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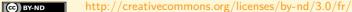
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## CONVERGENCE OF CSCK METRICS ON SMOOTH MINIMAL MODELS OF GENERAL TYPE

### by Wanxing LIU

ABSTRACT. — We consider constant scalar curvature Kähler metrics on a smooth minimal model of general type in a neighborhood of the canonical class, which is the perturbation of the canonical class by a fixed Kähler metric. We show that sequences of such metrics converge smoothly on compact subsets away from a subvariety to the singular Kähler–Einstein metric in the canonical class. This confirms partially a conjecture of Jian–Shi–Song about the convergence behavior of such sequences.

RÉSUMÉ. — On considère des métriques de Kähler à courbure scalaire constante sur une surface lissemodèle minimal de type général dans un voisinage de la classe canonique, qui estla perturbation de la classe canonique par une métrique de Kähler fixe. Nous montrons queles séquences de ces métriques convergent en douceur sur des sous-ensembles compacts loin d'un sous-ensemble. variété à la métrique singulière de Kähler Einstein dans la classe canonique. Cela confirme partiellement une conjecture de Jian-Shi-Song sur le comportement de convergence detels séquences.

#### 1. Introduction

Since the work of Calabi [5, 6] there has been much interest in the existence of constant scalar curvature Kähler (cscK) metrics. For the Kähler Einstein metric which is a special type of the cscK metric, Yau [53] and Aubin [2] established independently the existence of such a metric on Kähler manifolds of negative first Chern class, and Yau [53] also showed it for manifolds of zero first Chern class. For manifolds of positive first Chern class, the Yau-Tian-Donaldson conjecture predicted that the existence of a Kähler Einstein metric is equivalent to the K-stability. Chen-Donaldson-Sun [11, 12, 13] proved that the K-stability is sufficient for the existence of a Kähler Einstein metric (see also Tian [48]), while the necessity was shown by Tian [46], Donaldson [19], Stoppa [42] and Berman [3].

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For the cscK metric, Donaldson [20] considered the existence of cscK metrics on toric surfaces. Chen–Cheng [10] proved that the properness of the Mabuchi functional is sufficient for the existence of a cscK metric building on the work of [8, 14, 15]. The necessity was proven by Berman–Darvas–Lu [4]. We refer the interested reader to [16, 31, 32, 34] for surveys and related developments of this area. Following the breakthrough made by Chen–Cheng, Jian–Shi–Song [27] showed the following theorem.

THEOREM 1.1 ([27, Theorem 1.1]). — Let  $(M, \omega_0)$  be a compact Kähler manifold. If the canonical bundle  $K_M$  is semi-ample, then for any  $\varepsilon > 0$  small enough, there exists a unique cscK metric in the Kähler class  $-c_1(M) + \varepsilon[\omega_0] = c_1(K_M) + \varepsilon[\omega_0]$ .

Recently, using different tools, Sjöstrom Dyrefelt [36] and Song [37] strengthened this result to all compact Kähler manifolds with nef canonical bundle. We define the first Chern class  $c_1(M)$  to be

(1.1) 
$$c_1(M) = [\operatorname{Ric}(\omega_0)] = -[\sqrt{-1}\partial\bar{\partial}\log\det(g_0)],$$

where  $g_0$  is the metric tensor of  $\omega_0$ . Notice that this differs from the usual definition of  $c_1(M)$  by a factor of  $2\pi$ . Theorem 1.1 is a generalization of Arezzo-Pacard's [1] result on minimal surfaces of general type. The proof of this result is based on Chen-Cheng [10] and the properness criterions developed by Weinkove [52], Song-Weinkove [40] and Li-Shi-Yao [30], and Jian-Shi-Song also made the following conjecture.

Conjecture 1.2. — Let  $(M, \omega_0)$  be a compact Kähler manifold with semi-ample canonical bundle  $K_M$ . Then for any sequence  $\varepsilon_n$  going to zero, the corresponding sequence of cscK metrics in  $-c_1(M) + \varepsilon_n[\omega_0]$  converges to the twisted Kähler–Einstein metric  $g_{\rm can}$  on the canonical model  $M_{\rm can}$  of M. The convergence should be both global in Gromov–Hausdorff topology and local in smooth topology away from the singular fibres of the canonical map  $\Phi: M \to M_{\rm can}$ .

Jian–Shi–Song further noted that this conjecture can be understood from the perspective of slope stability introduced by Ross–Thomas [33] and is related to the result of Gross–Wilson [24] from the standpoint of the Strominger–Yau–Zaslow [43] conjecture in mirror symmetry. If the canonical model of M is smooth and the canonical morphism  $\Phi: M \to M_{\rm can}$  has no singular fibers, Conjecture 1.2 was shown to be true by Fine [23] (Theorem 8.1 and its proof), and the main purpose of the current paper is to show that the local smooth convergence of Conjecture 1.2 holds when  $-c_1(M)$  is big and nef.

We call a compact Kähler manifold M a smooth minimal model if  $-c_1(M)$  is nef, and a manifold of general type if  $-c_1(M)$  is big. Recall that  $-c_1(M)$  is said to be nef if for all  $\varepsilon > 0$ ,  $-c_1(M) > -\varepsilon[\omega_0]$  and big and nef if it is nef and  $(-c_1(M))^n > 0$ . One immediate benefit of assuming that  $-c_1(M)$  is big and nef is the existence of a semi-positive representative in  $-c_1(M)$ . Since  $-c_1(M)$  is big and nef, M is Moishezon which implies that M is projective. By Kawamata's base point free theorem,  $K_M$  is semi-ample. Hence there exists  $\eta \in -c_1(M)$ , which is some multiple of the pullback of the Fubini–Study metric through the canonical map  $\Phi: M \to \mathbb{P}^N$ . In particular,  $\eta$  is semi-positive, and we consider the sequence of reference Kähler metrics  $\omega_{\varepsilon} = \eta + \varepsilon \omega_0 > 0$ . Also notice that we can choose a volume form  $\Omega$  such that  $\operatorname{Ric}(\Omega) = -\eta$ , then by Yau's theorem [53] we can always choose  $\omega_0$  to be such that  $\operatorname{Ric}(\omega_0) = -\eta$ .

THEOREM 1.3. — Suppose that  $(M, \omega_0)$  is a compact Kähler manifold of dimension n with its canonical bundle being big and nef, and  $\operatorname{Ric}(\omega_0) = -\eta$ . There exists an effective divisor E such that for any  $\varepsilon_n$  converging to zero the corresponding sequence of  $\operatorname{csc} K$  metrics  $\omega_{\varphi_{\varepsilon_n}} \in -c_1(M) + \varepsilon_n[\omega_0]$ , given by Theorem 1.1, converges in  $C^{\infty}_{\operatorname{loc}}(M \setminus E)$  to the unique singular Kähler Einstein metric in  $-c_1(M)$ .

The singular Kähler Einstein metric was first constructed by Kobayashi [28] in complex dimension 2, and then it was constructed in all dimensions as the limit of the normalized Kähler Ricci flow. Consider the normalized Kähler-Ricci flow:

(1.2) 
$$\frac{\partial}{\partial t}\omega_t = -\operatorname{Ric}(\omega_t) - \omega_t, \ \omega|_{t=0} = \omega_0, \ \omega_t > 0.$$

The following theorem first appeared in the work of Tsuji [51], but later Tian–Zhang [49] extended it and clarified some parts of the proof (see also Song–Weinkove [41] or Tosatti [50] for nice expositions of this result).

THEOREM 1.4 ([49, Theorem 0.1]). — Let M be a smooth minimal model of general type. Then

- (1) The solution  $\omega = \omega(t)$  of the normalized Kähler–Ricci flow starting at any Kähler metric  $\omega_0$  on M exists for all time.
- (2) There exists an effective divisor E of M such that  $\omega(t)$  converges in  $C_{loc}^{\infty}(M\backslash E)$  to a smooth Kähler metric on  $M\backslash E$ .

Furthermore, the Kähler metric obtained above is the unique metric satisfying

(1)  $\operatorname{Ric}(\omega_{KE}) = -\omega_{KE}$  on  $M \setminus E$ .

(2) There exists a constant C such that

(1.3) 
$$C\omega_0^n \leqslant \omega_{KE}^n \leqslant \frac{1}{C}\omega_0^n.$$

 $\omega_{\rm KE}$  was also constructed by Eyssidieux–Guedj–Zeriahi [22] using pluri-potential theory.

As a consequence of Theorem 1.1, we can pick a unique sequence of cscK metrics  $\omega_{\varphi_{\varepsilon}} \in [\omega_{\varepsilon}]$ . The Kähler potentials of these metrics  $\varphi_{\varepsilon} \in \mathcal{H}_{\omega_{\varepsilon}}$  satisfy the following coupled equations:

(1.4) 
$$\frac{\frac{\omega_{\varphi_{\varepsilon}}^{n}}{\omega_{0}^{n}} = e^{F_{\varepsilon}}, \\ \Delta_{\omega_{\varphi_{\varepsilon}}} F_{\epsilon} = -\underline{R_{\varepsilon}} - \operatorname{tr}_{\omega_{\varphi_{\varepsilon}}} \eta$$

where  $\underline{R_{\epsilon}} = n \frac{c_1(M) \cdot [\omega_{\epsilon}]^{n-1}}{[\omega_{\epsilon}]^n}$ ,  $\underline{R_{\epsilon}} \to -n$  as  $\epsilon \to 0$ , and

(1.5) 
$$\mathcal{H}_{\omega_{\varepsilon}} = \left\{ v \in C^{\infty}(M) \,\middle|\, \omega_{\varepsilon} + \sqrt{-1} \partial \bar{\partial} v > 0, \sup_{M} v = 0 \right\}.$$

We stress that it is important to assume bigness of  $-c_1(M)$  in order for the limit of  $\underline{R_{\varepsilon}}$  to be -n, otherwise it is not true. Also notice that the equations (1.4) are slightly different from the coupled equations considered in [9]:

(1.6) 
$$\frac{\omega_{\varphi_{\varepsilon}}^{n}}{\omega_{\varepsilon}^{n}} = e^{F_{\varepsilon}},$$

$$\Delta_{\omega_{\varphi_{\varepsilon}}} F_{\epsilon} = -R_{\underline{\varepsilon}} + \operatorname{tr}_{\omega_{\varphi_{\varepsilon}}} \operatorname{Ric}(\omega_{\varepsilon}).$$

Specifically, we replace  $\omega_{\varepsilon}$  in the denominator of the first equation by  $\omega_0$  and adapt the second equation accordingly because  $\omega_{\varepsilon}$  is degenerating. We will see later that we will have to adjust the definition of the Mabuchi Energy to accommodate this change. The strategy of the proof of Theorem 1.3 consists of the following steps:

- (1) We establish through Sections 2-4 a degenerate version of the estimates in [9]. More specifically, we show that  $\varphi_{\varepsilon}$  is bounded in  $C_{\text{loc}}^{\infty}(M\backslash E)$  and the bound depends only on  $\int_{M} \log \frac{\omega_{\varphi_{\varepsilon}}^{n}}{\omega_{0}^{n}} \frac{\omega_{\varphi_{\varepsilon}}^{n}}{n!} = \int_{M} e^{F_{\varepsilon}} F_{\varepsilon} \frac{\omega_{0}^{n}}{n!}$  and  $(M, \omega_{0})$ . We will call  $\int_{M} \log \frac{\omega_{\varphi_{\varepsilon}}^{n}}{\omega_{0}^{n}} \frac{\omega_{\varphi_{\varepsilon}}^{n}}{n!}$  the entropy.
- (2) We show in Section 5 using a method of Dervan [18] that  $\int_M e^{F_{\varepsilon}} F_{\varepsilon} \frac{\omega_0^0}{n!}$  is uniformly bounded independent of  $\varepsilon$ , thus  $\varphi_{\varepsilon}$  is uniformly bounded in  $C_{\text{loc}}^{\infty}(M \setminus E)$  based on (1).
- (3) In Section 6 we use the estimates and an integration by part argument to conclude that  $\omega_{\varphi_{\varepsilon}}$  has to converge in  $C^{\infty}_{loc}(M\backslash E)$  to the unique singular Kähler Einstein metric in  $-c_1(M)$ .

We remark that Zheng [55] considered the problem of the  $L^1$  convergence of a sequence of smooth cscK metrics in a neighborhood of an arbitrary big class. There he also had to generalize Chen-Cheng's [9] original estimates. Our estimates are different in the sense that we are able to get full non-degenerate 0-th order estimates on  $F_{\varepsilon}$ . Furthermore, the estimates in Zheng [55] are proved with respect to a sequence of specially constructed reference metrics. Also, one critical element of his proof is a version of alpha invariant for any big class using machinery from pluripotential theory, but we do not need it here.

We conclude this section by mentioning that there are also generalizations of Chen-Cheng's estimates in other directions. For instance, Shen [35] generalizes them to the Hermitian setting, and He generalizes them to Sasaki manifolds [25], and extremal metrics [26].

