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NOTE ON COLEMAN'S FORMULA FOR THE ABSOLUTE FROBENIUS ON FERMAT CURVES

by Tomokazu KASHIO

Abstract. — Coleman calculated the absolute Frobenius on Fermat curves explicitly. In this paper we show that a kind of p-adic continuity implies a large part of his formula. To do this, we study a relation between functional equations of the gamma function, monomial relations on CM-periods, and their p-adic analogues.

Résumé. — Coleman a calculé explicitement le Frobenius absolu sur les courbes de Fermat. Dans cet article, nous montrons qu'une sorte de continuité p-adique implique une grande partie de sa formule. Pour ce faire, nous étudions une relation entre les équations fonctionnelles de la fonction gamma, les relations monomiales sur CM-périodes, et leurs analogues p-adiques.

1. Introduction

We modify Euler's gamma function $\Gamma(z)$ into

$$\Gamma_\infty(z) := \frac{\Gamma(z)}{\sqrt{2\pi}} = \exp(\zeta'(0,z)) \quad (z > 0)$$

and focus on its special values at rational numbers. Here we put $\zeta(s,z) := \sum_{k=0}^{\infty}(z+k)^{-s}$ to be the Hurwitz zeta function. The last equation is due to Lerch. One has a"simple proof" in [14, p. 17]. The gamma function enjoys some functional equations:

(1.1) Euler's Reflection formula: $\Gamma_\infty(z)\Gamma_\infty(1-z) = \dfrac{1}{2\sin \pi z}$,

(1.2) Gauss' Multiplication formula:

$$\prod_{k=0}^{d-1} \Gamma_\infty\left(z+\frac{k}{d}\right) = d^{\frac{1}{2}-dz}\Gamma_\infty(dz) \quad (d \in \mathbb{N}).$$

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For proofs, see [1, §3, 4]. The main topic of this paper is a relation between such functional equations and monomial relations of CM-periods, and its p-adic analogue. We introduce some notations.

Definition 1.1. — *Let K be a CM-field. We denote by I_K the $\mathbb{Q}$-vector space formally generated by all complex embeddings of K:*

$$I_K := \bigoplus_{\sigma\in \mathrm{Hom}(K,\mathbb{C})} \mathbb{Q}\cdot\sigma.$$

We identify a subset $S\subset \mathrm{Hom}(K,\mathbb{C})$ as an element $\sum_{\sigma\in S}\sigma\in I_K$. Shimura's period symbol is the bilinear map

$$p_K\colon I_K\times I_K\to \mathbb{C}^\times/\overline{\mathbb{Q}}^\times$$

characterized by the following properties (P1), (P2).

(P1) *Let A be an abelian variety defined over $\overline{\mathbb{Q}}$, having CM of type (K,Ξ). Namely, for each $\sigma\in\mathrm{Hom}(K,\mathbb{C})$, there exists a non-zero "K-eigen" differential form ω_σ of the second kind satisfying*

$$k^*(\omega_\sigma)=\sigma(k)\omega_\sigma\quad (k\in K),$$

where k^* denotes the action of $k\in K$ via $K\cong \mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$ on the de Rham cohomology $H^1_{\mathrm{d}R}(A,\mathbb{C})$. Then we have

$$\Xi=\{\sigma\in\mathrm{Hom}(K,\mathbb{C})\mid \omega_\sigma \text{ is holomorphic}\},$$

$$p_K(\sigma,\Xi)\equiv\begin{cases}\pi^{-1}\int_\gamma\omega_\sigma & (\sigma\in\Xi)\\ \int_\gamma\omega_\sigma & (\sigma\in\mathrm{Hom}(K,\mathbb{C})-\Xi)\end{cases}\quad \bmod \overline{\mathbb{Q}}^\times$$

for an arbitrary closed path $\gamma\subset A(\mathbb{C})$ satisfying $\int_\gamma\omega_\sigma\neq 0$.

(P2) *Let ρ be the complex conjugation. Then we have*

$$p_K(\sigma,\tau)p_K(\rho\circ\sigma,\tau)\equiv p_K(\sigma,\tau)p_K(\sigma,\rho\circ\tau)\equiv 1\mod \overline{\mathbb{Q}}^\times\qquad(\sigma,\tau\in\mathrm{Hom}(K,\mathbb{C})).$$

We note that, strictly speaking, Shimura's p_K in [13, §32] is a bilinear map on $\bigoplus_{\sigma\in\mathrm{Hom}(K,\mathbb{C})}\mathbb{Z}\cdot\sigma$. The period symbol also enjoys the following relations:

(P3) Let $\iota\colon K'\cong K$ be an isomorphism of CM-fields. Then we have

$$p_K(\sigma,\tau)\equiv p_{K'}(\sigma\circ\iota,\tau\circ\iota)\mod\overline{\mathbb{Q}}^\times\quad(\sigma,\tau\in\mathrm{Hom}(K,\mathbb{C})).$$

(P4) Let $K \subset L$ be a field extension of CM-fields. We define two linear maps defined as

$$\begin{aligned} &\mathrm{Res}\colon I_L \to I_K,\ \widetilde{\sigma} \mapsto \widetilde{\sigma}|_K \quad (\widetilde{\sigma} \in \mathrm{Hom}(L, \mathbb{C})), \\ &\mathrm{Inf}\colon I_K \to I_L,\ \sigma \mapsto \sum_{\substack{\widetilde{\sigma} \in \mathrm{Hom}(L,\mathbb{C}) \\ \widetilde{\sigma}|_K = \sigma}} \widetilde{\sigma} \quad (\sigma \in \mathrm{Hom}(K, \mathbb{C})). \end{aligned}$$

Then we have

$$p_K(\mathrm{Res}(X), Y) \equiv p_L(X, \mathrm{Inf}(Y)) \mod \overline{\mathbb{Q}}^\times \quad (X \in I_L,\ Y \in I_K).$$

The following results by Gross–Rohrlich and the above relations (P3), (P4) provide an explicit formula [14, Theorem 2.5, Chap. III] on p_K for $K = \mathbb{Q}(\zeta_N)$ ($\zeta_N = \mathrm{e}^{\frac{2\pi \mathrm{i}}{N}}$, $N \geqslant 3$). We can rewrite it in the form (1.5) by the arguments in [8, §6]. Let $\sigma_b \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ ($(b, N) = 1$) be defined by $\sigma_b(\zeta_N) \coloneqq \zeta_N^b$, $\langle \alpha \rangle \in (0, 1)$ denote the fraction part of $\alpha \in \mathbb{Q} - \mathbb{Z}$.

Theorem 1.2 ([5, Theorem in Appendix]). — *Let $F_N : x^N + y^N = 1$ be the Nth Fermat curve, $\eta_{r,s} \coloneqq x^{r-1}y^{s-N}\mathrm{d}x$ its differential forms of the second kind ($0 < r, s < N$, $r + s \neq N$). Then we have for any closed path γ on $F_N(\mathbb{C})$ with $\int_\gamma \eta_{r,s} \neq 0$*

$$\int_\gamma \eta_{r,s} \equiv \frac{\Gamma(\frac{r}{N})\Gamma(\frac{s}{N})}{\Gamma(\frac{r+s}{N})} \mod \mathbb{Q}(\zeta_N)^\times. \tag{1.3}$$

Theorem 1.3 ([5, §2], [14, §2, Chap. III]). — *The CM-type corresponding to $\eta_{r,s}$ is*

$$\Xi_{r,s} \coloneqq \left\{ \sigma_b \,\middle|\, \begin{array}{l} 1 \leqslant b \leqslant N,\ (b, N) = 1, \\ \left\langle \frac{br}{N} \right\rangle + \left\langle \frac{bs}{N} \right\rangle + \left\langle \frac{b(N-r-s)}{N} \right\rangle = 1 \end{array} \right\}. \tag{1.4}$$

That is, we have

$$p_{\mathbb{Q}(\zeta_N)}(\mathrm{id}, \Xi_{r,s}) \equiv \begin{cases} \pi^{-1} \int_\gamma \eta_{r,s} & (r + s < N) \\ \int_\gamma \eta_{r,s} & (r + s > N) \end{cases} \mod \overline{\mathbb{Q}}^\times.$$

Corollary 1.4 ([8, Theorem 3]). — *We have for any $\frac{a}{N} \in \mathbb{Q} - \mathbb{Z}$*

$$\Gamma_\infty\left(\frac{a}{N}\right) \equiv \pi^{\frac{1}{2} - \langle \frac{a}{N} \rangle} p_{\mathbb{Q}(\zeta_N)}\left(\mathrm{id}, \sum_{(b,N)=1} \left(\frac{1}{2} - \left\langle \frac{ab}{N} \right\rangle\right) \cdot \sigma_b\right) \mod \overline{\mathbb{Q}}^\times. \tag{1.5}$$

Here the sum runs over all b satisfying $1 \leqslant b \leqslant N$, $(b, N) = 1$.

