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Liviu Ornea & Misha Verbitsky

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MALL BUNDLES AND FLAT CONNECTIONS

by Liviu ORNEA & Misha VERBITSKY (*)

Abstract. — A *Mall bundle* on a Hopf manifold $H = \frac{\mathbb{C}^n\backslash 0}{\mathbb{Z}}$ is a holomorphic vector bundle whose pullback to $\mathbb{C}^n\backslash 0$ is trivial. We define *resonant* and *non-resonant* Mall bundles, generalizing the notion of the resonance in ODE, and prove that a non-resonant Mall bundle always admits a flat holomorphic connection. We use this observation to prove a version of Poincaré–Dulac linearization theorem, showing that any non-resonant invertible holomorphic contraction of $\mathbb{C}^n$ is linear in appropriate holomorphic coordinates. We define the notion of *resonance* in Hopf manifolds, and show that all non-resonant Hopf manifolds are linear; previously, this result was obtained by Kodaira using the Poincaré–Dulac theorem.

Résumé. — Un *fibré de Mall* sur une variété de Hopf $H = \frac{\mathbb{C}^n\backslash 0}{\mathbb{Z}}$ est un fibré vectoriel holomorphe dont le tiré en arrière sur $\mathbb{C}^n\backslash 0$ est trivial. Nous définissons des fibrés de Mall *résonants* et *non-résonants* en généralisant la notion de résonance dans les EDO et nous démontrons qu'un fibré de Mall non-résonant admet une connexion holomorphe plate. Nous employons cette observation pour démontrer une version du théorème de linéarisation de Poincaré–Dulac, en montrant que chaque contraction holomorphe bijective et non-résonante de $\mathbb{C}^n$ est linéaire dans certaines coordonnées holomorphes adaptées. Nous définissons la notion de *résonance* pour les variétés de Hopf et nous montrons que chaque variété de Hopf non-résonante est linéaire: ce résultat avait été déjà obtenu par Kodaira au moyen du théorème de Poincaré–Dulac.

Keywords: Holomorphic contraction, Hopf manifold, holomorphic bundle, holomorphic connection, affine manifold, resonance, coherent sheaf, Dolbeault cohomology.
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1. Introduction

1.1. Mall bundles: history and definition

Let $f: M \longrightarrow M$ be a continuous map fixing $x \in M$. We say that f is *a contraction centered in* x if for any compact set $K \subset M$ and any open set $U \subset M$ containing x, a sufficiently high power of f takes K to U.

Hopf manifolds are quotients of $\mathbb{C}^n \backslash 0$ by a holomorphic contraction centered at 0. In his seminal work [28], Daniel Mall computed the cohomology of a holomorphic vector bundle on a Hopf manifold such that its pullback to $\mathbb{C}^n \backslash 0$ can be extended to a holomorphic vector bundle on $\mathbb{C}^n$. We call such vector bundles *the Mall bundles* (Definition 4.1). The same argument was earlier applied by A. Haefliger, [18].

By the Oka–Grauert homotopy principle ([11, Theorem 5.3.1]), any vector bundle on $\mathbb{C}^n$ is trivial. Therefore, one could define the Mall bundles as holomorphic vector bundles on a Hopf manifold $H = \frac{\mathbb{C}^n \backslash 0}{\mathbb{Z}}$ such that their pullback to $\mathbb{C}^n \backslash 0$ is trivial (Remark 4.2).

Since 1990-ies, Mall's theorem was explored and generalized in many different directions; see, for example [25] and [13].

1.2. Mall theorem and its applications

The main utility of Mall bundles is Mall theorem, which allows one to compute the cohomology of a Mall bundle in terms of holomorphic sections. This theorem says that the cohomology $H^i(H, B)$ of a Mall bundle B on a Hopf manifold $H = \frac{\mathbb{C}^n \backslash 0}{\mathbb{Z}}$ vanish for $i \neq 0, 1, n-1, n$, and that $\dim H^0(H, B) = \dim H^1(H, B)$ and $\dim H^{n-1}(H, B) = \dim H^n(H, B)$. Serre duality gives an isomorphism $H^n(H, B) = H^0(H, B \otimes K_H)^*$, hence all cohomology of B can be expressed in terms of the holomorphic sections.

We use this observation on cotangent bundle and the tensor bundles of form $(\Omega^1 H)^{\otimes n} \otimes_{\mathcal{O}_H} TH$. It is not hard to see (Theorem 5.16, Step 2) that the obstruction to existence of a holomorphic connection belongs to $H^1(\Omega^1 H \otimes \operatorname{End}(B))$. Therefore, a Mall bundle admits a holomorphic connection whenever $\dim H^0(\Omega^1 H \otimes \operatorname{End}(B)) = 0$. A Mall bundle which satisfies this condition is called *non-resonant*. A non-resonant Mall bundle has a unique holomorphic connection, which is a posteriori flat (Theorem 5.16).

This is used to construct a torsion free flat affine connection on a non-resonant Hopf manifold, proving the Poincaré linearization theorem.

It would be interesting to access the existence of holomorphic connections on a resonant Mall bundle. We could not find a Mall bundle which does not admit a holomorphic connection. It is clear that a non-linearizable Hopf manifold cannot admit a flat, torsion-free holomorphic connection (Theorem 6.5). However, it is not clear whether the connection with non-zero holomorphic curvature can exist.

1.3. Mall bundles and holomorphic connections

In this paper we explore the geometric properties of the Mall bundles.

Let γ be an invertible holomorphic contraction of $\mathbb{C}^n$ centered in 0, and $H = \frac{\mathbb{C}^n \backslash 0}{\langle\gamma\rangle}$ the corresponding Hopf manifold. Since the pullback of a Mall bundle to $\mathbb{C}^n \backslash 0$ is a trivial bundle, the category of Mall bundles is equivalent to the category of γ-equivariant bundles on $\mathbb{C}^n$.

Let B be a complex vector bundle on a complex manifold M, and $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ a flat connection. The Hodge component $\bar{\partial} := \nabla^{0,1}$ is a holomorphic structure operator on B. By Koszul–Malgrange theorem ([20, Chapter I, Proposition 3.7.], [23]), the sheaf $\mathscr{B} := \ker \bar{\partial}$ is a holomorphic vector bundle on M, with $\mathscr{B} \otimes_{\mathcal{O}_M} C^\infty M = B$. In this situation we say that the flat connection ∇ *is compatible with the holomorphic structure on* B.

There are many examples of Mall bundles arising from the geometry of Hopf manifolds. All tensor bundles, all line bundles, and all extensions of Mall bundles are also Mall (Proposition 4.5). In many of those examples, the equivariant action of γ on B preserves a flat connection on B (for the line bundles, it follows from Proposition 5.13). In other words, these Mall bundles are obtained from flat bundles by taking the $(0,1)$-part of the connection. It turns out that this situation is quite general, and an arbitrary Mall bundle admits a compatible flat connection when the so-called "non-resonance" condition is satisfied.

This can be explained as follows. Recall that *a holomorphic connection* (Definition 5.9) on a holomorphic vector bundle B is a holomorphic differential operator $\nabla : B \longrightarrow B \otimes \Omega^1 M$ satisfying the Leibniz rule, $\nabla(fb) = f\nabla(b) + \mathrm{d}f \otimes \nabla(b)$ for any holomorphic $f \in \mathcal{O}_U$ and any $b \in H^0(U, B)$. We want to construct a holomorphic connection on a Mall bundle on a Hopf manifold; this is equivalent to having a γ-equivariant connection on its pullback to $\mathbb{C}^n \backslash 0$ considered as a γ-equivariant vector bundle.

Consider the space $\mathscr{A}$ of all holomorphic connections on a trivial γ-equivariant holomorphic vector bundle R on $\mathbb{C}^n$. This is an affine space modeled on the vector space $H^0(\Omega^1\mathbb{C}^n \otimes_{\mathcal{O}_{\mathbb{C}^n}} \operatorname{End}(R))$, and the equivariant action defines an affine endomorphism of $\mathscr{A}$. The linearization ρ of this action is a compact endomorphism of $H^0(\Omega^1\mathbb{C}^n \otimes_{\mathcal{O}_{\mathbb{C}^n}} \operatorname{End}(R))$ considered as a topological vector space with C^0-topology (Lemma 3.3). If ρ has all eigenvalues with absolute value < 1, Banach fixed point theorem would imply that R admits a γ-equivariant holomorphic connection. In fact, it would suffice to check that all eigenvalues λ_i of ρ are not equal to 1.

The "resonance" is a property of the eigenvalues of the γ-equivariant action; a γ-equivariant vector bundle R on $\mathbb{C}^n$ *has resonance* when the eigenvalues of $D\gamma$ on the fiber $R|_0$ are $\beta_1, \dots, \beta_m$, the eigenvalues of $D\gamma$ on $T_0\mathbb{C}^n$ are $\alpha_1, \dots, \alpha_n$, and there exists a relation of the form $\beta_p = \beta_q \prod_{i=1}^n \alpha_i^{k_i}$, with all k_i non-negative integers, and $\sum_i k_i > 0$, for some β_p, β_q, which are not necessarily distinct.

Let B be a vector bundle on a Hopf manifold, and R the extension of its pullback to $\mathbb{C}^n$. We prove that R has no resonance if and only if $H^0(H, \Omega^1_H \otimes \operatorname{End}(B)) = 0$ (Corollary 5.6). We also prove that any non-resonant holomorphic vector bundle on a Hopf manifold admits a flat connection compatible with the holomorphic structure (Theorem 5.16).

1.4. Flat affine structures and the development map

The notion of resonance is classical, and harks back to Poincaré, Lattès and Dulac, who discovered the resonance while working on the normal forms of ordinary differential equations. In the modern language, they were looking at the normal form of a real analytic or complex analytic vector field which has a simple zero at a given point.

The "normal form" is a classical notion, which is roughly equivalent, in the modern language, to "the moduli space", but includes a more explicit description in terms of coordinates.

