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SOME REMARKS ON THE SCHWEITZER COMPLEX

by Jonas STELZIG

Dedicated to the memory of J.-P. Demailly

ABSTRACT. — We prove that the Schweitzer complex is elliptic and its cohomologies define cohomological functors. As applications, we obtain finite dimensionality, a version of Serre duality, restrictions of the behaviour of cohomology in small deformations, and an index formula which turns out to be equivalent to the Hirzebruch–Riemann–Roch relations.

RÉSUMÉ. — Nous prouvons que le complexe de Schweitzer est elliptique et que ses cohomologies définissent des foncteurs cohomologiques. Comme applications, nous obtenons la dimensionnalité finie, une version de la dualité de Serre, des restrictions du comportement de la cohomologie dans les petites déformations, et une formule d'index qui s'avère être équivalente aux relations de Hirzebruch–Riemann–Roch.

1. Introduction

The Bott–Chern and Aeppli cohomology are classical and well-established invariants of complex manifolds [1, 5]. Given a complex manifold X, with double complex of forms $(\mathcal{A}_X(X), \partial, \bar\partial)$ they are defined as

$$H^{p,q}_{BC}(X) := \left(\frac{\ker\partial\cap\ker\bar\partial}{\operatorname{im}\partial\bar\partial}\right)^{p,q} \quad\text{and}\quad H^{p,q}_{A}(X) := \left(\frac{\ker\partial\bar\partial}{\operatorname{im}\partial+\operatorname{im}\bar\partial}\right)^{p,q}.$$

M. Schweitzer and J. P. Demailly [7, 14] have shown that they arise as the cohomology groups in certain degrees of a differential complex $(\mathcal{L}^\bullet_{p,q}(X), \mathrm{d}_{\mathcal{L}})$, the definition of which we recall below. The other cohomologies of this complex appear naturally in the classification of holomorphic string algebroids [9] or higher-length Aeppli–Bott–Chern Massey products [12], but have received relatively little attention otherwise. In this short note, we will establish some basic properties of the Schweitzer complex. More precisely, we show:

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Theorem A. — *Let X be a complex manifold and $(\mathcal{L}^\bullet_{p,q}(X), \mathrm{d}_{\mathcal{L}})$ its Schweitzer complex.*

(1) *The complex $(\mathcal{L}^\bullet_{p,q}(X), \mathrm{d}_{\mathcal{L}})$ is elliptic.*
(2) *For every $p, q, k \in \mathbb{Z}$, the assignement $X \mapsto H^k(\mathcal{L}^\bullet_{p,q}(X), \mathrm{d}_{\mathcal{L}})$ defines a cohomological functor in the sense of* [15, 16].

From now on, let us assume X to be compact and of dimension n. Then we obtain:

Corollary B. — *The dimensions $s^k_{p,q}(X) := \dim H^k(\mathcal{L}^\bullet_{p,q}(X), \mathrm{d}_{\mathcal{L}})$ are finite.*

This has also been shown by Demailly [7, Theorem 12.4], using a different argument. Further, we have

Corollary C. — *For X connected, wedge product and integration induce a perfect pairing*

$$H^k(\mathcal{L}^\bullet_{p,q}(X), \mathrm{d}_{\mathcal{L}}) \times H^{2n-k-1}(\mathcal{L}^\bullet_{n-p+1,n-q+1}(X), \mathrm{d}_{\mathcal{L}}) \longrightarrow \mathbb{C}.$$

Corollary D. — *The numbers $s^k_{p,q}(X)$ vary upper semi-continuously in families. I.e. given a differentiable family (cf.* [11]*) of compact complex manifolds $\{X_t\}_{t\in B}$ with $B \subseteq \mathbb{R}^m$, for any $t \in B$ sufficiently close to a given point $0 \in B$, there is an inequality $s^k_{p,q}(X_0) \geqslant s^k_{p,q}(X_t)$.*

We will illustrate that this last result gives new restrictions on the behaviour of multiplicities of indecomposable subcomplexes in of the double complex of smooth $\mathbb{C}$-valued forms on X under small deformations. In particular, one obtains semi-continuity results even for classical objects such as certain differentials in the Frölicher spectral sequence.

By ellipticity, we may apply the Atiyah–Singer index theorem and obtain equalities between the Euler-characteristics $\chi_{p,q}(X) := \sum_k (-1)^k s^k_{p,q}$ and certain expressions $td_{p,q}(X)$ in characteristic numbers (recalled below).

