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Note on Coleman’s formula for the absolute Frobenius on Fermat curves

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NOTE ON COLEMAN’S FORMULA FOR THE ABSOLUTE FROBENIUS ON FERMAT CURVES

by Tomokazu KASHIO

ABSTRACT. — Coleman calculated the absolute Frobenius on Fermat curves explicitly. In this paper we show that a kind of $p$-adic continuity implies a large part of his formula. To do this, we study a relation between functional equations of the gamma function, monomial relations on CM-periods, and their $p$-adic analogues.

RéSUMÉ. — Coleman a calculé explicitement le Frobenius absolu sur les courbes de Fermat. Dans cet article, nous montrons qu’une sorte de continuité $p$-adique implique une grande partie de sa formule. Pour ce faire, nous étudions une relation entre les équations fonctionnelles de la fonction gamma, les relations monomiales sur CM-périodes, et leurs analogues $p$-adiques.

1. Introduction

We modify Euler’s gamma function $\Gamma(z)$ into

$$\Gamma_{\infty}(z) := \frac{\Gamma(z)}{\sqrt{2\pi}} = \exp(\zeta'(0, z)) \quad (z > 0)$$

and focus on its special values at rational numbers. Here we put $\zeta(s, z) := \sum_{k=0}^{\infty} (z+k)^{-s}$ to be the Hurwitz zeta function. The last equation is due to Lerch. One has a “simple proof” in [14, p. 17]. The gamma function enjoys some functional equations:

$$\Gamma_{\infty}(z)\Gamma_{\infty}(1-z) = \frac{1}{2\sin \pi z}, \quad (1.1)$$

Euler’s Reflection formula:

$$\prod_{k=0}^{d-1} \Gamma_{\infty} \left( z + \frac{k}{d} \right) = d^{\frac{1}{2} - dz} \Gamma_{\infty} (dz) \quad (d \in \mathbb{N}). \quad (1.2)$$

Gauss’ Multiplication formula:

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For proofs, see [1, §3, 4]. The main topic of this paper is a relation between such functional equations and monomial relations of CM-periods, and its $p$-adic analogue. We introduce some notations.

**Definition 1.1.** — Let $K$ be a CM-field. We denote by $I_K$ the $\mathbb{Q}$-vector space formally generated by all complex embeddings of $K$:

$$I_K := \bigoplus_{\sigma \in \text{Hom}(K, \mathbb{C})} \mathbb{Q} \cdot \sigma.$$ 

We identify a subset $S \subset \text{Hom}(K, \mathbb{C})$ as an element $\sum_{\sigma \in S} \sigma \in I_K$. Shimura’s period symbol is the bilinear map

$$p_K : I_K \times I_K \rightarrow \mathbb{C}^\times / \mathbb{Q}^\times$$

characterized by the following properties (P1), (P2).

(P1) Let $A$ be an abelian variety defined over $\overline{\mathbb{Q}}$, having CM of type $(K, \Xi)$. Namely, for each $\sigma \in \text{Hom}(K, \mathbb{C})$, there exists a non-zero “$K$-eigen” differential form $\omega_\sigma$ of the second kind satisfying

$$k^* (\omega_\sigma) = \sigma (k) \omega_\sigma \ (k \in K),$$

where $k^*$ denotes the action of $k \in K$ via $K \cong \text{End}(A) \otimes_\mathbb{Z} \mathbb{Q}$ on the de Rham cohomology $H^1_{dR}(A, \mathbb{C})$. Then we have

$$\Xi = \{ \sigma \in \text{Hom}(K, \mathbb{C}) \mid \omega_\sigma \text{ is holomorphic} \},$$

$$p_K (\sigma, \Xi) \equiv \begin{cases} \pi^{-1} \int_{\gamma} \omega_\sigma & (\sigma \in \Xi) \\ \int_{\gamma} \omega_\sigma & (\sigma \in \text{Hom}(K, \mathbb{C}) - \Xi) \end{cases} \mod \mathbb{Q}^\times$$

for an arbitrary closed path $\gamma \subset A(\mathbb{C})$ satisfying $\int_{\gamma} \omega_\sigma \neq 0$.

(P2) Let $\rho$ be the complex conjugation. Then we have

$$p_K (\sigma, \tau) p_K (\rho \circ \sigma, \tau) \equiv p_K (\sigma, \tau) p_K (\sigma, \rho \circ \tau) \equiv 1 \mod \mathbb{Q}^\times$$

$$(\sigma, \tau \in \text{Hom}(K, \mathbb{C})).$$

We note that, strictly speaking, Shimura’s $p_K$ in [13, §32] is a bilinear map on $\bigoplus_{\sigma \in \text{Hom}(K, \mathbb{C})} \mathbb{Z} \cdot \sigma$. The period symbol also enjoys the following relations:

(P3) Let $\iota : K' \cong K$ be an isomorphism of CM-fields. Then we have

$$p_K (\sigma, \tau) \equiv p_{K'} (\sigma \circ \iota, \tau \circ \iota) \mod \mathbb{Q}^\times \ (\sigma, \tau \in \text{Hom}(K, \mathbb{C})).$$

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(P4) Let $K \subset L$ be a field extension of CM-fields. We define two linear maps defined as

$$\text{Res}: I_L \rightarrow I_K, \quad \tilde{\sigma} \mapsto \tilde{\sigma}|_K \quad (\tilde{\sigma} \in \text{Hom}(L, \mathbb{C})),$$

$$\text{Inf}: I_K \rightarrow I_L, \quad \sigma \mapsto \sum_{\tilde{\sigma} \in \text{Hom}(L, \mathbb{C})} \tilde{\sigma} \quad (\sigma \in \text{Hom}(K, \mathbb{C})).$$

Then we have

$$p_K(\text{Res}(X), Y) \equiv p_L(X, \text{Inf}(Y)) \mod \mathbb{Q}^\times \quad (X \in I_L, \ Y \in I_K).$$

The following results by Gross–Rohrlich and the above relations (P3), (P4) provide an explicit formula [14, Theorem 2.5, Chap. III] on $p_K$ for $K = \mathbb{Q}(\zeta_N)$ ($\zeta_N = e^{\frac{2\pi i}{N}}$, $N \geq 3$). We can rewrite it in the form (1.5) by the arguments in [8, §6]. Let $\sigma_b \in \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \ ((b, N) = 1)$ be defined by $\sigma_b(\zeta_N) := \zeta_b^N$, $(\alpha) \in (0, 1)$ denote the fraction part of $\alpha \in \mathbb{Q} - \mathbb{Z}$.

**Theorem 1.2** ([5, Theorem in Appendix]). — Let $F_N : x^N + y^N = 1$ be the $N$th Fermat curve, $\eta_{r,s} := x^{r-1}y^{s-N}dx$ its differential forms of the second kind $(0 < r, s < N, \ r + s \neq N)$. Then we have for any closed path $\gamma$ on $F_N(\mathbb{C})$ with $\int_{\gamma} \eta_{r,s} \neq 0$

$$\int_{\gamma} \eta_{r,s} \equiv \frac{\Gamma\left(\frac{r}{N}\right)\Gamma\left(\frac{s}{N}\right)}{\Gamma\left(\frac{r+s}{N}\right)} \mod \mathbb{Q}(\zeta_N)^\times. \quad (1.3)$$

**Theorem 1.3** ([5, §2], [14, §2, Chap. III]). — The CM-type corresponding to $\eta_{r,s}$ is

$$\Xi_{r,s} := \left\{ \sigma_b \mid 1 \leq b \leq N, \ (b, N) = 1, \left\langle \frac{br}{N} \right\rangle + \left\langle \frac{bs}{N} \right\rangle + \left\langle \frac{b(N-r-s)}{N} \right\rangle = 1 \right\}. \quad (1.4)$$

That is, we have

$$p_{\mathbb{Q}(\zeta_N)}(\text{id}, \Xi_{r,s}) \equiv \pi^{-1} \int_{\gamma} \eta_{r,s} \quad (r + s < N) \mod \mathbb{Q}^\times \quad \int_{\gamma} \eta_{r,s} \quad (r + s > N).$$

**Corollary 1.4** ([8, Theorem 3]). — We have for any $\frac{a}{N} \in \mathbb{Q} - \mathbb{Z}$

$$\Gamma_{\infty}\left(\frac{a}{N}\right) \equiv \pi^{\frac{1}{2} - \left(\frac{a}{N}\right)} p_{\mathbb{Q}(\zeta_N)}(\text{id}, \sum_{(b, N) = 1} \left(\frac{1}{2} - \left\langle \frac{ab}{N} \right\rangle\right) \cdot \sigma_b) \mod \mathbb{Q}^\times. \quad (1.5)$$

Here the sum runs over all $b$ satisfying $1 \leq b \leq N, \ (b, N) = 1$. 

