Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Wanxing Liu

Convergence of cscK metrics on smooth minimal models of general type

Article à paraître, mis en ligne le 17 mai 2024, 30 p.

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
Article à paraître
Mis en ligne le 17 mai 2024.

CONVERGENCE OF CSCK METRICS ON SMOOTH MINIMAL MODELS OF GENERAL TYPE

by Wanxing LIU

Abstract. — We consider constant scalar curvature Kähler metrics on a smooth minimal model of general type in a neighborhood of the canonical class, which is the perturbation of the canonical class by a fixed Kähler metric. We show that sequences of such metrics converge smoothly on compact subsets away from a subvariety to the singular Kähler–Einstein metric in the canonical class. This confirms partially a conjecture of Jian–Shi–Song about the convergence behavior of such sequences.

Résumé. — On considère des métriques de Kähler à courbure scalaire constante sur une surface lissemodèle minimal de type général dans un voisinage de la classe canonique, qui estla perturbation de la classe canonique par une métrique de Kähler fixe. Nous montrons queles séquences de ces métriques convergent en douceur sur des sous-ensembles compacts loin d'un sous-ensemble. variété à la métrique singuliére de Kähler Einstein dans la classe canonique. Cela confirme partiellement une conjecture de Jian–Shi–Song sur le comportement de convergence detels séquences.

1. Introduction

Since the work of Calabi [5, 6] there has been much interest in the existence of constant scalar curvature Kähler (cscK) metrics. For the Kähler Einstein metric which is a special type of the cscK metric, Yau [53] and Aubin [2] established independently the existence of such a metric on Kähler manifolds of negative first Chern class, and Yau [53] also showed it for manifolds of zero first Chern class. For manifolds of positive first Chern class, the Yau–Tian–Donaldson conjecture predicted that the existence of a Kähler Einstein metric is equivalent to the K-stability. Chen–Donaldson–Sun [11, 12, 13] proved that the K-stability is sufficient for the existence of a Kähler Einstein metric (see also Tian [48]), while the necessity was shown by Tian [46], Donaldson [19], Stoppa [42] and Berman [3].

Keywords: Kähler geometry, Constant Scalar Curvature Kähler metric.
2020 *Mathematics Subject Classification:* 32W20.

For the cscK metric, Donaldson [20] considered the existence of cscK metrics on toric surfaces. Chen–Cheng [10] proved that the properness of the Mabuchi functional is sufficient for the existence of a cscK metric building on the work of [8, 14, 15]. The necessity was proven by Berman–Darvas–Lu [4]. We refer the interested reader to [16, 31, 32, 34] for surveys and related developments of this area. Following the breakthrough made by Chen–Cheng, Jian–Shi–Song [27] showed the following theorem.

THEOREM 1.1 ([27, Theorem 1.1]). — *Let (M, ω_0) be a compact Kähler manifold. If the canonical bundle K_M is semi-ample, then for any $\varepsilon > 0$ small enough, there exists a unique cscK metric in the Kähler class $-c_1(M) + \varepsilon[\omega_0] = c_1(K_M) + \varepsilon[\omega_0]$.*

Recently, using different tools, Sjöstrom Dyrefelt [36] and Song [37] strengthened this result to all compact Kähler manifolds with nef canonical bundle. We define the first Chern class $c_1(M)$ to be

$$c_1(M) = [\mathrm{Ric}(\omega_0)] = -[\sqrt{-1}\partial\bar{\partial}\log\det(g_0)], \tag{1.1}$$

where g_0 is the metric tensor of ω_0. Notice that this differs from the usual definition of $c_1(M)$ by a factor of 2π. Theorem 1.1 is a generalization of Arezzo–Pacard's [1] result on minimal surfaces of general type. The proof of this result is based on Chen–Cheng [10] and the properness criterions developed by Weinkove [52], Song–Weinkove [40] and Li–Shi–Yao [30], and Jian–Shi–Song also made the following conjecture.

CONJECTURE 1.2. — *Let (M, ω_0) be a compact Kähler manifold with semi-ample canonical bundle K_M. Then for any sequence ε_n going to zero, the corresponding sequence of cscK metrics in $-c_1(M) + \varepsilon_n[\omega_0]$ converges to the twisted Kähler–Einstein metric g_{can} on the canonical model M_{can} of M. The convergence should be both global in Gromov–Hausdorff topology and local in smooth topology away from the singular fibres of the canonical map $\Phi : M \to M_{\mathrm{can}}$.*

Jian–Shi–Song further noted that this conjecture can be understood from the perspective of slope stability introduced by Ross–Thomas [33] and is related to the result of Gross–Wilson [24] from the standpoint of the Strominger–Yau–Zaslow [43] conjecture in mirror symmetry. If the canonical model of M is smooth and the canonical morphism $\Phi : M \to M_{\mathrm{can}}$ has no singular fibers, Conjecture 1.2 was shown to be true by Fine [23] (Theorem 8.1 and its proof), and the main purpose of the current paper is to show that the local smooth convergence of Conjecture 1.2 holds when $-c_1(M)$ is big and nef.

We call a compact Kähler manifold M a smooth minimal model if $-c_1(M)$ is nef, and a manifold of general type if $-c_1(M)$ is big. Recall that $-c_1(M)$ is said to be nef if for all $\varepsilon > 0$, $-c_1(M) > -\varepsilon[\omega_0]$ and big and nef if it is nef and $(-c_1(M))^n > 0$. One immediate benefit of assuming that $-c_1(M)$ is big and nef is the existence of a semi-positive representative in $-c_1(M)$. Since $-c_1(M)$ is big and nef, M is Moishezon which implies that M is projective. By Kawamata's base point free theorem, K_M is semi-ample. Hence there exists $\eta \in -c_1(M)$, which is some multiple of the pullback of the Fubini–Study metric through the canonical map $\Phi : M \to \mathbb{P}^N$. In particular, η is semi-positive, and we consider the sequence of reference Kähler metrics $\omega_\varepsilon = \eta + \varepsilon\omega_0 > 0$. Also notice that we can choose a volume form Ω such that $\mathrm{Ric}(\Omega) = -\eta$, then by Yau's theorem [53] we can always choose ω_0 to be such that $\mathrm{Ric}(\omega_0) = -\eta$.

Theorem 1.3. — *Suppose that (M, ω_0) is a compact Kähler manifold of dimension n with its canonical bundle being big and nef, and* $\mathrm{Ric}(\omega_0) = -\eta$*. There exists an effective divisor E such that for any ε_n converging to zero the corresponding sequence of cscK metrics $\omega_{\varphi_{\varepsilon_n}} \in -c_1(M) + \varepsilon_n[\omega_0]$, given by Theorem 1.1, converges in $C^\infty_{\mathrm{loc}}(M\backslash E)$ to the unique singular Kähler Einstein metric in $-c_1(M)$.*

The singular Kähler Einstein metric was first constructed by Kobayashi [28] in complex dimension 2, and then it was constructed in all dimensions as the limit of the normalized Kähler Ricci flow. Consider the normalized Kähler–Ricci flow:

$$\frac{\partial}{\partial t}\omega_t = -\mathrm{Ric}(\omega_t) - \omega_t,\ \omega|_{t=0} = \omega_0,\ \omega_t > 0. \tag{1.2}$$

The following theorem first appeared in the work of Tsuji [51], but later Tian–Zhang [49] extended it and clarified some parts of the proof (see also Song–Weinkove [41] or Tosatti [50] for nice expositions of this result).

Theorem 1.4 ([49, Theorem 0.1]). — *Let M be a smooth minimal model of general type. Then*

1. *The solution $\omega = \omega(t)$ of the normalized Kähler–Ricci flow starting at any Kähler metric ω_0 on M exists for all time.*
2. *There exists an effective divisor E of M such that $\omega(t)$ converges in $C^\infty_{\mathrm{loc}}(M\backslash E)$ to a smooth Kähler metric on $M\backslash E$.*

Furthermore, the Kähler metric obtained above is the unique metric satisfying

1. $\mathrm{Ric}(\omega_{\mathrm{KE}}) = -\omega_{\mathrm{KE}}$ *on $M\backslash E$.*

(2) *There exists a constant* C *such that*

$$C\omega_0^n \leqslant \omega_{\mathrm{KE}}^n \leqslant \frac{1}{C}\omega_0^n. \tag{1.3}$$

ω_{KE} was also constructed by Eyssidieux–Guedj–Zeriahi [22] using pluripotential theory.

As a consequence of Theorem 1.1, we can pick a unique sequence of cscK metrics $\omega_{\varphi_\varepsilon} \in [\omega_\varepsilon]$. The Kähler potentials of these metrics $\varphi_\varepsilon \in \mathcal{H}_{\omega_\varepsilon}$ satisfy the following coupled equations:

$$\begin{aligned} \frac{\omega_{\varphi_\varepsilon}^n}{\omega_0^n} &= \mathrm{e}^{F_\varepsilon}, \\ \Delta_{\omega_{\varphi_\varepsilon}} F_\epsilon &= -\underline{R_\varepsilon} - \mathrm{tr}_{\omega_{\varphi_\varepsilon}}\eta \end{aligned} \tag{1.4}$$

where $\underline{R_\epsilon} = n\frac{c_1(M)\cdot[\omega_\epsilon]^{n-1}}{[\omega_\varepsilon]^n}$, $\underline{R_\epsilon} \to -n$ as $\epsilon \to 0$, and

$$\mathcal{H}_{\omega_\varepsilon} = \left\{ v \in C^\infty(M) \,\middle|\, \omega_\varepsilon + \sqrt{-1}\partial\bar{\partial} v > 0, \sup_M v = 0 \right\}. \tag{1.5}$$

We stress that it is important to assume bigness of $-c_1(M)$ in order for the limit of $\underline{R_\varepsilon}$ to be $-n$, otherwise it is not true. Also notice that the equations (1.4) are slightly different from the coupled equations considered in [9]:

$$\begin{aligned} \frac{\omega_{\varphi_\varepsilon}^n}{\omega_\varepsilon^n} &= \mathrm{e}^{F_\varepsilon}, \\ \Delta_{\omega_{\varphi_\varepsilon}} F_\epsilon &= -\underline{R_\varepsilon} + \mathrm{tr}_{\omega_{\varphi_\varepsilon}}\mathrm{Ric}(\omega_\varepsilon). \end{aligned} \tag{1.6}$$

Specifically, we replace ω_ε in the denominator of the first equation by ω_0 and adapt the second equation accordingly because ω_ε is degenerating. We will see later that we will have to adjust the definition of the Mabuchi Energy to accommodate this change. The strategy of the proof of Theorem 1.3 consists of the following steps:

(1) We establish through Sections 2-4 a degenerate version of the estimates in [9]. More specifically, we show that φ_ε is bounded in $C^\infty_{\mathrm{loc}}(M\backslash E)$ and the bound depends only on $\int_M \log\frac{\omega_{\varphi_\varepsilon}^n}{\omega_0^n}\frac{\omega_{\varphi_\varepsilon}^n}{n!} = \int_M \mathrm{e}^{F_\varepsilon} F_\varepsilon \frac{\omega_0^n}{n!}$ and (M,ω_0). We will call $\int_M \log\frac{\omega_{\varphi_\varepsilon}^n}{\omega_0^n}\frac{\omega_{\varphi_\varepsilon}^n}{n!}$ the entropy.

(2) We show in Section 5 using a method of Dervan [18] that $\int_M \mathrm{e}^{F_\varepsilon} F_\varepsilon \frac{\omega_0^n}{n!}$ is uniformly bounded independent of ε, thus φ_ε is uniformly bounded in $C^\infty_{\mathrm{loc}}(M\backslash E)$ based on (1).

(3) In Section 6 we use the estimates and an integration by part argument to conclude that $\omega_{\varphi_\varepsilon}$ has to converge in $C^\infty_{\mathrm{loc}}(M\backslash E)$ to the unique singular Kähler Einstein metric in $-c_1(M)$.

We remark that Zheng [55] considered the problem of the L^1 convergence of a sequence of smooth cscK metrics in a neighborhood of an arbitrary big class. There he also had to generalize Chen–Cheng's [9] original estimates. Our estimates are different in the sense that we are able to get full non-degenerate 0-th order estimates on F_ε. Furthermore, the estimates in Zheng [55] are proved with respect to a sequence of specially constructed reference metrics. Also, one critical element of his proof is a version of alpha invariant for any big class using machinery from pluripotential theory, but we do not need it here.

We conclude this section by mentioning that there are also generalizations of Chen–Cheng's estimates in other directions. For instance, Shen [35] generalizes them to the Hermitian setting, and He generalizes them to Sasaki manifolds [25], and extremal metrics [26].

Acknowledgement

The author is grateful to his thesis advisor Ben Weinkove for his continued support, encouragement, countless discussions around these results as well as many valuable comments on the manuscript. The author would also like to thank Jian Song for pointing out an error in the original proof of Theorem 2.5. Finally, the author would like to thank the referee for carefully reading the manuscript and made numerous useful suggestions.

2. C^0 estimates

In this section we produce C^0 estimates for φ_ε and F_ε solving the following coupled equations:

$$\begin{aligned} \frac{\omega_{\varphi_\varepsilon}^n}{\omega_0^n} &= \mathrm{e}^{F_\varepsilon}, \\ \Delta_{\omega_{\varphi_\varepsilon}} F_\varepsilon &= -\underline{R_\varepsilon} - \operatorname{tr}_{\omega_{\varphi_\varepsilon}} \eta. \end{aligned} \tag{2.1}$$

We start with the C^0 estimate on φ_ε.