Theorem E. — *The relations $\chi_{p,q}(X) = td_{p,q}(X)$ are equivalent to the Hirzebruch–Riemann–Roch relations.*

Theorem E is in accord with a conjecture made in [16], stating that any universal linear relation between cohomological invariants and Chern numbers of compact complex manifolds of a given dimension is a consequence of the Hirzebruch–Riemann–Roch relations.

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2. Proofs of the main results

2.1. The Schweitzer complex

Throughout, we fix an n-dimensional complex manifold X and we denote by $\mathcal{A}_X^{p,q}$ the sheaf of smooth complex valued (p,q)-forms. Given any fixed pair of integers $p,q \in \mathbb{Z}$, Schweitzer and Demailly [7, 14], define a simple complex $\mathcal{L}_{p,q}^{\bullet}$ of locally free sheaves as follows:

$$\mathcal{L}_{p,q}^{k} \coloneqq \bigoplus_{\substack{r+s=k \\ r<p,s<q}} \mathcal{A}_X^{r,s} \quad \text{if } k \leqslant p+q-2$$

$$\mathcal{L}_{p,q}^{k} \coloneqq \bigoplus_{\substack{r+s=k+1 \\ r\geqslant p,s\geqslant q}} \mathcal{A}_X^{r,s} \quad \text{if } k \geqslant p+q-1$$

with differential $\mathrm{d}_{\mathcal{L}}$ given by:

$$\cdots \xrightarrow{\mathrm{pr}\, d} \mathcal{L}_{p,q}^{p+q-3} \xrightarrow{\mathrm{pr}\, d} \mathcal{L}_{p,q}^{p+q-2} \xrightarrow{\partial\bar{\partial}} \mathcal{L}_{p,q}^{p+q-1} \xrightarrow{d} \mathcal{L}_{p,q}^{p+q} \xrightarrow{d} \cdots,$$

where pr denotes projection from the sheaf of all forms in a given degree to the direct summand $\mathcal{L}_{p,q}^{k}$.

We illustrate which components of $\mathcal{A}_X$ contribute to $\mathcal{L}_{p,q}^{\bullet}$ for $n=3$, $(p,q)=(2,1)$:

$$\begin{array}{cccc} & & \mathcal{A}_X^{2,3} & \mathcal{A}_X^{3,3} \\ & & \mathcal{A}_X^{2,2} & \mathcal{A}_X^{3,2} \\ & & \mathcal{A}_X^{2,1} & \mathcal{A}_X^{3,1} \\ \mathcal{A}_X^{0,0} & \mathcal{A}_X^{1,0} & & \end{array}$$

By construction, one has

$$H_{BC}^{p,q}(X) = H^{p+q-1}(\mathcal{L}_{p,q}^{\bullet}(X), \mathrm{d}_{\mathcal{L}}) = \mathbb{H}^{p+q-1}(\mathcal{L}_{p,q}^{\bullet})$$

and

$$H_{A}^{p,q}(X) = H^{p+q}(\mathcal{L}_{p+1,q+1}^{\bullet}(X), \mathrm{d}_{\mathcal{L}}) \coloneqq \mathbb{H}^{p+q}(\mathcal{L}_{p+1,q+1}^{\bullet})$$

We will mainly be interested in the differential complex of global sections $\mathcal{L}_{p,q}(X) := (\mathcal{L}^\bullet_{p,q}(X), \mathrm{d}_{\mathcal{L}})$.[(1)]

Remark 2.1. — If X is compact Kähler (or more generally a $\partial\bar\partial$-manifold), the natural maps $H^{p,q}_{BC}(X) \cong H^{p,q}_{\bar\partial}(X)$ and $H^{p,q}_{\bar\partial}(X) \cong H^{p,q}_{A}(X)$ are isomorphisms. More generally, in that case all cohomology groups $H^k(\mathcal{L}_{p,q}(X))$ are isomorphic to direct sums of the Dolbeault cohomology groups in the relevant bidegrees, e.g. $H^{p+q}(\mathcal{L}_{p,q}(X)) \cong H^{p+1,q}_{\bar\partial}(X) \oplus H^{p,q+1}_{\bar\partial}(X)$ etc, as may be verified using the $\partial\bar\partial$-Lemma.