Note that (1.5) holds true even if $(a, N) > 1$, essentially due to (P4). Although the following is just a toy problem, we provide its proof by using the period symbol, in order to explain the theme of this paper: we may say that some functional equations of the gamma function "correspond" to some monomial relations of CM-periods.

Proposition 1.5 (A toy problem). — *The explicit formula* (1.5) *implies the following "functional equations* $\mathrm{mod}\overline{\mathbb{Q}}^{\times}$*" on* $\Gamma(\frac{a}{N})$*:*

"Reflection formula": $$\Gamma_{\infty}\left(\frac{a}{N}\right)\Gamma_{\infty}\left(\frac{N-a}{N}\right) \equiv 1 \mod \overline{\mathbb{Q}}^{\times},$$

"Multiplication formula": $$\prod_{k=0}^{d-1}\Gamma_{\infty}\left(\frac{a}{N}+\frac{k}{d}\right) \equiv \Gamma_{\infty}\left(\frac{da}{N}\right) \mod \overline{\mathbb{Q}}^{\times}.$$

Proof. — "Reflection formula" follows from (P2) immediately. Concerning "Multiplication formula", we may assume that $d \mid N$. Under the expression (1.5), "Multiplication formula" is equivalent to

$$\pi^{\sum_{k=0}^{d-1}\frac{1}{2}-\langle\frac{a}{N}+\frac{k}{d}\rangle} p_{\mathbb{Q}(\zeta_N)}\left(\mathrm{id}, \sum_{(b,N)=1}\left(\sum_{k=0}^{d-1}\frac{1}{2}-\left\langle\frac{ab}{N}+\frac{kb}{d}\right\rangle\right)\cdot\sigma_b\right)$$
$$\equiv \pi^{\frac{1}{2}-\langle\frac{ad}{N}\rangle} p_{\mathbb{Q}(\zeta_N)}\left(\mathrm{id}, \sum_{(b,N)=1}\left(\frac{1}{2}-\left\langle\frac{dab}{N}\right\rangle\right)\cdot\sigma_b\right).$$

This follows from the multiplication formula

$$\sum_{k=0}^{d-1}B_1\left(x+\frac{k}{d}\right) = B_1(dx) \tag{1.6}$$

for the 1st Bernoulli polynomial $B_1(x) = x - \frac{1}{2}$. □

The aim of this paper is to study a p-adic analogue of such "correspondence". More precisely, we shall characterize the p-adic gamma function by its functional equations and some special values. Then we show that the period symbol and its p-adic analogue satisfy the corresponding properties to such functional equations. As an application, we provide an alternative proof of a large part of Coleman's formula (Theorem 2.4(1)): originally, Coleman's formula was proved by calculating the absolute Frobenius on *all Fermat curves*. We shall see that it suffices to calculate it on *only one curve* (Remark 3.7).

Remark 1.6. — Yoshida and the author formulated conjectures in [8, 9, 10] which are generalizations of Coleman's formula, from cyclotomic fields to arbitrary CM-fields: Coleman's formula implies "the reciprocity law on cyclotomic units" [7] and "the Gross–Koblitz formula on Gauss sums" [3, 6] simultaneously. The author conjectured a generalization [8, Conjecture 4] of Coleman's formula which implies a part of Stark's conjecture and a generalization of (the rank 1 abelian) Gross–Stark conjecture simultaneously. The results in this paper (in particular Remark 3.7) are very important toward this generalization, since we know only a finite number of algebraic curves (e.g., [2]) whose Jacobian varieties have CM by CM-fields which are not abelian over $\mathbb{Q}$.

The outline of this paper is as follows. First we introduce Coleman's formula [4] for the absolute Frobenius on Fermat curves in Section 2. The author rewrote it in the form of Theorem 2.4: roughly speaking, we write Morita's p-adic gamma function Γ_p in terms of Shimura's period symbol p_K, its p-adic analogue $p_{K,p}$, and modified Euler's gamma function Γ_∞. In Section 3, we show that some functional equations almost characterize Γ_p (Corollary 3.3), and the corresponding properties ((3.7), Theorem 3.5) hold for $p_K, p_{K,p}, \Gamma_\infty$. Then we see that a large part (Corollary 3.6) of Coleman's formula follows automatically, without explicit computation, under assuming certain p-adic continuity properties. Unfortunately, our results have a root of unity ambiguity although the original formula is a complete equation, since some definitions are well-defined only up to roots of unity. In Section 4, we confirm that we can show (at least, a part of) needed p-adic continuity properties relatively easily.

2. Coleman's formula in terms of period symbols

Coleman explicitly calculated the absolute Frobenius on Fermat curves [4]. The author rewrote his formula in [7, 8] as follows.

2.1. p-adic period symbol

Let p be a rational prime, $\mathbb{C}_p$ the p-adic completion of the algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$, and μ_∞ the group of all roots of unity. For simplicity, we fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \mathbb{C}_p$ and consider any number field as a subfield of each of them. Let $B_{\mathrm{cris}} \subset B_{\mathrm{d}R}$ be Fontaine's p-adic period rings. We consider the composite ring $B_{\mathrm{cris}}\overline{\mathbb{Q}_p} \subset B_{\mathrm{d}R}$. Let A be an abelian variety with CM defined over $\overline{\mathbb{Q}}$, γ a closed path on $\subset A(\mathbb{C})$, and ω a differential form of the second kind of A. Then the p-adic period integral

$$\int_p : H_1^{\mathrm{B}}(A(\mathbb{C}),\mathbb{Q}) \times H^1_{\mathrm{d}R}(A,\overline{\mathbb{Q}}) \to B_{\mathrm{cris}}\overline{\mathbb{Q}_p},\ (\gamma,\omega)\mapsto \int_{\gamma,p}\omega$$

is defined by the comparison isomorphisms of p-adic Hodge theory, instead of the de Rham isomorphism (e.g., [8, §5.1], [7, §6]). Here H^B denotes the singular (Betti) homology. Then, in a similar manner to p_K, we can define the p-adic period symbol

$$p_{K,p}\colon I_K\times I_K \to (B_{\mathrm{cris}}\overline{\mathbb{Q}_p}-\{0\})^{\mathbb{Q}}/\overline{\mathbb{Q}}^{\times}$$

satisfying p-adic analogues of (P1), (P2), (P3), (P4). Here we put $(B_{\mathrm{cris}}\overline{\mathbb{Q}_p}-\{0\})^{\mathbb{Q}} := \{x\in B_{\mathrm{d}R} \mid \exists n\in\mathbb{N} \text{ s.t. } x^n \in B_{\mathrm{cris}}\overline{\mathbb{Q}_p}-\{0\}\}$. Moreover the "ratio"

$$\left[\int_\gamma \omega_\sigma : \int_{\gamma,p}\omega_\sigma\right] \in (\mathbb{C}^\times \times (B_{\mathrm{cris}}\overline{\mathbb{Q}_p}-\{0\}))/\overline{\mathbb{Q}}^{\times}$$

depends only on $\sigma\in\mathrm{Hom}(K,\mathbb{C})$ and the CM-type Ξ. That is, if we replace A,ω_σ,γ with $A',\omega'_\sigma,\gamma'$ for the same Ξ,σ, then we have

$$\frac{\int_{\gamma'}\omega'_\sigma}{\int_\gamma\omega_\sigma} = \frac{\int_{\gamma',p}\omega'_\sigma}{\int_{\gamma,p}\omega_\sigma}\in\overline{\mathbb{Q}}^{\times}.$$

Therefore we may consider the following ratio of the symbols $[p_K : p_{K,p}]$, which is well-defined up to μ_∞.

Proposition 2.1 ([8, Proposition 4]). — *There exists a bilinear map*

$$[p_K : p_{K,p}]\colon I_K\times I_K \to (\mathbb{C}^\times \times (B_{\mathrm{cris}}\overline{\mathbb{Q}_p}-\{0\})^{\mathbb{Q}})/(\mu_\infty\times\mu_\infty)\overline{\mathbb{Q}}^{\times}$$

satisfying the following.

