

ANNALES DE L'institut fourier

José Ángel PELÁEZ & Jouni RÄTTYÄ Weighted norm inequalities for derivatives on Bergman spaces

Tome 74, nº 4 (2024), p. 1721-1744.

https://doi.org/10.5802/aif.3632

Article mis à disposition par ses auteurs selon les termes de la licence CREATIVE COMMONS ATTRIBUTION – PAS DE MODIFICATION 3.0 FRANCE [CC] BY-ND http://creativecommons.org/licenses/by-nd/3.0/fr/



Les Annales de l'Institut Fourier sont membres du Centre Mersenne pour l'édition scientifique ouverte www.centre-mersenne.org e-ISSN : 1777-5310

WEIGHTED NORM INEQUALITIES FOR DERIVATIVES ON BERGMAN SPACES

by José Ángel PELÁEZ & Jouni RÄTTYÄ (*)

ABSTRACT. — An equivalent norm in the weighted Bergman space A_{ω}^{p} , induced by an ω in a certain large class of non-radial weights, is established in terms of higher order derivatives. Other Littlewood–Paley inequalities are also considered. On the way to the proofs, we characterize the *q*-Carleson measures for the weighted Bergman space A_{ω}^{p} and the boundedness of a Hörmander-type maximal function. Results obtained are further applied to describe the resolvent set of the integral operators $T_{g}(f)(z) = \int_{0}^{z} g'(\zeta) f(\zeta) d\zeta$ acting on A_{ω}^{p} .

RÉSUMÉ. — Nous construisons une norme équivalente, définie à l'aide des dérivées supérieures, dans un espace de Bergman pondéré A^p_{ω} où ω appartient à une large classe des poids non radiaux. Nous analysons aussi autres inégalités de Littlewood–Paley. Avant de démontrer les résultats principaux nous caractérisons les q-mesures de Carleson sur les espaces A^p_{ω} et montrons que la fonction maximale de Hörmander est bornée. En utilisant nos résultats nous pouvons décrire l'ensemble résolvant de l'opérateur intégral $T_g(f)(z) = \int_0^z g'(\zeta) f(\zeta) d\zeta$ agissant sur A^p_{ω} .

1. Introduction and main results

Let $\mathcal{H}(\mathbb{D})$ denote the algebra of all analytic functions in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ of the complex plane \mathbb{C} . A function $\omega : \mathbb{D} \to [0, \infty)$, integrable over \mathbb{D} , is called a weight. It is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ and $\int_0^1 \omega(s) \, ds < \infty$. For $0 and a weight <math>\omega$, the weighted Bergman space A^p_{ω} consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{A^p_{\omega}}^p = \int_{\mathbb{D}} |f(z)|^p \omega(z) \, \mathrm{d}A(z) < \infty,$$

Keywords: Bergman space, Carleson measure, integral operator, Littlewood–Paley inequality, Hörmander-type maximal function, resolvent set.

²⁰²⁰ Mathematics Subject Classification: 30H20, 47G10.

^(*) This research was supported in part by Ministerio de Ciencia Innovación y universidades, Spain, projects PGC2018-096166-B-100 and MTM2017-90584-REDT; La Junta de Andalucía, project FQM210; Academy of Finland 286877.

where $dA(z) = \frac{dx \, dy}{\pi}$ is the normalized Lebesgue area measure on \mathbb{D} .

In this paper we are interested in obtaining, for a large class of nonradial weights ω , equivalent norms of f in A^p_{ω} in terms of its higher order derivatives. This is a question that have been extensively studied for different classes of radial weights but it is not well-understood for general weights. See [1, 3, 5] for recent developments on the topic. These norms are extremely valuable within the theory of concrete operators acting on these spaces. To name a few instances, they are used; in the study of Volterratype operators, because they allow to get rid of the integral and they arise in a natural way in the description of its spectrum [1, 3, 4, 20], in order to get crucial estimates in the description of Schatten classes of Toeplitz operators [16, p. 356], in the boundedness of the Hilbert matrix [19, Proof of Theorem 2] or in obtaining norm estimates for the Bergman reproducing kernels in A^p_{ω} induced by radial weights [22, Proof of Theorem 1].

A well-known formula ensures that, for each $k \in \mathbb{N}$ and 0 , we have

(1.1)
$$||f||_{A^p_{\omega}}^p \asymp \int_{\mathbb{D}} |f^{(k)}(z)|^p (1-|z|)^{kp} \omega(z) \, \mathrm{d}A(z)$$

 $+ \sum_{j=0}^{k-1} |f^{(j)}(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}),$

if ω is a standard radial weight, that is, $\omega(z) = (\alpha + 1)(1 - |z|^2)^{\alpha}$ for some $-1 < \alpha < \infty$. Generalizations of this result for different classes of radial weights have been obtained in [4, 17, 24, 27]. In particular, it was recently proved [24, Theorem 5] that (1.1) holds for a radial weight ω if and only if $\omega \in \mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$. Recall that a radial weight ν belongs to $\widehat{\mathcal{D}}$ if there exists a constant $C = C(\nu) > 1$ such that the tail integral $\widehat{\nu}(z) = \int_{|z|}^{1} \nu(s) \, \mathrm{d}s$ satisfies the doubling condition $\widehat{\nu}(r) \leq C\widehat{\nu}(\frac{1+r}{2})$ for all $0 \leq r < 1$. Further, a radial weight ν belongs to $\check{\mathcal{D}}$ if there exists constants $K = K(\nu) > 1$ and $C = C(\nu) > 1$ such that $\widehat{\nu}(r) \geq C\widehat{\nu}(1 - \frac{1-r}{K})$ for all $0 \leq r < 1$.

For a given $a \in \mathbb{D} \setminus \{0\}$, consider the interval

$$I_a = \left\{ e^{i\theta} : |\arg(a e^{-i\theta})| \leqslant \frac{(1-|a|)}{2} \right\},\,$$

and let $S(a) = \{z \in \mathbb{D} : |z| \ge |a|, e^{it} \in I_a\}$ denote the Carleson square induced by a. We assume throughout the paper that $\omega(S(a)) > 0$ for all $a \in \mathbb{D} \setminus \{0\}$. If this is not the case and ω is radial, then $A^p_{\omega} = \mathcal{H}(\mathbb{D})$. For a weight ν , ω is a ν -weight if $\omega\nu$ is integrable. If $1 , a <math>\nu$ -weight ω belongs to the class $B_p(\nu)$ if there exists a constant $C = C(p, \nu, \omega)$ such that

(1.2)
$$\sup_{S} \frac{\left(\int_{S} \omega(z)\nu(z) \,\mathrm{d}A(z)\right)^{\frac{1}{p}} \left(\int_{S} \omega^{-\frac{p'}{p}}(z)\nu(z) \,\mathrm{d}A(z)\right)^{\frac{1}{p'}}}{\int_{S} \nu(z) \,\mathrm{d}A(z)} < \infty,$$

where the supremum is taken over all Carleson squares S. We denote $B_{\infty}(\nu) = \bigcup_{1 . It has recently been proved that the class <math>B_p(\nu)$ describes the weights ω such that Bergman projection P_{ν} , induced by a radial weight ν , is bounded on $L^p_{\omega\nu}$, $1 , whenever <math>\nu \in \mathcal{D}$ and the Bergman reproducing kernel of A^2_{ν} has a particular integral representation [26, Theorem 2]. This result is a natural extension of a classical result due to Bekollé and Bonami [6, 7] for standard weights. If $\nu(z) = (1 + \eta)(1 - |z|^2)^{\eta}$ we simply write $B_p(\eta)$ instead of $B_p((1 - |z|)^{\eta})$, $B_{\infty}(\eta) = B_{\infty}((1 - |z|)^{\eta})$ and $B_{\infty} = B_{\infty}(0)$. Nonnegative functions in the class $B_p(\eta)$ or $B_{\infty}(\eta)$ are usually called the Bekollé–Bonami weights. En route to describing the spectrum of the integral operator

$$T_g(f)(z) = \int_0^z f(\zeta) g'(\zeta) \,\mathrm{d}\zeta, \quad z \in \mathbb{D}, \quad g \in \mathcal{H}(\mathbb{D}),$$

on standard Bergman spaces it was shown that (1.1) is satisfied if there exists $\eta > -1$ such that $\frac{\omega}{(1-|z|)^{\eta}} \in B_{\infty}(\eta)$ [1, Theorem 3.1]. The first result of this study says that the hypothesis $\frac{\omega}{(1-|z|)^{\eta}} \in B_{\infty}(\eta)$ can be replaced by the weaker condition $\frac{\omega}{\nu} \in B_{\infty}(\nu)$ with $\nu \in \mathcal{D}$. To simplify the notation, we write $\mathcal{B}_{\infty}(\nu) = \bigcup_{1 and <math>\mathcal{B}_{\infty}(\mathcal{D}) = \bigcup_{\nu \in \mathcal{D}} \mathcal{B}_{\infty}(\nu)$. It is worth mentioning that $\mathcal{B}_{\infty}(\mathcal{D})$ is in a sense a much larger class than

$$\bigcup_{\eta>-1} \left\{ \omega \text{ is a weight} : \frac{\omega}{(1-|z|)^{\eta}} \in B_{\infty}(\eta) \right\}$$

because it contains weights with a strong oscillatory behaviour which may vanish on a hyperbolically bounded set of \mathbb{D} .

THEOREM 1.1. — Let $0 , <math>k \in \mathbb{N}$ and $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$. Then (1.1) holds.

Observe that the set of radial weights in $\mathcal{B}_{\infty}(\mathcal{D})$ coincides with \mathcal{D} , and hence [24, Theorem 5] implies that $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$ is also a necessary condition for (1.1) to hold if ω is radial. The class \mathcal{D} also appears innately in the study of classical questions related to the boundedness of the Bergman projection P_{ν} induced by a radial weight ν [24, Theorem 3 and Theorem 12], which is a frequently used tool in order to get Littlewood–Paley formulas in weighted Bergman spaces [1, 29].

It is also worth mentioning that (1.1) holds if and only if $\omega \in B_{\infty}$, when ω is essentially (or almost) constant in each hyperbolically bounded region [3, Corollary 4.4]. This last condition ensures that the inequality

(1.3)
$$\int_{\mathbb{D}} |f^{(k)}(z)|^{p} (1-|z|)^{kp} \omega(z) \, \mathrm{d}A(z) + \sum_{j=0}^{k-1} |f^{(j)}(0)|^{p} \lesssim \|f\|_{A^{p}_{\omega}}^{p}, \quad f \in \mathcal{H}(\mathbb{D}),$$

holds [1, Theorem 3.1], see also [5, Theorem A]. We prove the following result concerning this last inequality.