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Note that (1.5) holds true even if \((a, N) > 1\), essentially due to (P4). Although the following is just a toy problem, we provide its proof by using the period symbol, in order to explain the theme of this paper: we may say that some functional equations of the gamma function “correspond” to some monomial relations of CM-periods.

**Proposition 1.5** (A toy problem). — The explicit formula (1.5) implies the following “functional equations mod\( \overline{\mathbb{Q}}^\times \)” on \(\Gamma(\frac{a}{N})\):

- “Reflection formula”: \(\Gamma_\infty\left(\frac{a}{N}\right)\Gamma_\infty\left(\frac{N-a}{N}\right) \equiv 1 \mod \overline{\mathbb{Q}}^\times\),

- “Multiplication formula”: \(\prod_{k=0}^{d-1} \Gamma_\infty\left(\frac{a}{N} + \frac{k}{d}\right) \equiv \Gamma_\infty\left(\frac{da}{N}\right) \mod \overline{\mathbb{Q}}^\times\).

**Proof.** — “Reflection formula” follows from (P2) immediately. Concerning “Multiplication formula”, we may assume that \(d \mid N\). Under the expression (1.5), “Multiplication formula” is equivalent to

\[
\pi \sum_{k=0}^{d-1} \frac{1}{2} \left(\frac{a}{N} + \frac{k}{d}\right) p_{Q(\zeta_N)} \left(\text{id}, \sum_{(b, N) = 1} \left(\frac{d-1}{2} - \left\langle \frac{ab}{N} + \frac{kb}{d}\right\rangle\right) \cdot \sigma_b\right)
\]

\[
\equiv \pi \frac{1}{2} \left(\frac{ad}{N}\right) p_{Q(\zeta_N)} \left(\text{id}, \sum_{(b, N) = 1} \left(\frac{1}{2} - \left\langle \frac{dab}{N}\right\rangle\right) \cdot \sigma_b\right).
\]

This follows from the multiplication formula

\[
\sum_{k=0}^{d-1} B_1\left(x + \frac{k}{d}\right) = B_1(dx)
\]

for the 1st Bernoulli polynomial \(B_1(x) = x - \frac{1}{2}\).

The aim of this paper is to study a \(p\)-adic analogue of such “correspondence”. More precisely, we shall characterize the \(p\)-adic gamma function by its functional equations and some special values. Then we show that the period symbol and its \(p\)-adic analogue satisfy the corresponding properties to such functional equations. As an application, we provide an alternative proof of a large part of Coleman’s formula (Theorem 2.4(1)): originally, Coleman’s formula was proved by calculating the absolute Frobenius on all Fermat curves. We shall see that it suffices to calculate it on only one curve (Remark 3.7).

**Remark 1.6.** — Yoshida and the author formulated conjectures in [8, 9, 10] which are generalizations of Coleman’s formula, from cyclotomic fields.
to arbitrary CM-fields: Coleman’s formula implies “the reciprocity law on cyclotomic units” [7] and “the Gross–Koblitz formula on Gauss sums” [3, 6] simultaneously. The author conjectured a generalization [8, Conjecture 4] of Coleman’s formula which implies a part of Stark’s conjecture and a generalization of (the rank 1 abelian) Gross–Stark conjecture simultaneously. The results in this paper (in particular Remark 3.7) are very important toward this generalization, since we know only a finite number of algebraic curves (e.g., [2]) whose Jacobian varieties have CM by CM-fields which are not abelian over \( \mathbb{Q} \).

The outline of this paper is as follows. First we introduce Coleman’s formula [4] for the absolute Frobenius on Fermat curves in Section 2. The author rewrote it in the form of Theorem 2.4: roughly speaking, we write Morita’s \( p \)-adic gamma function \( \Gamma_p \) in terms of Shimura’s period symbol \( p_K \), its \( p \)-adic analogue \( p_{K,p} \), and modified Euler’s gamma function \( \Gamma_\infty \). In Section 3, we show that some functional equations almost characterize \( \Gamma_p \) (Corollary 3.3), and the corresponding properties ((3.7), Theorem 3.5) hold for \( p_K, p_{K,p}, \Gamma_\infty \). Then we see that a large part (Corollary 3.6) of Coleman’s formula follows automatically, without explicit computation, under assuming certain \( p \)-adic continuity properties. Unfortunately, our results have a root of unity ambiguity although the original formula is a complete equation, since some definitions are well-defined only up to roots of unity. In Section 4, we confirm that we can show (at least, a part of) needed \( p \)-adic continuity properties relatively easily.

### 2. Coleman’s formula in terms of period symbols

Coleman explicitly calculated the absolute Frobenius on Fermat curves [4]. The author rewrote his formula in [7, 8] as follows.

#### 2.1. \( p \)-adic period symbol

Let \( p \) be a rational prime, \( \mathbb{C}_p \) the \( p \)-adic completion of the algebraic closure \( \overline{\mathbb{Q}}_p \) of \( \mathbb{Q}_p \), and \( \mu_\infty \) the group of all roots of unity. For simplicity, we fix embeddings \( \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \mathbb{C}_p \) and consider any number field as a subfield of each of them. Let \( B_{\text{cris}} \subset B_{\text{dR}} \) be Fontaine’s \( p \)-adic period rings. We consider the composite ring \( B_{\text{cris}} \overline{\mathbb{Q}}_p \subset B_{\text{dR}} \). Let \( A \) be an abelian variety
with CM defined over \( \overline{\mathbb{Q}} \), \( \gamma \) a closed path on \( \subset A(\mathbb{C}) \), and \( \omega \) a differential form of the second kind of \( A \). Then the \( p \)-adic period integral

\[
\int_p : H^1_B(A(\mathbb{C}), \mathbb{Q}) \times H^1_{dR}(A, \overline{\mathbb{Q}}) \to B_{\text{cris}} \overline{\mathbb{Q}_p}, \ (\gamma, \omega) \mapsto \int_{\gamma,p} \omega
\]

is defined by the comparison isomorphisms of \( p \)-adic Hodge theory, instead of the de Rham isomorphism (e.g., [8, §5.1], [7, §6]). Here \( H^B \) denotes the singular (Betti) homology. Then, in a similar manner to \( p_K \), we can define the \( p \)-adic period symbol

\[
p_{K,p} : I_K \times I_K \to (B_{\text{cris}} \overline{\mathbb{Q}_p} - \{0\})^\mathbb{Q}/\mathbb{Q}^\times
\]

satisfying \( p \)-adic analogues of (P1), (P2), (P3), (P4). Here we put \( (B_{\text{cris}} \overline{\mathbb{Q}_p} - \{0\})^\mathbb{Q} := \{ x \in B_{dR} \mid \exists n \in \mathbb{N} \text{ s.t. } x^n \in B_{\text{cris}} \overline{\mathbb{Q}_p} - \{0\} \} \). Moreover the “ratio”

\[
\left[ \int_{\gamma} \omega_\sigma : \int_{\gamma,p} \omega_\sigma \right] \in (\mathbb{C}^\times \times (B_{\text{cris}} \overline{\mathbb{Q}_p} - \{0\}))/\mathbb{Q}^\times
\]

depends only on \( \sigma \in \text{Hom}(K, \mathbb{C}) \) and the CM-type \( \Xi \). That is, if we replace \( A, \omega_\sigma, \gamma \) with \( A', \omega'_\sigma, \gamma' \) for the same \( \Xi, \sigma, \gamma \), then we have

\[
\frac{\int_{\gamma'} \omega'_\sigma}{\int_{\gamma} \omega_\sigma} = \frac{\int_{\gamma'} \omega'_\sigma}{\int_{\gamma,p} \omega_\sigma} \in \overline{\mathbb{Q}}^\times.
\]

Therefore we may consider the following ratio of the symbols \([p_K : p_{K,p}]\), which is well-defined up to \( \mu_\infty \).