Theorem 2.1. — *There exists a constant $C > 0$ such that $\|\varphi_\varepsilon\|_{C^0(M)} \leqslant C$ where C is dependent on $\int_M \log \frac{\omega_{\varphi_\varepsilon}^n}{\omega_0^n} \frac{\omega_{\varphi_\varepsilon}^n}{n!}$.*

Before we prove this, we should make it clear that the techniques we use here follow exactly the same route laid out in the proof of theorem 5.2 in [9]. However the estimates in [9] depend on $\operatorname{Ric}(\omega_\varepsilon)$, but the metric is degenerating as $\varepsilon \to 0$, so we have to make essential modifications which involves clarifying that the $\alpha-$invariants can be controlled uniformly as shown in Lemma 2.3, specifying different choices of constants in Lemma 2.4, and checking everything still works as expected. It is during the course of this computation where we use the assumption that $\mathrm{Ric}(\omega_0) = -\eta$ and the fact that $\underline{R_\varepsilon} \to -n$ as $\varepsilon \to 0$, neither of which is needed in [9].

Let us start the proof with some preparations. Denote $\mathrm{Vol}(\omega_\varepsilon) = \int_M \omega_\varepsilon^n$, and as in [9] we define a smooth real-valued function ρ_ε to be the unique solution to the following equation using Yau's theorem [53]:

$$\begin{aligned} (\omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}\rho_\varepsilon)^n &= \frac{\mathrm{Vol}(\omega_\varepsilon)\,\mathrm{e}^{F_\varepsilon}\,\Phi(F_\varepsilon)\omega_0^n}{\int_M \mathrm{e}^{F_\varepsilon}\,\Phi(F_\varepsilon)\omega_0^n}, \\ \sup_M \rho_\varepsilon &= 0 \end{aligned} \tag{2.2}$$

where $\Phi(F_\varepsilon) = \sqrt{F_\varepsilon^2 + 1}$. Let us then recall the definition of Tian's α-invariant [45].

Proposition 2.2 (Tian's α-invariant). — *For any Kähler class $[\omega]$ on M, there exists an invariant $\alpha(M, [\omega]) > 0$ such that for any $\alpha < \alpha(M, [\omega])$, we have*

$$\int_M \mathrm{e}^{-\alpha v}\,\omega^n \leqslant C \tag{2.3}$$

for all $v \in \mathcal{H}_\omega$.

Lemma 2.3. — *Given $\alpha > 0$ with $\alpha < \alpha(M, [\eta + \omega_0])$, there is a uniform constant $C > 0$ such that for all $\varepsilon > 0$ with $\varepsilon \leqslant 1$ we have*

$$\int_M \mathrm{e}^{-\alpha v}\,\omega_0^n \leqslant C \tag{2.4}$$

for all $v \in \mathcal{H}_{\eta+\varepsilon\omega_0}$.

Proof. — The key observation is that when $\varepsilon > 0$ and $\varepsilon \leqslant 1$, $\eta + \varepsilon\omega_0 \leqslant \eta + \omega_0$ implies that $\mathcal{H}_{\eta+\varepsilon\omega_0} \subset \mathcal{H}_{\eta+\omega_0}$. Thus, for any $v \in \mathcal{H}_{\eta+\varepsilon\omega_0}$, we have for the fixed $\alpha > 0$ and $\alpha < \alpha(M, [\eta + \omega_0])$

$$\int_M \mathrm{e}^{-\alpha v}\,\omega_0^n \leqslant \int_M \mathrm{e}^{-\alpha v}(\eta + \omega_0)^n \leqslant C. \tag{2.5}$$ □

The most important estimate we need is the following Lemma analogous to [9, Theorem 5.2].

Lemma 2.4. — *There exists a constant C such that for all $\varepsilon > 0$ sufficiently small*

$$F_\varepsilon + \varepsilon\rho_\varepsilon - (1 + \varepsilon)\varphi_\varepsilon \leqslant C \tag{2.6}$$

where C only depends on $\int_M \log\frac{\omega_{\varphi_\varepsilon}^n}{\omega_0^n}\frac{\omega_{\varphi_\varepsilon}^n}{n!}$.

Proof. — Given a point $p_0 \in M$, $0 < d_0 < 1$, we start by choosing a smooth cut-off function f such that

$$\begin{gathered} 1-\theta \leqslant f \leqslant 1, \\ f(p_0)=1, f \equiv 1-\theta \text{ outside } B_{\frac{d_0}{2}}(p_0), \\ |\partial f|^2_{\omega_0} \leqslant \frac{4\theta^2}{d_0^2}, \quad |\partial^2 f|^2_{\omega_0} \leqslant \frac{4\theta}{d_0^2} \end{gathered} \tag{2.7}$$

where p_0 and $\theta > 0$ are going to be specified later, and d_0 is fixed to be a constant strictly larger than 0 and strictly less than 1. Let $\alpha > 0$ be a fixed constant strictly less than $\alpha(M, [\eta + \omega_0])$, and choose $\delta = \frac{\alpha}{4n}$. We will denote the metric tensor of $\omega_{\varphi_\varepsilon}$ by g_{φ_ε}, and suppress ε for simplicity of notation while carrying out calculations. Calculate

$$\begin{aligned} (2.8)\quad & \Delta_{\omega_\varphi}(\mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)} f) \\ &= \Delta_{\omega_\varphi}(\mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)})f + \mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}\,\Delta_{\omega_\varphi}(f) \\ &\qquad + 2\,\mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}\,\delta\mathrm{Re}(g_\varphi^{i\bar{j}}\partial_i(F+\varepsilon\rho-(1+\varepsilon)\varphi)\bar{\partial}_j f) \\ &= \mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}\, f(\delta^2|\partial(F+\varepsilon\rho-(1+\varepsilon)\varphi)|^2_{\omega_\varphi} \\ &\qquad + \delta\Delta_{\omega_\varphi}(F+\varepsilon\rho-(1+\varepsilon)\varphi)) + \mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}\,\Delta_{\omega_\varphi} f \\ &\qquad\quad + 2\,\mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}\,\delta\mathrm{Re}(g_\varphi^{i\bar{j}}\partial_i(F+\varepsilon\rho-(1+\varepsilon)\varphi)\bar{\partial}_j f). \end{aligned}$$

We estimate the terms involved in the above calculation as follows

$$\begin{aligned} (2.9)\quad & 2\delta\mathrm{Re}(g_\varphi^{i\bar{j}}\partial_i(F+\varepsilon\rho-(1+\varepsilon)\varphi)\bar{\partial}_j f) \\ &\geqslant -\delta^2 f|\partial(F+\varepsilon\rho-(1+\varepsilon)\varphi)|^2_{\omega_\varphi} - \frac{|\partial f|^2_{\omega_\varphi}}{f} \\ &\geqslant -\delta^2 f|\partial(F+\varepsilon\rho-(1+\varepsilon)\varphi)|^2_{\omega_\varphi} - \frac{|\partial f|^2_{\omega_0}\mathrm{tr}_{\omega_\varphi}\omega_0}{f} \\ &\geqslant -\delta^2 f|\partial(F+\varepsilon\rho-(1+\varepsilon)\varphi)|^2_{\omega_\varphi} - \frac{4\theta^2}{d_0^2(1-\theta)}\mathrm{tr}_{\omega_\varphi}\omega_0, \end{aligned}$$

$$\mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}\,\Delta_{\omega_\varphi} f \geqslant -\,\mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}\,\frac{4\theta}{d_0^2(1-\theta)} f\mathrm{tr}_{\omega_\varphi}\omega_0. \tag{2.10}$$

The key computation is the following:

$$\begin{aligned} (2.11)\quad & \Delta_{\omega_\varphi}(F+\varepsilon\rho-(1+\varepsilon)\varphi) \\ &= -(\underline{R}+(1+\varepsilon)n) - \mathrm{tr}_{\omega_\varphi}\eta + (1+\varepsilon)\mathrm{tr}_{\omega_\varphi}\omega + \varepsilon\Delta_{\omega_\varphi}\rho. \end{aligned}$$

Let $A_\Phi(F) = \int_M \mathrm{e}^F \Phi(F)\omega_0^n$, and notice by (2.2)

$$\begin{aligned}
\Delta_{\omega_\varphi}\rho &= \mathrm{tr}_{\omega_\varphi}(\omega + \sqrt{-1}\partial\overline{\partial}\rho) - \mathrm{tr}_{\omega_\varphi}\omega \\
&\geqslant n(\mathrm{e}^{-F}\,\mathrm{e}^F\,\mathrm{Vol}(\omega)\Phi(F)A_\Phi(F)^{-1})^{\frac{1}{n}} - \mathrm{tr}_{\omega_\varphi}\omega \\
&= n(\mathrm{Vol}(\omega)\Phi(F)A_\Phi(F)^{-1})^{\frac{1}{n}} - \mathrm{tr}_{\omega_\varphi}\omega.
\end{aligned} \tag{2.12}$$

Then we have

$$\begin{aligned}
&\Delta_{\omega_\varphi}(F + \varepsilon\rho - (1+\varepsilon)\varphi) \\
&\geqslant \left(-\underline{R} - (1+\varepsilon)n + \varepsilon n\mathrm{Vol}(\omega)^{\frac{1}{n}} A_\Phi(F)^{-\frac{1}{n}}\Phi(F)^{\frac{1}{n}}\right) + (1+\varepsilon)\mathrm{tr}_{\omega_\varphi}\omega \\
&\qquad - \varepsilon\mathrm{tr}_{\omega_\varphi}\omega - \mathrm{tr}_{\omega_\varphi}\eta \\
&= \left(-\underline{R} - (1+\varepsilon)n + \varepsilon n\mathrm{Vol}(\omega)^{\frac{1}{n}} A_\Phi(F)^{-\frac{1}{n}}\Phi(F)^{\frac{1}{n}}\right) + \varepsilon\mathrm{tr}_{\omega_\varphi}\omega_0.
\end{aligned} \tag{2.13}$$

Combining all these calculations, we conclude that

$$\begin{aligned}
&\Delta_{\omega_\varphi}(\mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}\,f) \\
&\geqslant \delta f\,\mathrm{e}^{\delta(F+\varepsilon\rho-(1+\epsilon)\varphi)}(-\underline{R} - (1+\varepsilon)n + \varepsilon n\mathrm{Vol}(\omega)^{\frac{1}{n}} A_\Phi(F)^{-\frac{1}{n}}\Phi(F)^{\frac{1}{n}}) \\
&\qquad + \mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}\left(\delta f\varepsilon - \frac{4\theta}{d_0^2(1-\theta)}f - \frac{4\theta^2}{d_0^2(1-\theta)^2}\right)\mathrm{tr}_{\omega_\varphi}\omega_0.
\end{aligned} \tag{2.14}$$

Choosing

$$\theta = \frac{\delta\varepsilon}{64}d_0^2 = \frac{\alpha\varepsilon}{256n}d_0^2,$$

notice that when ε is small enough we have $\theta < \frac{1}{2}$ and $\frac{\delta\varepsilon}{64} \leqslant \sqrt{\frac{\delta\varepsilon}{128}}$, so that because $f \leqslant 1$ we estimate

$$\begin{aligned}(2.15)\quad \delta f\epsilon - \frac{4\theta}{d_0^2(1-\theta)}f - \frac{4\theta^2}{d_0^2(1-\theta)^2} &\geqslant \delta(1-\theta)\epsilon - \frac{4\theta}{d_0^2(1-\theta)} - \frac{4\theta^2}{d_0^2(1-\theta)^2}\\ &\geqslant \frac{\delta\epsilon}{2} - \frac{8\theta}{d_0^2} - \frac{16\theta^2}{d_0^2}\\ &\geqslant \frac{\delta\varepsilon}{2} - \frac{\delta\varepsilon}{8} - \frac{\delta\varepsilon}{8} > 0.\end{aligned}$$

So the coefficient of $\mathrm{tr}_{\omega_\varphi}\omega_0$ in (2.14) is positive, and we throw it away and conclude

$$\begin{aligned}(2.16)\quad &\Delta_{\omega_\varphi}(\mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)} f)\\ &\geqslant \delta f\, \mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}(-\underline{R} - (1+\varepsilon)n + \varepsilon n \mathrm{Vol}(\omega)^{\frac{1}{n}} A_\Phi(F)^{-\frac{1}{n}}\Phi(F)^{\frac{1}{n}}).\end{aligned}$$

Let $u = \mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}$, and choose p_0 to be the maximum point of u. Applying the Alexandrov–Bakelman–Pucci maximum principle in $B_{d_0}(p_0)$ we get

$$\begin{aligned}(2.17)\quad &\sup_{B_{d_0}(p_0)} uf \leqslant \sup_{\partial B_{d_0}(p_0)} uf\\ &+C\left(\int_{B_{d_0}(p_0)} \frac{u^{2n}\Big((-\underline{R}-(1+\varepsilon)n+\varepsilon n\mathrm{Vol}(\omega)^{\frac{1}{n}}A_\Phi(F)^{-\frac{1}{n}}\Phi(F)^{\frac{1}{n}})^-\Big)^{2n}}{\mathrm{e}^{-2F}}\omega_0^n\right)^{\frac{1}{2n}}\end{aligned}$$

where C is a constant dependent on the dimension of the manifold n, d_0 and δ. Notice that the integral is only nonzero on the region where

$$(2.18)\qquad -\underline{R} - (1+\varepsilon)n + \varepsilon n \mathrm{Vol}(\omega)^{\frac{1}{n}} A_\Phi(F)^{-\frac{1}{n}}\Phi(F)^{\frac{1}{n}} < 0.$$

Observe that for ε sufficiently small,

$$\begin{aligned}(2.19)\quad -\underline{R} - (1+\epsilon)n &= \frac{n[\eta]\cdot\sum_{k=0}^{n-1}\binom{n-1}{k}\varepsilon^k[\omega_0]^k\cdot[\eta]^{n-1-k}}{\sum_{k=0}^{n}\binom{n}{k}\varepsilon^k[\omega_0]^k\cdot[\eta]^{n-k}} - (1+\epsilon)\\ &\geqslant -C\varepsilon.\end{aligned}$$