THEOREM 1.2. — Let $0 , <math>k \in \mathbb{N}$ and $\omega \in \mathcal{B}_{\infty}(\widehat{\mathcal{D}})$. Then (1.3) holds.

Obviously there are weights in $\mathcal{B}_{\infty}(\widehat{\mathcal{D}})$ which are not essentially constant in each hyperbolically bounded region. Moreover, since the restriction of $\mathcal{B}_{\infty}(\widehat{\mathcal{D}})$ to radial weights coincides with $\widehat{\mathcal{D}}$, the hypothesis $\omega \in \mathcal{B}_{\infty}(\widehat{\mathcal{D}})$ in Theorem 1.2 cannot be relaxed in the case of radial weights by [24, Theorem 6]. Observe that weights in $\widehat{\mathcal{D}}$, an consequently in $\mathcal{B}_{\infty}(\widehat{\mathcal{D}})$, may have a wild oscillatory behavior and they may even vanish on sets that are not hyperbolically uniformly bounded. Moreover, $\widehat{\mathcal{D}}$ is not closed by the multiplication of a standard weight induced by a positive parameter α . Illuminating examples of weights in the deceivingly simply looking class $\overline{\mathcal{D}}$ are given in [23, Proposition 10] and [25, Proposition 12].

The proofs of Theorems 1.1 and 1.2 have three key ingredients. The first of them provides a geometric description of the q-Carleson measures for A^p_{ω} , provided $q \ge p$ and $\omega \in \mathcal{B}_{\infty}(\widehat{\mathcal{D}})$. To state the result, for a given measure μ on \mathbb{D} , we write $\mu(E) = \int_E d\mu$ for each μ -measurable set $E \subset \mathbb{D}$. Further, for each $\varphi \in L^1_{\omega}$, the Hörmander-type maximal function [11] is defined by

$$M_{\omega}(\varphi)(z) = \sup_{z \in S} \frac{1}{\omega(S)} \int_{S} |\varphi(\zeta)| \omega(\zeta) \, \mathrm{d}A(\zeta), \quad z \in \mathbb{D}.$$

Our characterization of q-Carleson measures for A^p_{ω} reads as follows.

THEOREM 1.3. — Let $0 , <math>\omega \in \mathcal{B}_{\infty}(\widehat{\mathcal{D}})$ and let μ be a positive Borel measure on \mathbb{D} . Then the following statements are equivalent:

- (i) μ is a q-Carleson measure for A^p_{ω} ; (ii) $\left[M_{\omega}\left(|\cdot|^{\frac{1}{\alpha}}\right)\right]^{\alpha}: L^p_{\omega} \to L^q_{\mu}$ is bounded for each $\alpha > \frac{1}{p}$;
- (iii) μ satisfies

(1.4)
$$\sup_{S} \frac{\mu(S)}{(\omega(S))^{\frac{q}{p}}} < \infty$$

where the supremum runs over all the Carleson squares S in \mathbb{D} .

Moreover,

(1.5)
$$\|I_d\|^q_{A^p_{\omega} \to L^q_{\mu}} \asymp \left\| \left[M_{\omega} \left(|\cdot|^{\frac{1}{\alpha}} \right) \right]^{\alpha} \right\|^q_{L^p_{\omega} \to L^q_{\mu}} \asymp \sup_S \frac{\mu(S)}{\left(\omega(S) \right)^{\frac{q}{p}}}$$

Theorem 1.3 is a natural extension of [18, Theorem 3.3] and [20, Theorem 2.1] to non-radial weights.

A good understanding of the class of weights involved in Theorems 1.1 and 1.2 is needed. In particular, en route to the proofs, we show that

$$\omega \in \mathcal{B}_{\infty}(\mathcal{D}) \Rightarrow \omega_{[\beta]}(z) = (1 - |z|)^{\beta} \omega(z) \in \mathcal{B}_{\infty}(\mathcal{D}), \quad \text{for any } \beta > 0,$$

a fact which might be deceptively simple-looking. Indeed, the class $\mathcal{B}_{\infty}(\widehat{\mathcal{D}})$ does not have this property. See [23, Proposition 10] for the construction of a radial weight $\omega \in \mathcal{B}_{\infty}(\widehat{\mathcal{D}})$ such that $\omega_{[\beta]} \notin \mathcal{B}_{\infty}(\widehat{\mathcal{D}})$ for any $\beta > 0$.

The third key ingredient in the proofs of Theorems 1.1 and 1.2 concerns certain more smooth weights. Namely, each weight ω induces the nonnegative average function

$$\widetilde{\omega}(z) = \frac{\int_{S(z)} \omega(\zeta) \, \mathrm{d}A(\zeta)}{(1-|z|)^2}, \quad z \in \mathbb{D} \setminus \{0\}.$$

Which regard to this function we prove the following result.

THEOREM 1.4. — Let $0 , <math>k \in \mathbb{N}$ and $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$. Then

$$\|f\|_{A^p_{\omega}}^p \asymp \|f\|_{A^p_{\omega}}^p \asymp \int_{\mathbb{D}} |f^{(k)}(z)|^p (1-|z|)^{kp} \widetilde{\omega}(z) \,\mathrm{d}A(z) + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p,$$
$$f \in \mathcal{H}(\mathbb{D})$$

We emphasize that, under the hypothesis $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$, the weights ω and $\widetilde{\omega}$ are not pointwise equivalent, but $\|f\|_{A_{\omega}^{p}}^{p} \asymp \|f\|_{A_{\omega}^{p}}^{p}$ and $\widetilde{\omega}$ is essentially (or almost) constant in each hyperbolically bounded region. This together with the second equivalence in Theorem 1.4 and [3, Corollary 4.4-Theorem 1.7] implies that $\widetilde{\omega} \in B_{\infty}$. Therefore the study of certain type of questions on linear operators $T : \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$ on weighted Bergman spaces A_{ω}^{p} , with $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$, can be reduced to the particular case $\omega \in B_{\infty}$. We will make this statement precise in the case of some questions related to the integral operator T_{g} induced by $g \in \mathcal{H}(\mathbb{D})$. Indeed, Theorem 5.1 below describes the analytic symbols g such that $T_{g} : A_{\omega}^{p} \to A_{\omega}^{q}$ is bounded or compact. In particular, it shows that $T_{g} : A_{\omega}^{p} \to A_{\omega}^{p}$ is bounded if and only if g belongs to the classical space \mathcal{B} of Bloch functions. Further, by using ideas from studies [1, 2, 3], which link the resolvent set $\rho(T_{g}|A_{\omega}^{p})$ to the

theory of weighted norms in terms of derivatives, we obtain the following characterization of $\rho(T_q|A_{\omega}^p)$.

THEOREM 1.5. — Let $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$, $g \in \mathcal{B}$, $0 and <math>\lambda \in \mathbb{C} \setminus \{0\}$. Then the following statements are equivalent:

- (i) $\lambda \in \rho(T_g | A^p_\omega);$
- (ii) $||f||_{A^p_{\omega_{\lambda,g,p}}}^p \approx |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^p \omega_{\lambda,g,p}(z) \, \mathrm{d}A(z) \text{ for all } f \in \mathcal{H}(\mathbb{D}), \text{ where } \omega_{\lambda,g,p} = \omega \exp\left(p \operatorname{Re} \frac{g}{\lambda}\right);$ (iii) $\widetilde{\omega} \exp\left(n \operatorname{Re} \frac{g}{\lambda}\right) \in B$
- (iii) $\widetilde{\omega} \exp\left(p \operatorname{Re} \frac{g}{\lambda}\right) \in B_{\infty}.$

The remaining part of the paper is organized as follows. In Section 2 we state and prove some preliminary results on weights. Theorem 1.3 is proved in Section 3 while Section 4 is devoted to the proofs of Theorems 1.1, 1.2 and 1.4. In Section 5 we discuss some basic properties of the integral operator T_g acting on A^p_{ω} and then prove Theorem 1.5.

Before proceeding further, a word about notation used. The letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \leq b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \leq Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \leq b$ and $a \gtrsim b$, then we write $a \approx b$ and say that a and b are comparable. This notation has already been used above in the introduction.

2. Basic properties and lemmas on weights

The pseudohyperbolic distance between two points z_1 and z_2 in \mathbb{D} is $\rho(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|$. We say that a weight ω is essentially constant on each hyperbolically bounded region if there exist constants $r \in (0, 1)$ and $C = C(\omega, r)$ such that

$$C^{-1}\omega(z_2) \leqslant \omega(z_1) \leqslant C\omega(z_2), \quad \rho(z_1, z_2) < r.$$

This class of weights coincides with the weights satisfying [3, (1.6)] and has been also considered in [5].

In the classical setting, there are many equivalent conditions which describe the Muckenhoupt class $A_{\infty} = \bigcup_{1 \leq q < \infty} A_q$, see [10], [28, Chapter 5] or [9, p. 149]. However, this is no longer true for the class B_{∞} , that is, the corresponding conditions (defined on Carleson squares) do not coincide. This stems from the fact that B_{∞} -weights do not have the reverse Hölder property [3, 8]. It is worth mentioning that our definition of the class B_{∞} differs from the one provided in [3, (1.4)]. However, this does not cause any trouble because B_{∞} -weights, which are essentially constant in each hyperbolically bounded region, can be described in terms of a good number of conditions [3, Theorem 1.7], and in particular the definitions coincide under this extra hypotheses on the weight.

The main results of this paper are established under the hypothesis $\omega \in \mathcal{B}(\mathcal{D})$. Therefore we are interested in looking for neat conditions describing the class $\mathcal{B}_{\infty}(\nu)$ induced by $\nu \in \mathcal{D}$. In order to do this, for each weight ν , we say that a weight ω has the Kerman–Torchinsky $KT(\nu)$ -property if there exist constants $\delta \in (0, 1)$ and C > 0 such that

(2.1)
$$\frac{\nu(E)}{\nu(S)} \leqslant C \left(\frac{\omega(E)}{\omega(S)}\right)^{\delta}$$

for all Carleson squares $S \subset \mathbb{D}$ and measurable sets $E \subset S$. Here and from now on we write $\omega(E) = \int_E \omega(z) \, dA(z)$. If we replace ν by the Lebesgue measure in \mathbb{R}^n and Carleson squares by cubes Q in (2.1), we obtain a condition which describes the class A_{∞} of the classical Muckenhoupt weights [10, Theorem 3.1]. This condition was introduced by Kerman and Torchinsky [13, Proposition 1] in order to describe the Hardy–Littlewood maximal operators that are of restricted weak-type. The next result follows from [10, Theorem 3.1(c)] (which holds for general bases).