(1) *Let* $A,\Xi,\sigma,\omega_\sigma,\gamma$ *be as in* (P1). *Then*

$$[p_K : p_{K,p}](\sigma,\Xi) \equiv \begin{cases} [(2\pi\mathrm{i})^{-1}\int_\gamma\omega_\sigma : (2\pi\mathrm{i})_p^{-1}\int_{\gamma,p}\omega_\sigma] & (\sigma\in\Xi)\\ [\int_\gamma\omega_\sigma : \int_{\gamma,p}\omega_\sigma] & (\sigma\in\mathrm{Hom}(K,\mathbb{C})-\Xi)\end{cases} \mod (\mu_\infty\times\mu_\infty)\overline{\mathbb{Q}}^{\times}.$$

Here $(2\pi\mathrm{i})_p\in B_{\mathrm{cris}}$ *is the* p*-adic counterpart of* $2\pi\mathrm{i}$ *defined in, e.g.,* [8, §5.1].

(2) *We have for* $\sigma,\tau\in\mathrm{Hom}(K,\mathbb{C})$ *and for the complex conjugation* ρ

$$[p_K : p_{K,p}](\sigma,\tau)\cdot[p_K : p_{K,p}](\rho\circ\sigma,\tau)\equiv 1 \mod (\mu_\infty\times\mu_\infty)\overline{\mathbb{Q}}^{\times},$$

$$[p_K : p_{K,p}](\sigma,\tau)\cdot[p_K : p_{K,p}](\sigma,\rho\circ\tau)\equiv 1 \mod (\mu_\infty\times\mu_\infty)\overline{\mathbb{Q}}^{\times}.$$

(3) *Let* $\iota\colon K' \cong K$ *be an isomorphism of CM-fields. Then we have for* $\sigma, \tau \in \mathrm{Hom}(K, \mathbb{C})$

$$[p_K : p_{K,p}](\sigma, \tau) \equiv [p_{K'} : p_{K',p}](\sigma \circ \iota, \tau \circ \iota) \mod (\mu_\infty \times \mu_\infty)\overline{\mathbb{Q}}^\times.$$

(4) *Let* $K \subset L$ *be a field extension of CM-fields. Then we have for* $X \in I_L$, $Y \in I_K$

$$[p_K : p_{K,p}](\mathrm{Res}(X), Y) \equiv [p_L : p_{L,p}](X, \mathrm{Inf}(Y)) \mod (\mu_\infty \times \mu_\infty)\overline{\mathbb{Q}}^\times.$$

2.2. Coleman's formula

Theorem 2.4 below is essentially due to Coleman [4, Theorems 1.7, 3.13]. Note that the original formula does not have a root of unity ambiguity. First we prepare some notations. We assume that p is an odd prime.

Definition 2.2.

(1) *Let* $\mathbb{C}_p^1 := \{z \in \mathbb{C}_p^\times \mid |z|_p = 1\}$. *We fix a group homomorphism*

$$\exp_p\colon \mathbb{C}_p \to \mathbb{C}_p^1$$

which coincides with the usual power series $\exp_p(z) := \sum_{k=0}^\infty \frac{z^k}{k!}$ *on the convergence region. For* $\alpha \in \mathbb{C}_p^\times$, $\beta \in \mathbb{C}_p$, *we put*

$$\alpha^\beta := \exp_p(\beta \log_p \alpha)$$

with $\log_p$ *Iwasawa's* p*-adic* log *function.*

(2) *For* $z \in \mathbb{C}_p^\times$, *we put*

$$z^* := \exp_p(\log_p(z)), \qquad z^\flat := p^{\mathrm{ord}_p z} z^*.$$

Here we define $\mathrm{ord}_p z \in \mathbb{Q}$ *by* $|z|_p = |p|_p^{\mathrm{ord}_p z}$. *Note that* $z \equiv z^\flat \bmod \mu_\infty$ $(z \in \mathbb{C}_p^\times)$.

(3) *We define the* p*-adic gamma function on* $\mathbb{Q}_p$ *as follows.*

   (a) *On* $\mathbb{Z}_p$, $\Gamma_p(z)$ *denotes Morita's* p*-adic gamma function which is the unique continuous function* $\Gamma_p\colon \mathbb{Z}_p \to \mathbb{Z}_p^\times$ *satisfying*

$$\Gamma_p(n) := (-1)^n \prod_{1 \leqslant k \leqslant n-1,\ p \nmid k} k \quad (n \in \mathbb{N}).$$

(b) *On* $\mathbb{Q}_p - \mathbb{Z}_p$, *we use* $\Gamma_p\colon \mathbb{Q}_p - \mathbb{Z}_p \to \mathcal{O}_{\overline{\mathbb{Q}_p}}^{\times}$ *defined in* [7, Lemma 4.2], *which is a continuous function satisfying*

$$\Gamma_p(z+1) = z^*\Gamma_p(z),\ \Gamma_p(2z) = 2^{2z-\frac{1}{2}}\Gamma_p(z)\Gamma_p\left(z+\frac{1}{2}\right).$$

Such a continuous function on $\mathbb{Q}_p - \mathbb{Z}_p$ *is unique up to multiplication by* μ_∞.

(4) *For* $z \in \mathbb{Z}_p$, *we define* $z_0 \in \{1, 2, \dots, p\}$, $z_1 \in \mathbb{Z}_p$ *by*

$$z = z_0 + pz_1.$$

Note that when $p \mid z$, *we put* $z_0 = p$, *instead of* 0.

(5) *Let* W_p *be the Weil group defined as*

$$W_p := \{\tau \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \mid \tau|_{\mathbb{Q}_p^{ur}} = \sigma_p^{\deg\tau} \text{ with } \deg\tau \in \mathbb{Z}\}.$$

Here $\mathbb{Q}_p^{ur}$ *denotes the maximal unramified extension of* $\mathbb{Q}_p$, σ_p *the Frobenius automorphism on* $\mathbb{Q}_p^{ur}$.

(6) *We define the action of* W_p *on* $\mathbb{Q} \cap [0,1)$ *by identifying* $\mathbb{Q} \cap [0,1) = \mu_\infty$. *Namely*

$$\tau\left(\frac{a}{N}\right) := \frac{b}{N} \quad \text{if} \quad \tau(\zeta_N^a) = \zeta_N^b \quad (\tau \in W_p).$$

(7) *Let* Φ_{cris} *be the absolute Frobenius automorphism on* B_{cris}. *We consider the following action of* W_p *on* $B_{\mathrm{cris}}\overline{\mathbb{Q}_p} \cong B_{\mathrm{cris}} \otimes_{\mathbb{Q}^{ur}} \overline{\mathbb{Q}_p}$:

$$\Phi_\tau := \Phi_{\mathrm{cris}}^{\deg\tau} \otimes \tau \quad (\tau \in W_p).$$

(8) *For* $\frac{a}{N} \in \mathbb{Q} \cap (0,1)$ *we put*

$$P\left(\frac{a}{N}\right) := \frac{\Gamma_\infty(\frac{a}{N}) \cdot (2\pi \mathrm{i})_p^{\frac{1}{2}-\langle\frac{a}{N}\rangle} p_{\mathbb{Q}(\zeta_N),p}\left(\mathrm{id}, \sum_{(b,N)=1}\left(\frac{1}{2} - \langle\frac{ab}{N}\rangle\right)\sigma_b\right)}{(2\pi \mathrm{i})^{\frac{1}{2}-\langle\frac{a}{N}\rangle} p_{\mathbb{Q}(\zeta_N)}\left(\mathrm{id}, \sum_{(b,N)=1}\left(\frac{1}{2} - \langle\frac{ab}{N}\rangle\right)\sigma_b\right)} \in (B_{\mathrm{cris}}\overline{\mathbb{Q}_p} - \{0\})^{\mathbb{Q}}/\mu_\infty.$$

This definition makes sense since

$$\frac{\Gamma_\infty(\frac{a}{N})}{(2\pi \mathrm{i})^{\frac{1}{2}-\langle\frac{a}{N}\rangle} p_{\mathbb{Q}(\zeta_N)}\left(\mathrm{id}, \sum_{(b,N)=1}\left(\frac{1}{2} - \langle\frac{ab}{N}\rangle\right)\sigma_b\right)} \in \overline{\mathbb{Q}} \subset B_{\mathrm{cris}}\overline{\mathbb{Q}_p}$$

by (1.5) *and the ratio* $[p_K : p_{K,p}]$ *is well-defined up to* μ_∞ *by* Proposition 2.1.

Remarks 2.3.