So (2.18) is only possible if $F \leqslant C$ where C depends only on $A_\Phi(F)$, n and $A_\Phi(F)$ depends only on the entropy. So we get

$$\left(\int_{B_{d_0}(p_0)} \frac{u^{2n}\Big(-\underline{R} - (1+\varepsilon)n + \varepsilon n A_\Phi(F)^{-\frac{1}{n}}\Phi(F)^{\frac{1}{n}})^-\Big)^{2n}}{\mathrm{e}^{-2F}}\omega_0^n\right)^{\frac{1}{2n}} \tag{2.20}$$

$$= \left(\int_{B_{d_0}(p_0)\cap\{F\leqslant C\}} \frac{u^{2n}\Big(-\underline{R} - (1+\varepsilon)n + \varepsilon n A_\Phi(F)^{-\frac{1}{n}}\Phi(F)^{\frac{1}{n}})^-\Big)^{2n}}{\mathrm{e}^{-2F}}\omega_0^n\right)^{\frac{1}{2n}}$$

$$\leqslant \left(\int_{B_{d_0}(p_0)} \frac{u^{2n}((-\underline{R} - (1+\varepsilon)n)^-)^{2n}}{\mathrm{e}^{-2F}}\omega_0^n\right)^{\frac{1}{2n}}$$

$$\leqslant C\left(\int_{B_{d_0}(p_0)\cap\{F\leqslant C\}} \mathrm{e}^{2n\delta(F+\varepsilon\rho-(1+\epsilon)\varphi)}\,\mathrm{e}^{2F}\,\varepsilon^{2n}\omega_0^n\right)^{\frac{1}{2n}}$$

$$\leqslant C\varepsilon\left(\int_{B_{d_0}(p_0)} \mathrm{e}^{-(2n\delta(1+\varepsilon))\varphi}\,\omega_0^n\right)^{\frac{1}{2n}}$$

$$\leqslant C\varepsilon\left(\int_{B_{d_0}(p_0)} \mathrm{e}^{-\frac{\alpha}{2}\varphi}\,\omega_0^n\right)^{\frac{1}{2n}}$$

$$\leqslant C\varepsilon.$$

where for the last inequality we used Lemma 2.3. So

$$u(p_0) = \sup_M u \leqslant (1-\theta)\sup_M u + C\varepsilon, \tag{2.21}$$

and

$$u(p_0) \leqslant \frac{C\varepsilon}{\theta} = \frac{C\varepsilon}{\frac{\delta\varepsilon d_0^2}{64}} \leqslant C. \tag{2.22}$$

Thus we are able to conclude that for any ε sufficiently small,

$$F + \varepsilon\rho - (1+\varepsilon)\varphi \leqslant C. \tag{2.23}$$ □

With this estimate at our disposal, we now proceed to prove Theorem 2.1.

Proof of Theorem 2.1. — Let α be a positive constant strictly less than $\alpha(M, [\eta+\omega_0])$, the α-invariant associated with $[\eta+\omega_0]$, and $\varepsilon < \frac{\alpha}{2}$, we have

$$F + \frac{\alpha}{2}\rho - (1+\varepsilon)\varphi \leqslant F + \varepsilon\rho - (1+\varepsilon)\varphi \leqslant C \tag{2.24}$$

because $\rho \leqslant 0$. Thus for $\varepsilon < \frac{\alpha}{2}$,

$$\begin{aligned} C &\geqslant \int_M \mathrm{e}^{-\alpha\rho}\,\omega_0^n \geqslant \int_M \exp(2(F-(1+\varepsilon)\varphi - C))\omega_0^n \\ &\geqslant \int_M \exp(2(F-C))\omega_0^n \end{aligned} \tag{2.25}$$

where for the last inequality we used the fact that $\varphi \leqslant 0$. So we conclude that $\mathrm{e}^F \in L^2(M, \omega_0^n)$ when ε is small enough. Then the proof is done by applying a theorem first announced in Tian–Zhang [49] and later proved in Zhang [54] which asserts that if for some $p > 1$, $\mathrm{e}^F \in L^p(M, \omega_0^n)$, then the C^0 estimate on φ only depends on the $\| \mathrm{e}^F \|_{L^p(M,\omega_0^n)}$. This is a generalization of Kolodziej's [29] fundamental result to the degenerate setting. One can find more general versions of their result in Eyssidieux–Guedj–Zeriahi [21, 22] and Demailly–Pali [17] as well. □

To show that F_ε is uniformly bounded from below we recall a trick due to Song–Tian [38] which was pointed out to the author by Jian Song. We consider the following auxiliary complex Monge–Ampère equations:

$$(\omega_\varepsilon + \sqrt{-1}\partial\bar\partial h_\varepsilon)^n = \mathrm{e}^{h_\varepsilon} \frac{\omega_\varepsilon^n}{\omega_0^n}\omega_0^n. \tag{2.26}$$

Again, for each ε there exists a unique smooth real-valued h_ε solving (2.26) by Yau's theorem. At a maximum p_0 of h_ε we have

$$\mathrm{e}^{h_\varepsilon} = \frac{(\omega_\varepsilon + \sqrt{-1}\partial\bar\partial h_\varepsilon)^n}{\omega_0^n} \leqslant \frac{\omega_\varepsilon^n}{\omega_0^n} \leqslant C. \tag{2.27}$$

Thus, h_ε is uniformly bounded from above. Then by the theorem of Zhang [54] again, we have that $\|h_\varepsilon\|_{C^0(M)}$ is uniformly bounded.

Theorem 2.5. — *There exists a constant C depending only on $\|\varphi_\varepsilon\|_{C^0(M)}$ and $\|h_\varepsilon\|_{C^0(M)}$ such that*

$$F_\varepsilon \geqslant C \tag{2.28}$$

for ε sufficiently small.

Proof. — Compute at a minimum p_0 of $F + \varphi - h$, we get

$$\begin{aligned} 0 \leqslant \Delta_{\omega_\varphi}(F+\varphi-h) &= -\underline{R} + n - \mathrm{tr}_{\omega_\varphi}(\omega + \eta + \sqrt{-1}\partial\bar\partial h) \\ &\leqslant C - \mathrm{tr}_{\omega_\varphi}(\omega + \sqrt{-1}\partial\bar\partial h) \\ &\leqslant C - n\,\mathrm{e}^{\frac{h-F}{n}}\,. \end{aligned} \tag{2.29}$$

It implies that

$$F(p_0) \geqslant C. \tag{2.30}$$

So for any $p \in M$,

$$F(p)+\varphi(p)-h(p) \geqslant F(p_0)+\varphi(p_0)+h(p_0) \geqslant C+\varphi(p_0). \tag{2.31}$$ □

THEOREM 2.6. — *There exists a constant $C>0$ such that*

$$F_\varepsilon \leqslant C \tag{2.32}$$

for all ε sufficiently small.

Proof. — Using the L^p bound on e^F, the lower bound on F and the theorem of Zhang [54] again we can show that ρ_ε is uniformly bounded as well. Then the theorem follows immediately from Lemma 2.4. □

3. Degenerate bound on $|\partial\varphi_\varepsilon|^2_{\omega_0}$

Let us first recall the following the Kodaira's lemma, for handling degenerate reference metrics.

LEMMA 3.1. — *There exists an effective divisor E on M, a holomorphic section s of E which vanishes to order 1 along the divisor E, constants $\sigma>0, C>0$ such that for any $\delta' \in (0,\sigma]$, and any Hermitian metric h of $[E]$, where $[E]$ is the line bundle associated with E, we have*

$$\eta+\delta'\sqrt{-1}\partial\bar{\partial}\log|s|^2_h > C\delta'\omega_0. \tag{3.1}$$

Moreover, as a special case of the Kodaira lemma, we can and will choose E to be the exceptional locus of the canonical map $\Phi: M \to M_{\mathrm{can}}$, where Φ fails to be an isomorphism, so that η is Kähler outside E (see Song–Tian [39, Proposition 2.1]). The trick of applying Kodaira's lemma is commonly referred to as the *Tsuji's trick* in the literature, and the idea of it is straightforward. ω_ε is degenerate in the sense that it is tending to η which might be zero along E. By using the barrier function $\psi := \log|s|^2_h$ which is $-\infty$ along the singular set, one can carry out the usual maximum principle away from E. Equipped with this tool, we will show in this section a degenerate version of Theorem 2.2 in [9].

THEOREM 3.2. — *There exists constants $q>0$ and $C>0$ such that*

$$|\partial\varphi_\varepsilon|^2_{\omega_0} \leqslant C\frac{1}{|s|^{2q}_h} \tag{3.2}$$

where the constants depend only on $\|\varphi_\varepsilon\|_{C^0(M)}$, $\|F_\varepsilon\|_{C^0(M)}$ and (M,ω_0).

Proof. — We will suppress ε for simplicity and apply the maximum principle to

$$\mathrm{e}^{-(F+\lambda\varphi)+\frac{1}{2}\varphi^2+Q\delta'\psi}(|\partial\varphi|^2_{\omega_0}+K). \tag{3.3}$$

Here $\lambda>0, K>0, Q>0$ are constants to be determined later, and $\delta'\in(0,\min\{\sigma,1\}]$, where σ is the constant given in Lemma 3.1. Let $A(F,\varphi,\psi)=-(F+\lambda\varphi)+\frac{1}{2}\varphi^2+Q\delta'\psi$, and denote the metric tensor of ω_φ by g_φ, the metric tensor of ω by g and the metric tensor of ω_0 by g_0. Unless otherwise noted, we will always choose a normal holomorphic coordinate neighborhood for g_0 such that g_φ is diagonal.

$$\begin{aligned}&\Delta_{\omega_\varphi}(\mathrm{e}^A(|\partial\varphi|^2_{\omega_0}+K))\\ &\quad=\Delta_{\omega_\varphi}(\mathrm{e}^A)(|\partial\varphi|^2_{\omega_0}+K)+\mathrm{e}^A\,\Delta_{\omega_\varphi}(|\partial\varphi|^2_{\omega_0})+2g_\varphi^{i\bar{i}}\,\mathrm{e}^A\,\mathrm{Re}(A_i(|\partial\varphi|^2_{\omega_0})_{\bar{i}}).\end{aligned} \tag{3.4}$$

Let us estimate the three terms involved in the above equation separately

$$\begin{aligned}&\Delta_{\omega_\varphi}(\mathrm{e}^A)\\ &=\mathrm{e}^A\,g_\varphi^{i\bar{i}}|A_i|^2+\mathrm{e}^A(-\Delta_{\omega_\varphi}(F+\lambda\varphi)+\varphi\Delta_{\omega_\varphi}\varphi+Q\delta'\Delta_{\omega_\varphi}\psi)+\mathrm{e}^A\,g_\varphi^{i\bar{i}}|\varphi_i|^2\\ &=\mathrm{e}^A(\underline{R}-\lambda n+(\lambda-\varphi)\mathrm{tr}_{\omega_\varphi}\omega+\mathrm{tr}_{\omega_\varphi}\eta+n\varphi+Q\delta'\Delta_{\omega_\varphi}\psi)\\ &\qquad\qquad+\mathrm{e}^A\,g_\varphi^{i\bar{i}}|\varphi_i|^2+\mathrm{e}^A\,g_\varphi^{i\bar{i}}|A_i|^2,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\Delta_{\omega_\varphi}(|\partial\varphi|^2_{\omega_0})\\ &\quad=g_\varphi^{i\bar{i}}R_{\alpha\beta i\bar{i}}(g_0)\varphi_\alpha\varphi_{\overline{\beta}}+g_\varphi^{i\bar{i}}|\varphi_{\alpha i}|^2+g_\varphi^{i\bar{i}}|\varphi_{\alpha\bar{i}}|^2+g_\varphi^{i\bar{i}}(\varphi_{\alpha i\bar{i}}\varphi_{\overline{\alpha}}+\varphi_{\overline{\alpha}i\bar{i}}\varphi_\alpha).\end{aligned} \tag{3.6}$$

Differentiating

$$\log\frac{\omega_\varphi^n}{\omega_0^n}=F, \tag{3.7}$$

we get

$$g_\varphi^{i\bar{i}}(g_{i\bar{i},\alpha}+\varphi_{i\bar{i}\alpha})=F_\alpha \text{ and } g_\varphi^{i\bar{i}}(g_{i\bar{i},\overline{\alpha}}+\varphi_{i\bar{i}\overline{\alpha}})=F_{\overline{\alpha}}. \tag{3.8}$$

