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NOTE ON COLEMAN'S FORMULA FOR THE ABSOLUTE FROBENIUS ON FERMAT CURVES

by Tomokazu KASHIO

ABSTRACT. — Coleman calculated the absolute Frobenius on Fermat curves explicitly. In this paper we show that a kind of p-adic continuity implies a large part of his formula. To do this, we study a relation between functional equations of the gamma function, monomial relations on CM-periods, and their p-adic analogues.

RÉSUMÉ. — Coleman a calculé explicitement le Frobenius absolu sur les courbes de Fermat. Dans cet article, nous montrons qu'une sorte de continuité p-adique implique une grande partie de sa formule. Pour ce faire, nous étudions une relation entre les équations fonctionnelles de la fonction gamma, les relations monomiales sur CM-périodes, et leurs analogues p-adiques.

1. Introduction

We modify Euler's gamma function $\Gamma(z)$ into

$$\Gamma_{\infty}(z) := \frac{\Gamma(z)}{\sqrt{2\pi}} = \exp(\zeta'(0, z)) \quad (z > 0)$$

and focus on its special values at rational numbers. Here we put $\zeta(s,z) := \sum_{k=0}^{\infty} (z+k)^{-s}$ to be the Hurwitz zeta function. The last equation is due to Lerch. One has a "simple proof" in [14, p. 17]. The gamma function enjoys some functional equations:

- (1.1) Euler's Reflection formula: $\Gamma_{\infty}(z)\Gamma_{\infty}(1-z) = \frac{1}{2\sin \pi z}$,
- (1.2) Gauss' Multiplication formula:

$$\prod_{k=0}^{d-1} \Gamma_{\infty} \left(z + \frac{k}{d} \right) = d^{\frac{1}{2} - dz} \Gamma_{\infty} (dz) \quad (d \in \mathbb{N}).$$

Keywords: Absolute Frobenius, Fermat curves, Gross-Koblitz formula, p-adic gamma function, CM-periods, p-adic periods.

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For proofs, see $[1, \S 3, 4]$. The main topic of this paper is a relation between such functional equations and monomial relations of CM-periods, and its p-adic analogue. We introduce some notations.

DEFINITION 1.1. — Let K be a CM-field. We denote by I_K the \mathbb{Q} -vector space formally generated by all complex embeddings of K:

$$I_K \coloneqq \bigoplus_{\sigma \in \operatorname{Hom}(K,\mathbb{C})} \mathbb{Q} \cdot \sigma.$$

We identify a subset $S \subset \operatorname{Hom}(K,\mathbb{C})$ as an element $\sum_{\sigma \in S} \sigma \in I_K$. Shimura's period symbol is the bilinear map

$$p_K \colon I_K \times I_K \to \mathbb{C}^\times / \overline{\mathbb{Q}}^\times$$

characterized by the following properties (P1), (P2).

(P1) Let A be an abelian variety defined over $\overline{\mathbb{Q}}$, having CM of type (K,Ξ) . Namely, for each $\sigma \in \operatorname{Hom}(K,\mathbb{C})$, there exists a non-zero "K-eigen" differential form ω_{σ} of the second kind satisfying

$$k^*(\omega_\sigma) = \sigma(k)\omega_\sigma \quad (k \in K),$$

where k^* denotes the action of $k \in K$ via $K \cong \operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ on the de Rham cohomology $H^1_{\operatorname{dR}}(A,\mathbb{C})$. Then we have

 $\Xi = \{ \sigma \in \operatorname{Hom}(K, \mathbb{C}) \mid \omega_{\sigma} \text{ is holomorphic} \},$

$$p_K(\sigma,\Xi) \equiv \begin{cases} \pi^{-1} \int_{\gamma} \omega_{\sigma} & (\sigma \in \Xi) \\ \int_{\gamma} \omega_{\sigma} & (\sigma \in \operatorname{Hom}(K,\mathbb{C}) - \Xi) \end{cases} \mod \overline{\mathbb{Q}}^{\times}$$

for an arbitrary closed path $\gamma \subset A(\mathbb{C})$ satisfying $\int_{\gamma} \omega_{\sigma} \neq 0$.