(1) Let μ_{p-1} be the group of all $(p-1)$st roots of unity, $p^{\mathbb{Z}} := \{p^n \mid n \in \mathbb{Z}\}$, $1 + p\mathbb{Z}_p := \{1 + pz \mid z \in \mathbb{Z}_p\}$. Then we have the canonical decomposition

$$\begin{array}{ccccccc} \mathbb{Q}_p^\times & \to & \mu_{p-1} & \times & p^{\mathbb{Z}} & \times & 1 + p\mathbb{Z}_p, \\ z & \mapsto & (\omega(zp^{-\mathrm{ord}_p z}) & , & p^{\mathrm{ord}_p z} & , & z^*), \end{array}$$

where ω denotes the Teichmüller character. The maps $z \mapsto z^*, z^\flat$ provide a similar (but non-canonical) decomposition of $\mathbb{C}_p^\times$. Moreover, we note that the maps $z \mapsto \exp_p(z), z^*, z^\flat$ are continuous homomorphisms.

(2) We easily see that

$$\tau(z) = \langle pz \rangle, \ \tau^{-1}(z) = z_1 + 1 \quad (z \in \mathbb{Z}_{(p)} \cap (0,1), \ \tau \in W_p, \ \deg \tau = 1).$$

Theorem 2.4 ([8, Theorem 3]). — *Let p be an odd prime.*

(1) *Assume that $z \in \mathbb{Z}_{(p)} \cap (0,1)$. Then we have*

$$\Gamma_p(z) \equiv p^{\frac{1}{2} - \tau^{-1}(z)} \frac{P(z)}{\Phi_\tau(P(\tau^{-1}(z)))} \bmod \mu_\infty \quad (\tau \in W_p, \ \deg \tau = 1).$$

(2) *Assume that $z \in (\mathbb{Q} - \mathbb{Z}_{(p)}) \cap (0,1)$. Then we have*

$$\frac{\Gamma_p(\tau(z))}{\Gamma_p(z)} \equiv \frac{p^{(z-\tau(z))\mathrm{ord}_p z} P(\tau(z))}{\Phi_\tau(P(z))} \quad \bmod \mu_\infty \quad (\tau \in W_p).$$

Remark 2.5. — As a result, we see that the right-hand sides of Theorem 2.4(1), (2) are p-adic continuous on z, $(z, \tau(z))$ respectively, since the left-hand sides are so. We use only the p-adic continuity in the next section, in order to recover Theorem 2.4(1).

3. Main results

Morita's p-adic gamma function $\Gamma_p \colon \mathbb{Z}_p \to \mathbb{Z}_p^\times$ is the unique continuous function satisfying

$$\Gamma_p(0) = 1, \ \frac{\Gamma_p(z+1)}{\Gamma_p(z)} = \begin{cases} -z & (z \in \mathbb{Z}_p^\times), \\ -1 & (z \in p\mathbb{Z}_p). \end{cases} \tag{3.1}$$

In this section, we study other functional equations characterizing Γ_p and provide an alternative proof of Coleman's formula in the case $z \in \mathbb{Z}_{(p)}$. Strictly speaking, we only "assume" that the right-hand sides of Theorem 2.4(1), (2) are continuous on z, $(z, \tau(z))$ respectively (of course, this is correct). Then we can recover a "large part" (Corollary 3.6) of Theorem 2.4(1). We assume that p is an odd prime.

3.1. A characterization of Morita's p-adic gamma function

$\Gamma_p(z)$ satisfies the following p-adic analogues of multiplication formulas, which we consider only up to roots of unity in this paper. For the detailed formulation and its proof, see [11, "Basic properties of Γ_p" in Section 2 of Chapter IV].

Proposition 3.1. — *Let $d \in \mathbb{N}$ with $p \nmid d$. Then we have for $z \in \mathbb{Z}_p$*

$$\prod_{k=0}^{d-1} \Gamma_p\left(z+\frac{k}{d}\right) \equiv d^{1-dz+(dz)_1}\Gamma_p(dz) \mod \mu_\infty. \tag{3.2}$$

Note that if $p \mid d$, then $z+\frac{k}{d}$ is not in the domain of definition of Morita's Γ_p. In the rest of this subsection, we show that multiplication formulas (3.2) and some conditions characterize Morita's p-adic gamma function (at least up to μ_∞).

Proposition 3.2. — *Assume a continuous function $f(z)\colon \mathbb{Z}_p \to \mathbb{C}_p^\times$ satisfies*

$$\prod_{k=0}^{d-1} f\left(z+\frac{k}{d}\right) \equiv f(dz) \mod \mu_\infty \quad (p \nmid d). \tag{3.3}$$

Then the following holds.

(1) $\frac{f(z+1)}{f(z)} \mod \mu_\infty$ *depends only on* $\mathrm{ord}_p\, z$.

(2) *The values*

$$c_k := \left(\frac{f(p^k+1)}{f(p^k)}\right)^\flat$$

characterize the function $f(z)$ up to μ_∞. More precisely, for $z \in \mathbb{Z}_p$, we write the p-adic expansion of $z-1$ as

$$z-1=\sum_{k=0}^{\infty} x_k p^k \quad (x_k \in \{0,1,\dots,p-1\}).$$

Then we have

$$f(z) \equiv \prod_{k=0}^{\infty} \alpha_k^{x_k-\frac{p-1}{2}} \mod \mu_\infty \quad \textit{with} \quad \alpha_k := c_k \prod_{i=0}^{k-1} c_i^{p^{k-1-i}(p-1)}.$$

Conversely, assume that

$$f\left(1+\sum_{k=0}^{\infty} x_k p^k\right) \equiv \prod_{k=0}^{\infty} \alpha_k^{x_k-\frac{p-1}{2}} \mod \mu_\infty \quad (x_k \in \{0,1,\dots,p-1\}) \tag{3.4}$$

for constants $\alpha_k \in \mathbb{C}_p^\times$ satisfying $\alpha_k \to 1$ $(k \to \infty)$. Then $f(z)$ satisfies the functional equations (3.3).

Proof. — We suppress $\bmod \mu_\infty$. Assume (3.3). Replacing z with $z+\frac{1}{d}$, we obtain $\prod_{k=1}^{d} f(z+\frac{k}{d}) \equiv f(dz+1)$. It follows that $\frac{f(z+1)}{f(z)} \equiv \frac{f(dz+1)}{f(dz)}$. That is,

$$g(z) := \frac{f(z+1)}{f(z)} \equiv g(dz) \quad (p \nmid d \in \mathbb{N}).$$

Then the assertion (1) is clear. Let $c_k := (g(p^k))^\flat$, $a_n := x_0 + x_1 p + \cdots + x_n p^n$ $(0 \leqslant x_i \leqslant p-1)$. We easily see that

$$\#\{y=1,2,\ldots,a_n \mid \operatorname{ord}_p y = k\} = x_k + \sum_{i=k+1}^{n} x_i p^{i-k-1}(p-1) \quad (0 \leqslant k \leqslant n).$$

Then we can write

$$f(a_n+1)^\flat = (f(1)g(1)g(2)\cdots g(a_n))^\flat = f(1)^\flat \alpha_0^{x_0}\alpha_1^{x_1}\cdots\alpha_n^{x_n}$$

with $\alpha_k = c_k \prod_{i=0}^{k-1} c_i^{p^{k-1-i}(p-1)}$. Since $\lim_{n\to\infty} f(a_n+1)$ converges, so do $\lim_{n\to\infty} f(a_n+1)^\flat$ and $\prod_{k=0}^{\infty} \alpha_k^{x_k}$. Moreover we can write

$$f(z) \equiv f(1) \prod_{k=0}^{\infty} \alpha_k^{x_k}.$$

Consider the case of $d=2$, $z=\frac{1}{2}$ of (3.3): $f(\frac{1}{2})f(1) \equiv f(1)$. Therefore, noting that $-\frac{1}{2} = \sum_{k=0}^{\infty} \frac{p-1}{2} p^k$, we obtain

$$1 \equiv f\left(\frac{1}{2}\right) \equiv f(1) \prod_{k=0}^{\infty} \alpha_k^{\frac{p-1}{2}}, \quad \text{that is,} \quad f(1) \equiv \prod_{k=0}^{\infty} \alpha_k^{-\frac{p-1}{2}}.$$

Then the assertion (2) is also clear.