2.2. Cohomological functors

We recall (cf. [15, 16]) that a cohomological functor on the category of complex manifolds is a functor to the category of $\mathbb{C}$-vector spaces, which factors as $H \circ A$, where $A : X \mapsto A_X := (\mathcal{A}_X(X), \partial, \bar\partial)$ is the Dolbeault double complex functor and H is a linear functor which vanishes on direct sums of "squares", i.e. double complexes of the form

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\sim} & \mathbb{C} \\ {\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow \\ \mathbb{C} & \xrightarrow{\sim} & \mathbb{C} \end{array}.$$

Remark 2.2. — In [16], the definition was made only for compact manifolds, and is was further demanded that H takes values in the category of finite dimensional vector spaces. We can extend this finite-dimensionality condition to the non-compact situation by requiring $H(C)$ to be finite dimensional for finite-dimensional complexes. This recovers the condition in the compact case since for any compact complex manifold X, there exists a decomposition $A_X \cong A_X^{sq} \oplus A_X^{zig}$ where A_X^{sq} is a direct sum of squares and A_X^{zig} is finite-dimensional [15].

It is clear from the construction that $\mathcal{L}_{p,q}(X)$ depends only on A_X. For any double complex A, denote by $\mathcal{L}_{p,q}(A)$ the Schweitzer complex formed using A instead of $\mathcal{A}_X(X)$ and denote $H^k_{S_{p,q}}(A) := H^k(\mathcal{L}_{p,q}(A))$. It is then clear that $H^k_{S_{p,q}}(A)$ is a linear functor and that it commutes with arbitrary direct sums of double complexes. Further, note that $H^k_{S_{p,q}}(C)$, being a subquotient of C, is certainly finite dimensional for any finite dimensional double complex C. Thus, what remains to show is the following:

(1) Up to a shift in total degree, this complex is denoted $S_{p,q}(X)$ in [12]. Here we follow the indexing convention used in [7, 14].

Lemma 2.3. — *For any square $\square$ as above, we have $H^k_{S_{p,q}}(\square) = 0$ for all $p, q, k \in \mathbb{Z}$.*

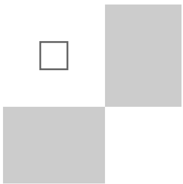
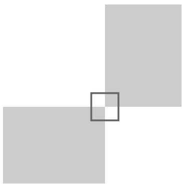
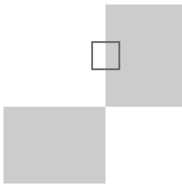
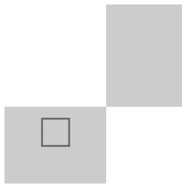
Proof. — Any square $\square$ is concentrated in four bidegrees:

$$S \coloneqq \operatorname{supp} \square \coloneqq \{(r, s), (r + 1, s), (r, s + 1), (r + 1, s + 1)\}.$$

The structure of $\mathcal{L}_{p,q}(\square)$ (and hence $H_{S_{p,q}}(\square)$) depends on the relative position of S and $T = T_A \cup T_B$ where

$$T_A \coloneqq \{(a, b) \in \mathbb{Z}^2 \mid a < p, b < q\}$$
$$T_B \coloneqq \{(a, b) \in \mathbb{Z}^2 \mid a \geqslant p, b \geqslant q\}.$$

There are three possibilities: $\#(S \cap T) \in \{0, 2, 4\}$, surveyed in the following table:

Case 1: $S \cap T = \emptyset$	Case 2: $\#(S \cap T) = 2$	Case 3: $\#(S \cap T) = 4$
	 or 	
$\mathcal{L}_{p,q}(\square) = 0$	$\mathcal{L}_{p,q}(\square) = \mathbb{C} \xrightarrow{\cong} \mathbb{C}$	$\mathcal{L}_{p,q}(\square) = \mathbb{C} \hookrightarrow \mathbb{C}^2 \twoheadrightarrow \mathbb{C}$

In each case, clearly $H^k_{S_{p,q}}(\square) = H^k(\mathcal{L}_{p,q}(\square)) = 0$ for all p, q, k. $\square$

Proof of Corollary B. — This is now a direct consequence of Lemma 2.3 and Remark 2.2. $\square$

Proof of Corollary C. — We recall from [15, Corollary 20] that for any cohomological functor H and compact complex manifold X, the integration pairing induces an isomorphism $H(A_X) \cong H(DA_X)$, where DA_X denotes the dual double complex as in [15]. The bigraded components of the dual complex are given by $(DA_X)^{p,q} = (A_X^{p,q})^\vee$ and the differentials (up to sign) by pullback with the differentials of A_X. Therefore, we have $\mathcal{L}^k_{p,q}(DA_X) = (\mathcal{L}^{2n-k-1}_{n-p+1,n-q+1}(A_X))^\vee$. But cohomology of a complex of vector spaces commutes with dualization, so

$$H^k(\mathcal{L}_{p,q}(A_X)) \cong H^k(\mathcal{L}_{p,q}(DA_X)) = H^{2n-k-1}(\mathcal{L}_{n-p+1,n-q+1}(X))^\vee. \quad \square$$