(P2) Let ρ be the complex conjugation. Then we have

$$p_K(\sigma,\tau)p_K(\rho\circ\sigma,\tau)\equiv p_K(\sigma,\tau)p_K(\sigma,\rho\circ\tau)\equiv 1 \mod \overline{\mathbb{Q}}^\times$$

$$(\sigma,\tau\in \operatorname{Hom}(K,\mathbb{C})).$$

We note that, strictly speaking, Shimura's p_K in [13, §32] is a bilinear map on $\bigoplus_{\sigma \in \operatorname{Hom}(K,\mathbb{C})} \mathbb{Z} \cdot \sigma$. The period symbol also enjoys the following relations:

(P3) Let $\iota \colon K' \cong K$ be an isomorphism of CM-fields. Then we have

$$p_K(\sigma, \tau) \equiv p_{K'}(\sigma \circ \iota, \tau \circ \iota) \mod \overline{\mathbb{Q}}^{\times} \quad (\sigma, \tau \in \text{Hom}(K, \mathbb{C})).$$

(P4) Let $K \subset L$ be a field extension of CM-fields. We define two linear maps defined as

Res:
$$I_L \to I_K$$
, $\widetilde{\sigma} \mapsto \widetilde{\sigma}|_K$ $(\widetilde{\sigma} \in \text{Hom}(L, \mathbb{C}))$,
Inf: $I_K \to I_L$, $\sigma \mapsto \sum_{\substack{\widetilde{\sigma} \in \text{Hom}(L, \mathbb{C}) \\ \widetilde{\sigma}|_K = \sigma}} \widetilde{\sigma}$ $(\sigma \in \text{Hom}(K, \mathbb{C}))$.

Then we have

$$p_K(\operatorname{Res}(X), Y) \equiv p_L(X, \operatorname{Inf}(Y)) \mod \overline{\mathbb{Q}}^{\times} \quad (X \in I_L, Y \in I_K).$$

The following results by Gross–Rohrlich and the above relations (P3), (P4) provide an explicit formula [14, Theorem 2.5, Chap. III] on p_K for $K = \mathbb{Q}(\zeta_N)$ ($\zeta_N = \mathrm{e}^{\frac{2\pi \mathrm{i}}{N}}$, $N \geq 3$). We can rewrite it in the form (1.5) by the arguments in [8, §6]. Let $\sigma_b \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ ((b, N) = 1) be defined by $\sigma_b(\zeta_N) := \zeta_N^b$, $\langle \alpha \rangle \in (0, 1)$ denote the fraction part of $\alpha \in \mathbb{Q} - \mathbb{Z}$.

THEOREM 1.2 ([5, Theorem in Appendix]). — Let $F_N: x^N + y^N = 1$ be the Nth Fermat curve, $\eta_{r,s} := x^{r-1}y^{s-N} dx$ its differential forms of the second kind $(0 < r, s < N, r + s \neq N)$. Then we have for any closed path γ on $F_N(\mathbb{C})$ with $\int_{\gamma} \eta_{r,s} \neq 0$

(1.3)
$$\int_{\gamma} \eta_{r,s} \equiv \frac{\Gamma(\frac{r}{N})\Gamma(\frac{s}{N})}{\Gamma(\frac{r+s}{N})} \mod \mathbb{Q}(\zeta_N)^{\times}.$$

THEOREM 1.3 ([5, §2], [14, §2, Chap. III]). — The CM-type corresponding to $\eta_{r,s}$ is

(1.4)
$$\Xi_{r,s} := \left\{ \sigma_b \middle| \begin{array}{l} 1 \leqslant b \leqslant N, \ (b,N) = 1, \\ \left\langle \frac{br}{N} \right\rangle + \left\langle \frac{bs}{N} \right\rangle + \left\langle \frac{b(N-r-s)}{N} \right\rangle = 1 \right\}.$$

