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WEIGHTED NORM INEQUALITIES FOR DERIVATIVES ON BERGMAN SPACES

by José Ángel PELÁEZ & Jouni RÄTTYÄ (*)

Abstract. — An equivalent norm in the weighted Bergman space A^p_ω, induced by an ω in a certain large class of non-radial weights, is established in terms of higher order derivatives. Other Littlewood–Paley inequalities are also considered. On the way to the proofs, we characterize the q-Carleson measures for the weighted Bergman space A^p_ω and the boundedness of a Hörmander-type maximal function. Results obtained are further applied to describe the resolvent set of the integral operators $T_g(f)(z) = \int_0^z g'(\zeta)f(\zeta)\,\mathrm{d}\zeta$ acting on A^p_ω.

Résumé. — Nous construisons une norme équivalente, définie à l'aide des dérivées supérieures, dans un espace de Bergman pondéré A^p_ω où ω appartient à une large classe des poids non radiaux. Nous analysons aussi autres inégalités de Littlewood–Paley. Avant de démontrer les résultats principaux nous caractérisons les q-mesures de Carleson sur les espaces A^p_ω et montrons que la fonction maximale de Hörmander est bornée. En utilisant nos résultats nous pouvons décrire l'ensemble résolvant de l'opérateur intégral $T_g(f)(z) = \int_0^z g'(\zeta)f(\zeta)\,\mathrm{d}\zeta$ agissant sur A^p_ω.

1. Introduction and main results

Let $\mathcal{H}(\mathbb{D})$ denote the algebra of all analytic functions in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$. A function $\omega : \mathbb{D} \to [0,\infty)$, integrable over $\mathbb{D}$, is called a weight. It is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ and $\int_0^1 \omega(s)\,\mathrm{d}s < \infty$. For $0 < p < \infty$ and a weight ω, the weighted Bergman space A^p_ω consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\,\mathrm{d}A(z) < \infty,$$

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where $\mathrm{d}A(z) = \frac{\mathrm{d}x\,\mathrm{d}y}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$.

In this paper we are interested in obtaining, for a large class of non-radial weights ω, equivalent norms of f in A^p_ω in terms of its higher order derivatives. This is a question that have been extensively studied for different classes of radial weights but it is not well-understood for general weights. See [1, 3, 5] for recent developments on the topic. These norms are extremely valuable within the theory of concrete operators acting on these spaces. To name a few instances, they are used; in the study of Volterra-type operators, because they allow to get rid of the integral and they arise in a natural way in the description of its spectrum [1, 3, 4, 20], in order to get crucial estimates in the description of Schatten classes of Toeplitz operators [16, p. 356], in the boundedness of the Hilbert matrix [19, Proof of Theorem 2] or in obtaining norm estimates for the Bergman reproducing kernels in A^p_ω induced by radial weights [22, Proof of Theorem 1].

A well-known formula ensures that, for each $k \in \mathbb{N}$ and $0 < p < \infty$, we have

$$\|f\|^p_{A^p_\omega} \asymp \int_{\mathbb{D}} |f^{(k)}(z)|^p (1-|z|)^{kp} \omega(z)\,\mathrm{d}A(z) + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}), \tag{1.1}$$

if ω is a standard radial weight, that is, $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$ for some $-1 < \alpha < \infty$. Generalizations of this result for different classes of radial weights have been obtained in [4, 17, 24, 27]. In particular, it was recently proved [24, Theorem 5] that (1.1) holds for a radial weight ω if and only if $\omega \in \mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$. Recall that a radial weight ν belongs to $\widehat{\mathcal{D}}$ if there exists a constant $C = C(\nu) > 1$ such that the tail integral $\widehat{\nu}(z) = \int_{|z|}^1 \nu(s)\,\mathrm{d}s$ satisfies the doubling condition $\widehat{\nu}(r) \leqslant C\widehat{\nu}(\frac{1+r}{2})$ for all $0 \leqslant r < 1$. Further, a radial weight ν belongs to $\check{\mathcal{D}}$ if there exist constants $K = K(\nu) > 1$ and $C = C(\nu) > 1$ such that $\widehat{\nu}(r) \geqslant C\widehat{\nu}(1 - \frac{1-r}{K})$ for all $0 \leqslant r < 1$.

For a given $a \in \mathbb{D} \setminus \{0\}$, consider the interval

$$I_a = \left\{ \mathrm{e}^{\mathrm{i}\theta} : |\arg(a\,\mathrm{e}^{-\mathrm{i}\theta})| \leqslant \frac{(1-|a|)}{2} \right\},$$

and let $S(a) = \{z \in \mathbb{D} : |z| \geqslant |a|, \mathrm{e}^{\mathrm{i}t} \in I_a\}$ denote the Carleson square induced by a. We assume throughout the paper that $\omega(S(a)) > 0$ for all $a \in \mathbb{D} \setminus \{0\}$. If this is not the case and ω is radial, then $A^p_\omega = \mathcal{H}(\mathbb{D})$. For a weight ν, ω is a ν-weight if $\omega\nu$ is integrable. If $1 < p < \infty$, a ν-weight ω belongs to the class $B_p(\nu)$ if there exists a constant $C = C(p, \nu, \omega)$ such that

$$\sup_S \frac{\left(\int_S \omega(z)\nu(z)\,\mathrm{d}A(z)\right)^{\frac{1}{p}}\left(\int_S \omega^{-\frac{p'}{p}}(z)\nu(z)\,\mathrm{d}A(z)\right)^{\frac{1}{p'}}}{\int_S \nu(z)\,\mathrm{d}A(z)} < \infty, \tag{1.2}$$

where the supremum is taken over all Carleson squares S. We denote $B_\infty(\nu) = \bigcup_{1<p<\infty} B_p(\nu)$. It has recently been proved that the class $B_p(\nu)$ describes the weights ω such that Bergman projection P_ν, induced by a radial weight ν, is bounded on $L^p_{\omega\nu}$, $1 < p < \infty$, whenever $\nu \in \mathcal{D}$ and the Bergman reproducing kernel of A^2_ν has a particular integral representation [26, Theorem 2]. This result is a natural extension of a classical result due to Bekollé and Bonami [6, 7] for standard weights. If $\nu(z) = (1+\eta)(1-|z|^2)^\eta$ we simply write $B_p(\eta)$ instead of $B_p((1-|z|)^\eta)$, $B_\infty(\eta) = B_\infty((1-|z|)^\eta)$ and $B_\infty = B_\infty(0)$. Nonnegative functions in the class $B_p(\eta)$ or $B_\infty(\eta)$ are usually called the Bekollé–Bonami weights. En route to describing the spectrum of the integral operator

$$T_g(f)(z) = \int_0^z f(\zeta)\, g'(\zeta)\,\mathrm{d}\zeta, \quad z \in \mathbb{D}, \quad g \in \mathcal{H}(\mathbb{D}),$$

on standard Bergman spaces it was shown that (1.1) is satisfied if there exists $\eta > -1$ such that $\frac{\omega}{(1-|z|)^\eta} \in B_\infty(\eta)$ [1, Theorem 3.1]. The first result of this study says that the hypothesis $\frac{\omega}{(1-|z|)^\eta} \in B_\infty(\eta)$ can be replaced by the weaker condition $\frac{\omega}{\nu} \in B_\infty(\nu)$ with $\nu \in \mathcal{D}$. To simplify the notation, we write $\mathcal{B}_\infty(\nu) = \bigcup_{1<p<\infty}\left\{\omega : \frac{\omega}{\nu} \in B_p(\nu)\right\}$, $\mathcal{B}_\infty(\widehat{\mathcal{D}}) = \bigcup_{\nu\in\widehat{\mathcal{D}}} \mathcal{B}_\infty(\nu)$ and $\mathcal{B}_\infty(\mathcal{D}) = \bigcup_{\nu\in\mathcal{D}} \mathcal{B}_\infty(\nu)$. It is worth mentioning that $\mathcal{B}_\infty(\mathcal{D})$ is in a sense a much larger class than

$$\bigcup_{\eta>-1}\left\{\omega \text{ is a weight} : \frac{\omega}{(1-|z|)^\eta} \in B_\infty(\eta)\right\}$$

because it contains weights with a strong oscillatory behaviour which may vanish on a hyperbolically bounded set of $\mathbb{D}$.

Theorem 1.1. — *Let $0 < p < \infty$, $k \in \mathbb{N}$ and $\omega \in \mathcal{B}_\infty(\mathcal{D})$. Then* (1.1) *holds.*

Observe that the set of radial weights in $\mathcal{B}_\infty(\mathcal{D})$ coincides with $\mathcal{D}$, and hence [24, Theorem 5] implies that $\omega \in \mathcal{B}_\infty(\mathcal{D})$ is also a necessary condition for (1.1) to hold if ω is radial. The class $\mathcal{D}$ also appears innately in the study of classical questions related to the boundedness of the Bergman projection P_ν induced by a radial weight ν [24, Theorem 3 and Theorem 12], which is a frequently used tool in order to get Littlewood–Paley formulas in weighted Bergman spaces [1, 29].

It is also worth mentioning that (1.1) holds if and only if $\omega \in B_\infty$, when ω is essentially (or almost) constant in each hyperbolically bounded region [3, Corollary 4.4]. This last condition ensures that the inequality

$$\int_{\mathbb{D}} |f^{(k)}(z)|^p(1-|z|)^{kp}\omega(z)\,\mathrm{d}A(z)+\sum_{j=0}^{k-1}|f^{(j)}(0)|^p \lesssim \|f\|^p_{A^p_\omega}, \quad f \in \mathcal{H}(\mathbb{D}), \tag{1.3}$$

holds [1, Theorem 3.1], see also [5, Theorem A]. We prove the following result concerning this last inequality.