2.3. Ellipticity

Let $x \in X$ be some fixed point in a complex manifold X. Denote by $\pi : TX^\vee \to X$ the projection map of the cotangent bundle and for any

$\xi = \xi_x \in TX_x^\vee$, denote by $(L^\bullet, \sigma) := (\pi^*\mathcal{L}^\bullet_{p,q}(X)_\xi, \sigma(d_{\mathcal{L}_{p,q}})(\xi))$ the symbol complex (cf. [2]). To show ellipticity, we have to prove:

Lemma 2.4. — *For any $\xi \neq 0$, the symbol complex $(L^\bullet, \sigma)$ is exact.*

Proof. — As is well-known, we have $\sigma(\partial)(\xi) = \xi^{1,0} \wedge \_$ and $\sigma(\bar\partial)(\xi) = \xi^{0,1} \wedge \_$, where the superscripts mean projection to the corresponding bidegree. Without loss of generality, we may pick a local coordinate system $z_1, ..., z_n$ around x such that $\xi = dz_1 + d\bar z_1$.

Let us check exactness of the following part of the complex. Exactness at stages of lower or higher degree is only notationally more cumbersome.

$$\begin{matrix} A^{p-3,q-1}_{X,x} \\ \oplus \\ A^{p-2,q-2}_{X,x} \\ \oplus \\ A^{p-1,q-3}_{X,x} \end{matrix} \longrightarrow \begin{matrix} A^{p-2,q-1}_{X,x} \\ \oplus \\ A^{p-1,q-2}_{X,x} \end{matrix} \longrightarrow A^{p-1,q-1}_{X,x} \longrightarrow A^{p,q}_{X,x} \longrightarrow \begin{matrix} A^{p,q+1}_{X,x} \\ \oplus \\ A^{p+1,q}_{X,x} \end{matrix} \longrightarrow \begin{matrix} A^{p,q+2}_{X,x} \\ \oplus \\ A^{p+1,q+1}_{X,x} \\ \oplus \\ A^{p,q+2}_{X,x} \end{matrix}.$$

The first map is given by

$$\sigma^{p+q-4} : \omega^{p-3,q-1} + \omega^{p-2,q-2} + \omega^{p-1,q-3} \longmapsto \begin{matrix} \xi^{1,0}\omega^{p-3,q-1} + \xi^{0,1}\omega^{p-2,q-2} \\ +\xi^{1,0}\omega^{p-2,q-2} + \xi^{0,1}\omega^{p-1,q-3} \end{matrix}.$$

The second map is given by

$$\sigma^{p+q-3} : \omega^{p-2,q-1} + \omega^{p-1,q-2} \longmapsto \xi^{1,0}\omega^{p-2,q-1} + \xi^{0,1}\omega^{p-1,q-2}.$$

The third map is given by

$$\sigma^{p+q-2} : \omega \longmapsto \xi^{1,0}\xi^{0,1}\omega.$$

The fourth and fifth maps are given by $\omega \longmapsto \xi\omega$.

Now assume $\omega = \omega^{p-2,q-1} + \omega^{p-1,q-2} \in \ker \sigma^{p+q-3}$. Write

$$\begin{aligned} \omega^{p-2,q-1} &= \xi^{1,0}\xi^{0,1}\omega_A + \xi^{1,0}\omega_B + \xi^{0,1}\omega_C + \omega_D \\ \omega^{p-1,q-2} &= \xi^{1,0}\xi^{0,1}\omega'_A + \xi^{1,0}\omega'_B + \xi^{0,1}\omega'_C + \omega'_D, \end{aligned}$$

with $\omega_A, \omega'_A, ...$ having no summand which contains a factor of $\xi^{1,0}$ or $\xi^{0,1}$. Then $\sigma(\omega) = 0$ translates into the three equations

$$\begin{gathered} \xi^{1,0}\xi^{0,1}\omega_C + \xi^{0,1}\xi^{1,0}\omega'_B = 0 \\ \xi^{1,0}\omega_D = \xi^{0,1}\omega'_D = 0. \end{gathered}$$

The second and third equation imply $\omega_D = \omega'_D = 0$ and the first implies $\omega_C = \omega'_B$. Hence, defining $\eta \in L_x^{p+q-4}$ as follows

$$\begin{aligned} \eta^{p-3,q-1} &:= \xi^{0,1}\omega_A + \omega_B \\ \eta^{p-2,q-2} &:= \omega_C \\ \eta^{p-1,q-3} &:= -\xi^{1,0}\omega'_A + \omega'_C, \end{aligned}$$

one obtains $\sigma(\eta) = \omega$.