**Proposition 2.1** ([8, Proposition 4]). — There exists a bilinear map

\[
[p_K : p_{K,p}] : I_K \times I_K \to (\mathbb{C}^\times \times (B_{\text{cris}} \overline{\mathbb{Q}_p} - \{0\}))/\mu_\infty \times \mu_\infty \overline{\mathbb{Q}}^\times
\]

satisfying the following.

1. Let \( A, \Xi, \sigma, \omega_\sigma, \gamma \) be as in (P1). Then

\[
[p_K : p_{K,p}] (\sigma, \Xi) \\
\equiv \left\{ \begin{array}{ll}
[(2\pi i)^{-1} \int_{\gamma} \omega_\sigma : (2\pi i)_p^{-1} \int_{\gamma,p} \omega_\sigma] & (\sigma \in \Xi) \\
[\int_{\gamma} \omega_\sigma : \int_{\gamma,p} \omega_\sigma] & (\sigma \in \text{Hom}(K, \mathbb{C}) - \Xi) \\
\end{array} \right. \mod (\mu_\infty \times \mu_\infty) \overline{\mathbb{Q}}^\times.
\]

Here \((2\pi i)_p \in B_{\text{cris}}\) is the \( p \)-adic counterpart of \( 2\pi i \) defined in, e.g., [8, §5.1].

2. We have for \( \sigma, \tau \in \text{Hom}(K, \mathbb{C}) \) and for the complex conjugation \( \rho \)

\[
[p_K : p_{K,p}] (\sigma, \tau) \cdot [p_K : p_{K,p}] (\rho \circ \sigma, \tau) \equiv 1 \mod (\mu_\infty \times \mu_\infty) \overline{\mathbb{Q}}^\times,
\]

\[
[p_K : p_{K,p}] (\sigma, \tau) \cdot [p_K : p_{K,p}] (\sigma, \rho \circ \tau) \equiv 1 \mod (\mu_\infty \times \mu_\infty) \overline{\mathbb{Q}}^\times.
\]
Let $\iota: K' \cong K$ be an isomorphism of CM-fields. Then we have for $\sigma, \tau \in \text{Hom}(K, \mathbb{C})$

$$[p_K : p_{K,p}](\sigma, \tau) \equiv [p_{K'} : p_{K',p}](\sigma \circ \iota, \tau \circ \iota) \mod (\mu_\infty \times \mu_\infty)\bar{\mathbb{Q}}^\times.$$  

Let $K \subset L$ be a field extension of CM-fields. Then we have for $X \in I_L, Y \in I_K$

$$[p_K : p_{K,p}](\text{Res}(X), Y) \equiv [p_L : p_{L,p}](X, \text{Inf}(Y)) \mod (\mu_\infty \times \mu_\infty)\bar{\mathbb{Q}}^\times.$$ 

\section{2.2. Coleman’s formula}

Theorem 2.4 below is essentially due to Coleman [4, Theorems 1.7, 3.13]. Note that the original formula does not have a root of unity ambiguity. First we prepare some notations. We assume that $p$ is an odd prime.

**Definition 2.2.**

1. Let $\mathbb{C}_p^1 := \{z \in \mathbb{C}_p^\times \mid |z|_p = 1\}$. We fix a group homomorphism

$$\exp_p: \mathbb{C}_p \to \mathbb{C}_p^1$$

which coincides with the usual power series $\exp_p(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$ on the convergence region. For $\alpha \in \mathbb{C}_p^\times, \beta \in \mathbb{C}_p$, we put

$$\alpha^\beta := \exp_p(\beta \log_p \alpha)$$

with $\log_p$ Iwasawa’s $p$-adic log function.

2. For $z \in \mathbb{C}_p^\times$, we put

$$z^* := \exp_p(\log_p(z)), \quad z^\beta := p^{\text{ord}_p z} z^*.$$ 

Here we define $\text{ord}_p z \in \mathbb{Q}$ by $|z|_p = |p|_p^{\text{ord}_p z}$. Note that $z \equiv z^\beta \mod \mu_\infty$ ($z \in \mathbb{C}_p^\times$).

3. We define the $p$-adic gamma function on $\mathbb{Q}_p$ as follows.

(a) On $\mathbb{Z}_p$, $\Gamma_p(z)$ denotes Morita’s $p$-adic gamma function which is the unique continuous function $\Gamma_p: \mathbb{Z}_p \to \mathbb{C}_p^\times$ satisfying

$$\Gamma_p(n) := (-1)^n \prod_{1 \leq k \leq n-1, p \nmid k} k \quad (n \in \mathbb{N}).$$
(b) On $\mathbb{Q}_p - \mathbb{Z}_p$, we use $\Gamma_p: \mathbb{Q}_p - \mathbb{Z}_p \to \mathcal{O}^\times_{\mathbb{Q}}$ defined in [7, Lemma 4.2], which is a continuous function satisfying

$$
\Gamma_p(z + 1) = z^* \Gamma_p(z), \quad \Gamma_p(2z) = 2^{2z} \frac{1}{2} \Gamma_p(z) \Gamma_p \left( z + \frac{1}{2} \right).
$$

Such a continuous function on $\mathbb{Q}_p - \mathbb{Z}_p$ is unique up to multiplication by $\mu_\infty$.

(4) For $z \in \mathbb{Z}_p$, we define $z_0 \in \{1, 2, \ldots, p\}$, $z_1 \in \mathbb{Z}_p$ by

$$
z = z_0 + pz_1.
$$

Note that when $p \mid z$, we put $z_0 = p$, instead of 0.

(5) Let $W_p$ be the Weil group defined as

$$
W_p := \{ \tau \in \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \mid \tau|_{\mathbb{Q}_p^{ur}} = \sigma_p^\deg \tau \text{ with } \deg \tau \in \mathbb{Z} \}.
$$

Here $\mathbb{Q}_p^{ur}$ denotes the maximal unramified extension of $\mathbb{Q}_p$, $\sigma_p$ the Frobenius automorphism on $\mathbb{Q}_p^{ur}$.

(6) We define the action of $W_p$ on $\mathbb{Q} \cap [0, 1)$ by identifying $\mathbb{Q} \cap [0, 1) = \mu_\infty$. Namely

$$
\tau \left( \frac{a}{N} \right) := \frac{b}{N} \quad \text{if} \quad \tau(\zeta_N^a) = \zeta_N^b \quad (\tau \in W_p).
$$

(7) Let $\Phi_{\text{cris}}$ be the absolute Frobenius automorphism on $B_{\text{cris}}$. We consider the following action of $W_p$ on $B_{\text{cris}} \otimes \mathbb{Q}_p \cong B_{\text{cris}} \otimes \mathbb{Q}_p$:

$$
\Phi_\tau := \Phi_{\text{cris}}^\deg \tau \otimes \tau \quad (\tau \in W_p).
$$

(8) For $\frac{a}{N} \in \mathbb{Q} \cap (0, 1)$ we put

$$
P \left( \frac{a}{N} \right) := \frac{\Gamma_\infty(\frac{a}{N}) \cdot (2\pi i)^{\frac{1}{2} - \left( \frac{N}{2} \right)} p_{\mathbb{Q}(\zeta_N), p} \left( \text{id}, \sum_{(b,N)=1} (\frac{1}{2} - \left( \frac{ab}{N} \right)) \sigma_b \right)}{(2\pi i)^{\frac{1}{2} - \left( \frac{N}{2} \right)} p_{\mathbb{Q}(\zeta_N)}} \left( \text{id}, \sum_{(b,N)=1} (\frac{1}{2} - \left( \frac{ab}{N} \right)) \sigma_b \right)
$$

$$
\in (B_{\text{cris}} \overline{\mathbb{Q}}_p - \{0\})^Q/\mu_\infty.
$$

This definition makes sense since

$$
\frac{\Gamma_\infty(\frac{a}{N})}{(2\pi i)^{\frac{1}{2} - \left( \frac{N}{2} \right)} p_{\mathbb{Q}(\zeta_N)} \left( \text{id}, \sum_{(b,N)=1} (\frac{1}{2} - \left( \frac{ab}{N} \right)) \sigma_b \right)} \in \overline{\mathbb{Q}} \subset B_{\text{cris}} \overline{\mathbb{Q}}_p
$$

by (1.5) and the ratio $[p_K : p_{K, p}]$ is well-defined up to $\mu_\infty$ by Proposition 2.1.
Remarks 2.3.