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### 2. $C^0$ estimates

In this section we produce  $C^0$  estimates for  $\varphi_{\varepsilon}$  and  $F_{\varepsilon}$  solving the following coupled equations:

(2.1) 
$$\frac{\frac{\omega_{\varphi_{\varepsilon}}^{n}}{\omega_{0}^{n}} = e^{F_{\varepsilon}},$$

$$\Delta_{\omega_{\varphi_{\varepsilon}}} F_{\varepsilon} = -R_{\varepsilon} - \operatorname{tr}_{\omega_{\varphi_{\varepsilon}}} \eta.$$

We start with the  $C^0$  estimate on  $\varphi_{\varepsilon}$ .

Theorem 2.1. — There exists a constant C > 0 such that  $\|\varphi_{\varepsilon}\|_{C^{0}(M)} \leq C$  where C is dependent on  $\int_{M} \log \frac{\omega_{\varepsilon_{\varepsilon}}^{n}}{\omega_{\varepsilon}^{n}} \frac{\omega_{\varepsilon_{\varepsilon}}^{n}}{n!}$ .

Before we prove this, we should make it clear that the techniques we use here follow exactly the same route laid out in the proof of theorem 5.2 in [9]. However the estimates in [9] depend on  $Ric(\omega_{\varepsilon})$ , but the metric is

degenerating as  $\varepsilon \to 0$ , so we have to make essential modifications which involves clarifying that the  $\alpha$ -invariants can be controlled uniformly as shown in Lemma 2.3, specifying different choices of constants in Lemma 2.4, and checking everything still works as expected. It is during the course of this computation where we use the assumption that  $\mathrm{Ric}(\omega_0) = -\eta$  and the fact that  $R_{\varepsilon} \to -n$  as  $\varepsilon \to 0$ , neither of which is needed in [9].

Let us start the proof with some preparations. Denote  $\operatorname{Vol}(\omega_{\varepsilon}) = \int_{M} \omega_{\varepsilon}^{n}$ , and as in [9] we define a smooth real-valued function  $\rho_{\varepsilon}$  to be the unique solution to the following equation using Yau's theorem [53]:

(2.2) 
$$(\omega_{\varepsilon} + \sqrt{-1}\partial\bar{\partial}\rho_{\varepsilon})^{n} = \frac{\operatorname{Vol}(\omega_{\varepsilon}) e^{F_{\varepsilon}} \Phi(F_{\varepsilon})\omega_{0}^{n}}{\int_{M} e^{F_{\varepsilon}} \Phi(F_{\varepsilon})\omega_{0}^{n}},$$

$$\sup_{M} \rho_{\varepsilon} = 0$$

where  $\Phi(F_{\varepsilon}) = \sqrt{F_{\varepsilon}^2 + 1}$ . Let us then recall the definition of Tian's  $\alpha$ -invariant [45].

PROPOSITION 2.2 (Tian's  $\alpha$ -invariant). — For any Kähler class  $[\omega]$  on M, there exists an invariant  $\alpha(M, [\omega]) > 0$  such that for any  $\alpha < \alpha(M, [\omega])$ , we have

(2.3) 
$$\int_{M} e^{-\alpha v} \omega^{n} \leqslant C$$

for all  $v \in \mathcal{H}_{\omega}$ .

LEMMA 2.3. — Given  $\alpha > 0$  with  $\alpha < \alpha(M, [\eta + \omega_0])$ , there is a uniform constant C > 0 such that for all  $\varepsilon > 0$  with  $\varepsilon \leqslant 1$  we have

(2.4) 
$$\int_{M} e^{-\alpha v} \, \omega_0^n \leqslant C$$

for all  $v \in \mathcal{H}_{\eta + \varepsilon \omega_0}$ .

*Proof.* — The key observation is that when  $\varepsilon > 0$  and  $\varepsilon \leqslant 1$ ,  $\eta + \varepsilon \omega_0 \leqslant \eta + \omega_0$  implies that  $\mathcal{H}_{\eta + \varepsilon \omega_0} \subset \mathcal{H}_{\eta + \omega_0}$ . Thus, for any  $v \in \mathcal{H}_{\eta + \varepsilon \omega_0}$ , we have for the fixed  $\alpha > 0$  and  $\alpha < \alpha(M, [\eta + \omega_0])$ 

(2.5) 
$$\int_{M} e^{-\alpha v} \omega_{0}^{n} \leqslant \int_{M} e^{-\alpha v} (\eta + \omega_{0})^{n} \leqslant C.$$

The most important estimate we need is the following Lemma analogous to [9, Theorem 5.2].

Lemma 2.4. — There exists a constant C such that for all  $\varepsilon > 0$  sufficiently small

(2.6) 
$$F_{\varepsilon} + \varepsilon \rho_{\varepsilon} - (1 + \varepsilon) \varphi_{\varepsilon} \leqslant C$$

where C only depends on  $\int_M \log \frac{\omega_{\varphi_{\varepsilon}}^n}{\omega_n^n} \frac{\omega_{\varphi_{\varepsilon}}^n}{n!}$ .

*Proof.* — Given a point  $p_0 \in M$ ,  $0 < d_0 < 1$ , we start by choosing a smooth cut-off function f such that

(2.7) 
$$1 - \theta \leqslant f \leqslant 1,$$

$$f(p_0) = 1, f \equiv 1 - \theta \text{ outside } B_{\frac{d_0}{2}}(p_0),$$

$$|\partial f|_{\omega_0}^2 \leqslant \frac{4\theta^2}{d_0^2}, \quad |\partial^2 f|_{\omega_0}^2 \leqslant \frac{4\theta}{d_0^2}$$

where  $p_0$  and  $\theta > 0$  are going to be specified later, and  $d_0$  is fixed to be a constant strictly larger than 0 and strictly less than 1. Let  $\alpha > 0$  be a fixed constant strictly less than  $\alpha(M, [\eta + \omega_0])$ , and choose  $\delta = \frac{\alpha}{4n}$ . We will denote the metric tensor of  $\omega_{\varphi_{\varepsilon}}$  by  $g_{\varphi_{\varepsilon}}$ , and suppress  $\varepsilon$  for simplicity of notation while carrying out calculations. Calculate

$$(2.8) \quad \Delta_{\omega_{\varphi}}(e^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}f)$$

$$= \Delta_{\omega_{\varphi}}(e^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)})f + e^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}\Delta_{\omega_{\varphi}}(f)$$

$$+ 2e^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}\delta\operatorname{Re}(g_{\varphi}^{i\bar{j}}\partial_{i}(F+\varepsilon\rho-(1+\varepsilon)\varphi)\bar{\partial}_{j}f)$$

$$= e^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}f(\delta^{2}|\partial(F+\varepsilon\rho-(1+\varepsilon)\varphi)|_{\omega_{\varphi}}^{2}$$

$$+ \delta\Delta_{\omega_{\varphi}}(F+\varepsilon\rho-(1+\varepsilon)\varphi)) + e^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}\Delta_{\omega_{\varphi}}f$$

$$+ 2e^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}\delta\operatorname{Re}(g_{\varphi}^{i\bar{j}}\partial_{i}(F+\varepsilon\rho-(1+\varepsilon)\varphi)\bar{\partial}_{j}f).$$

We estimate the terms involved in the above calculation as follows

$$(2.9) \quad 2\delta \operatorname{Re}(g_{\varphi}^{i\bar{j}}\partial_{i}(F+\varepsilon\rho-(1+\varepsilon)\varphi)\bar{\partial}_{j}f)$$

$$\geqslant -\delta^{2}f|\partial(F+\varepsilon\rho-(1+\varepsilon)\varphi)|_{\omega_{\varphi}}^{2} - \frac{|\partial f|_{\omega_{\varphi}}^{2}}{f}$$

$$\geqslant -\delta^{2}f|\partial(F+\varepsilon\rho-(1+\varepsilon)\varphi)|_{\omega_{\varphi}}^{2} - \frac{|\partial f|_{\omega_{0}}^{2}\operatorname{tr}_{\omega_{\varphi}}\omega_{0}}{f}$$

$$\geqslant -\delta^{2}f|\partial(F+\varepsilon\rho-(1+\varepsilon)\varphi)|_{\omega_{\varphi}}^{2} - \frac{4\theta^{2}}{d_{0}^{2}(1-\theta)}\operatorname{tr}_{\omega_{\varphi}}\omega_{0},$$

$$(2.10) \qquad \mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}\,\Delta_{\omega_{\varphi}}f\geqslant -\,\mathrm{e}^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}\,\frac{4\theta}{d_{0}^{2}(1-\theta)}f\mathrm{tr}_{\omega_{\varphi}}\omega_{0}.$$

The key computation is the following:

(2.11) 
$$\Delta_{\omega_{\varphi}}(F + \varepsilon \rho - (1 + \varepsilon)\varphi)$$

$$= -(\underline{R} + (1 + \varepsilon)n) - \operatorname{tr}_{\omega_{\varphi}} \eta + (1 + \varepsilon)\operatorname{tr}_{\omega_{\varphi}} \omega + \varepsilon \Delta_{\omega_{\varphi}} \rho.$$

Let  $A_{\Phi}(F) = \int_{M} e^{F} \Phi(F) \omega_{0}^{n}$ , and notice by (2.2)

(2.12) 
$$\Delta_{\omega_{\varphi}} \rho = \operatorname{tr}_{\omega_{\varphi}} (\omega + \sqrt{-1} \partial \bar{\partial} \rho) - \operatorname{tr}_{\omega_{\varphi}} \omega$$
$$\geqslant n(e^{-F} e^{F} \operatorname{Vol}(\omega) \Phi(F) A_{\Phi}(F)^{-1})^{\frac{1}{n}} - \operatorname{tr}_{\omega_{\varphi}} \omega$$
$$= n(\operatorname{Vol}(\omega) \Phi(F) A_{\Phi}(F)^{-1})^{\frac{1}{n}} - \operatorname{tr}_{\omega_{\varphi}} \omega.$$

Then we have

$$(2.13) \quad \Delta_{\omega_{\varphi}}(F + \varepsilon \rho - (1 + \varepsilon)\varphi)$$

$$\geqslant \left(-\underline{R} - (1 + \varepsilon)n + \varepsilon n \operatorname{Vol}(\omega)^{\frac{1}{n}} A_{\Phi}(F)^{-\frac{1}{n}} \Phi(F)^{\frac{1}{n}}\right) + (1 + \varepsilon) \operatorname{tr}_{\omega_{\varphi}} \omega$$

$$- \varepsilon \operatorname{tr}_{\omega_{\varphi}} \omega - \operatorname{tr}_{\omega_{\varphi}} \eta$$

$$= \left(-\underline{R} - (1 + \varepsilon)n + \varepsilon n \operatorname{Vol}(\omega)^{\frac{1}{n}} A_{\Phi}(F)^{-\frac{1}{n}} \Phi(F)^{\frac{1}{n}}\right) + \varepsilon \operatorname{tr}_{\omega_{\varphi}} \omega_{0}.$$

Combining all these calculations, we conclude that

$$\begin{split} (2.14) \quad \Delta_{\omega_{\varphi}} \big( \mathrm{e}^{\delta(F + \varepsilon \rho - (1 + \varepsilon)\varphi)} \, f \big) \\ \geqslant \delta f \, \mathrm{e}^{\delta(F + \varepsilon \rho - (1 + \varepsilon)\varphi)} \big( -\underline{R} - (1 + \varepsilon)n + \varepsilon n \mathrm{Vol}(\omega)^{\frac{1}{n}} A_{\Phi}(F)^{-\frac{1}{n}} \Phi(F)^{\frac{1}{n}} \big) \\ + \, \mathrm{e}^{\delta(F + \varepsilon \rho - (1 + \varepsilon)\varphi)} \left( \delta f \varepsilon - \frac{4\theta}{d_0^2 (1 - \theta)} f - \frac{4\theta^2}{d_0^2 (1 - \theta)^2} \right) \mathrm{tr}_{\omega_{\varphi}} \omega_0. \end{split}$$

Choosing

$$\theta = \frac{\delta \varepsilon}{64} d_0^2 = \frac{\alpha \varepsilon}{256n} d_0^2,$$

notice that when  $\varepsilon$  is small enough we have  $\theta < \frac{1}{2}$  and  $\frac{\delta \varepsilon}{64} \leqslant \sqrt{\frac{\delta \varepsilon}{128}}$ , so that because  $f \leqslant 1$  we estimate

$$(2.15) \quad \delta f \epsilon - \frac{4\theta}{d_0^2 (1 - \theta)} f - \frac{4\theta^2}{d_0^2 (1 - \theta)^2}$$

$$\geqslant \delta (1 - \theta) \epsilon - \frac{4\theta}{d_0^2 (1 - \theta)} - \frac{4\theta^2}{d_0^2 (1 - \theta)^2}$$

$$\geqslant \frac{\delta \epsilon}{2} - \frac{8\theta}{d_0^2} - \frac{16\theta^2}{d_0^2}$$

$$\geqslant \frac{\delta \epsilon}{2} - \frac{\delta \epsilon}{8} - \frac{\delta \epsilon}{8} > 0.$$