Now, let $\omega \in L_x^{p+q-2} = A_{X,x}^{p-1,q-1}$. Write $\omega = \xi^{1,0}\xi^{0,1}\omega_A + \xi^{1,0}\omega_B + \xi^{0,1}\omega_C + \omega_D$ as before. Then $0 = \sigma(\omega) = \xi^{1,0}\xi^{0,1}\omega$ implies $\omega_D = 0$. Hence, defining $\eta \in L_x^{p+q-3}$ by

$$\begin{aligned} \eta^{p-2,q-1} &:= \omega_B + \frac{1}{2}\xi^{0,1}\omega_A \\ \eta^{p-1,q-2} &:= \omega_C - \frac{1}{2}\xi^{1,0}\omega_A, \end{aligned}$$

we get $\sigma(\eta) = \omega$.

Next, let $\omega \in L_x^{p+q-1} = A_{X,x}^{p,q}$. Write $\omega = \xi^{1,0}\xi^{0,1}\omega_A + \xi^{1,0}\omega_B + \xi^{0,1}\omega_C + \omega_D$ as above. Then sorting $0 = \sigma(\omega) = \xi^{1,0}\omega + \xi^{0,1}\omega$ by bidegree yields

$$\begin{aligned} \xi^{1,0}\xi^{0,1}\omega_C + \xi^{1,0}\omega_D &= 0 \\ \xi^{0,1}\xi^{1,0}\omega_B + \xi^{0,1}\omega_D &= 0. \end{aligned}$$

In particular, $\omega_D = -\xi^{0,1}\omega_C$ and therefore $0 = \omega_D = \omega_C = \omega_B$. Thus, $\omega = \sigma(\omega_A)$. Finally, let $\omega = \omega^{p,q+1} + \omega^{p+1,q} \in A_{X,x}^{p,q+1} \oplus A_{X,x}^{p+1,q} = L_x^{p+q}$ s.t. $0 = \sigma(\omega) = \xi \wedge \omega$. Again, write

$$\begin{aligned} \omega^{p,q+1} &= \xi^{1,0}\xi^{0,1}\omega_A + \xi^{1,0}\omega_B + \xi^{0,1}\omega_C + \omega_D \\ \omega^{p+1,q} &= \xi^{1,0}\xi^{0,1}\omega'_A + \xi^{1,0}\omega'_B + \xi^{0,1}\omega'_C + \omega'_D. \end{aligned}$$

Then $\sigma(\omega) = 0$ translates into

$$\begin{aligned} 0 &= \xi^{0,1}\xi^{1,0}\omega_B + \xi^{0,1}\omega_D \\ 0 &= \xi^{1,0}\xi^{0,1}\omega_C + \xi^{1,0}\omega_D + \xi^{0,1}\xi^{1,0}\omega'_B + \xi^{0,1}\omega'_D \\ 0 &= \xi^{1,0}\xi^{0,1}\omega'_C + \xi^{1,0}\omega'_D. \end{aligned}$$

This implies $\omega_D = \omega_B = \omega'_C = \omega'_D = 0$ and $\omega_C = \omega'_B$. Defining

$$\eta := -\xi^{1,0}\omega_A + \omega_C + \xi^{0,1}\omega'_A \in L^{p+q-1} = A_{X,x}^{p,q},$$

we have $\sigma(\eta) = \omega$. □

Remark 2.5. — Ellipticity immediately gives a second proof of Corollary B, see [2, Proposition 6.5.]

2.4. Small deformations

Let us prove Corollary D:

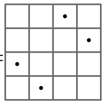
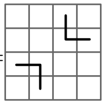
Proof. — Pick a hermitian metric. For any $k \neq p+q$, the order of $d^k_{\mathcal{L}_{p,q}}$ is 1 and thus for $k \neq p+q, p+q-1$, the operators $\Delta^k_{p,q} := (d^k_{\mathcal{L}_{p,q}})^* d^k_{\mathcal{L}_{p,q}} + d^{k-1}_{\mathcal{L}_{p,q}} (d^{k-1}_{\mathcal{L}_{p,q}})^*$ are elliptic and their kernel is isomorphic to $H^k(\mathcal{L}^\bullet_{p,q}(X))$. They vary smoothly in families and hence the result follows, cf. [11]. For $k = p+q, p+q-1$, i.e. for Bott–Chern and Aeppli cohomology, the Corollary is known. (In that case, $\Delta^k_{p,q}$ as defined above is not elliptic, but an appropriate modification of it is, see [2, Section 6] for a general statement or [7, 14] for an explicit construction in this case.) $\square$