Next, assume (3.4). When $\operatorname{ord}_p z = k$, we see that $\frac{f(z+1)}{f(z)} \equiv \frac{\alpha_k}{\alpha_{k-1}^{p-1}}$ (resp. α_0) if $k>0$ (resp. $k=0$). In particular, $g(z) := \frac{f(z+1)}{f(z)} \bmod \mu_\infty$ depends only on $\operatorname{ord}_p z$. When $z+z'=1$, the p-adic expansions $z-1 = \sum_{k=0}^{\infty} x_k p^k$, $z'-1 = \sum_{k=0}^{\infty} x'_k p^k$ satisfy $x_k + x'_k = p-1$ for any k. Then we have

$$f(z)f(z') \equiv \prod_{k=0}^{\infty} \alpha_k^0 = 1.$$

Therefore the case $z=0$ of (3.3) holds true since we have $\left(\prod_{k=1}^{d-1} f(\frac{k}{d})\right)^2 = \prod_{k=1}^{d-1} f(\frac{k}{d}) f(1-\frac{k}{d}) \equiv 1$. Then (3.3) for $z \in \mathbb{N}$ follows by mathematical induction on z noting that

$$\prod_{k=0}^{d-1} f\left(z+1+\frac{k}{d}\right) \equiv \prod_{k=0}^{d-1} f\left(z+\frac{k}{d}\right) g\left(z+\frac{k}{d}\right),$$

$$f(dz+d) \equiv f(dz)g(dz)\cdots g(dz+d-1),$$

$$\mathrm{ord}_p\,(dz+k) = \mathrm{ord}_p\left(z+\frac{k}{d}\right).$$

Since $\mathbb{N}$ is dense in $\mathbb{Z}_p$, we see that (3.3) holds for any $z \in \mathbb{Z}_p$. □

The following corollary provides a nice characterization of $\Gamma_p(z) \bmod \mu_\infty$ in terms of functional equations and one or two special values.

Corollary 3.3. — *Assume a continuous function $f(z)\colon \mathbb{Z}_p \to \mathbb{C}_p^\times$ satisfies*

$$\prod_{k=0}^{d-1} f\left(z+\frac{k}{d}\right) \equiv f(dz) \mod \mu_\infty \quad (p \nmid d)$$

and put

$$c_n := \left(\frac{f(p^n+1)}{f(p^n)}\right)^\flat .$$

Then the following equivalences hold:

(1) $c_0 = c_1 = \cdots \Leftrightarrow f(z) \equiv c_0^{z-\frac{1}{2}} \bmod \mu_\infty$.

(2) $c_1 = c_2 = \cdots \Leftrightarrow f(z) \equiv c_0^{z-\frac{1}{2}}(c_1/c_0)^{z_1+\frac{1}{2}} \bmod \mu_\infty$.

Proof. — We suppress modμ_∞. For (1), assume that $c_0 = c_1 = \cdots$. Then

$$\alpha_k := c_k \prod_{i=0}^{k-1} c_i^{p^{k-1-i}(p-1)} = c_0^{p^k}.$$

Hence we have by Proposition 3.2

$$f\left(1+\sum_{k=0}^{\infty} x_k p^k\right) \equiv \prod_{k=0}^{\infty} \alpha_k^{x_k-\frac{p-1}{2}} = c_0^{\sum_{k=0}^\infty x_k p^k - \frac{p-1}{2}p^k} = c_0^{z-1+\frac{1}{2}} = c_0^{z-\frac{1}{2}}.$$

The opposite direction is trivial by definition $c_n := (\frac{f(p^n+1)}{f(p^n)})^\flat$. For (2), the assumption $c_1 = c_2 = \cdots$ implies $\alpha_0 = c_0$, $\alpha_k = c_0^{p^k}(c_1/c_0)^{p^{k-1}}$ $(k \geqslant 1)$. In this case we have

$$\begin{aligned} f\left(1+\sum_{k=0}^{\infty} x_k p^k\right) &\equiv c_0^{\sum_{k=0}^\infty x_k p^k - \frac{p-1}{2}p^k} (c_1/c_0)^{\sum_{k=1}^\infty x_k p^{k-1} - \frac{p-1}{2}p^{k-1}} \\ &= c_0^{z-\frac{1}{2}}(c_1/c_0)^{z_1+\frac{1}{2}} \end{aligned}$$

since $\sum_{k=1}^\infty x_k p^{k-1} = \frac{z-1-x_0}{p} = z_1$. □

3.2. Alternative proof of a part of Coleman's formula

We fix $\tau \in W_p$ with $\deg \tau = 1$ and put

$$G_1(z) := \left(p^{\frac{1}{2}-\tau^{-1}(z)} \frac{P(z)}{\Phi_\tau(P(\tau^{-1}(z)))}\right)^\flat \quad (z \in \mathbb{Z}_{(p)} \cap (0,1)), \tag{3.5}$$

$$G_2(z) := \left(\frac{p^{(\tau^{-1}(z)-z)\operatorname{ord}_p z} P(z)}{\Phi_\tau(P(\tau^{-1}(z)))}\right)^\flat \quad (z \in (\mathbb{Q} - \mathbb{Z}_{(p)}) \cap (0,1)).$$

Here we added $(\ )^\flat$ to the right-hand sides of Coleman's formulas (Theorem 2.4), in order to resolve a root of unity ambiguity, only superficially. Note that G_2 corresponds to Theorem 2.4(2) replaced z with $\tau^{-1}(z)$.

By Theorem 2.4(1), we see that G_1 is continuous for the p-adic topology. G_2 is not p-adically continuous in the usual sense, on the whole of $(\mathbb{Q} - \mathbb{Z}_{(p)}) \cap (0,1)$ (for details, see Remark 3.8). Theorem 2.4(1) only implies the following "continuity":

(3.6) $G_1(z)$ is continuous for the relative topology
induced by $z \in (\mathbb{Q} - \mathbb{Z}_{(p)}) \cap (0,1) \hookrightarrow \mathbb{Q}_p \times \mathbb{Q}_p$, $z \mapsto (z, \tau^{-1}(z))$.

In Corollary 3.6, oppositely, we show that the p-adic continuity of G_1, G_2 implies a "large part"

$$G_1(z) \equiv a^{z-\frac{1}{2}} b^{z_1+\frac{1}{2}} \Gamma_p(z) \mod \mu_\infty \quad (a, b \in \mathbb{C}_p^\times)$$

of Theorem 2.4(1):

$$G_1(z) \equiv \Gamma_p(z) \mod \mu_\infty.$$

Besides we shall show the continuity of $G_1(z)$ in Section 4, independently of Theorem 2.4.

Hereinafter in this section, we forget Theorem 2.4. We assume the following Assumption instead.

Assumption 3.4. — *$G_1(z)$ is p-adically continuous and $G_2(z)$ is continuous in the sense of* (3.6). *In particular, we regard G_1 as a p-adic continuous function:*

$$G_1(z) \colon \mathbb{Z}_p \to \mathbb{C}_p.$$

First we derive "multiplication formula":

$$\prod_{k=0}^{d-1} G_1\left(z + \frac{k}{d}\right) \equiv d^{1-dz+(dz)_1} G_1(dz) \mod \mu_\infty \quad (p \nmid d \in \mathbb{N}) \tag{3.7}$$

independently of Theorem 2.4.

Proof of (3.7). — We suppress $\bmod \mu_\infty$. Let $z \in \mathbb{Z}_{(p)} \cap (0, \frac{1}{d})$. By Definition 2.2(8) and (3.5) we can write

$$\begin{aligned}
&\frac{\prod_{k=0}^{d-1} G_1(z+\frac{k}{d})}{G_1(dz)} \\
&\equiv \frac{\prod_{k=0}^{d-1} \Gamma_\infty(z+\frac{k}{d})}{\Gamma_\infty(dz)} \Phi_\tau \left(\frac{\Gamma_\infty(\tau^{-1}(dz))}{\prod_{k=0}^{d-1} \Gamma_\infty(\tau^{-1}(z+\frac{k}{d}))} \right) \frac{\prod_{k=0}^{d-1} p^{\frac{1}{2}-\tau^{-1}(z+\frac{k}{d})}}{p^{\frac{1}{2}-\tau^{-1}(dz)}} \\
&\quad \times \text{“products of classical or } p\text{-adic periods”},
\end{aligned}$$

where the “products of classical or p-adic periods” become trivial by (1.6), as we saw in the proof of Proposition 1.5. Besides we see that

$$\left\{ \tau^{-1}\left(z+\frac{k}{d}\right) \,\middle|\, k=0,\dots,d-1 \right\} = \left\{ \frac{\tau^{-1}(dz)}{d} + \frac{k}{d} \,\middle|\, k=0,\dots,d-1 \right\}.$$