So the coefficient of  $\operatorname{tr}_{\omega_{\varphi}}\omega_0$  in (2.14) is positive, and we throw it away and conclude

$$(2.16) \quad \Delta_{\omega_{\varphi}}(e^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}f)$$

$$\geqslant \delta f e^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}(-\underline{R}-(1+\varepsilon)n+\varepsilon n \operatorname{Vol}(\omega)^{\frac{1}{n}}A_{\Phi}(F)^{-\frac{1}{n}}\Phi(F)^{\frac{1}{n}}).$$

Let  $u = e^{\delta(F+\varepsilon\rho-(1+\varepsilon)\varphi)}$ , and choose  $p_0$  to be the maximum point of u. Applying the Alexandrov–Bakelman–Pucci maximum principle in  $B_{d_0}(p_0)$  we get

$$(2.17) \sup_{B_{d_0}(p_0)} uf \leqslant \sup_{\partial B_{d_0}(p_0)} uf$$

$$+C \left( \int_{B_{d_0}(p_0)} \frac{u^{2n} \left( (-\underline{R} - (1+\varepsilon)n + \varepsilon n \operatorname{Vol}(\omega)^{\frac{1}{n}} A_{\Phi}(F)^{-\frac{1}{n}} \Phi(F)^{\frac{1}{n}})^{-} \right)^{2n}}{\mathrm{e}^{-2F}} \omega_0^n \right)^{\frac{1}{2n}}$$

where C is a constant dependent on the dimension of the manifold  $n, d_0$  and  $\delta$ . Notice that the integral is only nonzero on the region where

$$(2.18) -\underline{R} - (1+\varepsilon)n + \varepsilon n \operatorname{Vol}(\omega)^{\frac{1}{n}} A_{\Phi}(F)^{-\frac{1}{n}} \Phi(F)^{\frac{1}{n}} < 0.$$

Observe that for  $\varepsilon$  sufficiently small,

$$(2.19) \qquad -\underline{R} - (1+\epsilon)n = \frac{n[\eta] \cdot \sum_{k=0}^{n-1} {n-1 \choose k} \varepsilon^k [\omega_0]^k \cdot [\eta]^{n-1-k}}{\sum_{k=0}^n {n \choose k} \varepsilon^k [\omega_0]^k \cdot [\eta]^{n-k}} - (1+\epsilon)$$
$$\geqslant -C\varepsilon.$$

So (2.18) is only possible if  $F \leq C$  where C depends only on  $A_{\Phi}(F)$ , n and  $A_{\Phi}(F)$  depends only on the entropy. So we get

$$(2.20) \qquad \left( \int_{B_{d_0}(p_0)} \frac{u^{2n} \left( -\underline{R} - (1+\varepsilon)n + \varepsilon nA_{\Phi}(F)^{-\frac{1}{n}}\Phi(F)^{\frac{1}{n}})^{-} \right)^{2n}}{e^{-2F}} \omega_0^n \right)^{\frac{1}{2n}}$$

$$= \left( \int_{B_{d_0}(p_0) \cap \{F \leqslant C\}} \frac{u^{2n} \left( -\underline{R} - (1+\varepsilon)n + \varepsilon nA_{\Phi}(F)^{-\frac{1}{n}}\Phi(F)^{\frac{1}{n}})^{-} \right)^{2n}}{e^{-2F}} \omega_0^n \right)^{\frac{1}{2n}}$$

$$\leqslant \left( \int_{B_{d_0}(p_0)} \frac{u^{2n} \left( (-\underline{R} - (1+\varepsilon)n)^{-})^{2n}}{e^{-2F}} \omega_0^n \right)^{\frac{1}{2n}}$$

$$\leqslant C \left( \int_{B_{d_0}(p_0) \cap \{F \leqslant C\}} e^{2n\delta(F+\varepsilon\rho - (1+\varepsilon)\varphi)} e^{2F} \varepsilon^{2n} \omega_0^n \right)^{\frac{1}{2n}}$$

$$\leqslant C\varepsilon \left( \int_{B_{d_0}(p_0)} e^{-(2n\delta(1+\varepsilon))\varphi} \omega_0^n \right)^{\frac{1}{2n}}$$

$$\leqslant C\varepsilon \left( \int_{B_{d_0}(p_0)} e^{-\frac{\alpha}{2}\varphi} \omega_0^n \right)^{\frac{1}{2n}}$$

$$\leqslant C\varepsilon.$$

where for the last inequality we used Lemma 2.3. So

(2.21) 
$$u(p_0) = \sup_{M} u \leqslant (1 - \theta) \sup_{M} u + C\varepsilon,$$

and

(2.22) 
$$u(p_0) \leqslant \frac{C\varepsilon}{\theta} = \frac{C\varepsilon}{\frac{\delta\varepsilon d_0^2}{64}} \leqslant C.$$

Thus we are able to conclude that for any  $\varepsilon$  sufficiently small,

$$(2.23) F + \varepsilon \rho - (1 + \varepsilon) \varphi \leqslant C. \Box$$

With this estimate at our disposal, we now proceed to prove Theorem 2.1.

*Proof of Theorem 2.1.* — Let  $\alpha$  be a positive constant strictly less than  $\alpha(M, [\eta + \omega_0])$ , the  $\alpha$ -invariant associated with  $[\eta + \omega_0]$ , and  $\varepsilon < \frac{\alpha}{2}$ , we have

(2.24) 
$$F + \frac{\alpha}{2}\rho - (1+\varepsilon)\varphi \leqslant F + \varepsilon\rho - (1+\varepsilon)\varphi \leqslant C$$

because  $\rho \leqslant 0$ . Thus for  $\varepsilon < \frac{\alpha}{2}$ ,

(2.25) 
$$C \geqslant \int_{M} e^{-\alpha \rho} \omega_{0}^{n} \geqslant \int_{M} \exp(2(F - (1 + \varepsilon)\varphi - C))\omega_{0}^{n}$$
$$\geqslant \int_{M} \exp(2(F - C))\omega_{0}^{n}$$

where for the last inequality we used the fact that  $\varphi \leq 0$ . So we conclude that  $e^F \in L^2(M, \omega_0^n)$  when  $\varepsilon$  is small enough. Then the proof is done by applying a theorem first announced in Tian–Zhang [49] and later proved in Zhang [54] which asserts that if for some p > 1,  $e^F \in L^p(M, \omega_0^n)$ , then the  $C^0$  estimate on  $\varphi$  only depends on the  $\|e^F\|_{L^p(M,\omega_0^n)}$ . This is a generalization of Kolodziej's [29] fundamental result to the degenerate setting. One can find more general versions of their result in Eyssidieux–Guedj–Zeriahi [21, 22] and Demailly–Pali [17] as well.

To show that  $F_{\varepsilon}$  is uniformly bounded from below we recall a trick due to Song-Tian [38] which was pointed out to the author by Jian Song. We consider the following auxiliary complex Monge-Ampère equations:

(2.26) 
$$(\omega_{\varepsilon} + \sqrt{-1}\partial \bar{\partial} h_{\varepsilon})^n = e^{h_{\varepsilon}} \frac{\omega_{\varepsilon}^n}{\omega_0^n} \omega_0^n.$$

Again, for each  $\varepsilon$  there exists a unique smooth real-valued  $h_{\varepsilon}$  solving (2.26) by Yau's theorem. At a maximum  $p_0$  of  $h_{\varepsilon}$  we have

(2.27) 
$$e^{h_{\varepsilon}} = \frac{(\omega_{\varepsilon} + \sqrt{-1}\partial\bar{\partial}h_{\varepsilon})^n}{\omega_0^n} \leqslant \frac{\omega_{\varepsilon}^n}{\omega_0^n} \leqslant C.$$

Thus,  $h_{\varepsilon}$  is uniformly bounded from above. Then by the theorem of Zhang [54] again, we have that  $||h_{\varepsilon}||_{C^{0}(M)}$  is uniformly bounded.

Theorem 2.5. — There exists a constant C depending only on  $\|\varphi_{\varepsilon}\|_{C^{0}(M)}$  and  $\|h_{\varepsilon}\|_{C^{0}(M)}$  such that

$$(2.28) F_{\varepsilon} \geqslant C$$

for  $\varepsilon$  sufficiently small.

*Proof.* — Compute at a minimum  $p_0$  of  $F + \varphi - h$ , we get

(2.29) 
$$0 \leqslant \Delta_{\omega_{\varphi}}(F + \varphi - h) = -\underline{R} + n - \operatorname{tr}_{\omega_{\varphi}}(\omega + \eta + \sqrt{-1}\partial\bar{\partial}h)$$
$$\leqslant C - \operatorname{tr}_{\omega_{\varphi}}(\omega + \sqrt{-1}\partial\bar{\partial}h)$$
$$\leqslant C - n e^{\frac{h - F}{n}}.$$

It implies that

$$(2.30) F(p_0) \geqslant C.$$

So for any  $p \in M$ ,

(2.31) 
$$F(p) + \varphi(p) - h(p) \geqslant F(p_0) + \varphi(p_0) + h(p_0) \geqslant C + \varphi(p_0).$$

Theorem 2.6. — There exists a constant C > 0 such that

$$(2.32) F_{\varepsilon} \leqslant C$$

for all  $\varepsilon$  sufficiently small.

*Proof.* — Using the  $L^p$  bound on  $e^F$ , the lower bound on F and the theorem of Zhang [54] again we can show that  $\rho_{\varepsilon}$  is uniformly bounded as well. Then the theorem follows immediately from Lemma 2.4.

## 3. Degenerate bound on $|\partial \varphi_{\varepsilon}|_{\omega_0}^2$

Let us first recall the following the Kodaira's lemma, for handling degenerate reference metrics.

LEMMA 3.1. — There exists an effective divisor E on M, a holomorphic section s of E which vanishes to order 1 along the divisor E, constants  $\sigma > 0, C > 0$  such that for any  $\delta' \in (0, \sigma]$ , and any Hermitian metric h of [E], where [E] is the line bundle associated with E, we have

(3.1) 
$$\eta + \delta' \sqrt{-1} \partial \bar{\partial} \log |s|_h^2 > C \delta' \omega_0.$$

Moreover, as a special case of the Kodaira lemma, we can and will choose E to be the exceptional locus of the canonical map  $\Phi: M \to M_{\text{can}}$ , where  $\Phi$  fails to be an isomorphism, so that  $\eta$  is Kähler outside E (see Song–Tian [39, Proposition 2.1]). The trick of applying Kodaira's lemma is commonly referred to as the Tsuji's trick in the literature, and the idea of it is straightforward.  $\omega_{\varepsilon}$  is degenerate in the sense that it is tending to  $\eta$  which might be zero along E. By using the barrier function  $\psi := \log |s|_h^2$  which is  $-\infty$  along the singular set, one can carry out the usual maximum principle away from E. Equipped with this tool, we will show in this section a degenerate version of Theorem 2.2 in [9].

THEOREM 3.2. — There exists constants q > 0 and C > 0 such that

$$(3.2) |\partial \varphi_{\varepsilon}|_{\omega_0}^2 \leqslant C \frac{1}{|s|_h^{2q}}$$

where the constants depend only on  $\|\varphi_{\varepsilon}\|_{C^{0}(M)}$ ,  $\|F_{\varepsilon}\|_{C^{0}(M)}$  and  $(M, \omega_{0})$ .

