49, 50]: In [48] a (homological) Morse–Thom–Smale complex computing singular homology (i.e., for a topologically normal space, intersection homology for the top perversity $\bar{t}$) has been constructed for compact abstract stratified spaces. In [49, 50] a (homological) Morse–Thom–Smale complex computing intersection homology has been constructed for spaces with isolated singularities and any perversity $\bar{p}$. This latter complex (for the middle perversity) is an essential ingredient in the generalisation of the Cheeger–Müller theorem to singular spaces with isolated conical singularities achieved in [51].

While the Morse–Bott type complex constructed in the first part of this article may be interesting in itself, as already hinted above, our main motivation is its application to *the comparison between analytic and topological torsion, aka Cheeger–Müller theorem, for spaces with conical singularities*. The Cheeger–Müller theorem for a smooth compact manifold, proved independently by Cheeger [16] and Müller [60], is one of the most important comparison theorems in global analysis. It states the equality between topological and analytic torsion of a smooth compact manifold equipped with a unitary flat vector bundle. In [61] Müller extended the result to the case of odd dimensional manifolds and unimodular flat vector bundles. In the same time, in [9], Bismut and Zhang combined the Witten deformation and local index techniques to generalise the result of Cheeger and Müller to arbitrary flat vector bundles with arbitrary Hermitian metrics, thus giving the most general version of the theorem on smooth compact manifolds. A crucial role in Bismut and Zhang's proof is played by the duality between the Morse–Thom–Smale complex and the Witten complex.

The Cheeger–Müller theorem has also been extended to the equivariant setting ([10, 45, 46]) and for manifolds with boundary ([12, 13, 46, 83]). In recent years the study of analytic torsion and of the Cheeger–Müller theorem for spaces with cuspidal singularities and locally symmetric spaces has seen a lot of activity (see e.g. [2, 3, 15, 55, 56, 62, 63, 64, 65, 68, 69, 70, 72]), development which has been in part motivated by the study of the torsion of the cohomology of locally symmetric spaces associated to arithmetic groups [8].

Let us now review in more detail the history of the study of analytic and topological torsion for spaces with *conical singularities*: It started with work of Dar [19], who proved that the analytic torsion is well-defined for spaces with isolated conical singularities. Dar also defined the (topological) Reidemeister torsion in the context of intersection homology. In case of an oriented even dimensional space, by Poincaré duality, both the analytic torsion as well as the intersection Reidemeister torsion with middle perversity are trivial.

Apart from the easy even dimensional case, the Cheeger–Müller theorem has resisted attempts of generalisation to spaces with conical singularities for a while. Note that, since the analytic torsion of a singular space is not a topological invariant in general, while, by a result of Dar, the intersection Reidemeister torsion is, an equality between the two cannot be expected in general.

An approach to the Cheeger–Müller theorem for spaces with isolated conical singularities suggested by Lesch [43, Problem 5.3] (see also [44], where this strategy has been explained in more generality) is to reduce the problem via the gluing formula for analytic torsion (see [12, 13, 44, 83]) to a comparison of torsions on a truncated cone. The analytic torsion of a truncated cone has been studied in [35, 36, 66, 82]. The recent preprint [37] seems to complete Lesch's programme, without however giving an interpretation of the analytic correction terms coming from the singularities.

The well-definedness of analytic torsion *for spaces with wedge singularities*, *i.e.* conical spaces with a single singular stratum of dimension $\geqslant 1$, is due to Mazzeo and Vertman [57]. In [4], Albin, Rochon and Sher establish an extension of the Cheeger–Müller theorem to spaces with wedge singularities with even codimensional singular stratum and unimodular flat vector bundles satisfying a strong acyclicity condition at infinity; this excludes the trivial bundle. The strategy in [4] is to study the analytic torsion via degeneration of smooth metrics into conical metrics. This strategy has been applied previously with success by the three authors for singular spaces with cuspidal singularities [2, 3].

In [51] the author has followed yet a third strategy for extending the Cheeger–Müller theorem to singular spaces with conical singularities, namely by adapting the approach of Bismut and Zhang to the singular situation. The main result in [51] is a comparison between the analytic torsion of a singular space with isolated conical singularities and a torsion, which, in the present article, will be called the Bismut–Zhang torsion.

The present article is part of the author's programme to extend the approach in [51] to general stratified spaces with iterated conical singularities. In the *second part of this article* we define the *Bismut–Zhang torsion*, which conjecturally will serve as the "topological side" in a Cheeger–Müller theorem for stratified spaces. There are two ingredients entering the definition of the Bismut–Zhang torsion: the Morse–Bott type cochain complex for the lower middle perversity $\overline{m}$ constructed in the first part of this article and the model Witten Laplacian appearing in the generalisation of the Witten deformation to stratified spaces in [49]. For clarity of the presentation, we only define the Bismut–Zhang torsion for spaces with two strata here. However, since both ingredients entering the definition of the Bismut–Zhang torsion are available in general, this definition can be extended directly to stratified pseudomanifolds of arbitrary depth (see Remark 6.4(b)).

The Bismut–Zhang torsion is not a topological invariant in general. However, we can prove the following

Main Theorem 2. — *Let X be an oriented compact pseudomanifold with two strata $Y < Z$ such that $\operatorname{codim}_X Y$ is even. Then the Bismut–Zhang torsion is a topological invariant. Moreover, the Bismut–Zhang torsion equals the intersection Reidemeister torsion defined by Aparna Dar* [19].

The Main Theorem 2 can be seen as the topological counterpart of a result in [57], which states that the analytic torsion on a wedge space with both strata of odd dimension is independent of the choice of a conical metric.

The article is organised as follows: In Section 2, to keep the article self-contained, we collect those basics on abstract stratified spaces, aka Thom–Mather stratified spaces, and on the intersection (co-)homology of pseudomanifolds as introduced by Goresky and MacPherson, which are used in the course of the article. In Section 3 we introduce the notion of a stratified anti-radial gradient-like vector field on an abstract stratified space. In Section 4 we define, for a stratified pseudomanifold X, a stratified anti-radial gradient-like vector field and a given perversity $\bar{p}$ in the sense of Goresky and MacPherson, the Morse–Bott type cochain complex advertised in the title, and prove the Main Theorem 1. In Section 5, adapting ideas from smooth Morse and Morse–Bott theory, we construct a perturbation of the Morse–Bott complex for a space with two strata. In the last section, Section 6, we define the Bismut–Zhang torsion for a stratified pseudomanifold with two strata, we recall the definition of intersection Reidemeister torsion from [19], and prove the Main Theorem 2.

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2. Preliminaries and Notation

In this article we use the language of abstract stratified spaces and control data as introduced by Thom [80] and Mather [53] (see also the reprint [54] of Mather's notes). A useful notion for carrying out analysis on singular spaces is that of a manifold with corners and iterated fibration structures as introduced by Melrose. For a dictionary between the two languages we refer the reader to [1, Section 6] and [21, Section 1].

Section 2 is organised as follows: In Section 2.1, for convenience of the reader, we recall some basic definitions from the theory of abstract stratified spaces. We also recall the notion of "déplissage élémentaire" (translated here as "unfolding") of a stratified space as introduced in [11, Section A.II].

In Section 2.2 we recall the definition and the local calculation for the intersection (co-)homology of a stratified pseudomanifold as introduced by Goresky and MacPherson [30, 31]. We also recall the complex of intersection differential forms introduced by Brylinksi in [14, Section 1.2], where it is attributed to Goresky and MacPherson. In this article we use the subcomplex of liftable intersection differential forms, which has been introduced in [11, Section B.IV] and which we also recall.

2.1. Stratified spaces and unfoldings

2.1.1. Abstract stratified spaces (aka Thom–Mather stratified spaces). Controlled maps

Let X be a topological space, Hausdorff, locally compact and with countable basis for its topology. A stratification $\mathcal{S}$ of the topological space X is a locally finite family of pairwise disjoint locally closed subsets $Z \subset X$, called strata, such that $X = \bigcup_{Z \in \mathcal{S}} Z$. The strata Z are smooth manifolds without boundary in the induced topology. We assume that the *frontier condition* is satisfied: For each pair of strata (Z, R), $Z \cap \overline{R} \neq \emptyset$ implies that $Z \leqslant R$, i.e. $Z \subset \overline{R}$.

A *tubular system* for the stratification $\mathcal{S}$ is a family of tuples $\{(T_Z, \pi_Z, \rho_Z)\}_{Z \in \mathcal{S}}$, where T_Z is an open neighbourhood of Z in X, $\pi_Z : T_Z \to Z$ is a continuous retraction and $\rho_Z : T_Z \to \mathbb{R}_{\geqslant 0}$ is continuous and satisfies $\rho_Z^{-1}(0) = Z$. We call a tubular system $\{(T_Z, \pi_Z, \rho_Z)\}_{Z \in \mathcal{S}}$ *control data* for the stratification $\mathcal{S}$ if the following conditions hold:

(a) For each pair of strata (Z, R), $T_Z \cap R \neq \emptyset$ implies that $Z \leqslant R$.
(b) For each pair of strata (Z, R) with $Z < R$ (i.e. $Z \leqslant R$, $Z \neq R$) the map

$$(\pi_Z, \rho_Z) : R \cap T_Z \longrightarrow Z \times \mathbb{R}_{>0} \tag{2.1}$$

is smooth and submersive.

(c) The tubular system is controlled, i.e. for each pair of strata (Z, R) with $Z < R$ and for all $x \in T_Z \cap T_R$ the following conditions are satisfied: $\pi_R(x) \in T_Z$ and

$$\pi_Z(\pi_R(x)) = \pi_Z(x), \tag{C1}$$

$$\rho_Z(\pi_R(x)) = \rho_Z(x). \tag{C2}$$

Definition 2.1. — *An abstract stratified space is a topological space X as above with a stratification $\mathcal{S}$ and control data $\{(T_Z, \pi_Z, \rho_Z)\}_{Z \in \mathcal{S}}$.*

Let X, Y be abstract stratified spaces with stratifications $\mathcal{S}$ resp. $\mathcal{S}'$. A continuous map $g : X \to Y$ is called a stratified map, if for every $Z \in \mathcal{S}$ there exists $W \in \mathcal{S}'$ such that $g(Z) \subset W$ and moreover $g(T_Z) \subset T_W$. We consider the following control conditions:

$$\pi_W(g(x)) = g(\pi_Z(x)), \qquad x \in T_Z, \tag{C3}$$

$$\rho_Z(x) = \rho_W(g(x)), \qquad x \in T_Z. \tag{C4}$$

A stratified map $g : X \to Y$ is called *weakly controlled (resp. controlled)* if for every $Z \in \mathcal{S}$ the restriction $g_{|Z}$ is smooth and the control condition (C3) (resp. the control conditions (C3) and (C4)) hold. A bijective controlled map $g : X \to Y$ is an *isomorphism of stratified spaces* if g sends each stratum Z of X diffeomorphically to a stratum of Y and moreover $g(T_Z) = T_{g(Z)}$. A controlled map $g : X \to M$ into a smooth manifold M is called a *controlled submersion*, if the restriction to each stratum is submersive.

The *depth of a stratum* Z, denoted by $\mathrm{depth}(Z)$, is the integer

$$\mathrm{depth}(Z) := \sup\{l \mid \exists \text{ a sequence of strata } Z = R_0 < \cdots < R_l\}. \tag{2.2}$$

The *depth of the stratified space* X is defined as

$$\mathrm{depth}(X) := \sup\{\mathrm{depth}(Z) \mid Z \in \mathcal{S}\}. \tag{2.3}$$

The reader is warned that, for the notion of depth, there are different conventions in the literature.

Let X be a stratified space with $\mathrm{depth}(X) = d$. By shrinking the tubes $\{T_Z\}_{Z\in\mathcal{S}}$, we can assume that two strata of the same depth have disjoint tubular neighbourhoods (see [54, page 492]). For inductive arguments it is convenient to work with the stratification of X by depth (associated to $\mathcal{S}$): The strata are given by $Z_i := \bigcup_{\mathrm{depth}(Z)=d-i} Z$, $i = 0, \ldots, d$. By taking the union of tubes, we get control data $\{(T_i, \pi_i, \rho_i)\}_{i=0,\ldots,d-1}$.

In the following we assume that X is connected. Hence $Z_0 < \cdots < Z_d$ and $X = \overline{Z_d}$.

2.1.2. Local triviality of the tubular neighbourhood T_i

Let $i \in \{0, \ldots, d-1\}$. For $x \in Z_i$ and $\epsilon > 0$ small enough, the spaces $T_i^\epsilon := \{w \in T_i \mid \rho_i(w) < \epsilon\}$, $S_i := \rho_i^{-1}(\epsilon)$ and $L_x := \pi_i^{-1}(x) \cap \rho_i^{-1}(\epsilon)$ are abstract stratified spaces, inheriting stratification and control data from those of X, and not depending (up to isomorphism of stratified spaces) on the choice of ϵ. The space L_x is called the *link* of X at x, it only depends on the connected component of Z_i containing x. We denote by $cL_x := (L_x \times [0,\infty))/_{(y,0)\sim(w,0)}$ the cone over L_x, it is also a stratified space.

The maps $\pi_i : T_i^\epsilon \to Z_i$ and $\pi_{i|S_i} : S_i \to Z_i$ are proper controlled submersions and therefore, by Thom's First Isotopy Lemma (see e.g. [81, Theorem 2.6]), locally trivial stratified fibre bundles. We denote by

$$c_\epsilon(S_i, \pi_i) := \big((S_i \times [0,\epsilon)) \sqcup (Z_i \times \{0\})\big)/_{(c,0)\sim(\pi_i(c),0)}, \tag{2.4}$$

the topological mapping cylinder of π_i. The mapping cylinder $c_\epsilon(S_i, \pi_i)$ inherits the structure of an abstract stratified space (see [81, Section 5.3.6] for more details) and there is an isomorphism of stratified spaces

$$\alpha_i : c_\epsilon(S_i, \pi_i) \longrightarrow T_i^\epsilon. \tag{2.5}$$

By composition of α_i with the canonical quotient map $S_i \times (-\epsilon, \epsilon) \to c_\epsilon(S_i, \pi_i)$, $(c, r) \mapsto [c, |r|]$, we get a map

$$\beta_i : S_i \times (\,-\epsilon, \epsilon) \longrightarrow T_i^\epsilon. \tag{2.6}$$

We denote by $(-\epsilon,\epsilon)^* \coloneqq (-\epsilon,\epsilon)\setminus\{0\}$. The restriction $\beta_{i|S_i\times(-\epsilon,\epsilon)^*} : S_i \times (-\epsilon,\epsilon)^* \to T_i^\epsilon \setminus Z_i$ is a twofold covering of stratified spaces.

2.1.3. Unfolding of X

In this section we recall the notions of *déplissage élémentaire* and *déplissage* as introduced in [11, Section A.II], and which are closely related to the notions of *decomposition* and *total decomposition* introduced by Verona (see [81, Sections 6.1.1 and 6.6.1]). Following [20] we translate *déplissage élémentaire* by *unfolding* in this article; other common translations in the literature are "doubling construction", "resolution" or "blow-up".

We assume that X as well as all links are connected.

To a fixed isomorphism of stratified spaces $\alpha_0 : c_\epsilon(S_0,\pi_0) \to T_0^\epsilon$ as in (2.5) (and the associated map $\beta_0 : S_0 \times (-\epsilon,\epsilon) \to T_0^\epsilon$ as in (2.6)) we define the *unfolding* $\widetilde{X}^0$ *of* X *along its minimal stratum* Z_0 as follows: $\widetilde{X}^0$ is the quotient of

$$\big((X\setminus Z_0)\times\{-1,1\}\big) \sqcup (S_0\times(-\epsilon,\epsilon)) \tag{2.7}$$

by the equivalence relation:

$$(x,j)\sim(c,r) \text{ if } |r| = jr \text{ and } x=\beta_0(c,r). \tag{2.8}$$

There is a natural folding map $\theta_0 : \widetilde{X}^0 \to X$. The space $\widetilde{X}^0$ inherits the structure of an abstract stratified space of $\mathrm{depth}(\widetilde{X}^0) = \mathrm{depth}(X) - 1$ (see [11, Proposition 3.1]). The restriction of the map θ_0 to $\widetilde{X}^0 \setminus \theta_0^{-1}(Z_0)$ is a twofold stratified covering of $X \setminus Z_0$. Moreover, we have a natural isomorphism of stratified spaces $\theta_0^{-1}(Z_0) \simeq S_0$.

Iterating the above unfolding procedure we get a sequence of unfoldings:

$$\widetilde{X} \coloneqq \widetilde{X}^{d-1} \xrightarrow{\theta_{d-1,d-2}} \widetilde{X}^{d-2} \xrightarrow{\theta_{d-2,d-3}} \cdots \xrightarrow{\theta_{1,0}} \widetilde{X}^0 \xrightarrow{\theta_0} X. \tag{2.9}$$

We denote by $\mathcal{S}^l \coloneqq \{Z_{l+1}^l < \cdots < Z_d^l\}$ the stratification by depth of $\widetilde{X}^l$, $l = 0,\dots,d-1$.

We call $\widetilde{X} \coloneqq \widetilde{X}^{d-1}$ the total unfolding of X (*déplissage* in [11]), it is a smooth manifold.

We also get unfoldings of T_i, $i = 0,\dots,d-1$, as well as of all links L_x, $x \in \overline{Z_{d-1}}$, which are compatible with the unfoldings of X, i.e. the inclusion $T_i \subset X$ induces an inclusion $\widetilde{T}_i \subset \widetilde{X}$, etc.

For $l \in \{1,\dots,d-1\}$, we denote by $\theta_{l,l-1} : \widetilde{X}^l \to \widetilde{X}^{l-1}$ the l-th step in the sequence of unfoldings (2.9) and by

$$\theta_l : \widetilde{X}^l \longrightarrow X \text{ the composition } \theta_l = \theta_{l,l-1}\circ\ldots\circ\theta_{1,0}\circ\theta_0. \tag{2.10}$$

For $i, j \in \{0, \dots, d-1\}$, $i < j$, we denote by

$$\theta_{j,i} : \widetilde{X}^j \longrightarrow \widetilde{X}^i \text{ the composition } \theta_{j,i} := \theta_{j,j-1} \circ \cdots \circ \theta_{i+1,i}. \tag{2.11}$$

We denote by $\theta = \theta_{d-1} : \widetilde{X} \to X$ the map from the total unfolding $\widetilde{X}$ to X. By construction $\widetilde{X} \setminus \theta^{-1}(\overline{Z_{d-1}})$ has 2^d connected components and $\theta_{|\widetilde{X}\setminus\theta^{-1}(\overline{Z_{d-1}})} : \widetilde{X} \setminus \theta^{-1}(\overline{Z_{d-1}}) \to X \setminus \overline{Z_{d-1}} = Z_d$ is a covering map. We fix one of these connected components and denote its closure in $\widetilde{X}$ by $\widehat{X}$. We call $\widehat{X}$ the marked leaf of $\widetilde{X}$, it is a smooth manifold with corners and iterated fibration structure (see [21, Section 1]).

Let $i \in \{0, \dots, d\}$, $l \in \{0, \dots, d-1\}$ and $x \in Z_i$. The fibre $\theta_l^{-1}(x)$ is one of the following:

- a discrete set of 2^{l+1} points, if $l < i$,
- a disjoint union of 2^l connected components isomorphic to L_x, if $l = i$,
- a disjoint union of 2^i connected components isomorphic to the $(l-i-1)$-th unfolding of L_x, if $l > i$.

2.1.4. Distinguished neighbourhoods

For each point $x \in Z_i$, $i = 0, \dots, d-1$, there is an open neighbourhood U_x and a homeomorphism of stratified spaces

$$\psi_x : U_x \simeq \mathbb{R}^{\dim Z_i} \times cL_x, \tag{2.12}$$

which is compatible with the isomorphisms (2.5), (2.6); here $\dim Z_i$ denotes the dimension of the connected component of Z_i containing x. We call U_x a distinguished neighbourhood and denote by (z, φ, r) the local coordinates in U_x. The coordinate change between two distinguished charts is of the form $(z, \varphi, r) \mapsto (z, h(z)(\varphi), r)$, where $h(z)$ is a stratified automorphism of the link L_x (see [11, Remark A.I.3]).

2.2. Intersection (co-)homology and intersection differential forms

In this article the intersection (co-)homology introduced by Goresky and MacPherson [30, 31] plays an important role. To be able to define it, we will assume in Sections 4-6, that the stratified space X is moreover a *pseudomanifold of dimension* n, i.e. if we denote by $X_i := \bigcup_{\dim Z \leqslant i} Z$, then $X = X_n$, $X_{n-1} = X_{n-2}$ and $X^{sm} := X \setminus X_{n-2}$ is open and dense in X. We call $\Sigma := X_{n-2}$ the singular set of X. Note that for X connected, $X^{sm} = Z_d$ and $\Sigma = \overline{Z_{d-1}}$.

When speaking of a *stratified pseudomanifold* X, in this article, we will always mean an abstract stratified space in the sense of Definition 2.1, which we assume in addition to be an *orientable, compact, connected pseudomanifold*. We moreover assume that *all links* are connected.

If X has only two strata and the singular set Σ is a manifold of dimension 0, we call X *a space with isolated singularities*.

2.2.1. Intersection (co-)homology

Let $\bar{p} = (p_2, \ldots, p_n)$ be a perversity function in the sense of Goresky and MacPherson, i.e. p_i are integers with $p_2 = 0$ and $p_j \leqslant p_{j+1} \leqslant p_j + 1$ for all $2 \leqslant j \leqslant n-1$. A subspace $Y \subset X$ is $(\bar{p}, i)$-allowable if $\dim(Y) \leqslant i$ and $\dim(Y \cap X_{n-k}) \leqslant i - k + p_k$ for $k \geqslant 2$.

Let $(IC^{\bar{p}}_{\bullet}(X, \mathbb{Z}), \partial_{\bullet})$ be the subcomplex of the complex of PL geometric chains of X consisting of intersection chains of perversity $\bar{p}$, i.e. $\sigma \in IC^{\bar{p}}_i(X, \mathbb{Z})$ iff $|\sigma|$ is $(\bar{p}, i)$-allowable and $|\partial\sigma|$ is $(\bar{p}, i-1)$-allowable. The intersection homology of X with perversity $\bar{p}$ is defined as the homology of the intersection chain complex $(IC^{\bar{p}}_{\bullet}(X, \mathbb{Z}), \partial_{\bullet})$,

$$IH^{\bar{p}}_{\bullet}(X, \mathbb{Z}) := H_{\bullet}(IC^{\bar{p}}_{\bullet}(X, \mathbb{Z}), \partial_{\bullet}). \tag{2.13}$$

The intersection cohomology $IH^{\bullet}_{\bar{p}}(X, \mathbb{Z})$ is defined as the cohomology of the dual complex. The intersection homology $IH^{\bar{p}}_{\bullet}(X)$ (resp. the intersection cohomology $IH^{\bullet}_{\bar{p}}(X)$) with real coefficients is defined in the same way, working with intersection chains with real coefficients instead. For $U \subset X$ open, one can also define the relative intersection homology $IH_{\bullet}(X, U)$ (see e.g. [40, Section 4.6] for more details). For a stratified pseudomanifold X with boundary ∂X, $IH_{\bullet}(X, \partial X)$ denotes the intersection homology of the interior of X modulo a collared neighbourhood of ∂X (see [32, Section 1.4]).

Let L be a stratified pseudomanifold of $\dim L = n-1$. For convenience of the reader we recall the local calculation for the intersection homology resp. for the relative intersection homology of a cone cL (see [31, Section 2.2] and also [40, Proposition 4.7.2]):

$$IH^{\bar{p}}_i(cL) = \begin{cases} IH^{\bar{p}}_i(L) & \text{for } i \leqslant n - p_n - 2, \\ 0 & \text{for } i > n - p_n - 2, \end{cases} \tag{2.14}$$

resp.

$$IH^{\bar{p}}_i(cL, L) := IH^{\bar{p}}_i(cL, cL \setminus \{0\}) = \begin{cases} IH^{\bar{p}}_{i-1}(L) & \text{for } i \geqslant n - p_n, \\ 0 & \text{for } i < n - p_n. \end{cases} \tag{2.15}$$

The formulas (2.14) and (2.15) also hold with coefficients in $\mathbb{Z}$.

Two perversities $\bar{p}$, $\bar{q}$ are called complementary if

$$\bar{p}+\bar{q}=\bar{t}:=(0,1,2,\dots,n-2). \tag{2.16}$$

In Section 6 of this article we will mainly focus on the lower middle perversity $\overline{m}$, $m_k=\lfloor k/2\rfloor-1$. Its complementary perversity is the upper middle perversity $\overline{n}$, $n_k=\lfloor (k-1)/2\rfloor$.

2.2.2. Intersection differential forms

Let M and N be smooth manifolds with $\dim M=m$, $\dim N=n$ and let $\pi:M\to N$ be a smooth submersion. A smooth j-form $\omega\in\Omega^j(M)$ has perversity $q\in\{0,\dots,m-n\}$ with respect to π if for every $(q+1)$-tuple of smooth vector fields $v_0,\dots,v_q$ tangent to the fibres of π (i.e. smooth sections of $\ker d\pi$), we have

$$\iota_{v_0}\cdots\iota_{v_q}\omega=0. \tag{2.17}$$

In other words, if the submersion $\pi:M\to N$ is represented locally as the projection $(z_1,\dots,z_n,\varphi_1,\dots,\varphi_{m-n})\mapsto(z_1,\dots,z_n)$, then a form of perversity q is a linear combination of forms $dz_I\wedge d\varphi_J$, where I and J are multi-indices with $|J|\leqslant q$.

Let now X be a stratified pseudomanifold of dimension n according to our convention at the beginning of Section 2.2. We denote by $(\Omega^\bullet(X),d)$ the de Rham complex of smooth forms on $X^{sm}=X\setminus\Sigma$. Let $\bar{p}$, $\bar{q}$ be two complementary perversities.

Definition 2.2.

(a) *A smooth differential j-form $\omega\in\Omega^j(X)$ has perversity $\bar{q}$ if for all strata Z the form $\omega_{|T_Z\setminus\Sigma}$ has perversity $q_{\operatorname{codim} Z}$ with respect to the submersion $\pi_Z:T_Z\setminus\Sigma\to Z$.*

(b) *A form $\omega\in\Omega^j(X)$ is an intersection differential form of perversity $\bar{q}$ if both ω and $d\omega$ have perversity $\bar{q}$. We denote by $(\Omega^\bullet_{\bar{q}}(X),d)$ the complex of intersection differential forms of perversity $\bar{q}$.*

The complex $(\Omega^\bullet_{\bar{q}}(X),d)$ computes the intersection cohomology of X with perversity $\bar{p}$ (see [14, Section 1.2]), $H^\bullet(\Omega^\bullet_{\bar{q}}(X),d)\simeq IH^\bullet_{\bar{p}}(X)$.