So

$$\begin{aligned}&\Delta_{\omega_\varphi}(|\partial\varphi|^2_{\omega_0})\\ &\qquad=g_\varphi^{i\bar{i}}R_{\alpha\overline{\beta}i\bar{i}}(g_0)\varphi_\alpha\varphi_{\overline{\beta}}+g_\varphi^{i\bar{i}}|\varphi_{\alpha i}|^2+g_\varphi^{i\bar{i}}|\varphi_{\alpha\bar{i}}|^2\\ &\qquad\qquad+F_\alpha\varphi_{\overline{\alpha}}+F_{\overline{\alpha}}\varphi_\alpha-2g_\varphi^{i\bar{i}}\mathrm{Re}(g_{i\bar{i},\alpha}\varphi_{\overline{\alpha}})\\ &\qquad\geqslant -C_1|\partial\varphi|^2_{\omega_0}\mathrm{tr}_{\omega_\varphi}\omega_0+g_\varphi^{i\bar{i}}|\varphi_{\alpha i}|^2+g_\varphi^{i\bar{i}}|\varphi_{\alpha\bar{i}}|^2-2\mathrm{Re}(A_\alpha\varphi_{\overline{\alpha}})\\ &\qquad\qquad-2(\lambda-\varphi)|\partial\varphi|^2_{\omega_0}+2Q\delta'\mathrm{Re}(\psi_\alpha\varphi_{\overline{\alpha}})-2g_\varphi^{i\bar{i}}\mathrm{Re}(g_{i\bar{i},\alpha}\varphi_{\overline{\alpha}}).\end{aligned} \tag{3.9}$$

where C_1 depends on a lower bound on the bisectional curvature of ω_0. Then we have

$$\begin{aligned}(3.10)\quad & \mathrm{e}^{-A}\,\Delta_{\omega_\varphi}(\mathrm{e}^A(|\partial\varphi|^2_{\omega_0}+K)) \\ &\geqslant (\underline{R}-\lambda n+(\lambda-\varphi)\mathrm{tr}_{\omega_\varphi}\omega+Q\delta'\Delta_{\omega_\varphi}\psi+\mathrm{tr}_{\omega_\varphi}\eta+n\varphi)(|\partial\varphi|^2_{\omega_0}+K) \\ &\quad + g_\varphi^{i\bar{i}}|A_i|^2(|\partial\varphi|^2_{\omega_0}+K)+|\partial\varphi|^2_{\omega_\varphi}(|\partial\varphi|^2_{\omega_0}+K) \\ &\quad - C_1|\partial\varphi|^2_{\omega_0}\mathrm{tr}_{\omega_\varphi}\omega_0+g_\varphi^{i\bar{i}}|\varphi_{\alpha i}|^2+g_\varphi^{i\bar{i}}|\varphi_{\alpha\bar{i}}|^2+(-2\lambda+2\varphi)|\partial\varphi|^2_{\omega_0} \\ &\quad - 2\mathrm{Re}(A_\alpha\varphi_{\bar{\alpha}})+2g_\varphi^{i\bar{i}}\mathrm{Re}(A_i(\varphi_{\alpha\bar{i}}\varphi_{\bar{\alpha}}+\varphi_{\bar{\alpha}\bar{i}}\varphi_\alpha)) \\ &\quad + 2Q\delta'\mathrm{Re}(\psi_\alpha\varphi_{\bar{\alpha}})-2g_\varphi^{i\bar{i}}\mathrm{Re}(g_{i\bar{i},\alpha}\varphi_{\bar{\alpha}}).\end{aligned}$$

Notice the following complete square in the above sum

$$(3.11)\quad g_\varphi^{i\bar{i}}|\varphi_{i\alpha}+A_i\varphi_\alpha|^2=g_\varphi^{i\bar{i}}|\varphi_{i\alpha}|^2+2g_\varphi^{i\bar{i}}\mathrm{Re}(A_i\varphi_\alpha\varphi_{\bar{\alpha}\bar{i}})+g_\varphi^{i\bar{i}}|A_i|^2|\partial\varphi|^2_{\omega_0}.$$

Also observe that

$$(3.12)\quad \begin{aligned}-A_\alpha\varphi_{\bar{\alpha}}+g_\varphi^{i\bar{i}}A_i\varphi_{\alpha\bar{i}}\varphi_{\bar{\alpha}}&=g_\varphi^{i\bar{i}}(A_i\varphi_{\alpha\bar{i}}\varphi_{\bar{\alpha}}-(g_\varphi)_{\alpha\bar{i}}A_i\varphi_{\bar{\alpha}}) \\ &=-g_\varphi^{i\bar{i}}g_{\alpha\bar{i}}A_i\varphi_{\bar{\alpha}}.\end{aligned}$$

In summary

$$\begin{aligned}(3.13)\quad & \Delta_\varphi(\mathrm{e}^A(|\partial\varphi|^2_{\omega_0}+K))\,\mathrm{e}^{-A} \\ &\geqslant Kg_\varphi^{i\bar{i}}|A_i|^2+g_\varphi^{i\bar{i}}|\varphi_i|^2(|\partial\varphi|^2_{\omega_0}+K) \\ &\quad + (\mathrm{tr}_{\omega_\varphi}(\lambda-\varphi)\omega+Q\delta'\Delta_{\omega_\varphi}\psi+\mathrm{tr}_{\omega_\varphi}\eta)(|\partial\varphi|^2_{\omega_0}+K) \\ &\quad + (\underline{R}-\lambda n+n\varphi)(|\partial\varphi|^2_{\omega_0}+K)-C_1|\partial\varphi|^2_{\omega_0}\mathrm{tr}_{\omega_\varphi}\omega_0+g_\varphi^{i\bar{i}}|\varphi_{\alpha\bar{i}}|^2 \\ &\quad + (-2\lambda+2\varphi)|\partial\varphi|^2_{\omega_0}-2g_\varphi^{i\bar{i}}\mathrm{Re}(g_{\alpha\bar{i}}A_i\varphi_{\bar{\alpha}}) \\ &\quad + 2Q\delta'\mathrm{Re}(\psi_\alpha\varphi_{\bar{\alpha}})-2\mathrm{Re}(g_\varphi^{i\bar{i}}g_{i\bar{i},\alpha}\varphi_{\bar{\alpha}}).\end{aligned}$$

Furthermore, by Young's inequality

$$(3.14)\quad 2\mathrm{Re}(g_\varphi^{i\bar{i}}g_{i\bar{i},\alpha}\varphi_{\bar{\alpha}})\leqslant C_2(\mathrm{tr}_{\omega_\varphi}\omega_0+|\partial\varphi|^2_{\omega_0}\mathrm{tr}_{\omega_\varphi}\omega_0),$$

and

$$(3.15)\quad 2\mathrm{Re}(g_\varphi^{i\bar{i}}g_{\alpha\bar{i}}A_i\varphi_{\bar{\alpha}})\leqslant C_3(g_\varphi^{i\bar{i}}|A_i|^2+|\partial\varphi|^2_{\omega_0}\mathrm{tr}_{\omega_\varphi}\omega_0).$$

Let $\lambda=(\frac{1}{\delta'}+\frac{1}{C_{\delta'}})(\|\varphi\|_{C^0(M)}+10+C_1+C_2+C_3)$, $Q=\lambda-\|\varphi\|_{C^0(M)}>\max\{\frac{10+C_1+C_2+C_3}{C_{\delta'}},\frac{2}{\delta'}\}$, $K=C_2+C_3$, so that $C_{\delta'}Q\geqslant 10+C_1+C_2+C_3$, $Q\delta'>1$ and recall that ψ is chosen such that

$$(3.16)\quad \omega+\delta'\sqrt{-1}\partial\bar{\partial}\psi\geqslant\eta+\delta'\sqrt{-1}\partial\bar{\partial}\psi>C_{\delta'}\omega_0.$$

So

$$(3.17)\quad (\mathrm{tr}_{\omega_\varphi}(\lambda-\varphi)\omega + Q\delta'\Delta_{\omega_\varphi}\psi + \mathrm{tr}_{\omega_\varphi}\eta)(|\partial\varphi|^2_{\omega_0}+K) \\ - C_1|\partial\varphi|^2_{\omega_0}\mathrm{tr}_{\omega_\varphi}\omega_0 - 2\mathrm{Re}(g_\varphi^{i\bar{i}}g_{i\bar{i},\alpha}\varphi_{\overline{\alpha}}) + Kg_\varphi^{i\bar{i}}|A_i|^2 + 2\mathrm{Re}(g_\varphi^{i\bar{i}}g_{\alpha\bar{i}}A_i\varphi_{\overline{\alpha}}) \\ \geqslant Q(\mathrm{tr}_{\omega_\varphi}\omega + \delta'\Delta_{\omega_\varphi}\psi)(|\partial\varphi|^2_{\omega_0}+K) - (C_1+C_2+C_3)|\partial\varphi|^2_{\omega_0}\mathrm{tr}_{\omega_\varphi}\omega_0 \\ - C_2\mathrm{tr}_{\omega_\varphi}\omega_0 + (K-C_3)g_\varphi^{i\bar{i}}|A_i|^2 \\ \geqslant QC_{\delta'}(|\partial\varphi|^2_{\omega_0}+K)\mathrm{tr}_{\omega_\varphi}\omega_0 - (C_1+C_2+C_3)|\partial\varphi|^2_{\omega_0}\mathrm{tr}_{\omega_\varphi}\omega_0 - C_2\mathrm{tr}_{\omega_\varphi}\omega_0 \\ \geqslant 10|\partial\varphi|^2_{\omega_0}\mathrm{tr}_{\omega_\varphi}\omega_0.$$

Recall that $\underline{R}\to -n$, so

$$(3.18)\qquad (\underline{R}-\lambda n + n\varphi)(|\partial\varphi|^2_{\omega_0}+K) \geqslant -C|\partial\varphi|^2_{\omega_0} - C,$$

and

$$(3.19)\qquad (-2\lambda+2\varphi)|\partial\varphi|^2_{\omega_0} \geqslant -C|\partial\varphi|^2_{\omega_0}$$

where the constants depend on the 0-th order estimate on φ. Also as an elementary consequence of the equation (see [9, p. 12])

$$(3.20)\qquad \frac{\omega_\varphi^n}{\omega_0^n} = \mathrm{e}^F$$

yields

$$(3.21)\qquad g_\varphi^{i\bar{i}}|\varphi_i|^2|\partial\varphi|^2_{\omega_0} + |\partial\varphi|^2_{\omega_0}\mathrm{tr}_{\omega_\varphi}\omega_0 \geqslant C|\partial\varphi|^{2+\frac{2}{n}}_{\omega_0}\,\mathrm{e}^{\frac{-F}{n}}\,.$$

Recall that $\psi = \log|s|^2_h$, where s is a holomorphic section vanishing on E. Then we get

$$(3.22)\qquad g_\varphi^{i\bar{i}}|\varphi_i|^2|\partial\varphi|^2_{\omega_0} + |\partial\varphi|^2_{\omega_0}\mathrm{tr}_{\omega_\varphi}\omega_0 \geqslant C|\partial\varphi|^{2+\frac{2}{n}}_{\omega_0}\,\mathrm{e}^{\frac{-F+Q\delta'\psi}{n}}\,.$$

We arrive at

$$(3.23)\quad \Delta_{\omega_\varphi}(\mathrm{e}^A(|\partial\varphi|^2_{\omega_0}+K)) \\ \geqslant \mathrm{e}^A(g_\varphi^{i\bar{i}}|\varphi_i|^2|\partial\varphi|^2_{\omega_0} + |\partial\varphi|^2_{\omega_0}\mathrm{tr}_{\omega_\varphi}\omega_0 - C|\partial\varphi|^2_{\omega_0} - C + 2Q\delta'\mathrm{Re}(\psi_\alpha\varphi_{\overline{\alpha}})) \\ \geqslant \mathrm{e}^A(C\,\mathrm{e}^{\frac{-F+Q\delta'\psi}{n}}(|\partial\varphi|^2_{\omega_0})^{1+\frac{1}{n}} - C|\partial\varphi|^2_{\omega_0} - C + 2Q\delta'\mathrm{Re}(\psi_\alpha\varphi_{\overline{\alpha}})) \\ \geqslant C(\mathrm{e}^{Q\delta'\psi}\,|\partial\varphi|^2_{\omega_0})^{1+\frac{1}{n}} - C\,\mathrm{e}^{Q\delta'\psi}\,|\partial\varphi|^2_{\omega_0} - C\,\mathrm{e}^{Q\delta'\psi} + 2\,\mathrm{e}^A\,Q\delta'\mathrm{Re}(\psi_\alpha\varphi_{\overline{\alpha}})$$

where we used the fact that $\|F\|_{C^0(M)}$ and $\|\varphi\|_{C^0(M)}$ are bounded. For the last term we have

$$(3.24)\qquad \psi_\alpha = \frac{\partial_\alpha(|s|^2_h)}{|s|^2_h},$$

So

$$2\,\mathrm{e}^{A}\,Q\delta'\mathrm{Re}(\psi_\alpha\varphi_\alpha) \geqslant -C\,\mathrm{e}^{2\psi}\,|\partial\psi|^2_{\omega_0} - C\,\mathrm{e}^{(2Q\delta'-2)\psi}\,|\partial\varphi|^2_{\omega_0} \geqslant -C - C\,\mathrm{e}^{Q\delta'\psi}\,|\partial\varphi|^2_{\omega_0}. \tag{3.25}$$

where we used the fact that Q is chosen such that $2Q\delta' - 2 > Q\delta'$ so $\mathrm{e}^{(2Q\delta'-2)\psi} \leqslant C\,\mathrm{e}^{Q\delta'\psi}$. Finally we reach

$$\Delta_{\omega_\varphi}(\mathrm{e}^{A}(|\partial\varphi|^2_{\omega_0} + K)) \geqslant C(\mathrm{e}^{Q\delta'\psi}\,|\partial\varphi|^2_{\omega_0})^{1+\frac{1}{n}} - C\,\mathrm{e}^{Q\delta'\psi}\,|\partial\varphi|^2_{\omega_0} - C\,\mathrm{e}^{Q\delta'\psi} - C. \tag{3.26}$$

At a maximum p of $\mathrm{e}^{A}(|\partial\varphi|^2_{\omega_0}+K)$, which is away from E because $A = -\infty$ on E, we have

$$C(\mathrm{e}^{Q\delta'\psi(p)}\,|\partial\varphi|^2_{\omega_0}(p))^{1+\frac{1}{n}} - C\,\mathrm{e}^{Q\delta'\psi(p)}\,|\partial\varphi|^2_{\omega_0}(p) \leqslant C\,\mathrm{e}^{Q\delta'\psi(p)} + C \leqslant C \tag{3.27}$$

where for the last inequality we used the fact that $\mathrm{e}^{P\psi}$ is bounded from above for any constant $P > 0$, and our choice that $Q\delta' > 1$. Thus, we conclude that $\mathrm{e}^{Q\delta'\psi}\,|\partial\varphi|^2_{\omega_0}(p)$ is bounded from above. As a consequence, $\mathrm{e}^{A}(|\partial\varphi|^2_{\omega_0} + K)(p)$ is also bounded from above. This concludes our proof. □

4. Degenerate L^p bound on $\mathrm{tr}_{\omega_0}\omega_{\varphi_\varepsilon}$ and $C^\infty_{\mathrm{loc}}(M\backslash E)$ bound on φ_ε

In this section we will first establish a degenerate version of Theorem 3.1 in [9].