Theorem 1.2. — *Let $0 < p < \infty$, $k \in \mathbb{N}$ and $\omega \in \mathcal{B}_\infty(\widehat{\mathcal{D}})$. Then* (1.3) *holds.*

Obviously there are weights in $\mathcal{B}_\infty(\widehat{\mathcal{D}})$ which are not essentially constant in each hyperbolically bounded region. Moreover, since the restriction of $\mathcal{B}_\infty(\widehat{\mathcal{D}})$ to radial weights coincides with $\widehat{\mathcal{D}}$, the hypothesis $\omega \in \mathcal{B}_\infty(\widehat{\mathcal{D}})$ in Theorem 1.2 cannot be relaxed in the case of radial weights by [24, Theorem 6]. Observe that weights in $\widehat{\mathcal{D}}$, an consequently in $\mathcal{B}_\infty(\widehat{\mathcal{D}})$, may have a wild oscillatory behavior and they may even vanish on sets that are not hyperbolically uniformly bounded. Moreover, $\widehat{\mathcal{D}}$ is not closed by the multiplication of a standard weight induced by a positive parameter α. Illuminating examples of weights in the deceivingly simply looking class $\widehat{\mathcal{D}}$ are given in [23, Proposition 10] and [25, Proposition 12].

The proofs of Theorems 1.1 and 1.2 have three key ingredients. The first of them provides a geometric description of the q-Carleson measures for A^p_ω, provided $q \geqslant p$ and $\omega \in \mathcal{B}_\infty(\widehat{\mathcal{D}})$. To state the result, for a given measure μ on $\mathbb{D}$, we write $\mu(E) = \int_E d\mu$ for each μ-measurable set $E \subset \mathbb{D}$. Further, for each $\varphi \in L^1_\omega$, the Hörmander-type maximal function [11] is defined by

$$M_\omega(\varphi)(z) = \sup_{z\in S} \frac{1}{\omega(S)} \int_S |\varphi(\zeta)|\omega(\zeta)\,\mathrm{d}A(\zeta), \quad z \in \mathbb{D}.$$

Our characterization of q-Carleson measures for A^p_ω reads as follows.

Theorem 1.3. — *Let $0 < p \leqslant q < \infty$, $\omega \in \mathcal{B}_\infty(\widehat{\mathcal{D}})$ and let μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) *μ is a q-Carleson measure for A^p_ω;*

(ii) $\left[M_\omega\left(|\cdot|^{\frac{1}{\alpha}}\right)\right]^\alpha : L^p_\omega \to L^q_\mu$ *is bounded for each $\alpha > \frac{1}{p}$;*

(iii) *μ satisfies*

$$\sup_S \frac{\mu(S)}{(\omega(S))^{\frac{q}{p}}} < \infty, \tag{1.4}$$

where the supremum runs over all the Carleson squares S in $\mathbb{D}$.

Moreover,

$$\|I_d\|^q_{A^p_\omega\to L^q_\mu} \asymp \left\| \left[M_\omega \left(|\cdot|^{\frac{1}{\alpha}} \right) \right]^\alpha \right\|^q_{L^p_\omega\to L^q_\mu} \asymp \sup_S \frac{\mu(S)}{(\omega(S))^{\frac{q}{p}}}. \tag{1.5}$$

Theorem 1.3 is a natural extension of [18, Theorem 3.3] and [20, Theorem 2.1] to non-radial weights.

A good understanding of the class of weights involved in Theorems 1.1 and 1.2 is needed. In particular, en route to the proofs, we show that

$$\omega \in \mathcal{B}_\infty(\mathcal{D}) \Rightarrow \omega_{[\beta]}(z) = (1-|z|)^\beta \omega(z) \in \mathcal{B}_\infty(\mathcal{D}), \quad \text{for any } \beta > 0,$$

a fact which might be deceptively simple-looking. Indeed, the class $\mathcal{B}_\infty(\widehat{\mathcal{D}})$ does not have this property. See [23, Proposition 10] for the construction of a radial weight $\omega \in \mathcal{B}_\infty(\widehat{\mathcal{D}})$ such that $\omega_{[\beta]} \notin \mathcal{B}_\infty(\widehat{\mathcal{D}})$ for any $\beta > 0$.

The third key ingredient in the proofs of Theorems 1.1 and 1.2 concerns certain more smooth weights. Namely, each weight ω induces the nonnegative average function

$$\widetilde{\omega}(z) = \frac{\int_{S(z)} \omega(\zeta)\,\mathrm{d}A(\zeta)}{(1-|z|)^2}, \quad z \in \mathbb{D}\setminus\{0\}.$$

Which regard to this function we prove the following result.

Theorem 1.4. — *Let $0 < p < \infty$, $k \in \mathbb{N}$ and $\omega \in \mathcal{B}_\infty(\mathcal{D})$. Then*

$$\|f\|^p_{A^p_\omega} \asymp \|f\|^p_{A^p_{\widetilde{\omega}}} \asymp \int_{\mathbb{D}} |f^{(k)}(z)|^p (1-|z|)^{kp} \widetilde{\omega}(z)\,\mathrm{d}A(z) + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p, \qquad f \in \mathcal{H}(\mathbb{D}).$$

We emphasize that, under the hypothesis $\omega \in \mathcal{B}_\infty(\mathcal{D})$, the weights ω and $\widetilde{\omega}$ are not pointwise equivalent, but $\|f\|^p_{A^p_\omega} \asymp \|f\|^p_{A^p_{\widetilde{\omega}}}$ and $\widetilde{\omega}$ is essentially (or almost) constant in each hyperbolically bounded region. This together with the second equivalence in Theorem 1.4 and [3, Corollary 4.4-Theorem 1.7] implies that $\widetilde{\omega} \in B_\infty$. Therefore the study of certain type of questions on linear operators $T : \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$ on weighted Bergman spaces A^p_ω, with $\omega \in \mathcal{B}_\infty(\mathcal{D})$, can be reduced to the particular case $\omega \in B_\infty$. We will make this statement precise in the case of some questions related to the integral operator T_g induced by $g \in \mathcal{H}(\mathbb{D})$. Indeed, Theorem 5.1 below describes the analytic symbols g such that $T_g : A^p_\omega \to A^q_\omega$ is bounded or compact. In particular, it shows that $T_g : A^p_\omega \to A^p_\omega$ is bounded if and only if g belongs to the classical space $\mathcal{B}$ of Bloch functions. Further, by using ideas from studies [1, 2, 3], which link the resolvent set $\rho(T_g|A^p_\omega)$ to the theory of weighted norms in terms of derivatives, we obtain the following characterization of $\rho\,(T_g|A^p_\omega)$.

THEOREM 1.5. — *Let $\omega \in \mathcal{B}_\infty(\mathcal{D})$, $g \in \mathcal{B}$, $0 < p < \infty$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then the following statements are equivalent:*

(i) $\lambda \in \rho\,(T_g|A^p_\omega)$;

(ii) $\|f\|^p_{A^p_{\omega_{\lambda,g,p}}} \asymp |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p(1-|z|)^p\omega_{\lambda,g,p}(z)\,\mathrm{d}A(z)$ *for all* $f \in \mathcal{H}(\mathbb{D})$, *where* $\omega_{\lambda,g,p} = \omega \exp\left(p\,\mathrm{Re}\,\frac{g}{\lambda}\right)$;

(iii) $\widetilde{\omega} \exp\left(p\,\mathrm{Re}\,\frac{g}{\lambda}\right) \in B_\infty$.

The remaining part of the paper is organized as follows. In Section 2 we state and prove some preliminary results on weights. Theorem 1.3 is proved in Section 3 while Section 4 is devoted to the proofs of Theorems 1.1, 1.2 and 1.4. In Section 5 we discuss some basic properties of the integral operator T_g acting on A^p_ω and then prove Theorem 1.5.

Before proceeding further, a word about notation used. The letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \leqslant Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we write $a \asymp b$ and say that a and b are comparable. This notation has already been used above in the introduction.

2. Basic properties and lemmas on weights

The pseudohyperbolic distance between two points z_1 and z_2 in $\mathbb{D}$ is $\rho(z_1, z_2) = \left|\frac{z_1-z_2}{1-\overline{z_1}z_2}\right|$. We say that a weight ω is essentially constant on each hyperbolically bounded region if there exist constants $r \in (0,1)$ and $C = C(\omega, r)$ such that

$$C^{-1}\omega(z_2) \leqslant \omega(z_1) \leqslant C\omega(z_2), \quad \rho(z_1, z_2) < r.$$

This class of weights coincides with the weights satisfying [3, (1.6)] and has been also considered in [5].

In the classical setting, there are many equivalent conditions which describe the Muckenhoupt class $A_\infty = \bigcup_{1\leqslant q<\infty} A_q$, see [10], [28, Chapter 5] or [9, p. 149]. However, this is no longer true for the class B_∞, that is, the corresponding conditions (defined on Carleson squares) do not coincide. This stems from the fact that B_∞-weights do not have the reverse Hölder property [3, 8]. It is worth mentioning that our definition of the class B_∞ differs from the one provided in [3, (1.4)]. However, this does not cause any trouble because B_∞-weights, which are essentially constant in each hyperbolically bounded region, can be described in terms of a good number of conditions [3, Theorem 1.7], and in particular the definitions coincide under this extra hypotheses on the weight.

The main results of this paper are established under the hypothesis $\omega \in \mathcal{B}(\mathcal{D})$. Therefore we are interested in looking for neat conditions describing the class $\mathcal{B}_\infty(\nu)$ induced by $\nu \in \mathcal{D}$. In order to do this, for each weight ν, we say that a weight ω has the Kerman–Torchinsky $KT(\nu)$-property if there exist constants $\delta \in (0,1)$ and $C > 0$ such that

$$\frac{\nu(E)}{\nu(S)} \leqslant C \left(\frac{\omega(E)}{\omega(S)}\right)^\delta \tag{2.1}$$

for all Carleson squares $S \subset \mathbb{D}$ and measurable sets $E \subset S$. Here and from now on we write $\omega(E) = \int_E \omega(z)\,\mathrm{d}A(z)$. If we replace ν by the Lebesgue measure in $\mathbb{R}^n$ and Carleson squares by cubes Q in (2.1), we obtain a condition which describes the class A_∞ of the classical Muckenhoupt weights [10, Theorem 3.1]. This condition was introduced by Kerman and Torchinsky [13, Proposition 1] in order to describe the Hardy–Littlewood maximal operators that are of restricted weak-type. The next result follows from [10, Theorem 3.1(c)] (which holds for general bases).