2.2.3. Intersection differential forms admitting a lift

We denote by $(\Omega^\bullet(\widetilde{X}),d)$ the de Rham complex of smooth differential forms on the smooth manifold $\widetilde{X}$. We denote by $(\Omega^\bullet_{\mathrm{lift}}(X),d)\subset(\Omega^\bullet(X),d)$

the subcomplex of smooth forms ω on the regular stratum X^{sm} admitting a lift $\widetilde{\omega} \in \Omega^\bullet(\widetilde{X})$, i.e. $(\theta_{|\theta^{-1}(X^{sm})})^*\omega = \widetilde{\omega}_{|\theta^{-1}(X^{sm})}$, where by $\widetilde{\omega}_{|\theta^{-1}(X^{sm})}$ we denote the pull back of $\widetilde{\omega}$ via the inclusion map $\theta^{-1}(X^{sm}) \subset \widetilde{X}$. Note that, by continuity of $\widetilde{\omega}$ and density of $\theta^{-1}(X^{sm})$ in $\widetilde{X}$, the lift $\widetilde{\omega}$ of a form $\omega \in \Omega^\bullet_{\mathrm{lift}}(X)$ is unique.

Definition 2.3. — *We denote by $(\mathcal{K}^\bullet_{\bar{q}}(X), d) \subset (\Omega^\bullet_{\bar{q}}(X), d)$ the subcomplex of intersection differential forms ω of perversity $\bar{q}$ which admit a lift, i.e. $\omega \in \Omega^\bullet_{\mathrm{lift}}(X)$.*

By associating to each form $\omega \in \mathcal{K}^\bullet_{\bar{q}}(X)$ its lift $\widetilde{\omega} \in \Omega^\bullet(\widetilde{X})$, the complex $(\mathcal{K}^\bullet_{\bar{q}}(X), d)$ can be seen as a subcomplex of the de Rham complex $(\Omega^\bullet(\widetilde{X}), d)$. The complex $(\mathcal{K}^\bullet_{\bar{q}}(X), d)$ computes the intersection cohomology of X with perversity $\bar{p}$ (see [11, page 212]),

$$H^\bullet(\mathcal{K}^\bullet_{\bar{q}}(X), d) \simeq IH^\bullet_{\bar{p}}(X). \tag{2.18}$$

Let L be a stratified pseudomanifold of dimension $n-1$. We denote by $(\mathcal{K}^\bullet_{\bar{q},tr}(L), d)$ the following subcomplex of $(\mathcal{K}^\bullet_{\bar{q}}(L), d)$:

$$\mathcal{K}^i_{\bar{q},tr}(L) = \begin{cases} \mathcal{K}^i_{\bar{q}}(L) & \text{if } i < q_n, \\ \ker d & \text{if } i = q_n, \\ 0 & \text{if } i > q_n. \end{cases} \tag{2.19}$$

By [11, Proposition C.IV.2.3 and page 212],

$$H^\bullet(\mathcal{K}^\bullet_{\bar{q},tr}(L), d) \simeq IH^\bullet_{\bar{p}}(cL). \tag{2.20}$$

Lemma 2.4. — *Let $\omega \in \Omega^\bullet(X)$ be a differential form admitting a lift $\omega_0 \in \Omega^\bullet(\widetilde{X}^0)$ to $\widetilde{X}^0$, i.e. $\omega_{0|\theta_0^{-1}(X^{sm})} = (\theta_{0|\theta_0^{-1}(X^{sm})})^*\omega$.*

(a) *If ω has perversity $\bar{q}$, then ω_0 also has perversity $\bar{q}$.*
(b) *If ω_0 has perversity $\bar{q}$, then ω satisfies the perversity condition for the perversity $\bar{q}$ and all non-minimal strata of X.*

Proof. — To prove that a form $\alpha \in \Omega^\bullet(\widetilde{X}^0)$ has perversity $\bar{q}$, by continuity, it is enough to check the perversity condition (2.17) on the dense subset $\widetilde{X}^0 \setminus \theta^{-1}(Z_0)$ of $\widetilde{X}^0$. The claim of the proposition follows from the fact that the restriction of the folding map θ_0 to a connected component of $\widetilde{X}_0 \setminus \theta_0^{-1}(Z_0)$ is an isomorphism of stratified spaces. □

3. Stratified anti-radial gradient-like vector fields

In this section we introduce the notion of a *stratified anti-radial gradient-like vector field* on an abstract stratified space X with unfolding $\widetilde{X}$. We assume in this section, that the abstract stratified space X is compact and connected.

Section 3 is organised as follows: In Section 3.1 we first recall some standard notions on stratified vector fields: the notion of a *controlled stratified vector field* is an important tool in Thom and Mather's theory; the notion of a *totally anti-radial vector field* is inspired from the radial vector fields as introduced by Marie-Hélène Schwartz [75] (for Whitney stratified spaces) in her study of characteristic classes for singular spaces. In Section 3.2 we introduce the notion of *unfolding* of a stratified vector field. In Section 3.3 we introduce the notion of a *stratified anti-radial gradient-like vector field* and study its main properties. The weaker notion of a stratified gradient-like vector field has been introduced and studied in [48]. The main additional feature of a stratified anti-radial gradient-like vector field is that it can be unfolded to a Morse–Bott vector field on the unfolding $\widetilde{X}$. This unfolding property will be crucial for the construction of the "singular" Morse–Bott type cochain complex figuring in the title of this article.

3.1. Controlled and totally anti-radial stratified vector fields

A stratified vector field ξ on an abstract stratified space X is a family $\{\xi_Z : Z \to TZ\}_{Z\in\mathcal{S}}$ of smooth vector fields.

Definition 3.1.

(a) *A stratified vector field on X is called controlled if for all pairs of strata (Z, R) with $Z < R$ and all $x \in T_Z \cap R$ the following control conditions are satisfied:*

$$d\pi_Z \xi_R(x) = \xi_Z(\pi_Z(x)), \tag{C5}$$

$$d\rho_Z \xi_R(x) = 0. \tag{C6}$$

A stratified vector field satisfying only the control condition (C5) *is called weakly controlled.*

(b) *A stratified vector field is called totally anti-radial if for each stratum Z there exists a bounded non-negative function $A_Z : Z \to [0, \infty)$, such that for all pairs of strata (Z, R) with $Z < R$ and all $x \in T_Z$,*

$$|d\rho_Z \xi_R(x)| \leqslant A_Z(\pi_Z(x))\rho_Z(x) \text{ and } d\rho_Z \xi_R(x) \leqslant 0. \tag{3.1}$$

There is no canonical notion of continuity for stratified vector fields on an abstract stratified space. However using the control conditions (C5) and (C6) one can show that a controlled stratified vector field is locally integrable and its (local) flow is continuous, strata-wise smooth and controlled (see [81, Lemma 2.3], [54, Proposition 10.1]). Weakly controlled, totally anti-radial vector fields are locally integrable to a (local) weakly controlled flow (see [71, Proposition 2.5.1]). Moreover total anti-radiality ensures that trajectories do not leave a stratum in finite time, and moreover, in positive infinite time flow lines can only go from a larger into a smaller stratum.

3.2. Stratified vector fields admitting a sequence of unfoldings

From now on, we fix a sequence of unfoldings (2.9). We denote by $E_0 := \theta_0^{-1}(Z_0) \simeq S_0$ the exceptional set of the folding map $\theta_0 : \widetilde{X}^0 \to X$.

Let $\xi = \{\xi_i\}_{i=0,\dots,d}$ be a stratified vector field on X. A stratified vector field $\widetilde{\xi}^0 = \{\widetilde{\xi}_i^0\}_{i=1,\dots,d}$ on $\widetilde{X}^0$ is called an unfolding of the vector field ξ under the folding map θ_0 if

- For $i = 1,\dots,d$, $\widetilde{\xi}_i^0$ is a smooth vector field on the stratum Z_i^0, tangent to the exceptional set $E_0 \cap Z_i^0$ and satisfying $d\theta_0 \widetilde{\xi}_i^0 = \xi_i \circ \theta_0$ over $Z_i^0 \setminus E_0$.
- $\widetilde{\xi}^0_{|E_0}$ is a controlled lift of ξ_0 under the proper controlled submersion $\theta_{0|E_0} : E_0 \to Z_0$, i.e. $\widetilde{\xi}^0_{|E_0}$ is a controlled vector field on the stratified space E_0 and, for each stratum W of E_0, $d\theta_{0|W}\widetilde{\xi}^0_W = \xi_0 \circ \theta_{0|W}$.

Not every stratified vector field on X admits an unfolding. However in case it exists, by density, the unfolding is unique. Moreover, if ξ satisfies the control condition (C5) (resp. (C6), resp. (3.1)) so does the unfolding $\widetilde{\xi}^0$.

Definition 3.2. — *We say that a stratified vector field ξ admits a sequence of unfoldings if there exist stratified vector fields $\widetilde{\xi}^l$ over $\widetilde{X}^l$, $l = 0,\dots,d-1$, such that $\widetilde{\xi}^l$ is an unfolding of $\widetilde{\xi}^{l-1}$ under the folding map $\theta_{l,l-1} : \widetilde{X}^l \to \widetilde{X}^{l-1}$.*

Remark 3.3. — Let $\lambda : X \to \mathbb{R}$ be a controlled cut-off function. By [11, C.II.Lemma 2.1], for $l = 0\dots,d-1$, the composition $\lambda \circ \theta_l : \widetilde{X}^l \to \mathbb{R}$ is also controlled. Let ξ be a weakly controlled vector field on X admitting a sequence of unfoldings $\{\widetilde{\xi}^l\}_l$. Then $\lambda{\cdot}\xi$ is also a weakly controlled vector field admitting a sequence of weakly controlled unfoldings, namely $\{(\lambda \circ \theta_l)\cdot\widetilde{\xi}^l\}_l$.

3.3. Stratified anti-radial gradient-like vector fields

Let $\xi = \{\xi_Z\}_{Z\in\mathcal{S}}$ be a stratified vector field on an abstract stratified space X. A point $x \in Z$ with $\xi_Z(x) = 0$ is called a singularity (or singular point) of ξ. We denote by $\mathrm{Crit}(\xi)$ the set of singular points of ξ.

Definition 3.4. — *Let ξ be a stratified vector field and $x \in Z \cap \mathrm{Crit}(\xi)$. The point x is called a singularity of strong standard form of ξ if there exists an open distinguished neighbourhood U_x of x in X and local coordinates $z_1, \dots, z_{\dim Z}$ of Z such that the vector field ξ has normal form*

$$\xi_{|U_x} = -r\frac{\partial}{\partial r} + \sum_{i=1}^{m} z_i \frac{\partial}{\partial z_i} - \sum_{i=m+1}^{\dim Z} z_i \frac{\partial}{\partial z_i} \tag{3.2}$$

in the coordinates (z, φ, r). We call $m = \mathrm{ind}(x) = \mathrm{ind}(x, \xi)$ the index of the singular point $x \in X$.

Let ξ be a stratified weakly controlled, totally anti-radial vector field with singularities of strong standard form. We denote by $\mathrm{Crit}_m(\xi)$, $m = 0, \dots, n$, the set of singular points of ξ of index m. Since X is compact and since singularities of strong standard form are isolated, the vector field ξ has only finitely many singularities. Using the results mentioned in Section 3.1 and the compactness of X, one can show that by integrating the vector field ξ one gets even a *globally* defined, continuous, strata-wise smooth and weakly controlled flow $\Phi : X \times \mathbb{R} \to X$.

A point of X is called a wandering point of the flow Φ if there exists an open neighbourhood V of the point as well as $t_0 \in \mathbb{R}$ such that $\Phi(t, V) \cap V = \emptyset$ for $|t| > t_0$. We denote by Ω_Φ the set of non-wandering points of Φ. Clearly $\mathrm{Crit}(\xi) \subset \Omega_\Phi$, but the converse does not hold in general.

Let us assume that $\mathrm{Crit}(\xi) = \Omega_\Phi$. Then, for $x \in Z \cap \mathrm{Crit}(\xi)$, the unstable set

$$W^u(x) := \left\{ y \in X \;\middle|\; \lim_{t\to-\infty} \Phi(y,t) = x \right\} \subset Z \tag{3.3}$$

is an embedded submanifold (see [48, Proposition 5.1]). The stable set

$$W^s(x) := \left\{ y \in X \;\middle|\; \lim_{t\to\infty} \Phi(y,t) = x \right\} \subset \bigcup_{Z\leqslant R} R \tag{3.4}$$

is an abstract stratified space with stratification and control data inherited from those of X (see [48, Proposition 5.2]).

It follows from the total anti-radiality of the vector field ξ that the strata of X are invariant for the flow Φ and that moreover in positive infinite time flow lines can only go from a larger into a smaller stratum. Therefore, for $x, y \in \mathrm{Crit}(\xi)$, $x \in Z$, $y \in R$,

$$W^u(x) \cap W^s(y) \neq \emptyset \Longrightarrow R \leqslant Z \text{ and } W^u(x) \cap W^s(y) \subset Z. \tag{3.5}$$

In view of (3.5) it is possible to define the Morse–Smale transversality condition, also in this stratified context: We say that ξ satisfies the Morse–Smale condition, if for all singular points $x \in Z \cap \mathrm{Crit}(\xi)$ and $y \in R \cap \mathrm{Crit}(\xi)$ $(R \leqslant Z)$ the manifolds $W^u(x)$ and $W^s(y) \cap Z$ intersect transversally, i.e. we have

$$T_w W^u(x) + T_w \left(W^s(y) \cap Z\right) = T_w Z \text{ for all points } w \in W^u(x) \cap W^s(y) \subset Z.$$

We denote by $\mathcal{M}(x, y) := \left(W^u(x) \cap W^s(y)\right)/\mathbb{R}$ the set of unparameterised trajectories starting in x and ending in y. The Morse–Smale transversality condition implies that $\mathcal{M}(x, y)$ is a smooth manifold of dimension $\mathrm{ind}(x) - \mathrm{ind}(y) - 1$. In particular $\mathcal{M}(x, y) = \emptyset$ if $\mathrm{ind}(x) \leqslant \mathrm{ind}(y)$.

Definition 3.5. — *A stratified vector field ξ on the abstract stratified space X is called anti-radial gradient-like if it is weakly controlled, totally anti-radial, with singularities of strong standard form and moreover the following three conditions hold:*

(a) $\Omega_\Phi = \mathrm{Crit}(\xi)$.
(b) *ξ satisfies the Morse–Smale condition.*
(c) *ξ admits a sequence of unfoldings $\{\widetilde{\xi}^l\}_{l \in \{0,\dots,d-1\}}$.*

Proposition 3.6. — *Let ξ be a stratified anti-radial gradient-like vector field. We denote by $\{\widetilde{\xi}^l\}_{l \in \{0,\dots,d-1\}}$ the sequence of unfoldings of ξ. Then*

(a) *For $l \in \{0, \dots, d-1\}$ the weakly controlled, totally anti-radial vector field $\widetilde{\xi}^l$ induces a weakly controlled flow $\widetilde{\Phi}^l : \widetilde{X}^l \times \mathbb{R} \to \widetilde{X}^l$, satisfying*

$$\theta_l(\widetilde{\Phi}^l(x, t)) = \Phi(\theta_l(x), t)) \text{ for all } x \in \widetilde{X}^l \text{ and all } t \in \mathbb{R}. \tag{3.6}$$

Moreover for $i, j \in \{0, \dots, d-1\}$, $i < j$,

$$\theta_{j,i}(\widetilde{\Phi}^j(x, t)) = \widetilde{\Phi}^i(\theta_{j,i}(x), t)) \text{ for all } x \in \widetilde{X}^j \text{ and all } t \in \mathbb{R}. \tag{3.7}$$

(b) *For $l \in \{0, \dots, d-1\}$ and $x \in \mathrm{Crit}(\xi)$, set $B_l(x) := \theta_l^{-1}(x)$. We have*

$$\mathrm{Crit}(\widetilde{\xi}^l) = \bigcup_{x \in \mathrm{Crit}(\xi)} B_l(x). \tag{3.8}$$

(c) *For* $l \in \{0,\dots,d-1\}$ *and* $x \in \mathrm{Crit}(\xi)$ *the stable resp. unstable set for* $B_l(x)$,

$$\widetilde{W}_l^{s/u}(x) := \Big\{ w \in \widetilde{X}^l \,\Big|\, \lim_{t\to\pm\infty} \widetilde{\Phi}^l(w,t) \in B_l(x) \Big\}, \tag{3.9}$$

are abstract stratified spaces.

Proof.

(a). — The unfolding $\widetilde{\xi}^0$ is weakly controlled and totally anti-radial with respect to the induced control data on $\widetilde{X}^0$, hence $\widetilde{\xi}^0$ is locally integrable to a weakly controlled, strata-wise smooth flow. Since the restriction of θ_0 to the complement of the exceptional set E_0 is a stratified covering and the flow Φ is globally defined, the flow $\widetilde{\Phi}^0$ induced from $\widetilde{\xi}^0$ is also globally defined on $\widetilde{X}^0 \setminus E_0$. On E_0 the vector field $\widetilde{\xi}^0$ is the controlled lift under a proper controlled submersion of a smooth vector field with global flow. By [81, Lemma 2.4] the flow of $\widetilde{\Phi}^0$ on E_0 then also exists for all time $t \in \mathbb{R}$. Thus $\widetilde{\xi}^0$ induces a globally defined stratified flow $\widetilde{\Phi}^0 : \widetilde{X} \times \mathbb{R} \to \widetilde{X}$ and E_0 is invariant under the flow $\widetilde{\Phi}^0$. The argument above can be iterated to show global existence of the flow $\widetilde{\Phi}^l : \widetilde{X}^l \times \mathbb{R} \to \widetilde{X}^l$ induced from $\widetilde{\xi}^l$, $l = 0,\dots,d-1$. By definition of the unfolding of a stratified vector field, the vector fields ξ, $\widetilde{\xi}^i$, $\widetilde{\xi}^j$ for $i,j \in \{0,\dots,d-1\}$ are related, hence (3.6) and (3.7) hold.

(b). — Singular points of $\widetilde{\xi}^l$ are precisely the fixed points of the flow $\widetilde{\Phi}^l$. Therefore, using the commutativity (3.6), $\mathrm{Crit}(\widetilde{\xi}^l) \subset \cup_{x\in \mathrm{Crit}(\xi)} B_l(x)$. Let $x \in \mathrm{Crit}(\xi) \cap Z_i$, $i > l$. For a small enough open neighbourhood $U_x \subset X$ of x the preimage $\theta_l^{-1}(U_x)$ consists of 2^{l+1} disjoint copies of U_x, $\theta_{l|\theta_l^{-1}(U_x)}$ is a covering map, and $d\theta_l \widetilde{\xi}^l = \xi \circ \theta_l$ on $\theta_l^{-1}(U_x)$. Therefore $B_l(x) \subset \mathrm{Crit}(\widetilde{\xi}^l)$. In case $x \in \mathrm{Crit}(\xi) \cap Z_i$, $i < l$ (resp. $i = l$), we have that for a distinguished neighbourhood U_x the preimage $\theta_l^{-1}(U_x)$ is a disjoint union of 2^i connected components, which are isomorphic (as stratified spaces) to $\mathbb{R}^{\dim Z_i} \times (-\epsilon,\epsilon) \times \widetilde{L}_x^{l-i-1}$ (resp. to $\mathbb{R}^{\dim Z_i} \times (-\epsilon,\epsilon) \times L_x$). It follows from the definition of a singularity of strong standard form that the restriction of $\widetilde{\xi}^l$ to these connected components can be written as in the normal form (3.2). Hence we deduce $B_l(x) \subset \mathrm{Crit}(\widetilde{\xi}^l)$.

(c). — The claim follows using the normal form (3.2) of a singularity of strong standard form and arguing as in the proof of [48, Propositions 5.1 and 5.2]. □

Let ξ be a stratified anti-radial gradient-like vector field. For $x,y \in \mathrm{Crit}(\xi)$ we denote by $\widetilde{\mathcal{M}}_l(x,y) := \big(\widetilde{W}_l^u(x) \cap \widetilde{W}_l^s(y)\big)/\mathbb{R}$ the trajectory space of unparameterised trajectories of $\widetilde{\Phi}^l$ starting in $B_l(x)$ and ending in $B_l(y)$. Following the flow as $t \to \pm\infty$, we get endpoint maps

$$\begin{aligned} \widetilde{\pi}^s_{x,y,l} &: \widetilde{\mathcal{M}}_l(x,y) \longrightarrow B_l(y), \\ \widetilde{\pi}^u_{x,y,l} &: \widetilde{\mathcal{M}}_l(x,y) \longrightarrow B_l(x). \end{aligned} \tag{3.10}$$

Similarly we have natural maps

$$\widetilde{\pi}^{s/u}_{x,l} : \widetilde{W}^{s/u}_l(x) \longrightarrow B_l(x). \tag{3.11}$$

Proposition 3.7. — *Let ξ be a stratified anti-radial gradient-like vector field on a compact abstract stratified space X. For $x, y \in \mathrm{Crit}(\xi)$ and all $w \in B_l(x)$, the intersection of $\widetilde{W}^s_l(y)$ and $(\widetilde{\pi}^u_{x,l})^{-1}(w)$ is transverse (in the stratum, where it takes place).*

Proof. — Since the flow preserves transverse intersections, it is enough to check the transversality condition in a neighbourhood of $B_l(x)$, where the vector field can be written in normal form (3.2). The latter follows easily using the Morse–Smale property for ξ. □

Similarly to the smooth situation (see [6, page 140]), the transversality condition in Proposition 3.7 together with the properties of the unfolding process, has the following important consequences

Corollary 3.8. — *The trajectory spaces $\widetilde{\mathcal{M}}_l(x,y)$ are stratified spaces with stratification and control data inherited from those of $\widetilde{X}^l$. Moreover, the endpoint maps $\widetilde{\pi}^u_{x,y,l}$ are locally trivial stratified fibre bundles.*

3.4. Existence of stratified anti-radial gradient-like vector fields. Morse and Morse–Bott function

An important tool in the theory of abstract stratified spaces is the existence of controlled lifts of smooth vector fields under controlled submersions (see [81, Lemma 2.4]). One can easily prove a stronger version of this lifting property, namely the existence of controlled lifts which in addition admit a sequence of unfoldings. Using this stronger lifting property and Remark 3.3 one can modify the inductive proof of the existence of stratified gradient-like vector fields [48, Proposition 6.4] to get

Proposition 3.9. — *Let X be a compact abstract stratified space. Then there exists a stratified anti-radial gradient-like vector field on X.*

For a strata-wise smooth function $f : X \to \mathbb{R}$ and $x \in Z$, we denote by $df_x := d(f_{|Z})_x$. Generalising [78, Theorem B] one can prove the existence of a self-indexing "Morse function" $f : X \to \mathbb{R}$ with negative pseudo-gradient vector field ξ:

Proposition 3.10. — *There exists a continuous strata-wise smooth function $f : X \to \mathbb{R}$ with the following properties:*

(a) *Let $x \in Z$. Then*

$$\begin{aligned} df_x(\xi_Z) &< 0 && \text{if } x \notin \mathrm{Crit}(\xi_Z), \\ df_x &= 0 && \text{if } x \in \mathrm{Crit}(\xi_Z). \end{aligned} \tag{3.12}$$

(b) *f is self-indexing, i.e. $f(x) = k$ for $x \in \mathrm{Crit}_k(\xi)$.*

(c) *$\widetilde{f} := f \circ \theta : \widetilde{X} \to \mathbb{R}$ is a smooth Morse–Bott function.*

4. The Morse–Bott type cochain complex computing intersection cohomology

In this section X denotes an n-dimensional compact stratified pseudo-manifold according to our conventions at the beginning of Section 2.2; ξ is a stratified anti-radial gradient-like vector field on X as in Definition 3.5. Moreover $\bar{p}$ and $\bar{q}$ denote complementary perversities.

The aim of this section is to construct the Morse–Bott type cochain complex $(C^\bullet_{\bar{p}}(X,\xi),\partial^\bullet)$ associated to ξ and the perversity $\bar{p}$, and which computes the intersection cohomology of X.

Section 4 is organised as follows: In view of the results in Sections 3.3 and 3.4 the unfolding vector field $\widetilde{\xi} := \widetilde{\xi}_{d-1}$ on $\widetilde{X}$ is a smooth Morse–Bott vector field satisfying the Morse–Bott transversality condition of [6]. In Section 4.1 we recall the definition of the Morse–Bott cochain complex $(C^\bullet(\widetilde{X},\widetilde{\xi}),\partial^\bullet)$ of Austin and Braam [6]: the complex $(C^\bullet(\widetilde{X},\widetilde{\xi}),\partial^\bullet)$ is generated by the de Rham complexes of the critical submanifolds of $\widetilde{\xi}$.

In Section 4.2, we construct the Morse–Bott type cochain complex $(C^\bullet_{\bar{p}}(X,\xi),\partial^\bullet)$. The main idea is to replace the de Rham complex of the critical submanifolds in Austin and Braam's construction with the complex of liftable intersection differential forms $(\mathcal{K}^\bullet_{\bar{q},tr}(L_x),d)$, $x \in \mathrm{Crit}(\xi)$. The main technical difficulty is to show the well-definedness of the boundary operator $\partial^\bullet$. For this we need to show that the boundary operator maps liftable intersection differential forms of perversity $\bar{q}$ to liftable intersection differential forms of perversity $\bar{q}$.

In Section 4.3 we prove the Main Theorem 1, i.e. that the Morse–Bott type cochain complex $(C^\bullet_{\bar{p}}(X,\xi),\partial^\bullet)$ computes the intersection cohomology $IH^\bullet_{\bar{p}}(X)$.