For vector fields without zeroes, the normal form is very simple: in an appropriate coordinate system, this vector field takes the form $\frac{\mathrm{d}}{\mathrm{d}x_1}$; this is called "straightening of a vector field". This result follows directly from the Peano and Picard theorems on the existence of solutions of ODE.

The normal form theorem for a vector field with a simple zero is known as the *Poincaré–Dulac theorem*, [2, 10, 24]. We give a general outline of this theory, following [32].

Let $\dot{x} = A(x) + u(x)$ be a formal differential equation, where $x(t) \in \mathbb{C}^n$ is a time-dependent point in $\mathbb{C}^n$, A a non-degenerate linear operator, and $u(x)$ a Taylor series starting from the second order terms. It is said that A

has a resonance if there is a relation of the form $\lambda_i = \sum_{j=0}^n m_j\lambda_j$, where $m_j \in \mathbb{Z}^{\geqslant 0}$ and $\sum_{j=0}^n m_j \geqslant 2$.[(1)]

If A has no resonance, then the normal form of this vector field is very simple: in appropriate coordinates $y_1,\dots,y_n$ it can be written as $\dot{y} = A(y)$. If A has a resonance, the vector field has a normal form, which is written in a coordinate system $y = (y_1,\dots,y_n)$ as follows. Choose y_i in such a way that A is upper triangular in the basis $\frac{\mathrm{d}}{\mathrm{d}y_i}$, and the diagonal terms corresponding to $\frac{\mathrm{d}}{\mathrm{d}y_i}$ are λ_i. Then the equation $\dot{x} = A(x)+w(x)$ has normal form $\dot{y} = A(y) + \sum e_i w_i(y)$, where $e_i = \frac{\mathrm{d}}{\mathrm{d}y_i}$ is the coordinate vector field, and $\sum e_i w_i(y)$ is a Taylor series obtained as a sum of *resonant monomials*. A coordinate monomial $e_i\prod_{j=1}^n y_j^{m_j}$ is *resonant* if $\lambda_i = \sum m_j\lambda_j$.

In general, it is hard to achieve convergence for these formal sums, even when the differential equation is analytic. However, if e^A is a contraction, the convergence is automatic, because the number of resonant monomials is finite. Indeed, e^A is a contraction if and only if $\operatorname{Re}\lambda_i < 0$ for all i, and the equation $\operatorname{Re}\lambda_i = \sum m_j \operatorname{Re}\lambda_j$, $m_i \in \mathbb{Z}^{\geqslant 0}$, implies that $m_i \leqslant \max_{j,l}\frac{\operatorname{Re}\lambda_j}{\operatorname{Re}\lambda_l}$.

A similar result is true for germs of biholomorphic contractions, due to S. Sternberg ([36]). However, in this case one should replace the linear resonance by multiplicative, $\lambda_i = \prod_{j=0}^n \lambda_j^{m_j}$, as in Definition 5.1.

An invertible holomorphic contraction gives rise to a Hopf manifold, and Sternberg's theorem can be interpreted as a structure theorem about Hopf manifolds; this is how Kodaira used it in [21].

In this paper, we use the flat connection inherent on Mall bundles to give a new proof of the non-resonant part of the Poincaré–Dulac theorem. Let γ be a germ of an invertible biholomorphic contraction of $\mathbb{C}^n$ with center in 0. We say that γ *is non-resonant* if the differential $D_0\gamma \in \operatorname{End}(\mathbb{C}^n)$ is a non-resonant matrix. Let H be the corresponding Hopf manifold; then the tangent bundle TH is non-resonant, which is equivalent to $H^0(H,\Omega^1 H\otimes \operatorname{End}(TH)) = 0$ (Corollary 5.6). This immediately implies that the flat connection in TH, given by Theorem 5.16, is torsion-free.

To prove the Poincaré–Dulac linearization theorem, we need to find the coordinates on $\mathbb{C}^n$ in which γ is linear. To produce the flat coordinates, we use the developing map defined in the framework of flat affine geometry (or, more generally, in Cartan geometries).

A manifold M is called *affine*, or *flat affine*, if it is equipped with an atlas of open sets, identified with open subsets in $\mathbb{R}^n$, with the transition

(1) Later in this chapter, we redefine this notion in such a way that this additive relation becomes multiplicative, $\lambda_i = \prod_{j=0}^n \lambda_j^{m_j}$; this is done because we work with holomorphic contractions and not with the vector fields.

functions affine. This is equivalent to having a torsion-free flat affine connection on M (Proposition 6.2). The study of compact flat affine manifolds is ongoing, with many conjectures still open. We refer to [1] for more details and open questions.

Each flat affine manifold is equipped with the natural flat, torsion-free connection ∇. Using this connection, the developing map can be defined as follows. Assume that (M, ∇) is a simply connected flat affine manifold. Let $\theta_1, \dots, \theta_n$ be parallel 1-forms which trivialize the bundle T^*M. Since ∇ is torsion-free, all the forms θ_i are closed; however, $H^1(M) = 0$, which implies that θ_i are exact, $\theta_i = dz_i$. The map $m \longrightarrow (z_1(m), \dots, z_n(m))$ is called *the developing map*.

We can also understand the developing map in terms of the geodesics, as the inverse of the *exponential map*. The exponential map in this context is a map taking a tangent vector to the point at time 1 on the geodesic tangent to this vector in time 0. This definition is equivalent to the one given above (Theorem 6.4). A flat affine structure is *complete* when the geodesic equation $\nabla_{\dot\gamma_t}\dot\gamma_t$ can be solved for all $t \in \mathbb{R}$ and all initial conditions $\gamma_0 \in M$, $\dot\gamma_0 \in T_{\gamma_0}M$.

The completeness condition is tricky and counter-intuitive; indeed, even a compact flat affine manifold is not necessarily complete. A textbook example of a non-complete flat affine manifold is a real linear Hopf manifold H, obtained as a quotient of $\mathbb{R}^n\backslash 0$ by a linear contraction. This manifold is compact, but its universal cover is $\mathbb{R}^n\backslash 0$, and the developing map is an open embedding $\mathbb{R}^n\backslash 0 \hookrightarrow \mathbb{R}^n$. To obtain a non-complete geodesic, one needs to start from a geodesic in $\mathbb{R}^n$ passing through 0; its image in H is manifestly non-complete.

Let M be a complete, simply connected affine manifold. Then the developing map $\mathsf{dev}: M \longrightarrow \mathbb{R}^n$ is an isomorphism of affine manifolds. This is a classical result by Auslander–Markus [4] that we prove in Theorem 6.4.

For our present purposes, we need a variation of this result, which ultimately implies the non-resonant case of the Poincaré–Dulac theorem. From Theorem 5.16, it follows that any non-resonant Hopf manifold $M = \frac{\mathbb{C}^n\backslash 0}{\langle A\rangle}$ is equipped with a unique torsion-free flat affine connection compatible with the complex structure. However, it is not complete, as we explained above. We prove that this flat affine connection lifted to the universal covering $\mathbb{C}^n\backslash 0$ of M can be extended to 0, resulting in a complete flat affine structure on $\mathbb{C}^n$. The corresponding developing map puts flat affine coordinates on $\mathbb{C}^n$, and the contraction A is affine in these coordinates, hence linear.

This gives a new proof of the non-resonant case of the Poincaré–Dulac theorem.

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2. Preliminaries on Banach spaces

We gather here what is needed in the sequel concerning compact operators on Banach spaces and the Riesz–Schauder theorem.

Recall that a subset X of a topological space Y is called *precompact*, or *relatively compact in* Y, if its closure is compact.

Definition 2.1. — *A subset* $K \subset V$ *of a topological vector space is called* bounded *if for any open set* $U \ni 0$*, there exists a number* $\lambda_U \in \mathbb{R}^{>0}$ *such that* $\lambda_U K \subset U$*.*

Definition 2.2. — *Let* V, W *be topological vector spaces, A continuous operator* $\varphi :\ V \longrightarrow W$ *is called* compact *if the image of any bounded set is precompact.*

Now Montel's theorem [37, Lemma 1.4] can be restated as follows:

Claim 2.3. — *Let* $V = H^0(\mathcal{O}_M)$ *be the space of holomorphic functions on a complex manifold* M *with* C^0*-topology. Then any bounded subset of* V *is precompact. In this case, the identity map is a compact operator.*

Remark 2.4. — It is not hard to deduce from Montel theorem that the space of bounded holomorphic functions on M is Banach (that is, complete as a metric space) with respect to the sup-norm.

The following theorem can be used to obtain a version of the Jordan normal form for a compact operator on a Banach space. Recall that the *spectrum* of a linear operator F is the set of all $\mu \in \mathbb{C}$ such that $F - \mu\,\mathrm{Id}$ is not invertible.

Theorem 2.5 (Riesz–Schauder, [12, Section 5.2]). — *Let* $F :\ V \longrightarrow V$ *be a compact operator on a Banach space. Then the spectrum* $\mathsf{Spec}\, F \subset \mathbb{C}$ *is compact and discrete outside of* $0 \in \mathbb{C}$*. Moreover, for each non-zero*

$\mu \in \mathsf{Spec}\, F$, there exists a sufficiently big number $N \in \mathbb{Z}^{\geqslant 0}$ such that for each $n > N$ one has

$$V = \ker(F - \mu\,\mathrm{Id})^n \oplus \overline{\mathrm{im}(F - \mu\,\mathrm{Id})^n},$$

where $\overline{\mathrm{im}(F - \mu\,\mathrm{Id})^n}$ is the closure of the image. Finally, the space $\ker(F - \mu\,\mathrm{Id})^n$ is finite-dimensional.