Theorem 4.1. — *For each $p > 0$, there exists a constant $\gamma(p) > 0$ such that*

$$\int_M \mathrm{e}^{\gamma(p)\psi}(\mathrm{tr}_{\omega_0}\omega_{\varphi_\varepsilon})^p\omega_0^n \leqslant C(p, (M,\omega_0), \|\varphi_\varepsilon\|_{C^0(M)}, \|F_\varepsilon\|_{C^0(M)}). \tag{4.1}$$

For simplicity, let $B_\varepsilon = F_\varepsilon + \lambda\varphi_\varepsilon - \lambda\delta'\psi$, where $\delta' < \min\{\frac{1}{2}, \sigma\}$ and σ is specified in Lemma 3.1. Moreover, let β be a constant greater than 1. We start by computing at a point away from E in a normal holomorphic coordinate neighborhood for g_0 such that g_φ is diagonal.

$$\Delta_{\omega_\varphi}(\mathrm{e}^{-\beta B}(\mathrm{tr}_{\omega_0}\omega_\varphi)) = \Delta_{\omega_\varphi}(\mathrm{e}^{-\beta B})\mathrm{tr}_{\omega_0}\omega_\varphi + \mathrm{e}^{-\beta B}\,\Delta_{\omega_\varphi}(\mathrm{tr}_{\omega_0}\omega_\varphi) - 2\beta\mathrm{Re}(\mathrm{e}^{-\beta B}\,g_\varphi^{i\bar{i}}(B_i\partial_{\bar{i}}\mathrm{tr}_{\omega_0}\omega_\varphi)), \tag{4.2}$$

and

$$\begin{aligned}(4.3)\quad &\Delta_{\omega_\varphi} \mathrm{e}^{-\beta B}\\ &= (\beta^2 g_\varphi^{i\bar{i}}|B_i|^2 + \beta(\underline{R} - \lambda n + \lambda \mathrm{tr}_{\omega_\varphi}(\omega + \delta'\sqrt{-1}\partial\bar{\partial}\psi) + \mathrm{tr}_{\omega_\varphi}\eta))\,\mathrm{e}^{-\beta B}\\ &\geqslant \beta^2 g_\varphi^{i\bar{i}}|B_i|^2 \mathrm{e}^{-\beta B} + \beta\,\mathrm{e}^{-\beta B}(\underline{R} - \lambda n) + C_{\delta'}\lambda\beta\,\mathrm{e}^{-\beta B}\,\mathrm{tr}_{\omega_\varphi}\omega_0.\end{aligned}$$

Now we estimate $\Delta_{\omega_\varphi}\mathrm{tr}_{\omega_0}\omega_\varphi$ by first recalling that from the calculation in Yau [53] (see also [44, Lemma 3.7] for an exposition) we have

$$\Delta_{\omega_\varphi}\log\mathrm{tr}_{\omega_0}\omega_\varphi \geqslant -C_1\mathrm{tr}_{\omega_\varphi}\omega_0 - \frac{g_0^{i\bar{j}}R_{i\bar{j}}(g_\varphi)}{\mathrm{tr}_{\omega_0}\omega_\varphi} \tag{4.4}$$

where C_1 only depends on a lower bound for the bisectional curvature of g_0. Recall that

$$\log\frac{\omega_\varphi^n}{\omega_0^n} = F. \tag{4.5}$$

Then we have

$$-\mathrm{tr}_{\omega_0}\mathrm{Ric}(\omega_\varphi) = \mathrm{tr}_{\omega_0}\eta + \Delta_{\omega_0}F. \tag{4.6}$$

Thus

$$\Delta_{\omega_\varphi}\mathrm{logtr}_{\omega_0}\omega_\varphi \geqslant -C_1\mathrm{tr}_{\omega_\varphi}\omega_0 + \frac{\Delta_{\omega_0}F}{\mathrm{tr}_{\omega_0}\omega_\varphi} + \frac{\mathrm{tr}_{\omega_0}\eta}{\mathrm{tr}_{\omega_0}\omega_\varphi}, \tag{4.7}$$

and

$$\Delta_{\omega_\varphi}\mathrm{tr}_{\omega_0}\omega_\varphi \geqslant -C_1\mathrm{tr}_{\omega_\varphi}\omega_0\mathrm{tr}_{\omega_0}\omega_\varphi + \frac{|\partial\mathrm{tr}_{\omega_0}\omega_\varphi|^2_{\omega_\varphi}}{\mathrm{tr}_{\omega_0}\omega_\varphi} + \Delta_{\omega_0}F + \mathrm{tr}_{\omega_0}\eta. \tag{4.8}$$

Then we are able to conclude that

$$\begin{aligned}(4.9)\quad &\mathrm{e}^{\beta B}\,\Delta_{\omega_\varphi}(\mathrm{e}^{-\beta B}(\mathrm{tr}_{\omega_0}\omega_\varphi))\\ &\geqslant (\beta^2 g_\varphi^{i\bar{i}}|B_i|^2 + \beta(\underline{R} - \lambda n))\mathrm{tr}_{\omega_0}\omega_\varphi + (C_{\delta'}\lambda\beta - C_1)\mathrm{tr}_{\omega_\varphi}\omega_0\mathrm{tr}_{\omega_0}\omega_\varphi\\ &\qquad + \frac{|\partial\mathrm{tr}_{\omega_0}\omega_\varphi|^2_{\omega_\varphi}}{\mathrm{tr}_{\omega_0}\omega_\varphi} + \Delta_{\omega_0}F + \mathrm{tr}_{\omega_0}\eta - 2\beta\mathrm{Re}(g_\varphi^{i\bar{i}}(B_i\partial_{\bar{i}}\mathrm{tr}_{\omega_0}\omega_\varphi))\\ &\geqslant \beta(\underline{R} - \lambda n)\mathrm{tr}_{\omega_0}\omega_\varphi + (C_{\delta'}\lambda\beta - C_1)\mathrm{tr}_{\omega_\varphi}\omega_0\mathrm{tr}_{\omega_0}\omega_\varphi + \Delta_{\omega_0}F + \mathrm{tr}_{\omega_0}\eta\end{aligned}$$

where we dropped the terms

$$\begin{aligned}(4.10)\quad &\beta^2 g_\varphi^{i\bar{i}}|B_i|^2\mathrm{tr}_{\omega_0}\omega_\varphi - 2\beta\mathrm{Re}(g_\varphi^{i\bar{i}}(B_i\partial_{\bar{i}}\mathrm{tr}_{\omega_0}\omega_\varphi)) + \frac{|\partial\mathrm{tr}_{\omega_0}\omega_\varphi|^2_{\omega_\varphi}}{\mathrm{tr}_{\omega_0}\omega_\varphi}\\ &= \mathrm{tr}_{\omega_0}\omega_\varphi g_\varphi^{i\bar{i}}(\beta^2|B_i|^2 - 2\beta\mathrm{Re}\left(\frac{B_i\partial_{\bar{i}}\mathrm{tr}_{\omega_0}\omega_\varphi}{\mathrm{tr}_{\omega_0}\omega_\varphi}\right) + \frac{|\partial_{\bar{i}}\mathrm{tr}_{\omega_0}\omega_\varphi|^2}{(\mathrm{tr}_{\omega_0}\omega_\varphi)^2}) \geqslant 0.\end{aligned}$$

Use

$$\operatorname{tr}_{\omega_\varphi}\omega_0 \operatorname{tr}_{\omega_0}\omega_\varphi \geqslant \mathrm{e}^{-\frac{F}{n-1}} (\operatorname{tr}_{\omega_0}\omega_\varphi)^{1+\frac{1}{n-1}}, \tag{4.11}$$

and choose $\lambda = \max(\frac{2C_1+2}{C_{\delta'}}, \frac{q}{2\delta'}+1, \frac{1}{2\delta'}+1)$, where q is specified in Theorem 3.2, so that $C_{\delta'}\lambda\beta \geqslant (2C_1+2)\beta > 2C_1+1$. We get

$$\begin{aligned} &\Delta_{\omega_\varphi}(\mathrm{e}^{-\beta B}(\operatorname{tr}_{\omega_0}\omega_\varphi)) \\ &\geqslant \beta(\underline{R}-\lambda n)\operatorname{tr}_{\omega_0}\omega_\varphi \,\mathrm{e}^{-\beta B} + (C_{\delta'}\lambda\beta - C_1)\,\mathrm{e}^{-\frac{F}{n-1}-\beta B}(\operatorname{tr}_{\omega_0}\omega_\varphi)^{1+\frac{1}{n-1}} \\ &\qquad + (\Delta_{\omega_0}F + \operatorname{tr}_{\omega_0}\eta)\,\mathrm{e}^{-\beta B} \\ &\geqslant \beta(\underline{R}-\lambda n)\operatorname{tr}_{\omega_0}\omega_\varphi \,\mathrm{e}^{-\beta B} + (C_{\delta'}\lambda\beta - C_1)\,\mathrm{e}^{-\frac{F}{n-1}-\beta B}(\operatorname{tr}_{\omega_0}\omega_\varphi)^{1+\frac{1}{n-1}} \\ &\qquad + \Delta_{\omega_0}F\,\mathrm{e}^{-\beta B} \end{aligned} \tag{4.12}$$

where for the last line we took advantage of the fact that $\eta \geqslant 0$. Let $u = \mathrm{e}^{-\beta B}\operatorname{tr}_{\omega_0}\omega_\varphi$, for any $p \geqslant 0$, we have

$$\begin{aligned} \frac{1}{2p+1}\Delta_{\omega_\varphi}u^{2p+1} &= 2pu^{2p-2}\,\mathrm{e}^{-\beta B}(\operatorname{tr}_{\omega_0}\omega_\varphi)|\partial u|^2_{\omega_\varphi} + u^{2p}\Delta_{\omega_\varphi}u \\ &\geqslant 2pu^{2p-2}|\partial u|^2_{\omega_0}\,\mathrm{e}^{-\beta B} + u^{2p}\Delta_{\omega_\varphi}u. \end{aligned} \tag{4.13}$$

Integrating both sides of (4.13) with respect to $\omega_\varphi^n = \mathrm{e}^F\omega_0^n$ and use (4.12) we get

$$\begin{aligned} &\int_M 2pu^{2p-2}|\partial u|^2_{\omega_0}\,\mathrm{e}^{-\beta B+F}\,\omega_0^n + \int_M \Delta_{\omega_0}F\,\mathrm{e}^{-\beta B+F}\,u^{2p}\omega_0^n \\ &\quad + \int_M u^{2p}\,\mathrm{e}^{-\beta B+\frac{n-2}{n-1}F}(C_{\delta'}\lambda\beta - C_1)(\operatorname{tr}_{\omega_0}\omega_\varphi)^{1+\frac{1}{n-1}}\omega_0^n \\ &\qquad\qquad \leqslant \int_M \beta(\lambda n - \underline{R})\operatorname{tr}_{\omega_0}\omega_\varphi\,\mathrm{e}^{-\beta B+F}\,u^{2p}\omega_0^n. \end{aligned} \tag{4.14}$$

Let us denote $\tilde{\varphi} = \varphi - \delta'\psi$, and to handle the term involving $\Delta_{\omega_0}F$, we apply integration by parts

$$\begin{aligned} &\int_M \mathrm{e}^{-\beta B+F}\,u^{2p}\Delta_{\omega_0}F\omega_0^n \\ &\quad = \int_M (\beta-1)\,\mathrm{e}^{(1-\beta)F-\beta\lambda\tilde{\varphi}}\,u^{2p}|\partial F|^2_{\omega_0}\omega_0^n \\ &\qquad + \int_M \beta\lambda\,\mathrm{e}^{(1-\beta)F-\lambda\beta\tilde{\varphi}}\,u^{2p}\langle\partial\tilde{\varphi},\partial F\rangle_{\omega_0}\omega_0^n \\ &\qquad\qquad - \int_M 2p\,\mathrm{e}^{(1-\beta)F-\beta\lambda\tilde{\varphi}}\,u^{2p-1}\langle\partial u,\partial F\rangle_{\omega_0}\omega_0^n, \end{aligned} \tag{4.15}$$

where $\langle\partial\tilde{\varphi},\partial F\rangle_{\omega_0} = g_0^{i\bar{j}}\tilde{\varphi}_i F_{\bar{j}}$, and $\langle\partial u,\partial F\rangle_{\omega_0} = g_0^{i\bar{j}}u_i F_{\bar{j}}$ in coordinates. Notice that by Young's inequality

$$|u^{2p-1}\langle\partial u,\partial F\rangle_{\omega_0}| \leqslant \frac{1}{2}u^{2p}|\partial F|^2_{\omega_0} + \frac{1}{2}u^{2p-2}|\partial u|^2_{\omega_0}, \tag{4.16}$$

and

$$\begin{aligned}(4.17)\quad & |\beta\lambda\,\mathrm{e}^{(1-\beta)F-\lambda\beta\tilde{\varphi}}\,u^{2p}\langle\partial\tilde{\varphi},\partial F\rangle_{\omega_0}| \\ &\leqslant |\beta\lambda\,\mathrm{e}^{(1-\beta)F-\lambda\beta\tilde{\varphi}}\,u^{2p}\langle\partial\varphi,\partial F\rangle_{\omega_0}| + |\delta'\beta\lambda\,\mathrm{e}^{(1-\beta)F-\lambda\beta\tilde{\varphi}}\,u^{2p}\langle\partial\psi,\partial F\rangle_{\omega_0}| \\ &\leqslant \frac{(\beta-1)}{2}u^{2p}|\partial F|^2_{\omega_0}\,\mathrm{e}^{(1-\beta)F-\lambda\beta\tilde{\varphi}} + |\partial\varphi|^2_{\omega_0}\frac{\beta^2\lambda^2}{(\beta-1)}u^{2p}\,\mathrm{e}^{(1-\beta)F-\lambda\beta\tilde{\varphi}} \\ &\qquad + \delta'^2|\partial\psi|^2_{\omega_0}\frac{\beta^2\lambda^2}{(\beta-1)}u^{2p}\,\mathrm{e}^{(1-\beta)F-\lambda\beta\tilde{\varphi}}.\end{aligned}$$