4.1. The smooth Morse–Bott cochain complex associated to $(\widetilde{X}, \widetilde{\xi})$

The vector field $\widetilde{\xi} := \widetilde{\xi}_{d-1}$ is a smooth vector field on the smooth manifold $\widetilde{X}$. We denote by $\widetilde{\Phi} := \widetilde{\Phi}^{d-1} : \widetilde{X} \times \mathbb{R} \to \widetilde{X}$ the smooth flow induced from $\widetilde{\xi}$. For $x \in \mathrm{Crit}(\xi)$ we write shortly $B(x) := B_{d-1}(x)$. We denote by $\widetilde{W}^{u/s}(x) := \widetilde{W}^{u/s}_{d-1}(x)$ the unstable/stable manifold of $B(x)$. For $x, y \in \mathrm{Crit}(\xi)$ we denote by $\widetilde{\mathcal{M}}(x,y) := \widetilde{\mathcal{M}}_{d-1}(x,y) = (\widetilde{W}^u(x) \cap \widetilde{W}^s(y))/\mathbb{R}$ the space of unparameterised trajectories of the flow $\widetilde{\Phi}$ starting in $B(x)$ and ending in $B(y)$ and by $\widetilde{\pi}^{u/s}_{x,y}$ the endpoint maps:

$$\begin{aligned} \widetilde{\pi}^u_{x,y} &:= \widetilde{\pi}^u_{x,y,d-1} : \widetilde{\mathcal{M}}(x,y) \longrightarrow B(x), \\ \widetilde{\pi}^s_{x,y} &:= \widetilde{\pi}^s_{x,y,d-1} : \widetilde{\mathcal{M}}(x,y) \longrightarrow B(y). \end{aligned} \tag{4.1}$$

In view of the results in Sections 3.3 and 3.4, all assumptions needed in [6] for the construction of the Morse–Bott cochain complex are satisfied (see [6, pages 140-141]), more precisely:

Proposition 4.1. — *The following holds for the flow $\widetilde{\Phi} : \widetilde{X} \times \mathbb{R} \to \widetilde{X}$ induced from $\widetilde{\xi}$:*

(a) *The set of singular points of $\widetilde{\xi}$ is given by*

$$\mathrm{Crit}(\widetilde{\xi}) := \left\{ w \in \widetilde{X} \;\middle|\; \widetilde{\xi}(w) = 0 \right\} = \bigcup_{x \in \mathrm{Crit}(\xi)} B(x). \tag{4.2}$$

For $x \in \mathrm{Crit}(\xi)$, the critical submanifold $B(x)$ is of Morse–Bott type with trivial unstable normal bundle of rank $\mathrm{ind}(x)$*. The critical submanifolds $B(x)$, $x \in \mathrm{Crit}(\xi)$, and their unstable normal bundles are orientable.*

(b) *For $x, y \in \mathrm{Crit}(\xi)$ with $\mathrm{ind}(x) \leqslant \mathrm{ind}(y)$, we have $\widetilde{\mathcal{M}}(x,y) = \emptyset$.*

(c) *For all $x, y \in \mathrm{Crit}(\xi)$ and all $w \in B(x)$, $(\widetilde{\pi}^u_x)^{-1}(w)$ and $\widetilde{W}^s(y)$ intersect tranversally.*

As a consequence of the transversality condition in Proposition 4.1(c) the trajectory space $\widetilde{\mathcal{M}}(x,y)$, $x, y \in \mathrm{Crit}(\xi)$, is a smooth manifold admitting a compactification (in the topology of convergence up to broken trajectories) to a manifold with corners (see [6, Lemma 3.3]) and moreover, the endpoint map $\widetilde{\pi}^u_{x,y} : \widetilde{\mathcal{M}}(x,y) \to B(x)$ is a locally trivial fibre bundle (see [6, page 140]). We fix orientations on all $B(x)$'s and their unstable normal bundles. Together with the flow, this induces orientations on the trajectory manifolds $\widetilde{\mathcal{M}}(x,y)$. Integration of smooth forms along the fibres of $\widetilde{\pi}^u_{x,y}$ is well-defined and will be denoted by $(\widetilde{\pi}^u_{x,y})_*$.

We now recall the construction of Austin and Braam [6]: For $x \in \mathrm{Crit}(\xi)$, we denote by $(\Omega^\bullet(B(x)), d)$ the de Rham complex of the smooth compact manifold $B(x)$. Let $x, y \in \mathrm{Crit}(\xi)$. Let $\widetilde{\sigma}_{xy}$ be the linear map defined by

$$\begin{aligned} \widetilde{\sigma}_{xy} : \Omega^\bullet(B(y)) &\longrightarrow \Omega^\bullet(B(x)) \\ \omega &\longmapsto \widetilde{\sigma}_{xy}(\omega) := (\widetilde{\pi}^u_{x,y})_*(\widetilde{\pi}^s_{x,y})^*\omega. \end{aligned} \tag{4.3}$$

For $\omega \in \Omega^j(B(y))$, we have that $\widetilde{\sigma}_{xy}(\omega) \in \Omega^{j-\mathrm{ind}(x)+\mathrm{ind}(y)+1}(B(x))$.

Definition 4.2. — *The Morse–Bott cochain complex $(C^\bullet(\widetilde{X}, \widetilde{\xi}), \partial^\bullet)$ associated to $\widetilde{\xi}$ is defined by*

$$C^k(\widetilde{X}, \widetilde{\xi}) := \bigoplus_{i+j=k} C^{i,j}(\widetilde{X}, \widetilde{\xi}) := \bigoplus_{i+j=k} \bigoplus_{\mathrm{ind}(x)=i} \Omega^j(B(x)). \tag{4.4}$$

The boundary operator $\partial^\bullet = \bigoplus \partial^\bullet_t$ is defined by

$$\partial_t : C^{i,j}(\widetilde{X}, \widetilde{\xi}) \longrightarrow C^{i+t,j-t+1}(\widetilde{X}, \widetilde{\xi}), \tag{4.5}$$

where for $y \in \mathrm{Crit}_i(\xi)$, $\omega \in \Omega^j(B(y))$,

$$\partial_t \omega = \begin{cases} d\omega & \text{for } t = 0, \\ (-1)^j \displaystyle\sum_{\mathrm{ind}(x)=i+t} \widetilde{\sigma}_{xy}(\omega) & \text{for } t > 0. \end{cases} \tag{4.6}$$

To prove that $(C^\bullet(\widetilde{X}, \widetilde{\xi}), \partial^\bullet)$ is indeed a complex, one has to show that $\partial^2 = 0$, see [6, Proposition 3.5]. More precisely using the compactification of the spaces of gradient lines, one can show that for $k \geqslant 0$,

$$\sum_{t=0}^{k} \partial_{k-t}\partial_t = 0. \tag{4.7}$$

Remark 4.3. — In case that all critical submanifolds of a smooth Morse–Bott vector field are just isolated singular points, i.e. the vector field is gradient-like, the Morse–Bott cochain complex is just the usual Morse–Thom–Smale cochain complex. In particular, in our situation, we have that, for $x, y \in \mathrm{Crit}(\xi) \cap Z_d$ the map (4.3) is the trivial map unless $\mathrm{ind}(x) = \mathrm{ind}(y) + 1$. In the latter case, $\widetilde{\sigma}_{xy}$ is just given by counting with signs trajectories of ξ starting in x and ending in y. More precisely, let us denote by $\widehat{B}(x) \simeq \{x\}$ (resp. by $\widehat{B}(y) \simeq \{y\}$) the connected component of $B(x)$ (resp. of $B(y)$) included in $\widehat{X}$. Then

$$\begin{aligned} \sigma_{xy} := \widetilde{\sigma}_{xy|\Omega^\bullet(\widehat{B}(y))} : \Omega^\bullet(\widehat{B}(y)) \simeq \mathbb{R} &\longrightarrow \Omega^\bullet(\widehat{B}(x)) \simeq \mathbb{R} \\ 1 &\longmapsto \widetilde{\sigma}_{xy}(1) = \pm n(x,y), \end{aligned} \tag{4.8}$$

where $n(x, y)$ is the number of trajectories between x and y counted with signs (see e.g. [42, Section c] for more details).

The Morse–Bott cochain complex $(C^\bullet(\widetilde{X},\widetilde{\xi}),\partial^\bullet)$ is filtered by the index of singular points of ξ,

$$0 \subset F_k C^k(\widetilde{X},\widetilde{\xi}) \subset \cdots \subset F_0 C^k(\widetilde{X},\widetilde{\xi}) = C^k(\widetilde{X},\widetilde{\xi}), \tag{4.9}$$

where

$$F_m C^k(\widetilde{X},\widetilde{\xi}) \coloneqq \bigoplus_{i+j=k, i\geqslant m} C^{i,j}(\widetilde{X},\widetilde{\xi}). \tag{4.10}$$

Let $f : X \to \mathbb{R}$ be a self-indexing function, s.th. $\widetilde{f} = f \circ \theta : \widetilde{X} \to \mathbb{R}$ is a smooth self-indexing Morse–Bott function as in Proposition 3.10. For $m \in \mathbb{N}_0$, set $\widetilde{Y}_m \coloneqq \widetilde{f}^{-1}((m-1/2,\infty)) \subset \widetilde{X}$. We denote by $(\Omega^\bullet_c(\widetilde{Y}_m), d)$ the de Rham complex of smooth compactly supported forms on $\widetilde{Y}_m$. The de Rham complex $(\Omega^\bullet(\widetilde{X}), d)$ computes the de Rham cohomology $H^\bullet_{dR}(\widetilde{X})$ and is filtered by

$$0 \subset \Omega^\bullet_c(\widetilde{Y}_n) \subset \Omega^\bullet_c(\widetilde{Y}_{n-1}) \subset \cdots \subset \Omega^\bullet_c(\widetilde{Y}_0) = \Omega^\bullet(\widetilde{X}). \tag{4.11}$$

For $x \in \mathrm{Crit}(\xi)$, the endpoint map $\widetilde{\pi}^u_x \coloneqq \widetilde{\pi}^u_{x,d-1} : \widetilde{W}^u(x) \to B(x)$, $\widetilde{\pi}^u_x(w) = \lim_{t\to-\infty} \widetilde{\Phi}(w,t)$, is a locally trivial fibre bundle. Actually in our situation it follows from the normal form (3.2) of a singularity of strong standard form and the unfolding process that the unstable fibre bundle is a trivial bundle. By [6, Lemma 3.3] the unstable manifold $\widetilde{W}^u(x)$ can be compactified to a manifold with corners. Integration of smooth forms along the fibres of the fibre bundle $\widetilde{\pi}^u_x$ is well-defined and will be denoted by $(\widetilde{\pi}^u_x)_*$. For $\omega \in \Omega^k(\widetilde{X})$, we denote by $\omega_{|\widetilde{W}^u(x)}$ the pullback of ω via the inclusion map $\widetilde{W}^u(x) \subset \widetilde{X}$. We define the linear map $\widetilde{\Psi}^i$ by

$$\begin{aligned} \widetilde{\Psi}^i : \Omega^k(\widetilde{X}) &\longrightarrow \widetilde{C}^{i,k-i}(\widetilde{X},\widetilde{\xi}) \\ \omega &\longmapsto \widetilde{\Psi}^i(\omega) = \sum_{x\in \mathrm{Crit}_i(\xi)} (\widetilde{\pi}^u_x)_*(\omega_{|\widetilde{W}^u(x)}). \end{aligned} \tag{4.12}$$

Note that for $\omega \in \Omega^\bullet_c(\widetilde{Y}_m)$ and $i < m$, we have $\widetilde{\Psi}^i(\omega) = 0$. Hence $(\oplus_i \widetilde{\Psi}^i)(\omega) \in F_m C^\bullet(\widetilde{X},\widetilde{\xi})$.

Theorem 4.4 ([6, Theorems 3.1 and 3.8, Lemma 3.6]). — *The map*

$$\widetilde{\Psi} = \bigoplus_i \widetilde{\Psi}^i : (\Omega^\bullet(\widetilde{X}), d) \longrightarrow (C^\bullet(\widetilde{X},\widetilde{\xi}),\partial^\bullet) \tag{4.13}$$

is a map of filtered cochain complexes inducing an isomorphism

$$H^\bullet_{dR}(\widetilde{X}) = H^\bullet(\Omega^\bullet(\widetilde{X}), d) \simeq H^\bullet(C^\bullet(\widetilde{X},\widetilde{\xi}),\partial^\bullet). \tag{4.14}$$

4.2. The Morse–Bott type cochain complex $(C^\bullet_{\bar p}(X,\xi),\partial^\bullet)$

4.2.1. Definition of the Morse–Bott type cochain complex $(C^\bullet_{\bar p}(X,\xi),\partial^\bullet)$

Before defining the complex $(C^\bullet_{\bar p}(X,\xi),\partial^\bullet)$ in Definition 4.8, we need several preparatory results (Lemmas 4.5 and 4.6, Proposition 4.7) to tackle the well-definedness of the boundary operator $\partial^\bullet$.

Let $x, y \in \mathrm{Crit}(\xi)$, with $x \neq y$, $x \in Z_l$ and $y \in Z_k$ and such that $\mathcal{M}(x,y) \neq \emptyset$. By total anti-radiality of the vector field ξ, we have $k \leqslant l$. We assume that $l < d$.

We consider the l-th unfolding $\widetilde{X}^l$ of X. Recall that by definition of unfoldings, $B_l(x)$ is a disjoint union of copies of L_x (see end of Section 2.1.3). If $k = l$, then $B_l(y)$ is a disjoint union of copies of L_y. If $k < l$, then $B_l(y)$ is a disjoint union of copies of the $(l-k-1)$-th unfolding of L_y. Let us denote by $\widehat{X}^l := \theta_{d-1,l}(\widehat{X})$ the marked leaf of $\widetilde{X}^l$. We denote by $\widehat{B}_l(x) \simeq L_x$ (resp. by $\widehat{B}_l(y)$) the connected component of $B_l(x)$ (resp. of $B_l(y)$) with $\widehat{B}_l(x) \subset \widehat{X}^l$ (resp. with $\widehat{B}_l(y) \cap \widehat{X}^l \neq \emptyset$). We have used the notation $\widehat{B}(x)$, $\widehat{B}(y)$ for the corresponding connected components of $B(x)$, $B(y)$ in $\widehat{X}$.

Recall that for a link L, we have an injective map of cochain complexes $(\Omega^\bullet_{\mathrm{lift}}(L),d) \to (\Omega^\bullet(\widetilde{L}),d)$ (resp. $(\mathcal{K}^\bullet_{\bar q}(L),d) \to (\Omega^\bullet(\widetilde{L}),d)$) by sending a liftable form ω to its unique lift $\widetilde{\omega}$. In the following, we will often tacitly identify $(\Omega^\bullet_{\mathrm{lift}}(L),d)$ (resp. $(\mathcal{K}^\bullet_{\bar q}(L),d)$) with the image subcomplex in $(\Omega^\bullet(\widetilde{L}),d)$. In particular, $(\Omega^\bullet_{\mathrm{lift}}(\widehat{B}_l(x)),d) = (\Omega^\bullet_{\mathrm{lift}}(L_x),d) \subset (\Omega^\bullet(\widehat{B}(x)),d) = (\Omega^\bullet(\widetilde{L}_x),d)$. Note also that, by uniqueness of lifts, we can identify liftable forms on a link L with liftable forms on some unfolding of L. Hence $(\Omega^\bullet_{\mathrm{lift}}(\widehat{B}_l(y)),d) = (\Omega^\bullet_{\mathrm{lift}}(L_y),d)$.

Lemma 4.5. — *The map $\widetilde{\sigma}_{xy}$ defined in* (4.3) *restricts to a linear map*

$$\widehat{\sigma}_{xy} : \Omega^\bullet_{\mathrm{lift}}(L_y) = \Omega^\bullet_{\mathrm{lift}}(\widehat{B}_l(y)) \longrightarrow \Omega^\bullet_{\mathrm{lift}}(L_x) = \Omega^\bullet_{\mathrm{lift}}(\widehat{B}_l(x)). \tag{4.15}$$

Proof. — The result is essentially a consequence of the construction of the unfolding (see Section 2.1.3) and of the commutativity of flows (3.7): The stratified spaces $\widetilde{X}^l$, $\widetilde{\mathcal{M}}_l(x,y)$, $B_l(x)$, $B_l(y)$ all carry stratifications and control data which are induced from the stratification of X and are therefore compatible. Denoting by E_l the exceptional set of the folding map $\theta_{l,l-1} : \widetilde{X}^l \to \widetilde{X}^{l-1}$, we have $B_l(x) \subset E_l$. We can also identify $\widetilde{\mathcal{M}}_l(x,y)$ with a stratified subset of $E_l \cap \widetilde{X}^l$, namely $\widetilde{\mathcal{M}}_l(x,y) \simeq \widetilde{W}^u_l(x) \cap \widetilde{W}^s_l(y) \cap (\theta_l \circ f)^{-1}(a)$, where f is a Morse function as in Section 3.4 and $a \in (f(y), f(x))$ is a regular level. The exceptional set E_l is flow invariant, the flow on E_l is controlled (not only weakly controlled). The endpoint maps $\widetilde{\pi}^{s/u}_{x,y,l}$ defined in (3.10) are strata-preserving and strata-wise smooth. Moreover, by Corollary 3.8, $\widetilde{\pi}^{u}_{x,y,l}$ is a locally trivial stratified fibre bundle.

We denote by $\widehat{\mathcal{M}}_l(x,y) := \widetilde{\mathcal{M}}_l(x,y) \cap \widehat{X}^l$, i.e. the space of trajectories between $\widehat{B}_l(x)$ and $\widehat{B}_l(y)$. We denote by $\widehat{\mathcal{M}}_l(x,y)^{sm} \subset \widehat{\mathcal{M}}_l(x,y)$ the top stratum of the stratified space $\widehat{\mathcal{M}}_l(x,y)$; $\widehat{\mathcal{M}}_l(x,y)^{sm}$ parameterises the trajectories between $\widehat{B}_l(x)$ and $\widehat{B}_l(y)$ which also lie in $\widetilde{X}^{l,sm}$. We denote by

$$\widehat{\pi}^{s}_{x,y,l} : \widehat{\mathcal{M}}_l(x,y)^{sm} \longrightarrow \widehat{B}_l(y) \cap \widetilde{X}^{l,sm} \tag{4.16}$$

resp. by

$$\widehat{\pi}^{u}_{x,y,l} : \widehat{\mathcal{M}}_l(x,y)^{sm} \longrightarrow \widehat{B}_l(x) \cap \widetilde{X}^{l,sm} \tag{4.17}$$

the restriction of the endpoint maps $\widetilde{\pi}^{s/u}_{x,y,l}$.

Similarly, we denote by $\widehat{\mathcal{M}}(x,y)$ the subset of $\widetilde{\mathcal{M}}(x,y)$ parameterising trajectories between $\widehat{B}(x)$ and $\widehat{B}(y)$; these trajectories lie in $\widehat{X}$. We denote by $\widehat{\pi}^{u}_{x,y} : \widehat{\mathcal{M}}(x,y) \to \widehat{B}(x)$, $\widehat{\pi}^{s}_{x,y} : \widehat{\mathcal{M}}(x,y) \to \widehat{B}(y)$ the restriction of the smooth endpoint maps $\widetilde{\pi}^{u/s}_{x,y}$. From (3.7), we get a commutative diagram

$$\begin{array}{lccc} \widehat{\pi}^{u}_{x,y} : & \widehat{\mathcal{M}}(x,y) & \longrightarrow & \widehat{B}(x) \\ & \Big\downarrow \theta_{d-1,l} & & \Big\downarrow \theta_{d-1,l} \\ \widetilde{\pi}^{u}_{x,y,l|\widehat{\mathcal{M}}_l(x,y)} : & \widehat{\mathcal{M}}_l(x,y) & \longrightarrow & \widehat{B}_l(x), \end{array} \tag{4.18}$$

and similarly for the stable endpoint maps. From the commutativity of the diagram (4.18) and by the well-definedness of integration along fibres of $\widehat{\pi}^{u}_{x,y}$, integration along fibres of $\widehat{\pi}^{u}_{x,y,l}$ is well-defined on forms on $\widehat{\mathcal{M}}_l(x,y)^{sm}$ admitting a smooth lift to $\widehat{\mathcal{M}}(x,y)$.

Let $\alpha \in \Omega^{\bullet}_{\text{lift}}(\widehat{B}_l(y)) = \Omega^{\bullet}_{\text{lift}}(L_y)$ and denote by $\widetilde{\alpha} \in \Omega^{\bullet}(\widehat{B}(y))$ its unique lift. By the commutativity of the diagram (4.18) and its stable counterpart, the form $(\widehat{\pi}^{u}_{x,y})_*(\widehat{\pi}^{s}_{x,y})^*\widetilde{\alpha} \in \Omega^{\bullet}(\widehat{B}(x))$ is the lift of the form $\widehat{\sigma}_{xy}(\alpha) := (\widehat{\pi}^{u}_{x,y,l})_*(\widehat{\pi}^{s}_{x,y,l})^*\alpha \in \Omega^{\bullet}(\widehat{B}_l(x)) = \Omega^{\bullet}(L_x)$. □

Let W be a stratum of $\widetilde{X}^l$ with tubular neighbourhood T_W and projection $\pi_W : T_W \to W$. Set $T_W^{sm} := T_W \cap \widetilde{X}^{l,sm}$. Since, the restriction of the flow to E_l is controlled, the sets $E_l \cap T_W$ and $E_l \cap T_W^{sm}$ are also preserved by the flow. Hence, again using the identification explained in the proof of Lemma 4.5 above, we have commutative diagrams

$$\begin{array}{ccc} T_W^{sm} \cap \widehat{\mathcal{M}}_l(x,y) & \xrightarrow{\pi_W} & W \cap \widehat{\mathcal{M}}_l(x,y) \\ \Big\downarrow \widehat{\pi}^{s}_{x,y,l} & & \Big\downarrow \widetilde{\pi}^{s}_{x,y,l} \\ T_W^{sm} \cap \widehat{B}_l(y) & \xrightarrow{\pi_W} & W \cap \widehat{B}_l(y) \end{array} \tag{4.19}$$

and

$$\begin{array}{ccc} T_W^{sm} \cap \widehat{\mathcal{M}}_l(x,y) & \xrightarrow{\pi_W} & W \cap \widehat{\mathcal{M}}_l(x,y) \\ \Big\downarrow \widehat{\pi}^u_{x,y,l} & & \Big\downarrow \widetilde{\pi}^u_{x,y,l} \\ T_W^{sm} \cap \widehat{B}_l(x) & \xrightarrow{\pi_W} & W \cap \widehat{B}_l(x). \end{array} \tag{4.20}$$

All maps in the unstable diagram (4.20) are locally trivial fibre bundles.

Lemma 4.6. — *Let $q \in \{0, \dots, \operatorname{codim} W\}$ and let $\alpha \in \Omega^\bullet_{\text{lift}}(\widehat{B}_l(y)) = \Omega^\bullet_{\text{lift}}(L_y)$ be a form of perversity q with respect to the smooth submersion $\pi_{W_{|\widehat{B}_l(y)\cap T_W^{sm}}}$. Then $\widehat{\sigma}_{xy}(\alpha) \in \Omega^\bullet_{\text{lift}}(\widehat{B}_l(x)) = \Omega^\bullet_{\text{lift}}(L_x)$ is a form of perversity q with respect to the smooth submersion $\pi_{W_{|\widehat{B}_l(x)\cap T_W^{sm}}}$.*

Proof. — We denote by $\Gamma(\ker(d\pi_W))$ the vector space of smooth sections of the vector bundle $\ker(d\pi_W)$. By commutativity of the diagram (4.19), we have $\pi_W \circ \widehat{\pi}^s_{x,y,l} = \widetilde{\pi}^s_{x,y,l} \circ \pi_W$. Therefore for $v_0, \dots, v_q \in \Gamma(\ker d\pi_W)$, also $d\widehat{\pi}^s_{x,y,l}(v_0), \dots d\widehat{\pi}^s_{x,y,l}(v_q) \in \Gamma(\ker(d\pi_W))$. By definition of the pullback of forms, and since the form $\alpha \in \Omega^\bullet(\widehat{B}_l(y))$ has perversity q with respect to $\pi_{W_{|\widehat{B}_l(y)\cap T_W^{sm}}}$ we get

$$\iota_{v_0} \dots \iota_{v_q} (\widehat{\pi}^s_{x,y,l})^* \alpha = \iota_{d\widehat{\pi}^s_{x,y,l}(v_0)} \cdots \iota_{d\widehat{\pi}^s_{x,y,l}(v_q)} \alpha = 0. \tag{4.21}$$

Hence $(\widehat{\pi}^s_{x,y,l})^*\alpha \in \Omega^\bullet_{\text{lift}}(\widehat{\mathcal{M}}_l(x,y))$ is a form of perversity q with respect to the submersion $\pi_{W_{|\widehat{\mathcal{M}}_l(x,y)\cap T_W^{sm}}}$. By commutativity of the diagram (4.20), integration along the fibres of $\widehat{\pi}^u_{x,y,l}$ preserves the perversity of a form. Hence $(\widehat{\pi}^u_{x,y,l})_*(\widehat{\pi}^s_{x,y,l})^*\alpha \in \Omega^\bullet_{\text{lift}}(\widehat{B}_l(x))$ is a form of perversity q with respect to the submersion $\pi_{W_{|\widehat{B}_l(x)\cap T_W^{sm}}}$. □

Proposition 4.7. — *The map* (4.15) *restricts to linear maps*

$$\sigma_{xy} : \mathcal{K}^\bullet_{\bar{q}}(L_y) \longrightarrow \mathcal{K}^\bullet_{\bar{q}}(L_x) \tag{4.22}$$

resp.

$$\sigma_{xy} : \mathcal{K}^\bullet_{\bar{q},tr}(L_y) \longrightarrow \mathcal{K}^\bullet_{\bar{q},tr}(L_x). \tag{4.23}$$

Proof. — The first claim follows from Lemmas 2.4 and 4.6, [6, Proposition 3.5] (more precisely using (4.7)) and the anti-radiality of ξ.

We now prove the second claim: We can assume w.l.o.g. that the strata Z_k and Z_l are connected. We have to distinguish two cases.

Case 1. — $k = l$, i.e. the critical points x and y lie in the same (connected) stratum. Therefore $L_x \simeq L_y$. Let $0 \neq \alpha \in \mathcal{K}^{\bullet}_{\bar{q},tr}(L_y)$. By definition (2.19) of $\mathcal{K}^{\bullet}_{\bar{q},tr}(L_y)$ we have that

$$\deg(\alpha) \leqslant q_{\dim L_y+1} \text{ and moreover } d\alpha = 0 \text{ in case } \deg(\alpha) = q_{\dim L_y+1}. \tag{4.24}$$

From (4.22) we know already that $\sigma_{xy}(\alpha) \in \mathcal{K}^{\bullet}_{\bar{q}}(L_x)$. In case $\mathrm{ind}(x) - \mathrm{ind}(y) = 1$, by [6, Proposition 3.5], the map σ_{xy} is a cochain map. Therefore condition (4.24) for the form α does imply that the same condition holds for $\sigma_{xy}(\alpha)$ as well, hence $\sigma_{xy}(\alpha) \in \mathcal{K}^{\bullet}_{\bar{q},tr}(L_x)$. In case $\mathrm{ind}(x) - \mathrm{ind}(y) > 1$, $\deg(\sigma_{xy}(\alpha)) < \deg(\alpha)$, and hence again $\sigma_{xy}(\alpha) \in \mathcal{K}^{\bullet}_{\bar{q},tr}(L_x)$.