*Proof.* — We will suppress  $\varepsilon$  for simplicity and apply the maximum principle to

(3.3) 
$$e^{-(F+\lambda\varphi)+\frac{1}{2}\varphi^2+Q\delta'\psi}(|\partial\varphi|^2_{\omega_0}+K).$$

Here  $\lambda > 0, K > 0, Q > 0$  are constants to be determined later, and  $\delta' \in (0, \min\{\sigma, 1\}]$ , where  $\sigma$  is the constant given in Lemma 3.1. Let  $A(F, \varphi, \psi) = -(F + \lambda \varphi) + \frac{1}{2}\varphi^2 + Q\delta'\psi$ , and denote the metric tensor of  $\omega_{\varphi}$  by  $g_{\varphi}$ , the metric tensor of  $\omega$  by g and the metric tensor of  $\omega_0$  by  $g_0$ . Unless otherwise noted, we will always choose a normal holomorphic coordinate neighborhood for  $g_0$  such that  $g_{\varphi}$  is diagonal.

$$(3.4) \quad \Delta_{\omega_{\varphi}}(e^{A}(|\partial\varphi|^{2}_{\omega_{0}}+K))$$

$$= \Delta_{\omega_{\varphi}}(e^{A})(|\partial\varphi|^{2}_{\omega_{0}}+K) + e^{A} \Delta_{\omega_{\varphi}}(|\partial\varphi|^{2}_{\omega_{0}}) + 2g_{\varphi}^{i\bar{i}} e^{A} \operatorname{Re}(A_{i}(|\partial\varphi|^{2}_{\omega_{0}})_{\bar{i}}).$$

Let us estimate the three terms involved in the above equation separately

$$(3.5) \quad \Delta_{\omega_{\varphi}}(\mathbf{e}^{A})$$

$$= \mathbf{e}^{A} g_{\varphi}^{i\bar{i}} |A_{i}|^{2} + \mathbf{e}^{A} (-\Delta_{\omega_{\varphi}}(F + \lambda\varphi) + \varphi \Delta_{\omega_{\varphi}}\varphi + Q\delta' \Delta_{\omega_{\varphi}}\psi) + \mathbf{e}^{A} g_{\varphi}^{i\bar{i}} |\varphi_{i}|^{2}$$

$$= \mathbf{e}^{A} (\underline{R} - \lambda n + (\lambda - \varphi) \operatorname{tr}_{\omega_{\varphi}}\omega + \operatorname{tr}_{\omega_{\varphi}}\eta + n\varphi + Q\delta' \Delta_{\omega_{\varphi}}\psi)$$

$$+ \mathbf{e}^{A} g_{\varphi}^{i\bar{i}} |\varphi_{i}|^{2} + \mathbf{e}^{A} g_{\varphi}^{i\bar{i}} |A_{i}|^{2},$$

and

$$(3.6) \quad \Delta_{\omega_{\varphi}}(|\partial\varphi|^{2}_{\omega_{0}})$$

$$= g_{\varphi}^{i\bar{i}} R_{\alpha\beta i\bar{i}}(g_{0}) \varphi_{\alpha} \varphi_{\bar{\beta}} + g_{\varphi}^{i\bar{i}} |\varphi_{\alpha i}|^{2} + g_{\varphi}^{i\bar{i}} |\varphi_{\alpha \bar{i}}|^{2} + g_{\varphi}^{i\bar{i}} (\varphi_{\alpha i\bar{i}} \varphi_{\bar{\alpha}} + \varphi_{\bar{\alpha} i\bar{i}} \varphi_{\alpha}).$$

Differentiating

(3.7) 
$$\log \frac{\omega_{\varphi}^{n}}{\omega_{0}^{n}} = F,$$

we get

$$(3.8) g_{\varphi}^{i\bar{i}}(g_{i\bar{i},\alpha} + \varphi_{i\bar{i}\alpha}) = F_{\alpha} \text{ and } g_{\varphi}^{i\bar{i}}(g_{i\bar{i},\bar{\alpha}} + \varphi_{i\bar{i}\bar{\alpha}}) = F_{\bar{\alpha}}.$$
So

$$(3.9) \quad \Delta_{\omega_{\varphi}}(|\partial\varphi|^{2}_{\omega_{0}})$$

$$= g_{\varphi}^{i\bar{i}}R_{\alpha\bar{\beta}i\bar{i}}(g_{0})\varphi_{\alpha}\varphi_{\bar{\beta}} + g_{\varphi}^{i\bar{i}}|\varphi_{\alpha i}|^{2} + g_{\varphi}^{i\bar{i}}|\varphi_{\alpha\bar{i}}|^{2}$$

$$+ F_{\alpha}\varphi_{\bar{\alpha}} + F_{\bar{\alpha}}\varphi_{\alpha} - 2g_{\varphi}^{i\bar{i}}\operatorname{Re}(g_{i\bar{i},\alpha}\varphi_{\bar{\alpha}})$$

$$\geqslant -C_{1}|\partial\varphi|^{2}_{\omega_{0}}\operatorname{tr}_{\omega_{\varphi}}\omega_{0} + g_{\varphi}^{i\bar{i}}|\varphi_{\alpha i}|^{2} + g_{\varphi}^{i\bar{i}}|\varphi_{\alpha\bar{i}}|^{2} - 2\operatorname{Re}(A_{\alpha}\varphi_{\bar{\alpha}})$$

$$-2(\lambda - \varphi)|\partial\varphi|^{2}_{\omega_{0}} + 2Q\delta'\operatorname{Re}(\psi_{\alpha}\varphi_{\bar{\alpha}}) - 2g_{\varphi}^{i\bar{i}}\operatorname{Re}(g_{i\bar{i},\alpha}\varphi_{\bar{\alpha}}).$$

where  $C_1$  depends on a lower bound on the bisectional curvature of  $\omega_0$ . Then we have

$$(3.10) \quad e^{-A} \Delta_{\omega_{\varphi}}(e^{A}(|\partial\varphi|_{\omega_{0}}^{2} + K))$$

$$\geqslant (\underline{R} - \lambda n + (\lambda - \varphi)\operatorname{tr}_{\omega_{\varphi}}\omega + Q\delta'\Delta_{\omega_{\varphi}}\psi + \operatorname{tr}_{\omega_{\varphi}}\eta + n\varphi)(|\partial\varphi|_{\omega_{0}}^{2} + K)$$

$$+ g_{\varphi}^{i\bar{i}}|A_{i}|^{2}(|\partial\varphi|_{\omega_{0}}^{2} + K) + |\partial\varphi|_{\omega_{\varphi}}^{2}(|\partial\varphi|_{\omega_{0}}^{2} + K)$$

$$- C_{1}|\partial\varphi|_{\omega_{0}}^{2}\operatorname{tr}_{\omega_{\varphi}}\omega_{0} + g_{\varphi}^{i\bar{i}}|\varphi_{\alpha i}|^{2} + g_{\varphi}^{i\bar{i}}|\varphi_{\alpha \bar{i}}|^{2} + (-2\lambda + 2\varphi)|\partial\varphi|_{\omega_{0}}^{2}$$

$$- 2\operatorname{Re}(A_{\alpha}\varphi_{\bar{\alpha}}) + 2g_{\varphi}^{i\bar{i}}\operatorname{Re}(A_{i}(\varphi_{\alpha \bar{i}}\varphi_{\bar{\alpha}} + \varphi_{\bar{\alpha}\bar{i}}\varphi_{\alpha}))$$

$$+ 2Q\delta'\operatorname{Re}(\psi_{\alpha}\varphi_{\bar{\alpha}}) - 2g_{\varphi}^{i\bar{i}}\operatorname{Re}(g_{i\bar{i}},\varphi_{\bar{\alpha}}).$$

Notice the following complete square in the above sum

$$(3.11) \quad g_{\varphi}^{i\bar{i}}|\varphi_{i\alpha} + A_i\varphi_{\alpha}|^2 = g_{\varphi}^{i\bar{i}}|\varphi_{i\alpha}|^2 + 2g_{\varphi}^{i\bar{i}}\operatorname{Re}(A_i\varphi_{\alpha}\varphi_{\bar{\alpha}\bar{i}}) + g_{\varphi}^{i\bar{i}}|A_i|^2|\partial\varphi|_{\omega_0}^2.$$

Also observe that

$$(3.12) -A_{\alpha}\varphi_{\overline{\alpha}} + g_{\varphi}^{i\overline{i}}A_{i}\varphi_{\alpha\overline{i}}\varphi_{\overline{\alpha}} = g_{\varphi}^{i\overline{i}}(A_{i}\varphi_{\alpha\overline{i}}\varphi_{\overline{\alpha}} - (g_{\varphi})_{\alpha\overline{i}}A_{i}\varphi_{\overline{\alpha}}) \\ = -g_{\varphi}^{i\overline{i}}g_{\alpha\overline{i}}A_{i}\varphi_{\overline{\alpha}}.$$

In summary

$$(3.13) \quad \Delta_{\varphi}(\mathbf{e}^{A}(|\partial\varphi|^{2}_{\omega_{0}}+K)) \, \mathbf{e}^{-A}$$

$$\geqslant Kg_{\varphi}^{i\bar{i}}|A_{i}|^{2} + g_{\varphi}^{i\bar{i}}|\varphi_{i}|^{2}(|\partial\varphi|^{2}_{\omega_{0}}+K)$$

$$+ (\operatorname{tr}_{\omega_{\varphi}}(\lambda-\varphi)\omega + Q\delta'\Delta_{\omega_{\varphi}}\psi + \operatorname{tr}_{\omega_{\varphi}}\eta)(|\partial\varphi|^{2}_{\omega_{0}}+K)$$

$$+ (\underline{R}-\lambda n + n\varphi)(|\partial\varphi|^{2}_{\omega_{0}}+K) - C_{1}|\partial\varphi|^{2}_{\omega_{0}}\operatorname{tr}_{\omega_{\varphi}}\omega_{0} + g_{\varphi}^{i\bar{i}}|\varphi_{\alpha\bar{i}}|^{2}$$

$$+ (-2\lambda + 2\varphi)|\partial\varphi|^{2}_{\omega_{0}} - 2g_{\varphi}^{i\bar{i}}\operatorname{Re}(g_{\alpha\bar{i}}A_{i}\varphi_{\bar{\alpha}})$$

$$+ 2Q\delta'\operatorname{Re}(\psi_{\alpha}\varphi_{\bar{\alpha}}) - 2\operatorname{Re}(g_{\varphi}^{i\bar{i}}q_{i\bar{i}},\varphi_{\bar{\alpha}}).$$

Furthermore, by Young's inequality

$$(3.14) 2\operatorname{Re}(g_{\varphi}^{i\bar{i}}g_{i\bar{i},\alpha}\varphi_{\bar{\alpha}}) \leqslant C_2(\operatorname{tr}_{\omega_{\varphi}}\omega_0 + |\partial\varphi|^2_{\omega_0}\operatorname{tr}_{\omega_{\varphi}}\omega_0),$$

and

$$(3.15) 2\operatorname{Re}(g_{\varphi}^{i\bar{i}}g_{\alpha\bar{i}}A_{i}\varphi_{\bar{\alpha}}) \leqslant C_{3}(g_{\varphi}^{i\bar{i}}|A_{i}|^{2} + |\partial\varphi|_{\omega_{0}}^{2}\operatorname{tr}_{\omega_{\varphi}}\omega_{0}).$$

Let  $\lambda = (\frac{1}{\delta'} + \frac{1}{C_{\delta'}})(\|\varphi\|_{C^0(M)} + 10 + C_1 + C_2 + C_3), \ Q = \lambda - \|\varphi\|_{C^0(M)} > \max\{\frac{10 + C_1 + C_2 + C_3}{C_{\delta'}}, \frac{2}{\delta'}\}, \ K = C_2 + C_3, \text{ so that } C_{\delta'}Q \geqslant 10 + C_1 + C_2 + C_3, \ Q\delta' > 1 \text{ and recall that } \psi \text{ is chosen such that}$ 

(3.16) 
$$\omega + \delta' \sqrt{-1} \partial \bar{\partial} \psi \geqslant \eta + \delta' \sqrt{-1} \partial \bar{\partial} \psi > C_{\delta'} \omega_0.$$

So

$$(3.17) \quad (\operatorname{tr}_{\omega_{\varphi}}(\lambda - \varphi)\omega + Q\delta'\Delta_{\omega_{\varphi}}\psi + \operatorname{tr}_{\omega_{\varphi}}\eta)(|\partial\varphi|^{2}_{\omega_{0}} + K) \\ - C_{1}|\partial\varphi|^{2}_{\omega_{0}}\operatorname{tr}_{\omega_{\varphi}}\omega_{0} - 2\operatorname{Re}(g^{i\bar{i}}_{\varphi}g_{i\bar{i},\alpha}\varphi_{\bar{\alpha}}) + Kg^{i\bar{i}}_{\varphi}|A_{i}|^{2} + 2\operatorname{Re}(g^{i\bar{i}}_{\varphi}g_{\alpha\bar{i}}A_{i}\varphi_{\bar{\alpha}}) \\ \geqslant Q(\operatorname{tr}_{\omega_{\varphi}}\omega + \delta'\Delta_{\omega_{\varphi}}\psi)(|\partial\varphi|^{2}_{\omega_{0}} + K) - (C_{1} + C_{2} + C_{3})|\partial\varphi|^{2}_{\omega_{0}}\operatorname{tr}_{\omega_{\varphi}}\omega_{0} \\ - C_{2}\operatorname{tr}_{\omega_{\varphi}}\omega_{0} + (K - C_{3})g^{i\bar{i}}_{\varphi}|A_{i}|^{2} \\ \geqslant QC_{\delta'}(|\partial\varphi|^{2}_{\omega_{0}} + K)\operatorname{tr}_{\omega_{\varphi}}\omega_{0} - (C_{1} + C_{2} + C_{3})|\partial\varphi|^{2}_{\omega_{0}}\operatorname{tr}_{\omega_{\varphi}}\omega_{0} - C_{2}\operatorname{tr}_{\omega_{\varphi}}\omega_{0} \\ \geqslant 10|\partial\varphi|^{2}_{\omega_{0}}\operatorname{tr}_{\omega_{\varphi}}\omega_{0}.$$

Recall that  $R \to -n$ , so

$$(3.18) (\underline{R} - \lambda n + n\varphi)(|\partial\varphi|^2_{\omega_0} + K) \geqslant -C|\partial\varphi|^2_{\omega_0} - C,$$

and

$$(3.19) \qquad (-2\lambda + 2\varphi)|\partial\varphi|_{\omega_0}^2 \geqslant -C|\partial\varphi|_{\omega_0}^2$$

where the constants depend on the 0-th order estimate on  $\varphi$ . Also as an elementary consequence of the equation (see [9, p. 12])

$$\frac{\omega_{\varphi}^n}{\omega_0^n} = e^F$$

yields

(3.21) 
$$g_{\omega}^{i\bar{i}}|\varphi_{i}|^{2}|\partial\varphi|_{\omega_{0}}^{2}+|\partial\varphi|_{\omega_{0}}^{2}\operatorname{tr}_{\omega_{\omega}}\omega_{0}\geqslant C|\partial\varphi|_{\omega_{0}}^{2+\frac{2}{n}}e^{\frac{-F}{n}}.$$