Remark 2.6. — Recall that *the root space of an operator $F \in \mathrm{End}(V)$, associated with an eigenvalue μ*, is $\bigcup_{n\in\mathbb{Z}} \ker(F - \mu\,\mathrm{Id})^n$. A vector v is called *a root vector* for the operator F if v lies in a root space of F, for some eigenvalue $\mu \in \mathbb{C}$. In the finite-dimensional case, the root spaces coincide with the Jordan cells of the corresponding matrix. Then Theorem 2.5 can be reformulated by saying that any compact operator $F \in \mathrm{End}(V)$ admits a Jordan cell decomposition, with V identified with a completed direct sum of the root spaces, which are all finite-dimensional; moreover, the eigenvalues of F converge to zero.

In the sequel, we shall use the following corollary of the Riesz–Schauder theorem.

Theorem 2.7. — *Let $F : V \longrightarrow V$ be a compact operator on a Banach space. Then the space generated by the root vectors is dense in V.*

3. Dolbeault cohomology of Hopf manifolds

3.1. Computation of $H^{0,p}(H)$ for a Hopf manifold

Dolbeault cohomology of Hopf manifolds is a classical subject, but we could not find the computation for the general case. For the classical Hopf manifold (a quotient of $\mathbb{C}^n\backslash 0$ by a constant times identity), an answer is given in [19]. More general Hopf surfaces were defined and classified by Kodaira [21]; he computed some of their cohomology in [22]. For a diagonal Hopf manifold, the Dolbeault cohomology was computed by D. Mall [27]. For a reference to other special cases of this theorem, see [9, 33].

We recall the statement of Mall's result:

Theorem 3.1 ([28]). — *Let $\pi : \mathbb{C}^n\backslash 0 \longrightarrow H$, $n \geqslant 3$, be the universal cover of a Hopf manifold, $j : \mathbb{C}^n\backslash 0 \hookrightarrow \mathbb{C}^n$ the standard embedding map, and B a holomorphic vector bundle over H such that $j_*\pi^*B$ is a locally* *trivial coherent sheaf on* $\mathbb{C}^n$.[2] *Then* $\dim H^0(H,B) = \dim H^1(H,B)$, *and this group is equal to the space of* $\mathbb{Z}$*-invariant sections of* $j_*\pi^*B$. *Moreover,* $H^i(H,B) = 0$ *for all* i *such that* $1 < i < n-1$.

Using Mall's theorem, one can easily compute the $(0,*)$-part of the Dolbeault cohomology of a Hopf n-manifold H. Indeed, $H^{0,1}(H) = H^1(\mathcal{O}_H)$ has the same rank as $H^0(\mathcal{O}_H) = \mathbb{C}$, and $H^{0,i}(H) = 0$ for $1 < i < n-1$. By Serre duality, $H^{0,n-i}(H) = H^i(K_H)^*$, where K is the canonical bundle, hence to prove that $H^{0,n-1}(H) = H^1(K_H)^*$ and $H^{0,n}(H) = H^0(K_H)^*$ vanishes, it would suffice to show that the canonical bundle on a Hopf manifold has no holomorphic sections.

Theorem 3.2. — *Let* $H = (\mathbb{C}^n\backslash 0)/\mathbb{Z}$ *be a Hopf manifold, that is, a quotient of* $\mathbb{C}^n\backslash 0$ *by a holomorphic contraction, Then* $H^i(\mathcal{O}_H) = 0$ *unless* $i = 0, 1$, *and* $\operatorname{rk} H^1(\mathcal{O}_H) = \operatorname{rk} H^0(\mathcal{O}_H) = 1$.

Proof. — By Mall's Theorem 3.1, $\operatorname{rk} H^1(H,\mathcal{O}_H) = \operatorname{rk} H^0(H,\mathcal{O}_H)$, and $H^i(H,\mathcal{O}_H) = 0$ for $1 < i < n-1$. However, $\operatorname{rk} H^0(H,\mathcal{O}_H) = 1$ because $\mathcal{O}_H$ is a trivial line bundle. To finish the proof, it remains only to show that $H^{n-1}(H,\mathcal{O}_H)$ and $H^n(H,\mathcal{O}_H)$ vanish. By Serre duality, these two spaces are dual to $H^0(H,K_H)$ and $H^1(H,K_H)$, which have the same rank by Mall's theorem again. It remains only to prove that $H^0(H,K_H) = 0$ for any Hopf manifold.

Suppose that η is a non-zero element in $H^0(H,K_H) = 0$; we consider η as a holomorphic volume form. Then $\mu := \eta \wedge \overline{\eta}$ is a measure on H, which is strictly positive outside of the zero divisor of η.

Consider the measure $\pi^*\mu := \pi^*\eta\wedge\pi^*\overline{\eta}$ on $\mathbb{C}^n\backslash 0$. Since $\pi^*\eta$ is $\mathbb{Z}$-invariant, the measure $\pi^*\mu$ is $\mathbb{Z}$-invariant as well. The canonical bundle of $\mathbb{C}^n$ is trivial, hence, by Hartogs theorem, $\pi^*\eta$ can be extended to a holomorphic section $j_*\pi^*\eta$ of $K_{\mathbb{C}^n}$. Denote by $j_*\pi^*\mu$ the measure $j_*\pi^*\mu := j_*\pi^*\eta\wedge j_*\pi^*\overline{\eta}$. This measure is finite on compacts, and is preserved by the contraction $\gamma :\ \mathbb{C}^n \longrightarrow \mathbb{C}^n$. This is impossible, unless $j_*\pi^*\mu = 0$, because any bounded set is mapped inside a given compact neighbourhood of 0 by a sufficiently big power of γ. This implies that $\eta = 0$, hence $H^0(H,K_H) = 0$. □

[2] By Oka–Grauert homotopy principle, [11, Theorem 5.3.1], any holomorphic vector bundle on $\mathbb{C}^n$ is trivial; thus, instead of local triviality, we could assume that $j_*\pi^*B$ is a trivial vector bundle.

3.2. Holomorphic differential forms on Hopf manifolds

The results of this subsection generalize the vanishing $H^0(H, K_H) = 0$ given in the proof of Theorem 3.2. It turns out that all holomorphic differential forms on Hopf manifolds vanish.

In [19] the vanishing of differential forms was proven for the classical Hopf manifold $\frac{\mathbb{C}^n\backslash 0}{\lambda\,\mathrm{Id}}$; we could not find other results in the literature, though the question seems to be elementary and classical.

We start from the following lemma.

Lemma 3.3. — *Let $\gamma :\ \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be an invertible holomorphic contraction centered in 0, and $D \subset \mathbb{C}^n$ an open set such that $\gamma(D)$ is precompact in D. Choose an Hermitian metric on $\mathbb{C}^n$, and define the norm on the space $H^0_b(D, \Omega^1 D)$ of bounded holomorphic 1-forms as $\|\eta\| := \mathsf{sup}_{x\in D}|\eta_x|$.*[(3)] *Then the operator $\gamma^* :\ H^0_b(D, \Omega^1 D) \longrightarrow H^0_b(D, \Omega^1 D)$ is compact, and all its eigenvalues are smaller than 1 in absolute value.*

For the proof, we shall need the following result about the compactness of contraction operators:

Theorem 3.4 ([30, Theorem 2.14]). — *Let X be a complex variety, and $\gamma :\ X \longrightarrow X$ a holomorphic contraction centered in $x \in X$ such that $\gamma(X)$ is precompact. Consider the Banach space $V = H^0_b(\mathcal{O}_X)$ of bounded holomorphic functions with the sup-norm, and let $V_x \subset V$ be the space of all $v \in V$ vanishing in x. Then the operator $\gamma^* :\ V \longrightarrow V$ is compact, and the eigenvalues of its restriction to V_x are strictly smaller than 1 in absolute value.*[(4)]

The differential of γ acts with all eigenvalues $|a_i| < 1$ on $T_0\mathbb{C}^n$, by Theorem 3.4. Then γ^* (the pullback operator on differential forms) acts on $T_0^*\mathbb{C}^n$ with all eigenvalues $|a_i| < 1$. Consider the Taylor expansion of a function f in 0. The chain rule and the estimate of the eigenvalues of γ^* on $T_0^*\mathbb{C}^n$ imply that γ^* acts on the non-constant Taylor coefficients of f with all eigenvalues $|a_i| < 1$.

Let a be an eigenvalue of the compact operator

$$\gamma^* :\ H^0_b(\Omega^1 D) \longrightarrow H^0_b(\Omega^1 D).$$

It remains to show that $|a| < 1$.

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(3) By Montel's theorem, $H^0_b(D, \Omega^1 D)$ with this norm is a Banach space.

(4) Since γ^* maps constants to constants identically, one cannot expect that the eigenvalues of γ^* satisfy $|a_i| < 1$ on V. However, if we add a condition which excludes constants, such as $v(x) = 0$, we immediately obtain $|a_i| < 1$.

The eigenvalues of γ^* on the vector space $\Omega^1\mathbb{C}^n\big|_0$ are powers of its action on the cotangent bundle, which are all smaller than 1 in absolute value, because γ is a contraction. This implies that the eigenvalues of γ^* on the degree 0 term of the Taylor expansion of η are also smaller than 1 in absolute value. Summarizing the above estimates, we obtain that $\lim_n (\gamma^*)^n\eta = 0$, hence γ^* acts on differential forms with all eigenvalues smaller than 1 in absolute value.

Proposition 3.5. — *Let H be a Hopf manifold, and B a tensor power of $\Omega^1 H$, $B = (\Omega^1 H)^{\otimes l}$. Then all holomorphic sections of B vanish.*

Proof.

Step 1. — The universal cover of the Hopf manifold is $\mathbb{C}^n\backslash 0$ equipped with the free and properly discontinuous holomorphic $\mathbb{Z}$-action. Choose precompact fundamental domains M_i of the $\mathbb{Z}$-action, with the generator of $\mathbb{Z}$ mapping M_i to M_{i+1}. Then we may assume that $\widetilde{M_0} := \{0\} \cup \bigcup_{i \leqslant 0} M_i$ is a bounded neighbourhood of 0 in $\mathbb{C}^n$.