PROPOSITION A. — Let ν be a weight. Then a weight ω belongs to $\mathcal{B}_{\infty}(\nu)$ if and only if it has the $KT(\nu)$ -property.

The K-top of a Carleson box S(a) is the polar rectangle $T_K(a) = \{r e^{it} : e^{it} \in I_a, |a| \leq r < 1 - \frac{1-|a|}{K}\}$. In some of the auxiliary results obtained en route to the main theorems the conditions $\mathcal{B}(\mathcal{D})$ and $\mathcal{B}(\widehat{\mathcal{D}})$ can be relaxed in the sense that (2.1) is only needed for K-tops or their complements $S \setminus T_K$. To be precise, we write $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$ if there exists $C = C(\omega) > 0$ such that $\omega(S(a)) \leq C\omega(S(\frac{1+|a|}{2} e^{i \arg a}))$ for all $a \in \mathbb{D} \setminus \{0\}$. It is easy to see that each $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$ satisfies $\omega(S(a')) \leq C(C+1)\omega(S(a))$ for all $a, a' \in \mathbb{D} \setminus \{0\}$ with |a'| = |a| and $\arg a' = \arg a \pm (1-|a|)$. Therefore $\omega(S(a)) \leq \omega(S(b))$ whenever $|b| = \frac{1+|a|}{2}$ and $S(b) \subset S(a)$. It is also obvious that radial weights in $\widehat{\mathcal{D}}(\mathbb{D})$ form the class $\widehat{\mathcal{D}}$, which plays a crucial role in the operator theory of Bergman spaces induced by radial weights [24]. Further, a weight ω on \mathbb{D} belongs to $\widetilde{\mathcal{D}}(\mathbb{D})$ if there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that

(2.2)
$$\omega(S(a)) \leq C\omega(T_K(a)), \quad a \in \mathbb{D} \setminus \{0\}.$$

It is clear that radial weights in $\check{\mathcal{D}}(\mathbb{D})$ form the class $\check{\mathcal{D}}$. Finally, we write $\mathcal{D}(\mathbb{D}) = \widehat{\mathcal{D}}(\mathbb{D}) \cap \check{\mathcal{D}}(\mathbb{D})$ for short.

In view of the above we have $\mathcal{B}_{\infty}(\widehat{\mathcal{D}}) \subset \widehat{\mathcal{D}}(\mathbb{D}), \ \mathcal{B}_{\infty}(\check{\mathcal{D}}) \subset \check{\mathcal{D}}(\mathbb{D})$ and $\mathcal{B}_{\infty}(\mathcal{D}) \subset \mathcal{D}(\mathbb{D})$. These embeddings, which will be used repeatedly throughout the paper, can also be proved by straightforward calculations which show that $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$ (resp. $\omega \in \check{\mathcal{D}}(\mathbb{D})$) if $\nu \in \widehat{\mathcal{D}}(\mathbb{D})$ (resp. $\nu \in \check{\mathcal{D}}(\mathbb{D})$) and $\omega \in \mathcal{B}_{\infty}(\nu)$. Therefore $\widehat{\mathcal{D}}$ and \mathcal{D} coincide with the radial weights in $\mathcal{B}_{\infty}(\widehat{\mathcal{D}})$ and $\mathcal{B}_{\infty}(\mathcal{D})$, respectively. However, $\mathcal{B}_{\infty}(\widehat{\mathcal{D}}) \subsetneq \widehat{\mathcal{D}}$. Namely, let

$$\Gamma(\zeta) = \left\{ z \in \mathbb{D} : |\arg \zeta - \arg z| < \frac{1}{2} (1 - |z|) \right\}, \quad \zeta \in \partial \mathbb{D},$$

and consider the weight $\omega = \chi_{\mathbb{D}\setminus\Gamma(1)}$. Then $\omega \notin \mathcal{B}_{\infty}(\widehat{\mathcal{D}})$ as is seen by considering the Carleson squares S(a) induced by $a \in (0, 1)$. But obviously there exists a constant C > 0 such that $\omega(S_a) \ge C|S_a|$ for all $a \in \mathbb{D} \setminus \{0\}$, and thus $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$.

The proof of the following result concerning the class $\widehat{\mathcal{D}}(\mathbb{D})$ can be found in [14, Lemma 14].

LEMMA B. — Let ω be a weight on \mathbb{D} . Then the following statements are equivalent:

(i) $\omega \in \widehat{\mathcal{D}}(\mathbb{D});$

(ii) there exist $\beta = \beta(\omega) > 0$ and $C = C(\omega) \ge 1$ such that

$$\frac{\omega(S(a))}{(1-|a|)^{\beta}} \leqslant C \frac{\omega(S(a'))}{(1-|a'|)^{\beta}}, \quad 0 < |a| \leqslant |a'| < 1, \quad \arg a = \arg a';$$

(iii) for some (equivalently for each) K > 0 there exists $C = C(\omega, K) > 0$ such that

$$\omega(S(a)) \leqslant C\omega\left(S\left(\frac{K+|a|}{K+1}e^{i\arg a}\right)\right), \quad a \in \mathbb{D} \setminus \{0\};$$

(iv) there exist $\eta = \eta(\omega) > 0$ and $C = C(\eta, \omega) > 0$ such that

$$\int_{\mathbb{D}} \frac{\omega(z)}{|1 - \overline{a}z|^{\eta}} \, \mathrm{d}A(z) \leqslant C \frac{\omega(S(a))}{(1 - |a|)^{\eta}}, \quad a \in \mathbb{D} \setminus \{0\}.$$

Observe that if the inequality in the case (ii) is satisfied for some $\beta > 0$, then it is certainly satisfied for every number larger than that because the quotient (1 - |a'|)/(1 - |a|) is at most one. A similar comment applies to the constant η appearing in the case (iv).

The following lemma gives an analogue of Lemma B(ii) for weights in $\check{\mathcal{D}}(\mathbb{D})$.

ANNALES DE L'INSTITUT FOURIER

LEMMA 2.1. — Let ω be a weight on \mathbb{D} . Then $\omega \in \check{\mathcal{D}}(\mathbb{D})$ if and only if there exist $K = K(\omega) > 1$ and $\beta_0 = \beta_0(\omega) > 0$ such that

(2.3)
$$\omega(S(a)) \ge \left(\frac{1-|a|}{1-|b|}\right)^{\beta} \omega(S(a) \setminus D(0,|b|)), \quad 1-\frac{1-|a|}{K} \le |b| < 1,$$

for all $0 < \beta \leq \beta_0$ and $a \in \mathbb{D} \setminus \{0\}$.

Proof. — First observe that $\omega \in \check{\mathcal{D}}(\mathbb{D})$ if and only if there exist $C = C(\omega) > 1$ and $K = K(\omega) > 1$ such that

(2.4)
$$\omega(S(a)) \ge C\omega(S(a) \setminus T_K(a)), \quad a \in \mathbb{D} \setminus \{0\}.$$

This is the characterization that we will use to prove the lemma.

The choice $b = 1 - \frac{1-|a|}{K}$ in (2.3) implies (2.4) with $C = K^{\beta}$, and therefore $\omega \in \check{\mathcal{D}}(\mathbb{D})$. To prove the converse implication, assume without loss of generality that $K \in \mathbb{N}$. Now divide $S(a) \setminus T_K(a)$ into K Carleson squares of equal size and apply (2.4) to each square to obtain $\omega(S(a)) \ge C^2 \omega(S(a) \setminus T_{K^2}(a))$ for all $a \in \mathbb{D} \setminus \{0\}$. Then divide $S(a) \setminus T_{K^2}(a)$ into K^2 squares and proceed. After $1 + K + K^2 + \cdots + K^{n-1}$ applications of (2.4) we obtain

(2.5)
$$\omega(S(a)) \ge C^n \omega(S(a) \setminus T_{K^n}(a)), \quad a \in \mathbb{D} \setminus \{0\}.$$

Now, for given $1 - \frac{1-|a|}{K} \leq |b| < 1$, pick up $n = n(a, b) \in \mathbb{N}$ such that

$$1 - \frac{1 - |a|}{K^n} \leqslant |b| < 1 - \frac{1 - |a|}{K^{n+1}} \quad \Longleftrightarrow \quad K^n \leqslant \frac{1 - |a|}{1 - |b|} < K^{n+1}.$$

Then (2.5) yields

$$\begin{split} \omega(S(a)) &\ge K^{n \log_{K} C} \omega\left(S(a) \setminus T_{K^{n}}(a)\right) \\ &> \left(\frac{1-|a|}{1-|b|}\right)^{\frac{n}{n+1}\log_{K} C} \omega\left(S(a) \setminus T_{\frac{1-|a|}{1-|b|}}(a)\right) \\ &\ge \left(\frac{1-|a|}{1-|b|}\right)^{\frac{1}{2}\log_{K} C} \omega\left(S(a) \setminus D(0,|b|)\right), \end{split}$$

which gives (2.3) for $\beta_0 = \frac{1}{2} \log_K C$.

For each $\epsilon \in (0, 1)$, a simple computation shows that the weight

$$W(r e^{i\theta}) = \begin{cases} \frac{1}{(1-r)^{1-\frac{\epsilon}{2}} |\theta|^{1-\frac{\epsilon}{2}}}, & \theta \neq 0\\ 1, & \theta = 0, \end{cases}$$

is a Bekollé–Bonami type weight such that $W(S(a)) \simeq (1-a)^{\epsilon}$ for all $a \in (0,1)$. This implies that $H^p \notin A^p_W$ for all 0 by the classical

TOME 74 (2024), FASCICULE 4

 \Box

Carleson embedding theorem. Let us compare this example with Theorem 1.4, which says, in particular, that $\tilde{\omega}$ is a weight whenever $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$. Therefore, despite of the fact that $\tilde{\omega}$ is a weight for all $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$, for each given $\varepsilon > 0$ there exist $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$ and a set $A \subset \mathbb{D}$, with |A| = 0 and $\overline{A} \cap \partial \mathbb{D} \neq \emptyset$, such that $\omega(S(a)) \asymp (1 - |a|)^{\epsilon}$, as $a \in A$ and $|a| \to 1^{-}$.

3. Carleson measures

Let X be a quasi-Banach space of analytic functions on \mathbb{D} . A positive Borel measure μ on \mathbb{D} is called a q-Carleson measure for X if the identity operator $I_d: X \to L^q_{\mu}$ is bounded. Moreover, if $I_d: X \to L^q_{\mu}$ is compact, then μ is a q-vanishing Carleson measure for X.