Recall that  $\psi = \log |s|_h^2$ , where s is a holomorphic section vanishing on E. Then we get

$$(3.22) g_{\omega}^{i\bar{i}} |\varphi_i|^2 |\partial \varphi|_{\omega_0}^2 + |\partial \varphi|_{\omega_0}^2 \operatorname{tr}_{\omega_{\omega}} \omega_0 \geqslant C |\partial \varphi|_{\omega_0}^{2 + \frac{2}{n}} e^{\frac{-F + Q\delta'\psi}{n}}$$

We arrive at

$$(3.23) \quad \Delta_{\omega_{\varphi}}(e^{A}(|\partial\varphi|_{\omega_{0}}^{2}+K))$$

$$\geqslant e^{A}(g_{\varphi}^{i\bar{i}}|\varphi_{i}|^{2}|\partial\varphi|_{\omega_{0}}^{2}+|\partial\varphi|_{\omega_{0}}^{2}\operatorname{tr}_{\omega_{\varphi}}\omega_{0}-C|\partial\varphi|_{\omega_{0}}^{2}-C+2Q\delta'\operatorname{Re}(\psi_{\alpha}\varphi_{\bar{\alpha}}))$$

$$\geqslant e^{A}(Ce^{\frac{-F+Q\delta'\psi}{n}}(|\partial\varphi|_{\omega_{0}}^{2})^{1+\frac{1}{n}}-C|\partial\varphi|_{\omega_{0}}^{2}-C+2Q\delta'\operatorname{Re}(\psi_{\alpha}\varphi_{\bar{\alpha}}))$$

$$\geqslant C(e^{Q\delta'\psi}|\partial\varphi|_{\omega_{0}}^{2})^{1+\frac{1}{n}}-Ce^{Q\delta'\psi}|\partial\varphi|_{\omega_{0}}^{2}-Ce^{Q\delta'\psi}+2e^{A}Q\delta'\operatorname{Re}(\psi_{\alpha}\varphi_{\bar{\alpha}})$$

where we used the fact that  $||F||_{C^0(M)}$  and  $||\varphi||_{C^0(M)}$  are bounded. For the last term we have

(3.24) 
$$\psi_{\alpha} = \frac{\partial_{\alpha}(|s|_{h}^{2})}{|s|_{h}^{2}},$$

So

(3.25) 
$$2 e^{A} Q \delta' \operatorname{Re}(\psi_{\alpha} \varphi_{\alpha}) \geqslant -C e^{2\psi} |\partial \psi|_{\omega_{0}}^{2} - C e^{(2Q\delta'-2)\psi} |\partial \varphi|_{\omega_{0}}^{2}$$
$$\geqslant -C - C e^{Q\delta'\psi} |\partial \varphi|_{\omega_{0}}^{2}.$$

where we used the fact that Q is chosen such that  $2Q\delta'-2>Q\delta'$  so  $e^{(2Q\delta'-2)\psi} \leq C e^{Q\delta'\psi}$ . Finally we reach

$$(3.26) \quad \Delta_{\omega_{\varphi}}(e^{A}(|\partial\varphi|^{2}_{\omega_{0}}+K))$$

$$\geqslant C(e^{Q\delta'\psi}|\partial\varphi|^{2}_{\omega_{0}})^{1+\frac{1}{n}}-Ce^{Q\delta'\psi}|\partial\varphi|^{2}_{\omega_{0}}-Ce^{Q\delta'\psi}-C.$$

At a maximum p of  $e^A(|\partial \varphi|^2_{\omega_0} + K)$ , which is away from E because  $A = -\infty$  on E, we have

$$(3.27) \quad C(e^{Q\delta'\psi(p)} |\partial\varphi|^2_{\omega_0}(p))^{1+\frac{1}{n}} - C e^{Q\delta'\psi(p)} |\partial\varphi|^2_{\omega_0}(p) \leqslant C e^{Q\delta'\psi(p)} + C \leqslant C$$

where for the last inequality we used the fact that  $e^{P\psi}$  is bounded from above for any constant P>0, and our choice that  $Q\delta'>1$ . Thus, we conclude that  $e^{Q\delta'\psi} |\partial\varphi|^2_{\omega_0}(p)$  is bounded from above. As a consequence,  $e^A(|\partial\varphi|^2_{\omega_0}+K)(p)$  is also bounded from above. This concludes our proof.

## 4. Degenerate $L^p$ bound on $\operatorname{tr}_{\omega_0}\omega_{\varphi_{\varepsilon}}$ and $C^{\infty}_{\operatorname{loc}}(M\backslash E)$ bound on $\varphi_{\varepsilon}$

In this section we will first establish a degenerate version of Theorem 3.1 in [9].

THEOREM 4.1. — For each p > 0, there exists a constant  $\gamma(p) > 0$  such that

$$(4.1) \qquad \int_{M} e^{\gamma(p)\psi} (\operatorname{tr}_{\omega_{0}} \omega_{\varphi_{\varepsilon}})^{p} \omega_{0}^{n} \leqslant C(p, (M, \omega_{0}), \|\varphi_{\varepsilon}\|_{C^{0}(M)}, \|F_{\varepsilon}\|_{C^{0}(M)}).$$

For simplicity, let  $B_{\varepsilon} = F_{\varepsilon} + \lambda \varphi_{\varepsilon} - \lambda \delta' \psi$ , where  $\delta' < \min\{\frac{1}{2}, \sigma\}$  and  $\sigma$  is specified in Lemma 3.1. Moreover, let  $\beta$  be a constant greater than 1. We start by computing at a point away from E in a normal holomorphic coordinate neighborhood for  $g_0$  such that  $g_{\varphi}$  is diagonal.

$$(4.2) \quad \Delta_{\omega_{\varphi}}(e^{-\beta B}(\operatorname{tr}_{\omega_{0}}\omega_{\varphi}))$$

$$= \Delta_{\omega_{\varphi}}(e^{-\beta B})\operatorname{tr}_{\omega_{0}}\omega_{\varphi} + e^{-\beta B}\Delta_{\omega_{\varphi}}(\operatorname{tr}_{\omega_{0}}\omega_{\varphi}) - 2\beta\operatorname{Re}(e^{-\beta B}g_{\varphi}^{i\bar{i}}(B_{i}\partial_{\bar{i}}\operatorname{tr}_{\omega_{0}}\omega_{\varphi})),$$

and

$$(4.3) \quad \Delta_{\omega_{\varphi}} e^{-\beta B}$$

$$= (\beta^{2} g_{\varphi}^{i\bar{i}} |B_{i}|^{2} + \beta(\underline{R} - \lambda n + \lambda \operatorname{tr}_{\omega_{\varphi}} (\omega + \delta' \sqrt{-1} \partial \overline{\partial} \psi) + \operatorname{tr}_{\omega_{\varphi}} \eta)) e^{-\beta B}$$

$$\geq \beta^{2} g_{\varphi}^{i\bar{i}} |B_{i}|^{2} e^{-\beta B} + \beta e^{-\beta B} (\underline{R} - \lambda n) + C_{\delta'} \lambda \beta e^{-\beta B} \operatorname{tr}_{\omega_{\varphi}} \omega_{0}.$$

Now we estimate  $\Delta_{\omega_{\varphi}} \operatorname{tr}_{\omega_0} \omega_{\varphi}$  by first recalling that from the calculation in Yau [53] (see also [44, Lemma 3.7] for an exposition) we have

(4.4) 
$$\Delta_{\omega_{\varphi}} \log \operatorname{tr}_{\omega_{0}} \omega_{\varphi} \geqslant -C_{1} \operatorname{tr}_{\omega_{\varphi}} \omega_{0} - \frac{g_{0}^{i\bar{j}} R_{i\bar{j}}(g_{\varphi})}{\operatorname{tr}_{\omega_{0}} \omega_{\varphi}}$$

where  $C_1$  only depends on a lower bound for the bisectional curvature of  $g_0$ . Recall that

(4.5) 
$$\log \frac{\omega_{\varphi}^n}{\omega_0^n} = F.$$

Then we have

$$-\operatorname{tr}_{\omega_0}\operatorname{Ric}(\omega_{\varphi}) = \operatorname{tr}_{\omega_0}\eta + \Delta_{\omega_0}F.$$

Thus

(4.7) 
$$\Delta_{\omega_{\varphi}} \operatorname{logtr}_{\omega_{0}} \omega_{\varphi} \geqslant -C_{1} \operatorname{tr}_{\omega_{\varphi}} \omega_{0} + \frac{\Delta_{\omega_{0}} F}{\operatorname{tr}_{\omega_{0}} \omega_{\varphi}} + \frac{\operatorname{tr}_{\omega_{0}} \eta}{\operatorname{tr}_{\omega_{0}} \omega_{\varphi}},$$

and

$$(4.8) \quad \Delta_{\omega_{\varphi}} \operatorname{tr}_{\omega_{0}} \omega_{\varphi} \geqslant -C_{1} \operatorname{tr}_{\omega_{\varphi}} \omega_{0} \operatorname{tr}_{\omega_{0}} \omega_{\varphi} + \frac{|\partial \operatorname{tr}_{\omega_{0}} \omega_{\varphi}|_{\omega_{\varphi}}^{2}}{\operatorname{tr}_{\omega_{0}} \omega_{\varphi}} + \Delta_{\omega_{0}} F + \operatorname{tr}_{\omega_{0}} \eta.$$

Then we are able to conclude that

$$(4.9) \quad e^{\beta B} \, \Delta_{\omega_{\varphi}} (e^{-\beta B} (\operatorname{tr}_{\omega_{0}} \omega_{\varphi}))$$

$$\geqslant (\beta^{2} g_{\varphi}^{i\bar{i}} |B_{i}|^{2} + \beta (\underline{R} - \lambda n)) \operatorname{tr}_{\omega_{0}} \omega_{\varphi} + (C_{\delta'} \lambda \beta - C_{1}) \operatorname{tr}_{\omega_{\varphi}} \omega_{0} \operatorname{tr}_{\omega_{0}} \omega_{\varphi}$$

$$+ \frac{|\partial \operatorname{tr}_{\omega_{0}} \omega_{\varphi}|_{\omega_{\varphi}}^{2}}{\operatorname{tr}_{\omega_{0}} \omega_{\varphi}} + \Delta_{\omega_{0}} F + \operatorname{tr}_{\omega_{0}} \eta - 2\beta \operatorname{Re}(g_{\varphi}^{i\bar{i}} (B_{i} \partial_{\bar{i}} \operatorname{tr}_{\omega_{0}} \omega_{\varphi}))$$

$$\geqslant \beta (\underline{R} - \lambda n) \operatorname{tr}_{\omega_{0}} \omega_{\varphi} + (C_{\delta'} \lambda \beta - C_{1}) \operatorname{tr}_{\omega_{\varphi}} \omega_{0} \operatorname{tr}_{\omega_{0}} \omega_{\varphi} + \Delta_{\omega_{0}} F + \operatorname{tr}_{\omega_{0}} \eta$$

where we dropped the terms

$$(4.10) \quad \beta^{2} g_{\varphi}^{i\bar{i}} |B_{i}|^{2} \operatorname{tr}_{\omega_{0}} \omega_{\varphi} - 2\beta \operatorname{Re}(g_{\varphi}^{i\bar{i}} (B_{i} \partial_{\bar{i}} \operatorname{tr}_{\omega_{0}} \omega_{\varphi})) + \frac{|\partial \operatorname{tr}_{\omega_{0}} \omega_{\varphi}|_{\omega_{\varphi}}^{2}}{\operatorname{tr}_{\omega_{0}} \omega_{\varphi}}$$

$$= \operatorname{tr}_{\omega_{0}} \omega_{\varphi} g_{\varphi}^{i\bar{i}} (\beta^{2} |B_{i}|^{2} - 2\beta \operatorname{Re}\left(\frac{B_{i} \partial_{\bar{i}} \operatorname{tr}_{\omega_{0}} \omega_{\varphi}}{\operatorname{tr}_{\omega_{0}} \omega_{\varphi}}\right) + \frac{|\partial_{\bar{i}} \operatorname{tr}_{\omega_{0}} \omega_{\varphi}|^{2}}{(\operatorname{tr}_{\omega_{0}} \omega_{\varphi})^{2}}) \geqslant 0.$$