(1) Let $\mu_{p-1}$ be the group of all $(p-1)$st roots of unity, $p^\mathbb{Z} := \{p^n \mid n \in \mathbb{Z}\}$, $1 + p\mathbb{Z}_p := \{1 + pz \mid z \in \mathbb{Z}_p\}$. Then we have the canonical decomposition

\[\mathbb{Q}_p^\times \to \mu_{p-1} \times p^\mathbb{Z} \times 1 + p\mathbb{Z}_p, z \mapsto (\omega(p^{-\mathrm{ord}_p z}), p^{\mathrm{ord}_p z}, z^*)\]

where $\omega$ denotes the Teichmüller character. The maps $z \mapsto z^*$, $z^\flat$ provide a similar (but non-canonical) decomposition of $\mathbb{C}_p^\times$. Moreover, we note that the maps $z \mapsto \exp(p(z), z^*, z^\flat)$ are continuous homomorphisms.

(2) We easily see that

\[\tau(z) = \langle pz \rangle, \quad \tau^{-1}(z) = z_1 + 1 \quad (z \in \mathbb{Z}_p(\mathbb{Q}) \cap (0, 1), \tau \in W_p, \deg \tau = 1).\]

Theorem 2.4 ([8, Theorem 3]). — Let $p$ be an odd prime.

(1) Assume that $z \in \mathbb{Z}_p(\mathbb{Q}) \cap (0, 1)$. Then we have

\[\Gamma_p(z) \equiv p^\frac{1}{2} - \tau^{-1}(z) - \frac{P(z)}{\Phi_\tau(P(\tau^{-1}(z)))} \mod \mu_\infty \quad (\tau \in W_p, \deg \tau = 1).\]

(2) Assume that $z \in (\mathbb{Q} - \mathbb{Z}_p(\mathbb{Q})) \cap (0, 1)$. Then we have

\[\frac{\Gamma_p(\tau(z))}{\Gamma_p(z)} \equiv p^{\tau(z) - \mathrm{ord}_p z} - \frac{P(\tau(z))}{\Phi_\tau(P(z))} \mod \mu_\infty \quad (\tau \in W_p).\]

Remark 2.5. — As a result, we see that the right-hand sides of Theorem 2.4(1), (2) are $p$-adic continuous on $z$, $(z, \tau(z))$ respectively, since the left-hand sides are so. We use only the $p$-adic continuity in the next section, in order to recover Theorem 2.4(1).

3. Main results

Morita’s $p$-adic gamma function $\Gamma_p: \mathbb{Z}_p \to \mathbb{Z}_p^\times$ is the unique continuous function satisfying

\[(3.1) \quad \Gamma_p(0) = 1, \quad \frac{\Gamma_p(z + 1)}{\Gamma_p(z)} = \begin{cases} -z & (z \in \mathbb{Z}_p^\times), \\ -1 & (z \in p\mathbb{Z}_p). \end{cases}\]

In this section, we study other functional equations characterizing $\Gamma_p$ and provide an alternative proof of Coleman’s formula in the case $z \in \mathbb{Z}_p(\mathbb{Q})$. Strictly speaking, we only “assume” that the right-hand sides of Theorem 2.4(1), (2) are continuous on $z$, $(z, \tau(z))$ respectively (of course, this is correct). Then we can recover a “large part” (Corollary 3.6) of Theorem 2.4(1). We assume that $p$ is an odd prime.
3.1. A characterization of Morita’s \( p \)-adic gamma function

\( \Gamma_p(z) \) satisfies the following \( p \)-adic analogues of multiplication formulas, which we consider only up to roots of unity in this paper. For the detailed formulation and its proof, see [11, “Basic properties of \( \Gamma_p \)” in Section 2 of Chapter IV].

**Proposition 3.1.** — Let \( d \in \mathbb{N} \) with \( p \nmid d \). Then we have for \( z \in \mathbb{Z}_p \)

\[
\prod_{k=0}^{d-1} \Gamma_p \left( z + \frac{k}{d} \right) \equiv d^{1-dz+(dz)}_1 \Gamma_p(dz) \mod \mu_\infty. \tag{3.2}
\]

Note that if \( p \mid d \), then \( z + \frac{k}{d} \) is not in the domain of definition of Morita’s \( \Gamma_p \). In the rest of this subsection, we show that multiplication formulas (3.2) and some conditions characterize Morita’s \( p \)-adic gamma function (at least up to \( \mu_\infty \)).

**Proposition 3.2.** — Assume a continuous function \( f(z): \mathbb{Z}_p \to \mathbb{C}_p^\times \) satisfies

\[
\prod_{k=0}^{d-1} f \left( z + \frac{k}{d} \right) \equiv f(dz) \mod \mu_\infty \quad (p \nmid d). \tag{3.3}
\]

Then the following holds.

1. \( \frac{f(z+1)}{f(z)} \mod \mu_\infty \) depends only on \( \text{ord}_p z \).
2. The values

\[
c_k := \left( \frac{f(p^k+1)}{f(p^k)} \right)^b
\]

characterize the function \( f(z) \) up to \( \mu_\infty \). More precisely, for \( z \in \mathbb{Z}_p \), we write the \( p \)-adic expansion of \( z - 1 \) as

\[
z - 1 = \sum_{k=0}^{\infty} x_k p^k \quad (x_k \in \{0, 1, \ldots, p-1\}).
\]

Then we have

\[
f(z) \equiv \prod_{k=0}^{\infty} \alpha_k x_k^{\frac{p-1}{2}} \mod \mu_\infty \quad \text{with} \quad \alpha_k := c_k \prod_{i=0}^{k-1} c_i^{p^{k-1-i}(p-1)}. \]

Conversely, assume that

\[
f \left( 1 + \sum_{k=0}^{\infty} x_k p^k \right) \equiv \prod_{k=0}^{\infty} \alpha_k x_k^{\frac{p-1}{2}} \mod \mu_\infty \quad (x_k \in \{0, 1, \ldots, p-1\}) \tag{3.4}
\]

for constants \( \alpha_k \in \mathbb{C}_p^\times \) satisfying \( \alpha_k \to 1 \ (k \to \infty) \). Then \( f(z) \) satisfies the functional equations (3.3).
Proof. — We suppress \( \mod \mu_\infty \). Assume (3.3). Replacing \( z \) with \( z + \frac{1}{d} \), we obtain \( \prod_{k=1}^{d} f(z + \frac{k}{d}) \equiv f(dz + 1) \). It follows that \( \frac{f(z+1)}{f(z)} \equiv \frac{f(dz+1)}{f(dz)} \). That is,
\[
g(z) := \frac{f(z+1)}{f(z)} \equiv g(dz) \quad (p \nmid d \in \mathbb{N}).
\]
Then the assertion (1) is clear. Let \( c_k := (g(p^k))^b, a_n := x_0 + x_1 p + \cdots + x_n p^n \) (\( 0 \leq x_i \leq p - 1 \)). We easily see that
\[
\# \{ y = 1, 2, \ldots, a_n \mid \text{ord}_p y = k \} = x_k + \sum_{i=k+1}^{n} x_i p^{i-k-1} (p - 1) \quad (0 \leq k \leq n).
\]
Then we can write
\[
f(a_n + 1)^b = (f(1)g(1)g(2) \cdots g(a_n))^b = f(1)^b \alpha_0^x \alpha_1^{x_1} \cdots \alpha_n^{x_n}
\]
with \( \alpha_k = c_k \prod_{i=0}^{k-1} \alpha_i^{p^k-i(p-1)} \). Since \( \lim_{n \to \infty} f(a_n + 1) \) converges, so do \( \lim_{n \to \infty} f(a_n + 1)^b \) and \( \prod_{k=0}^{\infty} \alpha_k^x \). Moreover we can write
\[
f(z) \equiv f(1) \prod_{k=0}^{\infty} \alpha_k^{x_k}.
\]
Consider the case of \( d = 2 \), \( z = \frac{1}{2} \) of (3.3): \( f(\frac{1}{2}) f(1) \equiv f(1) \). Therefore, noting that \( -\frac{1}{2} = \sum_{k=0}^{\infty} \frac{p-1}{2} p^k \), we obtain
\[
1 \equiv f \left( \frac{1}{2} \right) \equiv f(1) \prod_{k=0}^{\infty} \alpha_k^{\frac{p-1}{2}}, \quad \text{that is,} \quad f(1) \equiv \prod_{k=0}^{\infty} \alpha_k^{-\frac{p-1}{2}}.
\]
Then the assertion (2) is also clear.