We begin with the boundedness of the Hörmander-type maximal function on L^p_{ω} when $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$.

PROPOSITION 3.1. — Let $0 and <math>0 < \alpha < \infty$ such that $p\alpha > 1$. Let $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$ and let μ be a positive Borel measure on \mathbb{D} . Then $\left[M_{\omega}\left(|\cdot|\frac{1}{\alpha}\right)\right]^{\alpha} : L^{p}_{\omega} \to L^{q}_{\mu}$ is bounded if and only if μ satisfies (1.4). Moreover,

$$\left\| \left[M_{\omega}\left(|\cdot|^{\frac{1}{\alpha}}\right) \right]^{\alpha} \right\|_{L^{p}_{\omega} \to L^{q}_{\mu}}^{q} \asymp \sup_{S} \frac{\mu\left(S\right)}{\left(\omega\left(S\right)\right)^{\frac{q}{p}}}.$$

Proposition 3.1 can be established by following the lines of the proof of [21, Theorem 3]. We omit the details of the argument. A similar result was obtained in [12, Theorem 1.1] under stronger hypotheses on ω .

Proof of Theorem 1.3. — We will show first that (i) implies (iii) under the weaker hypothesis $0 < p, q < \infty$ and $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$. To see this, for $a \in \mathbb{D}$ and $0 < p, \gamma < \infty$, consider the test functions

(3.1)
$$F_{a,p,\gamma}(z) = \left(\frac{1-|a|^2}{1-\overline{a}z}\right)^{\frac{1}{p}}, \quad z \in \mathbb{D}.$$

Pick up $\gamma = \gamma(p, \omega) > 0$ sufficiently large such that $\frac{\gamma}{p} > \eta$, where $\eta = \eta(\omega) > 0$ is that of Lemma B(iv). Then

$$\mu(S(a)) \lesssim \int_{S(a)} |F_{a,p}(z)|^q \,\mathrm{d}\mu(z) \leqslant \int_{\mathbb{D}} |F_{a,p}(z)|^q \,\mathrm{d}\mu(z)$$
$$\lesssim \|F_{a,p}\|_{A^p_{\omega}}^q \lesssim \omega(S(a))^{\frac{q}{p}}, \quad a \in \mathbb{D},$$

and thus μ satisfies (iii).

The statements (ii) and (iii) are equivalent by Proposition 3.1. Hence, to complete the proof, it suffices to show that (iii) implies μ is a q-Carleson

measure for A^p_{ω} . Since $\frac{\omega}{\nu} \in B_{p_0}(\nu)$ for some $p_0 > 1$ and $\nu \in \widehat{\mathcal{D}}$ by the hypothesis $\omega \in \mathcal{B}_{\infty}(\widehat{\mathcal{D}})$, for any Carleson square S and any non-negative $\varphi \in L^{p_0}(\omega)$, Hölder's inequality yields

$$\frac{1}{\nu(S)} \int_{S} \varphi \nu \, \mathrm{d}A \leqslant \frac{1}{\nu(S)} \left(\int_{S} \varphi^{p_{0}} \omega \, \mathrm{d}A \right)^{\frac{1}{p_{0}}} \left(\int_{S} \left(\frac{\nu}{\omega^{\frac{1}{p_{0}}}} \right)^{p_{0}'} \, \mathrm{d}A \right)^{\frac{1}{p_{0}'}}$$
$$\lesssim \left(\frac{1}{\omega(S)} \int_{S} \varphi^{p_{0}} \omega \, \mathrm{d}A \right)^{\frac{1}{p_{0}}} \, .$$

It follows that $(M_{\nu}(\varphi))^{p_0} \leq M_{\omega}(\varphi^{p_0})$ on \mathbb{D} . This together with [18, Lemma 3.2] shows that for each s > 0 there exists a constant $C = C(s, \omega) > 0$ such that

(3.2)
$$|f(z)|^s \leq CM_{\omega}(f^s)(z), \quad z \in \mathbb{D}, \quad f \in \mathcal{H}(\mathbb{D}).$$

By choosing $s = \frac{1}{\alpha} < p$, and using the equivalence between (ii) and (iii) we deduce

$$\|f\|_{L^q_{\mu}}^q \lesssim \int_{\mathbb{D}} \left(M_{\omega}(f^{\frac{1}{\alpha}})(z) \right)^{q\alpha} \mathrm{d}\mu(z) \leqslant \|[M_{\omega}((\cdot)^{\frac{1}{\alpha}})]^{\alpha}\|_{L^p_{\omega} \to L^q_{\mu}}^q \|f\|_{A^p_{\omega}}^q.$$

To finish the proof of the theorem we observe that (1.5) follows from the arguments above.

For the sake of completeness we describe the q-vanishing Carleson measures for A^p_{ω} .

THEOREM 3.2. — Let $0 and <math>\omega \in \mathcal{B}_{\infty}(\widehat{\mathcal{D}})$, and let μ be a positive Borel measure on \mathbb{D} . Then $I_d : A^p_{\omega} \to L^q_{\mu}$ is compact if and only if

(3.3)
$$\lim_{|S|\to 0} \frac{\mu(S)}{\left(\omega(S)\right)^{\frac{q}{p}}} = 0.$$

Proof. — Let $0 and <math>\omega \in \mathcal{B}_{\infty}(\widehat{\mathcal{D}})$, and first that assume that $I_d : A^p_{\omega} \to L^q_{\mu}$ is compact. For each $a \in \mathbb{D}$, consider the function

(3.4)
$$f_{a,p,\gamma}(z) = F_{a,p,\gamma}(z)\omega\left(S(a)\right)^{-\frac{1}{p}}, \quad z \in \mathbb{D}$$

where $F_{a,p,\gamma}$ is the function defined in (3.1). Then by repeating the argument of [20, Theorem 2.1(ii)] and using Lemma B, we deduce

$$\lim_{|a| \to 1^{-}} \frac{\mu(S(a))}{(\omega(S(a)))^{\frac{q}{p}}} = 0,$$

and thus (3.3) is satisfied.

Conversely, assume that μ satisfies (3.3), and set

$$d\mu_r(z) = \chi_{\{r \leq |z| < 1\}}(z) d\mu(z), \quad z \in \mathbb{D}.$$

Then Theorem 1.3 implies

 $\|h\|_{L^q_{\mu_r}} \lesssim K_{\mu_r} \|h\|_{A^p_{\omega}}, \quad h \in A^p_{\omega},$

where $K_{\mu_r} = \sup_{a \in \mathbb{D} \setminus \{0\}} \frac{\mu_r(S(a))}{(\omega(S(a)))^{\frac{q}{p}}}$. We will prove next that

(3.5)
$$\lim_{r \to 1^-} K_{\mu_r} = 0,$$

and then the rest of the proof follows as that of [20, Theorem 2.1(ii)]. By the assumption, for a given $\varepsilon > 0$, there exists $r_0 \in (0, 1)$ such that

(3.6)
$$\sup_{a\in\mathbb{D}:\,|a|\geqslant r_0}\frac{\mu\left(S(a)\right)}{\left(\omega\left(S(a)\right)\right)^{\frac{q}{p}}}<\varepsilon.$$

Therefore for each $r \in (0, 1)$, we have

(3.7)
$$\sup_{a \in \mathbb{D}: |a| \ge r_0} \frac{\mu_r(S(a))}{(\omega(S(a)))^{\frac{q}{p}}} \le \sup_{a \in \mathbb{D}: |a| \ge r_0} \frac{\mu(S(a))}{(\omega(S(a)))^{\frac{q}{p}}} < \varepsilon$$

Next, if $|a| < r_0$, we choose $n \in \mathbb{N} \setminus \{1\}$ such that $(n-1)(1-r_0) < |I_a| \leq n(1-r_0)$. Let I_k be arcs on the boundary such that $|I_k| = 1 - r_0$ for all $k = 1, \ldots n$, and $I_a \subset \bigcup_{k=1}^n I_k \subset 2I_a$, where

$$2I_a = \left\{ e^{i\theta} : |\arg(a e^{-i\theta})| \leq (1 - |a|) \right\},\$$

where I_j and I_m , $j \neq m$, $j,m \in \{1,2,\ldots,n\}$ are disjoint or share an endpoint. Let $r \geq r_0$. Then, since $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$ by the hypothesis, (3.6) yields

$$\mu_r(S(a)) \leqslant \mu_{r_0}(S(a)) \leqslant \sum_{k=1}^n \mu(S(I_k)) \leqslant \varepsilon \sum_{k=1}^n \left(\omega(S(I_k))\right)^{\frac{q}{p}}$$
$$\leqslant \varepsilon \left(\sum_{k=1}^n \omega(S(I_k))\right)^{\frac{q}{p}} \leqslant \varepsilon \omega(S(2I_a))^{\frac{q}{p}} \lesssim \varepsilon \omega(S(a))^{\frac{q}{p}}$$

This together with (3.7) gives (3.5), and finishes the proof.

4. Littlewood–Paley inequalities

We begin with Theorem 1.4, splitting its proof in two parts. We first establish an equivalent norm to $\|\cdot\|_{A^p_{\omega}}$ and a Littlewood–Paley inequality in terms of the average weight

$$\omega_{h,r}(z) = \frac{\int_{\Delta(z,r)} \omega(\zeta) \, \mathrm{d}A(\zeta)}{(1-|z|)^2}, \quad z \in \mathbb{D},$$

where $r \in (0, 1)$, and $\Delta(z, r) = \{u \in \mathbb{D} : \rho(u, z) < r\}.$

ANNALES DE L'INSTITUT FOURIER

PROPOSITION 4.1. — Let $\omega \in \mathcal{B}_{\infty}(\widehat{\mathcal{D}})$, 0 < r < 1 and 0 . Then the following statements hold:

(i)
$$||f||_{A^{p}_{\omega}} \simeq ||f||_{A^{p}_{\omega_{h,r}}}$$
 for all $f \in \mathcal{H}(\mathbb{D})$;
(ii) $\int_{\mathbb{D}} |f^{(k)}(z)|^{p} (1-|z|)^{kp} \omega_{h,r}(z) \, \mathrm{d}A(z) + \sum_{j=0}^{k-1} |f^{(j)}(0)|^{p} \lesssim ||f||_{A^{p}_{\omega}}^{p}$ for all $f \in \mathcal{H}(\mathbb{D})$.

Proof.