To see this, it suffices to show that $\{\tau^{-1}(\zeta_N^a \zeta_d^k) \mid k=0,\dots,d-1\}$ and $\{\tau^{-1}(\zeta_N^{da})^{\frac{1}{d}} \zeta_d^k \mid k=0,\dots,d-1\}$ coincide with each other. We easily see that both of them are the inverse image of $\tau^{-1}(\zeta_N^{da})$ under the dth power map $\mu_\infty \to \mu_\infty$, $x \mapsto x^d$. Hence we obtain

$$\begin{aligned}
&\frac{\prod_{k=0}^{d-1} G_1(z+\frac{k}{d})}{G_1(dz)} \\
&\equiv \frac{\prod_{k=0}^{d-1} \Gamma_\infty(z+\frac{k}{d})}{\Gamma_\infty(dz)} \cdot \Phi_\tau \left(\frac{\Gamma_\infty(\tau^{-1}(dz))}{\prod_{k=0}^{d-1} \Gamma_\infty(\frac{\tau^{-1}(dz)}{d}+\frac{k}{d})} \right) \cdot \frac{\prod_{k=0}^{d-1} p^{\frac{1}{2}-(\frac{\tau^{-1}(dz)}{d}+\frac{k}{d})}}{p^{\frac{1}{2}-\tau^{-1}(dz)}} \\
&= d^{\frac{1}{2}-dz} \cdot \Phi_\tau(d^{\tau^{-1}(dz)-\frac{1}{2}}) \cdot 1 \equiv d^{\frac{1}{2}-dz} \cdot d^{\tau^{-1}(dz)-\frac{1}{2}}
\end{aligned}$$

by (1.2), (1.6). For the last “$\equiv$”, we note that Φ_τ acts on $\overline{\mathbb{Q}} \ni d^{\tau^{-1}(dz)-\frac{1}{2}}$ as τ. By Remark 2.3(2), we have $\tau^{-1}(dz) = (dz)_1 + 1$. Then the assertion is clear. □

Furthermore we can show that $c_n = \left(\frac{f(p^n+1)}{f(p^n)}\right)^\flat$ for $f(z) := \frac{G_1(z)}{\Gamma_p(z)}$ is constant, at least for $n \geqslant 1$.

THEOREM 3.5. — *We assume Assumption 3.4 and put* $f(z) := \frac{G_1(z)}{\Gamma_p(z)}$.

(1) *The following functional equations hold.*

$$\prod_{k=0}^{d-1} f\left(z+\frac{k}{d}\right) \equiv f(dz) \mod \mu_\infty \quad (p \nmid d).$$

(2) *We have* $c_1 = c_2 = \cdots$ *for* $c_n := \left(\frac{f(p^n+1)}{f(p^n)}\right)^\flat$.

Proof. — We suppress $\bmod \mu_\infty$. (1) follows from (3.2), (3.7). For (2), we need for $z \in p\mathbb{Z}_p$

$$\frac{G_1(pz)G_1(z+1)}{G_1(pz+1)G_1(z)} \equiv \frac{\Gamma_p(pz)\Gamma_p(z+1)}{\Gamma_p(pz+1)\Gamma_p(z)}.$$

Since the right-hand side is equal to $\begin{cases} l & (p|z) \\ z & (p\nmid z) \end{cases}$ by (3.1), it suffices to show that

$$\frac{G_1(pz)G_1(z+1)}{G_1(pz+1)G_1(z)} \equiv 1 \quad (z \in p\mathbb{Z}_p).$$

Note that we can not use the definition (3.5) directly since $z, z+1, pz, pz+1$ are not contained in $(0,1)$ simultaneously. Therefore a little complicated argument is needed as follows. Let $z \in \mathbb{Z}_{(p)} \cap (0, \frac{1}{p})$. By Remark 2.3(2), we have

$$\tau(z) = \langle pz \rangle = pz, \text{ hence } \tau^{-1}(pz) = z.$$

We can write

$$\begin{aligned} H_1(z) &:= \frac{G_1(z)G_2(z+\frac{1}{p})\cdots G_2(z+\frac{p-1}{p})}{G_1(pz)} \\ &\equiv p^{z+(z+\frac{1}{p})+\cdots+(z+\frac{p-1}{p})-\tau^{-1}(z)-\tau^{-1}(z+\frac{1}{p})-\cdots-\tau^{-1}(z+\frac{p-1}{p})} \\ &\quad\times \frac{P(z)P(z+\frac{1}{p})\cdots P(z+\frac{p-1}{p})}{P(pz)} \\ &\quad\times \Phi_\tau\left(\frac{P(z)}{P(\tau^{-1}(z))P(\tau^{-1}(z+\frac{1}{p}))\cdots P(\tau^{-1}(z+\frac{p-1}{p}))}\right). \end{aligned}$$

Here we note that $\operatorname{ord}_p(z+\frac{k}{p}) = -1$ for $k = 1, \ldots, p-1$. We have

$$\left\{\tau^{-1}\left(z+\frac{k}{p}\right) \,\middle|\, k = 0, \ldots, p-1\right\} = \left\{\frac{z+k}{p} \,\middle|\, k = 0, \ldots, p-1\right\} \tag{3.8}$$

since both of $\{\tau^{-1}(\zeta_N^a \zeta_p^k) \mid k = 0, \ldots, p-1\}$, $\{\zeta_{pN}^{a+Nk} \mid k = 0, \ldots, p-1\}$ are the set of the pth roots of ζ_N^a when $z = \frac{a}{N}$. Therefore the p-power parts of H_1 become

$$p^{z+(z+\frac{1}{p})+\cdots+(z+\frac{p-1}{p})-\frac{z}{p}-\frac{z+1}{p}-\cdots-\frac{z+p-1}{p}} = p^{(p-1)z}.$$

Moreover the "period parts" of H_1 become trivial by (1.6), (3.8). Namely we can write

$$H_1(z) \equiv p^{(p-1)z} \frac{\Gamma_\infty(z)\Gamma_\infty(z+\frac{1}{p})\cdots\Gamma_\infty(z+\frac{p-1}{p})}{\Gamma_\infty(pz)} \times \Phi_\tau\left(\frac{\Gamma_\infty(z)}{\Gamma_\infty(\frac{z}{p})\Gamma_\infty(\frac{z+1}{p})\cdots\Gamma_\infty(\frac{z+p-1}{p})}\right).$$

By using the original Multiplication formula (1.2) for Γ_∞, we obtain

$$H_1(z) \equiv p^{(p-1)z}p^{\frac{1}{2}-pz}p^{z-\frac{1}{2}} = 1.$$

Next, let $z = \frac{a}{N} \in \mathbb{Z}_{(p)} \cap (-\frac{1}{p}, 0)$. Then we have

- $\tau(z+1) = pz+1$. Hence $\tau^{-1}(pz+1) = z+1$.
- $\{\tau^{-1}(\zeta_N^a\zeta_p^k) \mid k = 1,\dots,p\} = \{\zeta \mid \zeta^p = \zeta_N^a\} = \{\zeta_{pN}^{a+Nk} \mid k = 1,\dots,p\}$. Hence $\{\tau^{-1}(z+\frac{k}{p}) \mid k = 1,\dots,p\} = \{\frac{z+k}{p} \mid k = 1,\dots,p\}$.

Then we can prove similarly that

$$\begin{aligned} H_2(z) &:= \frac{G_2(z+\frac{1}{p})\cdots G_2(z+\frac{p-1}{p})G_1(z+1)}{G_1(pz+1)} \\ &\equiv p^{(z+\frac{1}{p})+\cdots+(z+\frac{p-1}{p})+(z+1)-\tau^{-1}(z+\frac{1}{p})-\cdots-\tau^{-1}(z+\frac{p-1}{p})-\tau^{-1}(z+1)} \\ &\quad \times \frac{P(z+\frac{1}{p})\cdots P(z+\frac{p-1}{p})P(z+1)}{P(pz+1)} \\ &\quad \times \Phi_\tau\left(\frac{P(z+1)}{P(\tau^{-1}(z+\frac{1}{p}))\cdots P(\tau^{-1}(z+\frac{p-1}{p}))P(\tau^{-1}(z+1))}\right) \\ &\equiv p^{(z+\frac{1}{p})+\cdots+(z+\frac{p-1}{p})+(z+1)-\frac{z+1}{p}-\cdots-\frac{z+p-1}{p}-\frac{z+p}{p}}p^{\frac{1}{2}-(pz+1)}p^{z+1-\frac{1}{2}} = 1. \end{aligned}$$