Next, assume (3.4). When \( \text{ord}_p z = k \), we see that \( \frac{f(z+1)}{f(z)} \equiv \frac{\alpha_k}{\alpha_k^{p-1}} \) (resp. \( \alpha_0 \)) if \( k > 0 \) (resp. \( k = 0 \)). In particular, \( g(z) := \frac{f(z+1)}{f(z)} \mod \mu_\infty \) depends only on \( \text{ord}_p z \). When \( z + z' = 1 \), the \( p \)-adic expansions \( z - 1 = \sum_{k=0}^{\infty} x_k p^k, \) \( z' - 1 = \sum_{k=0}^{\infty} x'_k p^k \) satisfy \( x_k + x'_k = p - 1 \) for any \( k \). Then we have
\[
f(z) f(z') \equiv \prod_{k=0}^{\infty} \alpha_k^0 = 1.
\]
Therefore the case \( z = 0 \) of (3.3) holds true since we have \( \left( \prod_{k=1}^{d-1} f(\frac{k}{d}) \right)^2 = \prod_{k=1}^{d-1} f(\frac{k}{d}) f(1 - \frac{k}{d}) \equiv 1 \). Then (3.3) for \( z \in \mathbb{N} \) follows by mathematical
induction on $z$ noting that
\[
\prod_{k=0}^{d-1} f \left( z + \frac{k}{d} \right) \equiv \prod_{k=0}^{d-1} f \left( z + \frac{k}{d} \right) g \left( z + \frac{k}{d} \right),
\]
\[
f(dz + d) \equiv f(dz)g(dz) \cdots g(dz + d - 1),
\]
\[
\text{ord}_p (dz + k) = \text{ord}_p \left( z + \frac{k}{d} \right).
\]
Since $\mathbb{N}$ is dense in $\mathbb{Z}_p$, we see that (3.3) holds for any $z \in \mathbb{Z}_p$. \qed

The following corollary provides a nice characterization of $\Gamma_p(z)$ mod $\mu_\infty$ in terms of functional equations and one or two special values.

**Corollary 3.3.** — Assume a continuous function $f(z) : \mathbb{Z}_p \to \mathbb{C}_p^\times$ satisfies
\[
\prod_{k=0}^{d-1} f \left( z + \frac{k}{d} \right) \equiv f(dz) \mod \mu_\infty \quad (p \nmid d)
\]
and put
\[
c_n := \left( \frac{f(p^n + 1)}{f(p^n)} \right)^b.
\]
Then the following equivalences hold:

1. $c_0 = c_1 = \cdots \iff f(z) \equiv c_0^{z-\frac{1}{2}} \mod \mu_\infty$.
2. $c_1 = c_2 = \cdots \iff f(z) \equiv c_0^{z-\frac{1}{2}}(c_1/c_0)^{z_1+\frac{1}{2}} \mod \mu_\infty$.

**Proof.** — We suppress $\mod \mu_\infty$. For (1), assume that $c_0 = c_1 = \cdots$. Then
\[
\alpha_k := c_k \prod_{i=0}^{k-1} \frac{c_i}{c_0} = c_0^k.
\]

Hence we have by Proposition 3.2
\[
f\left( 1 + \sum_{k=0}^{\infty} x_k p^k \right) \equiv \prod_{k=0}^{\infty} \alpha_k = c_0^{\sum_{k=0}^{\infty} x_k p^k - \frac{p-1}{2} p^k} = c_0^{z - 1 + \frac{1}{2}} = c_0^{z-\frac{1}{2}}.
\]
The opposite direction is trivial by definition $c_n := \left( \frac{f(p^n+1)}{f(p^n)} \right)^b$. For (2), the assumption $c_1 = c_2 = \cdots$ implies $\alpha_0 = c_0$, $\alpha_k = c_0^k (c_1/c_0)^{z_1+\frac{1}{2}} (k \geq 1)$. In this case we have
\[
f\left( 1 + \sum_{k=0}^{\infty} x_k p^k \right) \equiv c_0^{\sum_{k=0}^{\infty} x_k p^k - \frac{p-1}{2} p^k} (c_1/c_0) \sum_{k=1}^{\infty} x_k p^{k-1} - \frac{p-1}{2} p^{k-1} = c_0^{z - 1 + \frac{1}{2}} (c_1/c_0)^{z_1+\frac{1}{2}}
\]

since $\sum_{k=1}^{\infty} x_k p^{k-1} = \frac{z-1-x_0}{p} = z_1$. \qed
3.2. Alternative proof of a part of Coleman’s formula

We fix $\tau \in W_p$ with $\text{deg} \tau = 1$ and put

\begin{align*}
G_1(z) &:= \left( p^{\frac{1}{2}\tau^{-1}(z)} \frac{P(z)}{\Phi_\tau(P(\tau^{-1}(z)))} \right)^b \quad (z \in \mathbb{Z}_p \cap (0,1)), \\
G_2(z) &:= \left( p^{(\tau^{-1}(z)-z)\text{ord}_p z P(z)} \frac{1}{\Phi_\tau(P(\tau^{-1}(z)))} \right)^b \quad (z \in (\mathbb{Q} - \mathbb{Z}_p) \cap (0,1)).
\end{align*}

Here we added \( (\quad)^b \) to the right-hand sides of Coleman’s formulas (Theorem 2.4), in order to resolve a root of unity ambiguity, only superficially. Note that $G_2$ corresponds to Theorem 2.4(2) replaced $z$ with $\tau^{-1}(z)$.

By Theorem 2.4(1), we see that $G_1$ is continuous for the $p$-adic topology. $G_2$ is not $p$-adically continuous in the usual sense, on the whole of $(\mathbb{Q} - \mathbb{Z}_p) \cap (0,1)$ (for details, see Remark 3.8). Theorem 2.4(1) only implies the following “continuity”:

\begin{align*}
G_1(z) &\quad \text{is continuous for the relative topology} \\
&\quad \text{induced by } z \in (\mathbb{Q} - \mathbb{Z}_p) \cap (0,1) \mapsto \mathbb{Q}_p \times \mathbb{Q}_p, \ z \mapsto (z, \tau^{-1}(z)).
\end{align*}

In Corollary 3.6, oppositely, we show that the $p$-adic continuity of $G_1, G_2$ implies a “large part”

\[ G_1(z) \equiv a^{z^{-\frac{1}{2}}b^{z_1+\frac{1}{2}}} \Gamma_p(z) \mod \mu_\infty \quad (a, b \in \mathbb{C}_p^\times) \]

of Theorem 2.4(1):

\[ G_1(z) \equiv \Gamma_p(z) \mod \mu_\infty. \]

Besides we shall show the continuity of $G_1(z)$ in Section 4, independently of Theorem 2.4.

Hereinafter in this section, we forget Theorem 2.4. We assume the following Assumption instead.