Let $\gamma \in \mathrm{Aut}(\mathbb{C}^n)$ be the generator of the $\mathbb{Z}$-action acting on $\mathbb{C}^n$ as a contraction. By Lemma 3.3, the action of γ^* on B defines a compact operator

$$H^0_b(\widetilde{M_0}, B) \longrightarrow H^0_b(\widetilde{M_0}, B),$$

with all eigenvalues smaller than 1.

Given a holomorphic differential form α on $\mathbb{C}^n$, we restrict it to $\widetilde{M_0}$ and observe that it is bounded because $\widetilde{M_0}$ is precompact. Using the compactness of γ^*-action and the estimate of its eigenvalues (Lemma 3.3), we show that the norm of $(\gamma^n)^*\alpha$ on $\widetilde{M_0}$ converges to 0.

Step 2. — Let $\pi : \mathbb{C}^n\backslash 0 \longrightarrow H$ be the universal cover of a Hopf manifold, and $\beta \in H^0(H, B)$ a section of B. Any holomorphic differential form on $\mathbb{C}^n\backslash 0$ can be extended to $\mathbb{C}^n$, when $n > 1$, by Hartogs theorem. Therefore, the section $\pi^*\beta$ can be holomorphically extended to 0, and this extension is γ^*-invariant. By the argument in Step 1, $\lim_n (\gamma^n)^*\alpha = 0$ for any holomorphic form α; applying this to $\pi^*\beta$, we conclude that $\beta = 0$. □

4. Mall bundles on Hopf manifolds

4.1. Mall bundles: definition and examples

We define *Mall bundles* on a Hopf manifold as bundles which satisfy the assumptions of Theorem 3.1.

Definition 4.1. — *Let* $\pi : \mathbb{C}^n\backslash 0 \longrightarrow H$ *be the universal cover of a Hopf manifold,* $j : \mathbb{C}^n\backslash 0 \hookrightarrow \mathbb{C}^n$ *the standard embedding map, and* B *a holomorphic vector bundle over* H *such that* $j_*\pi^*B$ *is a locally trivial coherent sheaf on* $\mathbb{C}^n$*, that is, a holomorphic vector bundle. Then* B *is called* a Mall bundle.

Note that any holomorphic vector bundle on $\mathbb{C}^n$ is trivial, as follows from the Oka–Grauert homotopy principle ([11, Theorem 5.3.1]).

Remark 4.2. — Let B be a Mall bundle on a Hopf manifold H. Then its pullback π^*B to $\mathbb{C}^n\backslash 0$ is extended to a trivial holomorphic bundle $\widehat{B}$, hence $\pi^*B = \widehat{B}\big|_{\mathbb{C}^n\backslash 0}$ is trivial. Conversely, if π^*B is trivial on $\mathbb{C}^n\backslash 0$, it can be extended to a bundle on $\mathbb{C}^n$, hence it is Mall. One could define Mall bundles as holomorphic bundles on H such that π^*B is a trivial bundle on $\mathbb{C}^n\backslash 0$.

Since j_* commutes with direct sums and tensor products, the following observation is clear.

Claim 4.3. — *A tensor product of Mall bundles, a direct sum of Mall bundles, and any direct sum component of a Mall bundle is again a Mall bundle.*

Proof. — For direct sums and tensor products the statement is obvious, because these operations commute with the functor $j_*\pi^*$. For a direct sum component, consider a coherent sheaf $\mathcal{F}$ on a complex manifold M, and let $\mathcal{F}|_x := \mathcal{F} \otimes_{\mathcal{O}_M} (\mathcal{O}_M/\mathfrak{m}_x)$. This is a finite-dimensional space; if $\mathcal{F}$ is a vector bundle, $\mathcal{F}|_x$ is the fiber of $\mathcal{F}$ in x. By Nakayama lemma, a coherent sheaf is generated by any collection $\{s_i\}$ of sections which generates $\mathcal{F}|_x = \mathcal{F} \otimes_{\mathcal{O}_M} (\mathcal{O}_M/\mathfrak{m}_x)$ for all $x \in M$. Indeed, let $\mathcal{F}_x$ be the stalk of $\mathcal{F}$ in x. Then $\mathcal{F}_x$ is a finitely-generated module over a local ring $\mathcal{O}_{M,x}$ of germs of holomorphic functions, which is Noetherian by Lasker theorem ([17, Chapter II, Theorem B.9]). Nakayama lemma tells that a finitely-generated module A over a Noetherian local ring is generated by any set of elements which generate A modulo the maximal ideal. Therefore, every stalk $\mathcal{F}_x$ of $\mathcal{F}$ is generated by the images of s_i in $\mathcal{F}_x$; by definition, this implies that $\mathcal{F}$ is generated by $\{s_i\}$.

We have shown that $\mathcal{F}$ is a vector bundle whenever $\operatorname{rk}\mathcal{F}_x$ is constant in x.

Given a direct sum decomposition $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$, we immediately obtain $\operatorname{rk}\mathcal{F}_x = \operatorname{rk}\mathcal{F}'_x + \operatorname{rk}\mathcal{F}''_x$. If $\operatorname{rk}\mathcal{F}|_x$ is constant in x, this implies that $\operatorname{rk}\mathcal{F}'_x = \mathsf{const}$ and $\operatorname{rk}\mathcal{F}''_x = \mathsf{const}$ because $\operatorname{rk}\mathcal{F}'_x$ is upper semicontinuous as a function of x by Nakayama lemma. This implies that for any direct sum decomposition $\mathscr{F} = \mathscr{F}' \oplus \mathscr{F}''$ of a vector bundle onto a direct sum of coherent sheaves, the summands are also vector bundles. □

For the next proposition, we need the following claim, which is almost trivial.

Claim 4.4. — *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of holomorphic bundles on a Hopf manifold. Assume that C and another of these bundles, B or A, are Mall. Then the third one is also Mall.*

Proof. — The functor π^* is exact, and j_* is left exact. Therefore, applying the functor $j_*\pi^*$ to this sequence, we obtain the following long exact sequence

$$0 \longrightarrow j_*\pi^*A \xrightarrow{\alpha} j_*\pi^*B \xrightarrow{\beta} j_*\pi^*C \longrightarrow R^1j_*\pi^*A^* \longrightarrow \cdots \tag{4.1}$$

where R^ij_* is the higher derived pushforward functor. If C is Mall, then the sheaf $j_*\pi^*C$ is locally free. Therefore, the map β has a section $\beta' : j_*\pi^*C \longrightarrow j_*\pi^*B$. This gives a subsheaf $\alpha(j_*\pi^*A) \oplus \beta'(j_*\pi^*C) \subset j_*\pi^*B$, which by construction coincides with $j_*\pi^*B$ outside of 0. By Hartogs, it has to coincide with $j_*\pi^*B$ in 0 as well. Then we have a direct sum decomposition $j_*\pi^*A \oplus \beta'(j_*\pi^*C) = j_*\pi^*B$. Therefore, one of the sheaves $j_*\pi^*A$ and $j_*\pi^*B$ is locally free when the other one is locally free (the sheaf $j_*\pi^*C = \beta'(j_*\pi^*C)$ is locally free by assumption). □

The next Proposition contains a list of examples of Mall bundles.

Proposition 4.5. — *Let H be a Hopf manifold. Then any line bundle on H, any tensor bundle $TH^{\otimes p} \otimes_{\mathcal{O}_H} T^*H^{\otimes q}$, their tensor products and direct sum components are Mall.*

Proof. — Tensor products and direct sum components were treated in Claim 4.3. Let L be a line bundle on a Hopf manifold, and $j_*\pi^*L$ the corresponding sheaf on $\mathbb{C}^n$ (Definition 4.1). By Siu's extension theorem, [35, Main Theorem], $j_*\pi^*L$ is a normal (and, therefore, reflexive) coherent sheaf of rank 1, hence it is locally free ([29, Chapter II, Lemma 1.1.15]). This implies that L is Mall.

The tangent bundle TH is Mall because π^*TH is the tangent bundle $T(\mathbb{C}^n\backslash 0)$; since the tangent bundle $T\mathbb{C}^n$ is trivial, $j_*\pi^*TH = T\mathbb{C}^n$ by Hartogs theorem. □

4.2. G-equivariant sheaves

Definition 4.6. — *Let M be a topological space equipped with the action of the group G by continuous maps.* A G-equivariant sheaf *is a sheaf*

$\mathcal{F}$ equipped with a collection of isomorphisms $R_g: \mathcal{F} \xrightarrow{\sim} g^*(\mathcal{F})$ for all $g \in G$, which defines a G-action on the étale space of $\mathcal{F}$.[(5)]

Remark 4.7. — Clearly, the equivariance of the G-action on $\mathcal{F}$ can be translated into the *equivariance relation* $R_{g_1 g_2} = g_2^*(R_{g_1})R_{g_2}$, for all $g_1, g_2 \in G$.

Theorem 4.8. — *Let M be a locally connected, locally simply connected topological space, and $\pi : M_1 \longrightarrow M$ a Galois cover, that is, a covering equipped with a free action of the group G such that $M = M_1/G$. Then the category of sheaves on M is equivalent to the category of G-equivariant sheaves on M_1.*

Proof. — Let $\mathcal{F}$ be a sheaf on M. Then the sheaf $\pi^*(\mathcal{F})$ is G-equivariant; indeed, $g \circ \pi = \pi$ for any $g \in G$, hence $\pi^*(\mathcal{F}) = g^*(\pi^*(\mathcal{F}))$, which defines a G-equivariant structure on $\pi^*(\mathcal{F})$.