As an example we use this result to obtain semicontinuity properties for a classical object: the Frölicher spectral sequence [8]. We denote by $e^{p,q}_r := \dim E^{p,q}_r$ the dimensions of the bigraded pieces on the r-th page of this spectral sequence. The dimensions on the first page $e^{p,q}_1 = h^{p,q}_{\bar\partial}$ are known to behave upper semi-continuously, but for later pages this is false in general (cf. [6, Corollary 4.9]). However, denoting by $FD^{p,q} := e^{p,q}_1 - e^{p,q}_\infty$, we may show:

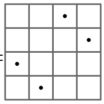
Corollary 2.6. — *Let X_t be a differentiable family of complex manifolds (cf. [11]) with $X_0 = X$ compact of dimension n. Then for t sufficiently close to zero, there are inequalities*

$$FD^{0,1}(X_0) \geqslant FD^{0,1}(X_t) \quad \text{and} \quad FD^{0,n-1}(X_0) \geqslant FD^{0,n-1}(X_t).$$

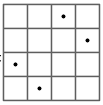
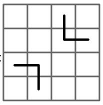
Proof. — In order to avoid heavy notation which obscures the idea of the proof, we do the proof in the case of 3-folds, where it is easier to draw all necessary diagrams. We use [10, 15] that A_X may be decomposed into a direct sum of indecomposable double complexes. In any such decomposition there appear finitely many zigzags (indecomposable double complexes concentrated in at most two neighboring total degrees) and (unless X has dimension 0) infinitely many squares (as in Lemma 2.3). The multiplicities $\operatorname{mult}_Z(A_X) =: \#Z(A_X)$ of these summands are an isomorphism invariant of A_X, [15]. Furthermore, for each zigzag, its images under the involutions $\tau : (p,q) \mapsto (q,p)$ "flipping along the diagonal" and $\sigma : (p,q) \mapsto (n-p, n-q)$ "flipping along the main antidiagonal" appear with the same multiplicity, so, denoting by Z' the sum of the elements in the $\langle\sigma,\tau\rangle$-orbit of Z we may write $\#Z' = \#Z$. Finally, no zigzag with nonzero multiplicity can have a nonzero component in degrees $(0,0), (n,0), (0,n), (n,n)$, unless it is one-dimensional and entirely concentrated in that bidegree ("only dots in the corners”), see [15, Chapter 4]. With this understood we compute:

$b_1(X) \coloneqq 2 \cdot \#$  $(A_X) + \#$  (A_X)

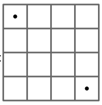
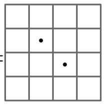
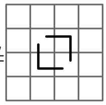
and

$h^{0,1}_{BC}(X) = \#$  (A_X),

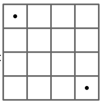
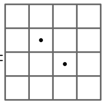
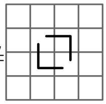
whereas

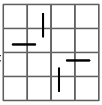
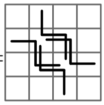
$e^{0,1}_{\infty}(X) = \#$  $(A_X) + \#$  (A_X).

Therefore $e^{0,1}_{\infty} = b_1 - h^{0,1}_{BC}$. Since b_1 stays constant in families and $h^{0,1}_{BC}$ behaves upper semi-continuously, the first inequality, $FD^{0,1}(X) \geqslant FD^{0,1}(X_t)$, follows. For the second one, we compute

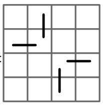
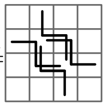
$b_3(X) = 2 \cdot \#$  $(A_X) + 2 \cdot \#$  $(A_X) + 2 \cdot \#$  (A_X)

and, writing as before $s^k_{p,q}(X) \coloneqq \dim H^k(\mathcal{L}_{p,q}(X))$,

$s^2_{1,0}(X) = \#$  $(A_X) + 2 \cdot \#$  $(A_X) + 2 \cdot \#$  $(A_X) +$

$\#$  $(A_X) + \#$  (A_X).

On the other hand, $FD^{0,2}(X) = h^{0,2}(X) - e^{0,2}_{\infty}(X)$ counts the dimensions of all differentials in the Frölicher spectral sequence starting at $(0,2)$, i.e.