\textbf{Assumption 3.4.} — $G_1(z)$ is $p$-adically continuous and $G_2(z)$ is continuous in the sense of (3.6). In particular, we regard $G_1$ as a $p$-adic continuous function:

\[ G_1(z): \mathbb{Z}_p \to \mathbb{C}_p. \]

First we derive “multiplication formula”:

\begin{align*}
\prod_{k=0}^{d-1} G_1 \left( z + \frac{k}{d} \right) &\equiv d^{1-dz+(dz)}; G_1(dz) \mod \mu_\infty \quad (p \nmid d \in \mathbb{N})
\end{align*}

independently of Theorem 2.4.
Proof of (3.7). — We suppress mod$\mu_\infty$. Let $z \in \mathbb{Z}_{(p)} \cap (0, \frac{1}{d})$. By Definition 2.2(8) and (3.5) we can write

$$\prod_{k=0}^{d-1} G_1(z + \frac{k}{d}) = \prod_{k=0}^{d-1} \Gamma_\infty(z + \frac{k}{d}) \Phi_\tau \left( \frac{\Gamma_\infty(\tau^{-1}(dz))}{\prod_{k=0}^{d-1} \Gamma_\infty(\tau^{-1}(z + \frac{k}{d}))} \right) \prod_{k=0}^{d-1} p^{\frac{1}{2} - \tau^{-1}(z + \frac{k}{d})}$$

$$\times \text{"products of classical or } p\text{-adic periods"},$$

where the "products of classical or } p\text{-adic periods" become trivial by (1.6), as we saw in the proof of Proposition 1.5. Besides we see that

$$\{ \tau^{-1}(z + \frac{k}{d}) \mid k = 0, \ldots, d-1 \} = \left\{ \frac{\tau^{-1}(dz)}{d} + \frac{k}{d} \mid k = 0, \ldots, d-1 \right\}.$$

To see this, it suffices to show that $\{ \tau^{-1}(\zeta_N^{a_d} \zeta_N^k) \mid k = 0, \ldots, d-1 \}$ and $\{ \tau^{-1}(\zeta_N^{d_a} \zeta_N^k) \mid k = 0, \ldots, d-1 \}$ coincide with each other. We easily see that both of them are the inverse image of $\tau^{-1}(\zeta_N^{d_a})$ under the $d$th power map $\mu_\infty \to \mu_\infty$, $x \mapsto x^d$. Hence we obtain

$$\prod_{k=0}^{d-1} G_1(z + \frac{k}{d}) = \prod_{k=0}^{d-1} \Gamma_\infty(z + \frac{k}{d}) \Phi_\tau \left( \frac{\Gamma_\infty(\tau^{-1}(dz))}{\prod_{k=0}^{d-1} \Gamma_\infty(\tau^{-1}(z + \frac{k}{d}))} \right) \prod_{k=0}^{d-1} p^{\frac{1}{2} - \tau^{-1}(z + \frac{k}{d})}$$

$$= d^{\frac{1}{2} - dz} \cdot \Phi_\tau(d^{-1}(dz) - \frac{1}{2}) \cdot 1 \equiv d^{\frac{1}{2} - dz} \cdot d^{-1}(dz) - \frac{1}{2}$$

by (1.2), (1.6). For the last "$\equiv\)" , we note that $\Phi_\tau$ acts on $\mathbb{Q}_l \supset d^{-1}(dz) - \frac{1}{2}$ as $\tau$. By Remark 2.3(2), we have $\tau^{-1}(dz) = (dz)_1 + 1$. Then the assertion is clear.

Furthermore we can show that $c_n = \left( \frac{f(p^{n+1})}{f(p^n)} \right)^b$ for $f(z) := \frac{G_1(z)}{\Gamma_p(z)}$ is constant, at least for $n \geq 1$.

Theorem 3.5. — We assume Assumption 3.4 and put $f(z) := \frac{G_1(z)}{\Gamma_p(z)}$.

1. The following functional equations hold.

$$\prod_{k=0}^{d-1} f\left( z + \frac{k}{d} \right) \equiv f(dz) \mod \mu_\infty \quad (p \nmid d).$$

2. We have $c_1 = c_2 = \cdots$ for $c_n := \left( \frac{f(p^{n+1})}{f(p^n)} \right)^b$.
**Proof.** — We suppress mod $\mu_\infty$. (1) follows from (3.2), (3.7). For (2), we need for $z \in p\mathbb{Z}_p$

\[
\frac{G_1(pz)G_1(z + 1)}{G_1(pz + 1)G_1(z)} = \frac{\Gamma_p(pz)\Gamma_p(z + 1)}{\Gamma_p(pz + 1)\Gamma_p(z)}.
\]

Since the right-hand side is equal to \( \left\{ \binom{l}{z} \right\}_{\substack{(p|z) \text{ by (3.1)}}} \), it suffices to show that

\[
\frac{G_1(pz)G_1(z + 1)}{G_1(pz + 1)G_1(z)} \equiv 1 \quad (z \in p\mathbb{Z}_p).
\]

Note that we can not use the definition (3.5) directly since $z, z + 1, pz, pz + 1$ are not contained in $(0, 1)$ simultaneously. Therefore a little complicated argument is needed as follows. Let $z \in \mathbb{Z}(p) \cap (0, 1)$. By Remark 2.3(2), we have

\[
\tau(z) = \langle pz \rangle = pz, \quad \text{hence } \tau^{-1}(pz) = z.
\]

We can write

\[
H_1(z) := \frac{G_1(z)G_2(z + \frac{1}{p}) \cdots G_2(z + \frac{p-1}{p})}{G_1(pz)}
\equiv p^{z+(z+\frac{1}{p})+\cdots+(z+\frac{p-1}{p})-\tau^{-1}(z)-\tau^{-1}(z+\frac{1}{p})-\cdots-\tau^{-1}(z+\frac{p-1}{p})}
\times \frac{P(z)P(z + \frac{1}{p}) \cdots P(z + \frac{p-1}{p})}{P(pz)}
\times \Phi_{\tau} \left( \frac{P(z)}{P(\tau^{-1}(z))P(\tau^{-1}(z + \frac{1}{p})) \cdots P(\tau^{-1}(z + \frac{p-1}{p}))} \right).
\]

Here we note that $\text{ord}_p(z + \frac{k}{p}) = -1$ for $k = 1, \ldots, p - 1$. We have

\[
(3.8) \quad \left\{ \tau^{-1} \left( z + \frac{k}{p} \right) \middle| k = 0, \ldots, p - 1 \right\} = \left\{ \frac{z + k}{p} \right\}_{\substack{k = 0, \ldots, p - 1}}
\]

since both of $\{\tau^{-1}(\zeta_{Np}^{a+rk}) \mid k = 0, \ldots, p - 1\}$, $\{\zeta_{Np}^{a+Nk} \mid k = 0, \ldots, p - 1\}$ are the set of the $p$th roots of $\zeta_N^a$ when $z = \frac{a}{N}$. Therefore the $p$-power parts of $H_1$ become

\[
p^{z+(z+\frac{1}{p})+\cdots+(z+\frac{p-1}{p})-\frac{z}{p}-\frac{z+1}{p}-\cdots-\frac{z+p-1}{p}} = p(p-1)z.
\]

*NOTE ON COLEMAN’S FORMULA*
Moreover the “period parts” of $H_1$ become trivial by (1.6), (3.8). Namely we can write

$$H_1(z) \equiv p^{(p-1)z} \frac{\Gamma_\infty(z) \Gamma_\infty(z + \frac{1}{p}) \cdots \Gamma_\infty(z + \frac{p-1}{p})}{\Gamma_\infty(pz)} \times \Phi_\tau \left( \frac{\Gamma_\infty(z)}{\Gamma_\infty(z + \frac{2}{p}) \cdots \Gamma_\infty(z + \frac{z+p-1}{p})} \right).$$

By using the original Multiplication formula (1.2) for $\Gamma_\infty$, we obtain

$$H_1(z) \equiv p^{(p-1)z} p^{\frac{1}{2} - \frac{pz}{p} - \frac{z}{2}} = 1.$$

Next, let $z = \frac{a}{N} \in \mathbb{Z}(p) \cap (-\frac{1}{p}, 0)$. Then we have

- $\tau(z + 1) = pz + 1$. Hence $\tau^{-1}(pz + 1) = z + 1$.
- $\{\tau^{-1}(\zeta_N^k) \mid k = 1, \ldots, p\} = \{\zeta \mid \zeta^p = \zeta_N\} = \{\zeta_{p_N}^a \mid k = 1, \ldots, p\}$. Hence $\{\tau^{-1}(z + \frac{a}{p}) \mid k = 1, \ldots, p\} = \{\frac{z+k}{p} \mid k = 1, \ldots, p\}$.