Case 2. — $k < l$. Let $\widetilde{X}^k$ be the k-th unfolding of X and let $\widehat{B}_k(y) \simeq L_y$ be the marked connected component of $B_k(y)$. We denote by $\widehat{W}^{s/u}_k(y)$ the stable resp. unstable set of $\widehat{B}_k(y)$ w.r.t. the flow $\widetilde{\Phi}^k$. Since $y \in \mathrm{Crit}(\xi)$ is a singularity of strong standard form, there exists an open neighbourhood $U \simeq \mathbb{R}^{\mathrm{ind}(y)} \times \mathbb{R}^{\dim Z_k - \mathrm{ind}(y)} \times L_y \times \mathbb{R}$ of $\widehat{B}_k(y)$ in $\widetilde{X}^k$ such that $\widehat{W}^s_k(y) \cap U \simeq \{0\} \times \mathbb{R}^{\dim Z_k - \mathrm{ind}(y)} \times L_y \times \mathbb{R}$ and $\widehat{B}_k(y) \simeq \{0\} \times \{0\} \times L_y \times \{0\}$. Let $W \coloneqq L_y \cap Z_l$ be the stratum of L_y corresponding to the stratum Z_l of X. For a point in W there exists a distinguished neighbourhood $V \subset L_y$, with $V \simeq \mathbb{R}^{\dim Z_l - \dim Z_k - 1} \times cL_x$. We denote by (z^u, z^s, φ, r) the local coordinates in $U' \subset U$, $U' \simeq \mathbb{R}^{\mathrm{ind}(y)} \times \mathbb{R}^{\dim Z_k - \mathrm{ind}(y)} \times V \times \mathbb{R}$, where $z = (z^u, z^s)$ are the coordinates in $\mathbb{R}^{\mathrm{ind}(y)} \times \mathbb{R}^{\dim Z_k - \mathrm{ind}(y)}$, r is the distance from the stratum Z_k and $\varphi = (z^l, \varphi^l, r^l)$ are the local coordinates in V. A form of perversity $\bar{q}$ on the top stratum of $\widehat{W}^s_k(y)$ can be written (locally in $U' \cap \widehat{W}^s_k(y)$) as a linear combination of forms $dz^s_I \wedge d\varphi_J \wedge dr$ and $dz^s_I \wedge d\varphi_J$; where $d\varphi_J$ is of the form $dz^l_{J_1} \wedge d\varphi^l_{J_2} \wedge dr^l$ (resp. $dz^l_{J_1} \wedge d\varphi^l_{J_2}$) and I, J, J_1, J_2 are multi-indices. The perversity condition imposes restrictions on the multi-indices J, J_1, J_2. In particular, the fact that the form $dz^s_I \wedge d\varphi_J$ with $d\varphi_J = dz^l_{J_1} \wedge d\varphi^l_{J_2} \wedge dr^l$ (resp. $d\varphi_J = dz^l_{J_1} \wedge d\varphi^l_{J_2}$) has perversity $q_{\mathrm{codim}\, Z^k_l} = q_{\mathrm{codim}\, W} = q_{\dim L_x+1}$ with respect to the projection $T_{Z^k_l} \cap \widehat{W}^s_k(y) \cap \widetilde{X}^{k,sm} \to Z^k_l \cap \widehat{W}^s_k(y)$, implies that

$$|J_2| + 1 \leqslant q_{\mathrm{codim}\, W} = q_{\dim L_x+1} \text{ (resp. } |J_2| \leqslant q_{\mathrm{codim}\, W} = q_{\dim L_x+1}). \tag{4.25}$$

Note that under integration of $dz^s_I \wedge d\varphi_J \wedge dr$ (resp. of $dz^s_I \wedge d\varphi_J$) along the fibres of $\widehat{\pi}^u_{x,y,l}$, only the (lift of the) form $d\varphi^l_{J_2}$ survives.

Let now $\alpha \in \mathcal{K}^{\bullet}_{\bar{q}}(L_y)$; we still denote by the same letter its lift to $\widehat{B}_l(y)$. As in the proof of Lemma 4.6, one can show that $(\widehat{\pi}^s_{x,y,l})^*(\alpha)$, $(\widehat{\pi}^s_{x,y,l})^*(d\alpha)$, $\sigma_{xy}(\alpha)$ and $\sigma_{xy}(d\alpha)$ also have perversity $\bar{q}$. From the above discussion (in particular from (4.25)) and the definition of σ_{xy}, we deduce that $\deg\sigma_{xy}(\alpha) \leqslant q_{\dim L_x+1}$ and $\deg d\sigma_{xy}(\alpha) = \deg\sigma_{xy}(d\alpha) \leqslant q_{\dim L_x+1}$, which implies $\sigma_{xy}(\alpha) \in \mathcal{K}^{\bullet}_{\bar{q},tr}(L_x)$. □

For $x \in \mathrm{Crit}(\xi)\cap Z_d$ we have used the notation $\widehat{B}(x) := \theta^{-1}(x)\cap\widehat{X} \simeq \{x\}$. We set $\mathcal{K}^{\bullet}_{\bar{q},tr}(L_x) = \mathcal{K}^{0}_{\bar{q},tr}(L_x) := \Omega^0(\widehat{B}(x)) = \mathbb{R}$. For $x, y \in \mathrm{Crit}(\xi) \cap Z_d$ we have defined $\sigma_{xy} : \mathcal{K}^{\bullet}_{\bar{q},tr}(L_y) = \Omega^{\bullet}(\widehat{B}(y)) \to \mathcal{K}^{\bullet}_{\bar{q},tr}(L_x) = \Omega^{\bullet}(\widehat{B}(x))$ as the restriction of the map $\widetilde{\sigma}_{xy}$ (see (4.8)). By Remark 4.3, σ_{xy} is just given by counting trajectories of $\widetilde{\Phi}$ between x and y with sign.

Definition 4.8. — *To the stratified anti-radial gradient-like vector field ξ on X and the perversity $\bar{p}$ we associate the well-defined complex $(C^{\bullet}_{\bar{p}}(X,\xi), \partial^{\bullet})$, defined by*

$$C^{k}_{\bar{p}}(X,\xi) := \bigoplus_{i+j=k} C^{i,j}_{\bar{p}}(X,\xi) := \bigoplus_{i+j=k}\ \bigoplus_{\mathrm{ind}(x)=i} \mathcal{K}^{j}_{\bar{q},tr}(L_x), \tag{4.26}$$

for $k = 0,\ldots,n$. The boundary operator $\partial^{\bullet} = \oplus\partial^{\bullet}_t$ is defined by

$$\partial_t : C^{i,j}_{\bar{p}}(X,\xi) \longrightarrow C^{i+t,j-t+1}_{\bar{p}}(X,\xi), \tag{4.27}$$

where for $\alpha \in \mathcal{K}^{j}_{\bar{q},tr}(L_y)$, $\mathrm{ind}(y) = i$,

$$\partial_t\alpha = \begin{cases} d\alpha & \text{for } t = 0,\\ (-1)^j \displaystyle\sum_{\mathrm{ind}(x)=i+t} \sigma_{xy}(\alpha) & \text{for } t > 0. \end{cases} \tag{4.28}$$

Proof. — Let us denote by $(\widetilde{C}^{\bullet}_{\bar{p}}, \partial^{\bullet})$ the subcomplex of $(C^{\bullet}(\widetilde{X},\widetilde{\xi}), \partial^{\bullet})$ generated by $\mathcal{K}^{\bullet}_{\bar{q},tr}(L_x) \subset \Omega^{\bullet}(B'(x))$, where $x \in \mathrm{Crit}(\xi)$ and $B'(x)$ runs over all connected components of $B(x)$. The fact, that $(\widetilde{C}^{\bullet}_{\bar{p}}, \partial^{\bullet})$ is a subcomplex, follows arguing as in Lemmas 4.5 and 4.6 and Proposition 4.7. By total anti-radiality of ξ, there is a subcomplex $(D^{\bullet}, \partial^{\bullet})$ of $(\widetilde{C}^{\bullet}_{\bar{p}}, \partial^{\bullet})$, which is generated by $\mathcal{K}^{\bullet}_{\bar{q},tr}(L_x) \subset \Omega^{\bullet}(B'(x))$, where $x \in \mathrm{Crit}(\xi)$, and $B'(x)$ now only runs over all connected components of $B(x)$ in $\widetilde{X} \setminus \widehat{X}$. The complex $(C^{\bullet}_{\bar{p}}(X,\xi), \partial^{\bullet})$ can be identified with the quotient complex $\widetilde{C}^{\bullet}_{\bar{p}}/D^{\bullet}$. □

Remark 4.9. — As mentioned in the introduction, for spaces with isolated singularities, a (homological) complex $(C^{\bar{q}}_{\bullet}(X,-\xi), \partial_{\bullet})$ associated to the radial gradient-like vector field $-\xi$ and a perversity $\bar{q}$ has already been defined in [49, 50]. The complex $(C^{\bar{q}}_{\bullet}(X,-\xi), \partial_{\bullet})$ defined in [49, 50] is a finite dimensional subcomplex of the complex constructed here, $(C^{\bar{q}}_{\bullet}(X,-\xi), \partial_{\bullet}) \subset (C^{n-\bullet}_{\bar{p}}(X,\xi), \partial^{n-\bullet})$.

4.2.2. Filtration of the complex $(C^{\bullet}_{\bar{p}}(X,\xi),\partial^{\bullet})$

The complex $(C^{\bullet}_{\bar{p}}(X,\xi),\partial^{\bullet})$ inherits a filtration by the index of singular points of ξ,

$$0\subset F_k C^k_{\bar{p}}(X,\xi)\subset\cdots\subset F_0 C^k_{\bar{p}}(X,\xi)=C^k_{\bar{p}}(X,\xi), \tag{4.29}$$

where

$$F_m C^k(X,\xi):=\bigoplus_{i+j=k,i\geqslant m} C^{i,j}_{\bar{p}}(X,\xi). \tag{4.30}$$

The graded complex associated to the filtered complex $(C^{\bullet}_{\bar{p}}(X,\xi),\partial^{\bullet})$ is

$$\begin{aligned} G^k_m:=F_m C^k_{\bar{p}}(X,\xi)/F_{m+1}C^k_{\bar{p}}(X,\xi)=C^{m,k-m}_{\bar{p}}(X,\xi)\\ =\bigoplus_{x\in\mathrm{Crit}_m(\xi)}\mathcal{K}^{k-m}_{\bar{q},tr}(L_x),\end{aligned} \tag{4.31}$$

with boundary given by the de Rham differential. From (2.20) and (4.31), we get for the E_1-term of the spectral sequence associated to the filtered chain complex $(C^{\bullet}_{\bar{p}}(X,\xi),\partial^{\bullet})$

$$\begin{aligned} E^{m,k}_1=H^{m+k}(G^{\bullet}_m,d)=\left(\bigoplus_{x\in\mathrm{Crit}_m(\xi)}H^k(\mathcal{K}^{\bullet}_{\bar{q},tr}(L_x),d)\right)\\ \simeq\left(\bigoplus_{x\in\mathrm{Crit}_m(\xi)\cap\Sigma}IH^k_{\bar{p}}(cL_x)\right)\oplus\left(\bigoplus_{x\in\mathrm{Crit}_m(\xi)\cap X^{sm}}H^k(\{x\})\right).\end{aligned} \tag{4.32}$$

4.3. The Morse–Bott type cochain complex $(C^{\bullet}_{\bar{p}}(X,\xi),\partial^{\bullet})$ computes intersection cohomology

In this section, by adapting the proof of [6, Theorems 3.8] to the singular situation and taking into account the properties of intersection cohomology, we prove the Main Theorem 1, i.e. that the Morse–Bott type cochain complex $(C^{\bullet}_{\bar{p}}(X,\xi),\partial^{\bullet})$ computes the intersection cohomology $IH^{\bullet}_{\bar{p}}(X)$.

4.3.1. Filtration of the complex $(\mathcal{K}^{\bullet}_{\bar{q}}(X),d)$

For $m\in\mathbb{N}_0$, we denote by $N_m:=f^{-1}((-\infty,m-1/2))$ and by $Y_m:=f^{-1}((m-1/2,\infty))=X\setminus\overline{N_m}$. The complex of liftable intersection forms $(\mathcal{K}^{\bullet}_{\bar{q}}(X),d)$ is filtered by

$$0\subset\mathcal{K}^k_{\bar{q},c}(Y_n)\subset\mathcal{K}^k_{\bar{q},c}(Y_{n-1})\subset\cdots\subset\mathcal{K}^k_{\bar{q},c}(Y_0)=\mathcal{K}^k_{\bar{q}}(X), \tag{4.33}$$

where $(\mathcal{K}^{\bullet}_{\bar{q},c}(Y_m), d) \subset (\mathcal{K}^{\bullet}_{\bar{q}}(X), d)$ denotes the subcomplex of the complex of intersection forms, whose lifts are compactly supported in $\theta^{-1}(Y_m)$. We denote by $G'^{\bullet}_m = \mathcal{K}^{\bullet}_{\bar{q},c}(Y_m)/\mathcal{K}^{\bullet}_{\bar{q},c}(Y_{m+1})$, $m \in \mathbb{N}_0$, the graded complex associated to the filtration (4.33). The short exact sequence of cochain complexes

$$0 \longrightarrow (\mathcal{K}^{\bullet}_{\bar{q},c}(Y_{m+1}), d) \longrightarrow (\mathcal{K}^{\bullet}_{\bar{q},c}(Y_m), d) \longrightarrow (G'^{\bullet}_m, d) \longrightarrow 0 \tag{4.34}$$

induces a long exact sequence in cohomology related to the exact sequence in intersection cohomology of the triple (X, N_{m+1}, N_m) as follows

$$\begin{array}{ccccccc} \dots H^i(\mathcal{K}^{\bullet}_{\bar{q},c}(Y_{m+1}), d) & \longrightarrow & H^i(\mathcal{K}^{\bullet}_{\bar{q},c}(Y_m), d) & \longrightarrow & H^i(G'^{\bullet}_m, d) & \dots \\ \downarrow & & \downarrow & & \downarrow & \\ \dots \quad IH^i_{\bar{p}}(X, N_{m+1}) & \longrightarrow & IH^i_{\bar{p}}(X, N_m) & \longrightarrow & IH^i_{\bar{p}}(N_{m+1}, N_m) & \dots \end{array} \tag{4.35}$$

The first two vertical maps in (4.35) are isomorphisms and hence, by Five Lemma, $H^i(G'^{\bullet}_m, d) \simeq IH^i_{\bar{p}}(N_{m+1}, N_m)$.

Let $x \in \mathrm{Crit}(\xi)$ and Z be the stratum containing x. Let $D_x \subset Z$ be a small open ball in Z centred around x, such that $\pi_{Z|\pi_Z^{-1}(D_x)}$ is trivial, i.e. we have an isomorphism of stratified spaces $\pi_Z^{-1}(D_x) \simeq D_x \times cL_x$ and $\xi_{|\pi_Z^{-1}(D_x)}$ has normal form (3.2). We denote by $D^u_x := W^u(x) \cap D_x \subset Z$ the local unstable disc of x. Using the stratified flow Φ we get a stratified deformation retract of the pair (N_{m+1}, N_m) into the pair

$$(N_m \cup H_m, N_m)\,, \tag{4.36}$$

where

$$H_m \simeq \left(\bigcup_{x \in \mathrm{Crit}_m(\xi) \cap \Sigma} (D^u_x \times cL_x) \right) \cup \left(\bigcup_{x \in \mathrm{Crit}_m(\xi) \cap X^{sm}} D^u_x \right). \tag{4.37}$$

Hence, by the excision formula for intersection cohomology (see [32, Section 1.5]),

$$\begin{aligned} H^{\bullet}(G'^{\bullet}_m, d) &\simeq IH^{\bullet}_{\bar{p}}(N_{m+1}, N_m) \\ &\simeq \left(\bigoplus_{x \in \mathrm{Crit}_m(\xi) \cap \Sigma} IH^{\bullet}_{\bar{p}}(D^u_x \times cL_x, \partial D^u_x \times cL_x) \right) \\ &\qquad \oplus \left(\bigoplus_{x \in \mathrm{Crit}_m(\xi) \cap X^{sm}} H^{\bullet}(D^u_x, \partial D^u_x) \right). \end{aligned} \tag{4.38}$$

4.3.2. The map of filtered complexes $\Psi : (\mathcal{K}^\bullet_{\bar{q}}(X), d) \to (C^\bullet_{\bar{p}}(X,\xi), \partial^\bullet)$

Let $x \in \mathrm{Crit}(\xi)$. Denote by $\widehat{W}^u(x)$ the unstable set of $\widehat{B}(x) \subset \widehat{X}$. The endpoint map $\widehat{\pi}^u : \widehat{W}^u(x) \to \widehat{B}(x)$, $\widehat{\pi}^u(w) := \lim_{t\to-\infty} \Phi(w,t)$, is a locally trivial fibration. Actually in our situation this fibration is even trivial, due to the normal form (3.2), compatible with the unfolding process.

Lemma 4.10. — *Let $x \in \mathrm{Crit}(\xi)$. Let $\omega \in \mathcal{K}^\bullet_{\bar{q}}(X)$ with lift $\widetilde{\omega} \in \Omega^\bullet(\widetilde{X})$. Then*

$$(\widehat{\pi}^u_x)_* \widetilde{\omega}_{|\widehat{W}^u(x)} \in \mathcal{K}^\bullet_{\bar{q},tr}(L_x). \tag{4.39}$$

Proof. — The claim follows with similar arguments as in the proof of Proposition 4.7. □

Proposition 4.11. — *The map of filtered complexes $\widetilde{\Psi}$ defined in* (4.12), (4.13) *induces a map of filtered complexes*

$$\Psi := \oplus_i \Psi^i : (\mathcal{K}^\bullet_{\bar{q}}(X), d) \longrightarrow (C^\bullet_{\bar{p}}(X,\xi), \partial^\bullet), \tag{4.40}$$

where

$$\begin{aligned} \Psi^i : \mathcal{K}^k_{\bar{q}}(X) &\longrightarrow C^{i,k-i}_{\bar{p}}(X,\xi) \\ \omega &\longmapsto \Psi^i(\omega) = \sum_{x\in \mathrm{Crit}_i(\xi)} (\widehat{\pi}^u_x)_* \widetilde{\omega}_{|\widehat{W}^u(x)}. \end{aligned} \tag{4.41}$$

Proof. — The well-definedness of the map follows from Lemma 4.10. □

4.3.3. Proof of Main Theorem 1

Proof of Main Theorem 1. — We follow closely the proof of [6, Theorem 3.8], whose statement has been recalled in Theorem 4.4. We will prove that the map induced from the map of filtered cochain complexes $\Psi : (\mathcal{K}^\bullet_{\bar{q}}(X), d) \to (C^\bullet_{\bar{p}}(X,\xi), \partial^\bullet)$ on the E_1-terms of the spectral sequences associated to the two filtered complexes is an isomorphism. Then, by [58, Theorem 3.2], the induced morphism of cohomologies is an isomorphism, $H^\bullet(\mathcal{K}^\bullet_{\bar{q}}(X), d) \simeq H^\bullet(C^\bullet_{\bar{p}}(X,\xi), \partial^\bullet)$. The claim then follows using the isomorphism (2.18).

The map induced from the map of filtered cochain complexes Ψ on the E_1-terms of the associated spectral sequences, factors as

$$
\begin{aligned}
E_1'^{m,k} &\simeq H^{m+k}(G_m'^{\bullet}, d) \\
&\longrightarrow \left(\bigoplus_{\substack{x\in \mathrm{Crit}_m(\xi)\\ x\in\Sigma}} H^{m+k}(\mathcal{K}^{\bullet}_{\bar q}(D^u_x \times cL_x, \partial D^u_x \times cL_x), d) \right) \\
&\qquad \oplus \left(\bigoplus_{\substack{x\in \mathrm{Crit}_m(\xi)\\ x\in X^{sm}}} H^{m+k}(D^u_x, \partial D^u_x) \right) \\
&\longrightarrow E_1^{m,k} \simeq \left(\bigoplus_{x\in \mathrm{Crit}_m(\xi)} H^k(\mathcal{K}^{\bullet}_{\bar q, tr}(L_x), d) \right) \\
&\qquad \simeq \left(\bigoplus_{\substack{x\in \mathrm{Crit}_m(\xi)\\ x\in\Sigma}} IH^k_{\bar p}(cL_x) \right) \oplus \left(\bigoplus_{\substack{x\in \mathrm{Crit}_m(\xi)\\ x\in X^{sm}}} H^k(\{x\}) \right).
\end{aligned}
\tag{4.42}
$$

The first map in (4.42) is induced from the restriction of forms, the second map is induced from integration along the fibres of the unstable fibration. More precisely, for $x \in \mathrm{Crit}(\xi) \cap \Sigma$ it is given by the composition

$$
\omega \longmapsto \widetilde{\omega} \longmapsto \widetilde{\omega}_{|\widehat{W}^u(x)} \longmapsto (\widehat{\pi}^u_x)_* \widetilde{\omega}_{|\widehat{W}^u(x)}, \tag{4.43}
$$

where $\widetilde{\omega}$ denotes the lift of ω to the total unfolding $D^u_x \times \widetilde{L}_x \times \mathbb{R}$ of $D^u_x \times cL_x$.

It is not difficult to prove by mimicking the smooth proofs, that

$$
H^{\bullet}(\mathcal{K}^{\bullet}_{\bar q}(D^u_x \times cL_x, \partial D^u_x \times cL_x), d) \simeq IH^{\bullet}_{\bar p}(D^u_x \times cL_x, \partial D^u_x \times cL_x). \tag{4.44}
$$

Hence, from (4.38) and (4.44) we conclude that the first map in the factorisation (4.42) is an isomorphism. By (4.44) and the Künneth formula for intersection cohomology (see [32, Section 1.6]),

$$
\begin{aligned}
IH^{m+k}_{\bar p}(D^u_x \times cL_x, \partial D^u_x \times cL_x) &\simeq H^m(D^u_x, \partial D^u_x) \otimes IH^k_{\bar p}(cL_x) \\
&\simeq IH^k_{\bar p}(cL_x),
\end{aligned}
\tag{4.45}
$$

the second map in (4.42) is also an isomorphism. □

5. Perturbation of the local Morse–Bott type cochain complex

In this section, let X be an n-dimensional oriented pseudomanifold with 2 strata $\{Y, Z \coloneqq X^{sm}\}$, $Y < Z$. For simplicity we assume that the singular stratum Y is connected. We denote by L its link and by $T \coloneqq \{x \in T_Y \mid \rho_Y(x) \leqslant \epsilon\}$ its (closed) tubular neighbourhood with boundary $S_Y = \{x \in T_Y \mid \rho_Y(x) = \epsilon\}$ and projection $\pi = \pi_Y : T \to Y$. We denote by $\widetilde{T} \simeq S_Y \times [-\epsilon, \epsilon]$ the unfolding of the tubular neighbourhood T and by $\theta : \widetilde{T} \to T$ the folding map. Set $c \coloneqq \operatorname{codim}_X Y$.

Let $\xi = \{\xi_Y, \xi_Z\}$ be a stratified anti-radial gradient-like vector field on X. Let $\bar{p}$ and $\bar{q}$ be complementary perversities.

Since here the link L is a smooth manifold of dimension $c-1$, the complex $(\mathcal{K}^\bullet_{\bar{q},tr}(L), \partial^\bullet)$ defined in (2.19) and used in the definition of the Morse–Bott type cochain complex $(C^\bullet_{\bar{p}}(X, \xi), \partial^\bullet)$, is just the truncated de Rham complex $(\Omega^\bullet_{\bar{q},tr}(L), d) \subset (\Omega^\bullet(L), d)$,

$$\Omega^k_{\bar{q},tr}(L) \coloneqq \begin{cases} \Omega^k(L) & \text{if } k < q_c, \\ \ker d & \text{if } k = q_c, \\ 0 & \text{if } k > q_c. \end{cases} \tag{5.1}$$

Let $(C^\bullet(Z, \xi_Z), \partial^\bullet)$ be the subcomplex of $(C^\bullet_{\bar{p}}(X, \xi), \partial^\bullet)$ generated by the singular points of ξ_Z. We denote by $(C^\bullet_{\bar{p},\mathrm{loc}}(T, \xi), \partial^\bullet)$ the quotient complex,

$$C^k_{\bar{p},\mathrm{loc}}(T, \xi) = \sum_{i+j=k} \sum_{x \in \mathrm{Crit}_i(\xi_Y)} \Omega^j_{\bar{q},tr}(L_x), \text{ for } k = 0, \dots, n, \tag{5.2}$$

and call it the local Morse–Bott type cochain complex. The complex $(C^\bullet_{\bar{p},\mathrm{loc}}(T, \xi), \partial^\bullet)$ is filtered by the index of critical points of ξ_Y.

In this section, by adapting a construction in smooth Morse and Morse–Bott theory, we construct a perturbation of the local Morse–Bott type cochain complex $(C^\bullet_{\bar{p},\mathrm{loc}}(T, \xi), \partial^\bullet)$. We can assume w.l.o.g. that in the tubular neighbourhood T of Y the stratified anti-radial gradient-like vector field ξ is of the form $-r\partial_r + \xi_Y$. The idea of the construction presented in this section, is to produce a perturbation $\widetilde{\xi}_\eta$ of the unfolded vector field $\widetilde{\xi}$ on $\widetilde{T}$, which has isolated singularities only. The perturbed complex is a subcomplex of the Morse–Thom–Smale complex associated to $\widetilde{\xi}_\eta$, quasi-isomorphic to $(C^\bullet_{\bar{p},\mathrm{loc}}(T, \xi), \partial^\bullet)$.

The results of this section will be used in the proof of the Main Theorem 2 (in particular in Sections 6.3.2 and 6.4.2). A further application of the construction in this section, handled in a sequel paper, is to the comparison between the Morse–Bott type complex $(C^\bullet_{\overline{m}}(X,\xi),\partial^\bullet)$ and the "singular" Witten complex established in [49].