That is, we have

$$p_{\mathbb{Q}(\zeta_N)}(\mathrm{id},\Xi_{r,s}) \equiv \begin{cases} \pi^{-1} \int_{\gamma} \eta_{r,s} & (r+s < N) \\ \int_{\gamma} \eta_{r,s} & (r+s > N) \end{cases} \mod \overline{\mathbb{Q}}^{\times}.$$

Corollary 1.4 ([8, Theorem 3]). — We have for any $\frac{a}{N} \in \mathbb{Q} - \mathbb{Z}$

$$(1.5) \quad \Gamma_{\infty}\left(\frac{a}{N}\right) \equiv \pi^{\frac{1}{2} - \left\langle \frac{a}{N} \right\rangle} p_{\mathbb{Q}(\zeta_N)} \left(\mathrm{id}, \sum_{(b,N)=1} \left(\frac{1}{2} - \left\langle \frac{ab}{N} \right\rangle \right) \cdot \sigma_b \right) \mod \overline{\mathbb{Q}}^{\times}.$$

Here the sum runs over all b satisfying $1 \leq b \leq N$, (b, N) = 1.

Note that (1.5) holds true even if (a, N) > 1, essentially due to (P4). Although the following is just a toy problem, we provide its proof by using the period symbol, in order to explain the theme of this paper: we may say that some functional equations of the gamma function "correspond" to some monomial relations of CM-periods.

PROPOSITION 1.5 (A toy problem). — The explicit formula (1.5) implies the following "functional equations $\operatorname{mod} \overline{\mathbb{Q}}^{\times}$ " on $\Gamma(\frac{a}{N})$:

"Reflection formula":
$$\Gamma_{\infty}\left(\frac{a}{N}\right)\Gamma_{\infty}\left(\frac{N-a}{N}\right)\equiv 1 \mod \overline{\mathbb{Q}}^{\times},$$
 "Multiplication formula":
$$\prod_{k=0}^{d-1}\Gamma_{\infty}\left(\frac{a}{N}+\frac{k}{d}\right)\equiv \Gamma_{\infty}\left(\frac{da}{N}\right) \mod \overline{\mathbb{Q}}^{\times}.$$

Proof. — "Reflection formula" follows from (P2) immediately. Concerning "Multiplication formula", we may assume that $d \mid N$. Under the expression (1.5), "Multiplication formula" is equivalent to

$$\pi^{\sum_{k=0}^{d-1} \frac{1}{2} - \langle \frac{a}{N} + \frac{k}{d} \rangle} p_{\mathbb{Q}(\zeta_N)} \left(id, \sum_{(b,N)=1} \left(\sum_{k=0}^{d-1} \frac{1}{2} - \left\langle \frac{ab}{N} + \frac{kb}{d} \right\rangle \right) \cdot \sigma_b \right)$$

$$\equiv \pi^{\frac{1}{2} - \langle \frac{ad}{N} \rangle} p_{\mathbb{Q}(\zeta_N)} \left(id, \sum_{(b,N)=1} \left(\frac{1}{2} - \left\langle \frac{dab}{N} \right\rangle \right) \cdot \sigma_b \right).$$

This follows from the multiplication formula

(1.6)
$$\sum_{k=0}^{d-1} B_1 \left(x + \frac{k}{d} \right) = B_1(dx)$$

for the 1st Bernoulli polynomial $B_1(x) = x - \frac{1}{2}$.

The aim of this paper is to study a p-adic analogue of such "correspondence". More precisely, we shall characterize the p-adic gamma function by its functional equations and some special values. Then we show that the period symbol and its p-adic analogue satisfy the corresponding properties to such functional equations. As an application, we provide an alternative proof of a large part of Coleman's formula (Theorem 2.4(1)): originally, Coleman's formula was proved by calculating the absolute Frobenius on all Fermat curves. We shall see that it suffices to calculate it on only one curve (Remark 3.7).