Then substitute this back into (4.15), we get

$$\begin{aligned}(4.18)\quad & \int_M \mathrm{e}^{(1-\beta)F-\beta\lambda\tilde{\varphi}}\,u^{2p}\Delta_{\omega_0}F\omega_0^n \\ &\geqslant -\int_M |\partial\psi|^2_{\omega_0}\frac{\delta'^2\beta^2\lambda^2}{(\beta-1)}\,\mathrm{e}^{(1-\beta)F-\lambda\beta\tilde{\varphi}}\,u^{2p}\omega_0^n \\ &\quad -\int_M p\,\mathrm{e}^{(1-\beta)F-\lambda\beta\tilde{\varphi}}\,u^{2p-2}|\partial u|^2_{\omega_0}\omega_0^n \\ &\quad +\int_M\left(\frac{\beta-1}{2}-p\right)\mathrm{e}^{(1-\beta)F-\beta\lambda\tilde{\varphi}}\,u^{2p}|\partial F|^2_{\omega_0}\omega_0^n \\ &\quad -\int_M |\partial\varphi|^2_{\omega_0}\frac{\beta^2\lambda^2}{(\beta-1)}\,\mathrm{e}^{(1-\beta)F-\lambda\beta\tilde{\varphi}}\,u^{2p}\omega_0^n.\end{aligned}$$

Substituting (4.18) back into (4.14), we conclude

$$\begin{aligned}(4.19)\quad & \int_M pu^{2p-2}|\partial u|^2_{\omega_0}\,\mathrm{e}^{(1-\beta)F-\lambda\beta\tilde{\varphi}}\,\omega_0^n \\ &+\int_M\left(\frac{\beta-1}{2}-p\right)\mathrm{e}^{(1-\beta)F-\beta\lambda\tilde{\varphi}}\,u^{2p}|\partial F|^2_{\omega_0}\omega_0^n \\ &+\int_M u^{2p}\,\mathrm{e}^{-(\beta-\frac{n-2}{n-1})F-\beta\lambda\tilde{\varphi}}(C_{\delta'}\beta\lambda-C_1)(\mathrm{tr}_{\omega_0}\omega_\varphi)^{1+\frac{1}{n-1}}\omega_0^n \\ &\leqslant \int_M |\partial\varphi|^2_{\omega_0}\frac{\beta^2\lambda^2}{(\beta-1)}\,\mathrm{e}^{(1-\beta)F-\lambda\beta\tilde{\varphi}}\,u^{2p}\omega_0^n \\ &\quad+\int_M |\partial\psi|^2_{\omega_0}\frac{\delta'^2\beta^2\lambda^2}{(\beta-1)}\,\mathrm{e}^{(1-\beta)F-\lambda\beta\tilde{\varphi}}\,u^{2p}\omega_0^n \\ &\quad+\int_M \beta(\lambda n-\underline{R})\mathrm{tr}_{\omega_0}\omega_\varphi\,\mathrm{e}^{(1-\beta)F-\beta\lambda\tilde{\varphi}}\,u^{2p}\omega_0^n.\end{aligned}$$

Now choose $\beta > 2p+1$, and recall that $\lambda = \max(\frac{2C_1+2}{C_{\delta'}}, \frac{q}{2\delta'}+1, \frac{1}{2\delta'}+1)$, where q is specified in Theorem 3.2, so that $C_{\delta'}\lambda\beta \geqslant (2C_1+2)\beta > 2C_1+1$. Because $\operatorname{tr}_{\omega_0}\omega_\varphi \geqslant n\,\mathrm{e}^{\frac{F}{n}}$ by the arithmetic-geometric means inequality, we have

$$\begin{aligned}(4.20)\quad &\int_M u^{2p}\,\mathrm{e}^{-(\beta-\frac{n-2}{n-1})F-\beta\lambda\tilde\varphi}(\operatorname{tr}_{\omega_0}\omega_\varphi)^{1+\frac{1}{n-1}}\omega_0^n\\ &\leqslant C\int_M \operatorname{tr}_{\omega_0}\omega_\varphi\,\mathrm{e}^{(1-\beta)F-\beta\lambda\tilde\varphi}\,u^{2p}\omega_0^n + C\int_M u^{2p}\,\mathrm{e}^{(1-\beta)F-\beta\lambda\tilde\varphi}\,\omega_0^n\\ &\quad + C\int_M |\partial\varphi|^2_{\omega_0}\frac{\beta^2}{\beta-1}\,\mathrm{e}^{(1-\beta)F-\lambda\beta\tilde\varphi}\,u^{2p}\omega_0^n\\ &\quad + C\int_M |\partial\psi|^2_{\omega_0}\frac{\delta'^2\beta^2}{\beta-1}\,\mathrm{e}^{(1-\beta)F-\lambda\beta\tilde\varphi}\,u^{2p}\omega_0^n.\end{aligned}$$

For $p=0$, $\beta=2$ we have

$$\begin{aligned}(4.21)\quad &\int_M \mathrm{e}^{2\lambda\delta'\psi}(\operatorname{tr}_{\omega_0}\omega_\varphi)^{1+\frac{1}{n-1}}\omega_0^n\\ &\leqslant C\int_M \mathrm{e}^{-(2-\frac{n-2}{n-1})F-2\lambda\tilde\varphi}(\operatorname{tr}_{\omega_0}\omega_\varphi)^{1+\frac{1}{n-1}}\omega_0^n\\ &\leqslant C\int_M \operatorname{tr}_{\omega_0}\omega_\varphi\,\mathrm{e}^{2\lambda\delta'\psi}\,\omega_0^n + C\int_M \mathrm{e}^{2\lambda\delta'\psi}\,\omega_0^n\\ &\quad + C\int_M |\partial\varphi|^2_{\omega_0}\,\mathrm{e}^{-2\lambda\varphi+2\lambda\delta'\psi}\,\omega_0^n + C\int_M |\partial\psi|^2_{\omega_0}\,\mathrm{e}^{-2\lambda\varphi+2\lambda\delta'\psi}\,\omega_0^n\\ &\leqslant C\int_M \operatorname{tr}_{\omega_0}\omega_\varphi\omega_0^n + C + C\int_M |\partial\varphi|^2_{\omega_0}\,\mathrm{e}^{2\lambda\delta'\psi}\,\omega_0^n\\ &\quad + C\int_M |\partial\psi|^2_{\omega_0}\,\mathrm{e}^{2\lambda\delta'\psi}\,\omega_0^n,\end{aligned}$$

where we used the fact that φ and F are bounded. Also notice that

$$(4.22)\qquad \int_M \operatorname{tr}_{\omega_0}\omega_\varphi\omega_0^n = \int_M \operatorname{tr}_{\omega_0}\omega\omega_0^n + \int_M \Delta_{\omega_0}\varphi\omega_0^n \leqslant C.$$

For the last two terms

$$(4.23)\qquad \int_M |\partial\varphi|^2_{\omega_0}\,\mathrm{e}^{2\lambda\delta'\psi}\,\omega_0^n \leqslant C\int_M |s|_h^{4\lambda\delta'-2q}\omega_0^n \leqslant C,$$

and

$$(4.24)\qquad \int_M |\partial\psi|^2_{\omega_0}\,\mathrm{e}^{2\lambda\delta'\psi}\,\omega_0^n \leqslant C\int_M |s|_h^{4\lambda\delta'-2}\omega_0^n \leqslant C.$$

Recall that we choose $\lambda > \max\{\frac{q}{2\delta'}, \frac{1}{2\delta'}\}$, so we can bound the last two terms since $|s|_h^2$ is bounded. As a result, we obtained a bound for

$$\int_M \mathrm{e}^{2\lambda\delta'\psi}(\mathrm{tr}_{\omega_0}\omega_\varphi)^{\frac{n}{n-1}}\omega_0^n.$$

Now it suffices to show that for each given $p > 0$,

$$\int_M (\mathrm{tr}_{\omega_0}\omega_\varphi)^p\, \mathrm{e}^{\lambda(2p+2)(\delta'(p-1)+1)\psi}\,\omega_0^n$$

(recall that $\delta' < \frac{1}{2}$, so that $\delta'(p-1)+1 > \delta'$ when $p = \frac{n}{n-1}$) being bounded above implies that for $p + \frac{1}{n-1}$,

$$\int_M (\mathrm{tr}_{\omega_0}\omega_\varphi)^{p+\frac{1}{n-1}}\, \mathrm{e}^{\lambda(2p+2)(\delta'(p+\frac{1}{n-1}-1)+1)\psi}\,\omega_0^n$$

is bounded above, then we are done by induction. Given $p > 1$, let $\tilde{p} = \frac{p-1}{2}$, $\beta = 2p+2 > 2\tilde{p}+1$ and take $\tilde{p}, \beta$ into our inequality (4.20), we get

$$\begin{aligned}
\int_M & \mathrm{e}^{\lambda(2p+2)(\delta'(p+\frac{1}{n+1}-1)+1)\psi}(\mathrm{tr}_{\omega_0}\omega_\varphi)^{p+\frac{1}{n-1}}\omega_0^n \\
&\leqslant C\int_M \mathrm{e}^{-(2p+2-\frac{n-2}{n-1})F-(2p+2)\lambda\tilde{\varphi}-2\tilde{p}(2p+2)B}(\mathrm{tr}_{\omega_0}\omega_\varphi)^{p+\frac{1}{n-1}}\omega_0^n \\
&\leqslant C\int_M \mathrm{e}^{\lambda(2p+2)(\delta'(p-1)+1)\psi}(\mathrm{tr}_{\omega_0}\omega_\varphi)^p\omega_0^n \\
&\qquad + C\int_M \mathrm{e}^{\lambda(2p+2)(\delta'(p-1)+1)\psi}(\mathrm{tr}_{\omega_0}\omega_\varphi)^{p-1}\omega_0^n \\
&\qquad\quad + C\int_M |\partial\varphi|_{\omega_0}^2\, \mathrm{e}^{\lambda(2p+2)(\delta'(p-1)+1)\psi}(\mathrm{tr}_{\omega_0}\omega_\varphi)^{p-1}\omega_0^n \\
&\qquad\qquad + C\int_M |\partial\psi|_{\omega_0}^2\, \mathrm{e}^{\lambda(2p+2)(\delta'(p-1)+1)\psi}(\mathrm{tr}_{\omega_0}\omega_\varphi)^{p-1}\omega_0^n.
\end{aligned} \tag{4.25}$$

The first two terms on the right hand side are bounded due to the inductive hypothesis and the 0-th order bound on F and φ. For the last two terms we can use the same idea as in the case where $p = 0, \beta = 2$, to bound them. Then we are done by induction. We now need the following proposition from [9] [(1)] to show that φ_ε is locally uniformly smoothly bounded away from E.

Proposition 4.2. — *Let K be a compact subset of M, suppose that $\mathrm{tr}_{\omega_0}\omega_{\varphi_\varepsilon} \in L^p(K)$, and $\mathrm{tr}_{\omega_{\varphi_\varepsilon}}\omega_0 \in L^p(K)$ for some $p > 3n(n-1)$ (n is the dimension of M), then for any $m \in \mathbb{N}, m \geqslant 2$, $\varphi_\varepsilon \in C^m(K, \omega_0)$ with the*

(1) The authors of [9] decided to omit this result in the published version of the article and it can be found in Corollary 6.2 of the arXiv version (`https://arxiv.org/abs/1712.06697`) of [9].

bound only depends on p, $\|\mathrm{tr}_{\omega_{\varphi_\varepsilon}}\omega_0\|_{L^p(K')}$, $\|\mathrm{tr}_{\omega_0}\omega_{\varphi_\varepsilon}\|_{L^p(K')}$, $\|\varphi_\varepsilon\|_{C^0(K')}$, m, and (M,ω_0), where K' is a larger compact set containing K.