Section 5 is organised as follows: In Section 5.1 we explain the construction for spaces with isolated singularities, in Section 5.2 we extend the construction to the case $\dim Y \geqslant 1$. In Section 5.3 we prove that the perturbed complex and the local Morse–Bott cochain complex $(C^\bullet_{\bar{p},\mathrm{loc}}(T,\xi),\partial^\bullet)$ are quasi-isomorphic; both complexes compute the intersection cohomology $IH^\bullet_{\bar{p}}(T)$. Note that the argument given in Section 5.3 can be used to give an alternative proof of the Main Theorem 1 in the case of a pseudomanifold with two strata.

5.1. Perturbation of the local Morse–Bott type complex: Isolated singularities

In this section we explain the construction of the perturbed complex for a space X with isolated singularities. Let L be the smooth compact connected oriented link manifold of dimension $\dim L = c-1 = n-1$. Here $T \simeq L \times [0,\epsilon]/_{(y,0)\sim(w,0)}$ and $\xi = -r\partial_r$.

Let η be a smooth gradient-like vector field on L. We define the smooth gradient-like vector field

$$\widetilde{\xi}_\eta := -r\partial_r + \eta \tag{5.3}$$

on the manifold with boundary $\widetilde{T} \simeq L \times [-\epsilon,\epsilon]$. The singularities of the vector field $\widetilde{\xi}_\eta$ on $L\times[-\epsilon,\epsilon]$ are in one-to-one correspondence with the singularities of the vector field η on L. We orient all unstable manifolds of $\widetilde{\xi}_\eta$. Using the orientations of $L\times[-\epsilon,\epsilon]$ and of the unstable manifolds of $\widetilde{\xi}_\eta$, we get induced orientations on all stable manifolds of $\widetilde{\xi}_\eta$; note that the stable manifolds of $\widetilde{\xi}_\eta$ are the unstable manifolds of $-\widetilde{\xi}_\eta$. We denote by $(C_\bullet(L\times[-\epsilon,\epsilon],-\widetilde{\xi}_\eta,\mathbb{Z}),\partial_\bullet)$ the homological Morse–Thom–Smale complex with integer coefficients, associated to the manifold with boundary $L\times[-\epsilon,\epsilon]$ and the vector field $-\widetilde{\xi}_\eta$, generated by the unstable manifolds of the singular points of the vector field $-\widetilde{\xi}_\eta$ (see e.g. [67, Section 3.4]); it computes the relative homology $H_\bullet(L\times[-\epsilon,\epsilon],\partial(L\times[-\epsilon,\epsilon]),\mathbb{Z})$. We denote by $(C^{\bar{q},\mathrm{pert}}_\bullet(cL,-\xi,-\eta,\mathbb{Z}),\partial_\bullet) \subset (C_\bullet(L\times[-\epsilon,\epsilon],-\widetilde{\xi}_\eta,\mathbb{Z}),\partial_\bullet)$ the subcomplex defined by

$$C^{\bar{q},\mathrm{pert}}_k(cL,-\xi,-\eta,\mathbb{Z}) := \{\sigma \mid \theta(\sigma) \in IC^{\bar{q}}_k(cL,L,\mathbb{Z})\}. \tag{5.4}$$

In the following we tacitly identify $\sigma \in C^{\bar{q},\mathrm{pert}}_k(cL,-\xi,-\eta,\mathbb{Z})$ with the cone $\theta(\sigma)$.

Proposition 5.1. — *The complex* $(C_\bullet^{\bar{q},\mathrm{pert}}(cL,-\xi,-\eta,\mathbb{Z}),\partial_\bullet)$ *computes the relative intersection homology of the cone cL with integer coefficients,*

$$H_\bullet(C_\bullet^{\bar{q},\mathrm{pert}}(cL,-\xi,-\eta,\mathbb{Z}),\partial_\bullet)\simeq IH_\bullet^{\bar{q}}(cL,L,\mathbb{Z}). \tag{5.5}$$

Moreover

$$H_\bullet(C_\bullet^{\bar{q},\mathrm{pert}}(cL,-\xi,-\eta,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R},\partial_\bullet)\simeq IH_\bullet^{\bar{q}}(cL,L). \tag{5.6}$$

Proof. — We denote by $(C_\bullet(L,-\eta,\mathbb{Z}),\partial_\bullet)$ the homological Morse–Thom–Smale complex with integer coefficients associated to the smooth manifold L and the vector field $-\eta$, generated by the unstable manifolds of singular points of $-\eta$ (see e.g. [42, Section (c)]). Recall, that the complex $(C_\bullet(L,-\eta,\mathbb{Z}),\partial_\bullet)$ computes the (singular) homology of L,

$$H_\bullet(C_\bullet(L,-\eta,\mathbb{Z}),\partial_\bullet)\simeq H_\bullet(L,\mathbb{Z}). \tag{5.7}$$

We denote by $(C_\bullet^{\bar{q},tr}(L,-\eta,\mathbb{Z}),\partial_\bullet)\subset(C_\bullet(L,-\eta,\mathbb{Z}),\partial_\bullet)$ the subcomplex defined by

$$C_k^{\bar{q},tr}(L,-\eta,\mathbb{Z}):=\begin{cases}C_k(L,-\eta,\mathbb{Z}) & \text{for } k\geqslant c-q_c,\\ \ker\partial_k & \text{for } k=c-q_c-1,\\ 0 & \text{else.}\end{cases} \tag{5.8}$$

There is an isomorphism of chain complexes

$$(C_\bullet^{\bar{q},tr}(L,-\eta,\mathbb{Z}),\partial_\bullet)\longrightarrow(C_{\bullet+1}^{\bar{q},\mathrm{pert}}(cL,-\xi,-\eta,\mathbb{Z}),\partial_{\bullet+1}),\ \tau\longmapsto c\tau, \tag{5.9}$$

where for $\tau\subset L$ we denote by $c\tau\subset cL$ the cone over τ.

From (5.7), (5.8), (5.9) and the local calculation for relative intersection homology (recalled in (2.15)), we have

$$\begin{aligned}H_\bullet(C_\bullet^{\bar{q},\mathrm{pert}}(cL,-\xi,-\eta,\mathbb{Z}),\partial_\bullet)&\simeq H_\bullet(C_{\bullet-1}^{\bar{q},tr}(L,-\eta,\mathbb{Z}),\partial_{\bullet-1})\\ &\simeq IH_\bullet^{\bar{q}}(cL,L,\mathbb{Z}).\end{aligned} \tag{5.10}$$

Since $\mathbb{R}$ is a field, the isomorphism (5.6) follows from (5.5) and the universal coefficient theorem. □

Set $(C^\bullet_{\bar{p},\mathrm{pert}}(cL,\xi,\eta),\partial^\bullet):=(C_{c-\bullet}^{\bar{q},\mathrm{pert}}(cL,-\xi,-\eta,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R},(-1)^{\bullet+1}\partial_{c-\bullet})$. By Poincaré duality for intersection (co-)homology (see [30, Section 3.3]) and Proposition 5.1,

$$H^\bullet(C^\bullet_{\bar{p},\mathrm{pert}}(cL,\xi,\eta),\partial^\bullet)\simeq IH^\bullet_{\bar{p}}(cL). \tag{5.11}$$

5.2. Perturbation of the local Morse–Bott type complex: General case

Let now $\dim Y \geqslant 1$. We start by constructing a perturbation of the unfolded vector field $\widetilde{\xi}$ on $\widetilde{T}$. We essentially follow the construction in [67, Section 3]: We choose pairwise disjoint, open neighbourhoods $V_x \subset Y$, $x \in \mathrm{Crit}(\xi_Y)$. The sets $\widetilde{U}_x := \theta^{-1}(\pi^{-1}(V_x)) \subset \widetilde{T}$ are pairwise disjoint with $\widetilde{U}_x \simeq V_x \times L_x \times [-\epsilon, \epsilon]$. We denote by $\pi_{L_x} : \widetilde{U}_x \to L_x$ the canonical projection into the second factor. Let $\lambda_x : Y \to \mathbb{R}_{\geqslant 0}$ be a bump-function with compact support in V_x which is equal to 1 in a neighbourhood of x. Let η be a gradient-like vector field on L, and denote by η_x the corresponding vector field on $\theta^{-1}(x) \simeq L_x \simeq L$. We define a smooth vector field $\widetilde{\xi}_\eta$ on $\widetilde{T}$ as follows

$$\widetilde{\xi}_\eta(w) := \widetilde{\xi}(w) + \sum_{x \in \mathrm{Crit}(\xi_Y)} \lambda_x(\pi(\theta(w)))\eta_x(\pi_{L_x}(w)),\ w \in \widetilde{T}. \tag{5.12}$$

We have

$$\theta_* \widetilde{\xi}_\eta = \xi; \tag{5.13}$$

therefore flow lines for $\widetilde{\xi}_\eta$ project to flow lines for ξ (under the folding map θ). The singular points of $\widetilde{\xi}_\eta$ are precisely the singular points of η in the fibres of θ lying over the singular points $x \in \mathrm{Crit}(\xi_Y)$:

$$\mathrm{Crit}(\widetilde{\xi}_\eta) = \{\alpha \in \widetilde{T} \mid \theta(\alpha) = x \in \mathrm{Crit}(\xi_Y) \text{ and } \pi_{L_x}(\alpha) \in \mathrm{Crit}(\eta_x)\}. \tag{5.14}$$

All singular points of the smooth vector field $\widetilde{\xi}_\eta$ on $\widetilde{T}$ are of standard form. By genericity of the Morse–Smale condition, after possibly perturbing the pair $(\widetilde{\xi}, \eta)$ we can achieve that the vector field $\widetilde{\xi}_\eta$ is gradient-like.

We choose orientations of the unstable manifolds of η_x, $x \in \mathrm{Crit}(\xi_Y)$. Together with the orientations on the unstable normal bundles of L_x, $x \in \mathrm{Crit}(\xi_Y)$, we get orientations on all unstable manifolds of $\widetilde{\xi}_\eta$. Using the orientations of $\widetilde{T}$ and of the unstable manifolds, we get induced orientations on all stable manifolds of $\widetilde{\xi}_\eta$ as well.

For a singular point $\alpha \in \mathrm{Crit}(\widetilde{\xi}_\eta)$ with $\theta(\alpha) = x$, we denote by $W^u(\alpha, -\widetilde{\xi}_\eta) \subset \widetilde{T}$ the unstable manifold of α w.r.t. the flow induced from $-\widetilde{\xi}_\eta$. We denote by $W^u(\alpha, \pm\eta_x) \subset L_x$ the unstable manifold of $\pi_{L_x}(\alpha)$ w.r.t. the flow induced from $\pm\eta_x$ on L_x.

We denote by $(C_\bullet(\widetilde{T}, -\widetilde{\xi}_\eta, \mathbb{Z}), \partial_\bullet)$ the homological Morse–Thom–Smale complex with integer coefficients associated to the manifold with boundary $\widetilde{T}$ and the vector field $-\widetilde{\xi}_\eta$, generated by the unstable manifolds of $-\widetilde{\xi}_\eta$ on $\widetilde{T}$ (see e.g. [67, Section 3.4]). The complex $(C_\bullet(\widetilde{T}, -\widetilde{\xi}_\eta, \mathbb{Z}), \partial_\bullet)$ computes the relative homology $H_\bullet(\widetilde{T}, \partial\widetilde{T}, \mathbb{Z})$. It is filtered by the index of critical points of $-\xi_Y$, more precisely

$$F_m(C_\bullet(\widetilde{T}, -\widetilde{\xi}_\eta, \mathbb{Z}), \partial_\bullet) = \bigoplus_{\operatorname{ind}(\theta(\alpha), -\xi_Y) \leqslant m} \mathbb{Z} \cdot [W^u(\alpha, -\widetilde{\xi}_\eta)]. \tag{5.15}$$

Let $\alpha \in \operatorname{Crit}_k(-\widetilde{\xi}_\eta)$ with $\theta(\alpha) \in \operatorname{Crit}_m(-\xi_Y)$, and let $\beta \in \operatorname{Crit}_{k-1}(-\widetilde{\xi}_\eta)$. By Morse–Smale transversality for the vector field $\widetilde{\xi}_\eta$ there is only a finite number of trajectories γ for the flow induced from the vector field $-\widetilde{\xi}_\eta$, which start in α and end in β. Set $n(\alpha, \beta) \coloneqq \sum_\gamma n_\gamma(\alpha, \beta)$, where $n_\gamma(\alpha, \beta) \in \{\pm 1\}$ is the sign obtained by comparison of orientations (see e.g. [42, Section (c)] for more details). Note that, if $\operatorname{ind}(\theta(\beta), -\xi_Y) \geqslant m$, by (5.13) and Morse–Smale transversality for ξ, there are no trajectories between α and β unless $\theta(\alpha) = \theta(\beta) = x$. In the latter case, all trajectories between α and β lie in the fibre $\theta^{-1}(x)$. From the above discussion we get the following formula for the boundary operator of the complex $(C_\bullet(\widetilde{T}, -\widetilde{\xi}_\eta, \mathbb{Z}), \partial_\bullet)$ (see e.g. [67, Section 3.3.3]):

$$\begin{aligned}
\partial[W^u(\alpha, -\widetilde{\xi}_\eta)] &= \sum_{\operatorname{ind}(\beta, -\widetilde{\xi}_\eta) = k-1} n(\alpha, \beta)[W^u(\beta, -\widetilde{\xi}_\eta)] \\
&= \sum_{i \leqslant m} \sum_{\substack{\operatorname{ind}(\beta, -\widetilde{\xi}_\eta) = k-1 \\ \operatorname{ind}(\theta(\beta), -\xi_Y) = i}} n(\alpha, \beta)[W^u(\beta, -\widetilde{\xi}_\eta)] \\
&= \sum_{\substack{\operatorname{ind}(\beta, -\widetilde{\xi}_\eta) = k-1 \\ \theta(\beta) = \theta(\alpha)}} n(\alpha, \beta)[W^u(\beta, -\widetilde{\xi}_\eta)] \\
&\qquad + \sum_{i < m} \sum_{\substack{\operatorname{ind}(\beta, -\widetilde{\xi}_\eta) = k-1 \\ \operatorname{ind}(\theta(\beta), -\xi_Y) = i}} n(\alpha, \beta)[W^u(\beta, -\widetilde{\xi}_\eta)].
\end{aligned} \tag{5.16}$$

From (5.16) we get, that the graded complex associated to the filtered chain complex $(C_\bullet(\widetilde{T}, -\widetilde{\xi}_\eta, \mathbb{Z}), \partial_\bullet)$ (together with its boundary) can be identified with the sum of the Morse–Thom–Smale complexes "in the fibre" $(C_{\bullet-m}(L_x \times [-\epsilon, \epsilon], r\partial_r - \eta_x, \mathbb{Z}), \partial_{\bullet-m})$, $x \in \operatorname{Crit}_m(-\xi_Y)$.

We denote by $(C_\bullet^{\bar{q},\mathrm{pert}}(T, -\xi, -\eta, \mathbb{Z}), \partial_\bullet)$ the subcomplex of the complex $(C_\bullet(\widetilde{T}, -\widetilde{\xi}_\eta, \mathbb{Z}), \partial_\bullet)$ defined by

$$\begin{aligned}
C_k^{\bar{q},\mathrm{pert}}&(T, -\xi, -\eta, \mathbb{Z}) \\
&\coloneqq \{\sigma \in C_k(\widetilde{T}, -\widetilde{\xi}_\eta, \mathbb{Z}) \mid \theta(\sigma) \in IC_k^{\bar{q}}(T, \partial T, \mathbb{Z})\}.
\end{aligned} \tag{5.17}$$

Note that, for $\alpha \in \mathrm{Crit}_k(-\widetilde{\xi}_\eta)$ with $\theta(\alpha) = x \in \mathrm{Crit}(\xi_Y)$,

$$\theta(W^u(\alpha, -\widetilde{\xi}_\eta)) \text{ is } (\bar{q}, k)\text{-allowable} \Longleftrightarrow \mathrm{ind}(\pi_{L_x}(\alpha), \eta_x) \leqslant q_c. \tag{5.18}$$

Let $\sigma = \sum_{\alpha, \mathrm{ind}(\theta(\alpha), -\xi_Y) = m} a_\alpha [W^u(\alpha, -\widetilde{\xi}_\eta)] \in C^{\bar{q},\mathrm{pert}}_{m+c-q_c}(T, -\xi, -\eta, \mathbb{Z})$. Denote by $(\partial\sigma)_m$ the part of $\partial\sigma$ lying in $\bigoplus_{\mathrm{ind}(\theta(\beta), -\xi_Y) = m} \mathbb{Z} \cdot [W^u(\beta, -\widetilde{\xi}_\eta)]$. Using (5.16) and (5.18) we get that

$$Y \cap \theta((\partial\sigma)_m) = \emptyset \tag{5.19}$$

is equivalent to

(5.20)

$$\text{for all } x \in \mathrm{Crit}_m(-\xi_Y) : \{x\} \cap \theta\left(\sum_{\alpha, \beta \in \theta^{-1}(x)} a_\alpha n(\alpha, \beta) [W^u(\beta, -\widetilde{\xi}_\eta)]\right) = \emptyset$$
$$\Longleftrightarrow \text{ for all } x \in \mathrm{Crit}_m(-\xi_Y) : \partial\left(\sum_{\theta(\alpha) = x} a_\alpha [W^u(\alpha, -\eta_x)]\right) = 0.$$

Let us now consider the subcomplex

$$(C^{\bar{q},\mathrm{pert}}_\bullet(T, -\xi, -\eta, \mathbb{Z}), \partial_\bullet) \subset (C_\bullet(\widetilde{T}, -\widetilde{\xi}_\eta, \mathbb{Z}), \partial_\bullet) \tag{5.21}$$

with its filtration induced from the filtration (5.15). Using (5.8), (5.9), (5.16), (5.18), (5.19), (5.20), we conclude for the associated graded complex,

$$\begin{aligned} F_m(C^{\bar{q},\mathrm{pert}}_\bullet(T, -\xi, -\eta, \mathbb{Z}), \partial_\bullet) / F_{m-1}(C^{\bar{q},\mathrm{pert}}_\bullet(T, -\xi, -\eta, \mathbb{Z}), \partial_\bullet) \\ \simeq \bigoplus_{x \in \mathrm{Crit}_m(-\xi_Y)} C^{\bar{q},\mathrm{pert}}_{\bullet - m}(cL_x, r\partial_r, -\eta_x, \mathbb{Z}); \end{aligned} \tag{5.22}$$

i.e. it is the sum of the complexes "in the fibre" studied in Section 5.1, $(C^{\bar{q},\mathrm{pert}}_\bullet(cL_x, r\partial_r, -\eta_x, \mathbb{Z}), \partial_\bullet)$.

Proposition 5.2. — *The complex* $(C^{\bar{q},\mathrm{pert}}_\bullet(T, -\xi, -\eta, \mathbb{Z}), \partial_\bullet)$ *computes the relative intersection homology of the tubular neighbourhood T with integer coefficients,*

$$H_\bullet(C^{\bar{q},\mathrm{pert}}_\bullet(T, -\xi, -\eta, \mathbb{Z}), \partial_\bullet) \simeq IH^{\bar{q}}_\bullet(T, \partial T, \mathbb{Z}). \tag{5.23}$$

Moreover

$$H_\bullet(C^{\bar{q},\mathrm{pert}}_\bullet(T, -\xi, -\eta, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}, \partial_\bullet) \simeq IH^{\bar{q}}_\bullet(T, \partial T). \tag{5.24}$$

Proof. — The idea of the proof is to adapt the Leray–Serre spectral sequence in smooth Morse homology (see e.g. [29, 39, 67]) to the singular situation; here we rely on the version in [67]. The second ingredient in this proof is the Leray–Serre spectral sequence for singular fibrations in intersection homology due to Friedman (see [26, 27, 28]). Here we use the version in [26, Section 5.8], which we now recall: Let us fix a triangulation K of Y, which is sufficiently fine, such that each simplex is contained in a neighbourhood, where the fibre bundle $T \to Y$ is trivial. We denote by $Y^m \subset Y$ the open regular neighbourhood (in the barycentric subdivision of K) of the simplicial m-skeleton of Y. This provides a filtration of the space Y as well as an induced filtration of the space T by $T^m := \pi^{-1}(Y^m)$. The complex of PL intersection chains $(IC^{\bar{q}}_\bullet(T, \partial T, \mathbb{Z}), \partial_\bullet)$ is filtered by $F_m(IC^{\bar{q}}_\bullet(T, \partial T, \mathbb{Z}), \partial_\bullet) := \operatorname{im}(IC^{\bar{q}}_\bullet(T^m, \partial T^m, \mathbb{Z}) \to IC^{\bar{q}}_\bullet(T, \partial T, \mathbb{Z}))$. The spectral sequence associated to the filtration described above abuts to $IH^{\bar{q}}_\bullet(T, \partial T, \mathbb{Z})$. We denote by $I\mathcal{H}^{\bar{q}}_\bullet(cL, L, \mathbb{Z})$ the local system on Y with fibre $I\mathcal{H}^{\bar{q}}_\bullet(cL, L, \mathbb{Z})_x = IH^{\bar{q}}_\bullet(cL_x, L_x, \mathbb{Z})$. The E^1-term of the filtered complex described above can be identified with the cellular complex $(C_\bullet(Y, I\mathcal{H}^{\bar{q}}_\bullet(cL, L, \mathbb{Z})), \partial_\bullet)$, $E^1_{m,k} \simeq C_m(Y, I\mathcal{H}^{\bar{q}}_k(cL, L, \mathbb{Z}))$. For the E^2-term we get $E^2_{m,k} \simeq H_m(Y, I\mathcal{H}^{\bar{q}}_k(cL, L, \mathbb{Z}))$, where $H_\bullet(Y, I\mathcal{H}^{\bar{q}}_k(cL, L, \mathbb{Z}))$ denotes the singular homology of Y with values in the local system $I\mathcal{H}^{\bar{q}}_k(cL, L, \mathbb{Z})$.

We denote by $(C_\bullet(Y, -\xi_Y, I\mathcal{H}^{\bar{q}}_\bullet(cL, L, \mathbb{Z})), \partial_\bullet)$ the homological Morse–Thom–Smale complex on Y associated to the vector field $-\xi_Y$ with values in the local system $I\mathcal{H}^{\bar{q}}_\bullet(cL, L, \mathbb{Z})$ (see e.g. [67, Section 3.2.3], [9, Section I(c)]). Using (5.22) and following the arguments in [67, Sections 3.3 and 3.4] we can prove that the E^1-term in the spectral sequence associated to the filtered complex $(C^{\bar{q},\mathrm{pert}}_\bullet(T, -\xi, -\eta, \mathbb{Z}), \partial_\bullet)$ can be identified with $(C_\bullet(Y, -\xi_Y, I\mathcal{H}^{\bar{q}}_\bullet(cL, L, \mathbb{Z})), \partial_\bullet)$, $E^1_{m,k} \simeq C_m(Y, -\xi_Y, I\mathcal{H}^{\bar{q}}_k(cL, L, \mathbb{Z}))$. For the E^2-term we get again $E^2_{m,k} \simeq H_m(Y, I\mathcal{H}^{\bar{q}}_k(cL, L, \mathbb{Z}))$.

We now choose the triangulation K of Y such that in addition it is compatible with the stratification of Y by the unstable manifolds of the gradient-like vector field $-\xi_Y$. This is possible by the triangulability of abstract stratified spaces (see [81]) and since, by [42, Proposition 2], the manifold Y with the unstable cell decomposition is an abstract stratified space. We then have a natural map of filtered chain complexes

$$(C^{\bar{q},\mathrm{pert}}_\bullet(T, -\xi, -\eta, \mathbb{Z}), \partial_\bullet) \longrightarrow (IC^{\bar{q}}_\bullet(T, \partial T, \mathbb{Z}), \partial_\bullet). \tag{5.25}$$

From the above discussion, the induced map on the E^1-terms of the associated spectral sequences is the natural quasi-isomorphism

$$(C_\bullet(Y, -\xi_Y, I\mathcal{H}^{\bar{q}}_k(cL, L, \mathbb{Z})), \partial_\bullet) \longrightarrow (C_\bullet(Y, I\mathcal{H}^{\bar{q}}_k(cL, L, \mathbb{Z})), \partial_\bullet), \tag{5.26}$$

and the induced map on the E_2-term is an isomorphism.

The isomorphism (5.23) now follows using [58, Theorem 3.2]. The isomorphism (5.24) follows from (5.23) and the universal coefficient theorem. □

We denote by

$$(5.27)\quad (C^{\bullet}_{\overline{p},\mathrm{pert}}(T,\xi,\eta),\partial^{\bullet}) := (C^{\overline{q},\mathrm{pert}}_{n-\bullet}(T,-\xi,-\eta,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R},(-1)^{\bullet+1}\partial_{n-\bullet}).$$

By Poincaré duality for intersection (co-)homology (see [30, Section 3.3]) and Proposition 5.2,

$$(5.28)\qquad H^{\bullet}(C^{\bullet}_{\overline{p},\mathrm{pert}}(T,\xi,\eta),\partial^{\bullet}) \simeq IH^{\bullet}_{\overline{p}}(T).$$

5.3. The cohomology of the local Morse–Bott type cochain complex

The situation is as in the beginning of Section 5. We denote by $(C^{\bullet}(\widetilde{T},\widetilde{\xi}),\partial^{\bullet})$ the smooth Morse–Bott complex associated to the Morse–Bott vector field $\widetilde{\xi}$ on $\widetilde{T}$; it is generated by the de Rham complexes $(\Omega^{\bullet}(L_x),d)$, $x\in\mathrm{Crit}(\xi_Y)$. By [6, Proposition 3.10] we have a quasi-isomorphism of chain complexes

$$(5.29)\qquad F:(C^{\bullet}(\widetilde{T},\widetilde{\xi}),\partial^{\bullet}) \longrightarrow (C^{\bullet}(\widetilde{T},\widetilde{\xi}_{\eta}),\partial^{\bullet})$$

defined as follows: For $x\in\mathrm{Crit}_i(\xi_Y)$ and $\omega\in\Omega^j(L_x)\subset C^{i,j}(\widetilde{T},\widetilde{\xi})\subset C^{i+j}(\widetilde{T},\widetilde{\xi})$ we define (5.29) by

$$(5.30)\qquad F(\omega)=\sum_{\substack{\alpha\in\mathrm{Crit}_{i+j}(\widetilde{\xi}_{\eta})\\ \theta(\alpha)=x}}\left(\int_{\overline{W^u(\alpha,\eta_x)}}\omega\right)[W^u(\alpha,\widetilde{\xi}_{\eta})]^*.$$

Proposition 5.3. — *The map* (5.29) *restricts to a map, still denoted by* F,

$$(5.31)\qquad F:(C^{\bullet}_{\overline{p},\mathrm{loc}}(T,\xi),\partial^{\bullet}) \longrightarrow (C^{\bullet}_{\overline{p},\mathrm{pert}}(T,\xi,\eta),\partial^{\bullet}).$$

Proof. — Let $x\in\mathrm{Crit}_i(\xi_Y)$ and $\omega\in\Omega^j(L_x)$. We denote by $F(\omega)^s\in C_{n-(i+j)}(\widetilde{T},-\widetilde{\xi}_{\eta},\mathbb{Z})\otimes\mathbb{R}$ the element in $C_{n-(i+j)}(\widetilde{T},-\widetilde{\xi}_{\eta},\mathbb{Z})\otimes\mathbb{R}$ corresponding to $F(\omega)\in C^{i+j}(\widetilde{T},\widetilde{\xi}_{\eta})$.