Then we can prove similarly that

$$H_2(z) := \frac{G_2(z + \frac{1}{p}) \cdots G_2(z + \frac{p-1}{p}) G_1(z + 1)}{G_1(pz + 1)}$$

$$\equiv p^{(z+\frac{1}{p}) + \cdots + (z+\frac{p-1}{p}) + (z+1) - \tau^{-1}(z + \frac{1}{p}) - \cdots - \tau^{-1}(z + \frac{p-1}{p}) - \tau^{-1}(z+1)}$$

$$\times \frac{P(z + \frac{1}{p}) \cdots P(z + \frac{p-1}{p}) P(z + 1)}{P(pz + 1)}$$

$$\times \Phi_\tau \left( \frac{P(z + 1)}{P(\tau^{-1}(z + \frac{1}{p}) \cdots P(\tau^{-1}(z + \frac{p-1}{p})) P(\tau^{-1}(z + 1))} \right)$$

$$\equiv p^{(z+\frac{1}{p}) + \cdots + (z+\frac{p-1}{p}) + (z+1) - \frac{s+1}{p} - \cdots - \frac{s+p-1}{p} - \frac{s+p}{p} p^{\frac{1}{2} - (pz+1)} p^{z+1} - \frac{1}{2}} = 1.$$
Let \( z \in p\mathbb{Z}_p \). Then there exist \( z_n^+ \in p\mathbb{Z}_p \cap (0, \frac{1}{p}) \), \( z_n^- \in p\mathbb{Z}_p \cap (-\frac{1}{p}, 0) \) which converge to \( z \) when \( n \to \infty \) respectively. Then we can write
\[
\frac{G_1(pz)}{G_1(z)} = \lim_{n \to \infty} \frac{G_1(pz_n^+)}{G_1(z_n^+)} = \lim_{n \to \infty} G_2 \left( z_n^+ + \frac{1}{p} \right) \cdots G_2 \left( z_n^+ + \frac{p-1}{p} \right),
\]
\[
\frac{G_1(pz+1)}{G_1(z+1)} = \lim_{n \to \infty} \frac{G_1(pz_n^-+1)}{G_1(z_n^-+1)} = \lim_{n \to \infty} G_2 \left( z_n^- + \frac{1}{p} \right) \cdots G_2 \left( z_n^- + \frac{p-1}{p} \right).
\]
Recall that \( G_2(z) \) is continuous in the sense of (3.6). Clearly we have for \( k = 1, \ldots, p-1 \)
\[
z_n^\pm + \frac{k}{p} \to z + \frac{k}{p} \ (n \to \infty).
\]
Additionally we see that
\[
\tau^{-1} \left( z_n^+ + \frac{k}{p} \right) = \frac{z_n^\pm}{p} + \tau^{-1} \left( \frac{k}{p} \right) \to \frac{z}{p} + \tau^{-1} \left( \frac{k}{p} \right) \ (n \to \infty)
\]
by noting that \( \tau^{-1}(z + z') \equiv \tau^{-1}(z) + \tau^{-1}(z') \mod \mathbb{Z} \ (\forall z, z') \), \( \tau^{-1}(z) \equiv \frac{z}{p} \mod \mathbb{Z} \) if \( p \mid z, \frac{z_n^+}{p} \in (-\frac{1}{p}, \frac{1}{p}), \frac{z_n^-+1}{p} \in [\frac{1}{p}, \frac{p-1}{p}] \). It follows that
\[
\lim_{n \to \infty} G_2 \left( z_n^+ + \frac{k}{p} \right) = \lim_{n \to \infty} G_2 \left( z_n^- + \frac{k}{p} \right).
\]
Then the assertion is clear. \( \square \)

**Corollary 3.6.** Assume Assumption 3.4. Then there exist constants \( a, b \) satisfying
\[
G_1(z) \equiv a^{\frac{n-2}{4}} b^{\frac{1}{2}} p^1 \Gamma_p(z) \mod \mu_\infty.
\]

**Remark 3.7.** In addition to the above results, by computing the absolute Frobenius on only one Fermat curve, we obtain Coleman’s formula \( G_1(z) \equiv \Gamma_p(z) \mod \mu_\infty \). For example, when \( p = 3 \), we obtain it for \( z = \frac{1}{5}, \frac{2}{5} \) by the computation on \( F_5 \). It follows that \( a^{\frac{3}{14}} b^{\frac{1}{14}} \equiv a^{\frac{1}{14}} b^{\frac{1}{14}} \equiv 1 \), hence \( a \equiv b \equiv 1 \).

**Remark 3.8.** We used the assumption \( p \mid z \) only in the last paragraph of the proof for Theorem 3.5 because \( G_2 \) is not \( p \)-adically continuous on the whole of \((Q - \mathbb{Z}_p) \cap (0, 1) \). For example, we put
\[
z_n := \frac{1}{p^2} + \frac{p^{n+1}}{p^{n+2} + (1-p)^n} \in (Q - \mathbb{Z}_p) \cap (0, 1) \ (n \in \mathbb{N})
\]
and take \( \tau \in W_p \) with \( \deg \tau = 1 \) so that
\[
\tau(\zeta_p^2) = \zeta_p^{-1}.
\]
In particular we see that

\[ z_n \to \frac{1}{p^2} \text{ for the } p\text{-adic topology.} \]

On the other hand we see that

\[
\tau^{-1}(z_n) = \tau^{-1} \left( \frac{1}{p^2} \right) + \tau^{-1} \left( \frac{p^{n+1}}{p^{n+2} + (1-p)^n} \right) \mod \mathbb{Z}
\]

\[
= \frac{p^2 - 1}{p^2} + \frac{p^n}{p^{n+2} + (1-p)^n} = 1 - \frac{(1-p)^n}{p^2(p^{n+2} + (1-p)^n)},
\]

\[
1 - \frac{(1-p)^n}{p^2(p^{n+2} + (1-p)^n)} \in \begin{cases} (1,2) & \text{if } n \text{ is odd,} \\ (0,1) & \text{if } n \text{ is even.} \end{cases}
\]

Hence we have

\[
\tau^{-1}(z_n) = \begin{cases} -\frac{(1-p)^n}{p^2(p^{n+2} + (1-p)^n)} & \to -\frac{1}{p^2} \text{ if } n = 2k + 1, k \to \infty, \\ 1 - \frac{(1-p)^n}{p^2(p^{n+2} + (1-p)^n)} & \to 1 - \frac{1}{p^2} \text{ if } n = 2k, k \to \infty. \end{cases}
\]

Then, by Theorem 2.4(2), we see that \( G_2(z_n) = (\Gamma_p(z_n)/\Gamma_p(\tau^{-1}(z_n)))^b \) does not converge \( p \)-adically although \( z_n \) does.

4. On the \( p \)-adic continuity

In the previous section, we showed that the \( p \)-adic continuity of the right-hand sides of Theorem 2.4(1), (2) implies a large part of Theorem 2.4(1) itself. In this section, we see that it is relatively easy to show such \( p \)-adic continuity properties, without explicit computation. For simplicity, we consider only the case \( z \in \mathbb{Z}_p \). Assume that \( p \nmid N \).

**Lemma 4.1 ([3, §VI]).** — Let \( 1 \leq r, s < N \) with \( r + s \neq N \). We consider the formal expansion of the differential form \( \eta_{r,s} = x^r y^{s-N} \frac{dx}{x} \) on \( F_N : x^N + y^N = 1 \) at \((x, y) = (0, 1)\):

\[
\eta_{r,s} = \sum_{n=0}^{\infty} b_{r,s}(n) x^n \frac{dx}{x},
\]

\[
b_{r,s}(n) := \begin{cases} (-1)^{n+r} \left( \frac{s}{N} - 1 \right) & (n \equiv r \mod N), \\ 0 & (n \not\equiv r \mod N). \end{cases}
\]
Let $\Phi$ be the absolute Frobenius on $H^1_{\dR}(F_N, \mathbb{Q}_p)$. Then there exists $\alpha_{r',s'} \in \mathbb{Q}_p$ satisfying

$$\Phi(\eta_{r,s}) = \alpha_{r',s'} \eta_{r',s'}$$

for $r', s'$ with $1 \leq r', s' < N$, $pr \equiv r'$ mod $N$, $ps \equiv s'$ mod $N$.

Then we have

$$\alpha_{r',s'} = \lim_{n \to (0 \mod N)} \frac{pb_{r,s}(n)}{b_{r',s'}(pn)}$$

(4.1)

$$= \lim_{n \to \frac{1}{N}} (-1)^{(p-1)k/p_{r'-r'}} \frac{p\left(\frac{s}{N} - 1\right)}{(pk + \frac{p_{r'-r'}}{N})}.$$  

We note that $\alpha_{r',s'}$ depends only on $(\frac{r'}{N}, \frac{s'}{N})$. That is $\alpha_{r',s'}$ with $N = N_1$ is equal to $\alpha_{t'r',ts'}$ with $N = tN_1$.