Conversely, let $\mathcal{F}_1$ be a G-equivariant sheaf on M_1, $U \subset M$, and $U_1 := \pi^{-1}(U)$. The equivariant structure on $\mathcal{F}_1$ defines a G-action on the space of sections $\mathcal{F}_1(U_1)$. Indeed, $g^*(\mathcal{F}_1)(U_1)$ is by definition isomorphic to $\mathcal{F}_1(U_1)$, and the isomorphisms $R_g: g^*(\mathcal{F}_1)(U_1) \longrightarrow \mathcal{F}_1(U_1)$ can be interpreted as self-maps on this space indexed by $g \in G$. The equivariance relation $R_{g_1 g_2} = g_2^*(R_{g_1})R_{g_2}$ implies that the composition of these self-maps is compatible with the multiplication in G.

Consider the sheaf $\mathcal{F}_1^G$ on M with the space of sections $\mathcal{F}_1^G(U)$ equal to the space of G-invariant sections of $\mathcal{F}_1(\pi^{-1}(U))$. We claim that this functor from the category of G-equivariant sheaves on M_1 to the category of sheaves on M is inverse to the pullback functor defined above.

Clearly, any section of $\mathcal{F}$ on U can be interpreted as a G-invariant section of $\pi^*(\mathcal{F})$ on $\pi^{-1}(U)$, and any G-invariant section of $\pi^*(\mathcal{F})$ comes from $\mathcal{F}(U)$. Therefore, the composition of π^* and $(\cdot)^G$ is equivalent to identity. To prove that π^* and $(\cdot)^G$ are inverse, it remains to show that the functor $\mathcal{F}_1 \longrightarrow \pi^*(\mathcal{F}_1^G)$ is naturally isomorphic to identity ([26, Section IV.4]). We leave this as an exercise to the reader. □

5. Resonance in Mall bundles

5.1. Resonant matrices

Definition 5.1. — *Let $A \in \mathsf{GL}(n, \mathbb{C})$ be a matrix with eigenvalues $\alpha_1, \ldots, \alpha_n$. This matrix is called* resonant *if there exists a relation $\alpha =$*

(5) This notion is called a *G-sheaf* in [16].

$\prod_{i=1}^n \alpha_i^{k_i}$, *with* α *an eigenvalue of* A *and* $k_i \in \mathbb{Z}^{\geqslant 0}$, $\sum_i k_i \geqslant 2$, *and* non-resonant *otherwise.*

The reason we need this definition is the following elementary lemma.

LEMMA 5.2. — *Let* $\gamma : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ *be a germ of a holomorphic diffeomorphism in 0. Let* $B := T\mathbb{C}^n \bigotimes_{\mathcal{O}_{\mathbb{C}^n}} (T^*\mathbb{C}^n)^{\otimes k}$, $k > 1$. *Diffeomorphisms of* $\mathbb{C}^n$ *can be naturally extended to* $T\mathbb{C}^n$ *and to its tensor powers. Let* B_0 *denote the space of germs of sections of* B *in 0; clearly,* γ *induces a natural automorphism of* B_0. *Assume that the differential* $A := D_0\gamma$ *of* γ *in 0 is non-resonant. Then any* γ*-invariant germ* $v \in B_0$ *vanishes.*

Proof. — Let $t_1, \dots, t_n$ be coordinates on $\mathbb{C}^n$. Consider the Taylor series decomposition for v:

$$v = \sum_i v_i P_i(t_1, \dots, t_n)$$

where P_i is a homogeneous polynomial of degree i and $v_i \in V \otimes (V^*)^{\otimes k}$, where $V = T_0\mathbb{C}^n$. Let l be the smallest integer such that $P_l \neq 0$. The corresponding Taylor term can be considered as an element of $V \otimes (V^*)^{\otimes k} \otimes \mathsf{Sym}^l(V^*)$. By the chain rule, this tensor is coordinate-independent, hence it is also A-invariant. Let $\alpha_1, \dots, \alpha_n$ be the eigenvalues of A on $T_0\mathbb{C}^n$. Then $\alpha_1^{-1}, \dots, \alpha_n^{-1}$ are the eigenvalues of A on $T_0^*\mathbb{C}^n$. This is clear, because A preserves the pairing between $T\mathbb{C}^n$ and $T^*\mathbb{C}^n$: the differential $D\gamma$ acts covariantly on vector field and contravariantly on 1-forms.

We obtain that the eigenvalues of A on $V \otimes (V^*)^{\otimes k} \otimes \mathsf{Sym}^l(V^*)$ are products of the form $\alpha_l \prod_{i=1}^k \alpha_{j_i}^{-1} \prod_{i=1}^l \alpha_{s_i}^{-1}$. Unless A acts on $V \otimes (V^*)^{\otimes k} \otimes \mathsf{Sym}^l(V^*)$ with an eigenvalue 1, there would be no A-invariant vectors. Therefore $v = 0$ unless $\alpha_l = \prod_{i=1}^n \alpha_i^{k_i}$, where $\sum_i k_i \geqslant 2$, for some eigenvalue α_l. □

5.2. Resonant equivariant bundles

DEFINITION 5.3. — *Let* γ *be an invertible holomorphic contraction on* $\mathbb{C}^n$, *centered in 0, and* B *a* γ*-equivariant holomorphic vector bundle on* $\mathbb{C}^n$. *Let* $\alpha_1, \dots, \alpha_n$ *be the eigenvalues of the differential* $A = D_0\gamma$ *of* γ *in 0, and* $\beta_1, \dots, \beta_m$ *the eigenvalues of the differential of the equivariant action of* γ *on the fiber* $B\big|_0$. *We say that* B *is* non-resonant *if there is no multiplicative relation of the form* $\beta_i = \beta_j \prod_{l=1}^k \alpha_{i_l}$ *for some integer* $k \geqslant 1$, *where* $\{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ *is any collection of* k *eigenvalues, possibly repeating, and* $\beta_i, \beta_j \in \{\beta_1, \dots, \beta_m\}$ *any two eigenvalues, possibly the same.*

Remark 5.4. — A linear map with eigenvalues $\alpha_1, \dots, \alpha_n$ is resonant if one of the eigenvalues is a product of two or more eigenvalues. The data associated with a γ-equivariant bundle consists of linear operators, the differential $D\gamma\big|_{T_0\mathbb{C}^n}$ with the eigenvalues α_i and the differential of the equivariant action of γ on the fiber $B\big|_0$, with the eigenvalues β_j. It has resonance when one of β_j is a product of another and one or more α_i's. The resonance in the bundle $T\mathbb{C}^n$ with the natural γ-equivariant structure is the same as the resonance in the linear operator $D\gamma\big|_{T_0\mathbb{C}^n}$.

In other words, if B is $T\mathbb{C}^n$ with the standard γ-equivariant structure, the relation $\beta_i = \beta_j \prod_{l=1}^k \alpha_{i_l}$ becomes $\alpha_i = \alpha_j \prod_{l=1}^k \alpha_{i_l}$, $k \geqslant 1$, that is, $B = T\mathbb{C}^n$ is non-resonant if and only if the differential $A = D_0\gamma$ is non-resonant.

Resonant automorphisms can be characterized in terms of invariant 1-forms with coefficients in endomorphisms of B, as follows.

Theorem 5.5. — *Let $\gamma : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be an invertible holomorphic contraction centered in 0, and B a γ-equivariant vector bundle. Then B is resonant if and only if there exists a non-zero γ-invariant section of the bundle $\Omega^1\mathbb{C}^n \otimes \operatorname{End}(B)$.*

Proof.

Step 1. — Let R be a γ-invariant section of $\Omega^1\mathbb{C}^n \otimes \operatorname{End}(B)$. We are going to prove that B is resonant. By [11, Theorem 5.3.1], B is trivial. Choose a basis $b_1, \dots, b_m$ of B. Let $b_{ij} \in \operatorname{End}(B)$ be the corresponding elementary matrices, defining a basis in $\operatorname{End}(B)$. We write $R = \sum_{i,j,l} f_{ijl}\, \mathrm{d}z_l \otimes b_{ij}$, where z_i are coordinates in $\mathbb{C}^n$, and $f_{ijl} \in \mathcal{O}_{\mathbb{C}^n}$ a function. We write each f_{ijl} as Taylor series, $f_{ijl} = \sum_s P_s^{ijl}$, where $P_s^{ijl}(z_1, \dots, z_n)$ is a homogeneous polynomial of coordinate functions z_i of degree s. Let d be the smallest number for which not all P_d^{ijl} vanish. Using the chain rule again, we obtain that $\sum_{i,j} P_d^{ijl} dz_l \otimes b_{ij}$ is $D_0\gamma$-invariant. This is a $D_0\gamma$-invariant vector in the space $W^* \otimes \operatorname{End}(B|_0) \otimes \mathsf{Sym}^d(W^*)$, where $W = T\mathbb{C}^n$, with the action of $D_0\gamma$ which comes from the tensor product.

The eigenvalues of $D_0\gamma$ on the space $W^* \otimes \operatorname{End}(B|_0) \otimes \mathsf{Sym}^d(W^*)$ are eigenvalues of $D_0\gamma$ on W^* times the eigenvalues on $\operatorname{End}(B|_0)$ times a product of d eigenvalues of $D_0\gamma$ on W^*, that is, a numbers of form $\beta_u\beta_v^{-1}\alpha_l^{-1}$.

Since $\sum_{i,j} P_d^{ijl} dz_l \otimes b_{ij} \in W^* \otimes \operatorname{End}(B|_0) \otimes \mathsf{Sym}^d(W^*)$ is $D_0\gamma$-invariant, one of these numbers is 1. This gives a relation $\beta_u = \beta_v \alpha_l \prod_{l=1}^d \alpha_{i_l}$.