Proposition A. — *Let ν be a weight. Then a weight ω belongs to $\mathcal{B}_\infty(\nu)$ if and only if it has the $KT(\nu)$-property.*

The K-top of a Carleson box $S(a)$ is the polar rectangle $T_K(a) = \{r\,\mathrm{e}^{\mathrm{i}t} : \mathrm{e}^{\mathrm{i}t} \in I_a,\ |a| \leqslant r < 1 - \frac{1-|a|}{K}\}$. In some of the auxiliary results obtained en route to the main theorems the conditions $\mathcal{B}(\mathcal{D})$ and $\mathcal{B}(\widehat{\mathcal{D}})$ can be relaxed in the sense that (2.1) is only needed for K-tops or their complements $S \setminus T_K$. To be precise, we write $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$ if there exists $C = C(\omega) > 0$ such that $\omega(S(a)) \leqslant C\omega(S(\frac{1+|a|}{2}\,\mathrm{e}^{\mathrm{i}\arg a}))$ for all $a \in \mathbb{D} \setminus \{0\}$. It is easy to see that each $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$ satisfies $\omega(S(a')) \leqslant C(C+1)\omega(S(a))$ for all $a, a' \in \mathbb{D} \setminus \{0\}$ with $|a'| = |a|$ and $\arg a' = \arg a \pm (1-|a|)$. Therefore $\omega(S(a)) \lesssim \omega(S(b))$ whenever $|b| = \frac{1+|a|}{2}$ and $S(b) \subset S(a)$. It is also obvious that radial weights in $\widehat{\mathcal{D}}(\mathbb{D})$ form the class $\widehat{\mathcal{D}}$, which plays a crucial role in the operator theory of Bergman spaces induced by radial weights [24]. Further, a weight ω on $\mathbb{D}$ belongs to $\check{\mathcal{D}}(\mathbb{D})$ if there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that

$$\omega(S(a)) \leqslant C\omega(T_K(a)), \quad a \in \mathbb{D} \setminus \{0\}. \tag{2.2}$$

It is clear that radial weights in $\check{\mathcal{D}}(\mathbb{D})$ form the class $\check{\mathcal{D}}$. Finally, we write $\mathcal{D}(\mathbb{D}) = \widehat{\mathcal{D}}(\mathbb{D}) \cap \check{\mathcal{D}}(\mathbb{D})$ for short.

In view of the above we have $\mathcal{B}_\infty(\widehat{\mathcal{D}}) \subset \widehat{\mathcal{D}}(\mathbb{D})$, $\mathcal{B}_\infty(\check{\mathcal{D}}) \subset \check{\mathcal{D}}(\mathbb{D})$ and $\mathcal{B}_\infty(\mathcal{D}) \subset \mathcal{D}(\mathbb{D})$. These embeddings, which will be used repeatedly throughout the paper, can also be proved by straightforward calculations which show that $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$ (resp. $\omega \in \check{\mathcal{D}}(\mathbb{D})$) if $\nu \in \widehat{\mathcal{D}}(\mathbb{D})$ (resp. $\nu \in \check{\mathcal{D}}(\mathbb{D})$) and $\omega \in \mathcal{B}_\infty(\nu)$. Therefore $\widehat{\mathcal{D}}$ and $\mathcal{D}$ coincide with the radial weights in $\mathcal{B}_\infty(\widehat{\mathcal{D}})$ and $\mathcal{B}_\infty(\mathcal{D})$, respectively. However, $\mathcal{B}_\infty(\widehat{\mathcal{D}}) \subsetneq \widehat{\mathcal{D}}$. Namely, let

$$\Gamma(\zeta) = \left\{ z \in \mathbb{D} : \, |\arg \zeta - \arg z| < \frac{1}{2}(1-|z|) \right\}, \quad \zeta \in \partial\mathbb{D},$$

and consider the weight $\omega = \chi_{\mathbb{D}\setminus\Gamma(1)}$. Then $\omega \notin \mathcal{B}_\infty(\widehat{\mathcal{D}})$ as is seen by considering the Carleson squares $S(a)$ induced by $a \in (0,1)$. But obviously there exists a constant $C > 0$ such that $\omega(S_a) \geqslant C|S_a|$ for all $a \in \mathbb{D} \setminus \{0\}$, and thus $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$.

The proof of the following result concerning the class $\widehat{\mathcal{D}}(\mathbb{D})$ can be found in [14, Lemma 14].

Lemma B. — *Let ω be a weight on $\mathbb{D}$. Then the following statements are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$;

(ii) *there exist $\beta = \beta(\omega) > 0$ and $C = C(\omega) \geqslant 1$ such that*

$$\frac{\omega(S(a))}{(1-|a|)^\beta} \leqslant C \frac{\omega(S(a'))}{(1-|a'|)^\beta}, \quad 0 < |a| \leqslant |a'| < 1, \quad \arg a = \arg a';$$

(iii) *for some (equivalently for each) $K > 0$ there exists $C = C(\omega, K) > 0$ such that*

$$\omega(S(a)) \leqslant C\omega\left(S\left(\frac{K+|a|}{K+1} \mathrm{e}^{\mathrm{i}\arg a} \right) \right), \quad a \in \mathbb{D}\setminus\{0\};$$

(iv) *there exist $\eta = \eta(\omega) > 0$ and $C = C(\eta, \omega) > 0$ such that*

$$\int_{\mathbb{D}} \frac{\omega(z)}{|1-\bar{a}z|^\eta} \, \mathrm{d}A(z) \leqslant C \frac{\omega(S(a))}{(1-|a|)^\eta}, \quad a \in \mathbb{D}\setminus\{0\}.$$

Observe that if the inequality in the case (ii) is satisfied for some $\beta > 0$, then it is certainly satisfied for every number larger than that because the quotient $(1-|a'|)/(1-|a|)$ is at most one. A similar comment applies to the constant η appearing in the case (iv).

The following lemma gives an analogue of Lemma B(ii) for weights in $\check{\mathcal{D}}(\mathbb{D})$.

Lemma 2.1. — *Let ω be a weight on $\mathbb{D}$. Then $\omega \in \check{\mathcal{D}}(\mathbb{D})$ if and only if there exist $K = K(\omega) > 1$ and $\beta_0 = \beta_0(\omega) > 0$ such that*

$$\omega(S(a)) \geqslant \left(\frac{1-|a|}{1-|b|}\right)^{\beta} \omega\left(S(a) \setminus D(0,|b|)\right), \quad 1 - \frac{1-|a|}{K} \leqslant |b| < 1, \tag{2.3}$$

for all $0 < \beta \leqslant \beta_0$ and $a \in \mathbb{D} \setminus \{0\}$.

Proof. — First observe that $\omega \in \check{\mathcal{D}}(\mathbb{D})$ if and only if there exist $C = C(\omega) > 1$ and $K = K(\omega) > 1$ such that

$$\omega(S(a)) \geqslant C\omega\left(S(a) \setminus T_K(a)\right), \quad a \in \mathbb{D} \setminus \{0\}. \tag{2.4}$$

This is the characterization that we will use to prove the lemma.

The choice $b = 1 - \frac{1-|a|}{K}$ in (2.3) implies (2.4) with $C = K^{\beta}$, and therefore $\omega \in \check{\mathcal{D}}(\mathbb{D})$. To prove the converse implication, assume without loss of generality that $K \in \mathbb{N}$. Now divide $S(a) \setminus T_K(a)$ into K Carleson squares of equal size and apply (2.4) to each square to obtain $\omega(S(a)) \geqslant C^2\omega\left(S(a) \setminus T_{K^2}(a)\right)$ for all $a \in \mathbb{D} \setminus \{0\}$. Then divide $S(a) \setminus T_{K^2}(a)$ into K^2 squares and proceed. After $1 + K + K^2 + \cdots + K^{n-1}$ applications of (2.4) we obtain

$$\omega(S(a)) \geqslant C^n\omega\left(S(a) \setminus T_{K^n}(a)\right), \quad a \in \mathbb{D} \setminus \{0\}. \tag{2.5}$$

Now, for given $1 - \frac{1-|a|}{K} \leqslant |b| < 1$, pick up $n = n(a, b) \in \mathbb{N}$ such that

$$1 - \frac{1-|a|}{K^n} \leqslant |b| < 1 - \frac{1-|a|}{K^{n+1}} \quad \Longleftrightarrow \quad K^n \leqslant \frac{1-|a|}{1-|b|} < K^{n+1}.$$

Then (2.5) yields

$$\begin{aligned}\omega(S(a)) &\geqslant K^{n\log_K C}\omega\left(S(a) \setminus T_{K^n}(a)\right)\\ &> \left(\frac{1-|a|}{1-|b|}\right)^{\frac{n}{n+1}\log_K C} \omega\left(S(a) \setminus T_{\frac{1-|a|}{1-|b|}}(a)\right)\\ &\geqslant \left(\frac{1-|a|}{1-|b|}\right)^{\frac{1}{2}\log_K C} \omega\left(S(a) \setminus D(0,|b|)\right),\end{aligned}$$

which gives (2.3) for $\beta_0 = \frac{1}{2}\log_K C$. □

For each $\epsilon \in (0, 1)$, a simple computation shows that the weight

$$W(r\,\mathrm{e}^{\mathrm{i}\theta}) = \begin{cases} \frac{1}{(1-r)^{1-\frac{\epsilon}{2}}|\theta|^{1-\frac{\epsilon}{2}}}, & \theta \neq 0 \\ 1, & \theta = 0, \end{cases}$$

is a Bekollé–Bonami type weight such that $W(S(a)) \asymp (1-a)^{\epsilon}$ for all $a \in (0, 1)$. This implies that $H^p \not\subset A^p_W$ for all $0 < p < \infty$ by the classical

Carleson embedding theorem. Let us compare this example with Theorem 1.4, which says, in particular, that $\widetilde{\omega}$ is a weight whenever $\omega \in \mathcal{B}_\infty(\mathcal{D})$. Therefore, despite of the fact that $\widetilde{\omega}$ is a weight for all $\omega \in \mathcal{B}_\infty(\mathcal{D})$, for each given $\varepsilon > 0$ there exist $\omega \in \mathcal{B}_\infty(\mathcal{D})$ and a set $A \subset \mathbb{D}$, with $|A| = 0$ and $\overline{A} \cap \partial\mathbb{D} \neq \emptyset$, such that $\omega(S(a)) \asymp (1-|a|)^\epsilon$, as $a \in A$ and $|a| \to 1^-$.