Remark 1.6. — Yoshida and the author formulated conjectures in [8, 9, 10] which are generalizations of Coleman's formula, from cyclotomic fields

to arbitrary CM-fields: Coleman's formula implies "the reciprocity law on cyclotomic units" [7] and "the Gross–Koblitz formula on Gauss sums" [3, 6] simultaneously. The author conjectured a generalization [8, Conjecture 4] of Coleman's formula which implies a part of Stark's conjecture and a generalization of (the rank 1 abelian) Gross–Stark conjecture simultaneously. The results in this paper (in particular Remark 3.7) are very important toward this generalization, since we know only a finite number of algebraic curves (e.g., [2]) whose Jacobian varieties have CM by CM-fields which are not abelian over \mathbb{Q} .

The outline of this paper is as follows. First we introduce Coleman's formula [4] for the absolute Frobenius on Fermat curves in Section 2. The author rewrote it in the form of Theorem 2.4: roughly speaking, we write Morita's p-adic gamma function Γ_p in terms of Shimura's period symbol p_K , its p-adic analogue $p_{K,p}$, and modified Euler's gamma function Γ_{∞} . In Section 3, we show that some functional equations almost characterize Γ_p (Corollary 3.3), and the corresponding properties ((3.7), Theorem 3.5) hold for $p_K, p_{K,p}, \Gamma_{\infty}$. Then we see that a large part (Corollary 3.6) of Coleman's formula follows automatically, without explicit computation, under assuming certain p-adic continuity properties. Unfortunately, our results have a root of unity ambiguity although the original formula is a complete equation, since some definitions are well-defined only up to roots of unity. In Section 4, we confirm that we can show (at least, a part of) needed p-adic continuity properties relatively easily.

2. Coleman's formula in terms of period symbols

Coleman explicitly calculated the absolute Frobenius on Fermat curves [4]. The author rewrote his formula in [7, 8] as follows.

2.1. p-adic period symbol

Let p be a rational prime, \mathbb{C}_p the p-adic completion of the algebraic closure $\overline{\mathbb{Q}_p}$ of \mathbb{Q}_p , and μ_{∞} the group of all roots of unity. For simplicity, we fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, \mathbb{C}_p and consider any number field as a subfield of each of them. Let $B_{\text{cris}} \subset B_{\text{d}R}$ be Fontaine's p-adic period rings. We consider the composite ring $B_{\text{cris}}\overline{\mathbb{Q}_p} \subset B_{\text{d}R}$. Let A be an abelian variety

with CM defined over $\overline{\mathbb{Q}}$, γ a closed path on $\subset A(\mathbb{C})$, and ω a differential form of the second kind of A. Then the p-adic period integral

$$\int_{p} : H_{1}^{\mathrm{B}}(A(\mathbb{C}), \mathbb{Q}) \times H_{\mathrm{d}R}^{1}(A, \overline{\mathbb{Q}}) \to B_{\mathrm{cris}} \overline{\mathbb{Q}_{p}}, \ (\gamma, \omega) \mapsto \int_{\gamma, p} \omega$$

is defined by the comparison isomorphisms of p-adic Hodge theory, instead of the de Rham isomorphism (e.g., [8, §5.1], [7, §6]). Here H^B denotes the singular (Betti) homology. Then, in a similar manner to p_K , we can define the p-adic period symbol

$$p_{K,p} \colon I_K \times I_K \to (B_{\operatorname{cris}} \overline{\mathbb{Q}_p} - \{0\})^{\mathbb{Q}} / \overline{\mathbb{Q}}^{\times}$$

satisfying p-adic analogues of (P1), (P2), (P3), (P4). Here we put $(B_{\text{cris}}\overline{\mathbb{Q}_p} - \{0\})^{\mathbb{Q}} := \{x \in B_{dR} \mid \exists n \in \mathbb{N} \text{ s.t. } x^n \in B_{\text{cris}}\overline{\mathbb{Q}_p} - \{0\}\}$. Moreover the "ratio"

$$\left[\int_{\gamma} \omega_{\sigma} : \int_{\gamma, p} \omega_{\sigma}\right] \in (\mathbb{C}^{\times} \times (B_{\mathrm{cris}}\overline{\mathbb{Q}_{p}} - \{0\}))/\overline{\mathbb{Q}}^{\times}$$

depends only on $\sigma \in \text{Hom}(K,\mathbb{C})$ and the CM-type Ξ . That is, if we replace $A, \omega_{\sigma}, \gamma$ with $A', \omega'_{\sigma}, \gamma'$ for the same Ξ, σ , then we have

$$\frac{\int_{\gamma'} \omega'_{\sigma}}{\int_{\gamma} \omega_{\sigma}} = \frac{\int_{\gamma', p} \omega'_{\sigma}}{\int_{\gamma, p} \omega_{\sigma}} \in \overline{\mathbb{Q}}^{\times}.$$