**Proposition 4.2.** — $\alpha_{r',s'}$ is $p$-adically continuous on $(\frac{r'}{N}, \frac{s'}{N}) \in (\mathbb{Z}(p) \cap (0,1))^2$.

**Proof.** — It suffices to show that $\alpha_{r_1',s_1'}$ with $N = N_1$ is close to $\alpha_{r_2',s_2'}$ with $N = N_2$ when $\frac{r_1'}{N_1}$ is close to $\frac{r_2'}{N_2}$ and $\frac{s_1'}{N_1}$ is close to $\frac{s_2'}{N_2}$. We may assume $N := N_1 = N_2$ by considering $N = N_1N_2$. First we fix $r' := r_1' = r_2'$ and assume that $s_1'$ is close to $s_2'$. Then we can take the same $k$ for the limit expressions (4.1) of $\alpha_{r',s_1'}$, $\alpha_{r',s_2'}$. We easily see that if $p^j \mid (s_1' - s_2')$, then $p^{j-1} \mid (s_1 - s_2)$. In fact, we can write $s_i' = ps_i - l_iN$ with $l_i = 0, 1, \ldots, p - 1$ since $0 < s_i, s_i' < N$ for $i = 1, 2$. If $p \mid (s_1' - s_2')$, then we have $p \mid (l_1 - l_2)$, so $l_1 = l_2$. Therefore we obtain $s_1 - s_2 = \frac{s_1' - s_2'}{p}$. It follows that $s_1$ also is close to $s_2$. Hence the continuity on $\frac{s}{N}$ is clear since the numerator (resp. the denominator) of the expression (4.1) is a polynomial on $\frac{s}{N}$ (resp. $\frac{s'}{N}$).

For the variable $\frac{r'}{N}$, we replace $x$ with $y$. In other words, replace the point $(x, y) = (0, 1)$ for the expansion with $(1, 0)$. Then the continuity on $\frac{s'}{N}$ also follows from the same argument. $\square$

**Corollary 4.3.** — $G_1(z)$ defined in (3.5) is $p$-adically continuous on $z \in \mathbb{Z}(p) \cap (0,1)$. In particular, we may regard $G_1(z)$ as a continuous function on $\mathbb{Z}_p$.

**Proof.** — CM-types $\Xi_{r,s}$ of (1.4), corresponding to $\eta_{r,s}$, generate the $\mathbb{Q}$-vector space $\{\sum_{\sigma} c_{\sigma} \cdot \sigma \mid c_{\sigma} + c_{\rho_{\sigma}}$ is a constant$\}$. More explicitly, we claim
that
\[
\sum_{(b,N)=1} \left( \frac{1}{2} - \langle \frac{ab}{N} \rangle \right) \sigma_b = \frac{1}{N} \sum_{1 \leq s < N, \ a+s \neq N} \Xi_{a,s} \sigma_b - \frac{N-2}{2N} \sum_{(b,N)=1} \sigma_b,
\]
where \( s \) runs over \( 1 \leq s < N \) with \( a+s \neq N \) in the first sum of the right-hand side. By the definition (1.4), \( \sigma_b \in \Xi_{a,s} \) if and only if \( \langle \frac{s}{N} \rangle = \frac{1}{N}, \ldots, \frac{1}{N} - \langle \frac{ab}{N} \rangle \). The number of such \( b \) is congruent to \(-1-ab \mod N\). Hence we have
\[
\frac{1}{N} \sum_{1 \leq s < N, \ a+s \neq N} \Xi_{a,s} = \sum_{(b,N)=1} \left( \frac{-1-ab}{N} \right) \sigma_b = \sum_{(b,N)=1} \left( 1 - \frac{1}{N} - \langle \frac{ab}{N} \rangle \right) \sigma_b.
\]
Here we note that \( ab \not\equiv 0 \mod N \) since \( (b,N) = 1 \), \( a \not\equiv 0 \mod N \). Then the above claim follows. By substituting this into Definition 2.2(8), we can write
\[
P\left( \frac{a}{N} \right) \equiv \frac{\Gamma(\frac{a}{N})(2\pi i)^{\frac{1}{2} - \frac{N}{2}}}{\pi} \prod_{1 \leq s < N, \ a+s \neq N} \left( (2\pi i)^{e_s} \int_{\gamma_0, \gamma} \eta_{a,s} \right)^{\frac{1}{N}} \mod \mu_{\infty},
\]
where we put
\[
e_s := \begin{cases} -1 & (a+s < N) \\ 0 & (a+s > N) \end{cases}
\]
since the part \( \sum_{(b,N)=1} \sigma_b \) becomes trivial by Proposition 2.1(2). We can strengthen the congruence relation \( \equiv \) of the formula (1.3) into an equality \( = \), by selecting a specific closed path \( \gamma_0 \) (e.g., \( \gamma_0 = N\gamma \) with \( \gamma \) in [12, Proposition 4.9]). Then we have
\[
P\left( \frac{a}{N} \right) \equiv c \cdot (2\pi i)^{\frac{1}{2} + \frac{N}{2}} \prod_{1 \leq s < N, \ a+s \neq N} \left( \int_{\gamma_0, \gamma} \eta_{a,s} \right)^{\frac{1}{N}} \mod \mu_{\infty},
\]
where we put
\[
c := \frac{\Gamma(\frac{a}{N})}{(2\pi i)^{\frac{N}{2}}} \left( \prod_{1 \leq s < N, \ a+s \neq N} \frac{\Gamma(\frac{a+s}{N})}{\Gamma(\frac{a}{N})\Gamma(\frac{s}{N})} \right)^{\frac{1}{N}}.
\]
Since (1.2) implies that
\[
\prod_{1 \leq s \leq N} \frac{\Gamma(\frac{a+s}{N})}{\Gamma(\frac{a}{N})\Gamma(\frac{s}{N})} = \frac{N^{-a}a!}{\Gamma(\frac{a}{N})^{N}},
\]
we obtain
\[
c = \frac{\Gamma(\frac{a}{N})}{(2\pi i)^{\frac{N}{2}}} \left( \frac{\Gamma(\frac{a}{N})\Gamma(\frac{N-a}{N})}{\Gamma(1)} \cdot \frac{\Gamma(\frac{a}{N})\Gamma(\frac{N}{N})}{\Gamma(\frac{a+N}{N}) \Gamma(\frac{N}{N})} \right)^{\frac{1}{N}} = \left( \frac{N^{1-a}(a-1)!}{2\sin(\frac{a}{N}\pi)} \right)^{\frac{1}{N}}.
\]
For the last equality we used (1.1) and the difference equation \( \Gamma(z + 1) = z\Gamma(z) \). Take \( \tau \in W_p \) with \( \deg \tau = 1 \). Then we have

\[
G_1 \left( \frac{a'}{N} \right) \equiv p^{\frac{1}{2}-\frac{s}{2}} \frac{P \left( \frac{a'}{N} \right)}{\Phi \left( \frac{a'}{N} \right)}
\]
\[
\equiv \left( \frac{N^{a-a'}(a'-1)!}{p^{a-N-1}(a-1)!} \prod_{1 \leq s < N, \ a+s \neq N} \alpha_{a',s}^{-1} \right)^{\frac{1}{N}} \mod \mu_\infty,
\]

by noting that \( \Phi \left( (2\pi i)_p \right) = p(2\pi i)_p \) and \( \Phi \left( \sin \left( \frac{a}{N} \pi \right) \right) = \tau(\sin \left( \frac{a}{N} \pi \right)) = \pm \sin \left( \frac{a'}{N} \pi \right) \). Here \( a', s' \) denote integers satisfying \( 1 \leq a', s' < N, \ pa \equiv a' \mod N, \ ps \equiv s' \mod N \) as above. By Proposition 4.2, \( \alpha_{a',s'} \) are continuous for \( a' \). When \( a \) is in a small open ball, as we saw in the proof of Proposition 4.2, we may write \( a' = pa - M \) for a fixed \( M \) (\( M \in \mathbb{N} \) in the proof of Proposition 4.2). Then the remaining part becomes

\[
\frac{N^{a-a'}(a'-1)!}{p^{a-N-1}(a-1)!} = \pm \Gamma_p(a' + M + 1) \frac{p^N N^{(1-p)a'+M}}{a'(a'+1)(a'+2) \cdots (a'+M)},
\]

which is also continuous as desired. \( \square \)

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