3. Carleson measures

Let X be a quasi-Banach space of analytic functions on $\mathbb{D}$. A positive Borel measure μ on $\mathbb{D}$ is called a q-Carleson measure for X if the identity operator $I_d : X \to L^q_\mu$ is bounded. Moreover, if $I_d : X \to L^q_\mu$ is compact, then μ is a q-vanishing Carleson measure for X.

We begin with the boundedness of the Hörmander-type maximal function on L^p_ω when $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$.

Proposition 3.1. — *Let $0 < p \leqslant q < \infty$ and $0 < \alpha < \infty$ such that $p\alpha > 1$. Let $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$ and let μ be a positive Borel measure on $\mathbb{D}$. Then $\left[M_\omega\left(|\cdot|^{\frac{1}{\alpha}}\right)\right]^\alpha : L^p_\omega \to L^q_\mu$ is bounded if and only if μ satisfies* (1.4). *Moreover,*

$$\left\|\left[M_\omega\left(|\cdot|^{\frac{1}{\alpha}}\right)\right]^\alpha\right\|^q_{L^p_\omega \to L^q_\mu} \asymp \sup_S \frac{\mu(S)}{(\omega(S))^{\frac{q}{p}}}.$$

Proposition 3.1 can be established by following the lines of the proof of [21, Theorem 3]. We omit the details of the argument. A similar result was obtained in [12, Theorem 1.1] under stronger hypotheses on ω.

Proof of Theorem 1.3. — We will show first that (i) implies (iii) under the weaker hypothesis $0 < p, q < \infty$ and $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$. To see this, for $a \in \mathbb{D}$ and $0 < p, \gamma < \infty$, consider the test functions

$$F_{a,p,\gamma}(z) = \left(\frac{1-|a|^2}{1-\overline{a}z}\right)^{\frac{\gamma}{p}}, \quad z \in \mathbb{D}. \tag{3.1}$$

Pick up $\gamma = \gamma(p,\omega) > 0$ sufficiently large such that $\frac{\gamma}{p} > \eta$, where $\eta = \eta(\omega) > 0$ is that of Lemma B(iv). Then

$$\begin{aligned}\mu(S(a)) &\lesssim \int_{S(a)} |F_{a,p}(z)|^q \,\mathrm{d}\mu(z) \leqslant \int_{\mathbb{D}} |F_{a,p}(z)|^q \,\mathrm{d}\mu(z)\\ &\lesssim \|F_{a,p}\|^q_{A^p_\omega} \lesssim \omega\left(S(a)\right)^{\frac{q}{p}}, \quad a \in \mathbb{D},\end{aligned}$$

and thus μ satisfies (iii).

The statements (ii) and (iii) are equivalent by Proposition 3.1. Hence, to complete the proof, it suffices to show that (iii) implies μ is a q-Carleson measure for A^p_ω. Since $\frac{\omega}{\nu} \in B_{p_0}(\nu)$ for some $p_0 > 1$ and $\nu \in \widehat{\mathcal{D}}$ by the hypothesis $\omega \in \mathcal{B}_\infty(\widehat{\mathcal{D}})$, for any Carleson square S and any non-negative $\varphi \in L^{p_0}(\omega)$, Hölder's inequality yields

$$\begin{aligned}\frac{1}{\nu(S)}\int_S \varphi\nu\,\mathrm{d}A &\leqslant \frac{1}{\nu(S)}\left(\int_S \varphi^{p_0}\omega\,\mathrm{d}A\right)^{\frac{1}{p_0}}\left(\int_S\left(\frac{\nu}{\omega^{\frac{1}{p_0}}}\right)^{p_0'}\mathrm{d}A\right)^{\frac{1}{p_0'}}\\ &\lesssim \left(\frac{1}{\omega(S)}\int_S \varphi^{p_0}\omega\,\mathrm{d}A\right)^{\frac{1}{p_0}}.\end{aligned}$$

It follows that $(M_\nu(\varphi))^{p_0} \lesssim M_\omega(\varphi^{p_0})$ on $\mathbb{D}$. This together with [18, Lemma 3.2] shows that for each $s > 0$ there exists a constant $C = C(s,\omega) > 0$ such that

$$|f(z)|^s \leqslant CM_\omega(f^s)(z), \quad z \in \mathbb{D}, \quad f \in \mathcal{H}(\mathbb{D}). \tag{3.2}$$

By choosing $s = \frac{1}{\alpha} < p$, and using the equivalence between (ii) and (iii) we deduce

$$\|f\|^q_{L^q_\mu} \lesssim \int_{\mathbb{D}}\left(M_\omega(f^{\frac{1}{\alpha}})(z)\right)^{q\alpha}\mathrm{d}\mu(z) \leqslant \|[M_\omega((\cdot)^{\frac{1}{\alpha}})]^\alpha\|^q_{L^p_\omega\to L^q_\mu}\|f\|^q_{A^p_\omega}.$$

To finish the proof of the theorem we observe that (1.5) follows from the arguments above. □

For the sake of completeness we describe the q-vanishing Carleson measures for A^p_ω.

Theorem 3.2. — *Let $0 < p \leqslant q < \infty$ and $\omega \in \mathcal{B}_\infty(\widehat{\mathcal{D}})$, and let μ be a positive Borel measure on $\mathbb{D}$. Then $I_d : A^p_\omega \to L^q_\mu$ is compact if and only if*

$$\lim_{|S|\to 0}\frac{\mu(S)}{(\omega(S))^{\frac{q}{p}}} = 0. \tag{3.3}$$

Proof. — Let $0 < p \leqslant q < \infty$ and $\omega \in \mathcal{B}_\infty(\widehat{\mathcal{D}})$, and first that assume that $I_d : A^p_\omega \to L^q_\mu$ is compact. For each $a \in \mathbb{D}$, consider the function

$$f_{a,p,\gamma}(z) = F_{a,p,\gamma}(z)\omega(S(a))^{-\frac{1}{p}}, \quad z \in \mathbb{D}, \tag{3.4}$$

where $F_{a,p,\gamma}$ is the function defined in (3.1). Then by repeating the argument of [20, Theorem 2.1(ii)] and using Lemma B, we deduce

$$\lim_{|a|\to 1^-}\frac{\mu(S(a))}{(\omega(S(a)))^{\frac{q}{p}}} = 0,$$

and thus (3.3) is satisfied.

Conversely, assume that μ satisfies (3.3), and set

$$\mathrm{d}\mu_r(z) = \chi_{\{r\leqslant|z|<1\}}(z)\,\mathrm{d}\mu(z), \quad z \in \mathbb{D}.$$

Then Theorem 1.3 implies

$$\|h\|_{L^q_{\mu_r}} \lesssim K_{\mu_r}\|h\|_{A^p_\omega}, \quad h \in A^p_\omega,$$

where $K_{\mu_r} = \sup_{a\in\mathbb{D}\setminus\{0\}} \frac{\mu_r(S(a))}{(\omega(S(a)))^{\frac{q}{p}}}$. We will prove next that

$$\lim_{r\to 1^-} K_{\mu_r} = 0, \tag{3.5}$$

and then the rest of the proof follows as that of [20, Theorem 2.1(ii)]. By the assumption, for a given $\varepsilon > 0$, there exists $r_0 \in (0,1)$ such that

$$\sup_{a\in\mathbb{D}\colon |a|\geqslant r_0} \frac{\mu\left(S(a)\right)}{\left(\omega\left(S(a)\right)\right)^{\frac{q}{p}}} < \varepsilon. \tag{3.6}$$

Therefore for each $r \in (0,1)$, we have

$$\sup_{a\in\mathbb{D}\colon |a|\geqslant r_0} \frac{\mu_r\left(S(a)\right)}{\left(\omega\left(S(a)\right)\right)^{\frac{q}{p}}} \leqslant \sup_{a\in\mathbb{D}\colon |a|\geqslant r_0} \frac{\mu\left(S(a)\right)}{\left(\omega\left(S(a)\right)\right)^{\frac{q}{p}}} < \varepsilon. \tag{3.7}$$

Next, if $|a| < r_0$, we choose $n \in \mathbb{N}\setminus\{1\}$ such that $(n-1)(1-r_0) < |I_a| \leqslant n(1-r_0)$. Let I_k be arcs on the boundary such that $|I_k| = 1-r_0$ for all $k = 1,\dots n$, and $I_a \subset \bigcup_{k=1}^n I_k \subset 2I_a$, where

$$2I_a = \left\{ \mathrm{e}^{\mathrm{i}\theta} : |\arg(a\,\mathrm{e}^{-\mathrm{i}\theta})| \leqslant (1-|a|) \right\},$$

where I_j and I_m, $j \neq m$, $j,m \in \{1,2,\dots,n\}$ are disjoint or share an endpoint. Let $r \geqslant r_0$. Then, since $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$ by the hypothesis, (3.6) yields

$$\begin{aligned}\mu_r\left(S(a)\right) &\leqslant \mu_{r_0}\left(S(a)\right) \leqslant \sum_{k=1}^n \mu\left(S(I_k)\right) \leqslant \varepsilon \sum_{k=1}^n \left(\omega\left(S(I_k)\right)\right)^{\frac{q}{p}} \\ &\leqslant \varepsilon \left(\sum_{k=1}^n \omega\left(S(I_k)\right)\right)^{\frac{q}{p}} \leqslant \varepsilon\omega\left(S(2I_a)\right)^{\frac{q}{p}} \lesssim \varepsilon\omega\left(S(a)\right)^{\frac{q}{p}}.\end{aligned}$$

This together with (3.7) gives (3.5), and finishes the proof. □

4. Littlewood–Paley inequalities

We begin with Theorem 1.4, splitting its proof in two parts. We first establish an equivalent norm to $\|\cdot\|_{A^p_\omega}$ and a Littlewood–Paley inequality in terms of the average weight

$$\omega_{h,r}(z) = \frac{\int_{\Delta(z,r)} \omega(\zeta)\,\mathrm{d}A(\zeta)}{(1-|z|)^2}, \quad z \in \mathbb{D},$$

where $r \in (0,1)$, and $\Delta(z,r) = \{u \in \mathbb{D} : \rho(u,z) < r\}$.