(i). — Let 0 < r < 1 be fixed. Then Fubini's theorem yields

$$\begin{split} \int_{S(a)} \frac{\omega(\Delta(\zeta, r))}{(1 - |\zeta|)^2} \, \mathrm{d}A(\zeta) \\ &= \int_{\{z \in \mathbb{D}: S(a) \cap \Delta(z, r) \neq \emptyset\}} \left(\int_{S(a) \cap \Delta(z, r)} \frac{\mathrm{d}A(\zeta)}{(1 - |\zeta|)^2} \right) \omega(z) \, \mathrm{d}A(z) \\ &\leqslant \int_{S(b)} \left(\int_{\Delta(z, r)} \frac{\mathrm{d}A(\zeta)}{(1 - |\zeta|)^2} \right) \omega(z) \, \mathrm{d}A(z) \asymp \omega(S(b)), \quad |a| > r, \end{split}$$

where $b = b(a, r) \in \mathbb{D}$ satisfies $\arg b = \arg a$ and $1 - |b| \approx 1 - |a|$ for all $a \in \mathbb{D} \setminus \overline{D(0, r)}$. Since $\mathcal{B}_{\infty}(\widehat{\mathcal{D}}) \subset \widehat{\mathcal{D}}(\mathbb{D}), \, \omega(S(b)) \lesssim \omega(S(a))$ by Lemma B(ii), and therefore Theorem 1.3 yields

(4.1)
$$\|f\|_{A^p_{\omega_{h,r}}}^p \lesssim \|f\|_{A^p_{\omega}}^p, \quad f \in \mathcal{H}(\mathbb{D}).$$

To see the converse inequality, use the subharmonicity of $|f|^p$ and Fubini's theorem to deduce

$$\|f\|_{A^p_{\omega}}^p \lesssim \int_{\mathbb{D}} \omega(\zeta) \left(\int_{\Delta(\zeta,r)} \frac{|f(z)|^p}{(1-|z|)^2} \, \mathrm{d}A(z) \right) \mathrm{d}A(\zeta) = \|f\|_{A^p_{\omega_{h,r}}}^p, \quad f \in \mathcal{H}(\mathbb{D}).$$

Thus (i) is proved.

(ii). — Let 0 < r < 1 be fixed. It is well known that, for each $0 , <math>k \in \mathbb{N}$ and 0 < s < 1, we have

(4.2)
$$|f^{(k)}(z)|^p \lesssim \frac{1}{(1-|z|)^{2+kp}} \int_{\Delta(z,s)} |f(\zeta)|^p \, \mathrm{d}A(\zeta), \ z \in \mathbb{D}, \ f \in \mathcal{H}(\mathbb{D}),$$

see, for example, [15, Lemma 2.1] for details. Fix now $s = s(r) \in (0, 1)$ and $R = R(r) \in (r, 1)$ such that $\Delta(z, r) \subset \Delta(\zeta, R)$ for all $z \in \Delta(\zeta, s)$. Then an

application of (4.2), Fubini's theorem and Part (i) give

$$\begin{split} \int_{\mathbb{D}} |f^{(k)}(z)|^p (1-|z|)^{kp} \omega_{h,r}(z) \, \mathrm{d}A(z) \\ &\lesssim \int_{\mathbb{D}} \left(\int_{\Delta(z,s)} \frac{|f(\zeta)|^p}{(1-|\zeta|)^2} \, \mathrm{d}A(\zeta) \right) \omega_{h,r}(z) \, \mathrm{d}A(z) \\ &= \int_{\mathbb{D}} \frac{|f(\zeta)|^p}{(1-|\zeta|)^2} \left(\int_{\Delta(\zeta,s)} \omega_{h,r}(z) \, \mathrm{d}A(z) \right) \, \mathrm{d}A(\zeta) \\ &\lesssim \|f\|_{A^p_{\omega_{h,R}}}^p \asymp \|f\|_{A^p_{\omega}}^p, \quad f \in \mathcal{H}(\mathbb{D}). \end{split}$$

Moreover, for each $j \in \mathbb{N}$, $|f^{(j)}(0)|^p \lesssim \int_{D(0,\frac{1}{2})} |f|^p \, \mathrm{d}A$ by the subharmonicity of $|f|^p$, and therefore Theorem 1.3 implies $|f^{(j)}(0)|^p \lesssim ||f||^p_{A^p_\omega}$ once we show that $\int_{S \cap D(0,\frac{1}{2})} \mathrm{d}A \lesssim \omega(S)$ for all Carleson squares S. This last inequality is obviously valid if S = S(a) with $|a| \ge \frac{1}{2}$ because in this case the left hand side equals zero. For $|a| \le \frac{1}{2}$ we have

$$\int_{S(a)\cap D(0,\frac{1}{2})}\,\mathrm{d} A\leqslant \frac{1}{8}\leqslant \frac{1}{8}\frac{\omega(S(a))}{\inf_{a\in\overline{D(0,\frac{1}{2})}}\omega(S(a))}\lesssim \omega(S(a)).$$

 \square

This finishes the proof.

Given a weight ω and $\beta \in \mathbb{R}$, we denote $\omega_{[\beta]}(z) = (1 - |z|)^{\beta} \omega(z)$ for all $z \in \mathbb{D}$. We will use this definition to shorten the notation in several instances in the proofs from here after.

PROPOSITION 4.2. — Let $k \in \mathbb{N}$. Then the following statements hold:

(i) If $0 and <math>\omega \in \mathcal{B}_{\infty}(\widehat{\mathcal{D}})$, then

$$\|f\|_{A^{p}_{\omega}}^{p} \lesssim \int_{\mathbb{D}} |f^{(k)}(z)|^{p} (1-|z|)^{kp} \widetilde{\omega}(z) \, \mathrm{d}A(z) + \sum_{j=0}^{k-1} |f^{(j)}(0)|^{p}, \quad f \in \mathcal{H}(\mathbb{D}).$$

(ii) If
$$1 and $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$, then$$

$$\|f\|_{A^{p}_{\omega}}^{p} \lesssim \int_{\mathbb{D}} |f^{(k)}(z)|^{p} (1-|z|)^{kp} \widetilde{\omega}(z) \, \mathrm{d}A(z) + \sum_{j=0}^{k-1} |f^{(j)}(0)|^{p}, \quad f \in \mathcal{H}(\mathbb{D}).$$

Proof.

(i). — Let $0 . First observe that <math>\omega \in \mathcal{B}_{\infty}(\widehat{\mathcal{D}}) \subset \widehat{\mathcal{D}}(\mathbb{D})$, and hence, by Lemma B(ii) and Theorem 1.3, there exists $\beta_0 = \beta_0(\omega, p) > 0$ such that A^p_{ω} is continuously embedded into $A^1_{\beta-1}$ for all $\beta \geq \beta_0$. A wellknown reproducing formula for functions in $A^1_{\beta-1}$ [29, Proposition 4.27] now guarantees the estimate

.

(4.3)
$$\left| f(z) - \sum_{j=0}^{k-1} f^{(j)}(0) \right| \lesssim \int_{\mathbb{D}} \left| \frac{f^{(k)}(\zeta)}{(1 - \overline{z}\zeta)^{1+\beta}} \right| (1 - |\zeta|)^{\beta+k-1} \, \mathrm{d}A(\zeta), \quad z \in \mathbb{D}.$$

Fix $\beta \ge \beta_0$ sufficiently large such that $p(1+\beta) \ge \eta$, where $\eta = \eta(\omega) > 0$ is that of Lemma B(iv), and $\alpha = p(\beta + k + 1) - 2 > -1$. Then $A^p_{\alpha} \subset A^1_{\beta+k-1}$ by a well-known embedding that can be also deduced from Theorem 1.3, and hence

$$\left| f(z) - \sum_{j=0}^{k-1} f^{(j)}(0) \right|^p \lesssim \int_{\mathbb{D}} \left| f^{(k)}(\zeta) \right|^p \frac{(1-|\zeta|)^{\alpha}}{|1-\overline{z}\zeta|^{p(1+\beta)}} \,\mathrm{d}A(\zeta), \quad z \in \mathbb{D}.$$

Therefore Fubini's theorem and Lemma B(iv) yield

$$(4.4) \quad \left\| f - \sum_{j=0}^{k-1} f^{(j)}(0) \right\|_{A^p_{\omega}}^p$$

$$\lesssim \int_{\mathbb{D}} \left| f^{(k)}(\zeta) \right|^p (1 - |\zeta|)^{\alpha} \left(\int_{\mathbb{D}} \frac{\omega(z)}{|1 - \overline{z}\zeta|^{p(1+\beta)}} \, \mathrm{d}A(z) \right) \, \mathrm{d}A(\zeta)$$

$$\lesssim \int_{\mathbb{D}} \left| f^{(k)}(\zeta) \right|^p (1 - |\zeta|)^{kp} \frac{\omega(S(\zeta))}{(1 - |\zeta|)^2} \, \mathrm{d}A(\zeta), \quad f \in \mathcal{H}(\mathbb{D}).$$

Thus (i) is proved.

(ii). — Let now $1 . Observe that <math>\omega \in \mathcal{B}(\mathcal{D}) \subset \mathcal{D}(\mathbb{D})$. We begin with showing that for each $\omega \in \mathcal{D}(\mathbb{D})$ there exists $\varepsilon_0 > 0$ such that $\omega_{[-\varepsilon]} \in \mathcal{D}(\mathbb{D})$ for all $0 < \varepsilon < \varepsilon_0$. To see this first note that by Lemma 2.1 there exists $\beta = \beta(\omega) > 0$ such that $\omega(\mathbb{D} \setminus D(0,r)) \leq (1-r)^{\beta}$ for all $0 \leq r < 1$. This and Fubini's theorem yield

$$\begin{split} \int_{\mathbb{D}\setminus D(0,\frac{1}{2})} \frac{\omega(z)}{(1-|z|)^{\varepsilon}} \, \mathrm{d}A(z) \\ & \asymp \int_{\mathbb{D}\setminus D(0,\frac{1}{2})} \omega(z) \left(\int_{D(0,|z|)} \frac{\mathrm{d}A(\zeta)}{(1-|\zeta|)^{1+\varepsilon}} \right) \, \mathrm{d}A(z) \\ & = \int_{\mathbb{D}} \frac{1}{(1-|\zeta|)^{1+\varepsilon}} \left(\int_{\mathbb{D}\setminus D\left(0,\max\left\{\frac{1}{2},|\zeta|\right\}\right)} \omega(z) \, \mathrm{d}A(z) \right) \, \mathrm{d}A(\zeta) \\ & \lesssim \int_{\mathbb{D}} \frac{\mathrm{d}A(\zeta)}{(1-|\zeta|)^{1+\varepsilon-\beta}} < \infty \end{split}$$