For $\deg(\omega)=j\leqslant q_c$ we have

$$(5.32)\qquad \dim(\theta(F(\omega)^s)\cap Y)\leqslant(n-c)-i\leqslant n-(i+j)-c+q_c.$$

Hence $\theta(F(\omega)^s)$ is $(\overline{q},n-(i+j))$-allowable. This shows that, for $\deg(\omega)<q_c$, $F(\omega)\in C^{\bullet}_{\overline{p},\mathrm{pert}}(T,\xi,\eta)$.

Let now $\deg(\omega)=j=q_c$ and $d\omega=0$ (see (5.1), (5.2)). We have, using the definition of the boundary operator in the various complexes involved, as well as Stokes' theorem,

$$
\begin{aligned}
\partial^\bullet F(\omega) &= \pm \sum_{\substack{\alpha\in \mathrm{Crit}_{i+j}(\widetilde{\xi}_\eta)\\ \theta(\alpha)=x}} \left(\int_{\overline{W^u(\alpha,\eta_x)}} \omega\right) \\
&\qquad\qquad \sum_{\beta\in \mathrm{Crit}_{i+j+1}(\widetilde{\xi}_\eta)} n(\alpha,\beta)[W^u(\beta,\widetilde{\xi}_\eta)]^* \\
&= \pm \sum_{\beta,\theta(\beta)=x} \left(\int_{\sum_\alpha n(\alpha,\beta)\overline{W^u(\alpha,\eta_x)}} \omega\right)[W^u(\beta,\widetilde{\xi}_\eta)]^* \\
&\qquad\qquad \pm \sum_{\substack{\alpha,\beta\\ \theta(\beta)\neq x}} n(\alpha,\beta)\left(\int_{\overline{W^u(\alpha,\eta_x)}} \omega\right)[W^u(\beta,\widetilde{\xi}_\eta)]^* \\
&= \pm \sum_{\beta,\theta(\beta)=x} \left(\int_{\partial_\bullet\overline{W^u(\beta,\eta_x)}} \omega\right)[W^u(\beta,\widetilde{\xi}_\eta)]^* \\
&\qquad\qquad \pm \sum_{\substack{\alpha,\beta\\ \theta(\beta)\neq x}} n(\alpha,\beta)\left(\int_{\overline{W^u(\alpha,\eta_x)}} \omega\right)[W^u(\beta,\widetilde{\xi}_\eta)]^* \\
&= \pm \sum_{\beta,\theta(\beta)=x} \left(\int_{\overline{W^u(\beta,\eta_x)}} d\omega\right)[W^u(\beta,\widetilde{\xi}_\eta)]^* \\
&\qquad\qquad \pm \sum_{\substack{\alpha,\beta\\ \theta(\beta)\neq x}} n(\alpha,\beta)\left(\int_{\overline{W^u(\alpha,\eta_x)}} \omega\right)[W^u(\beta,\widetilde{\xi}_\eta)]^* \\
&= \pm \sum_{\substack{\alpha,\beta\\ \theta(\beta)\neq x}} n(\alpha,\beta)\left(\int_{\overline{W^u(\alpha,\eta_x)}} \omega\right)[W^u(\beta,\widetilde{\xi}_\eta)]^*.
\end{aligned}
\tag{5.33}
$$

Note that for all β in the above sum, by Morse–Smale transversality, we have $\mathrm{ind}(\theta(\beta),\xi_Y)\geqslant i+1$. Therefore from (5.33)

$$
\dim(\partial\theta(F(\omega)^s)\cap Y)\leqslant (n-c)-(i+1)=n-(i+j+1)-c+q_c. \tag{5.34}
$$

This shows that $\partial\theta(F(\omega)^s)$ is $(\bar{q},n-(i+j+1))$-allowable, hence $F(\omega)\in C^\bullet_{\overline{p},\mathrm{pert}}(T,\xi,\eta)$. □

Corollary 5.4. — *The map of filtered cochain complexes* (5.31) *induces an isomorphism on the E_1-terms of the spectral sequences associated to the two filtered complexes. Hence F is a quasi-isomorphism and*

$$
H^\bullet(C^\bullet_{\overline{p},\mathrm{loc}}(T,\xi),\partial^\bullet)\simeq H^\bullet(C^\bullet_{\overline{p},\mathrm{pert}}(T,\xi,\eta),\partial^\bullet)\simeq IH^\bullet_{\overline{p}}(T). \tag{5.35}
$$

Proof. — The map induced from F on the graded complexes is essentially given by the sum of the "truncated" de Rham maps in smooth Morse theory for the gradient-like vector field η_x on the link L_x, $x \in \mathrm{Crit}(\xi_Y)$,

$$(5.36)\qquad \begin{aligned} (\Omega^\bullet_{\overline{q},tr}(L_x), d) &\longrightarrow (C^\bullet_{\overline{p},\mathrm{pert}}(cL_x, -r\partial r, \eta_x), \partial^\bullet) \\ \omega &\longmapsto \sum_{\alpha \in \mathrm{Crit}(\eta_x)} \left(\int_{\overline{W^u(\alpha,\eta_x)}} \omega \right) [W^u(\alpha, \eta_x)]^*. \end{aligned}$$

It is easy to check using smooth Morse theory (see e.g. [9, Theorem 2.9]), that (5.36) is a quasi-isomorphism. Hence the map induced from F on the E_1-term of the spectral sequences associated to the two filtered complexes is an isomorphism. The isomorphism (5.35) follows from the first claim, (5.11), (5.28) and [58, Theorem 3.2]. □

6. Bismut–Zhang Torsion

The aim of this section is to define the *Bismut–Zhang torsion* of a compact stratified pseudomanifold for the lower middle perversity $\overline{m}$. The definition of the Bismut–Zhang torsion uses the Morse–Bott type cochain complex $(C^\bullet_{\overline{m}}(X, \xi), \partial^\bullet)$ for the lower middle perversity $\overline{m}$ defined in Section 4.2. The second ingredient is the model Witten Laplacian appearing in the generalisation of the Witten deformation to stratified spaces in [49]. Conjecturally the Bismut–Zhang torsion will serve as the "topological" side in a Cheeger–Müller theorem for singular spaces with iterated conical singularities. Indeed, for Witt spaces with isolated conical singularities, the identity between analytic and Bismut–Zhang torsion has been established in [51].

The Bismut–Zhang torsion for the upper middle perversity $\overline{n}$ can be defined analogously (see Remarks 6.2 and 6.4(a)). In case of a Witt space (see [77, Definition 2.1]), hence in particular for a space having strata of even codimension only, the Bismut–Zhang torsion for upper and lower middle perversity coincide.

Section 6 is organised as follows: In Section 6.1, for convenience of the reader, we recall some preliminaries on torsion. In Section 6.2 we define the Bismut–Zhang torsion for a stratified pseudomanifold with two strata. The Bismut–Zhang torsion is not a topological invariant in general. The aim of Sections 6.3 and 6.4 is the proof of Main Theorem 2, i.e. we prove that, in case of even-codimensional singular stratum, the Bismut–Zhang torsion is a topological invariant of the pseudomanifold X and moreover is equal to the intersection Reidemeister torsion defined by Aparna Dar [19].

Here, for the clarity of presentation, we give the definition of the Bismut–Zhang torsion only for pseudomanifolds with two strata. However, as mentioned in the introduction, the Bismut–Zhang torsion can be defined in a completely analogous way for stratified pseudomanifolds of arbitrary depth (see Remark 6.4(b)). The results in Sections 6.3 and 6.4 can not be extended beyond the case of two strata at the present: they rely on the Cheeger–Müller theorem for the smooth links.

6.1. Preliminaries on Torsion

In this section for convenience of the reader, we recall some basics about torsion: In Section 6.1.1 we recall the definition of the torsion of a finite Hilbert cochain complex, i.e. a cochain complex $(C^\bullet, \partial^\bullet)$ of finite dimensional Hilbert spaces, such that $C^i = 0$ for $|i| \geqslant N$ for some $N \in \mathbb{N}$. We also recall the definition of the *Milnor- and Reidemeister torsion* of a smooth manifold. Moreover, we recall the concept of a *simple structure on an infinite dimensional cochain complex* as introduced by Lück, Schick and Thielmann in [47, Section 2]. In Section 6.1.2 we recall results of [47] on the torsion of a filtered complex; they will be used in the definition of the Bismut–Zhang torsion in Section 6.2 and for the proof of its topological invariance in Sections 6.3 and 6.4.

6.1.1. The torsion of a cochain complex

For $f : V \to W$ a linear isomorphism of real finite-dimensional Hilbert spaces, set

$$\llbracket f \rrbracket \coloneqq \frac{1}{2} \ln(|\det(f^* f)|) \in \mathbb{R}. \tag{6.1}$$

For an acyclic finite Hilbert cochain complex $(C^\bullet, \partial^\bullet)$, define

$$\rho(C^\bullet, \partial^\bullet) \coloneqq \llbracket (\partial^\bullet + \Upsilon^\bullet) : C^{\mathrm{ev}} \longrightarrow C^{\mathrm{odd}} \rrbracket \in \mathbb{R}, \tag{6.2}$$

where $\Upsilon^\bullet$ is a cochain contraction. Definition (6.2) is independent of the choice of the cochain contraction $\Upsilon^\bullet$. For a cochain homotopy equivalence of finite Hilbert complexes $f : C^\bullet \to D^\bullet$, the cone $(\mathrm{Cone}(f)^\bullet, \partial^\bullet)$ is the acyclic cochain complex with n-th differential

$$\begin{pmatrix} \partial^n_{C^\bullet} & 0 \\ f^n & -\partial^{n-1}_{D^\bullet} \end{pmatrix} : C^n \oplus D^{n-1} \longrightarrow C^{n+1} \oplus D^n. \tag{6.3}$$

Let $(C^\bullet, \partial^\bullet)$ be a finite Hilbert cochain complex such that its cohomology $H^\bullet(C^\bullet, \partial^\bullet)$ carries a Hilbert structure. We can consider $H^\bullet(C^\bullet, \partial^\bullet)$ as a cochain complex with trivial differential. There is up to homotopy precisely one cochain map $i : H^\bullet(C^\bullet, \partial^\bullet) \to (C^\bullet, \partial^\bullet)$ with $H^\bullet(i) = \mathrm{id}$. Define

$$\rho(C^\bullet, \partial^\bullet) := -\rho(\mathrm{Cone}(i)^\bullet, \partial^\bullet) \in \mathbb{R}. \tag{6.4}$$

The minus sign in (6.4) ensures that the definitions (6.2) and (6.4) coincide for acyclic complexes. If we fix an orthonormal basis of $(C^\bullet, \partial^\bullet)$ and of $H^\bullet(C^\bullet, \partial^\bullet)$, then the logarithm of the torsion defined in [59, page 365] is equal to (6.4).

The Milnor and Reidemeister torsion of a smooth compact manifold Y are defined as follows: Let $(\mathcal{F}, \nabla^{\mathcal{F}}, g^{\mathcal{F}})$ be a flat unimodular vector bundle, i.e. $\mathcal{F}$ is a flat vector bundle, $\nabla^{\mathcal{F}}$ is the canonical flat connection and $g^{\mathcal{F}}$ is a Hermitian metric on $\mathcal{F}$, such that the induced metric on the flat line bundle $\det \mathcal{F}$ is flat. Let ξ_Y be a smooth gradient-like vector field on Y. We denote by $(C^\bullet(Y, \xi_Y, \mathcal{F}), \partial^\bullet)$ the Morse–Thom–Smale cochain complex (associated to the manifold Y and the vector field ξ_Y) with values in the flat vector bundle $\mathcal{F}$ (see e.g. [9, Section I (c)]). The Hermitian metric $g^{\mathcal{F}}$ induces a Hilbert structure on the Morse–Thom–Smale complex $(C^\bullet(Y, \xi_Y, \mathcal{F}), \partial^\bullet)$ (see [9, Section I (d)]). We equip the cohomology $H^\bullet(C^\bullet(Y, \xi_Y, \mathcal{F}), \partial^\bullet) \simeq H^\bullet(Y, \mathcal{F})$ with a metric h. The Milnor torsion of the manifold Y with coefficients in $\mathcal{F}$ (see [59, Section 8]) is defined as

$$\rho_M(Y, \mathcal{F}, h) := \rho(C^\bullet(Y, \xi_Y, \mathcal{F}), \partial^\bullet). \tag{6.5}$$

The Reidemeister torsion $\rho_R(Y, \mathcal{F}, h)$ (see [23, 74]) can be defined analogously to the Milnor torsion, using the complex of simplicial cochains w.r.t. a triangulation of Y instead of the Morse–Thom–Smale complex. For a flat unimodular vector bundle $(\mathcal{F}, \nabla^{\mathcal{F}}, g^{\mathcal{F}})$, the Reidemeister torsion $\rho_R(Y, \mathcal{F}, h)$ is a combinatorial invariant of Y (see [59, Theorem 7.1 and Section 8] and [9, Remark 1.10]). Moreover, Milnor and Reidemeister torsion are equal (see [59, Theorem 9.3] and [9, Remark 1.10]). Hence, the Milnor torsion $\rho_M(Y, \mathcal{F}, h)$ is a combinatorial invariant of Y and in particular does not dependent on the choice of the gradient-like vector field ξ_Y (see also [9, Theorem 16.1]). In case $\mathcal{F} = \mathbb{R}$, i.e. $\mathcal{F}$ is the trivial vector bundle of rank 1 equipped with the trivial connection and the canonical metric, we write simply $\rho_M(Y, h)$ (resp. $\rho_R(Y, h)$) for the Milnor torsion (resp. the Reidemeister torsion).

The concept of torsion of a cochain complex has been extended to infinite dimensional cochain complexes by Lück, Schick and Thielmann [47, Section 2] as follows: A simple structure on a cochain complex $(C^\bullet, \partial^\bullet)$ is an equivalence class of cochain homotopy equivalences $u : (\overline{C}^\bullet, \partial^\bullet) \to (C^\bullet, \partial^\bullet)$ with a finite Hilbert cochain complex as source. Hereby two cochain homotopy equivalences $u : (\overline{C}^\bullet, \partial^\bullet) \to (C^\bullet, \partial^\bullet)$ and $v : \big(\overline{\overline{C}}^\bullet, \partial^\bullet\big) \to (C^\bullet, \partial^\bullet)$ are equivalent if $\rho(\mathrm{Cone}^\bullet(v^{-1} \circ u), \partial^\bullet) = 0$.

Let $(C^\bullet, \partial^\bullet)$ be a cochain complex with a simple structure and such that $H^\bullet(C^\bullet, \partial^\bullet)$ carries a Hilbert structure. Define

$$\rho(C^\bullet, \partial^\bullet) := \rho(\overline{C}^\bullet, \partial^\bullet), \tag{6.6}$$

for any representative of the simple structure $u : (\overline{C}^\bullet, \partial^\bullet) \to (C^\bullet, \partial^\bullet)$, where we use the Hilbert structure on $H^\bullet(\overline{C}^\bullet, \partial^\bullet)$ for which $H^\bullet(u)$ is an isometry.

Let $0 \to (C^\bullet, \partial^\bullet) \to (D^\bullet, \partial^\bullet) \to (E^\bullet, \partial^\bullet) \to 0$ be an exact sequence of finite Hilbert cochain complexes. Then $C^n \to D^n \to E^n$, $n \in \mathbb{Z}$, can be seen as an acyclic finite Hilbert cochain complex and we define

$$\rho(C^\bullet \longrightarrow D^\bullet \longrightarrow E^\bullet) := \sum_{n\in\mathbb{Z}} (-1)^n \rho(C^n \longrightarrow D^n \longrightarrow E^n). \tag{6.7}$$

The definition (6.7) can be extended to an exact sequence of cochain complexes with simple structures in the obvious way (see [47, (2.5)]).

6.1.2. Torsion and spectral sequences

The study of torsion for filtered complexes is due to Freed [25] and to Lück, Schick and Thielmann [47]; here we follow [47, Section 4]. In Theorem 6.1 below we recall a result of [47, Section 4] on the torsion of a filtered cochain complex, which will be useful in the sequel, more precisely in Proposition 6.8 and Section 6.4.2.

Let $(C^\bullet, \partial^\bullet)$ be a cochain complex with filtration

$$0 = F_N^\bullet \subset \cdots \subset F_0^\bullet = C^\bullet. \tag{6.8}$$

The associated spectral cohomology sequence $(E_\bullet^{\bullet,\bullet}, d_\bullet^{\bullet,\bullet})$ converges to the cohomology $H^\bullet(C^\bullet, \partial^\bullet)$.

Assume that the following data are given

(a) Simple structures on the graded complexes $F_m^\bullet / F_{m+1}^\bullet$, $m \in \mathbb{N}_0$,
(b) a Hilbert structure on the cohomology $H^\bullet(C^\bullet, \partial^\bullet)$.

Using the data (a), we can define inductively simple structures on the complexes $(F_m^\bullet, \partial^\bullet)$, $m \in \mathbb{N}_0$, by requiring that

$$\rho(F_{m+1}^\bullet \longrightarrow F_m^\bullet \longrightarrow F_m^\bullet / F_{m+1}^\bullet) = 0, \tag{6.9}$$

(see [47, Section 4.1]). Hence we get in particular a simple structure on $(C^\bullet, \partial^\bullet)$ induced from the data (a). Using (a) and (b) we can therefore define $\rho(C^\bullet, \partial^\bullet)$.

Set $K^{m,k} := \mathrm{im}(H^{m+k}(F^\bullet_m, \partial^\bullet) \to H^{m+k}(C^\bullet, \partial^\bullet))$. There is a natural isomorphism

$$\Psi^{m,k} : K^{m,k}/K^{m+1,k-1} \longrightarrow E^{m,k}_\infty. \tag{6.10}$$

We equip $K^{m,k} \subset H^{m+k}(C^\bullet, \partial^\bullet)$ with the Hilbert substructure and the quotient $K^{m,k}/K^{m+1,k-1}$ with the Hilbert quotient structure. The same can be done iteratively for $E^{m,k}_r$ and $H^\bullet(E_r^{m+r\bullet,k-(r-1)\bullet})$.

Theorem 6.1. — *In the situation described above, we have*

$$\rho(C^\bullet, \partial^\bullet) = \sum_m \rho(F^\bullet_m/F^\bullet_{m+1}) + \sum_k (-1)^k \rho(E_1^{\bullet,k}) + \rho^{\geqslant 2}(C^\bullet, \partial^\bullet), \tag{6.11}$$

where

$$\begin{aligned}\rho^{\geqslant 2}(C^\bullet, \partial^\bullet) := &\sum_{r\geqslant 2} \sum_{m=0}^{r-1} \sum_k (-1)^{m+k} \rho(E_r^{m+r\bullet,k-(r-1)\bullet}) \\ &- \sum_{m,k} (-1)^{m+k} [\![\Psi^{m,k}]\!].\end{aligned} \tag{6.12}$$

6.2. Definition of the Bismut–Zhang torsion

6.2.1. The model Witten Laplacian

Next to the Morse–Bott type cochain complex constructed in Section 4.2, the second ingredient needed for the definition of the Bismut–Zhang torsion is the model Witten Laplacian, which we recall in this subsection.

The smooth model Witten Laplacian appears as model operator in the Witten deformation of the de Rham complex of a smooth compact manifold using a smooth Morse function (see [38] and [84]).

The generalisation of the "easy" part of Witten's programme, i.e. the analytic proof of the Morse inequalities, to spaces with iterated conical singularities and radial/anti-radial Morse functions has been studied in [5] and [49]; the main idea is to deform the complex of L^2-forms (instead of the de Rham complex). In [49] also the "hard" part of Witten's programme, i.e. the comparison between the Witten complex and an appropriate singular Morse–Thom–Smale complex, has been addressed for spaces with isolated conical singularities.

The model Witten Laplacian and its torsion zeta function. The smooth case. We start by recalling the well known model Witten Laplacian $\mathbf{\Delta}_{W,d,m}$ for a smooth Morse critical point of index m on $\mathbb{R}^d$ (see [84], [9, Proposition 8.2]). We equip the Euclidean space $\mathbb{R}^d$ with the Morse function

$$f_{d,m} = \frac{1}{2}(-z_1^2 - \cdots - z_m^2 + z_{m+1}^2 + \cdots + z_d^2) \tag{6.13}$$

and the Euclidean metric $dz_1^2 + \cdots + dz_d^2$. We denote by $T^*\mathbb{R}^d$ the cotangent bundle of $\mathbb{R}^d$. The Euclidean metric induces an L^2-metric on sections of $\Lambda^\bullet(T^*\mathbb{R}^d)$; we denote by $\mathrm{L}^2(\Lambda^\bullet(T^*\mathbb{R}^d))$ the vector space of L^2-sections of $\Lambda^\bullet(T^*\mathbb{R}^d)$.

Let $0 \leqslant l \leqslant d$ and let $I = (i_1, \ldots, i_l) \in \{1, \ldots, d\}^l$ be a multi-index of length l. Set $dz_I \coloneqq dz_{i_1} \wedge \cdots \wedge dz_{i_l}$. We denote by N^+ resp. by N^- the number operators defined by

$$N^+ dz_I \coloneqq \#\{s \mid m+1 \leqslant i_s \leqslant d\} \cdot dz_I, \quad N^- dz_I \coloneqq (l - N^+) \cdot dz_I. \tag{6.14}$$

We denote by $\Delta_{\mathbb{R}^d} = -\sum_{j=1}^d \frac{\partial^2}{\partial z_j^2}$ the (geometric) Laplacian acting on smooth compactly supported sections of $\Lambda^\bullet(T^*\mathbb{R}^d)$. For $z \in \mathbb{R}^d$, set $|z| \coloneqq (z_1^2 + \cdots + z_d^2)^{1/2}$. The model Witten Laplacian $\mathbf{\Delta}_{W,d,m}$ is defined as the closure in $\mathrm{L}^2(\Lambda^\bullet(T^*\mathbb{R}^d))$ of the following operator acting on smooth compactly supported sections of $\Lambda^\bullet(T^*\mathbb{R}^d)$:

$$\begin{aligned} \Delta_{\mathbb{R}^d} + |z|^2 - d + 2(m + N^+ - N^-) \\ = \sum_{j=1}^d \left(-\frac{\partial^2}{\partial z_j^2} + z_j^2 \right) - d + 2(m + N^+ - N^-). \end{aligned} \tag{6.15}$$

We denote by $\mathbf{\Delta}_{W,d,m}^{(k)}$ the restriction of $\mathbf{\Delta}_{W,d,m}$ to k-forms, $k = 0, \ldots, d$. By [84] (see also [9, Proposition 8.2]), $\operatorname{spec}(\mathbf{\Delta}_{W,d,m}) = 2\mathbb{N}_0$ and

$$\ker(\mathbf{\Delta}_{W,d,m}^{(k)}) \simeq H^k(D^m, S^{m-1}), \; k = 0, \ldots, d, \tag{6.16}$$

where D^m denotes an m-dimensional closed ball and S^{m-1} denotes its boundary. We explain the isomorphism (6.16) in more detail: Both sides of (6.16) are non-trivial in degree m only. An orthonormal basis of $\ker(\mathbf{\Delta}_{W,d,m}^{(m)})$ w.r.t. the L^2-metric (induced from the Euclidean metric on $\mathbb{R}^d$) is given by

$$\alpha^{d,m} \coloneqq \pi^{-d/4} e^{-|z|^2/2} dz_1 \wedge \cdots \wedge dz_m. \tag{6.17}$$

The isomorphism $\ker(\mathbf{\Delta}_{W,d,m}^{(m)}) \simeq H^m(D^m, S^{m-1}) \simeq \mathbb{R}$ is given by multiplication with $e^{f_{d,m}}$ followed by integration (along the unstable manifold of the critical point 0 w.r.t. the flow induced from the negative gradient vector field $-\nabla f_{d,m}$):

$$\alpha^{d,m} \longmapsto e^{f_{d,m}}\alpha^{d,m} \longmapsto \int_{\mathbb{R}^m} e^{f_{d,m}}\alpha^{d,m} = \pi^{(2m-d)/4}. \tag{6.18}$$

We denote by $\mathbf{\Delta}^{\perp}_{W,d,m}$ the restriction of $\mathbf{\Delta}_{W,d,m}$ to the orthogonal complement of $\ker(\mathbf{\Delta}_{W,d,m})$. We denote by $N := N^+ + N^-$ the number operator acting on sections of $\Lambda^\bullet(T^*\mathbb{R}^d)$ by multiplication by the form degree. For $\Re(s) > d/2$, the zeta function

$$s \longmapsto \zeta_{d,m}(s) := -\mathrm{Tr_s}\left[N\left(\mathbf{\Delta}^{\perp}_{W,d,m}\right)^{-s}\right] \tag{6.19}$$

is a well-defined holomorphic function (see [9], [76, Section 2]). Moreover $\zeta_{d,m}$ extends to a meromorphic function on the whole complex plane, which is holomorphic at $s=0$. We have (see [9], [76, Section 2])

$$\zeta'_{d,m}(0) = (-1)^m\left(m - \frac{d}{2}\right)\ln\pi. \tag{6.20}$$

The model Witten Laplacian and its torsion zeta function. The case of isolated singularities. In this paragraph we explain the model Witten Laplacian for an infinite cone cL over a smooth compact connected manifold L of dimension $\dim L = c-1$. We equip $(cL)^{sm}$ with the radial Morse function $f_{cL} = \frac{1}{2}r^2$ and the conical Riemannian metric

$$dr^2 + r^2 g^{TL}, \tag{6.21}$$

where g^{TL} is a Riemannian metric on the smooth link manifold L not depending on r. We denote by T^*cL the cotangent bundle of $(cL)^{sm}$ and by $\mathrm{L}^2(\Lambda^\bullet(T^*cL))$ the vector space of L^2-sections of $\Lambda^\bullet(T^*cL)$ w.r.t. the L^2-metric induced from the Riemannian metric (6.21).