Corollary 4.3. — *Given any compact subset K of $M\backslash E$, φ_ε is uniformly bounded in $C^\infty(K)$.*

Proof. — Given a compact subset $K\subset M\backslash E$, the $L^p(K)$ bound on $\mathrm{tr}_{\omega_0}\omega_{\varphi_\varepsilon}$ is a direct consequence of Theorem 4.1. Furthermore, notice that

$$\begin{aligned}\mathrm{tr}_{\omega_{\varphi_\varepsilon}}\omega_0 &= \sum_i \frac{1}{(g_{\varphi_\varepsilon})_{i\bar{i}}} = \sum_i \frac{1}{\prod_m (g_{\varphi_\varepsilon})_{m\bar{m}}}\prod_{j\neq i}(g_{\varphi_\varepsilon})_{j\bar{j}} \\ &= \mathrm{e}^{-F}\prod_{j\neq i}(g_{\varphi_\varepsilon})_{j\bar{j}} \leqslant \mathrm{e}^{-F}(\mathrm{tr}_{\omega_0}\omega_{\varphi_\varepsilon})^{n-1}.\end{aligned} \tag{4.26}$$

So we are able to obtain the $L^p(K)$ bound on $\mathrm{tr}_{\omega_\varphi}\omega_0$ by using the $C^0(M)$ bound on F and Theorem 4.1 again. We conclude the proof by recalling that we previously showed that $\|\varphi_\varepsilon\|_{C^0(M)}$ is uniformly bounded in Section 2 and $|\partial\varphi_\varepsilon|_{\omega_0}$ is uniformly bounded on K in Section 3. □

5. Uniform upper bound on the Mabuchi energy and the entropy

All the estimates we proved in the previous sections depend on the entropy

$$\int_M \log\frac{\omega_{\varphi_\varepsilon}^n}{\omega_0^n}\frac{\omega_{\varphi_\varepsilon}^n}{n!}, \tag{5.1}$$

thus in order to obtain uniform estimates, it is essential to show that the entropies are uniformly bounded from above independent of ε. To achieve that, we need to study the Mabuchi energy which is closely related to the entropy. Recall that $\omega_\varepsilon = \eta + \varepsilon\omega_0$, and we will continue suppressing ε for simplicity of notation. The Mabuchi energy is defined to be the functional

$$\mathcal{M}_\omega(\theta) = \int_M \log\frac{\omega_\theta^n}{\omega^n}\frac{\omega_\theta^n}{n!} + J_{-\mathrm{Ric}(\omega)}(\theta) \tag{5.2}$$

where $\theta\in\mathcal{H}_\omega := \{\theta\in C^\infty(M)|\omega+\sqrt{-1}\partial\bar\partial\theta>0, \sup_M\theta=0\}$. For any real $(1,1)$ form χ, J_χ is a functional on $\mathcal{H}_\omega$ defined through its variation

$$\frac{\mathrm{d}J_\chi}{\mathrm{d}t} = \int_M \frac{\partial\theta}{\partial t}(\mathrm{tr}_{\omega_\theta}\chi - \underline{\chi})\frac{\omega_\theta^n}{n!} \tag{5.3}$$

where $\underline{\chi} = n\frac{[\chi]\cdot[\omega]^{n-1}}{[\omega]^n}$. Following Chen [7] (see also [10, Section 2.1]), we have the following explicit formula for J_χ

$$(5.4)\quad J_\chi(\theta) = -\frac{\underline{\chi}}{(n+1)!}\int_M \theta\left(\sum_{i=0}^{n}\omega^i\wedge\omega_\theta^{n-i}\right) + \frac{1}{n!}\int_M \theta\chi\wedge\left(\sum_{i=0}^{n-1}\omega^i\wedge\omega_\theta^{n-1-i}\right),$$

so $J_\chi(0) = 0$. Furthermore, the variation of the Mabuchi functional is given by

$$\frac{\mathrm{d}\mathcal{M}_\omega}{\mathrm{d}t} = \int_M \frac{\partial\theta}{\partial t}(-\mathrm{tr}_{\omega_\theta}\mathrm{Ric}(\omega_\theta) + \underline{\mathrm{Ric}(\omega)})\frac{\omega_\theta^n}{n!}. \tag{5.5}$$

The first term appearing in the Mabuchi functional is the same as what we referred to as the entropy term before except that the leading term in the integral has its denominator as ω^n while ours is ω_0^n. Thus, we need to adjust the Mabuchi energy slighly to match our entropy term. We define our modified Mabuchi energy to be

$$E_\omega(\theta) = \int_M \log\frac{\omega_\theta^n}{\omega_0^n}\frac{\omega_\theta^n}{n!} + J_\eta(\theta). \tag{5.6}$$

It turns out that E_ω only differs from $\mathcal{M}_\omega$ by a term equal to $\int_M \log\frac{\omega^n}{\omega_0^n}\frac{\omega^n}{n!}$. Observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M \log\frac{\omega_\theta^n}{\omega_0^n}\frac{\omega_\theta^n}{n!} = \int_M \frac{\partial\theta}{\partial t}(-\mathrm{tr}_{\omega_\theta}\eta - \mathrm{tr}_{\omega_\theta}\mathrm{Ric}(\omega_\theta))\frac{\omega_\theta^n}{n!}. \tag{5.7}$$

So we know that

$$\begin{aligned}\frac{\mathrm{d}E_\omega}{\mathrm{d}t} &= \int_M \frac{\partial\theta}{\partial t}(-\mathrm{tr}_{\omega_\theta}\mathrm{Ric}(\omega_\theta) - \underline{\eta})\frac{\omega_\theta^n}{n!}\\ &= \int_M \frac{\partial\theta}{\partial t}(-\mathrm{tr}_{\omega_\theta}\mathrm{Ric}(\omega_\theta) + \underline{\mathrm{Ric}(\omega)})\frac{\omega_\theta^n}{n!}.\end{aligned} \tag{5.8}$$

Thus the variation of E_ω is actually the same as that of the usual Mabuchi energy $\mathcal{M}_\omega$. We know that the potentials of cscK metrics are minimizers of $\mathcal{M}_\omega$ and we want to show that the same is true for E_ω. We have just showed that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_\omega = \frac{\mathrm{d}}{\mathrm{d}t}E_\omega$. Taking a path $t\theta$ in $\mathcal{H}_\omega$, we have

$$\mathcal{M}_\omega(\theta) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}E_\omega\,\mathrm{d}t + \mathcal{M}_\omega(0) = E_\omega(\theta) - \int_M \log\frac{\omega^n}{\omega_0^n}\frac{\omega^n}{n!}. \tag{5.9}$$

We know that

$$\mathcal{M}_\omega(\theta) - \mathcal{M}_\omega(\varphi) \geqslant 0 \tag{5.10}$$

for all $\theta\in\mathcal{H}_\omega$ and φ such that $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is a cscK metric thus,

$$E_\omega(\theta) - E_\omega(\varphi) = \mathcal{M}_\omega(\theta) - \mathcal{M}_\omega(\varphi) \geqslant 0. \tag{5.11}$$

So we reach the conclusion that the potentials of cscK metrics are still minimizers of E_ω. One immediate consequence is the following lemma.

Lemma 5.1. — *Let $\varphi_\varepsilon \in \mathcal{H}_\omega$ be a smooth real-valued function such that $\omega_{\varphi_\varepsilon} = \omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon$ is cscK, then*

$$E_{\omega_\varepsilon}(\varphi_\varepsilon) \leqslant E_{\omega_\varepsilon}(0) = \int_M \log \frac{\omega_\varepsilon^n}{\omega_0^n} \frac{\omega_\varepsilon^n}{n!} \leqslant C \tag{5.12}$$

where C is independent of ε.

We are now ready to show that the entropies are uniformly bounded from above. One possible approach is to first establish a uniform lower bound on the J functionals since we already know from Lemma 5.1 that $E_\omega(\varphi)$ is uniformly bounded from above. However, this approach requires one to study the solutions of the J equations assoicated with the J functionals. For more details related to this route, see [27, 30, 52]. Instead of following this path, we turn to an approach introduced by Dervan [18] which generates explicit bounds on the functionals involved.

To begin with, we estimate the entropy term in the Mabuchi energy using the α-invariant, and this technique should be well-known to the experts in this field (see for example Tian [47]).

Lemma 5.2. — *for all $\varepsilon \in [0,1]$, and α such that $\alpha > 0$ and $\alpha < \alpha(M, [\eta + \omega_0])$, we have*

$$\int_M \log \frac{\omega_\theta^n}{\omega_0^n} \frac{\omega_\theta^n}{n!} \geqslant -\alpha \int_M \theta \frac{\omega_\theta^n}{n!} - C \tag{5.13}$$

where C is a positive constant independent of ε.

Proof. — First notice that

$$\int_M \mathrm{e}^{-\alpha\theta} \frac{\omega_0^n}{n!} = \int_M \mathrm{e}^{-\log \frac{\omega_\theta^n}{\omega_0^n} - \alpha\theta} \frac{\omega_\theta^n}{n!} \leqslant C \tag{5.14}$$

where we used Lemma 2.3 again. Then by Jensen's inequality,

$$\int_M \left(-\log \frac{\omega_\theta^n}{\omega_0^n} - \alpha\theta \right) \frac{\omega_\theta^n}{n!} \leqslant C \int_M \frac{\omega^n}{n!} \leqslant C \tag{5.15}$$

where $\int_M \frac{\omega^n}{n!}$ arises because we need a probability measure in order to apply Jensen's inequality. Thus,

$$\int_M \log \frac{\omega_\theta^n}{\omega_0^n} \frac{\omega_\theta^n}{n!} \geqslant -\alpha \int_M \theta \frac{\omega_\theta^n}{n!} - C. \tag{5.16}$$

□

We will need the following lemmas due to [18], and we include brief proofs of them here for the reader's convenience.

Lemma 5.3 ([18, Lemma 2.9]). — *We have*

$$-n\int_M \theta\omega_\theta^n \geqslant -\sum_{i=1}^n \int_M \theta\omega^i\wedge\omega_\theta^{n-i}. \tag{5.17}$$

Proof.

$$-n\int_M \theta\omega_\theta^n + \sum_{i=1}^n \int_M \theta\omega^i\wedge\omega_\theta^{n-i} = \sum_{i=1}^n \int_M \theta\omega_\theta^{n-i}\wedge(\omega^i-\omega_\theta^i). \tag{5.18}$$

We only need to show that each summand is positive separately

$$\begin{aligned}\int_M \theta\omega_\theta^{n-i}\wedge(\omega^i-\omega_\theta^i) &= \int_M \sqrt{-1}\partial\theta\wedge\bar\partial\theta\wedge\omega_\theta^{n-i}\wedge\left(\sum_{j=1}^i \omega^{i-j}\wedge\omega_\theta^{j-1}\right)\\ &\geqslant 0.\end{aligned} \tag{5.19}$$

□

Lemma 5.4. — *We have*

$$\begin{aligned}&-\frac{\eta}{(n+1)!}\int_M \theta\left(\sum_{i=0}^n \omega^i\wedge\omega_\theta^{n-i}\right)\\ &\qquad\geqslant -\frac{\eta}{n!n}\int_M \theta\omega\wedge\left(\sum_{i=0}^{n-1}\omega^i\wedge\omega_\theta^{n-i-1}\right).\end{aligned} \tag{5.20}$$

Proof. — We calculate

$$\begin{aligned}&-\frac{\eta}{(n+1)!}\int_M \theta\left(\sum_{i=0}^n \omega^i\wedge\omega_\theta^{n-i}\right)\\ &= -n\frac{\eta}{n(n+1)!}\int_M \theta\omega_\theta^n - \frac{\eta}{(n+1)!}\int_M \theta\omega\wedge\left(\sum_{i=0}^{n-1}\omega\wedge\omega_\theta^{n-i-1}\right)\\ &\geqslant -\frac{\eta}{(n+1)!n}\int_M \sum_{i=1}^n \theta\omega^i\wedge\omega_\theta^{n-i} - \frac{\eta}{(n+1)!}\int_M \theta\omega\wedge\left(\sum_{i=0}^{n-1}\omega^i\wedge\omega_\theta^{n-i-1}\right)\\ &= -\frac{\eta}{n!n}\int_M \theta\omega\wedge\left(\sum_{i=0}^{n-1}\omega^i\wedge\omega_\theta^{n-i-1}\right)\end{aligned} \tag{5.21}$$

where for the inequality we used Lemma 5.3. □

Theorem 5.5. — *There exists $C > 0$, such that*

$$\int_M \log\frac{\omega_{\varphi_\epsilon}^n}{\omega_0^n}\frac{\omega_0^n}{n!} \leqslant C. \tag{5.22}$$

Proof. — Again, we will suppress ε for brevity. Fix $\alpha > 0$ with $\alpha < \alpha(M, [\eta + \omega_0])$, then we get

$$\begin{aligned} (5.23)\quad & E_\omega(\varphi) \\ & \geqslant \frac{1}{2}\int_M \log\frac{\omega_\varphi^n}{\omega_0^n}\frac{\omega_\varphi^n}{n!} - \frac{\alpha}{2}\int_M \varphi\frac{\omega_\varphi^n}{n!} - \frac{\underline{\eta}}{(n+1)!}\int_M \varphi\left(\sum_{i=0}^{n}\omega^i\wedge\omega_\varphi^{n-i}\right) \\ & \qquad + \frac{1}{n!}\int_M \varphi\eta\wedge\left(\sum_{i=0}^{n-1}\omega^i\wedge\omega_\varphi^{n-1-i}\right) - C \\ & \geqslant \frac{1}{2}\int_M \log\frac{\omega_\varphi^n}{\omega_0^n}\frac{\omega_\varphi^n}{n!} - \frac{1}{n!}\int_M \varphi\left(\frac{\alpha}{2n}\omega - \eta + \frac{1}{n}\underline{\eta}\omega\right) \\ & \qquad \wedge\left(\sum_{i=0}^{n-1}\omega^i\wedge\omega_\varphi^{n-1-i}\right) - C \end{aligned}$$

where for the first inequality we used Lemma 5.2, and the second inequality we used Lemma 5.3 and Lemma 5.4. Notice that

$$(5.24)\qquad \frac{\alpha}{2n}\omega - \eta + \frac{1}{n}\underline{\eta}\omega = \left(\frac{\alpha}{2n} - 1 + \frac{1}{n}\underline{\eta}\right)\eta + \varepsilon\left(\frac{\alpha}{2n} + \frac{1}{n}\underline{\eta}\right)\omega_0 > 0$$

for $\varepsilon > 0$ small enough because $\underline{\eta} = n\frac{[\eta]\cdot[\omega_\varepsilon]^{n-1}}{[\omega_\varepsilon]^n} \to n$ as $\varepsilon \to 0$. Thus for ε sufficiently small, we have

$$\begin{aligned} (5.25)\quad & E_\omega(\varphi) \\ & \geqslant \frac{1}{2}\int_M \log\frac{\omega_\varphi^n}{\omega_0^n}\frac{\omega_\varphi^n}{n!} - \int_M \varphi\left(\frac{\alpha}{2n} - \eta + \frac{1}{n}\underline{\eta}\omega\right)\wedge\left(\sum_{i=0}^{n-1}\omega^i\wedge\omega_\varphi^{n-1-i}\right) - C \\ & \geqslant \frac{1}{2}\int_M \log\frac{\omega_\varphi^n}{\omega_0^n}\frac{\omega_\varphi^n}{n!} - C, \end{aligned}$$

since we assume that $\varphi \leqslant 0$. We conclude the proof by recalling that $E_\omega(\varphi) \leqslant E_\omega(0) \leqslant C$ according to Lemma 5.1. □