Proposition 4.1. — *Let $\omega \in \mathcal{B}_\infty(\widehat{\mathcal{D}})$, $0<r<1$ and $0<p<\infty$. Then the following statements hold:*

(i) $\|f\|_{A^p_\omega} \asymp \|f\|_{A^p_{\omega_{h,r}}}$ *for all* $f \in \mathcal{H}(\mathbb{D})$;

(ii) $\displaystyle\int_{\mathbb{D}} |f^{(k)}(z)|^p(1-|z|)^{kp}\omega_{h,r}(z)\,\mathrm{d}A(z) + \sum_{j=0}^{k-1}|f^{(j)}(0)|^p \lesssim \|f\|^p_{A^p_\omega}$ *for all* $f \in \mathcal{H}(\mathbb{D})$.

Proof.

(i). — Let $0<r<1$ be fixed. Then Fubini's theorem yields

$$\begin{aligned}\int_{S(a)} \frac{\omega(\Delta(\zeta,r))}{(1-|\zeta|)^2}\,\mathrm{d}A(\zeta) & \\ &= \int_{\{z\in\mathbb{D}:S(a)\cap\Delta(z,r)\neq\emptyset\}} \left(\int_{S(a)\cap\Delta(z,r)} \frac{\mathrm{d}A(\zeta)}{(1-|\zeta|)^2}\right)\omega(z)\,\mathrm{d}A(z)\\ &\leqslant \int_{S(b)} \left(\int_{\Delta(z,r)} \frac{\mathrm{d}A(\zeta)}{(1-|\zeta|)^2}\right)\omega(z)\,\mathrm{d}A(z) \asymp \omega(S(b)), \quad |a|>r,\end{aligned}$$

where $b=b(a,r)\in\mathbb{D}$ satisfies $\arg b = \arg a$ and $1-|b| \asymp 1-|a|$ for all $a \in \mathbb{D}\setminus\overline{D(0,r)}$. Since $\mathcal{B}_\infty(\widehat{\mathcal{D}}) \subset \widehat{\mathcal{D}}(\mathbb{D})$, $\omega(S(b)) \lesssim \omega(S(a))$ by Lemma B(ii), and therefore Theorem 1.3 yields

$$\|f\|^p_{A^p_{\omega_{h,r}}} \lesssim \|f\|^p_{A^p_\omega}, \quad f\in\mathcal{H}(\mathbb{D}). \tag{4.1}$$

To see the converse inequality, use the subharmonicity of $|f|^p$ and Fubini's theorem to deduce

$$\|f\|^p_{A^p_\omega} \lesssim \int_{\mathbb{D}} \omega(\zeta)\left(\int_{\Delta(\zeta,r)} \frac{|f(z)|^p}{(1-|z|)^2}\,\mathrm{d}A(z)\right)\mathrm{d}A(\zeta) = \|f\|^p_{A^p_{\omega_{h,r}}}, \quad f\in\mathcal{H}(\mathbb{D}).$$

Thus (i) is proved.

(ii). — Let $0<r<1$ be fixed. It is well known that, for each $0<p<\infty$, $k\in\mathbb{N}$ and $0<s<1$, we have

$$|f^{(k)}(z)|^p \lesssim \frac{1}{(1-|z|)^{2+kp}}\int_{\Delta(z,s)} |f(\zeta)|^p\,\mathrm{d}A(\zeta), \quad z\in\mathbb{D}, \quad f\in\mathcal{H}(\mathbb{D}), \tag{4.2}$$

see, for example, [15, Lemma 2.1] for details. Fix now $s=s(r)\in(0,1)$ and $R=R(r)\in(r,1)$ such that $\Delta(z,r)\subset\Delta(\zeta,R)$ for all $z\in\Delta(\zeta,s)$. Then an application of (4.2), Fubini's theorem and Part (i) give

$$\begin{aligned}
\int_{\mathbb{D}} |f^{(k)}(z)|^p(1-|z|)^{kp}\omega_{h,r}(z)\,\mathrm{d}A(z) \\
&\lesssim \int_{\mathbb{D}}\left(\int_{\Delta(z,s)}\frac{|f(\zeta)|^p}{(1-|\zeta|)^2}\,\mathrm{d}A(\zeta)\right)\omega_{h,r}(z)\,\mathrm{d}A(z)\\
&= \int_{\mathbb{D}}\frac{|f(\zeta)|^p}{(1-|\zeta|)^2}\left(\int_{\Delta(\zeta,s)}\omega_{h,r}(z)\,\mathrm{d}A(z)\right)\mathrm{d}A(\zeta)\\
&\lesssim \|f\|^p_{A^p_{\omega_{h,R}}} \asymp \|f\|^p_{A^p_\omega}, \quad f\in\mathcal{H}(\mathbb{D}).
\end{aligned}$$

Moreover, for each $j\in\mathbb{N}$, $|f^{(j)}(0)|^p \lesssim \int_{D(0,\frac12)}|f|^p\,\mathrm{d}A$ by the subharmonicity of $|f|^p$, and therefore Theorem 1.3 implies $|f^{(j)}(0)|^p\lesssim\|f\|^p_{A^p_\omega}$ once we show that $\int_{S\cap D(0,\frac12)}\mathrm{d}A\lesssim\omega(S)$ for all Carleson squares S. This last inequality is obviously valid if $S=S(a)$ with $|a|\geqslant\frac12$ because in this case the left hand side equals zero. For $|a|\leqslant\frac12$ we have

$$\int_{S(a)\cap D(0,\frac12)}\mathrm{d}A\leqslant\frac18\leqslant\frac18\frac{\omega(S(a))}{\inf_{a\in\overline{D(0,\frac12)}}\omega(S(a))}\lesssim\omega(S(a)).$$

This finishes the proof. □

Given a weight ω and $\beta\in\mathbb{R}$, we denote $\omega_{[\beta]}(z)=(1-|z|)^\beta\omega(z)$ for all $z\in\mathbb{D}$. We will use this definition to shorten the notation in several instances in the proofs from here after.

Proposition 4.2. — *Let $k\in\mathbb{N}$. Then the following statements hold:*

(i) *If $0<p\leqslant1$ and $\omega\in\mathcal{B}_\infty(\widehat{\mathcal{D}})$, then*

$$\|f\|^p_{A^p_\omega}\lesssim\int_{\mathbb{D}}|f^{(k)}(z)|^p(1-|z|)^{kp}\widetilde{\omega}(z)\,\mathrm{d}A(z)+\sum_{j=0}^{k-1}|f^{(j)}(0)|^p,\quad f\in\mathcal{H}(\mathbb{D}).$$

(ii) *If $1<p<\infty$ and $\omega\in\mathcal{B}_\infty(\mathcal{D})$, then*

$$\|f\|^p_{A^p_\omega}\lesssim\int_{\mathbb{D}}|f^{(k)}(z)|^p(1-|z|)^{kp}\widetilde{\omega}(z)\,\mathrm{d}A(z)+\sum_{j=0}^{k-1}|f^{(j)}(0)|^p,\quad f\in\mathcal{H}(\mathbb{D}).$$

Proof.

(i). — Let $0<p\leqslant1$. First observe that $\omega\in\mathcal{B}_\infty(\widehat{\mathcal{D}})\subset\widehat{\mathcal{D}}(\mathbb{D})$, and hence, by Lemma B(ii) and Theorem 1.3, there exists $\beta_0=\beta_0(\omega,p)>0$ such that A^p_ω is continuously embedded into $A^1_{\beta-1}$ for all $\beta\geqslant\beta_0$. A well-known reproducing formula for functions in $A^1_{\beta-1}$ [29, Proposition 4.27] now guarantees the estimate

$$\left|f(z)-\sum_{j=0}^{k-1}f^{(j)}(0)\right|\lesssim\int_{\mathbb{D}}\left|\frac{f^{(k)}(\zeta)}{(1-\overline{z}\zeta)^{1+\beta}}\right|(1-|\zeta|)^{\beta+k-1}\,\mathrm{d}A(\zeta),\quad z\in\mathbb{D}. \tag{4.3}$$

Fix $\beta\geqslant\beta_0$ sufficiently large such that $p(1+\beta)\geqslant\eta$, where $\eta=\eta(\omega)>0$ is that of Lemma B(iv), and $\alpha=p(\beta+k+1)-2>-1$. Then $A^p_\alpha\subset A^1_{\beta+k-1}$ by a well-known embedding that can be also deduced from Theorem 1.3, and hence

$$\left|f(z)-\sum_{j=0}^{k-1}f^{(j)}(0)\right|^p\lesssim\int_{\mathbb{D}}\left|f^{(k)}(\zeta)\right|^p\frac{(1-|\zeta|)^\alpha}{|1-\overline{z}\zeta|^{p(1+\beta)}}\,\mathrm{d}A(\zeta),\quad z\in\mathbb{D}.$$

Therefore Fubini's theorem and Lemma B(iv) yield

$$\begin{aligned}\left\|f-\sum_{j=0}^{k-1}f^{(j)}(0)\right\|^p_{A^p_\omega}&\\ \lesssim\int_{\mathbb{D}}&\left|f^{(k)}(\zeta)\right|^p(1-|\zeta|)^\alpha\left(\int_{\mathbb{D}}\frac{\omega(z)}{|1-\overline{z}\zeta|^{p(1+\beta)}}\,\mathrm{d}A(z)\right)\mathrm{d}A(\zeta)\\ \lesssim\int_{\mathbb{D}}&\left|f^{(k)}(\zeta)\right|^p(1-|\zeta|)^{kp}\frac{\omega(S(\zeta))}{(1-|\zeta|)^2}\,\mathrm{d}A(\zeta),\quad f\in\mathcal{H}(\mathbb{D}).\end{aligned} \tag{4.4}$$

Thus (i) is proved.