Let $\mathbf{\Delta}_{cL}$ be the Laplacian acting on smooth compactly supported sections of $\Lambda^\bullet(T^*cL)$. Recall that, unlike in the smooth situation, $\mathbf{\Delta}_{cL}$ admits several closed self adjoint extensions in $\mathrm{L}^2(\Lambda^\bullet(T^*cL))$. We denote by d (resp. by δ) the outer differential (resp. its adjoint w.r.t. the L^2-metric induced from the conical metric (6.21)) acting on smooth compactly supported sections of $\Lambda^\bullet(T^*cL)$. Their minimal resp. maximal extensions are denoted by $d_{\min/\max}$, $\delta_{\min/\max}$. We denote by N the number operator acting on sections of $\Lambda^\bullet(T^*cL)$ by multiplication by the form degree. The model Witten Laplacian on cL is defined as

$$\mathbf{\Delta}_{W,cL} := \mathbf{\Delta}_{cL} - (c - 2N) + r^2, \tag{6.22}$$

with domain the following subspace of $\mathrm{L}^2(\Lambda^\bullet(T^*cL))$

$$\operatorname{dom}(\mathbf{\Delta}_{W,cL}) = \left\{ \omega \;\middle|\; \begin{array}{l} \omega \in \operatorname{dom}(d_{\max}) \cap \operatorname{dom}(\delta_{\min}), d_{\max}\omega \in \operatorname{dom}\delta_{\min}, \\ \delta_{\min}\omega \in \operatorname{dom} d_{\max} \text{ locally near the cone point} \end{array} \right\}. \tag{6.23}$$

(The reader is encouraged to compare the definition of the singular model Witten Laplacian (6.22) with the definition of the smooth model Witten Laplacian (6.15) in case $m = 0$.)

By [52, Section 4] the model Witten Laplacian has discrete spectrum. We denote by $\mathbf{\Delta}^{(k)}_{W,cL}$ the restriction of $\mathbf{\Delta}_{W,cL}$ to k-forms. We denote by $\operatorname{Harm}^\bullet(L)$ the vector space of harmonic forms on (L, g^{TL}). By [49, Theorem 4.2], the local calculation for intersection (co-)homology with lower middle perversity (recalled in (2.14)) and the Hodge theorem for the smooth manifold (L, g^{TL}),

$$\begin{aligned} \ker(\mathbf{\Delta}^{(k)}_{W,cL}) \simeq IH^k_{\overline{m}}(cL_x) &\simeq \begin{cases} H^k(L) & \text{for } k < \frac{c}{2}, \\ 0 & \text{else}, \end{cases} \\ &\simeq \begin{cases} \operatorname{Harm}^k(L) & \text{for } k < \frac{c}{2}, \\ 0 & \text{else}. \end{cases} \end{aligned} \tag{6.24}$$

For later use (see the proof of Proposition 6.6), we explain the isomorphism (6.24) in more detail: Let $\{\beta^k_i\}_i$ be an ON-basis (w.r.t. the L^2-metric induced from g^{TL}) of $\operatorname{Harm}^k(L)$. For $0 \leqslant k < c/2$, an ON-basis of $\ker(\mathbf{\Delta}^{(k)}_{W,cL})$ w.r.t. the L^2-metric induced from the conical metric (6.21) is given by

$$\left\{ \gamma^k_i := \left(\Gamma(c/2 - k)/2\right)^{-1/2} e^{-r^2/2} \beta^k_i \;\middle|\; i = 1, \dots, \dim H^k(L) \right\}. \tag{6.25}$$

For $0 \leqslant k < c/2$, the isomorphism (6.24) is given by multiplication with $e^{f_{cL}}$ followed by evaluation at $r = 0$,

$$\gamma^k_i \longmapsto e^{f_{cL}} \gamma^k_i \longmapsto \left(\Gamma(c/2 - k)/2\right)^{-1/2} \beta^k_i. \tag{6.26}$$

We denote by $\mathbf{\Delta}^{\perp}_{W,cL}$ the restriction of $\mathbf{\Delta}_{W,cL}$ to the orthogonal complement of $\ker(\mathbf{\Delta}_{W,cL})$. By [52, Theorem I], for $\Re(s) \gg 0$, the torsion zeta function

$$s \longmapsto \zeta_{cL}(s) := -\operatorname{Tr_s}\left[N \left(\mathbf{\Delta}^{\perp}_{W,cL} \right)^{-s} \right] \tag{6.27}$$

is a well-defined holomorphic function, which can be described explicitly in terms of the spectrum of the Laplacian on L. Moreover ζ_{cL} extends to a meromorphic function on the whole complex plane, which is holomorphic at $s = 0$.

(In [52] the Witt condition and a spectral Witt condition are assumed, however the cited results also hold without these assumptions.)

Remark 6.2. — To define the Bismut–Zhang torsion for the upper middle perversity $\bar{n}$ we have to use a different self-adjoint extension $\mathbf{\Delta}'_{W,cL}$ of (6.22), namely the operator with domain the following subset of $\mathrm{L}^2(T^*cL)$:

$$\begin{aligned}(6.28)\quad &\mathrm{dom}(\mathbf{\Delta}'_{W,cL})\\ &= \left\{\omega \;\middle|\; \begin{array}{l}\omega \in \mathrm{dom}(d_{\min}) \cap \mathrm{dom}(\delta_{\max}), d_{\min}\omega \in \mathrm{dom}\,\delta_{\max},\\ \delta_{\max}\omega \in \mathrm{dom}\, d_{\min} \text{ locally near the cone point}\end{array}\right\}.\end{aligned}$$

The model Witten Laplacian and its torsion zeta function. The general case. Let L be a smooth compact connected manifold of dimension $\dim L = c-1$. In this paragraph, we explain the model Witten Laplacian for a critical point of index m of a radial Morse function on $\mathbb{R}^{n-c} \times cL$, i.e. we equip $\mathbb{R}^{n-c} \times cL$ with the Morse function

$$\begin{aligned}(6.29)\qquad f_{\mathrm{loc}}(z,\varphi,r) &= \frac{1}{2}(r^2 - z_1^2 - \cdots - z_m^2 + z_{m+1}^2 + \cdots + z_{n-c}^2)\\ &= f_{cL}(r) + f_{n-c,m}(z)\end{aligned}$$

and with a wedge metric $dz_1^2 + \cdots + dz_{n-c}^2 + dr^2 + r^2 g^{TL}$. Note that the negative gradient vector field $-\nabla f_{\mathrm{loc}}$ of the function f_{loc} with respect to the wedge metric is a vector field with a singularity of strong standard form at 0 in the sense of Definition 3.4.

The model Witten Laplacian is the self-adjoint operator

$$(6.30)\qquad \mathbf{\Delta}_W := \mathrm{Id}_{\mathrm{L}^2(T^*\mathbb{R}^{n-c})} \otimes \mathbf{\Delta}_{W,cL} + \mathbf{\Delta}_{W,n-c,m} \otimes \mathrm{Id}_{\mathrm{L}^2(T^*cL)},$$

with domain

$$\begin{aligned}(6.31)\quad &\mathrm{dom}(\mathbf{\Delta}_{W,n-c,m}) \otimes \mathrm{dom}(\mathbf{\Delta}_{W,cL})\\ &\subset \mathrm{L}^2(\Lambda^\bullet(T^*\mathbb{R}^{n-c})) \otimes \mathrm{L}^2(\Lambda^\bullet(T^*cL))\\ &= \mathrm{L}^2(\Lambda^\bullet(T^*(\mathbb{R}^{n-c} \times cL))).\end{aligned}$$

Again, we denote by $\mathbf{\Delta}_W^{(k)}$ the restriction of the model Witten Laplacian to k-forms. By [49, Section 4.4], the local calculation for intersection (co-)homology (recalled in (2.14)) and the Hodge theorem for the smooth manifold (L, g^{TL}),

$$(6.32)\qquad \ker(\mathbf{\Delta}_W^{(k+m)}) \simeq IH^k(cL) \simeq \begin{cases}\mathrm{Harm}^k(L) & \text{for } 0 \leqslant k < \frac{c}{2},\\ 0 & \text{else.}\end{cases}$$

For later use (see the proof of Proposition 6.8), we explain the isomorphism (6.32) in more detail. Let $\alpha^{n-c,m} \in \ker(\mathbf{\Delta}_{W,n-c,m})$ denote the generator of $\ker(\mathbf{\Delta}_{W,n-c,m})$ defined in (6.17). Let $\{\beta_i^k\}_i$ (resp. $\{\gamma_i^k\}_i$) be the ON-basis of $\mathrm{Harm}^k(L)$ (resp. of $\ker(\mathbf{\Delta}_{W,cL}^{(k)})$) described in the previous paragraph (see in particular (6.25)). For $0 \leqslant k < c/2$, an ON-basis of $\ker(\mathbf{\Delta}_W^{(k+m)})$ is given by

$$\left\{\omega_i^{k+m} \coloneqq \alpha^{n-c,m} \otimes \gamma_i^k \mid i = 1, \dots, \dim H^k(L)\right\}_i. \tag{6.33}$$

For $0 \leqslant k < c/2$, the isomorphism (6.32) is given by multiplication with $e^{f_{\mathrm{loc}}}$ followed by integration along the fibres of the unstable endpoint map for $-\nabla f_{\mathrm{loc}}$,

$$\begin{aligned} \omega_i^{k+m} \longmapsto e^{f_{\mathrm{loc}}}\omega_i^{k+m} \longmapsto \left(\int_{\mathbb{R}^m} e^{f_{n-c,m}}\alpha^{n-c,m}\right)\left(e^{f_{cL}}\gamma_i^k\right)_{r=0} \\ = \pi^{(2m+c-n)/4}\left(\Gamma(c/2-k)/2\right)^{-1/2}\beta_i^k. \end{aligned} \tag{6.34}$$

We denote by $\mathbf{\Delta}_W^{\perp}$ the restriction of $\mathbf{\Delta}_W$ to the orthogonal complement of $\ker(\mathbf{\Delta}_W)$. We denote by N the number operator acting on sections of $\Lambda^\bullet(T^*(\mathbb{R}^{n-c} \times cL))$ by multiplication by the form degree. By the results of the previous two paragraphs, for $\Re(s) >> 0$, the zeta function

$$s \longmapsto \zeta(s) \coloneqq -\mathrm{Tr_s}\left[N\left(\mathbf{\Delta}_W^{\perp}\right)^{-s}\right], \tag{6.35}$$

is a well-defined holomorphic function. Moreover ζ extends to a meromorphic function on the whole complex plane, which is holomorphic at $s = 0$. Using (6.16), (6.20), (6.24) and the product formula for the analytic torsion [73, Theorem 2.5], we have

$$\zeta'(0) = (-1)^m\zeta'_{cL}(0) + (-1)^m I\chi_{\overline{m}}(cL)\left(m - \frac{n-c}{2}\right)\ln\pi, \tag{6.36}$$

where $I\chi_{\overline{m}}(cL) \coloneqq \sum_{k=0}^{c}(-1)^k \dim IH^k_{\overline{m}}(cL)$ denotes the intersection Euler characteristic of cL.

6.2.2. Definition of the Bismut–Zhang torsion $\rho_{BZ}(X, \xi, g^{TL}, h_X)$

From now on, let X be an n-dimensional compact pseudomanifold with two strata $\{Y, Z \coloneqq X^{sm}\}$, $Y < Z$. For simplicity we assume that the singular stratum Y is connected. We denote by L its link and by T its closed tubular neighbourhood. Set $c \coloneqq \mathrm{codim}_X Y$.

Let $\xi = \{\xi_Y, \xi_Z\}$ be a stratified anti-radial gradient-like vector field on X. For all $x \in \mathrm{Crit}(\xi_Y)$, we fix a Riemannian metric g^{TL_x} on the smooth link manifold L_x. For $x \in \mathrm{Crit}(\xi_Y)$, we denote by $\mathbf{\Delta}_{W,x}$ (resp. by ζ_x) the model Witten Laplacian (resp. its torsion zeta function) defined in the last paragraph of Subsection 6.2.1. For $x \in \mathrm{Crit}_m(\xi_Z)$, we denote by $\boldsymbol{\Delta}_{W,x}$ (resp. by ζ_x) the smooth model Witten Laplacian for a smooth Morse critical point of index m on $\mathbb{R}^n$ (resp. its torsion zeta function).

(a) Let $(C^\bullet_{\overline{m}}(X,\xi), \partial^\bullet)$ be the filtered Morse–Bott type cochain complex associated to the stratified anti-radial gradient-like vector field ξ and the lower middle perversity (see Definition 4.8) with associated graded complex $G^\bullet_m$, $m \in \mathbb{N}_0$ (see Section 4.2.2). For $m \in \mathbb{N}_0$, we define a finite dimensional cochain complex

$$\begin{aligned}\overline{G}^\bullet_m &:= H^\bullet(G^\bullet_m) \\ &\simeq \left(\bigoplus_{x\in\mathrm{Crit}_m(\xi_Y)} IH^{\bullet-m}_{\overline{m}}(cL_x)\right) \oplus \left(\bigoplus_{x\in\mathrm{Crit}_m(\xi_Z)} H^\bullet(D^u_x, \partial D^u_x)\right),\end{aligned} \tag{6.37}$$

with trivial coboundary operator. We equip the cohomology of the complex $\overline{G}^\bullet_m$ with the Hilbert structure induced from the L^2-metric on $\oplus_{x\in\mathrm{Crit}_m(\xi)} \ker(\boldsymbol{\Delta}_{W,x})$ via (6.16) resp. the first isomorphism in (6.32). We equip the finite complex $\overline{G}^\bullet_m$ with a Hilbert structure, such that

$$\rho(\overline{G}^\bullet_m) = -\sum_{x\in\mathrm{Crit}_m(\xi)} \frac{1}{2}\zeta'_x(0). \tag{6.38}$$

We have an up to homotopy unique cochain homotopy equivalence $u : \overline{G}^\bullet_m \to G^\bullet_m$ with $H^\bullet(u) = \mathrm{id}$. This equips the graded complexes $G^\bullet_m$, $m \in \mathbb{N}_0$, with simple structures.

(b) We equip the cohomology $H^\bullet(C^\bullet_{\overline{m}}(X,\xi), \partial^\bullet) \simeq IH^\bullet_{\overline{m}}(X)$ with a Hilbert structure h_X.

As explained in Section 6.1.2, from the above data, we get a preferred simple structure $(\overline{C}^\bullet, \overline{\partial}^\bullet)$ on $(C^\bullet_{\overline{m}}(X,\xi), \partial^\bullet)$.

Definition 6.3. — *Let h_X be a Hilbert structure on $H^\bullet(C^\bullet_{\overline{m}}(X,\xi), \partial^\bullet) \simeq IH^\bullet_{\overline{m}}(X)$. Moreover, let $(C^\bullet_{\overline{m}}(X,\xi), \partial^\bullet)$ be equipped with the preferred simple structure $(\overline{C}^\bullet, \overline{\partial}^\bullet)$ as explained above. The Bismut–Zhang torsion of X is defined by*

$$\rho_{BZ}(X, g^{TL}, \xi, h_X) := \rho(C^\bullet_{\overline{m}}(X,\xi), \partial^\bullet) = \rho(\overline{C}^\bullet, \overline{\partial}^\bullet). \tag{6.39}$$

Remarks 6.4.

(a) The Bismut–Zhang torsion for the upper middle perversity $\overline{n}$ is defined analogously, using the complex $(C^\bullet_{\overline{n}}(X,\xi), \partial^\bullet)$ and the model Witten Laplacian from Remark 6.2. In case of a Witt space, the

Bismut–Zhang torsion for lower middle and upper middle perversity coincide, since the two ingredients used for their definition do coincide.

(b) Let L be a stratified pseudomanifold equipped with an iterated conical Riemannian metric g^{TL}. The model Witten Laplacian on cL resp. on $\mathbb{R}^{n-c} \times cL$ has also been studied in this more general situation [49, Sections 4.3 and 4.4]. One can study the associated zeta function, compute it (in terms of the spectrum of the transverse Laplacian) and prove holomorphicity in 0. In Section 4, the Morse–Bott type complex $(C^{\bullet}_{\overline{m}}(X,\xi), \partial^{\bullet})$ has been constructed for stratified pseudomanifolds of arbitrary depth. Hence both ingredients in the definition of the Bismut–Zhang torsion are available for stratified pseudomanifolds of arbitrary depth. The above definition of the Bismut–Zhang torsion can be generalised directly to the case of stratified pseudomanifolds of arbitrary depth.

(c) By definition, the Bismut–Zhang torsion depends on the anti-radial gradient-like vector field ξ and on the metrics g^{TL_x}, $x \in \mathrm{Crit}(\xi_Y)$. However, using the homotopy principle in smooth Morse theory as in [9, Section XIV], one can prove that

$$\rho_{BZ}(X, g^{TL}, \xi, h_X) = \rho_{BZ}(X, g^{TL}, \xi', h_X), \tag{6.40}$$

for a stratified anti-radial gradient-like vector field ξ' with $\xi' = \xi$ on T. Moreover (6.40) also holds, if we perturb ξ to $\xi' = -r\partial_r + \xi_Y$ in the tubular neighbourhood T.

(d) The definition of the Bismut–Zhang torsion bears some similarity to the definition of the smooth Milnor torsion. Using (6.18), (6.20) one can prove that the singular points of ξ_Z do contribute to ρ_{BZ} as in the classical definition of the Milnor torsion. However, the contribution of the singular points of ξ_Y is given by the analytic torsion of the model Witten Laplacian.

(e) Let (X, g^{TX}) be a singular space with isolated conical singularities, i.e. X^{sm} carries a Riemannian metric g^{TX} which, near the singularities, is of the form (6.21). By work of Cheeger [17, Section 1 and Theorem 5.1] and of Cheeger–Goresky–MacPherson [18, Section 3.4], we have the following Hodge-de Rham theorem:

$$IH^{\bullet}_{\overline{m}}(X) \simeq H^{\bullet}_{(2)}(X) \simeq \mathrm{Harm}^{\bullet}_{(2)}(X), \tag{6.41}$$

where $H^{\bullet}_{(2)}(X)$ denotes the L^2-cohomology of X and $\mathrm{Harm}^{\bullet}_{(2)}(X)$ denotes the vector space of L^2-harmonic forms on X. Let h_X be the metric on $H^{\bullet}_{(2)}(X)$ induced from the L^2-metric on $\mathrm{Harm}^{\bullet}_{(2)}(X)$

via the isomorphism (6.41). We denote by $|\ \ |^{RS}_{\det H^\bullet_{(2)}(X)}$ the Ray–Singer metric induced on the determinant line $\det H^\bullet_{(2)}(X)$ from h_X. The Bismut–Zhang metric $\|\ \ \|^{\xi,g}_{\det H^\bullet_{(2)}(X)}$ defined in [51] relates as follows to the Bismut–Zhang torsion defined here:

$$\rho^{BZ}(X, g^{TL}, \xi, h_X) = \pm \ln \left(\frac{\|\ \ \|^{\xi,g}_{\det H^\bullet_{(2)}(X)}}{|\ \ |^{RS}_{\det H^\bullet_{(2)}(X)}} \right). \tag{6.42}$$

Note that the general assumption in [51] is that X satisfies the Witt and a spectral Witt condition (see [51, Section 2.3]).

6.2.3. Gluing formula for the Bismut–Zhang torsion

The complexes $(C^\bullet(Z, \xi_Z), \partial^\bullet)$ and $(C^\bullet_{\overline{m},\mathrm{loc}}(T, \xi), \partial^\bullet)$ have been introduced at the beginning of Section 5. Adapting the construction of Section 6.2.2, we can equip the complex $(C^\bullet_{\overline{m},\mathrm{loc}}(T, \xi), \partial^\bullet)$ with a simple structure. We equip $H^\bullet(C^\bullet_{\overline{m},\mathrm{loc}}(T, \xi), \partial^\bullet) \simeq IH^\bullet_{\overline{m}}(T)$ (see Corollary 5.4) with a Hilbert structure h_T and define the torsion $\rho(C^\bullet_{\overline{m},\mathrm{loc}}(T, \xi), \partial^\bullet)$ proceeding as in Section 6.2.2.

We have a short exact sequence of complexes

$$0 \longrightarrow (C^\bullet(Z, \xi_Z), \partial^\bullet) \longrightarrow (C^\bullet_{\overline{m}}(X, \xi), \partial^\bullet) \longrightarrow (C^\bullet_{\overline{m},\mathrm{loc}}(T, \xi), \partial^\bullet) \longrightarrow 0. \tag{6.43}$$

Set $M := \overline{X \setminus T}$, which is a smooth manifold with boundary $\partial M = \partial T$. By smooth Morse theory, the Main Theorem 1 and Corollary 5.4 the long exact sequence in cohomology induced from (6.43) is isomorphic to

$$(\mathcal{L}^\bullet, \partial^\bullet) : \cdots \longrightarrow H^\bullet(M, \partial M) \longrightarrow IH^\bullet_{\overline{m}}(X) \longrightarrow IH^\bullet_{\overline{m}}(T) \longrightarrow \cdots . \tag{6.44}$$

We equip $IH^\bullet_{\overline{m}}(X)$ (resp. $H^\bullet(M, \partial M)$) with a metric h_X (resp. $h_{(M,\partial M)}$). We denote by $\rho(\mathcal{L}^\bullet, \partial^\bullet)$ the torsion of the long exact sequence (6.44).

Proposition 6.5.

$$\begin{aligned} \rho_{BZ}(X, g^{TL}, \xi, h_X) \\ = \rho(C^\bullet_{\overline{m},\mathrm{loc}}(T, \xi), \partial^\bullet) + \rho_R(M, \partial M, h_{(M,\partial M)}) + \rho(\mathcal{L}^\bullet, \partial^\bullet). \end{aligned} \tag{6.45}$$

Proof. — The claim follows by applying the gluing formula for torsion to the exact sequence (6.43) and using Remark 6.4(d). The gluing formula for torsion is due to Milnor [59, Theorem 3.2] for finite Hilbert cochain complexes; here we use the extension of Milnor's result to cochain complexes with simple structures in [47, Section 2.6]). □

6.3. The Bismut–Zhang torsion for pseudomanifolds with singular stratum of even codimension

Topological invariance of the Bismut–Zhang metric for oriented spaces with isolated conical singularities of even dimension has been proved already in [51, Section 8]. In Section 6.3.1 we recall some of the arguments of the proof for convenience of the reader–the arguments in the proof of Proposition 6.6 reappear in the proof of Proposition 6.8. In Section 6.3.2 we give a spectral sequence formula for the Bismut–Zhang torsion, for the case $\dim Y \geqslant 1$. The results of this section will be used in the proof of the Main Theorem 2 in Section 6.4.2.

6.3.1. Vanishing of the Bismut–Zhang torsion for even dimensional spaces with isolated singularities

Let (L, g^{TL}) be a smooth compact connected oriented Riemannian manifold of dimension $c-1$. From Corollary 5.4, the local calculation for intersection (co-)homology (recalled in (2.14)) and the Hodge theorem for the link manifold (L, g^{TL}),

$$H^k(C^\bullet_{\overline{m},\mathrm{loc}}(cL, -r\partial_r), \partial^\bullet) \simeq IH^k_{\overline{m}}(cL) \simeq \begin{cases} H^k(L) & \text{for } k < c/2, \\ 0 & \text{else,} \end{cases} \simeq \begin{cases} \mathrm{Harm}^k(L) & \text{for } k < c/2, \\ 0 & \text{else.} \end{cases} \tag{6.46}$$

We equip $H^\bullet(C^\bullet_{\overline{m},\mathrm{loc}}(cL, -r\partial_r), \partial^\bullet)$ with the metric $h_{\mathrm{Harm}(L)}$ induced from the L^2-metric on harmonic forms on (L, g^{TL}) via the isomorphism (6.46).

Proposition 6.6. — *Let (L, g^{TL}) be a smooth compact connected oriented odd dimensional Riemannian manifold. Then,*

$$\rho(C^\bullet_{\overline{m},\mathrm{loc}}(cL, -r\partial_r), \partial^\bullet) = \frac{1}{2}\rho_R(L, h_{\mathrm{Harm}(L)}). \tag{6.47}$$

Proof. — We denote by $h_{\ker(\mathbf{\Delta}_{W,cL})}$ the metric induced on $IH^\bullet_{\overline{m}}(cL)$ from the L^2-metric on forms on the cone cL (restricted to $\ker(\mathbf{\Delta}_{W,cL})$) via the first isomorphism in (6.24). By the definition of $\rho(C^\bullet_{\overline{m},\mathrm{loc}}(cL, -r\partial_r), \partial^\bullet)$ in Section 6.2.3, we have

$$\rho(C^\bullet_{\overline{m},\mathrm{loc}}(cL, -r\partial_r), \partial^\bullet) = -\frac{1}{2}\zeta'_{cL}(0) - \sum_{0 \leqslant k < \frac{c}{2}} (-1)^k [\![\mathrm{id}^k]\!], \tag{6.48}$$

where we abbreviated

$$[\![\mathrm{id}^k]\!] := [\![\mathrm{id}^k : (IH^k_{\bar{m}}(cL), h_{\mathrm{Harm}(L)}) \longrightarrow (IH^k_{\bar{m}}(cL), h_{\ker \mathbf{\Delta}_{W,cL}})]\!], \tag{6.49}$$

Set $b^k(L) := \dim H^k(L)$, $k = 0, \dots, c-1$. Using (6.24), (6.26), for $0 \leqslant k < c/2$,

$$[\![\mathrm{id}^k]\!] = \frac{1}{2} b^k(L) \ln\left(\frac{\Gamma(c/2-k)}{2}\right). \tag{6.50}$$

We denote by $\theta_L(s) = -\mathrm{Tr_s}[N(\Delta_L^{\perp})^{-s}]$ the torsion zeta function of the smooth link manifold L; here Δ_L denotes the Laplace–Beltrami operator on the link manifold L. Comparing the explicit computation of the zeta function ζ_{cL} in [52, Theorem I] with the computation of the analytic torsion of a truncated cone in [82] and using [36, Theorem 1.1],

$$\zeta'_{cL}(0) = \frac{1}{2}\theta'_L(0) - \sum_{0 \leqslant k < \frac{c}{2}} (-1)^k b^k(L) \ln\left(\frac{\Gamma(c/2-k)}{2}\right). \tag{6.51}$$

By the Cheeger–Müller theorem [16, 60],

$$\rho_R(L, h_{\mathrm{Harm}(L)}) = -\frac{1}{2}\theta'_L(0). \tag{6.52}$$

The claim of the proposition follows putting together (6.48)–(6.52). □

Let now X be an even dimensional oriented space with isolated singularities, $\dim X = n = c$. An even dimensional space with isolated singularities is a Witt space, hence the intersection cohomology with lower middle and upper middle perversity coincide (see [77, Section 3]) and Poincaré duality holds for $IH^{\bullet}_{\bar{m}}(X)$ (see [30, Section 3.3]).