6. Convergence

Now we use the estimates developed in previous sections to show that away from E, $\omega_{\varphi_\varepsilon}$ converges to the singular Kähler Einstein metric ω_{KE} on $M\backslash E$ given by Theorem 1.4. Fix a connected compact subset $K \subset M$, and a subsequence ε_i such that φ_{ε_i} converges smoothly to φ_∞, and F_{ε_i}

converges smoothly to F_∞ (such a subsequence always exists by Arzela–Ascoli). Recall that our equations read

$$\frac{(\omega_{\varepsilon_i}+\sqrt{-1}\partial\bar\partial\varphi_{\varepsilon_i})^n}{\omega_0^n}=\mathrm{e}^{F_{\varepsilon_i}},\qquad \Delta_{\omega_{\varphi_{\varepsilon_i}}}F_{\epsilon_i}=-\mathrm{tr}_{\omega_{\varphi_{\varepsilon_i}}}\eta-\underline{R_{\varepsilon_i}}. \tag{6.1}$$

Let us compute

$$\begin{aligned}\int_K |\partial(F_{\varepsilon_i}-\varphi_{\varepsilon_i})|^2_{\omega_{\varphi_{\varepsilon_i}}}\omega^n_{\varphi_{\varepsilon_i}} &\leqslant \int_M |\partial(F_{\varepsilon_i}-\varphi_{\varepsilon_i})|^2_{\omega_{\varphi_{\varepsilon_i}}}\omega^n_{\varphi_{\varepsilon_i}}\\ &= -\int_M (F_{\varepsilon_i}-\varphi_{\varepsilon_i})\Delta_{\omega_{\varphi_{\varepsilon_i}}}(F_{\varepsilon_i}-\varphi_{\varepsilon_i})\omega^n_\varphi\\ &= (\underline{R}+n)\int_M (F_{\varepsilon_i}-\varphi_{\varepsilon_i})\omega^n_{\varphi_{\varepsilon_i}} - \varepsilon_i\int_M (F_{\varepsilon_i}-\varphi_{\varepsilon_i})\mathrm{tr}_{\omega_{\varphi_{\varepsilon_i}}}\omega_0\omega^n_{\varphi_{\varepsilon_i}}\\ &\leqslant C(\underline{R}+n)\int_M \omega^n_{\varphi_{\varepsilon_i}} + C\varepsilon_i n\int_M \omega^{n-1}_{\varphi_{\varepsilon_i}}\wedge\omega_0 \to 0\end{aligned} \tag{6.2}$$

where for the last line we used the C^0 estimates on F and φ. So $\partial F_\infty=\partial\varphi_\infty$ on K which implies that $F_\infty=\varphi_\infty+C$ on K. Furthermore,

$$\mathrm{Ric}(\omega_{\varphi_\infty})=\mathrm{Ric}(\omega_0)-\sqrt{-1}\partial\bar\partial F_\infty=-\eta-\sqrt{-1}\partial\bar\partial\varphi_\infty=-\omega_{\varphi_\infty} \tag{6.3}$$

on K. Now, we can take a countable compact increasing exhaustion U_n of $M\backslash E$ such that $\cup_{n\in\mathbb{N}}K_n=M\backslash E$, and $K_i\subset K_{i+1},\forall i$. Then by using the usual diagonal argument, we can get a subsequence of φ_{ε_i} so that passing to its limit defines a φ_∞ on $M\backslash E$ and

$$\mathrm{Ric}(\omega_{\varphi_\infty})=-\omega_{\varphi_\infty}. \tag{6.4}$$

By Theorem 1.4, the uniqueness of such singular Kähler Einstein metric, all convergent subsequences on K we initially start with have to converge to the same limit. If for each sequence we have a further subsequence converging to the same limit, then the sequence itself has to converge.

BIBLIOGRAPHY

[1] C. Arezzo & F. Pacard, "Blowing up and desingularizing constant scalar curvature Kähler manifolds", *Acta Math.* **196** (2006), no. 2, p. 179-228.

[2] T. Aubin, "Équations du type Monge–Ampère sur les variétés kählériennes compactes", *Bull. Sci. Math.* **102** (1978), no. 1, p. 63-95.

[3] R. J. Berman, "K-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler–Einstein metrics", *Invent. Math.* **203** (2016), no. 3, p. 973-1025.

[4] R. J. Berman, T. Darvas & C. H. Lu, "Regularity of weak minimizers of the K-energy and applications to properness and K-stability", *Ann. Sci. Éc. Norm. Supér.* **53** (2020), no. 2, p. 267-289.

[5] E. Calabi, "Extremal Kähler metrics", in *Seminar on Differential Geometry*, Annals of Mathematics Studies, vol. 102, Princeton University Press, 1982, p. 259-290.

[6] ———, "Extremal Kähler metrics. II", in *Differential geometry and complex analysis*, Springer, 1985, p. 95-114.

[7] X. Chen, "On the lower bound of the Mabuchi energy and its application", *Int. Math. Res. Not.* (2000), no. 12, p. 607-623.

[8] ———, "On the existence of constant scalar curvature Kähler metric: a new perspective", *Ann. Math. Qué.* **42** (2018), no. 2, p. 169-189.

[9] X. Chen & J. Cheng, "On the constant scalar curvature Kähler metrics (I): A priori estimates", *J. Am. Math. Soc.* **34** (2021), no. 4, p. 909-936.

[10] ———, "On the constant scalar curvature Kähler metrics (II): Existence results", *J. Am. Math. Soc.* **34** (2021), no. 4, p. 937-1009.

[11] X. Chen, S. K. Donaldson & S. Sun, "Kähler–Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities", *J. Am. Math. Soc.* **28** (2015), no. 1, p. 183-197.

[12] ———, "Kähler–Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π", *J. Am. Math. Soc.* **28** (2015), no. 1, p. 199-234.

[13] ———, "Kähler–Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof", *J. Am. Math. Soc.* **28** (2015), no. 1, p. 235-278.

[14] T. Darvas, "The Mabuchi geometry of finite energy classes", *Adv. Math.* **285** (2015), p. 182-219.

[15] ———, "The Mabuchi completion of the space of Kähler potentials", *Am. J. Math.* **139** (2017), no. 5, p. 1275-1313.

[16] T. Darvas & Y. A. Rubinstein, "Tian's properness conjectures and Finsler geometry of the space of Kähler metrics", *J. Am. Math. Soc.* **30** (2017), no. 2, p. 347-387.

[17] J.-P. Demailly & N. Pali, "Degenerate complex Monge–Ampère equations over compact Kähler manifolds", *Int. J. Math.* **21** (2010), no. 3, p. 357-405.

[18] R. Dervan, "Alpha invariants and coercivity of the Mabuchi functional on Fano manifolds", *Ann. Fac. Sci. Toulouse, Math.* **25** (2016), no. 4, p. 919-934.

[19] S. K. Donaldson, "Scalar curvature and projective embeddings. I", *J. Differ. Geom.* **59** (2001), no. 3, p. 479-522.

[20] ———, "Constant scalar curvature metrics on toric surfaces", *Geom. Funct. Anal.* **19** (2009), no. 1, p. 83-136.

[21] P. Eyssidieux, V. Guedj & A. Zeriahi, "A priori L^∞-estimates for degenerate complex Monge–Ampère equations", *Int. Math. Res. Not.* (2008), article no. rnn070 (8 pages).

[22] ———, "Singular Kähler-Einstein metrics", *J. Am. Math. Soc.* **22** (2009), no. 3, p. 607-639.

[23] J. Fine, "Constant scalar curvature Kähler metrics on fibred complex surfaces", *J. Differ. Geom.* **68** (2004), no. 3, p. 397-432.

[24] M. Gross & P. M. H. Wilson, "Large complex structure limits of $K3$ surfaces", *J. Differ. Geom.* **55** (2000), no. 3, p. 475-546.

[25] W. He, "Scalar curvature and properness on Sasaki manifolds", `https://arxiv.org/abs/1802.03841`, 2018.

[26] ———, "On Calabi's extremal metric and properness", *Trans. Am. Math. Soc.* **372** (2019), no. 8, p. 5595-5619.

[27] W. Jian, Y. Shi & J. Song, "A remark on constant scalar curvature Kähler metrics on minimal models", *Proc. Am. Math. Soc.* **147** (2019), no. 8, p. 3507-3513.

[28] R. Kobayashi, "Einstein–Kähler V-metrics on open Satake V-surfaces with isolated quotient singularities", *Math. Ann.* **272** (1985), no. 3, p. 385-398.

[29] S. a. Koł odziej, "The complex Monge–Ampère equation", *Acta Math.* **180** (1998), no. 1, p. 69-117.

[30] H. Li & Y. Shi, "A criterion for the properness of the K-energy in a general Kähler class (II)", *Commun. Contemp. Math.* **18** (2016), no. 6, article no. 1550071 (15 pages).

[31] D. H. Phong, J. Song & J. Sturm, "Complex Monge–Ampère equations", in *Surveys in differential geometry. Vol. XVII*, Surveys in Differential Geometry, vol. 17, International Press, 2012, p. 327-410.

[32] D. H. Phong & J. Sturm, "Lectures on stability and constant scalar curvature", in *Handbook of geometric analysis, No. 3*, Advanced Lectures in Mathematics, vol. 14, International Press, 2010, p. 357-436.

[33] J. Ross & R. Thomas, "An obstruction to the existence of constant scalar curvature Kähler metrics", *J. Differ. Geom.* **72** (2006), no. 3, p. 429-466.

[34] Y. A. Rubinstein, "Tian's properness conjectures: an introduction to Kähler geometry", `https://arxiv.org/abs/1807.00928`, 2018.

[35] X. S. Shen, "Estimates for metrics of constant Chern scalar curvature", *Math. Res. Lett.* **28** (2021), no. 5, p. 1525-1550.

[36] Z. Sjöström Dyrefelt, "Existence of cscK metrics on smooth minimal models", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **23** (2022), no. 1, p. 223-232.

[37] J. Song, "Nakai-Moishezon criterions for complex Hessian equations", `https://arxiv.org/abs/2012.07956`, 2020.

[38] J. Song & G. Tian, "Bounding scalar curvature for global solutions of the Kähler–Ricci flow", *Am. J. Math.* **138** (2016), no. 3, p. 683-695.

[39] ———, "The Kähler–Ricci flow through singularities", *Invent. Math.* **207** (2017), no. 2, p. 519-595.

[40] J. Song & B. Weinkove, "On the convergence and singularities of the J-flow with applications to the Mabuchi energy", *Commun. Pure Appl. Math.* **61** (2008), no. 2, p. 210-229.

[41] ———, "An introduction to the Kähler–Ricci flow", in *An introduction to the Kähler–Ricci flow*, Lecture Notes in Mathematics, vol. 2086, Springer, 2013, p. 89-188.

[42] J. Stoppa, "K-stability of constant scalar curvature Kähler manifolds", *Adv. Math.* **221** (2009), no. 4, p. 1397-1408.

[43] A. Strominger, S.-T. Yau & E. Zaslow, "Mirror symmetry is T-duality", *Nucl. Phys., B* **479** (1996), no. 1-2, p. 243-259.

[44] G. Székelyhidi, *An introduction to extremal Kähler metrics*, Graduate Studies in Mathematics, vol. 152, American Mathematical Society, 2014, xvi+192 pages.

[45] G. Tian, "On Kähler–Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$", *Invent. Math.* **89** (1987), no. 2, p. 225-246.

[46] ———, "Kähler-Einstein metrics with positive scalar curvature", *Invent. Math.* **130** (1997), no. 1, p. 1-37.

[47] ———, *Canonical metrics in Kähler geometry. Notes taken by Meike Akveld*, Lectures in Mathematics, ETH Zürich, Birkhäuser, 2000, vi+101 pages.

[48] ———, "K-stability and Kähler-Einstein metrics", *Commun. Pure Appl. Math.* **68** (2015), no. 7, p. 1085-1156.

[49] G. Tian & Z. Zhang, "On the Kähler–Ricci flow on projective manifolds of general type", *Chin. Ann. Math., Ser. B* **27** (2006), no. 2, p. 179-192.

[50] V. Tosatti, "KAWA lecture notes on the Kähler–Ricci flow", *Ann. Fac. Sci. Toulouse, Math.* **27** (2018), no. 2, p. 285-376.

[51] H. Tsuji, "Existence and degeneration of Kähler–Einstein metrics on minimal algebraic varieties of general type", *Math. Ann.* **281** (1988), no. 1, p. 123-133.

[52] B. Weinkove, "On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy", *J. Differ. Geom.* **73** (2006), no. 2, p. 351-358.

[53] S. T. Yau, "On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I", *Commun. Pure Appl. Math.* **31** (1978), no. 3, p. 339-411.

[54] Z. Zhang, "Degenerate Monge–Ampère equations over projective manifolds", PhD Thesis, Massachusetts Institute of Technology, 2006.

[55] K. Zheng, "Existence of constant scalar curvature Kähler cone metrics, properness and geodesic stability", `https://arxiv.org/abs/1803.09506`, 2018.

Manuscrit reçu le 23 mars 2021,
révisé le 4 novembre 2022,
accepté le 28 novembre 2022.

Wanxing LIU
Department of Mathematics,
Northwestern University,
2033 Sheridan Road,
Evanston, IL 60208 (USA)
WanxingLiu2023@u.northwestern.edu