Use

$$(4.11) tr_{\omega_{\varphi}}\omega_{0}tr_{\omega_{0}}\omega_{\varphi} \geqslant e^{-\frac{F}{n-1}}(tr_{\omega_{0}}\omega_{\varphi})^{1+\frac{1}{n-1}},$$

and choose  $\lambda = \max(\frac{2C_1+2}{C_{\delta'}}, \frac{q}{2\delta'}+1, \frac{1}{2\delta'}+1)$ , where q is specified in Theorem 3.2, so that  $C_{\delta'}\lambda\beta \geqslant (2C_1+2)\beta > 2C_1+1$ . We get

$$(4.12) \quad \Delta_{\omega_{\varphi}}(e^{-\beta B}(\operatorname{tr}_{\omega_{0}}\omega_{\varphi}))$$

$$\geqslant \beta(\underline{R} - \lambda n)\operatorname{tr}_{\omega_{0}}\omega_{\varphi} e^{-\beta B} + (C_{\delta'}\lambda\beta - C_{1}) e^{-\frac{F}{n-1} - \beta B}(\operatorname{tr}_{\omega_{0}}\omega_{\varphi})^{1 + \frac{1}{n-1}} + (\Delta_{\omega_{0}}F + \operatorname{tr}_{\omega_{0}}\eta) e^{-\beta B}$$

$$\geqslant \beta(\underline{R} - \lambda n)\operatorname{tr}_{\omega_{0}}\omega_{\varphi} e^{-\beta B} + (C_{\delta'}\lambda\beta - C_{1}) e^{-\frac{F}{n-1} - \beta B}(\operatorname{tr}_{\omega_{0}}\omega_{\varphi})^{1 + \frac{1}{n-1}} + \Delta_{\omega_{0}}F e^{-\beta B}$$

where for the last line we took advantage of the fact that  $\eta \geqslant 0$ . Let  $u = e^{-\beta B} \operatorname{tr}_{\omega_0} \omega_{\varphi}$ , for any  $p \geqslant 0$ , we have

$$(4.13) \qquad \frac{1}{2p+1} \Delta_{\omega_{\varphi}} u^{2p+1} = 2pu^{2p-2} e^{-\beta B} (\operatorname{tr}_{\omega_{0}} \omega_{\varphi}) |\partial u|_{\omega_{\varphi}}^{2} + u^{2p} \Delta_{\omega_{\varphi}} u$$
$$\geqslant 2pu^{2p-2} |\partial u|_{\omega_{0}}^{2} e^{-\beta B} + u^{2p} \Delta_{\omega_{\varphi}} u.$$

Integrating both sides of (4.13) with respect to  $\omega_{\varphi}^{n} = e^{F} \omega_{0}^{n}$  and use (4.12) we get

$$(4.14) \int_{M} 2pu^{2p-2} |\partial u|_{\omega_{0}}^{2} e^{-\beta B+F} \omega_{0}^{n} + \int_{M} \Delta_{\omega_{0}} F e^{-\beta B+F} u^{2p} \omega_{0}^{n}$$

$$+ \int_{M} u^{2p} e^{-\beta B+\frac{n-2}{n-1}F} (C_{\delta'} \lambda \beta - C_{1}) (\operatorname{tr}_{\omega_{0}} \omega_{\varphi})^{1+\frac{1}{n-1}} \omega_{0}^{n}$$

$$\leq \int_{M} \beta (\lambda n - \underline{R}) \operatorname{tr}_{\omega_{0}} \omega_{\varphi} e^{-\beta B+F} u^{2p} \omega_{0}^{n}.$$

Let us denote  $\tilde{\varphi} = \varphi - \delta' \psi$ , and to handle the term involving  $\Delta_{\omega_0} F$ , we apply integration by parts

$$(4.15) \int_{M} e^{-\beta B + F} u^{2p} \Delta_{\omega_{0}} F \omega_{0}^{n}$$

$$= \int_{M} (\beta - 1) e^{(1-\beta)F - \beta \lambda \tilde{\varphi}} u^{2p} |\partial F|_{\omega_{0}}^{2} \omega_{0}^{n}$$

$$+ \int_{M} \beta \lambda e^{(1-\beta)F - \lambda \beta \tilde{\varphi}} u^{2p} \langle \partial \tilde{\varphi}, \partial F \rangle_{\omega_{0}} \omega_{0}^{n}$$

$$- \int_{M} 2p e^{(1-\beta)F - \beta \lambda \tilde{\varphi}} u^{2p-1} \langle \partial u, \partial F \rangle_{\omega_{0}} \omega_{0}^{n},$$

where  $\langle \partial \tilde{\varphi}, \partial F \rangle_{\omega_0} = g_0^{i\bar{j}} \tilde{\varphi}_i F_{\bar{j}}$ , and  $\langle \partial u, \partial F \rangle_{\omega_0} = g_0^{i\bar{j}} u_i F_{\bar{j}}$  in coordinates. Notice that by Young's inequality

$$(4.16) |u^{2p-1}\langle \partial u, \partial F \rangle_{\omega_0}| \leq \frac{1}{2}u^{2p}|\partial F|_{\omega_0}^2 + \frac{1}{2}u^{2p-2}|\partial u|_{\omega_0}^2,$$

and

$$(4.17) \quad |\beta\lambda e^{(1-\beta)F - \lambda\beta\tilde{\varphi}} u^{2p} \langle \partial\tilde{\varphi}, \partial F \rangle_{\omega_0}|$$

$$\leq |\beta\lambda e^{(1-\beta)F - \lambda\beta\tilde{\varphi}} u^{2p} \langle \partial\varphi, \partial F \rangle_{\omega_0}| + |\delta'\beta\lambda e^{(1-\beta)F - \lambda\beta\tilde{\varphi}} u^{2p} \langle \partial\psi, \partial F \rangle_{\omega_0}|$$

$$\leq \frac{(\beta-1)}{2} u^{2p} |\partial F|_{\omega_0}^2 e^{(1-\beta)F - \lambda\beta\tilde{\varphi}} + |\partial\varphi|_{\omega_0}^2 \frac{\beta^2\lambda^2}{(\beta-1)} u^{2p} e^{(1-\beta)F - \lambda\beta\tilde{\varphi}}$$

$$+ \delta'^2 |\partial\psi|_{\omega_0}^2 \frac{\beta^2\lambda^2}{(\beta-1)} u^{2p} e^{(1-\beta)F - \lambda\beta\tilde{\varphi}}.$$

Then substitute this back into (4.15), we get

$$(4.18) \int_{M} e^{(1-\beta)F - \beta\lambda\tilde{\varphi}} u^{2p} \Delta_{\omega_{0}} F \omega_{0}^{n}$$

$$\geqslant -\int_{M} |\partial\psi|_{\omega_{0}}^{2} \frac{\delta'^{2}\beta^{2}\lambda^{2}}{(\beta - 1)} e^{(1-\beta)F - \lambda\beta\tilde{\varphi}} u^{2p} \omega_{0}^{n}$$

$$-\int_{M} p e^{(1-\beta)F - \lambda\beta\tilde{\varphi}} u^{2p-2} |\partial u|_{\omega_{0}}^{2} \omega_{0}^{n}$$

$$+\int_{M} \left(\frac{\beta - 1}{2} - p\right) e^{(1-\beta)F - \beta\lambda\tilde{\varphi}} u^{2p} |\partial F|_{\omega_{0}}^{2} \omega_{0}^{n}$$

$$-\int_{M} |\partial\varphi|_{\omega_{0}}^{2} \frac{\beta^{2}\lambda^{2}}{(\beta - 1)} e^{(1-\beta)F - \lambda\beta\tilde{\varphi}} u^{2p} \omega_{0}^{n}.$$

Substituting (4.18) back into (4.14), we conclude

$$(4.19) \int_{M} pu^{2p-2} |\partial u|_{\omega_{0}}^{2} e^{(1-\beta)F - \lambda\beta\tilde{\varphi}} \omega_{0}^{n}$$

$$+ \int_{M} \left(\frac{\beta - 1}{2} - p\right) e^{(1-\beta)F - \beta\lambda\tilde{\varphi}} u^{2p} |\partial F|_{\omega_{0}}^{2} \omega_{0}^{n}$$

$$+ \int_{M} u^{2p} e^{-(\beta - \frac{n-2}{n-1})F - \beta\lambda\tilde{\varphi}} (C_{\delta'}\beta\lambda - C_{1}) (\operatorname{tr}_{\omega_{0}}\omega_{\varphi})^{1 + \frac{1}{n-1}} \omega_{0}^{n}$$

$$\leq \int_{M} |\partial \varphi|_{\omega_{0}}^{2} \frac{\beta^{2}\lambda^{2}}{(\beta - 1)} e^{(1-\beta)F - \lambda\beta\tilde{\varphi}} u^{2p} \omega_{0}^{n}$$

$$+ \int_{M} |\partial \psi|_{\omega_{0}}^{2} \frac{\delta'^{2}\beta^{2}\lambda^{2}}{(\beta - 1)} e^{(1-\beta)F - \lambda\beta\tilde{\varphi}} u^{2p} \omega_{0}^{n}$$

$$+ \int_{M} \beta(\lambda n - \underline{R}) \operatorname{tr}_{\omega_{0}} \omega_{\varphi} e^{(1-\beta)F - \beta\lambda\tilde{\varphi}} u^{2p} \omega_{0}^{n}.$$

Now choose  $\beta > 2p+1$ , and recall that  $\lambda = \max(\frac{2C_1+2}{C_{\delta'}}, \frac{q}{2\delta'}+1, \frac{1}{2\delta'}+1)$ , where q is specified in Theorem 3.2, so that  $C_{\delta'}\lambda\beta \geqslant (2C_1+2)\beta > 2C_1+1$ . Because  $\operatorname{tr}_{\omega_0}\omega_{\varphi} \geqslant n\operatorname{e}^{\frac{F}{n}}$  by the arithmetic-geometric means inequality, we have

$$(4.20) \int_{M} u^{2p} e^{-(\beta - \frac{n-2}{n-1})F - \beta\lambda\tilde{\varphi}} (\operatorname{tr}_{\omega_{0}}\omega_{\varphi})^{1 + \frac{1}{n-1}} \omega_{0}^{n}$$

$$\leqslant C \int_{M} \operatorname{tr}_{\omega_{0}}\omega_{\varphi} e^{(1-\beta)F - \beta\lambda\tilde{\varphi}} u^{2p} \omega_{0}^{n} + C \int_{M} u^{2p} e^{(1-\beta)F - \beta\lambda\tilde{\varphi}} \omega_{0}^{n}$$

$$+ C \int_{M} |\partial\varphi|_{\omega_{0}}^{2} \frac{\beta^{2}}{\beta - 1} e^{(1-\beta)F - \lambda\beta\tilde{\varphi}} u^{2p} \omega_{0}^{n}$$

$$+ C \int_{M} |\partial\psi|_{\omega_{0}}^{2} \frac{\delta'^{2}\beta^{2}}{\beta - 1} e^{(1-\beta)F - \lambda\beta\tilde{\varphi}} u^{2p} \omega_{0}^{n}.$$

For p = 0,  $\beta = 2$  we have

$$(4.21) \int_{M} e^{2\lambda\delta'\psi} (\operatorname{tr}_{\omega_{0}}\omega_{\varphi})^{1+\frac{1}{n-1}}\omega_{0}^{n}$$

$$\leq C \int_{M} e^{-(2-\frac{n-2}{n-1})F-2\lambda\tilde{\varphi}} (\operatorname{tr}_{\omega_{0}}\omega_{\varphi})^{1+\frac{1}{n-1}}\omega_{0}^{n}$$

$$\leq C \int_{M} \operatorname{tr}_{\omega_{0}}\omega_{\varphi} e^{2\lambda\delta'\psi} \omega_{0}^{n} + C \int_{M} e^{2\lambda\delta'\psi} \omega_{0}^{n}$$

$$+ C \int_{M} |\partial\varphi|_{\omega_{0}}^{2} e^{-2\lambda\varphi+2\lambda\delta'\psi} \omega_{0}^{n} + C \int_{M} |\partial\psi|_{\omega_{0}}^{2} e^{-2\lambda\varphi+2\lambda\delta'\psi} \omega_{0}^{n}$$

$$\leq C \int_{M} \operatorname{tr}_{\omega_{0}}\omega_{\varphi}\omega_{0}^{n} + C + C \int_{M} |\partial\varphi|_{\omega_{0}}^{2} e^{2\lambda\delta'\psi} \omega_{0}^{n}$$

$$+ C \int_{M} |\partial\psi|_{\omega_{0}}^{2} e^{2\lambda\delta'\psi} \omega_{0}^{n},$$

where we used the fact that  $\varphi$  and F are bounded. Also notice that

$$(4.22) \qquad \int_{M} \operatorname{tr}_{\omega_{0}} \omega_{\varphi} \omega_{0}^{n} = \int_{M} \operatorname{tr}_{\omega_{0}} \omega \omega_{0}^{n} + \int_{M} \Delta_{\omega_{0}} \varphi \omega_{0}^{n} \leqslant C.$$

For the last two terms

$$(4.23) \qquad \int_{M} |\partial \varphi|_{\omega_0}^2 e^{2\lambda \delta' \psi} \,\omega_0^n \leqslant C \int_{M} |s|_h^{4\lambda \delta' - 2q} \omega_0^n \leqslant C,$$

and

$$(4.24) \qquad \int_{M} |\partial \psi|_{\omega_{0}}^{2} e^{2\lambda \delta' \psi} \omega_{0}^{n} \leqslant C \int_{M} |s|_{h}^{4\lambda \delta' - 2} \omega_{0}^{n} \leqslant C.$$