$FD^{0,2}(X) = \#$  $(A_X) + \#$  (A_X).

Therefore,

$$FD^{0,2} = h^{0,3}_{BC} + s^2_{1,0} - b_3,$$

which is upper semi-continuous since the first two terms are and b_3 is constant. □

Remark 2.7. — The proof also shows that the quantity $e_\infty^{0,1}$ behaves lower semi-continuously. The result should be compared to the fact that $e_\infty^k = \bigoplus_{p+q=k} e_\infty^{p,q} = b_k$ is constant along deformations and the total Frölicher defect $FD^k = \bigoplus_{p+q=k} FD^{p,q} = h_{\bar\partial}^k - b_k$ is upper semi-continuous.

2.5. Index formulae

Denote by $\chi_p(X) := \sum_q (-1)^q h_{\bar\partial}^{p,q}(X)$ the analytical index of $(\mathcal{A}_X^{p,\bullet}, \bar\partial)$ and its topological index by

$$td_p(X) := \int_X \mathrm{Td}(X)\,\mathrm{ch}(\Omega_X^p) = \int_X \frac{\sum_k (-1)^k \,\mathrm{Td}(TX \otimes \mathbb{C})\,\mathrm{ch}(\mathcal{A}_X^{p,k})}{c_n(X)},$$

where $\mathrm{Td}(X) = \mathrm{Td}(TX^{1,0})$ denotes the Todd class, ch the Chern character and Ω_X^p the holomorphic p-forms. The quotient on the right hand can be computed from the universal expression on the classifying space of $U(n)$, even if $c_n(X) = 0$, see [3, Section 3], [13, Chapter 3]. The Hirzebruch–Riemann–Roch relations can be expressed as $\chi_p(X) = td_p(X)$ for all $p \in \mathbb{Z}$, see [3].

Similarly, let us denote by $\chi_{p,q}(X) := \sum_k (-1)^k \dim H^k(\mathcal{L}_{p,q}^\bullet(X))$ the analytical index of $\mathcal{L}_{p,q}^\bullet$ and its topological index by

$$td_{p,q}(X) := \int_X \frac{\sum_k (-1)^k \,\mathrm{Td}(TX \otimes \mathbb{C})\,\mathrm{ch}(\mathcal{L}_{p,q}^k)}{c_n(X)}.$$

The Atiyah–Singer index theorem [3, 4] then yields:

Theorem 2.8 (ABC index formulae). — *For any compact complex manifold X and $p, q \in \mathbb{Z}$,*

$$\chi_{p,q}(X) = td_{p,q}(X).$$

Remark 2.9. — Strictly speaking, [3, 4] treat only elliptic complexes where all operators have order one, which is not the case for $\mathcal{L}_{p,q}(X)$. However, the validity of Theorem 2.8 also follows a posteriori from Theorem E.

Proof of Theorem E. — Recall from [7, 14] that there are subcomplexes $(\mathcal{S}_p^\bullet, \partial)$ and $(\bar{\mathcal{S}}_q^\bullet, \bar\partial)$ of $\mathcal{L}_{p,q}$ defined as follows (if $p, q \geqslant 1$):

$$\mathcal{S}_p^k := \begin{cases} \Omega_X^k & \text{if } 0 \leqslant k \leqslant p-1 \\ 0 & \text{else,} \end{cases} \qquad \bar{\mathcal{S}}_q^k := \begin{cases} \bar\Omega_X^k & \text{if } 0 \leqslant k \leqslant q-1 \\ 0 & \text{else.} \end{cases}$$

If $p = 0$ or $q = 0$, one sets instead $\mathcal{S}_p^0 = \mathbb{C}$, resp. $\bar{\mathcal{S}}_q^0 = \mathbb{C}$ and all other components equal to 0. Set $\mathcal{S}_{p,q}^k \coloneqq \mathcal{S}_p^k + \bar{\mathcal{S}}_q^k$, the sum being direct except for $k = 0$. It is then known [7, 14] that $\mathcal{S}_{p,q} \hookrightarrow \mathcal{L}_{p,q}$ is a quasi-isomorphism. Further, there is a short exact sequence of complexes

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{S}_p^\bullet \oplus \bar{\mathcal{S}}_q^\bullet \longrightarrow \mathcal{S}_{p,q}^\bullet \longrightarrow 0.$$