TOME 74 (2024), FASCICULE 4

for each $\varepsilon < \beta$. Thus $\omega_{[-\varepsilon]}$ is a weight for all $0 < \varepsilon < \beta$. Further, for such an ε , the inequality (2.3) and the hypothesis $\omega \in \check{\mathcal{D}}(\mathbb{D})$ yield

$$\begin{split} \int_{S(a)} \frac{\omega(z)}{(1-|z|)^{\varepsilon}} \, \mathrm{d}A(z) \\ &= \int_{S(a)} \omega(z) \left(\varepsilon \int_{0}^{|a|} \frac{\mathrm{d}t}{(1-t)^{\varepsilon+1}} + \varepsilon \int_{|a|}^{|z|} \frac{\mathrm{d}t}{(1-t)^{\varepsilon+1}} + 1 \right) \, \mathrm{d}A(z) \\ &= \frac{\omega(S(a))}{(1-|a|)^{\varepsilon}} + \varepsilon \int_{|a|}^{1} \left(\int_{S(a) \setminus D(0,t)} \omega(z) \, \mathrm{d}A(z) \right) \frac{\mathrm{d}t}{(1-t)^{\varepsilon+1}} \\ &\lesssim \frac{\omega(S(a))}{(1-|a|)^{\varepsilon}} + \frac{\omega(S(a))}{(1-|a|)^{\beta}} \varepsilon \int_{|a|}^{1} \frac{\mathrm{d}t}{(1-t)^{\varepsilon+1-\beta}} \\ &= \frac{\omega(S(a))}{(1-|a|)^{\varepsilon}} + \frac{\omega(S(a))}{(1-|a|)^{\varepsilon}} \frac{\varepsilon}{\beta-\varepsilon} \\ &\lesssim \frac{\omega(T_{K}(a))}{(1-|a|)^{\varepsilon}} \leqslant \int_{T_{K}(a)} \frac{\omega(z)}{(1-|z|)^{\varepsilon}} \, \mathrm{d}A(z), \quad a \in \mathbb{D} \setminus \{0\}, \end{split}$$

and thus $\omega_{[-\varepsilon]} \in \check{\mathcal{D}}(\mathbb{D})$, provided $\omega \in \check{\mathcal{D}}(\mathbb{D})$ and $0 < \varepsilon < \beta$. Further, since $\omega \in \check{\mathcal{D}}(\mathbb{D})$, there exists $K = K(\omega, \varepsilon) > 1$ such that $\omega_{[-\varepsilon]}(S(a)) \lesssim \omega_{[-\varepsilon]}(T_K(a))$ and $\omega(S(a)) \lesssim \omega(T_K(a))$ for all $a \in \mathbb{D} \setminus \{0\}$. Write $a' = \frac{1+|a|}{2} e^{i \arg a}$ for short. Then the inequalities just obtained and the hypothesis $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$ yield

(4.5)

$$\omega_{[-\varepsilon]}(S(a)) \lesssim \omega_{[-\varepsilon]}(T_K(a)) \leqslant \frac{K^{\varepsilon}\omega(T_K(a))}{(1-|a|)^{\varepsilon}} \leqslant \frac{K^{\varepsilon}\omega(S(a))}{(1-|a|)^{\varepsilon}}$$

$$\lesssim \frac{\omega(S(a'))}{(1-|a'|)^{\varepsilon}} \lesssim \frac{\omega(T_K(a'))}{(1-|a'|)^{\varepsilon}} \leqslant \omega_{[-\varepsilon]}(T_K(a'))$$

$$\leqslant \omega_{[-\varepsilon]}(S(a')), \quad a \in \mathbb{D} \setminus \{0\},$$

and hence $\omega_{[-\varepsilon]} \in \widehat{\mathcal{D}}(\mathbb{D})$, for all $0 < \varepsilon < \beta$. Therefore we have shown that $\omega_{[-\varepsilon]} \in \mathcal{D}(\mathbb{D})$ for all $0 < \varepsilon < \varepsilon_0 = \beta$.

To prove the statement (ii) of the proposition, fix $\alpha_2 \in \left(\frac{2}{p'}, \frac{2}{p'} + \frac{\varepsilon_0}{p}\right)$, where $\varepsilon_0 = \varepsilon_0(\omega) > 0$ is the constant we just found. Then $\varepsilon = \frac{p}{p'}(p'\alpha_2 - 2) \in (0, \varepsilon_0)$, and thus $\omega_{[-\varepsilon]} \in \widehat{\mathcal{D}}(\mathbb{D})$. Let now $\beta > \max\{\beta_0, (\eta + 2(p-1) + \varepsilon_0)/p - 1\}$, where $\eta = \eta(\omega_{[-\varepsilon]}) > 0$ is that of Lemma B(iv) and $\beta_0 = \beta_0(\omega, 1)$ is that of Part (i). Finally, write $1 + \beta = \alpha_1 + \alpha_2$. Then the estimate (4.3) and Hölder's inequality imply

$$\begin{split} \left| f(z) - \sum_{j=0}^{k-1} f^{(j)}(0) \right|^p \\ \lesssim \int_{\mathbb{D}} |f^{(k)}(\zeta)|^p \frac{(1-|\zeta|)^{p(\beta+k-1)}}{|1-\overline{z}\zeta|^{p\alpha_1}} \, \mathrm{d}A(\zeta) \left(\int_{\mathbb{D}} \frac{\mathrm{d}A(\zeta)}{|1-\overline{z}\zeta|^{p'\alpha_2}} \right)^{\frac{p}{p'}} \\ \lesssim \int_{\mathbb{D}} |f^{(k)}(\zeta)|^p \frac{(1-|\zeta|)^{p(\beta+k-1)}}{|1-\overline{z}\zeta|^{p\alpha_1}} \, \mathrm{d}A(\zeta)(1-|z|)^{(2-p'\alpha_2)\frac{p}{p'}}, \end{split}$$

because $p'\alpha_2 > 2$. By using this and Fubini's theorem we deduce

$$\left\| f - \sum_{j=0}^{k-1} f^{(j)}(0) \right\|_{A^p_{\omega}}^p \lesssim \int_{\mathbb{D}} |f^{(k)}(\zeta)|^p (1 - |\zeta|)^{p(\beta+k-1)} \\ \cdot \left(\int_{\mathbb{D}} \frac{\omega(z)(1 - |z|)^{(2-p'\alpha_2)\frac{p}{p'}}}{|1 - \overline{z}\zeta|^{p\alpha_1}} \, \mathrm{d}A(z) \right) \mathrm{d}A(\zeta)$$

for all $f \in \mathcal{H}(\mathbb{D})$. Since $\varepsilon = \frac{p}{p'}(p'\alpha_2 - 2) \in (0, \varepsilon_0)$ and $p\alpha_1 > \eta$ by our choices, we may apply Lemma B(iv) to the inner integral above. This together with (4.5) imply

$$\begin{split} \left\| f - \sum_{j=0}^{k-1} f^{(j)}(0) \right\|_{A_{\omega}^{p}}^{p} \lesssim \int_{\mathbb{D}} |f^{(k)}(\zeta)|^{p} (1 - |\zeta|)^{p(\beta+k-1)} \\ \cdot \left(\int_{\mathbb{D}} \frac{\omega(z)(1 - |z|)^{(2-p'\alpha_{2})\frac{p}{p'}}}{|1 - \overline{z}\zeta|^{p\alpha_{1}}} \, \mathrm{d}A(z) \right) \mathrm{d}A(\zeta) \\ \lesssim \int_{\mathbb{D}} |f^{(k)}(\zeta)|^{p} (1 - |\zeta|)^{p(\beta+k-1)} \frac{\omega_{[-\varepsilon]}(S(\zeta))}{(1 - |\zeta|)^{p\alpha_{1}}} \, \mathrm{d}A(\zeta) \\ \asymp \int_{\mathbb{D}} |f^{(k)}(\zeta)|^{p} (1 - |\zeta|)^{kp} \frac{\omega(S(\zeta))}{(1 - |\zeta|)^{2}} \, \mathrm{d}A(\zeta), \quad f \in \mathcal{H}(\mathbb{D}), \end{split}$$

and finishes the proof of (ii).

With these preparations we can deduce Theorem 1.4. Namely, it is easy to see that for each $\omega \in \mathcal{D}(\mathbb{D})$, and in particular for each $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$, there exists $r_0 = r_0(\omega) \in (0, 1)$ such that $\omega_{h,r} \simeq \tilde{\omega}$ in \mathbb{D} for each $r \ge r_0$. Therefore Theorem 1.4 follows from Propositions 4.1 and 4.2.

We proceed to prove Theorems 1.2 and 1.1 in the said order.

 \square

Proof of Theorem 1.2. — Let 0 < r < 1 be fixed. The inequality (4.2), Fubini's theorem and Proposition 4.1(i) yield

$$\begin{split} \int_{\mathbb{D}} |f^{(k)}(z)|^{p} (1-|z|)^{kp} \omega(z) \, \mathrm{d}A(z) \\ &\lesssim \int_{\mathbb{D}} \left(\int_{\Delta(z,r)} |f(\zeta)|^{p} \, \mathrm{d}A(\zeta) \right) \frac{\omega(z)}{(1-|z|)^{2}} \, \mathrm{d}A(z) \\ &\asymp \int_{\mathbb{D}} |f(\zeta)|^{p} \frac{\omega(\Delta(\zeta,r))}{(1-|\zeta|)^{2}} \, \mathrm{d}A(\zeta) \\ &= \|f\|_{A^{p}_{\omega_{h,r}}}^{p} \asymp \|f\|_{A^{p}_{\omega}}^{p}, \quad f \in \mathcal{H}(\mathbb{D}). \end{split}$$

Moreover, arguing as in the proof of Proposition 4.1(ii) we deduce

$$\sum_{j=0}^{k-1} |f^{(j)}(0)|^p \lesssim ||f||_{A^p_{\omega}}^p, \quad f \in \mathcal{H}(\mathbb{D}).$$

By combining the above estimates we get the assertion.