Step 2. — Let B be a resonant bundle; we are going to produce an invariant section of $\Omega^1\mathbb{C}^n\otimes\mathrm{End}(B)$. We follow the standard scheme (see Theorem 3.4, [31, Theorem 3.1]), using the Riesz–Schauder theorem and compactness of the action of holomorphic contractions on holomorphic functions; this time, the contraction acts on the sections of an equivariant bundle. Let $M_0 \subset \mathbb{C}^n$ be a subset which satisfies $\gamma(M_0) \Subset M_0$. We equip B with a Hermitian metric, and notice that the space $H^0_b(M_0, \Omega^1 M_0 \otimes \mathrm{End}(B))$ with sup-norm is Banach, by Montel theorem. Denote by V the bundle $\Omega^1 M_0 \otimes \mathrm{End}(B)$. By the standard argument (Theorem 3.4), the operator $\gamma^* :\ H^0_b(M_0, V) \longrightarrow H^0_b(M_0, V)$ is compact.

Consider the filtration on $H^0_b(M_0, V)$ by the powers of the maximal ideal $\mathfrak{m}_0$ of zero,

$$H^0_b(M_0, V) \supset H^0_b(M_0, \mathfrak{m}_0 V) \supset H^0_b(M_0, \mathfrak{m}_0^2 V) \supset \cdots$$

The finite-dimensional space $\frac{H^0_b(M_0,V)}{H^0_b(M_0,\mathfrak{m}_0^k V)}$ is interpreted as the space of $(k-1)$-jets of the sections of V in 0.

Using the integral Cauchy formula, any derivative of a function in a point can be expressed through a certain integral of this function. Therefore, the restriction map

$$H^0_b(M_0, V) \longrightarrow \frac{H^0_b(M_0, V)}{H^0_b(M_0, \mathfrak{m}_0^k V)} = V\big|_0$$

taking a section of V to its $(k-1)$-jet is also continuous in sup-norm. This implies also that the subspaces $H^0_b(M_0, \mathfrak{m}_0^k V) \subset H^0_b(M_0, V)$ of sections with vanishing $(k-1)$-jet are closed.

The map $\gamma^* :\ H^0_b(M_0, V) \longrightarrow H^0_b(M_0, V)$ preserves this filtration. Therefore, the eigenvalues of γ^* on $H^0_b(M_0, V)$ and on the associated graded space $\bigoplus_i \frac{H^0_b(M_0,\mathfrak{m}^{i-1}V)}{H^0_b(M_0,\mathfrak{m}_0^i V)}$ are equal. However, the space $\frac{H^0_b(M_0,\mathfrak{m}^{d-1}V)}{H^0_b(M_0,\mathfrak{m}_0^d V)}$ is naturally identified with $W^* \otimes_{\mathbb{C}} \mathsf{Sym}^d W^* \otimes_{\mathbb{C}} \mathrm{End}(B|_0)$, where $W = T_0\mathbb{C}^n$ (Step 1). In Step 1, we identified this space with the space of $(d-1)$-th Taylor coefficients of a section of V.

The eigenvalues of γ^* on this space are $\beta_1\beta_2^{-1}\prod_{i=1}^{d+1}\alpha_i^{-1}$, where β_1, β_2 denotes some eigenvalues of the equivariant action of γ^* on $B|_0$, possibly equal, and $\alpha_1, \dots, \alpha_{d+1}$ is a collection of some eigenvalues of $D_0\gamma$ on W, also possibly equal. The existence of resonance on B implies that $\prod_{i=1}^{d+1}\alpha_i^{-1}\beta_1\beta_2^{-1} = 1$ for an appropriate choice of the eigenvalues $\alpha_1, \dots, \alpha_{d+1},\ \beta_1, \beta_2$, and this is equivalent to γ^* having eigenvalue 1 on some γ^*-invariant quotient of $H^0_B(M_0, V)$, hence on $H^0_B(M_0, V)$, too. □

We can rewrite this theorem as a result about Mall bundles.

Corollary 5.6. — *Let B_H be a Mall bundle on a Hopf manifold $H = \frac{\mathbb{C}^n\backslash 0}{\langle\gamma\rangle}$, and $B := j_*\pi^* B_H$ the corresponding γ-equivariant bundle on $\mathbb{C}^n$. Then B is resonant if and only if $H^0(H, \Omega^1 H \otimes_{\mathcal{O}_H} (\operatorname{End} B_H)) \neq 0$.*

When $B = \Omega^1\mathbb{C}^n$, the equivariant bundle B is resonant if and only if the action of $D_0\gamma^*$ on $T_0\mathbb{C}^n$ is resonant:

Corollary 5.7. — *Let $\gamma : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be an invertible contraction centered in zero, $W := T_0\mathbb{C}^n$, and $A = D_0\gamma \in \operatorname{End} W$ its differential in zero. Then the following are equivalent.*

(i) *The matrix A is resonant.*
(ii) *There exists a non-zero γ^*-invariant section of $\Omega^1(\mathbb{C}^n)\otimes\operatorname{End}(T\mathbb{C}^n)$.*
(iii) *The bundle $T\mathbb{C}^n$ with the natural equivariant structure induced by the action of γ is resonant.*

Proof. — From the definition it is clear that A is resonant if and only if the γ-equivariant bundle $B = \Omega^1\mathbb{C}^n$ is resonant. By Theorem5.5, this is equivalent to the existence of non-zero γ^*-invariant sections of $\Omega^1(\mathbb{C}^n) \otimes \operatorname{End}(T\mathbb{C}^n)$. □

Consider an invertible contraction $\gamma : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ centered in zero, and let B be a γ-equivariant vector bundle. Let $R = (\Omega^1\mathbb{C}^n)^{\otimes k}$ be the bundle of k-multilinear forms on $\mathbb{C}^n$. We consider R as a γ-equivariant vector bundle as well. Then the set of eigenvalues of the γ^*-action on $H^0_b(M_0, R\otimes\operatorname{End} B)$ is equal to

$$\left\{\beta_1\beta_2^{-1}\prod_{i=1}^{d+k}\alpha_i^{-1}\right\}$$

Here $d \in \mathbb{Z}^{\geqslant 0}$, $\alpha_1, \dots, \alpha_{d+k}$ is any collection of eigenvalues of $D_0\gamma^*$ on $W = T_0\mathbb{C}^n$, and β_1, β_2 some eigenvalues of γ^* on $B|_0$. This is proven by the same argument as proves Theorem 5.5, Step 2. We obtained the following corollary.

Corollary 5.8. — *Let $\gamma : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be an invertible contraction of $\mathbb{C}^n$ centered in zero, and B a γ-equivariant vector bundle. Assume that B is non-resonant. Consider $R = (\Omega^1\mathbb{C}^n)^{\otimes k}$, $k \geqslant 1$, as a γ-equivariant bundle. Then the space of γ^*-invariant sections of $R \otimes_{\mathcal{O}_{\mathbb{C}^n}} B$ is empty.*

5.3. Holomorphic connections on vector bundles

To go on, we need the notion of a holomorphic connection.

Definition 5.9. — *Let B be a holomorphic vector bundle on a complex manifold, and $\nabla : B \longrightarrow B \otimes_{\mathcal{O}_M} \Omega^1 M$ a differential operator which satisfies $\nabla(fb) = \mathrm{d}f \otimes b + f\nabla b$ for any locally defined holomorphic function f and any local section b of B. Then ∇ is called* a holomorphic connection. *Using the Leibniz rule, the connection can be extended to a map on the B-valued differential forms,*

$$B \xrightarrow{\nabla} B \otimes_{\mathcal{O}_M} \Omega^1 M \xrightarrow{d_\nabla} B \otimes_{\mathcal{O}_M} \Omega^2 M \xrightarrow{d_\nabla} B \otimes_{\mathcal{O}_M} \Omega^3 M \xrightarrow{d_\nabla} \cdots$$

Its curvature *can be defined as the map $b \mapsto \nabla^2(b)$ taking $b \in B$ to $d_\nabla(\nabla b) \in B \otimes_{\mathcal{O}_M} \Omega^2 M$; since this map is $\mathcal{O}_M$-linear, it can be considered as an* $\mathrm{End}(B)$*-valued holomorphic 2-form, $\nabla^2 \in \Omega^2 M \otimes_{\mathcal{O}_M} \mathrm{End}(B)$.*

The notion of a holomorphic connection was introduced by M. Atiyah in [3]; for more results and references, see [5, 6, 7].

Every flat connection is holomorphic with respect to the holomorphic structure induced by this connection, but there are more holomorphic connections than there are flat connections. Indeed, holomorphic connections can be realized as objects of differential geometry, as follows.

Let (B, ∇) be a complex vector bundle with connection on a complex manifold, and $\bar\partial = \nabla^{0,1}$ the corresponding $\bar\partial$-operator. By Koszul–Malgrange theorem ([20, Chapter I, Proposition 3.7], [23]), $\bar\partial$ defines a holomorphic structure on B if and only if $\bar\partial^2 = 0$, or, equivalently, when $\nabla^2 \in [\Lambda^{2,0}(M) \oplus \Lambda^{1,1}(M)] \otimes \mathrm{End}(B)$.

Proposition 5.10. — *Let (B, ∇) be a complex vector bundle with connection on a complex manifold, and $\bar\partial = \nabla^{0,1}$ the corresponding $\bar\partial$-operator. Assume that $\bar\partial^2 = 0$, and let $\mathcal{B} = \ker \bar\partial$ be the holomorphic vector bundle obtained from $\bar\partial$ using* [20, *Chapter I, Proposition 3.7*], [23]*. Then the following assertions are equivalent.*

(i) *The operator $\nabla^{1,0}$ is a holomorphic connection operator on $\mathcal{B}$.*
(ii) $\nabla^2 \in \Lambda^{2,0}(M) \otimes \mathrm{End}(B)$.