For a complex C of abelian sheaves on X, denote by

$$\chi(C) \coloneqq \sum_k (-1)^k \dim \mathbb{H}^k(C)$$

its hypercohomology Euler characteristic. Then, using $\chi_p = (-1)^n \chi_{n-p}$, there is an equality of Euler characteristics:

$$\begin{aligned}
\chi_{p,q}(X) &= \chi(\mathcal{S}_{p,q}^\bullet) \\
&= \chi(\mathcal{S}_p^\bullet) + \chi(\overline{\mathcal{S}_{\mathrm{II}}}^\bullet) - \chi(\mathbb{C}) \\
&= \sum_{k=0}^{p-1} (-1)^k \chi_k + \sum_{k=0}^{q-1} (-1)^k \chi_k - \sum_{k=0}^{n} (-1)^k \chi_k \\
&= \sum_{k=0}^{p-1} (-1)^k \chi_k + \sum_{k=n-q}^{n} (-1)^k \chi_k - \sum_{k=0}^{n} (-1)^k \chi_k \\
&= \sum_{k=p}^{n-q} (-1)^{k+1} \chi_k
\end{aligned}$$

To identify the characteristic number expressions, we instead identify the K-theory classes of the relevant complexes

$$[\mathcal{A}_X^{p,\bullet}] \coloneqq \sum_k (-1)^k [\mathcal{A}_X^{p,k}] \in K(X) \text{ and } [\mathcal{L}_{p,q}^\bullet] \coloneqq \sum_k (-1)^k [\mathcal{L}_{p,q}^k] \in K(X),$$

using only the relation $(\mathcal{A}_X^{p,q})^\vee \cong \mathcal{A}_X^{n-q,n-p}$. Assume for simplicity $p+q \leqslant n$ and $p > q$, the other cases are similar. The following chain of equalities in $K(X)$, where we drop the square brackets, might be easier to follow with the following picture in mind, which illustrates the case $n = 3$, $(p, q) = (2, 1)$:

$$\begin{array}{|c|c|c|c|}\hline & & + & - \\ \hline & & - & + \\ \hline & & + & - \\ \hline + & - & & \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|}\hline & & + & \\ \hline & & - & \\ \hline & & + & - \\ \hline & & & \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|}\hline & & + & \\ \hline & & - & \\ \hline & & + & \\ \hline & & - & \\ \hline\end{array}$$

$$\begin{aligned}
\mathcal{L}^{\bullet}_{p,q} &= \sum_{k\in\mathbb{Z}}(-1)^k\mathcal{L}^k_{p,q}\\
&= \sum_{k\leqslant p+q-2}(-1)^k\sum_{\substack{r+s=k\\ r<p,s<q}}\mathcal{A}^{r,s}_X + \sum_{k\geqslant p+q-1}(-1)^k\sum_{\substack{r+s=k+1\\ r\geqslant p,s\geqslant q}}\mathcal{A}^{r,s}_X\\
&= \sum_{r=0}^{p-1}\sum_{s=0}^{q-1}(-1)^{r+s}\mathcal{A}^{r,s}_X + \sum_{r=p}^{n}\sum_{s=q}^{n}(-1)^{r+s-1}\mathcal{A}^{r,s}_X\\
&= \sum_{r=n-q+1}^{n}\sum_{s=n-p+1}^{n}(-1)^{r+s}\mathcal{A}^{r,s}_X + \sum_{r=p}^{n}\sum_{s=q}^{n}(-1)^{r+s-1}\mathcal{A}^{r,s}_X\\
&= \sum_{r=p}^{n-q}\sum_{s=q}^{n}(-1)^{r+s-1}\mathcal{A}^{r,s}_X + \sum_{r=n-q+1}^{n}\sum_{s=q}^{n-p}(-1)^{r+s-1}\mathcal{A}^{r,s}_X\\
&= \sum_{r=p}^{n-q}\sum_{s=0}^{n}(-1)^{r+s-1}\mathcal{A}^{r,s}_X = \sum_{r=p}^{n-q}(-1)^{r+1}\mathcal{A}^{r,\bullet}_X.
\end{aligned}$$

And therefore:

$$td_{p,q} = \sum_{r=p}^{n-q}(-1)^{r+1}td_p,$$

which implies the theorem since $\chi_{p,q} = td_{p,q}$ for all p,q if and only if $\chi_p = td_p$ for all p. □

Remark 2.10. — In particular, we have shown the following relation between Dolbeault and Schweitzer cohomologies:

$$\chi_{p,q}(X) = \sum_{k=p}^{n-q}(-1)^{k+1}\chi_p(X).$$

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