Therefore we may consider the following ratio of the symbols $[p_K : p_{K,p}]$, which is well-defined up to μ_{∞} .

Proposition 2.1 ([8, Proposition 4]). — There exists a bilinear map

$$[p_K:p_{K,p}]:I_K\times I_K\to (\mathbb{C}^\times\times (B_{\mathrm{cris}}\overline{\mathbb{Q}_p}-\{0\})^{\mathbb{Q}})/(\mu_\infty\times\mu_\infty)\overline{\mathbb{Q}}^\times$$
 satisfying the following.

(1) Let $A, \Xi, \sigma, \omega_{\sigma}, \gamma$ be as in (P1). Then

()

$$[p_K : p_{K,p}](\sigma, \Xi)$$

$$\equiv \begin{cases} [(2\pi i)^{-1} \int_{\gamma} \omega_{\sigma} : (2\pi i)_p^{-1} \int_{\gamma,p} \omega_{\sigma}] & (\sigma \in \Xi) \\ [\int_{\gamma} \omega_{\sigma} : \int_{\gamma,p} \omega_{\sigma}] & (\sigma \in \operatorname{Hom}(K, \mathbb{C}) - \Xi) \end{cases}$$

$$\mod (\mu_{\infty} \times \mu_{\infty}) \overline{\mathbb{Q}}^{\times}.$$

Here $(2\pi i)_p \in B_{cris}$ is the p-adic counterpart of $2\pi i$ defined in, e.g., [8, §5.1].

(2) We have for $\sigma, \tau \in \text{Hom}(K, \mathbb{C})$ and for the complex conjugation ρ $[p_K : p_{K,p}](\sigma, \tau) \cdot [p_K : p_{K,p}](\rho \circ \sigma, \tau) \equiv 1 \mod (\mu_{\infty} \times \mu_{\infty}) \overline{\mathbb{Q}}^{\times},$ $[p_K : p_{K,p}](\sigma, \tau) \cdot [p_K : p_{K,p}](\sigma, \rho \circ \tau) \equiv 1 \mod (\mu_{\infty} \times \mu_{\infty}) \overline{\mathbb{Q}}^{\times}.$

(3) Let $\iota \colon K' \cong K$ be an isomorphism of CM-fields. Then we have for $\sigma, \tau \in \operatorname{Hom}(K, \mathbb{C})$

$$[p_K:p_{K,p}](\sigma,\tau) \equiv [p_{K'}:p_{K',p}](\sigma \circ \iota,\tau \circ \iota) \mod (\mu_\infty \times \mu_\infty) \overline{\mathbb{Q}}^{\times}.$$

(4) Let $K \subset L$ be a field extension of CM-fields. Then we have for $X \in I_L, Y \in I_K$

$$[p_K: p_{K,p}](\operatorname{Res}(X), Y) \equiv [p_L: p_{L,p}](X, \operatorname{Inf}(Y)) \mod (\mu_{\infty} \times \mu_{\infty}) \overline{\mathbb{Q}}^{\times}.$$

2.2. Coleman's formula

Theorem 2.4 below is essentially due to Coleman [4, Theorems 1.7, 3.13]. Note that the original formula does not have a root of unity ambiguity. First we prepare some notations. We assume that p is an odd prime.

Definition 2.2.