Proof. — If $\nabla^{1,0}$ is a holomorphic connection operator, it commutes with $\bar\partial$, hence the $(1,1)$-part of the curvature of ∇ vanishes. Conversely, if $\nabla^2 \in \Lambda^{2,0}(M) \otimes \mathrm{End}(B)$, this implies that $\nabla^{1,0}$ commutes with $\bar\partial$, hence $\nabla^{1,0}$ maps holomorphic sections of B to the holomorphic sections of $B \otimes \Lambda^{1,0}(M)$. This implies that $\nabla^{1,0}$ is a holomorphic connection. □

Remark 5.11. — A holomorphic vector bundle B has Chern classes represented by closed forms of type (p,p), because it admits Chern connections. However, the Chern classes of a bundle equipped with a holomorphic connection are represented by holomorphic forms of type $(2p, 0)$. On a Kähler manifold (or any other manifold admitting the (p,q)-decomposition in cohomology), this is impossible, unless $c_i(B)=0$ for all i. This is why holomorphic connections rarely occur in Kähler geometry. However, on non-Kähler manifolds they don't seem that rare.

The Picard group $\operatorname{Pic}(M)$ of a complex manifold M (that is, the group of line bundles, with the group operation defined by the tensor multiplication) is naturally identified with $H^1(M,\mathcal{O}_M^*)$. Similarly, one could indentify the group of 1-dimensional local systems with $H^1(M,\mathbb{C}_M^*)$. Here $\mathbb{C}_M^*$ denotes the constant sheaf with the space of sections $H^0(U,\mathbb{C}_M^*)=\mathbb{C}^*$ for each connected open set $U\subset M$. The natural map $H^1(M,\mathbb{C}_M^*)\longrightarrow H^1(M,\mathcal{O}_M^*)$ can be interpreted as a forgetful map, taking a flat line bundle (L,∇) to the holomorphic line bundle $(L,\nabla^{0,1})$.

For Kähler manifolds, the map $H^1(M,\mathbb{C}_M^*)\longrightarrow H^1(M,\mathcal{O}_M^*)$ is never an isomorphism, unless $b_1(M)=0$. Indeed, from the exponential exact sequence it follows that $\dim_{\mathbb{C}} H^1(M,\mathbb{C}_M^*)=b_1(M)$ and $\dim_{\mathbb{C}} H^1(M,\mathcal{O}_M^*)=\frac{1}{2}b_1(M)$. However, on a Hopf manifold, these two groups are isomorphic, that is, every holomorphic line bundle admits a unique flat connection (Proposition 5.13).

For Kähler manifold, one could characterize line bundles admitting a holomorphic connection in terms of the Chern class.

Claim 5.12. — *Let L be a holomorphic line bundle on a compact Kähler manifold. Then L admits a holomorphic connection if and only if its first Chern class is a torsion.*

Proof. — Indeed, if $c_1(L)$ is a torsion, L admits a flat connection by dd^c-lemma [15]. Conversely, if L is a holomorphic line bundle admitting a holomorphic connection, then, by Chern–Weil theory, its first Chern class has Hodge type $(2,0)$; however, the Chern classes of a holomorphic bundle have type (p,p), hence the de Rham representative of $c_1(L)$ vanishes. □

For the sake of completeness, we give the proof of the following result of Kodaira:

Proposition 5.13 ([22, page 57]). — *Let H be a Hopf manifold, and $H^1(H,\mathbb{C}_H^*)$ the cohomology with coefficients in the constant sheaf $\mathbb{C}_H^*$. Then the natural map $H^1(H,\mathbb{C}_H^*)\longrightarrow H^1(H,\mathcal{O}_H^*)$ to the Picard group is an isomorphism.*

Proof.

Step 1. — Consider the exponential exact sequences

$$0\longrightarrow \mathbb{Z}_H\longrightarrow \mathcal{O}_H\longrightarrow \mathcal{O}_H^*\longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z}_H \longrightarrow \mathbb{C}_H \longrightarrow \mathbb{C}_H^* \longrightarrow 0.$$

The corresponding long exact sequences of cohomology give

$$0 \longrightarrow H^1(\mathbb{Z}_H) \longrightarrow H^1(\mathcal{O}_H) \longrightarrow H^1(\mathcal{O}_H^*) \longrightarrow H^2(\mathbb{Z}_H) = 0$$

and

$$0 \longrightarrow H^1(\mathbb{Z}_H) \longrightarrow H^1(\mathbb{C}_H) \longrightarrow H^1(\mathbb{C}_H^*) \longrightarrow H^2(\mathbb{Z}_H) = 0.$$

It remains to show that the natural map $\nu : \ H^1(\mathbb{C}_H) \longrightarrow H^1(\mathcal{O}_H)$ is an isomorphism; however, both groups are equal to $\mathbb{C}$ (Theorem 3.2), hence it would suffice to show that $\nu \neq 0$. The relevant E_2-term of the Dolbeault spectral sequence is $(H^{0,1}(H) = \mathbb{C}; H^{1,0}(H) = 0)$ (Theorem 3.2, Proposition 3.5). The sum of dimensions of these spaces is 1. However, $b_1(H) = 1$, hence the higher differentials of this spectral sequence vanish on $E_2^{1,0} + E_2^{0,1}$ and it degenerates in the $E_2^{1,0} + E_2^{0,1}$-term. This implies that the first de Rham cohomology $H^1(H)$ is equal to the first Dolbeault cohomology $H^{0,1}(H) \oplus H^{1,0}(H)$.

Step 2. — The standard map $\nu : \ H^1(\mathbb{C}_H) \longrightarrow H^1(\mathcal{O}_H)$ is the natural map from the $E_\infty^{0,1}$-term of this spectral sequence to $E_2^{0,1}$, which is an isomorphism because the spectral sequence degenerates. □

5.4. The flat connection on a non-resonant Mall bundle

Definition 5.14. — *Let B_H be a Mall bundle on a Hopf manifold $H = \frac{\mathbb{C}^n \backslash 0}{\langle \gamma \rangle}$, and $B := j_* \pi^* B_H$ the corresponding γ-equivariant bundle on $\mathbb{C}^n$. We call B_H* resonant *if $H^0(H, \Omega_H^1 \otimes \operatorname{End} B_H) \neq 0$, or, equivalently, when $B = j_* \pi^* B_H$ is a resonant γ-equivariant bundle on $\mathbb{C}^n$ (Corollary 5.6).*

Remark 5.15. — Let B_H be a non-resonant Mall bundle on a Hopf manifold. Since the difference of two connections is a holomorphic 1-form with coefficients in endomorphisms, and $H^0(H, \Omega_H^1 \otimes \operatorname{End} B_H) = 0$, a holomorphic connection on B_H is unique, if it exists.

Theorem 5.16. — *Let B be a holomorphic vector bundle over a Hopf manifold H, $\dim_{\mathbb{C}} H \geqslant 3$. Assume that B admits a flat connection compatible with the holomorphic structure. Then B is Mall.*(6) *Conversely, any non-resonant Mall bundle on H admits a flat connection ∇.*(7)

Proof.

Step 1. — Let (B, ∇) be a holomorphic bundle with a flat connection on H, and $\pi : \mathbb{C}^n \backslash 0 \longrightarrow H$ the universal cover. Since $\pi_1(\mathbb{C}^n \backslash 0) = 0$, the flat bundle $\pi^* B$ is trivial, hence B is Mall.

Step 2. — Let B be a non-resonant Mall bundle on a Hopf manifold H. We are going to show that B admits a holomorphic connection. Locally, a holomorphic connection always exists, and the difference between two holomorphic connections is a section of $\Omega^1 U \otimes_{\mathcal{O}_U} \operatorname{End}(B)$. Therefore, the obstruction to the existence of a holomorphic connection belongs to $H^1(H, \Omega^1 H \otimes_{\mathcal{O}_H} \operatorname{End}(B))$. Since B is Mall, we have

$$\operatorname{rk} H^1(H, \Omega^1 H \otimes_{\mathcal{O}_H} \operatorname{End}(B)) = \operatorname{rk} H^0(H, \Omega^1 H \otimes_{\mathcal{O}_H} \operatorname{End}(B))$$

(Theorem 3.1). When B has no resonance, $\operatorname{rk} H^0(H, \Omega^1 H \otimes_{\mathcal{O}_H} \operatorname{End}(B)) = 0$ hence B admits a connection.

Step 3. — To finish Theorem 5.16, it remains to show that any holomorphic connection ∇ on a non-resonant Mall bundle B is flat. However, its curvature is a holomorphic 2-form, and the space of holomorphic 2-forms with coefficients in $\operatorname{End}(B)$ vanishes by Corollary 5.8. □

6. Flat connections on Hopf manifolds

6.1. Developing map for flat affine manifolds

For an introduction to flat affine manifolds, see [1, 14, 34] and the references therein. Recall that *an affine function* on a vector space is a linear function plus constant.

(6) By a theorem of Buchdahl and Harris, any bundle B on H admitting a holomorphic connection is Mall. Indeed, a bundle on $\mathbb{C}^n \backslash 0$ with a holomorphic connection can be extended to $\mathbb{C}^n$ by [8, Theorem 1.2], and any holomorphic bundle on $\mathbb{C}^n$ is trivial by Grauert–Oka principle, [11, Theorem 5.3.1].

(7) All flat connections are holomorphic (Proposition 5.10). By Remark 5.15, B admits a unique holomorphic connection, hence ∇ is unique.

Definition 6.1. — *Let M be a manifold equipped with a sheaf $\mathcal{F} \subset \mathbb{C}^\infty M$. We say that $\mathcal{F}$* defines a flat affine structure on M *if for each $x \in M$ there exists a neighbourhood diffeomorphic to the ball $B \subset \mathbb{R}^n$ such that in this coordinate patch, the sheaf $\mathcal{F}|_B$ is the sheaf of affine functions on B. In other words, M is a flat affine manifold if there exists an open cover $\{U_i\}$ with all U_i diffeomorphic to an open ball in $\mathbb{R}^n$ and all transition maps are affine. Such a cover is called* an affine atlas *of M. The sheaf $\mathcal{F}$ is called* the sheaf of affine functions *on the flat affine manifold M.*

Flat affine structures can be equivalently described in terms of torsion-free, flat connections. The following proposition is well-known; we include its proof for completeness.