(ii). — Let now $1<p<\infty$. Observe that $\omega\in\mathcal{B}(\mathcal{D})\subset\mathcal{D}(\mathbb{D})$. We begin with showing that for each $\omega\in\mathcal{D}(\mathbb{D})$ there exists $\varepsilon_0>0$ such that $\omega_{[-\varepsilon]}\in\mathcal{D}(\mathbb{D})$ for all $0<\varepsilon<\varepsilon_0$. To see this first note that by Lemma 2.1 there exists $\beta=\beta(\omega)>0$ such that $\omega(\mathbb{D}\setminus D(0,r))\lesssim(1-r)^\beta$ for all $0\leqslant r<1$. This and Fubini's theorem yield

$$\begin{aligned}\int_{\mathbb{D}\setminus D(0,\frac12)}&\frac{\omega(z)}{(1-|z|)^\varepsilon}\,\mathrm{d}A(z)\\ &\asymp\int_{\mathbb{D}\setminus D(0,\frac12)}\omega(z)\left(\int_{D(0,|z|)}\frac{\mathrm{d}A(\zeta)}{(1-|\zeta|)^{1+\varepsilon}}\right)\mathrm{d}A(z)\\ &=\int_{\mathbb{D}}\frac{1}{(1-|\zeta|)^{1+\varepsilon}}\left(\int_{\mathbb{D}\setminus D\left(0,\max\left\{\frac12,|\zeta|\right\}\right)}\omega(z)\,\mathrm{d}A(z)\right)\mathrm{d}A(\zeta)\\ &\lesssim\int_{\mathbb{D}}\frac{\mathrm{d}A(\zeta)}{(1-|\zeta|)^{1+\varepsilon-\beta}}<\infty\end{aligned}$$

for each $\varepsilon < \beta$. Thus $\omega_{[-\varepsilon]}$ is a weight for all $0 < \varepsilon < \beta$. Further, for such an ε, the inequality (2.3) and the hypothesis $\omega \in \check{\mathcal{D}}(\mathbb{D})$ yield

$$\begin{aligned}
&\int_{S(a)} \frac{\omega(z)}{(1-|z|)^{\varepsilon}}\,\mathrm{d}A(z)\\
&\quad= \int_{S(a)} \omega(z)\left(\varepsilon\int_0^{|a|}\frac{\mathrm{d}t}{(1-t)^{\varepsilon+1}} + \varepsilon\int_{|a|}^{|z|}\frac{\mathrm{d}t}{(1-t)^{\varepsilon+1}} + 1\right)\mathrm{d}A(z)\\
&\quad= \frac{\omega(S(a))}{(1-|a|)^{\varepsilon}} + \varepsilon\int_{|a|}^{1}\left(\int_{S(a)\setminus D(0,t)}\omega(z)\,\mathrm{d}A(z)\right)\frac{\mathrm{d}t}{(1-t)^{\varepsilon+1}}\\
&\quad\lesssim \frac{\omega(S(a))}{(1-|a|)^{\varepsilon}} + \frac{\omega(S(a))}{(1-|a|)^{\beta}}\varepsilon\int_{|a|}^{1}\frac{\mathrm{d}t}{(1-t)^{\varepsilon+1-\beta}}\\
&\quad= \frac{\omega(S(a))}{(1-|a|)^{\varepsilon}} + \frac{\omega(S(a))}{(1-|a|)^{\varepsilon}}\frac{\varepsilon}{\beta-\varepsilon}\\
&\quad\lesssim \frac{\omega(T_K(a))}{(1-|a|)^{\varepsilon}} \leqslant \int_{T_K(a)}\frac{\omega(z)}{(1-|z|)^{\varepsilon}}\,\mathrm{d}A(z), \quad a\in\mathbb{D}\setminus\{0\},
\end{aligned}$$

and thus $\omega_{[-\varepsilon]} \in \check{\mathcal{D}}(\mathbb{D})$, provided $\omega \in \check{\mathcal{D}}(\mathbb{D})$ and $0 < \varepsilon < \beta$. Further, since $\omega \in \check{\mathcal{D}}(\mathbb{D})$, there exists $K = K(\omega, \varepsilon) > 1$ such that $\omega_{[-\varepsilon]}(S(a)) \lesssim \omega_{[-\varepsilon]}(T_K(a))$ and $\omega(S(a)) \lesssim \omega(T_K(a))$ for all $a \in \mathbb{D}\setminus\{0\}$. Write $a' = \frac{1+|a|}{2}\mathrm{e}^{\mathrm{i}\arg a}$ for short. Then the inequalities just obtained and the hypothesis $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$ yield

$$\begin{aligned}
\omega_{[-\varepsilon]}(S(a)) &\lesssim \omega_{[-\varepsilon]}(T_K(a)) \leqslant \frac{K^{\varepsilon}\omega(T_K(a))}{(1-|a|)^{\varepsilon}} \leqslant \frac{K^{\varepsilon}\omega(S(a))}{(1-|a|)^{\varepsilon}}\\
&\lesssim \frac{\omega(S(a'))}{(1-|a'|)^{\varepsilon}} \lesssim \frac{\omega(T_K(a'))}{(1-|a'|)^{\varepsilon}} \leqslant \omega_{[-\varepsilon]}(T_K(a'))\\
&\leqslant \omega_{[-\varepsilon]}(S(a')), \quad a\in\mathbb{D}\setminus\{0\},
\end{aligned}\tag{4.5}$$

and hence $\omega_{[-\varepsilon]} \in \widehat{\mathcal{D}}(\mathbb{D})$, for all $0 < \varepsilon < \beta$. Therefore we have shown that $\omega_{[-\varepsilon]} \in \mathcal{D}(\mathbb{D})$ for all $0 < \varepsilon < \varepsilon_0 = \beta$.

To prove the statement (ii) of the proposition, fix $\alpha_2 \in \left(\frac{2}{p'}, \frac{2}{p'} + \frac{\varepsilon_0}{p}\right)$, where $\varepsilon_0 = \varepsilon_0(\omega) > 0$ is the constant we just found. Then $\varepsilon = \frac{p}{p'}(p'\alpha_2 - 2) \in (0, \varepsilon_0)$, and thus $\omega_{[-\varepsilon]} \in \widehat{\mathcal{D}}(\mathbb{D})$. Let now $\beta > \max\{\beta_0, (\eta + 2(p-1) + \varepsilon_0)/p - 1\}$, where $\eta = \eta(\omega_{[-\varepsilon]}) > 0$ is that of Lemma B(iv) and $\beta_0 = \beta_0(\omega, 1)$ is that of Part (i). Finally, write $1 + \beta = \alpha_1 + \alpha_2$. Then the estimate (4.3) and Hölder's inequality imply

$$\begin{aligned}\left|f(z)-\sum_{j=0}^{k-1}f^{(j)}(0)\right|^p&\\ &\lesssim\int_{\mathbb{D}}|f^{(k)}(\zeta)|^p\frac{(1-|\zeta|)^{p(\beta+k-1)}}{|1-\overline{z}\zeta|^{p\alpha_1}}\,\mathrm{d}A(\zeta)\left(\int_{\mathbb{D}}\frac{\mathrm{d}A(\zeta)}{|1-\overline{z}\zeta|^{p'\alpha_2}}\right)^{\frac{p}{p'}}\\ &\lesssim\int_{\mathbb{D}}|f^{(k)}(\zeta)|^p\frac{(1-|\zeta|)^{p(\beta+k-1)}}{|1-\overline{z}\zeta|^{p\alpha_1}}\,\mathrm{d}A(\zeta)(1-|z|)^{(2-p'\alpha_2)\frac{p}{p'}},\end{aligned}$$

because $p'\alpha_2>2$. By using this and Fubini's theorem we deduce

$$\begin{aligned}\left\|f-\sum_{j=0}^{k-1}f^{(j)}(0)\right\|_{A^p_\omega}^p\lesssim&\int_{\mathbb{D}}|f^{(k)}(\zeta)|^p(1-|\zeta|)^{p(\beta+k-1)}\\ &\cdot\left(\int_{\mathbb{D}}\frac{\omega(z)(1-|z|)^{(2-p'\alpha_2)\frac{p}{p'}}}{|1-\overline{z}\zeta|^{p\alpha_1}}\,\mathrm{d}A(z)\right)\mathrm{d}A(\zeta)\end{aligned}$$

for all $f\in\mathcal{H}(\mathbb{D})$. Since $\varepsilon=\frac{p}{p'}(p'\alpha_2-2)\in(0,\varepsilon_0)$ and $p\alpha_1>\eta$ by our choices, we may apply Lemma B(iv) to the inner integral above. This together with (4.5) imply