(1) Let $\mathbb{C}_p^1 \coloneqq \{z \in \mathbb{C}_p^{\times} \mid |z|_p = 1\}$. We fix a group homomorphism $\exp_n \colon \mathbb{C}_p \to \mathbb{C}_n^1$

which coincides with the usual power series $\exp_p(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$ on the convergence region. For $\alpha \in \mathbb{C}_p^{\times}$, $\beta \in \mathbb{C}_p$, we put

$$\alpha^{\beta} \coloneqq \exp_p(\beta \log_p \alpha)$$

with \log_p Iwasawa's p-adic log function.

(2) For $z \in \mathbb{C}_p^{\times}$, we put

$$z^* := \exp_p(\log_p(z)), \qquad \qquad z^{\flat} := p^{\operatorname{ord}_p z} z^*.$$

Here we define $\operatorname{ord}_p z \in \mathbb{Q}$ by $|z|_p = |p|_p^{\operatorname{ord}_p z}$. Note that $z \equiv z^{\flat} \mod \mu_{\infty}$ $(z \in \mathbb{C}_p^{\times})$.

- (3) We define the p-adic gamma function on \mathbb{Q}_p as follows.
 - (a) On \mathbb{Z}_p , $\Gamma_p(z)$ denotes Morita's p-adic gamma function which is the unique continuous function $\Gamma_p \colon \mathbb{Z}_p \to \mathbb{Z}_p^{\times}$ satisfying

$$\Gamma_p(n) \coloneqq (-1)^n \prod_{1 \leqslant k \leqslant n-1, \ p \nmid k} k \quad (n \in \mathbb{N}).$$

(b) On $\mathbb{Q}_p - \mathbb{Z}_p$, we use $\Gamma_p \colon \mathbb{Q}_p - \mathbb{Z}_p \to \mathcal{O}_{\mathbb{Q}_i}^{\times}$ defined in [7, Lemma 4.2], which is a continuous function satisfying

$$\Gamma_p(z+1) = z^* \Gamma_p(z), \ \Gamma_p(2z) = 2^{2z - \frac{1}{2}} \Gamma_p(z) \Gamma_p\left(z + \frac{1}{2}\right).$$

Such a continuous function on $\mathbb{Q}_p - \mathbb{Z}_p$ is unique up to multiplication by μ_{∞} .

(4) For $z \in \mathbb{Z}_p$, we define $z_0 \in \{1, 2, \dots, p\}$, $z_1 \in \mathbb{Z}_p$ by

$$z = z_0 + pz_1.$$

Note that when $p \mid z$, we put $z_0 = p$, instead of 0.

(5) Let W_p be the Weil group defined as

$$W_p := \{ \tau \in \operatorname{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \mid \tau|_{\mathbb{Q}_p^{ur}} = \sigma_p^{\deg \tau} \text{ with } \deg \tau \in \mathbb{Z} \}.$$

Here \mathbb{Q}_p^{ur} denotes the maximal unramified extension of \mathbb{Q}_p , σ_p the Frobenius automorphism on \mathbb{Q}_p^{ur} .

(6) We define the action of W_p on $\mathbb{Q} \cap [0,1)$ by identifying $\mathbb{Q} \cap [0,1) = \mu_{\infty}$. Namely

$$\tau\left(\frac{a}{N}\right) := \frac{b}{N} \quad \text{if} \quad \tau(\zeta_N^a) = \zeta_N^b \quad (\tau \in W_p).$$

(7) Let Φ_{cris} be the absolute Frobenius automorphism on B_{cris} . We consider the following action of W_p on $B_{\text{cris}}\overline{\mathbb{Q}_p} \cong B_{\text{cris}} \otimes_{\mathbb{Q}^{ur}} \overline{\mathbb{Q}_1}$:

$$\Phi_{\tau} := \Phi_{\operatorname{cris}}^{\operatorname{deg} \tau} \otimes \tau \quad (\tau \in W_p).$$