Here $H_i(z) \equiv 1 \bmod \mu_\infty$ implies $H_i(z) = 1$ $(i = 1, 2)$ since we have $x^\flat = \exp_p(\log_p x) = \exp_p(0) = 1$ for $x \in \mu_\infty$. ($G_1(z), G_2(z)$ are in the image under $(\ )^\flat$ by definition, so are $H_i(z)$.) In particular, we have

$$\begin{aligned} \frac{G_1(pz)}{G_1(z)} &= G_2\left(z+\frac{1}{p}\right)\cdots G_2\left(z+\frac{p-1}{p}\right) \quad \left(z \in \mathbb{Z}_{(p)} \cap \left(0, \frac{1}{p}\right)\right), \\ \frac{G_1(pz+1)}{G_1(z+1)} &= G_2\left(z+\frac{1}{p}\right)\cdots G_2\left(z+\frac{p-1}{p}\right) \quad \left(z \in \mathbb{Z}_{(p)} \cap \left(-\frac{1}{p}, 0\right)\right). \end{aligned}$$

Let $z \in p\mathbb{Z}_{(p)}$. Then there exist $z_n^+ \in p\mathbb{Z}_{(p)} \cap (0, \frac{1}{p})$, $z_n^- \in p\mathbb{Z}_{(p)} \cap (-\frac{1}{p}, 0)$ which converge to z when $n \to \infty$ respectively. Then we can write

$$\frac{G_1(pz)}{G_1(z)} = \lim_{n\to\infty} \frac{G_1(pz_n^+)}{G_1(z_n^+)} = \lim_{n\to\infty} G_2\left(z_n^+ + \frac{1}{p}\right) \cdots G_2\left(z_n^+ + \frac{p-1}{p}\right),$$
$$\frac{G_1(pz+1)}{G_1(z+1)} = \lim_{n\to\infty} \frac{G_1(pz_n^- +1)}{G_1(z_n^- +1)} = \lim_{n\to\infty} G_2\left(z_n^- + \frac{1}{p}\right) \cdots G_2\left(z_n^- + \frac{p-1}{p}\right).$$

Recall that $G_2(z)$ is continuous in the sense of (3.6). Clearly we have for $k = 1, \ldots, p-1$

$$z_n^\pm + \frac{k}{p} \to z + \frac{k}{p} \quad (n \to \infty).$$

Additionally we see that

$$\tau^{-1}\left(z_n^\pm + \frac{k}{p}\right) = \frac{z_n^\pm}{p} + \tau^{-1}\left(\frac{k}{p}\right) \to \frac{z}{p} + \tau^{-1}\left(\frac{k}{p}\right) \quad (n \to \infty)$$

by noting that $\tau^{-1}(z + z') \equiv \tau^{-1}(z) + \tau^{-1}(z') \bmod \mathbb{Z}$ $(\forall z, z')$, $\tau^{-1}(z) \equiv \frac{z}{p} \bmod \mathbb{Z}$ if $p \mid z$, $\frac{z_n^\pm}{p} \in (-\frac{1}{p}, \frac{1}{p})$, $\tau^{-1}(\frac{k}{p}) \in [\frac{1}{p}, \frac{p-1}{p}]$. It follows that

$$\lim_{n\to\infty} G_2\left(z_n^+ + \frac{k}{p}\right) = \lim_{n\to\infty} G_2\left(z_n^- + \frac{k}{p}\right).$$

Then the assertion is clear. □

By Corollary 3.3, we obtain the following.

Corollary 3.6. — *Assume* Assumption 3.4. *Then there exist constants a, b satisfying*

$$G_1(z) \equiv a^{z-\frac{1}{2}} b^{z_1+\frac{1}{2}} \Gamma_p(z) \mod \mu_\infty.$$

Remark 3.7. — In addition to the above results, by computing the absolute Frobenius on only one Fermat curve, we obtain Coleman's formula $G_1(z) \equiv \Gamma_p(z) \bmod \mu_\infty$. For example, when $p = 3$, we obtain it for $z = \frac{1}{5}, \frac{2}{5}$ by the computation on F_5. It follows that $a^{\frac{-3}{10}} b^{\frac{-1}{10}} \equiv a^{\frac{-1}{10}} b^{\frac{3}{10}} \equiv 1$, hence $a \equiv b \equiv 1$.

Remark 3.8. — We used the assumption $p \mid z$ only in the last paragraph of the proof for Theorem 3.5 because G_2 is not p-adically continuous on the whole of $(\mathbb{Q} - \mathbb{Z}_{(p)}) \cap (0, 1)$. For example, we put

$$z_n := \frac{1}{p^2} + \frac{p^{n+1}}{p^{n+2} + (1-p)^n} \in (\mathbb{Q} - \mathbb{Z}_{(p)}) \cap (0, 1) \quad (n \in \mathbb{N})$$

and take $\tau \in W_p$ with $\deg \tau = 1$ so that

$$\tau(\zeta_{p^2}) = \zeta_{p^2}^{-1}.$$

In particular we see that

$$z_n \to \frac{1}{p^2} \text{ for the } p\text{-adic topology.}$$

On the other hand we see that

$$\begin{aligned}\tau^{-1}(z_n) &\equiv \tau^{-1}\left(\frac{1}{p^2}\right) + \tau^{-1}\left(\frac{p^{n+1}}{p^{n+2}+(1-p)^n}\right) \mod \mathbb{Z}\\ &= \frac{p^2-1}{p^2} + \frac{p^n}{p^{n+2}+(1-p)^n} = 1 - \frac{(1-p)^n}{p^2(p^{n+2}+(1-p)^n)},\\ 1 - &\frac{(1-p)^n}{p^2(p^{n+2}+(1-p)^n)} \in \begin{cases}(1,2) & \text{if } n \text{ is odd,}\\ (0,1) & \text{if } n \text{ is even.}\end{cases}\end{aligned}$$

Hence we have

$$\tau^{-1}(z_n) = \begin{cases} -\frac{(1-p)^n}{p^2(p^{n+2}+(1-p)^n)} \to -\frac{1}{p^2} & \text{if } n = 2k+1,\ k \to \infty,\\ 1 - \frac{(1-p)^n}{p^2(p^{n+2}+(1-p)^n)} \to 1 - \frac{1}{p^2} & \text{if } n = 2k,\ k \to \infty.\end{cases}$$

Then, by Theorem 2.4(2), we see that $G_2(z_n) = (\Gamma_p(z_n)/\Gamma_p(\tau^{-1}(z_n)))^\flat$ does not converge p-adically although z_n does.

4. On the p-adic continuity

In the previous section, we showed that the p-adic continuity of the right-hand sides of Theorem 2.4(1), (2) implies a large part of Theorem 2.4(1) itself. In this section, we see that it is relatively easy to show such p-adic continuity properties, without explicit computation. For simplicity, we consider only the case $z \in \mathbb{Z}_p$. Assume that $p \nmid N$.

Lemma 4.1 ([3, §VI]). — *Let $1 \leqslant r, s < N$ with $r + s \neq N$. We consider the formal expansion of the differential form $\eta_{r,s} = x^r y^{s-N} \frac{dx}{x}$ on $F_N \colon x^N + y^N = 1$ at $(x, y) = (0, 1)$:*

$$\eta_{r,s} = \sum_{n=0}^{\infty} b_{r,s}(n) x^n \frac{dx}{x},$$

$$b_{r,s}(n) := \begin{cases} (-1)^{\frac{n-r}{N}} \binom{\frac{s}{N}-1}{\frac{n-r}{N}} & (n \equiv r \mod N),\\ 0 & (n \not\equiv r \mod N).\end{cases}$$

Let Φ be the absolute Frobenius on $H^1_{\mathrm{d}R}(F_N, \mathbb{Q}_p)$. Then there exists $\alpha_{r',s'} \in \mathbb{Q}_p$ satisfying

$$\Phi(\eta_{r,s}) = \alpha_{r',s'}\eta_{r',s'}$$
$$\text{for } r', s' \text{ with } 1 \leqslant r', s' < N,\ pr \equiv r' \bmod N,\ ps \equiv s' \bmod N.$$

Then we have

$$\begin{aligned}\alpha_{r',s'} &= \lim_{\substack{\mathbb{N}\ni n\mapsto 0\\ n\equiv r \bmod N}} \frac{pb_{r,s}(n)}{b_{r',s'}(pn)}\\ &= \lim_{\mathbb{N}\ni k\to -\frac{r}{N}} (-1)^{(p-1)k+\frac{pr-r'}{N}} \frac{p\binom{\frac{s}{N}-1}{k}}{\binom{\frac{s'}{N}-1}{pk+\frac{pr-r'}{N}}}.\end{aligned} \tag{4.1}$$

We note that $\alpha_{r',s'}$ depends only on $(\frac{r'}{N}, \frac{s'}{N})$. That is $\alpha_{r',s'}$ with $N = N_1$ is equal to $\alpha_{tr',ts'}$ with $N = tN_1$.