$$\begin{aligned}\left\|f-\sum_{j=0}^{k-1}f^{(j)}(0)\right\|_{A^p_\omega}^p&\lesssim\int_{\mathbb{D}}|f^{(k)}(\zeta)|^p(1-|\zeta|)^{p(\beta+k-1)}\\ &\qquad\cdot\left(\int_{\mathbb{D}}\frac{\omega(z)(1-|z|)^{(2-p'\alpha_2)\frac{p}{p'}}}{|1-\overline{z}\zeta|^{p\alpha_1}}\,\mathrm{d}A(z)\right)\mathrm{d}A(\zeta)\\ &\lesssim\int_{\mathbb{D}}|f^{(k)}(\zeta)|^p(1-|\zeta|)^{p(\beta+k-1)}\frac{\omega_{[-\varepsilon]}(S(\zeta))}{(1-|\zeta|)^{p\alpha_1}}\,\mathrm{d}A(\zeta)\\ &\asymp\int_{\mathbb{D}}|f^{(k)}(\zeta)|^p(1-|\zeta|)^{kp}\frac{\omega(S(\zeta))}{(1-|\zeta|)^2}\,\mathrm{d}A(\zeta),\quad f\in\mathcal{H}(\mathbb{D}),\end{aligned}$$

and finishes the proof of (ii). □

With these preparations we can deduce Theorem 1.4. Namely, it is easy to see that for each $\omega\in\mathcal{D}(\mathbb{D})$, and in particular for each $\omega\in\mathcal{B}_\infty(\mathcal{D})$, there exists $r_0=r_0(\omega)\in(0,1)$ such that $\omega_{h,r}\asymp\widetilde{\omega}$ in $\mathbb{D}$ for each $r\geqslant r_0$. Therefore Theorem 1.4 follows from Propositions 4.1 and 4.2.

We proceed to prove Theorems 1.2 and 1.1 in the said order.

Proof of Theorem 1.2. — Let $0 < r < 1$ be fixed. The inequality (4.2), Fubini's theorem and Proposition 4.1(i) yield

$$\begin{aligned}\int_{\mathbb{D}} |f^{(k)}(z)|^p(1-|z|)^{kp}\omega(z)\,\mathrm{d}A(z)& \\ &\lesssim \int_{\mathbb{D}}\left(\int_{\Delta(z,r)}|f(\zeta)|^p\,\mathrm{d}A(\zeta)\right)\frac{\omega(z)}{(1-|z|)^2}\,\mathrm{d}A(z)\\ &\asymp \int_{\mathbb{D}}|f(\zeta)|^p\frac{\omega(\Delta(\zeta,r))}{(1-|\zeta|)^2}\,\mathrm{d}A(\zeta)\\ &= \|f\|^p_{A^p_{\omega_{h,r}}} \asymp \|f\|^p_{A^p_\omega},\quad f\in\mathcal{H}(\mathbb{D}).\end{aligned}$$

Moreover, arguing as in the proof of Proposition 4.1(ii) we deduce

$$\sum_{j=0}^{k-1}|f^{(j)}(0)|^p \lesssim \|f\|^p_{A^p_\omega},\quad f\in\mathcal{H}(\mathbb{D}).$$

By combining the above estimates we get the assertion. □

Proof of Theorem 1.1. — By Theorem 1.2 it suffices to prove

$$\int_{\mathbb{D}}|f^{(k)}(z)|^p(1-|z|)^{kp}\omega(z)\,\mathrm{d}A(z)+\sum_{j=0}^{k-1}|f^{(j)}(0)|^p \gtrsim \|f\|^p_{A^p_\omega},\quad f\in\mathcal{H}(\mathbb{D}).$$

To see this, we will need to know more about the weights involved. In particular, we want to show that $\omega_{[kp]} \in B_\infty(\mathcal{D})$ for each $k \in \mathbb{N}$ and $0 < p < \infty$. We will deduce this in several steps. First observe that if $\omega \in \check{\mathcal{D}}(\mathbb{D})$ and $\beta > 0$, then $\omega_{[\beta]} \in \check{\mathcal{D}}(\mathbb{D})$. Namely, if $\omega \in \check{\mathcal{D}}(\mathbb{D})$ there exists $K = K(\omega) > 1$ such that for each $\beta > 0$ we have

$$\begin{aligned}\omega_{[\beta]}(T_K(a)) &\geqslant \frac{(1-|a|)^\beta}{K^\beta}\omega(T_K(a)) \gtrsim (1-|a|)^\beta\omega(S(a))\\ &\geqslant \omega_{[\beta]}(S(a)),\quad a\in\mathbb{D}\setminus\{0\},\end{aligned}$$

and hence $\omega_{[\beta]} \in \check{\mathcal{D}}(\mathbb{D})$. If in addition $\omega \in \mathcal{D}(\mathbb{D})$, then $\omega_{[\beta]} \in \widehat{\mathcal{D}}(\mathbb{D})$. To see this, let $a \in \mathbb{D}\setminus\{0\}$ and write $a' = \frac{1+|a|}{2}\,\mathrm{e}^{\mathrm{i}\arg a}$ for short. Since $\omega \in \check{\mathcal{D}}(\mathbb{D})$, there exists $K = K(\omega) > 1$ such that $\omega(T_K(a)) \asymp \omega(S(a))$ for all $a \in \mathbb{D}\setminus\{0\}$. By using this and the hypothesis $\omega \in \widehat{\mathcal{D}}(\mathbb{D})$ we deduce

$$\begin{aligned}\omega_{[\beta]}(S(a)) &\leqslant (1-|a|)^\beta\omega(S(a)) \asymp (1-|a|)^\beta\,\omega(S(a'))\\ &\asymp \left(1-\left(1-\frac{1-|a'|}{K}\right)\right)^\beta\omega(T_K(a'))\\ &\leqslant \omega_{[\beta]}(S(a')),\quad a\in\mathbb{D}\setminus\{0\}.\end{aligned}$$

and thus $\omega_{[\beta]} \in \widehat{\mathcal{D}}(\mathbb{D})$. Therefore we have shown that $\omega_{[\beta]} \in \mathcal{D}(\mathbb{D})$, provided $\omega \in \mathcal{D}(\mathbb{D})$ and $\beta > 0$.

The other property we need to know is that if $\omega \in \mathcal{B}_\infty(\mathcal{D})$, then $\omega_{[\beta]} \in \mathcal{B}_\infty(\mathcal{D})$ for all $\beta > 0$. We will use the fact we just proved to see this and the fact that $\frac{\omega}{\nu} \in B_p(\nu)$ if and only if there exists a constant $C = C(\omega, \nu) > 0$ such that

$$\left(\frac{\int_S |f(z)|\nu(z)\,\mathrm{d}A(z)}{\nu(S)}\right)^p \leqslant C\frac{\int_S |f(z)|^p\omega(z)\,\mathrm{d}A(z)}{\omega(S)} \tag{4.6}$$

for all Carleson squares S and all measurable functions f on $\mathbb{D}$. Next observe that if $\nu \in \mathcal{D}(\mathbb{D})$, $1 < p < \infty$ and $\frac{\omega}{\nu} \in B_p(\nu)$, then $\frac{\omega_{[\beta]}}{\nu_{[\beta]}} \in B_p(\nu_{[\beta]})$ for all $0 < \beta < \infty$. Namely, if $\frac{\omega}{\nu} \in B_p(\nu)$, then (4.6) yields

$$\begin{aligned}
&\left(\frac{\int_{S(a)} |f(z)|\nu_{[\beta]}(z)\,\mathrm{d}A(z)}{\nu_{[\beta]}(S(a))}\right)^p \\
&\qquad\leqslant C\frac{\nu(S(a))^p}{\nu_{[\beta]}(S(a))^p}\frac{\int_{S(a)} |f(z)|^p(1-|z|)^{p\beta}\omega(z)\,\mathrm{d}A(z)}{\omega(S(a))} \\
&\qquad\leqslant C\frac{\nu(S(a))^p}{\nu_{[\beta]}(S(a))^p}(1-|a|)^{(p-1)\beta}\frac{\int_{S(a)} |f(z)|^p\omega_{[\beta]}(z)\,\mathrm{d}A(z)}{\omega(T_K(a))} \\
&\qquad\leqslant C\frac{(1-|a|)^{p\beta}\nu(S(a))^p}{\nu_{[\beta]}(S(a))^p}\frac{\int_{S(a)} |f(z)|^p\omega_{[\beta]}(z)\,\mathrm{d}A(z)}{\omega_{[\beta]}(T_K(a))}
\end{aligned}$$

for all $a \in \mathbb{D}\setminus\{0\}$ and all measurable functions f on $\mathbb{D}$. Since $\nu \in \mathcal{D}(\mathbb{D})$ by the hypothesis, then $\omega \in \mathcal{D}(\mathbb{D})$, and hence $\omega_{[\beta]} \in \mathcal{D}(\mathbb{D})$. Therefore $\omega_{[\beta]}(T_K(a)) \asymp \omega_{[\beta]}(S(a))$ for all $a \in \mathbb{D}\setminus\{0\}$, provided $K = K(\omega, \beta) > 1$ is sufficiently large. Moreover, for $M = M(\nu) > 1$ sufficiently large, we have

$$\begin{aligned}
\nu_{[\beta]}(S(a)) &\leqslant (1-|a|)^\beta\nu(S(a)) \asymp (1-|a|)^\beta\nu(T_M(a)) \\
&\leqslant M^\beta\nu_{[\beta]}(T_M(a)) \leqslant M^\beta\nu_{[\beta]}(S(a)), \quad a \in \mathbb{D}\setminus\{0\}.
\end{aligned}$$

It follows that

$$\left(\frac{\int_{S(a)} |f(z)|\nu_{[\beta]}(z)\,\mathrm{d}A(z)}{\nu_{[\beta]}(S(a))}\right)^p \lesssim \frac{\int_{S(a)} |f(z)|^p\omega_{[\beta]}(z)\,\mathrm{d}A(z)}{\omega_{[\beta]}(S(a))}, \quad a \in \mathbb{D}\setminus\{0\},$$

for all measurable functions f on $\mathbb{D}$. This shows that $\frac{\omega_{[\beta]}}{\nu_{[\beta]}} \in B_p(\nu_{[\beta]})$.