Proposition 6.2. — *Let $(M, \mathcal{F})$ be a flat affine manifold. Then M admits a unique torsion-free, flat connection ∇ such that the sections f of $\mathcal{F}$ satisfy $\nabla(\mathrm{d}f) = 0$. Conversely, if ∇ is a torsion-free flat connection on M, then the sheaf $\{f \in C^\infty M \mid \nabla(\mathrm{d}f) = 0\}$ defines a flat affine structure.*

Proof. — Let $(M, \mathcal{F})$ be a flat affine manifold, and $\{U_i\}$ its affine atlas. Each U_i admits a connection $\nabla : TM \longrightarrow TM \otimes \Lambda^1 M$ with

$$\nabla\left(\sum_j f_j \frac{\partial}{\partial x_j}\right) = \sum_j \mathrm{d}f_j \otimes \frac{\partial}{\partial x_j} \tag{6.1}$$

where x_j are coordinate functions; this connection is clearly flat and torsion-free, and $\ker(\nabla \mathrm{d})$ is the sheaf of affine functions on U_i. We call (6.1) *the standard flat connection* on $U \subset \mathbb{R}^n$.

Conversely, let ∇ be a torsion-free affine connection on M. Locally in an open subset $U \subset M$, the bundle $\Lambda^1 M$ admits a basis $\theta_1, \dots, \theta_n$ of ∇-parallel sections, which are closed because ∇ is torsion-free (here we use the relation $\mathrm{d}\theta = \mathrm{Alt}(\nabla\theta)$, which holds for any torsion-free connection). This implies that $\theta_i = \mathrm{d}x_i$ whenever U is simply connected. The functions x_i give a coordinate system on U, because $\mathrm{d}x_i$ are linearly independent, and in this coordinate system the affine functions are those which satisfy $\nabla(\mathrm{d}f) = 0$. $\square$

Further on, we will also call a torsion-free flat connection *an affine structure* and a pair (M, ∇) *an affine manifold*, or *a flat affine manifold*.

Definition 6.3. — *Let ∇ be a connection on TM, where M is a smooth manifold. We say that (M, ∇) is* complete *if for any $x \in M$ and any $v \in T_x M$, there exists a solution $\gamma_v : \mathbb{R} \longrightarrow M$ of the geodesic equation $\nabla_{\gamma_v'(t)} \gamma_v'(t) = 0$ for all $t \in]-\infty, \infty[$.* The exponential map *is the map* $\exp_x$ *: $T_x M \longrightarrow M$ taking $v \in T_x M$ to $\gamma_v(1)$. Clearly,* $\exp_x$ *is a* *diffeomorphism in a neighbourhood U of* $0 \in T_xM$. The developing map $\mathsf{dev}: \exp_x(U) \longrightarrow U$ *is the inverse of* $\exp_x$*; in general, it is defined only in a neighbourhood of* $x \in M$.

The following classical theorem, due to Auslander and Markus, is well known. It can be considered as an alternative definition of the complete flat affine manifold.

THEOREM 6.4 ([4, 14]). — *Let (M, ∇) be a simply connected, connected flat affine manifold. Then the developing map can be extended to an affine map* $\mathsf{dev}: M \longrightarrow T_xM$, *also called* the developing map. *If, in addition, M is complete, the developing map is a diffeomorphism.*

6.2. Flat affine connections on a Hopf manifold

The following result will be used further on in our proof of the classical Poincaré linearization theorem.

THEOREM 6.5. — *Let $H = \mathbb{C}^n \backslash 0/\mathbb{Z}$ be a Hopf manifold. Assume that TH admits a torsion-free, flat connection ∇ compatible with the holomorphic structure. Then H is isomorphic to a linear Hopf manifold $\mathbb{C}^n \backslash 0/\langle A\rangle$, where $A \in \mathsf{GL}(n, \mathbb{C})$ is a linear endomorphism.*

Proof.

Step 1. — The universal cover $\widetilde{H}$ of H is biholomorphic to $\mathbb{C}^n \backslash 0$. To avoid confusion, we denote its completion $\mathbb{C}^n$ by M. By a theorem of Buchdahl–Harris ([8, Theorem 1.2]), any bundle with a holomorphic connection, defined on the complement $M \backslash Z$ to a codimension $\geqslant 2$ set Z, can be extended over this set to a bundle with holomorphic connection on M.(8) Therefore, the flat connection ∇ on TH can be extended to the flat, torsion-free connection on the completion M of the universal cover of H, identified with $\mathbb{C}^n$. Denote by ∇_M the flat, torsion-free connection on M given by Buchdahl–Harris theorem.

To finish the proof of Theorem 6.5, it would suffice to show that the flat affine manifold (M, ∇_M) is isomorphic to $\mathbb{C}^n$ with the standard flat connection. Since the $\mathbb{Z}$-action on M preserves the affine structure and fixes 0, it is defined by a linear endomorphism, and $(M\backslash 0)/\mathbb{Z}$ is a linear Hopf manifold.

(8) We are much grateful to the referee for a reference to this superb paper.

It remains to show that the developing map $\mathsf{dev} : M \longrightarrow \mathbb{C}^n$ associated to ∇_M is an isomorphism. This would follow from Theorem 6.4 if we could prove that ∇_M is complete, but we have no direct control over the connection form of ∇_M, hence the completeness is not obvious.

Step 2. — Completeness of M is implied by A-equivariance of the development map.

Denote by A the generator of the $\mathbb{Z}$-action on M contracting M to 0 and let $A_0 \in \operatorname{End}(T_0M)$ be its differential. By construction, ∇_M is A-invariant. Therefore, A maps the parallel 1-forms to parallel 1-forms. Fix a basis $\lambda_1, \dots, \lambda_n$ of parallel 1-forms on M, and let $f_1, \dots, f_n$ be its antiderivatives, $df_i = \lambda_i$ vanishing in the origin $0 \in M = \mathbb{C}^n$ By definition, the development map dev takes $m \in M$ to $(f_1(m), \dots, f_n(m)$. Since A acts linearly on the space $\langle \lambda_1, \dots, \lambda_n \rangle$, it takes f_i to a linear combination of $f_1, \dots, f_n$ plus a constant term; however, we chose $f_i(0) = 0$, hence A acts on $\langle f_1, \dots, f_n \rangle$ linearly.

This gives a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\mathsf{dev}} & \mathbb{C}^n \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A} \\ M & \xrightarrow{\mathsf{dev}} & \mathbb{C}^n \end{array}$$

This implies that $\mathsf{dev}(M)$ is an $\mathbb{Z}$-invariant neighbourhood of $0 \in \mathbb{C}^n$, where the action of $\mathbb{Z} = \langle A_0 \rangle$ is generated by A_0. Since $\bigcup_n A_0^{-n}(V) = \mathbb{C}^n$ for any neighbourhood V of 0, any $\langle A_0 \rangle$-invariant neighbourhood of $0 \in \mathbb{C}^n$ is equal to $\mathbb{C}^n$. Therefore, the development map is surjective, and, moreover, every geodesics in M passing through $0 \in M$ can be extended indefinitely in both directions. This implies that M is complete as a flat affine manifold. □

6.3. A new proof of Poincaré theorem about linearization of non-resonant contractions

The Poincaré–Dulac theorem [2] gives a normal form of a smooth (or analytic) contraction; its non-resonant case is sometimes called *the Poincaré theorem*. It proves that a contraction (or a germ of a contraction), which is non-resonant, becomes linear after an appropriate coordinate change. We give a new proof of this theorem based on complex geometry. Note that the assumption $n \geqslant 3$ below is unnecessary; we leave the case $n = 2$ for the reader as an exercise.

Theorem 6.6. — *Let γ be an invertible holomorphic contraction of $\mathbb{C}^n$ centered in 0, $n \geqslant 3$. Assume that the differential $D_0\gamma \in \mathsf{GL}(T_0\mathbb{C}^n)$ is non-resonant. Then there exists a holomorphic diffeomorphism $U : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $U\gamma U^{-1}$ is linear.*(9)

Proof. — Let $H := \mathbb{C}^n\backslash 0/\langle\gamma\rangle$ be the Hopf manifold associated with γ, and $\pi : \mathbb{C}^n\backslash 0 \longrightarrow H$ the universal covering map. By Proposition 4.5, the tangent bundle TH is Mall. By Corollary 5.7, it is non-resonant. By Theorem 5.16, TH admits a flat holomorphic connection ∇. Since $\Omega^2 H \otimes TH$ is a direct sum component of $\Omega^1 H \otimes \operatorname{End}(TH)$, and TH is non-resonant,

$$H^0(H, \Omega^2 H \otimes TH) \subset H^0(H, \Omega^1 H \otimes \operatorname{End}(TH)) = 0,$$

(Corollary 5.7), hence ∇ is torsion-free.

By Theorem 6.5, the universal cover $\mathbb{C}^n\backslash 0$ of H admits flat coordinates such that γ is linear in these coordinates. This proves Theorem 6.6. □

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(9) The same statement can be stated for a germ of a holomorphic diffeomorphism; the proof will be essentially the same. We leave the required changes to the reader.

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Liviu ORNEA
University of Bucharest, Faculty of Mathematics and Informatics,
14 Academiei str., 70109 Bucharest (Romania)
Institute of Mathematics “Simion Stoilow” of the Romanian Academy,
21, Calea Grivitei Str.010702-Bucharest (Romania)
lornea@fmi.unibuc.ro
liviu.ornea@imar.ro

Misha VERBITSKY
Instituto Nacional de Matemática Pura e Aplicada (IMPA)
Estrada Dona Castorina, 110
Jardim Botânico, CEP 22460-320
Rio de Janeiro, RJ (Brasil)
Laboratory of Algebraic Geometry,
Faculty of Mathematics,
National Research University Higher School of Economics,
6 Usacheva Str. Moscow, (Russia)
verbit@impa.br