Proposition 4.2. — *$\alpha_{r',s'}$ is p-adically continuous on $(\frac{r'}{N}, \frac{s'}{N}) \in (\mathbb{Z}_{(p)} \cap (0,1))^2$.*

Proof. — It suffices to show that $\alpha_{r'_1,s'_1}$ with $N = N_1$ is close to $\alpha_{r'_2,s'_2}$ with $N = N_2$ when $\frac{r'_1}{N_1}$ is close to $\frac{r'_2}{N_2}$ and $\frac{s'_1}{N_1}$ is close to $\frac{s'_2}{N_2}$. We may assume $N \coloneqq N_1 = N_2$ by considering $N = N_1N_2$. First we fix $r' \coloneqq r'_1 = r'_2$ and assume that s'_1 is close to s'_2. Then we can take the same k for the limit expressions (4.1) of α_{r',s'_1}, α_{r',s'_2}. We easily see that if $p^l \mid (s'_1 - s'_2)$, then $p^{l-1} \mid (s_1 - s_2)$. In fact, we can write $s'_i = ps_i - l_iN$ with $l_i = 0, 1, \ldots, p-1$ since $0 < s_i, s'_i < N$ for $i = 1, 2$. If $p \mid (s'_1 - s'_2)$, then we have $p \mid (l_1 - l_2)$, so $l_1 = l_2$. Therefore we obtain $s_1 - s_2 = \frac{s'_1 - s'_2}{p}$. It follows that s_1 also is close to s_2. Hence the continuity on $\frac{s'}{N}$ is clear since the numerator (resp. the denominator) of the expression (4.1) is a polynomial on $\frac{s}{N}$ (resp. $\frac{s'}{N}$).

For the variable $\frac{r'}{N}$, we replace x with y. In other words, replace the point $(x, y) = (0, 1)$ for the expansion with $(1, 0)$. Then the continuity on $\frac{r'}{N}$ also follows from the same argument. ☐

Corollary 4.3. — *$G_1(z)$ defined in (3.5) is p-adically continuous on $z \in \mathbb{Z}_{(p)} \cap (0, 1)$. In particular, we may regard $G_1(z)$ as a continuous function on $\mathbb{Z}_p$.*

Proof. — CM-types $\Xi_{r,s}$ of (1.4), corresponding to $\eta_{r,s}$, generate the $\mathbb{Q}$-vector space $\{\sum_\sigma c_\sigma \cdot \sigma \mid c_\sigma + c_{\rho\circ\sigma}$ is a constant$\}$. More explicitly, we claim that

$$\sum_{(b,N)=1}\left(\frac{1}{2}-\left\langle\frac{ab}{N}\right\rangle\right)\sigma_b=\frac{1}{N}\sum_{1\leqslant s<N,\ a+s\neq N}\Xi_{a,s}-\frac{N-2}{2N}\sum_{(b,N)=1}\sigma_b,$$

where s runs over $1\leqslant s<N$ with $a+s\neq N$ in the first sum of the right-hand side. By the definition (1.4), $\sigma_b\in\Xi_{a,s}$ if and only if $\langle\frac{ab}{N}\rangle+\langle\frac{sb}{N}\rangle<1$. Namely $\langle\frac{sb}{N}\rangle=\frac{1}{N},\frac{2}{N},\dots,1-\frac{1}{N}-\langle\frac{ab}{N}\rangle$. The number of such b is congruent to $-1-ab \bmod N$. Hence we have

$$\frac{1}{N}\sum_{\substack{1\leqslant s<N,\\ a+s\neq N}}\Xi_{a,s}=\sum_{(b,N)=1}\left\langle\frac{-1-ab}{N}\right\rangle\sigma_b=\sum_{(b,N)=1}\left(1-\frac{1}{N}-\left\langle\frac{ab}{N}\right\rangle\right)\sigma_b.$$

Here we note that $ab\not\equiv 0 \bmod N$ since $(b,N)=1$, $a\not\equiv 0 \bmod N$. Then the above claim follows. By substituting this into Definition 2.2(8), we can write

$$P\left(\frac{a}{N}\right)\equiv\frac{\Gamma_\infty(\frac{a}{N})(2\pi\mathrm{i})_p^{\frac{1}{2}-\frac{a}{N}}\prod_{1\leqslant s<N,\ a+s\neq N}\left((2\pi\mathrm{i})_p^{e_s}\int_{\gamma,p}\eta_{a,s}\right)^{\frac{1}{N}}}{(2\pi\mathrm{i})^{\frac{1}{2}-\frac{a}{N}}\prod_{1\leqslant s<N,\ a+s\neq N}\left((2\pi\mathrm{i})^{e_s}\int_{\gamma}\eta_{a,s}\right)^{\frac{1}{N}}}\mod\mu_\infty,$$

$$e_s:=\begin{cases}-1 & (a+s<N)\\ 0 & (a+s>N)\end{cases}$$

since the part $\sum_{(b,N)=1}\sigma_b$ becomes trivial by Proposition 2.1(2). We can strengthen the congruence relation $\equiv$ of the formula (1.3) into an equality $=$, by selecting a specific closed path γ_0 (e.g., $\gamma_0=N\gamma_N$ with γ_N in [12, Proposition 4.9]). Then we have

$$P\left(\frac{a}{N}\right)\equiv c\cdot(2\pi\mathrm{i})_p^{\frac{-1}{2}+\frac{1}{N}}\prod_{1\leqslant s<N,\ a+s\neq N}\left(\int_{\gamma_0,p}\eta_{a,s}\right)^{\frac{1}{N}}\mod\mu_\infty,$$

where we put

$$c:=\frac{\Gamma(\frac{a}{N})}{(2\pi)^{\frac{1}{N}}}\left(\prod_{1\leqslant s<N,\ a+s\neq N}\frac{\Gamma(\frac{a+s}{N})}{\Gamma(\frac{a}{N})\Gamma(\frac{s}{N})}\right)^{\frac{1}{N}}.$$

Since (1.2) implies that

$$\prod_{1\leqslant s\leqslant N}\frac{\Gamma(\frac{a+s}{N})}{\Gamma(\frac{a}{N})\Gamma(\frac{s}{N})}=\frac{N^{-a}a!}{\Gamma(\frac{a}{N})^N},$$

we obtain

$$c=\frac{\Gamma(\frac{a}{N})}{(2\pi)^{\frac{1}{N}}}\left(\frac{\Gamma(\frac{a}{N})\Gamma(\frac{N-a}{N})}{\Gamma(1)}\frac{\Gamma(\frac{a}{N})\Gamma(\frac{N}{N})}{\Gamma(\frac{a+N}{N})}\frac{N^{-a}a!}{\Gamma(\frac{a}{N})^N}\right)^{\frac{1}{N}}=\left(\frac{N^{1-a}(a-1)!}{2\sin(\frac{a}{N}\pi)}\right)^{\frac{1}{N}}.$$

For the last equality we used (1.1) and the difference equation $\Gamma(z+1) = z\Gamma(z)$. Take $\tau \in W_p$ with $\deg\tau = 1$. Then we have

$$G_1\left(\frac{a'}{N}\right) \equiv p^{\frac{1}{2}-\frac{a}{N}} \frac{P(\frac{a'}{N})}{\Phi_\tau(P(\frac{a}{N}))}$$
$$\equiv \left(\frac{N^{a-a'}(a'-1)!}{p^{a-N-1}(a-1)!} \prod_{1\leqslant s<N,\ a+s\neq N} \alpha_{a',s'}^{-1}\right)^{\frac{1}{N}} \mod \mu_\infty,$$

by noting that $\Phi_\tau((2\pi \mathrm{i})_p) = p(2\pi\mathrm{i})_p$ and $\Phi_\tau(\sin(\frac{a}{N}\pi)) = \tau(\sin(\frac{a}{N}\pi)) = \pm\sin(\frac{a'}{N}\pi)$. Here a', s' denote integers satisfying $1 \leqslant a', s' < N$, $pa \equiv a' \bmod N$, $ps \equiv s' \bmod N$ as above. By Proposition 4.2, $\alpha_{a',s'}$ are continuous for a'. When a is in a small open ball, as we saw in the proof of Proposition 4.2, we may write $a' = pa - M$ for a fixed M (M is lN in the proof of Proposition 4.2). Then the remaining part becomes

$$\frac{N^{a-a'}(a'-1)!}{p^{a-N-1}(a-1)!} = \pm\Gamma_p(a'+M+1)\frac{p^N N^{\frac{(1-p)a'+M}{p}}(a'+M)}{a'(a'+1)(a'+2)\cdots(a'+M)},$$

which is also continuous as desired. □

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