Finally, by the hypothesis $\omega \in \mathcal{B}_\infty(\mathcal{D})$, and hence there exist $1 < p < \infty$ and $\nu \in \mathcal{D}$ such that $\frac{\omega}{\nu} \in B_p(\nu)$. Therefore $\frac{\omega_{[\beta]}}{\nu_{[\beta]}} \in B_p(\nu_{[\beta]})$ for all $0 < \beta < \infty$. Moreover, $\nu_{[\beta]} \in \mathcal{D}$, and hence $\omega_{[\beta]} \in \mathcal{B}_\infty(\mathcal{D})$.

Now we can proceed to prove the statement of the theorem. Recall that we just showed that $\omega_{[kp]} \in B_\infty(\mathcal{D})$ for each $k \in \mathbb{N}$ and $0 < p < \infty$ by the hypothesis $\omega \in B_\infty(\mathcal{D})$. In particular $\omega_{[kp]} \in \mathcal{D}(\mathbb{D})$, and hence the inequality we are after now follows from Proposition 4.2 and Theorem 1.3 if we show that

$$\int_S (1-|\zeta|)^{kp} \frac{\omega(S(\zeta))}{(1-|\zeta|)^2}\,\mathrm{d}A(\zeta) \lesssim \int_S (1-|\zeta|)^{kp}\omega(\zeta)\,\mathrm{d}A(\zeta), \quad S \subset \mathbb{D}. \tag{4.7}$$

Since $\omega \in \check{\mathcal{D}}(\mathbb{D})$, there exists $r = r(\omega) \in (0,1)$ sufficiently large such that $\omega(S(\zeta)) \lesssim \omega(\Delta(\zeta,r))$ for all $\zeta \in \mathbb{D}$. Therefore

$$\begin{aligned}
&\int_{S(a)} (1-|\zeta|)^{kp} \frac{\omega(S(\zeta))}{(1-|\zeta|)^2}\,\mathrm{d}A(\zeta)\\
&\quad \lesssim \int_{S(a)} (1-|\zeta|)^{kp} \frac{\omega(\Delta(\zeta,r))}{(1-|\zeta|)^2}\,\mathrm{d}A(\zeta)\\
&\quad = \int_{\{z:S(a)\cap\Delta(z,r)\neq\emptyset\}} \omega(z)\left(\int_{S(a)\cap\Delta(z,r)} (1-|\zeta|)^{kp-2}\,\mathrm{d}A(\zeta)\right)\mathrm{d}A(z)\\
&\quad \lesssim \int_{\{z:S(a)\cap\Delta(z,r)\neq\emptyset\}} (1-|z|)^{kp}\omega(z)\,\mathrm{d}A(z)\\
&\quad \leqslant \int_{S(a')} (1-|z|)^{kp}\omega(z)\,\mathrm{d}A(z), \quad a \in \mathbb{D}\setminus\{0\},
\end{aligned}$$

where $\arg a' = \arg a$ and $1-|a'| \asymp 1-|a|$ for all $a \in \mathbb{D}\setminus\{0\}$. Moreover, since $\omega_{[kp]} \in \widehat{\mathcal{D}}(\mathbb{D})$, we have $\omega_{[kp]}(S(a')) \lesssim \omega_{[kp]}(S(a))$ for all $a \in \mathbb{D}\setminus\{0\}$. This gives (4.7) and finishes the proof. □

5. Spectra of integration operator

Let $\mathcal{B}$ denote the classical space of Bloch functions, $\mathcal{B}_0$ the little Bloch space and $D_a = \left\{z \in \mathbb{D} : |z-a| < \frac{1-|a|}{2}\right\}$ for all $a \in \mathbb{D}$. Recall that $\widetilde{\omega}$ is essentially constant in each hyperbolically bounded region if $\omega \in \mathcal{B}_\infty(\mathcal{D})$. This together with Theorem 1.4 and [3, Corollary 4.4-Theorem 1.7] implies that $\widetilde{\omega} \in B_\infty$. Therefore the next result follows from [1, Theorem 4.1], Theorem 1.4 and the fact that

$$\widetilde{\omega}(D_a) \asymp \omega(S(a)), \quad a \in \mathbb{D}\setminus\{0\},$$

provided $\omega \in \mathcal{B}_\infty(\mathcal{D})$.

THEOREM 5.1. — *Let $\omega \in \mathcal{B}_\infty(\mathcal{D})$ and $0 < p, q < \infty$. Then the following statements hold:*

(i) *If $0 < p \leqslant q < \infty$, then $T_g : A^p_\omega \to A^q_\omega$ is bounded if and only if*

$$\sup_{a\in\mathbb{D}}(1-|a|)|g'(a)|\left(\omega(S(a))\right)^{\frac{1}{q}-\frac{1}{p}} < \infty.$$

(ii) *If $0 < p \leqslant q < \infty$, then $T_g : A^p_\omega \to A^q_\omega$ is compact if and only if*

$$\lim_{|a|\to 1^-}(1-|a|)|g'(a)|\left(\omega(S(a))\right)^{\frac{1}{q}-\frac{1}{p}} = 0.$$

(iii) *If $0 < q < p < \infty$, then $T_g : A^p_\omega \to A^q_\omega$ is bounded (equivalently compact) if and only if $g \in A^s_\omega$, where $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$.*

Theorem 5.1 shows, in particular, that $T_g : A^p_\omega \to A^p_\omega$ is bounded (resp. compact) if and only if $g \in \mathcal{B}$ (resp. $g \in \mathcal{B}_0$), provided $\omega \in \mathcal{B}_\infty(\mathcal{D})$.

We will next study the spectrum of T_g acting on A^p_ω, when $\omega \in \mathcal{B}_\infty(\mathcal{D})$. We begin with noticing that T_g has no eigenvalues [1, Proposition 5.1], and hence its spectrum is nothing else but $\{0\}$ if $g \in \mathcal{B}_0$. The proof of Theorem 1.5 follows ideas from the papers [1, 2, 3], where the approach used reveals a natural connection to weighted norm inequalities for derivatives. This general idea applies to our context as well. Indeed, a simple computation shows that for a given analytic function h in $\mathbb{D}$ and $\lambda \in \mathbb{C}\setminus\{0\}$, the equation

$$\lambda f - T_g f = h$$

has the unique solution $f = \frac{1}{\lambda} R_{\lambda,g} h$, where

$$R_{\lambda,g}h(z) = h(0)\,\mathrm{e}^{\frac{g(z)}{\lambda}} + \mathrm{e}^{\frac{g(z)}{\lambda}} \int_0^z \mathrm{e}^{-\frac{g(\xi)}{\lambda}}\, h'(\xi)\,\mathrm{d}\xi, \quad z \in \mathbb{D}. \tag{5.1}$$

Thus λ belongs the resolvent set $\rho\,(T_g|A^p_\omega)$ if and only if $R_{\lambda,g}$ is a bounded invertible operator on A^p_ω.

proof of Theorem 1.5. — The equivalence between (i) and (ii) follows by arguing as in the proof of [1, Theorem 5.1] and applying Theorems 1.1 and 5.1.

To see that (ii) and (iii) are equivalent, observe that since $\widetilde{\omega}$ is essentially constant in each hyperbolically bounded region, the proof of [3, Proposition 2.1(b)] shows that there exists a differentiable weight W such that $\widetilde{\omega} \asymp W$ on $\mathbb{D}$, and

$$|\nabla W(z)| \lesssim (1-|z|)W(z), \quad z \in \mathbb{D}. \tag{5.2}$$

Therefore, by arguing as in the first part of the proof, but applying Theorem 1.4 instead of Theorem 1.1, we deduce that

$$\lambda \in \rho\left(T_g \mid A^p_\omega\right) = \rho\left(T_g \,\middle|\, A^p_{\widetilde{\omega}}\right) = \rho\left(T_g \mid A^p_W\right)$$

if and only if

$$\|f\|^p_{A^p_{W_{\lambda,g,p}}} \asymp |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^p W_{\lambda,g,p}(z)\,\mathrm{d}A(z), \quad f \in \mathcal{H}(\mathbb{D}),$$

where $W_{\lambda,g,p} = W \exp\left(p \operatorname{Re} \frac{g}{\lambda}\right)$. Since $g \in \mathcal{B}$, also the weight $W_{\lambda,g,p}$ satisfies (5.2). Indeed,

$$\begin{aligned}\left|\nabla W \exp\left(p \operatorname{Re} \frac{g}{\lambda}\right)(z)\right| &\lesssim \left(|\nabla W(z)| + \frac{p|g'(z)|W(z)}{|\lambda|}\right) \exp\left(p \operatorname{Re} \frac{g}{\lambda}\right)(z) \\ &\lesssim \frac{W \exp\left(p \operatorname{Re} \frac{g}{\lambda}\right)(z)}{1-|z|}, \quad z \in \mathbb{D}.\end{aligned}$$

Hence $W_{\lambda,g,p}$ is essentially constant in each hyperbolically bounded region by [3, Proposition 2.1(i)]. Therefore $\lambda \in \rho(T_g \mid A^p_\omega) = \rho(T_g \mid A^p_{\widetilde{\omega}}) = \rho\left(T_g \mid A^p_W\right)$ if and only if $W_{\lambda,g,p} \in B_\infty$ by [3, Corollary 4.4]. This is equivalent to $\widetilde{\omega} \exp\left(p \operatorname{Re} \frac{g}{\lambda}\right) \in B_\infty$. □

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