Proof of Theorem 1.1. — By Theorem 1.2 it suffices to prove

$$\int_{\mathbb{D}} |f^{(k)}(z)|^p (1-|z|)^{kp} \omega(z) \, \mathrm{d}A(z) + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p \gtrsim ||f||_{A^p_{\omega}}^p, \quad f \in \mathcal{H}(\mathbb{D}).$$

To see this, we will need to know more about the weights involved. In particular, we want to show that $\omega_{[kp]} \in B_{\infty}(\mathcal{D})$ for each $k \in \mathbb{N}$ and 0 . We will deduce this in several steps. First observe that if $<math>\omega \in \check{\mathcal{D}}(\mathbb{D})$ and $\beta > 0$, then $\omega_{[\beta]} \in \check{\mathcal{D}}(\mathbb{D})$. Namely, if $\omega \in \check{\mathcal{D}}(\mathbb{D})$ there exists $K = K(\omega) > 1$ such that for each $\beta > 0$ we have

$$\omega_{[\beta]}(T_K(a)) \ge \frac{(1-|a|)^{\beta}}{K^{\beta}} \omega(T_K(a)) \gtrsim (1-|a|)^{\beta} \omega(S(a))$$
$$\ge \omega_{[\beta]}(S(a)), \quad a \in \mathbb{D} \setminus \{0\},$$

and hence $\omega_{[\beta]} \in \check{\mathcal{D}}(\mathbb{D})$. If in addition $\omega \in \mathcal{D}(\mathbb{D})$, then $\omega_{[\beta]} \in \widehat{\mathcal{D}}(\mathbb{D})$. To see this, let $a \in \mathbb{D} \setminus \{0\}$ and write $a' = \frac{1+|a|}{2} e^{i \arg a}$ for short. Since $\omega \in \check{\mathcal{D}}(\mathbb{D})$, there exists $K = K(\omega) > 1$ such that $\omega(T_K(a)) \asymp \omega(S(a))$ for all $a \in \mathbb{D} \setminus \{0\}$. By using this and the hypothesis $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$ we deduce

$$\omega_{[\beta]}(S(a)) \leq (1 - |a|)^{\beta} \omega(S(a)) \asymp (1 - |a|)^{\beta} \omega(S(a'))$$
$$\asymp \left(1 - \left(1 - \frac{1 - |a'|}{K}\right)\right)^{\beta} \omega(T_K(a'))$$
$$\leq \omega_{[\beta]}(S(a')), \quad a \in \mathbb{D} \setminus \{0\}.$$

ANNALES DE L'INSTITUT FOURIER

1738

 \Box

and thus $\omega_{[\beta]} \in \widehat{\mathcal{D}}(\mathbb{D})$. Therefore we have shown that $\omega_{[\beta]} \in \mathcal{D}(\mathbb{D})$, provided $\omega \in \mathcal{D}(\mathbb{D})$ and $\beta > 0$.

The other property we need to know is that if $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$, then $\omega_{[\beta]} \in \mathcal{B}_{\infty}(\mathcal{D})$ for all $\beta > 0$. We will use the fact we just proved to see this and the fact that $\frac{\omega}{\nu} \in B_p(\nu)$ if and only if there exists a constant $C = C(\omega, \nu) > 0$ such that

(4.6)
$$\left(\frac{\int_{S} |f(z)|\nu(z) \, \mathrm{d}A(z)}{\nu(S)}\right)^{p} \leqslant C \frac{\int_{S} |f(z)|^{p} \omega(z) \, \mathrm{d}A(z)}{\omega(S)}$$

for all Carleson squares S and all measurable functions f on \mathbb{D} . Next observe that if $\nu \in \mathcal{D}(\mathbb{D})$, $1 and <math>\frac{\omega}{\nu} \in B_p(\nu)$, then $\frac{\omega_{[\beta]}}{\nu_{[\beta]}} \in B_p(\nu_{[\beta]})$ for all $0 < \beta < \infty$. Namely, if $\frac{\omega}{\nu} \in B_p(\nu)$, then (4.6) yields

$$\begin{split} \left(\frac{\int_{S(a)} |f(z)|\nu_{[\beta]}(z) \, \mathrm{d}A(z)}{\nu_{[\beta]}(S(a))}\right)^p \\ &\leqslant C \frac{\nu(S(a))^p}{\nu_{[\beta]}(S(a))^p} \frac{\int_{S(a)} |f(z)|^p (1-|z|)^{p\beta} \omega(z) \, \mathrm{d}A(z)}{\omega(S(a))} \\ &\leqslant C \frac{\nu(S(a))^p}{\nu_{[\beta]}(S(a))^p} (1-|a|)^{(p-1)\beta} \frac{\int_{S(a)} |f(z)|^p \omega_{[\beta]}(z) \, \mathrm{d}A(z)}{\omega(T_K(a))} \\ &\leqslant C \frac{(1-|a|)^{p\beta} \nu(S(a))^p}{\nu_{[\beta]}(S(a))^p} \frac{\int_{S(a)} |f(z)|^p \omega_{[\beta]}(z) \, \mathrm{d}A(z)}{\omega_{[\beta]}(T_K(a))} \end{split}$$

for all $a \in \mathbb{D} \setminus \{0\}$ and all measurable functions f on \mathbb{D} . Since $\nu \in \mathcal{D}(\mathbb{D})$ by the hypothesis, then $\omega \in \mathcal{D}(\mathbb{D})$, and hence $\omega_{[\beta]} \in \mathcal{D}(\mathbb{D})$. Therefore $\omega_{[\beta]}(T_K(a)) \asymp \omega_{[\beta]}(S(a))$ for all $a \in \mathbb{D} \setminus \{0\}$, provided $K = K(\omega, \beta) > 1$ is sufficiently large. Moreover, for $M = M(\nu) > 1$ sufficiently large, we have

$$\nu_{[\beta]}(S(a)) \leqslant (1-|a|)^{\beta} \nu(S(a)) \asymp (1-|a|)^{\beta} \nu(T_M(a))$$
$$\leqslant M^{\beta} \nu_{[\beta]}(T_M(a)) \leqslant M^{\beta} \nu_{[\beta]}(S(a)), \quad a \in \mathbb{D} \setminus \{0\}.$$

It follows that

$$\left(\frac{\int_{S(a)} |f(z)|\nu_{[\beta]}(z) \,\mathrm{d}A(z)}{\nu_{[\beta]}(S(a))}\right)^p \lesssim \frac{\int_{S(a)} |f(z)|^p \omega_{[\beta]}(z) \,\mathrm{d}A(z)}{\omega_{[\beta]}(S(a))}, \quad a \in \mathbb{D} \setminus \{0\},$$

for all measurable functions f on \mathbb{D} . This shows that $\frac{\omega_{[\beta]}}{\nu_{[\beta]}} \in B_p(\nu_{[\beta]})$.

Finally, by the hypothesis $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$, and hence there exist 1 $and <math>\nu \in \mathcal{D}$ such that $\frac{\omega}{\nu} \in B_p(\nu)$. Therefore $\frac{\omega_{[\beta]}}{\nu_{[\beta]}} \in B_p(\nu_{[\beta]})$ for all $0 < \beta < \infty$. Moreover, $\nu_{[\beta]} \in \mathcal{D}$, and hence $\omega_{[\beta]} \in \mathcal{B}_{\infty}(\mathcal{D})$.

Now we can proceed to prove the statement of the theorem. Recall that we just showed that $\omega_{[kp]} \in B_{\infty}(\mathcal{D})$ for each $k \in \mathbb{N}$ and 0 by the hypothesis $\omega \in B_{\infty}(\mathcal{D})$. In particular $\omega_{[kp]} \in \mathcal{D}(\mathbb{D})$, and hence the inequality we are after now follows from Proposition 4.2 and Theorem 1.3 if we show that

(4.7)
$$\int_{S} (1-|\zeta|)^{kp} \frac{\omega(S(\zeta))}{(1-|\zeta|)^2} \,\mathrm{d}A(\zeta) \lesssim \int_{S} (1-|\zeta|)^{kp} \omega(\zeta) \,\mathrm{d}A(\zeta), \quad S \subset \mathbb{D}.$$

Since $\omega \in \check{\mathcal{D}}(\mathbb{D})$, there exists $r = r(\omega) \in (0, 1)$ sufficiently large such that $\omega(S(\zeta)) \leq \omega(\Delta(\zeta, r))$ for all $\zeta \in \mathbb{D}$. Therefore

$$\begin{split} &\int_{S(a)} (1-|\zeta|)^{kp} \frac{\omega(S(\zeta))}{(1-|\zeta|)^2} \,\mathrm{d}A(\zeta) \\ &\lesssim \int_{S(a)} (1-|\zeta|)^{kp} \frac{\omega(\Delta(\zeta,r))}{(1-|\zeta|)^2} \,\mathrm{d}A(\zeta) \\ &= \int_{\{z:S(a)\cap\Delta(z,r)\neq\emptyset\}} \omega(z) \left(\int_{S(a)\cap\Delta(z,r)} (1-|\zeta|)^{kp-2} \,\mathrm{d}A(\zeta) \right) \,\mathrm{d}A(z) \\ &\lesssim \int_{\{z:S(a)\cap\Delta(z,r)\neq\emptyset\}} (1-|z|)^{kp} \omega(z) \,\mathrm{d}A(z) \\ &\leqslant \int_{S(a')} (1-|z|)^{kp} \omega(z) \,\mathrm{d}A(z), \quad a \in \mathbb{D} \setminus \{0\}, \end{split}$$

where $\arg a' = \arg a$ and $1 - |a'| \simeq 1 - |a|$ for all $a \in \mathbb{D} \setminus \{0\}$. Moreover, since $\omega_{[kp]} \in \widehat{\mathcal{D}}(\mathbb{D})$, we have $\omega_{[kp]}(S(a')) \lesssim \omega_{[kp]}(S(a))$ for all $a \in \mathbb{D} \setminus \{0\}$. This gives (4.7) and finishes the proof.

5. Spectra of integration operator

Let \mathcal{B} denote the classical space of Bloch functions, \mathcal{B}_0 the little Bloch space and $D_a = \{z \in \mathbb{D} : |z - a| < \frac{1 - |a|}{2}\}$ for all $a \in \mathbb{D}$. Recall that $\widetilde{\omega}$ is essentially constant in each hyperbolically bounded region if $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$. This together with Theorem 1.4 and [3, Corollary 4.4-Theorem 1.7] implies that $\widetilde{\omega} \in B_{\infty}$. Therefore the next result follows from [1, Theorem 4.1], Theorem 1.4 and the fact that

$$\widetilde{\omega}(D_a) \asymp \omega(S(a)), \quad a \in \mathbb{D} \setminus \{0\},\$$

provided $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$.