Corollary 6.7. — *Let X be an even dimensional oriented space with isolated singularities, ξ a stratified anti-radial gradient-like vector field on X and $\{g^{TL_x}, x \in \Sigma\}$ Riemannian metrics on the link manifolds. We equip $IH^{\bullet}_{\bar{m}}(X)$ with a Hilbert structure h_X compatible with Poincaré duality. Then*

$$\rho_{BZ}(X, g^{TL}, \xi, h_X) = 0. \tag{6.53}$$

Proof. — The claim has been proved in [51, Section 8] and relies on Propositions 6.5 and 6.6. □

6.3.2. Spectral sequence formula for the Bismut–Zhang torsion in case $\dim Y \geqslant 1$, $\mathrm{codim}_X Y$ even

In this section we assume that X is a compact oriented pseudomanifold with two strata $\{Y, Z = X^{sm}\}$ with $\dim Y \geqslant 1$ and $c = \mathrm{codim}_X Y$ even.

The assumptions and notations are as in the beginning of Section 6.2.2. Let $\xi = \{\xi_Y, \xi_Z\}$ be a stratified anti-radial gradient-like vector field on X. We may moreover assume that ξ is of the form $\xi = -r\partial_r + \xi_Y$ on T (see Remark 6.4(c)).

We denote by $\mathcal{H}^k(L)$ (resp. by $I\mathcal{H}^k_{\overline{m}}(cL)$) the flat vector bundle on Y with fibre $H^k(L_x)$, $x \in Y$ (resp. $IH^k_{\overline{m}}(cL_x)$, $x \in Y$). By [47, Lemma 5.2], $\otimes_k \det(\mathcal{H}^k(L))^{\otimes(-1)^k}$ is unimodular. Since L is odd dimensional, using Poincaré duality on the link and the local calculation for intersection homology (see (6.24)), we have that $\otimes_k \det(I\mathcal{H}^\bullet_{\overline{m}}(cL))^{\otimes(-1)^k}$ is unimodular as well. Hence, we can equip the flat vector bundles $I\mathcal{H}^k(cL)$ (and hence $\mathcal{H}^k_{\overline{m}}(L)$) with Hermitian metrics $\overline{h}$, such that the induced metric on $\otimes_k \det(I\mathcal{H}^k(cL))^{\otimes(-1)^k}$ (resp. on $\otimes_k \det(\mathcal{H}^\bullet_{\overline{m}}(L))^{\otimes(-1)^k}$) is flat.

The Morse–Thom–Smale complex $(C^\bullet(Y, \xi_Y, I\mathcal{H}^k_{\overline{m}}(cL)), \partial^\bullet)$ on Y associated to the vector field ξ_Y with values in the flat vector bundle $I\mathcal{H}^k_{\overline{m}}(cL)$, carries a metric induced from the metric on $I\mathcal{H}^k_{\overline{m}}(cL)$. Moreover we fix a Hilbert structure on $H^\bullet(C^\bullet(Y, \xi_Y, I\mathcal{H}^k_{\overline{m}}(cL)), \partial^\bullet) \simeq H^\bullet(Y, I\mathcal{H}^k_{\overline{m}}(cL))$. For the E_1-term resp. the E_2-term of the spectral sequence associated to the filtered chain complex $(C^\bullet_{\overline{m},\mathrm{loc}}(T, \xi), \partial^\bullet)$ we have, by the results in Section 5,

$$\begin{aligned} U_1^{\bullet,k} &: (C^\bullet(Y, \xi_Y, I\mathcal{H}^k_{\overline{m}}(cL)), \partial^\bullet) \simeq E_1^{\bullet,k}, \\ U_2^{\bullet,k} &: H^\bullet(Y, I\mathcal{H}^k_{\overline{m}}(cL)) \simeq E_2^{\bullet,k}. \end{aligned} \tag{6.54}$$

Set $\rho_R(L, \overline{h}) \coloneqq \rho_R(L, \overline{h}_x)$, which does not depend on $x \in Y$, since the metric induced on $\otimes_k \det(\mathcal{H}^k(L))^{\otimes(-1)^k}$ from $\overline{h}$ is flat. We denote by $\chi(Y)$ the Euler characteristic of the manifold Y.

Proposition 6.8. — *Let $c = \operatorname{codim}_X Y$ be even. Then*

$$\begin{aligned} \rho(C^\bullet_{\overline{m},\mathrm{loc}}(T, \xi), \partial^\bullet) = {} & \frac{1}{2}\chi(Y)\rho_R(L, \overline{h}) \\ & + \sum_k (-1)^k \rho(C^\bullet(Y, \xi_Y, I\mathcal{H}^k_{\overline{m}}(cL)), \partial^\bullet) \\ & + \rho^{\geqslant 2}(C^\bullet_{\overline{m},\mathrm{loc}}(T, \xi), \partial^\bullet) + \sum_{m,k} (-1)^{m+k} [\![U_2^{m,k}]\!]. \end{aligned} \tag{6.55}$$

Proof. — In the sequel, we write $(F^\bullet_{\mathrm{loc},m}/F^\bullet_{\mathrm{loc},m+1})$, $m \in \mathbb{N}_0$, for the graded complex associated to the filtered complex $(C^\bullet_{\overline{m},\mathrm{loc}}(T, \xi), \partial^\bullet)$. From the definition of $\rho(C^\bullet_{\overline{m},\mathrm{loc}}(T, \xi), \partial^\bullet)$ in Section 6.2.3 and the transformation formula [47, Section 2.6] we get

$$\begin{aligned}(6.56)\quad & \rho(F^\bullet_{\mathrm{loc},m}/F^\bullet_{\mathrm{loc},m+1}) \\ &= -\sum_{x\in \mathrm{Crit}_m(\xi_Y)}\left[\frac{1}{2}\zeta_x'(0)+\sum_{0\leqslant k<c/2}(-1)^{m+k}[\![\mathrm{id}^k_x]\!]\right] \\ &\quad +\sum_k (-1)^{m+k}[\![U_1^{m,k}]\!],\end{aligned}$$

where we abbreviated

$$(6.57)\qquad [\![\mathrm{id}^k_x]\!] \coloneqq [\![\mathrm{id}^k : (IH^k_{\overline{m}}(cL_x),\overline{h}_x)\longrightarrow (IH^k_{\overline{m}}(cL_x), h_{\ker \boldsymbol{\Delta}_{W,x}})]\!].$$

From the proof of Proposition 6.6 (see in particular (6.51) and (6.52)) and the product formula (6.36),

$$\begin{aligned}\zeta_x'(0) &= (-1)^m\zeta'_{cL_x}(0)+(-1)^m I\chi_{\overline{m}}(cL_x)\left(m-\frac{\dim Y}{2}\right)\ln\pi \\ &= -(-1)^m\rho_R(L_x,h_{\mathrm{Harm}(L_x)}) \\ (6.58)\qquad &\quad -(-1)^m\sum_{0\leqslant k<c/2}(-1)^k b^k(L_x)\ln\left(\frac{\Gamma(c/2-k)}{2}\right) \\ &\quad +(-1)^m I\chi_{\overline{m}}(cL_x)\left(m-\frac{\dim Y}{2}\right)\ln\pi.\end{aligned}$$

Using (6.32), (6.34), for $x\in \mathrm{Crit}_m(\xi_Y)$, $0\leqslant k<\frac{c}{2}$,

$$\begin{aligned}(6.59)\quad [\![\mathrm{id}^k_x]\!] &= [\![\mathrm{id}^k : (IH^k_{\overline{m}}(cL_x),\overline{h}_x)\longrightarrow (IH^k_{\overline{m}}(cL_x), h_{\mathrm{Harm}(L_x)})]\!] \\ &\quad + b^k(L_x)\left(\frac{1}{2}\ln\left(\frac{\Gamma\,(c/2-k)}{2}\right)-\frac{1}{2}\left(m-\frac{\dim Y}{2}\right)\ln\pi\right).\end{aligned}$$

From (6.56)–(6.59) and the Poincaré–Hopf theorem for the vector field ξ_Y on Y,

$$\begin{aligned}(6.60)\quad & \sum_m \rho(F^\bullet_{\mathrm{loc},m}/F^\bullet_{\mathrm{loc},m+1}) \\ &= \sum_m(-1)^m\sum_{x\in\mathrm{Crit}_m(\xi_Y)}\frac{1}{2}\rho_R(L_x,\overline{h}_x)+\sum_{m,k}(-1)^{m+k}[\![U_1^{m,k}]\!] \\ &= \frac{1}{2}\chi(Y)\cdot\rho_R(L,\overline{h})+\sum_{m,k}(-1)^{m+k}[\![U_1^{m,k}]\!].\end{aligned}$$

From (6.54) and the transformation formula [47, Section 2.6],

$$(6.61)\quad \sum_k (-1)^k \rho(E_1^{\bullet,k}) = \sum_k (-1)^k \rho(C^\bullet(Y,\xi_Y, I\mathcal{H}^k_{\overline{m}}(cL)),\partial^\bullet) - \sum_{m,k}(-1)^{m+k}[\![U_1^{m,k}]\!] + \sum_{m,k}(-1)^{m+k}[\![U_2^{m,k}]\!].$$

The claim of the proposition follows applying Theorem 6.1 to the filtered complex $(C^\bullet_{\overline{m},\mathrm{loc}}(T,\xi),\partial^\bullet)$ and using (6.60) and (6.61). □

Corollary 6.9. — *Let $c = \operatorname{codim}_X Y$ be even. Then the torsion $\rho(C^\bullet_{\overline{m},\mathrm{loc}}(T,\xi),\partial^\bullet)$ is independent of the choice of the vector field ξ_Y and the metrics g^{TL_x}, $x \in \mathrm{Crit}(\xi_Y)$.*

Proof. — Let ξ_Y, ξ'_Y be two smooth Morse–Smale vector fields on Y and let $\xi = -r\partial_r + \xi_Y$, $\xi' = -r\partial_r + \xi'_Y$ be the corresponding stratified anti-radial gradient-like vector fields on T. Denote by $U_2^{m,k}$ (resp. $U_2'^{m,k}$) the isomorphism in (6.54) for the E_2-term of the spectral sequence associated to the filtered complex $(C^\bullet_{\overline{m},\mathrm{loc}}(T,\xi),\partial^\bullet)$ (resp. $(C^\bullet_{\overline{m},\mathrm{loc}}(T,\xi'),\partial^\bullet)$). Obviously, the first term in the formula (6.55) does not depend on the choice of the vector field ξ_Y and the choice of metrics g^{TL_x}, $x \in \mathrm{Crit}(\xi_Y)$. Since the Milnor torsion for a smooth manifold with coefficients in a unimodular flat vector bundle is a combinatorial invariant (see Section 6.1),

$$(6.62)\quad \sum_k (-1)^k \rho(C^\bullet(Y,\xi_Y, I\mathcal{H}^k(cL)),\partial^\bullet) = \sum_k (-1)^k \rho(C^\bullet(Y,\xi'_Y, I\mathcal{H}^k(cL)),\partial^\bullet).$$

From the anomaly formula for the $\rho^{\geqslant 2}$-term in the formula for the torsion of a spectral sequence (see [47, Remark 4.3]),

$$(6.63)\quad \rho^{\geqslant 2}(C^\bullet_{\overline{m},\mathrm{loc}}(T,\xi),\partial^\bullet) + \sum_{m,k}(-1)^{m+k}[\![U_2^{m,k}]\!] = \rho^{\geqslant 2}(C^\bullet_{\overline{m},\mathrm{loc}}(T,\xi'),\partial^\bullet) + \sum_{m,k}(-1)^{m+k}[\![U_2'^{m,k}]\!].$$

The claim follows from (6.55), (6.62) and (6.63). □

6.4. Comparison between the Bismut–Zhang and the intersection Reidemeister torsion for pseudomanifolds with singular stratum of even codimension

In Section 6.4.1 we recall the definition of intersection Reidemeister torsion given in [19], and show that the perturbed complex constructed in Section 5 computes the intersection Reidemeister torsion of T. We prove Main Theorem 2 in Section 6.4.2.

6.4.1. Intersection Reidemeister torsion

Let X be a stratified pseudomanifold as defined in Section 2.2, $\bar{p}$ a perversity in the sense of Goresky and MacPherson.

Let K be a triangulation of X compatible with the stratification, i.e. the closure of the X_k's are subcomplexes of K. We assume moreover, that the triangulation is flaglike with respect to the stratification, i.e. for any simplex σ, the intersection of σ with the closure of an X_k is a single face of σ. An easy way to achieve a flaglike triangulation is to take the first barycentric subdivision of a given compatible triangulation.

We consider the complex $(IC^{\bar{p},K}_\bullet(X,\mathbb{Z}),\partial_\bullet)$ of intersection chains of perversity $\bar{p}$ on X, which are simplicial with respect to K. The complex $(IC^{\bar{p},K}_\bullet(X,\mathbb{Z}),\partial_\bullet)$ is a complex of finitely generated free abelian groups. Moreover $H_\bullet(IC^{\bar{p},K}_\bullet(X,\mathbb{Z}),\partial_\bullet)\simeq IH^{\bar{p}}_\bullet(X,\mathbb{Z})$ (see [33]), and hence by [34, Theorem 8.1], $H_\bullet(IC^{\bar{p},K}_\bullet(X,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R},\partial_\bullet\otimes_{\mathbb{Z}}\mathbb{R})\simeq IH^{\bar{p}}_\bullet(X)$.

Let us denote by $(\mathcal{D}^\bullet_{\bar{p},K}(X),\partial^\bullet)$ the complex dual to the complex $(IC^{\bar{p},K}_\bullet(X,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R},\partial_\bullet\otimes_{\mathbb{Z}}\mathbb{R})$. Then

$$H^\bullet(\mathcal{D}^\bullet_{\bar{p},K}(X),\partial^\bullet)\simeq IH^\bullet_{\bar{p}}(X). \tag{6.64}$$

The complex $(\mathcal{D}^\bullet_{\bar{p},K}(X),\partial^\bullet)$ carries a natural Hilbert structure induced from its integer structure. Let us fix a Hilbert structure h_X on its cohomology $H^\bullet(\mathcal{D}^\bullet_{\bar{p},K}(X),\partial^\bullet)\simeq IH^\bullet_{\bar{p}}(X)$. The intersection Reidemeister torsion with perversity $\bar{p}$ of X is defined as

$$\rho^{\bar{p}}_{IR}(X,h_X)\coloneqq\rho(\mathcal{D}^\bullet_{\bar{p},K}(X),\partial^\bullet). \tag{6.65}$$

Definition (6.65) is dual to the definition of A. Dar [19, Definition 2.5], which is formulated in intersection homology.

If the Hilbert structure h_X on $H^\bullet(\mathcal{D}^\bullet_{\bar{p},K}(X),\partial^\bullet)\simeq IH^\bullet_{\bar{p}}(X)$ is also induced from the integer structure of intersection cohomology, we have by [16, Example 1.3] (see also [19, Example 2.1.9]):

$$\rho^{\bar{p}}_{IR}(X,h_X)=\sum_k(-1)^{k+1}\ln|IH^k_{\bar{p}}(X,\mathbb{Z})_{\mathrm{tors}}|, \tag{6.66}$$

where $IH^{\bullet}_{\bar{p}}(X,\mathbb{Z})_{\mathrm{tors}}$ denotes the torsion subgroup of $IH^{\bullet}_{\bar{p}}(X,\mathbb{Z})$. Using (6.66) one can show, that the definition (6.65) is independent of the chosen flaglike triangulation.

Let $(C^{\bullet}_{\bar{p},\mathrm{pert}}(T,\xi,\eta),\partial^{\bullet})$ be the perturbed complex introduced in Section 5 with the Hilbert structure induced from its integer structure. Let h_T be a Hilbert structure on $H^{\bullet}(C^{\bullet}_{\bar{p},\mathrm{pert}}(T,\xi,\eta),\partial^{\bullet}) \simeq IH^{\bullet}_{\bar{p}}(T)$. Using (5.28), (6.66) (for T instead of X) and applying Cheeger's result in [16, Example 1.3] also to the complex $(C^{\bullet}_{\bar{p},\mathrm{pert}}(T,\xi,\eta),\partial^{\bullet})$, we conclude

Proposition 6.10. — *Under the above assumptions we have*

$$\rho^{\bar{p}}_{IR}(T,h_T) = \rho(C^{\bullet}_{\bar{p},\mathrm{pert}}(T,\xi,\eta),\partial^{\bullet}). \tag{6.67}$$

In particular, in case of an isolated singularity,

$$\rho^{\bar{p}}_{IR}(cL,h_{cL}) = \rho(C^{\bullet}_{\bar{p},\mathrm{pert}}(cL,-r\partial_r,\eta),\partial^{\bullet}). \tag{6.68}$$

In the following we are only interested in the case where $\bar{p}=\overline{m}$ is the lower middle perversity.

Let L be a smooth compact connected oriented manifold. Using the isomorphism (6.32), we equip $IH^{\bullet}_{\overline{m}}(cL)$ with the metric $h_{\mathrm{Harm}(L)}$ induced from the metric on harmonic forms on the link manifold L. Using the Poincaré duality for the intersection Reidemeister torsion [19, Theorem 2.8] and the combinatorial gluing formula (applied to the exact sequence of intersection chain complexes associated to the pair (cL,L)), one can prove the following

Proposition 6.11. — *Let L be a smooth compact connected oriented odd dimensional manifold. Then*

$$\rho^{\overline{m}}_{IR}(cL,h_{\mathrm{Harm}(L)}) = \frac{1}{2}\rho_R(L,h_{\mathrm{Harm}(L)}). \tag{6.69}$$

6.4.2. Proof of Main Theorem 2

Case 1. — Let X be an even dimensional oriented space with isolated conical singularities. The first claim of Main Theorem 2 follows directly from Corollary 6.7. Let h_X be a Hilbert structure on $IH^{\bullet}_{\overline{m}}(X)$ compatible with Poincaré duality. By the duality for the intersection Reidemeister torsion (see [19, Theorem 2.8]) and Corollary 6.7, we get

$$\rho_{BZ}(X,g^{TL},\xi,h_X) = \rho^{\overline{m}}_{IR}(X,h_X) = 0. \tag{6.70}$$

Case 2. — Let now X be a space with two strata, $\dim Y \geqslant 1$ and $c = \operatorname{codim}_X Y$ even. The fact, that $\rho_{BZ}(X,g^{TL},\xi,h_X)$ is independent on the stratified anti-radial gradient-like vector field ξ and the Riemannian metrics

g^{TL_x}, $x \in \mathrm{Crit}(\xi_Y)$, follows from Corollary 6.9, topological invariance of the smooth Milnor torsion, and the gluing formula for the Bismut–Zhang torsion, Proposition 6.5. This shows the first claim of the Main Theorem 2.

Using the gluing formula for torsion [59, Theorem 3.2] one can prove a gluing formula for the intersection Reidemeister torsion. Using the gluing formula for the Bismut–Zhang torsion in Proposition 6.5 as well as Proposition 6.10, the proof of the equality of intersection Reidemeister and Bismut–Zhang torsion is now reduced to the proof of the following proposition:

Proposition 6.12. — *In case $c = \mathrm{codim}_X Y$ even, we have*

$$\rho(C^\bullet_{\overline{m},\mathrm{pert}}(T,\xi,\eta),\partial^\bullet) = \rho(C^\bullet_{\overline{m},\mathrm{loc}}(T,\xi),\partial^\bullet). \tag{6.71}$$

Proof. — We proceed similarly to the proof of Proposition 6.8 and establish a spectral sequence formula for $\rho(C^\bullet_{\overline{m},\mathrm{pert}}(T,\xi,\eta),\partial^\bullet)$. The complex $(C^\bullet_{\overline{m},\mathrm{pert}}(T,\xi,\eta),\partial^\bullet)$ is filtered by the index of critical points of ξ_Y in Y. For the associated graded complex we have (compare (5.22)),

$$\begin{aligned} G^\bullet_{\xi,\eta,m} &:= F_m(C^\bullet_{\overline{m},\mathrm{pert}}(T,\xi,\eta),\partial^\bullet)/F_{m+1}(C^\bullet_{\overline{m},\mathrm{pert}}(T,\xi,\eta),\partial^\bullet) \\ &\simeq \bigoplus_{x\in\mathrm{Crit}_m(\xi_Y)} C^{\bullet-m}_{\overline{m},\mathrm{pert}}(cL_x,-r\partial_r,\eta). \end{aligned} \tag{6.72}$$

As in Section 6.3.2 we denote by $I\mathcal{H}^k_{\overline{m}}(cL)$ the local system on Y with fibre $IH^k(cL_x)$, $x \in Y$. We again equip the flat vector bundles $I\mathcal{H}^k(cL)$ (and hence $\mathcal{H}^k_{\overline{m}}(L)$) with Hermitian metrics $\overline{h}$, such that the induced metric on $\otimes_k \det(I\mathcal{H}^k(cL))^{\otimes(-1)^k}$ (resp. on $\otimes_k \det(\mathcal{H}^\bullet_{\overline{m}}(L))^{\otimes(-1)^k}$) is flat; as already explained, this is possible since c is even. The Morse–Thom–Smale complex $(C^\bullet(Y,\xi_Y,I\mathcal{H}^k_{\overline{m}}(cL)),\partial^\bullet)$ carries the metric induced from $\overline{h}$. Moreover we fix a Hilbert structure on $H^\bullet(C^\bullet(Y,\xi_Y,I\mathcal{H}^k_{\overline{m}}(cL)),\partial^\bullet) \simeq H^\bullet(Y,I\mathcal{H}^k_{\overline{m}}(cL))$.

From (6.72) (see also the proof of Proposition 5.2), we deduce for the E_1- resp. the E_2-term in the associated spectral sequence, which we denote by $E_1'^{\bullet,k}$ resp. $E_2'^{\bullet,k}$,

$$\begin{aligned} V_1^{\bullet,k} &: (C^\bullet(Y,\xi_Y,I\mathcal{H}^k_{\overline{m}}(cL)),\partial^\bullet) \simeq E_1'^{\bullet,k}, \\ V_2^{\bullet,k} &: H^\bullet(Y,I\mathcal{H}^k_{\overline{m}}(cL)) \simeq E_2'^{\bullet,k}. \end{aligned} \tag{6.73}$$

From Proposition 6.11, (6.68), (6.72) and the transformation formula [47, (2.6)] we get

$$\begin{aligned}\sum_m \rho(G^\bullet_{\xi,\eta,m}) = \sum_m (-1)^m \sum_{x\in \mathrm{Crit}_m(\xi_Y)} \rho^{\overline{m}}_{IR}(cL_x, \overline{h}_x) \\ + \sum_{m,k}(-1)^{m+k}[\![V_1^{m,k}]\!] \\ = \frac{1}{2}\chi(Y)\cdot \rho_R(L,\overline{h}) + \sum_{m,k}(-1)^{m+k}[\![V_1^{m,k}]\!].\end{aligned} \tag{6.74}$$

From (6.73) and the transformation formula [47, (2.6)] we get

$$\begin{aligned}\sum_k (-1)^k \rho(E_1'^{\bullet,k}) = \sum_k (-1)^k \rho(C^\bullet(Y,\xi_Y, I\mathcal{H}^k_{\overline{m}}(cL)),\partial^\bullet) \\ - \sum_{m,k}(-1)^{m+k}[\![V_1^{m,k}]\!] \\ + \sum_{m,k}(-1)^{m+k}[\![V_2^{m,k}]\!].\end{aligned} \tag{6.75}$$

Applying Theorem 6.1 to the filtered complex $(C^\bullet_{\overline{m},\mathrm{pert}}(T,\xi,\eta),\partial^\bullet)$ and using (6.74) and (6.75),

$$\begin{aligned}\rho(C^\bullet_{\overline{m},\mathrm{pert}}(T,\xi,\eta),\partial^\bullet) \\ = \sum_m \rho(G^\bullet_{\xi,\eta,,m}) + \sum_k (-1)^k \rho(E_1'^{\bullet,k}) \\ + \rho^{\geqslant 2}(C^\bullet_{\overline{m},\mathrm{pert}}(T,\xi),\partial^\bullet) \\ = \frac{1}{2}\chi(Y)\cdot\rho_R(L,\overline{h}) \\ + \sum_k (-1)^k \rho(C^\bullet(Y,\xi_Y,I\mathcal{H}^k_{\overline{m}}(cL)),\partial^\bullet) \\ + \rho^{\geqslant 2}(C^\bullet_{\overline{m},\mathrm{pert}}(T,\xi,\eta),\partial^\bullet) + \sum_{m,k}(-1)^{m+k}[\![V_2^{m,k}]\!].\end{aligned} \tag{6.76}$$

From the anomaly formula for the $\rho^{\geqslant 2}$-term (see [47, Remark 4.3]), we have

$$\begin{aligned}\rho^{\geqslant 2}(C^\bullet_{\overline{m},\mathrm{loc}}(T,\xi),\partial^\bullet) + \sum_{m,k}(-1)^{m+k}[\![U_2^{m,k}]\!] \\ = \rho^{\geqslant 2}(C^\bullet_{\overline{m},\mathrm{pert}}(T,\xi),\partial^\bullet) + \sum_{m,k}(-1)^{m+k}[\![V_2^{m,k}]\!].\end{aligned} \tag{6.77}$$

From Proposition 6.8 and (6.76) and (6.77) we get

$$\begin{aligned}(6.78)\qquad &\rho(C^\bullet_{\overline{m},\mathrm{pert}}(T,\xi,\eta),\partial^\bullet)\\ &=\frac{1}{2}\chi(Y)\cdot\rho_R(L,\overline{h})\\ &\quad+\sum_k(-1)^k\rho(C^\bullet(Y,\xi_Y,I\mathcal{H}^k_{\overline{m}}(cL)),\partial^\bullet)\\ &\quad+\rho^{\geqslant 2}(C^\bullet_{\overline{m},\mathrm{loc}}(T,\xi),\partial^\bullet)+\sum_{m,k}(-1)^{m+k}[\![U_2^{m,k}]\!]\\ &=\rho(C^\bullet_{\overline{m},\mathrm{loc}}(T,\xi),\partial^\bullet).\end{aligned}$$

□

Remark 6.13. — By combinatorial invariance of the intersection Reidemeister torsion, the second statement of the Main Theorem 2 implies the first. However, here we have also given an independent proof of the first statement.

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