Recall that we choose  $\lambda > \max\{\frac{q}{2\delta'}, \frac{1}{2\delta'}\}$ , so we can bound the last two terms since  $|s|_h^2$  is bounded. As a result, we obtained a bound for

$$\int_{M} e^{2\lambda \delta' \psi} (\operatorname{tr}_{\omega_0} \omega_{\varphi})^{\frac{n}{n-1}} \omega_0^n.$$

Now it suffices to show that for each given p > 0,

$$\int_{M} (\operatorname{tr}_{\omega_0} \omega_{\varphi})^p e^{\lambda(2p+2)(\delta'(p-1)+1)\psi} \omega_0^n$$

(recall that  $\delta' < \frac{1}{2}$ , so that  $\delta'(p-1)+1 > \delta'$  when  $p = \frac{n}{n-1}$ ) being bounded above implies that for  $p + \frac{1}{n-1}$ ,

$$\int_{M} (\operatorname{tr}_{\omega_{0}} \omega_{\varphi})^{p + \frac{1}{n-1}} e^{\lambda(2p+2)(\delta'(p + \frac{1}{n-1} - 1) + 1)\psi} \omega_{0}^{n}$$

is bounded above, then we are done by induction. Given p>1, let  $\tilde{p}=\frac{p-1}{2}$ ,  $\beta=2p+2>2\tilde{p}+1$  and take  $\tilde{p},\beta$  into our inequality (4.20), we get

$$(4.25) \int_{M} e^{\lambda(2p+2)(\delta'(p+\frac{1}{n+1}-1)+1)\psi} (\operatorname{tr}_{\omega_{0}}\omega_{\varphi})^{p+\frac{1}{n-1}}\omega_{0}^{n}$$

$$\leq C \int_{M} e^{-(2p+2-\frac{n-2}{n-1})F-(2p+2)\lambda\tilde{\varphi}-2\tilde{p}(2p+2)B} (\operatorname{tr}_{\omega_{0}}\omega_{\varphi})^{p+\frac{1}{n-1}}\omega_{0}^{n}$$

$$\leq C \int_{M} e^{\lambda(2p+2)(\delta'(p-1)+1)\psi} (\operatorname{tr}_{\omega_{0}}\omega_{\varphi})^{p}\omega_{0}^{n}$$

$$+ C \int_{M} e^{\lambda(2p+2)(\delta'(p-1)+1)\psi} (\operatorname{tr}_{\omega_{0}}\omega_{\varphi})^{p-1}\omega_{0}^{n}$$

$$+ C \int_{M} |\partial\varphi|_{\omega_{0}}^{2} e^{\lambda(2p+2)(\delta'(p-1)+1)\psi} (\operatorname{tr}_{\omega_{0}}\omega_{\varphi})^{p-1}\omega_{0}^{n}$$

$$+ C \int_{M} |\partial\psi|_{\omega_{0}}^{2} e^{\lambda(2p+2)(\delta'(p-1)+1)\psi} (\operatorname{tr}_{\omega_{0}}\omega_{\varphi})^{p-1}\omega_{0}^{n}.$$

The first two terms on the right hand side are bounded due to the inductive hypothesis and the 0-th order bound on F and  $\varphi$ . For the last two terms we can use the same idea as in the case where  $p=0, \beta=2$ , to bound them. Then we are done by induction. We now need the following proposition from [9]  $^{(1)}$  to show that  $\varphi_{\varepsilon}$  is locally uniformly smoothly bounded away from E.

PROPOSITION 4.2. — Let K be a compact subset of M, suppose that  $\operatorname{tr}_{\omega_0}\omega_{\varphi_{\varepsilon}}\in L^p(K)$ , and  $\operatorname{tr}_{\omega_{\varphi_{\varepsilon}}}\omega_0\in L^p(K)$  for some p>3n(n-1)(n is the dimension of M), then for any  $m\in\mathbb{N}, m\geqslant 2$ ,  $\varphi_{\varepsilon}\in C^m(K,\omega_0)$  with the

<sup>(1)</sup> The authors of [9] decided to omit this result in the published version of the article and it can be found in Corollary 6.2 of the arXiv version (https://arxiv.org/abs/1712.06697) of [9].

bound only depends on p,  $\|\operatorname{tr}_{\omega_{\varphi_{\varepsilon}}}\omega_0\|_{L^p(K')}$ ,  $\|\operatorname{tr}_{\omega_0}\omega_{\varphi_{\varepsilon}}\|_{L^p(K')}$ ,  $\|\varphi_{\varepsilon}\|_{C^0(K')}$ , m, and  $(M, \omega_0)$ , where K' is a larger compact set containing K.

COROLLARY 4.3. — Given any compact subset K of  $M \setminus E$ ,  $\varphi_{\varepsilon}$  is uniformly bounded in  $C^{\infty}(K)$ .

*Proof.* — Given a compact subset  $K \subset M \setminus E$ , the  $L^p(K)$  bound on  $\operatorname{tr}_{\omega_0} \omega_{\varphi_{\varepsilon}}$  is a direct consequence of Theorem 4.1. Furthermore, notice that

(4.26) 
$$\operatorname{tr}_{\omega_{\varphi_{\varepsilon}}}\omega_{0} = \sum_{i} \frac{1}{(g_{\varphi_{\varepsilon}})_{i\bar{i}}} = \sum_{i} \frac{1}{\prod_{m} (g_{\varphi_{\varepsilon}})_{m\overline{m}}} \prod_{j \neq i} (g_{\varphi_{\varepsilon}})_{j\bar{j}}$$
$$= e^{-F} \prod_{j \neq i} (g_{\varphi_{\varepsilon}})_{j\bar{j}} \leqslant e^{-F} (\operatorname{tr}_{\omega_{0}}\omega_{\varphi_{\varepsilon}})^{n-1}.$$

So we are able to obtain the  $L^p(K)$  bound on  $\operatorname{tr}_{\omega_{\varphi}}\omega_0$  by using the  $C^0(M)$  bound on F and Theorem 4.1 again. We conclude the proof by recalling that we previously showed that  $\|\varphi_{\varepsilon}\|_{C^0(M)}$  is uniformly bounded in Section 2 and  $|\partial \varphi_{\varepsilon}|_{\omega_0}$  is uniformly bounded on K in Section 3.

## 5. Uniform upper bound on the Mabuchi energy and the entropy

All the estimates we proved in the previous sections depend on the entropy

$$\int_{M} \log \frac{\omega_{\varphi_{\varepsilon}}^{n}}{\omega_{0}^{n}} \frac{\omega_{\varphi_{\varepsilon}}^{n}}{n!},$$

thus in order to obtain uniform estimates, it is essential to show that the entropies are uniformly bounded from above independent of  $\varepsilon$ . To achieve that, we need to study the Mabuchi energy which is closely related to the entropy. Recall that  $\omega_{\varepsilon} = \eta + \varepsilon \omega_0$ , and we will continue suppressing  $\varepsilon$  for simplicity of notation. The Mabuchi energy is defined to be the functional

(5.2) 
$$\mathcal{M}_{\omega}(\theta) = \int_{M} \log \frac{\omega_{\theta}^{n}}{\omega^{n}} \frac{\omega_{\theta}^{n}}{n!} + J_{-\operatorname{Ric}(\omega)}(\theta)$$

where  $\theta \in \mathcal{H}_{\omega} := \{\theta \in C^{\infty}(M) | \omega + \sqrt{-1}\partial \bar{\partial}\theta > 0, \sup_{M} \theta = 0\}$ . For any real (1,1) form  $\chi$ ,  $J_{\chi}$  is a functional on  $\mathcal{H}_{\omega}$  defined through its variation

(5.3) 
$$\frac{\mathrm{d}J_{\chi}}{\mathrm{d}t} = \int_{M} \frac{\partial \theta}{\partial t} (\mathrm{tr}_{\omega_{\theta}} \chi - \underline{\chi}) \frac{\omega_{\theta}^{n}}{n!}$$

where  $\underline{\chi} = n \frac{[\underline{\chi}] \cdot [\omega]^{n-1}}{[\omega]^n}$ . Following Chen [7] (see also [10, Section 2.1]), we have the following explicit formula for  $J_{\chi}$ 

$$(5.4) \quad J_{\chi}(\theta) = -\frac{\underline{\chi}}{(n+1)!} \int_{M} \theta \left( \sum_{i=1}^{n} \omega^{i} \wedge \omega_{\theta}^{n-i} \right) + \frac{1}{n!} \int_{M} \theta \chi \wedge \left( \sum_{i=1}^{n-1} \omega^{i} \wedge \omega_{\theta}^{n-1-i} \right),$$

so  $J_{\chi}(0)=0$ . Furthermore, the variation of the Mabuchi functional is given by

(5.5) 
$$\frac{\mathrm{d}\mathcal{M}_{\omega}}{\mathrm{d}t} = \int_{\mathcal{M}} \frac{\partial \theta}{\partial t} (-\mathrm{tr}_{\omega_{\theta}} \operatorname{Ric}(\omega_{\theta}) + \underline{\operatorname{Ric}(\omega)}) \frac{\omega_{\theta}^{n}}{n!}.$$

The first term appearing in the Mabuchi functional is the same as what we referred to as the entropy term before except that the leading term in the integral has its denominator as  $\omega^n$  while ours is  $\omega_0^n$ . Thus, we need to adjust the Mabuchi energy slighly to match our entropy term. We define our modified Mabuchi energy to be

(5.6) 
$$E_{\omega}(\theta) = \int_{M} \log \frac{\omega_{\theta}^{n}}{\omega_{0}^{n}} \frac{\omega_{\theta}^{n}}{n!} + J_{\eta}(\theta).$$

It turns out that  $E_{\omega}$  only differs from  $\mathcal{M}_{\omega}$  by a term equal to  $\int_{M} \log \frac{\omega^{n}}{\omega_{0}^{n}} \frac{\omega^{n}}{n!}$ . Observe that

(5.7) 
$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{M} \log \frac{\omega_{\theta}^{n}}{\omega_{0}^{n}} \frac{\omega_{\theta}^{n}}{n!} = \int_{M} \frac{\partial \theta}{\partial t} (-\mathrm{tr}_{\omega_{\theta}} \eta - \mathrm{tr}_{\omega_{\theta}} \operatorname{Ric}(\omega_{\theta})) \frac{\omega_{\theta}^{n}}{n!}.$$

So we know that

(5.8) 
$$\frac{\mathrm{d}E_{\omega}}{\mathrm{d}t} = \int_{M} \frac{\partial \theta}{\partial t} (-\mathrm{tr}_{\omega_{\theta}} \operatorname{Ric}(\omega_{\theta}) - \underline{\eta}) \frac{\omega_{\theta}^{n}}{n!} \\ = \int_{M} \frac{\partial \theta}{\partial t} (-\mathrm{tr}_{\omega_{\theta}} \operatorname{Ric}(\omega_{\theta}) + \underline{\operatorname{Ric}}(\underline{\omega})) \frac{\omega_{\theta}^{n}}{n!}.$$

Thus the variation of  $E_{\omega}$  is actually the same as that of the usual Mabuchi energy  $\mathcal{M}_{\omega}$ . We know that the potentials of cscK metrics are minimizers of  $\mathcal{M}_{\omega}$  and we want to show that the same is true for  $E_{\omega}$ . We have just showed that  $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_{\omega}=\frac{\mathrm{d}}{\mathrm{d}t}E_{\omega}$ . Taking a path  $t\theta$  in  $\mathcal{H}_{\omega}$ , we have

(5.9) 
$$\mathcal{M}_{\omega}(\theta) = \int_{0}^{1} \frac{\mathrm{d}}{\mathrm{d}t} E_{\omega} \, \mathrm{d}t + \mathcal{M}_{\omega}(0) = E_{\omega}(\theta) - \int_{M} \log \frac{\omega^{n}}{\omega_{0}^{n}} \frac{\omega^{n}}{n!}.$$

We know that

(5.10) 
$$\mathcal{M}_{\omega}(\theta) - \mathcal{M}_{\omega}(\varphi) \geqslant 0$$

for all  $\theta \in \mathcal{H}_{\omega}$  and  $\varphi$  such that  $\omega_{\varphi} = \omega + \sqrt{-1}\partial \bar{\partial} \varphi$  is a cscK metric thus,

(5.11) 
$$E_{\omega}(\theta) - E_{\omega}(\varphi) = \mathcal{M}_{\omega}(\theta) - \mathcal{M}_{\omega}(\varphi) \geqslant 0.$$

So we reach the conclusion that the potentials of cscK metrics are still minimizers of  $E_{\omega}$ . One immediate consequence is the following lemma.

LEMMA 5.1. — Let  $\varphi_{\varepsilon} \in \mathcal{H}_{\omega}$  be a smooth real-valued function such that  $\omega_{\varphi_{\varepsilon}} = \omega_{\varepsilon} + \sqrt{-1}\partial \bar{\partial} \varphi_{\varepsilon}$  is cscK, then

(5.12) 
$$E_{\omega_{\varepsilon}}(\varphi_{\varepsilon}) \leqslant E_{\omega_{\varepsilon}}(0) = \int_{M} \log \frac{\omega_{\varepsilon}^{n}}{\omega_{0}^{n}} \frac{\omega_{\varepsilon}^{n}}{n!} \leqslant C$$

where C is independent of  $\varepsilon$ .