ANNALES DE L'INSTITUT FOURIER

THEOREM 5.1. — Let $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$ and $0 < p, q < \infty$. Then the following statements hold:

(i) If $0 , then <math>T_g : A^p_\omega \to A^q_\omega$ is bounded if and only if

$$\sup_{a\in\mathbb{D}}(1-|a|)|g'(a)|\left(\omega(S(a))^{\frac{1}{q}-\frac{1}{p}}<\infty\right)$$

- (ii) If $0 , then <math>T_g : A^p_\omega \to A^q_\omega$ is compact if and only if $\lim_{|a| \to 1^-} (1 - |a|)|g'(a)| (\omega(S(a))^{\frac{1}{q} - \frac{1}{p}} = 0.$
- (iii) If $0 < q < p < \infty$, then $T_g : A^p_\omega \to A^q_\omega$ is bounded (equivalently compact) if and only if $g \in A^s_\omega$, where $\frac{1}{s} = \frac{1}{p} \frac{1}{q}$.

Theorem 5.1 shows, in particular, that $T_g : A^p_\omega \to A^p_\omega$ is bounded (resp. compact) if and only if $g \in \mathcal{B}$ (resp. $g \in \mathcal{B}_0$), provided $\omega \in \mathcal{B}_\infty(\mathcal{D})$.

We will next study the spectrum of T_g acting on A^p_{ω} , when $\omega \in \mathcal{B}_{\infty}(\mathcal{D})$. We begin with noticing that T_g has no eigenvalues [1, Proposition 5.1], and hence its spectrum is nothing else but $\{0\}$ if $g \in \mathcal{B}_0$. The proof of Theorem 1.5 follows ideas from the papers [1, 2, 3], where the approach used reveals a natural connection to weighted norm inequalities for derivatives. This general idea applies to our context as well. Indeed, a simple computation shows that for a given analytic function h in \mathbb{D} and $\lambda \in \mathbb{C} \setminus \{0\}$, the equation

$$\lambda f - T_g f = h$$

has the unique solution $f = \frac{1}{\lambda} R_{\lambda,g} h$, where

(5.1)
$$R_{\lambda,g}h(z) = h(0) e^{\frac{g(z)}{\lambda}} + e^{\frac{g(z)}{\lambda}} \int_0^z e^{-\frac{g(\xi)}{\lambda}} h'(\xi) d\xi, \quad z \in \mathbb{D}.$$

Thus λ belongs the resolvent set $\rho(T_g|A^p_{\omega})$ if and only if $R_{\lambda,g}$ is a bounded invertible operator on A^p_{ω} .

proof of Theorem 1.5. — The equivalence between (i) and (ii) follows by arguing as in the proof of [1, Theorem 5.1] and applying Theorems 1.1 and 5.1.

To see that (ii) and (iii) are equivalent, observe that since $\tilde{\omega}$ is essentially constant in each hyperbolically bounded region, the proof of [3, Proposition 2.1(b)] shows that there exists a differentiable weight W such that $\tilde{\omega} \simeq W$ on \mathbb{D} , and

(5.2)
$$|\nabla W(z)| \lesssim (1-|z|)W(z), \quad z \in \mathbb{D}.$$

TOME 74 (2024), FASCICULE 4

Therefore, by arguing as in the first part of the proof, but applying Theorem 1.4 instead of Theorem 1.1, we deduce that

$$\lambda \in \rho\left(T_g \mid A^p_{\omega}\right) = \rho\left(T_g \mid A^p_{\widetilde{\omega}}\right) = \rho\left(T_g \mid A^p_W\right)$$

if and only if

$$\|f\|_{A^p_{W_{\lambda,g,p}}}^p \asymp |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^p W_{\lambda,g,p}(z) \,\mathrm{d}A(z), \quad f \in \mathcal{H}(\mathbb{D}),$$

where $W_{\lambda,g,p} = W \exp\left(p \operatorname{Re} \frac{g}{\lambda}\right)$. Since $g \in \mathcal{B}$, also the weight $W_{\lambda,g,p}$ satisfies (5.2). Indeed,

$$\begin{split} \left| \nabla W \exp\left(p \operatorname{Re} \frac{g}{\lambda} \right)(z) \right| &\lesssim \left(|\nabla W(z)| + \frac{p |g'(z)| W(z)}{|\lambda|} \right) \exp\left(p \operatorname{Re} \frac{g}{\lambda} \right)(z) \\ &\lesssim \frac{W \exp\left(p \operatorname{Re} \frac{g}{\lambda} \right)(z)}{1 - |z|}, \quad z \in \mathbb{D}. \end{split}$$

Hence $W_{\lambda,g,p}$ is essentially constant in each hyperbolically bounded region by [3, Proposition 2.1(i)]. Therefore $\lambda \in \rho(T_g \mid A^p_{\omega}) = \rho(T_g \mid A^p_{\omega}) = \rho(T_g \mid A^p_{\omega})$ if and only if $W_{\lambda,g,p} \in B_{\infty}$ by [3, Corollary 4.4]. This is equivalent to $\widetilde{\omega} \exp\left(p \operatorname{Re} \frac{g}{\lambda}\right) \in B_{\infty}$.

BIBLIOGRAPHY

- A. ALEMAN & O. CONSTANTIN, "Spectra of integration operators on weighted Bergman spaces", J. Anal. Math. 109 (2009), p. 199-231.
- [2] A. ALEMAN & J. Á. PELÁEZ, "Spectra of integration operators and weighted square functions", Indiana Univ. Math. J. 61 (2012), no. 2, p. 775-793.
- [3] A. ALEMAN, S. POTT & M. C. REGUERA, "Characterizations of a limiting class B_{∞} of Békollé–Bonami weights", *Rev. Mat. Iberoam.* **35** (2019), no. 6, p. 1677-1692.
- [4] A. ALEMAN & A. G. SISKAKIS, "Integration operators on Bergman spaces", Indiana Univ. Math. J. 46 (1997), no. 2, p. 337-356.
- [5] G. BAO, H. WULAN & K. ZHU, "A Hardy–Littlewood theorem for Bergman spaces", Ann. Acad. Sci. Fenn., Math. 43 (2018), no. 2, p. 807-821.
- [6] D. BEKOLLÉ, "Inégalité à poids pour le projecteur de Bergman dans la boule unité de Cⁿ", Stud. Math. 71 (1982), no. 3, p. 305-323.
- [7] D. BEKOLLÉ & A. BONAMI, "Inégalités à poids pour le noyau de Bergman", C. R. Math. Acad. Sci. Paris 286 (1978), no. 18, p. A775-A778.
- [8] A. BORICHEV, "On the Bekollé–Bonami condition", Math. Ann. 328 (2004), no. 3, p. 389-398.
- [9] J. DUOANDIKOETXEA, Fourier analysis, Graduate Studies in Mathematics, vol. 29, American Mathematical Society, 2001, xviii+222 pages.
- [10] J. DUOANDIKOETXEA, F. J. MARTÍN-REYES & S. OMBROSI, "On the A_{∞} conditions for general bases", Math. Z. **282** (2016), no. 3-4, p. 955-972.

- [11] L. HÖRMANDER, "L^p estimates for (pluri-) subharmonic functions", Math. Scand. 20 (1967), p. 65-78.
- [12] C. D. KENFACK & B. F. SEHBA, "Maximal function and Carleson measures in the theory of Békollé–Bonami weights", Colloq. Math. 142 (2016), no. 2, p. 211-226.
- [13] R. A. KERMAN & A. TORCHINSKY, "Integral inequalities with weights for the Hardy maximal function", Stud. Math. 71 (1981/82), no. 3, p. 277-284.
- [14] T. KORHONEN & J. RÄTTYÄ, "Zero sequences, factorization and sampling measures for weighted Bergman spaces", Math. Z. 291 (2019), no. 3-4, p. 1145-1173.
- [15] D. H. LUECKING, "Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives", Am. J. Math. 107 (1985), no. 1, p. 85-111.
- [16] —, "Trace ideal criteria for Toeplitz operators", J. Funct. Anal. 73 (1987), no. 2, p. 345-368.
- [17] M. PAVLOVIĆ & J. Á. PELÁEZ, "An equivalence for weighted integrals of an analytic function and its derivative", Math. Nachr. 281 (2008), no. 11, p. 1612-1623.
- [18] J. Á. PELÁEZ, "Small weighted Bergman spaces", in Proceedings of the Summer School in Complex and Harmonic Analysis, and Related Topics, Publ. Univ. East. Finl. Rep. Stud. For. Nat. Sci., vol. 22, Univ. East. Finl., Fac. Sci. For., Joensuu, 2016, p. 29-98.
- [19] J. Á. PELÁEZ & J. RÄTTYÄ, "Generalized Hilbert operators on weighted Bergman spaces", Adv. Math. 240 (2013), p. 227-267.
- [20] ——, Weighted Bergman spaces induced by rapidly increasing weights, vol. 227, Memoirs of the American Mathematical Society, no. 1066, American Mathematical Society, 2014, vi+124 pages.
- [21] ——, "Embedding theorems for Bergman spaces via harmonic analysis", Math. Ann. 362 (2015), no. 1-2, p. 205-239.
- [22] ——, "Two weight inequality for Bergman projection", J. Math. Pures Appl. 105 (2016), no. 1, p. 102-130.
- [23] —, "Harmonic conjugates on Bergman spaces induced by doubling weights", Anal. Math. Phys. 10 (2020), no. 2, article no. 18 (22 pages).
- [24] —, "Bergman projection induced by radial weight", Adv. Math. 391 (2021), article no. 107950 (70 pages).
- [25] ——, "Bergman projection and BMO in hyperbolic metric: improvement of classical result", Math. Z. **305** (2023), no. 2, article no. 19 (9 pages).
- [26] J. Á. PELÁEZ, J. RÄTTYÄ & B. D. WICK, "Bergman projection induced by kernel with integral representation", J. Anal. Math. 138 (2019), no. 1, p. 325-360.
- [27] A. G. SISKAKIS, "Weighted integrals of analytic functions", Acta Sci. Math. 66 (2000), no. 3-4, p. 651-664.
- [28] E. M. STEIN, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Mathematical Series, vol. 43, Princeton University Press, 1993, xiv+695 pages.
- [29] K. ZHU, Operator theory in function spaces, second ed., Mathematical Surveys and Monographs, vol. 138, American Mathematical Society, 2007, xvi+348 pages.

Manuscrit reçu le 3 juillet 2020, révisé le 11 juillet 2022, accepté le 17 novembre 2022.

José Ángel PELÁEZ Departamento de Análisis Matemático, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, (Spain) japelaez@uma.es

Jouni RÄTTYÄ University of Eastern Finland, P.O.Box 111, 80101 Joensuu (Finland) jouni.rattya@uef.fi