We are now ready to show that the entropies are uniformly bounded from above. One possible approach is to first establish a uniform lower bound on the J functionals since we already know from Lemma 5.1 that  $E_{\omega}(\varphi)$  is uniformly bounded from above. However, this approach requires one to study the solutions of the J equations assoicated with the J functionals. For more details related to this route, see [27, 30, 52]. Instead of following this path, we turn to an approach introduced by Dervan [18] which generates explicit bounds on the functionals involved.

To begin with, we estimate the entropy term in the Mabuchi energy using the  $\alpha$ -invariant, and this technique should be well-known to the experts in this field (see for example Tian [47]).

Lemma 5.2. — for all  $\varepsilon \in [0,1]$ , and  $\alpha$  such that  $\alpha > 0$  and  $\alpha < \alpha(M, [\eta + \omega_0])$ , we have

(5.13) 
$$\int_{M} \log \frac{\omega_{\theta}^{n}}{\omega_{0}^{n}} \frac{\omega_{\theta}^{n}}{n!} \geqslant -\alpha \int_{M} \theta \frac{\omega_{\theta}^{n}}{n!} - C$$

where C is a positive constant independent of  $\varepsilon$ .

*Proof.* — First notice that

(5.14) 
$$\int_{M} e^{-\alpha \theta} \frac{\omega_{0}^{n}}{n!} = \int_{M} e^{-\log \frac{\omega_{0}^{n}}{\omega_{0}^{n}} - \alpha \theta} \frac{\omega_{\theta}^{n}}{n!} \leqslant C$$

where we used Lemma 2.3 again. Then by Jensen's inequality,

(5.15) 
$$\int_{M} \left( -\log \frac{\omega_{\theta}^{n}}{\omega_{0}^{n}} - \alpha \theta \right) \frac{\omega_{\theta}^{n}}{n!} \leqslant C \int_{M} \frac{\omega^{n}}{n!} \leqslant C$$

where  $\int_M \frac{\omega^n}{n!}$  arises because we need a probability measure in order to apply Jensen's inequality. Thus,

(5.16) 
$$\int_{M} \log \frac{\omega_{\theta}^{n}}{\omega_{0}^{n}} \frac{\omega_{\theta}^{n}}{n!} \ge -\alpha \int_{M} \theta \frac{\omega_{\theta}^{n}}{n!} - C. \qquad \Box$$

We will need the following lemmas due to [18], and we include brief proofs of them here for the reader's convenience.

LEMMA 5.3 ([18, Lemma 2.9]). — We have

(5.17) 
$$-n\int_{M}\theta\omega_{\theta}^{n}\geqslant -\sum_{i=1}^{n}\int_{M}\theta\omega^{i}\wedge\omega_{\theta}^{n-i}.$$

Proof.

$$(5.18) \quad -n \int_{M} \theta \omega_{\theta}^{n} + \sum_{i=1}^{n} \int_{M} \theta \omega^{i} \wedge \omega_{\theta}^{n-i} = \sum_{i=1}^{n} \int_{M} \theta \omega_{\theta}^{n-i} \wedge (\omega^{i} - \omega_{\theta}^{i}).$$

We only need to show that each summand is positive separately

$$(5.19) \int_{M} \theta \omega_{\theta}^{n-i} \wedge (\omega^{i} - \omega_{\theta}^{i}) = \int_{M} \sqrt{-1} \partial \theta \wedge \bar{\partial} \theta \wedge \omega_{\theta}^{n-i} \wedge \left( \sum_{j=1}^{i} \omega^{i-j} \wedge \omega_{\theta}^{j-1} \right)$$

$$\geqslant 0.$$

Lemma 5.4. — We have

$$(5.20) \quad -\frac{\underline{\eta}}{(n+1)!} \int_{M} \theta \left( \sum_{i=0}^{n} \omega^{i} \wedge \omega_{\theta}^{n-i} \right)$$

$$\geqslant -\frac{\underline{\eta}}{n!n} \int_{M} \theta \omega \wedge \left( \sum_{i=0}^{n-1} \omega^{i} \wedge \omega_{\theta}^{n-i-1} \right).$$

Proof. — We calculate

$$(5.21) \quad -\frac{\underline{\eta}}{(n+1)!} \int_{M} \theta \left( \sum_{i=0}^{n} \omega^{i} \wedge \omega_{\theta}^{n-i} \right)$$

$$= -n \frac{\underline{\eta}}{n(n+1)!} \int_{M} \theta \omega_{\theta}^{n} - \frac{\underline{\eta}}{(n+1)!} \int_{M} \theta \omega \wedge \left( \sum_{i=0}^{n-1} \omega \wedge \omega_{\theta}^{n-i-1} \right)$$

$$\geqslant -\frac{\underline{\eta}}{(n+1)!n} \int_{M} \sum_{i=1}^{n} \theta \omega^{i} \wedge \omega_{\theta}^{n-i} - \frac{\underline{\eta}}{(n+1)!} \int_{M} \theta \omega \wedge \left( \sum_{i=0}^{n-1} \omega^{i} \wedge \omega_{\theta}^{n-i-1} \right)$$

$$= -\frac{\underline{\eta}}{n!n} \int_{M} \theta \omega \wedge \left( \sum_{i=0}^{n-1} \omega^{i} \wedge \omega_{\theta}^{n-i-1} \right)$$

where for the inequality we used Lemma 5.3.

Theorem 5.5. — There exists C > 0, such that

$$\int_{M} \log \frac{\omega_{\varphi_{\epsilon}}^{n}}{\omega_{0}^{n}} \frac{\omega_{0}^{n}}{n!} \leqslant C.$$

*Proof.* — Again, we will suppress  $\varepsilon$  for brevity. Fix  $\alpha > 0$  with  $\alpha < \alpha(M, [\eta + \omega_0])$ , then we get

$$(5.23) \quad E_{\omega}(\varphi)$$

$$\geqslant \frac{1}{2} \int_{M} \log \frac{\omega_{\varphi}^{n}}{\omega_{0}^{n}} \frac{\omega_{\varphi}^{n}}{n!} - \frac{\alpha}{2} \int_{M} \varphi \frac{\omega_{\varphi}^{n}}{n!} - \frac{\underline{\eta}}{(n+1)!} \int_{M} \varphi \left( \sum_{i=0}^{n} \omega^{i} \wedge \omega_{\varphi}^{n-i} \right) + \frac{1}{n!} \int_{M} \varphi \eta \wedge \left( \sum_{i=0}^{n-1} \omega^{i} \wedge \omega_{\varphi}^{n-1-i} \right) - C$$

$$\geqslant \frac{1}{2} \int_{M} \log \frac{\omega_{\varphi}^{n}}{\omega_{0}^{n}} \frac{\omega_{\varphi}^{n}}{n!} - \frac{1}{n!} \int_{M} \varphi \left( \frac{\alpha}{2n} \omega - \eta + \frac{1}{n} \underline{\eta} \omega \right) + C$$

$$\wedge \left( \sum_{i=0}^{n-1} \omega^{i} \wedge \omega_{\varphi}^{n-1-i} \right) - C$$

where for the first inequality we used Lemma 5.2, and the second inequality we used Lemma 5.3 and Lemma 5.4. Notice that

$$(5.24) \qquad \frac{\alpha}{2n}\omega - \eta + \frac{1}{n}\underline{\eta}\omega = \left(\frac{\alpha}{2n} - 1 + \frac{1}{n}\underline{\eta}\right)\eta + \varepsilon\left(\frac{\alpha}{2n} + \frac{1}{n}\underline{\eta}\right)\omega_0 > 0$$

for  $\varepsilon > 0$  small enough because  $\underline{\eta} = n \frac{[\underline{\eta}] \cdot [\omega_{\varepsilon}]^{n-1}}{[\omega_{\varepsilon}]^n} \to n$  as  $\varepsilon \to 0$ . Thus for  $\varepsilon$  sufficiently small, we have

$$(5.25) \quad E_{\omega}(\varphi)$$

$$\geqslant \frac{1}{2} \int_{M} \log \frac{\omega_{\varphi}^{n}}{\omega_{0}^{n}} \frac{\omega_{\varphi}^{n}}{n!} - \int_{M} \varphi \left( \frac{\alpha}{2n} - \eta + \frac{1}{n} \underline{\eta} \omega \right) \wedge \left( \sum_{i=0}^{n-1} \omega^{i} \wedge \omega_{\varphi}^{n-1-i} \right) - C$$

$$\geqslant \frac{1}{2} \int_{M} \log \frac{\omega_{\varphi}^{n}}{\omega_{0}^{n}} \frac{\omega_{\varphi}^{n}}{n!} - C,$$

since we assume that  $\varphi \leq 0$ . We conclude the proof by recalling that  $E_{\omega}(\varphi) \leq E_{\omega}(0) \leq C$  according to Lemma 5.1.

### 6. Convergence

Now we use the estimates developed in previous sections to show that away from E,  $\omega_{\varphi_{\varepsilon}}$  converges to the singular Kähler Einstein metric  $\omega_{\text{KE}}$  on  $M \setminus E$  given by Theorem 1.4. Fix a connected compact subset  $K \subset M$ , and a subsequence  $\varepsilon_i$  such that  $\varphi_{\varepsilon_i}$  converges smoothly to  $\varphi_{\infty}$ , and  $F_{\varepsilon_i}$ 

converges smoothly to  $F_{\infty}$  (such a subsequence always exists by Arzela–Ascoli). Recall that our equations read

(6.1) 
$$\frac{(\omega_{\varepsilon_i} + \sqrt{-1}\partial\bar{\partial}\varphi_{\varepsilon_i})^n}{\omega_0^n} = e^{F_{\varepsilon_i}}, \\ \Delta_{\omega_{\varphi_{\varepsilon_i}}} F_{\varepsilon_i} = -\text{tr}_{\omega_{\varphi_{\varepsilon_i}}} \eta - \underline{R_{\varepsilon_i}}.$$

Let us compute

$$(6.2) \int_{K} |\partial(F_{\varepsilon_{i}} - \varphi_{\varepsilon_{i}})|^{2}_{\omega_{\varphi_{\varepsilon_{i}}}} \omega_{\varphi_{\varepsilon_{i}}}^{n}$$

$$\leq \int_{M} |\partial(F_{\varepsilon_{i}} - \varphi_{\varepsilon_{i}})|^{2}_{\omega_{\varphi_{\varepsilon_{i}}}} \omega_{\varphi_{\varepsilon_{i}}}^{n}$$

$$= -\int_{M} (F_{\varepsilon_{i}} - \varphi_{\varepsilon_{i}}) \Delta_{\omega_{\varphi_{\varepsilon_{i}}}} (F_{\varepsilon_{i}} - \varphi_{\varepsilon_{i}}) \omega_{\varphi}^{n}$$

$$= (\underline{R} + n) \int_{M} (F_{\varepsilon_{i}} - \varphi_{\varepsilon_{i}}) \omega_{\varphi_{\varepsilon_{i}}}^{n} - \varepsilon_{i} \int_{M} (F_{\varepsilon_{i}} - \varphi_{\varepsilon_{i}}) \operatorname{tr}_{\omega_{\varphi_{\varepsilon_{i}}}} \omega_{0} \omega_{\varphi_{\varepsilon_{i}}}^{n}$$

$$\leq C(\underline{R} + n) \int_{M} \omega_{\varphi_{\varepsilon_{i}}}^{n} + C\varepsilon_{i} n \int_{M} \omega_{\varphi_{\varepsilon_{i}}}^{n-1} \wedge \omega_{0} \to 0$$

where for the last line we used the  $C^0$  estimates on F and  $\varphi$ . So  $\partial F_{\infty} = \partial \varphi_{\infty}$  on K which implies that  $F_{\infty} = \varphi_{\infty} + C$  on K. Furthermore,

(6.3) 
$$\operatorname{Ric}(\omega_{\varphi_{\infty}}) = \operatorname{Ric}(\omega_0) - \sqrt{-1}\partial\bar{\partial}F_{\infty} = -\eta - \sqrt{-1}\partial\bar{\partial}\varphi_{\infty} = -\omega_{\varphi_{\infty}}$$

on K. Now, we can take a countable compact increasing exhaustion  $U_n$  of  $M \setminus E$  such that  $\bigcup_{n \in \mathbb{N}} K_n = M \setminus E$ , and  $K_i \subset K_{i+1}, \forall i$ . Then by using the usual diagonal argument, we can get a subsequence of  $\varphi_{\varepsilon_i}$  so that passing to its limit defines a  $\varphi_{\infty}$  on  $M \setminus E$  and

(6.4) 
$$\operatorname{Ric}(\omega_{\varphi_{\infty}}) = -\omega_{\varphi_{\infty}}.$$

By Theorem 1.4, the uniqueness of such singular Kähler Einstein metric, all convergent subsequences on K we initially start with have to converge to the same limit. If for each sequence we have a further subsequence converging to the same limit, then the sequence itself has to converge.

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