(8) For $\frac{a}{N} \in \mathbb{Q} \cap (0,1)$ we put

$$\begin{split} P\left(\frac{a}{N}\right) \coloneqq \frac{\Gamma_{\infty}(\frac{a}{N}) \cdot \left(2\pi\mathrm{i}\right)_{p}^{\frac{1}{2} - \left\langle\frac{a}{N}\right\rangle} p_{\mathbb{Q}(\zeta_{N}), p}\left(\mathrm{id}, \sum_{(b, N) = 1} \left(\frac{1}{2} - \left\langle\frac{ab}{N}\right\rangle\right) \sigma_{b}\right)}{\left(2\pi\mathrm{i}\right)^{\frac{1}{2} - \left\langle\frac{a}{N}\right\rangle} p_{\mathbb{Q}(\zeta_{N})}\left(\mathrm{id}, \sum_{(b, N) = 1} \left(\frac{1}{2} - \left\langle\frac{ab}{N}\right\rangle\right) \sigma_{b}\right)} \\ & \in \left(B_{\mathrm{cris}} \overline{\mathbb{Q}_{p}} - \{0\}\right)^{\mathbb{Q}} / \mu_{\infty}. \end{split}$$

This definition makes sense since

$$\frac{\Gamma_{\infty}(\frac{a}{N})}{(2\pi i)^{\frac{1}{2} - \langle \frac{a}{N} \rangle} p_{\mathbb{Q}(\zeta_N)} \left(id, \sum_{(b,N)=1} \left(\frac{1}{2} - \langle \frac{ab}{N} \rangle \right) \sigma_b \right)} \in \overline{\mathbb{Q}} \subset B_{\mathrm{cris}} \overline{\mathbb{Q}_p}$$

by (1.5) and the ratio $[p_K: p_{K,p}]$ is well-defined up to μ_{∞} by Proposition 2.1.

Remarks 2.3.

(1) Let μ_{p-1} be the group of all (p-1)st roots of unity, $p^{\mathbb{Z}} := \{p^n \mid n \in \mathbb{Z}\}, 1 + p\mathbb{Z}_p := \{1 + pz \mid z \in \mathbb{Z}_p\}$. Then we have the canonical decomposition

$$\begin{array}{ccccc}
\mathbb{Q}_p^{\times} & \to & \mu_{p-1} & \times & p^{\mathbb{Z}} & \times & 1 + p\mathbb{Z}_p, \\
z & \mapsto & (\omega(zp^{-\operatorname{ord}_p z}) & , & p^{\operatorname{ord}_p z} & , & z^*),
\end{array}$$

where ω denotes the Teichmüller character. The maps $z \mapsto z^*, z^{\flat}$ provide a similar (but non-canonical) decomposition of \mathbb{C}_p^{\times} . Moreover, we note that the maps $z \mapsto \exp_p(z), z^*, z^{\flat}$ are continuous homomorphisms.

(2) We easily see that

$$\tau(z) = \langle pz \rangle, \ \tau^{-1}(z) = z_1 + 1 \quad (z \in \mathbb{Z}_{(p)} \cap (0, 1), \ \tau \in W_p, \ \deg \tau = 1).$$

THEOREM 2.4 ([8, Theorem 3]). — Let p be an odd prime.

(1) Assume that $z \in \mathbb{Z}_{(p)} \cap (0,1)$. Then we have

$$\Gamma_p(z) \equiv p^{\frac{1}{2} - \tau^{-1}(z)} \frac{P(z)}{\Phi_{\tau}(P(\tau^{-1}(z)))} \mod \mu_{\infty} \quad (\tau \in W_p, \ \deg \tau = 1).$$

(2) Assume that $z \in (\mathbb{Q} - \mathbb{Z}_{(p)}) \cap (0,1)$. Then we have

$$\frac{\Gamma_p(\tau(z))}{\Gamma_n(z)} \equiv \frac{p^{(z-\tau(z))\operatorname{ord}_p z} P(\tau(z))}{\Phi_{\tau}(P(z))} \mod \mu_{\infty} \quad (\tau \in W_p).$$

Remark 2.5. — As a result, we see that the right-hand sides of Theorem 2.4(1), (2) are p-adic continuous on z, $(z, \tau(z))$ respectively, since the left-hand sides are so. We use only the p-adic continuity in the next section, in order to recover Theorem 2.4(1).

3. Main results

Morita's p-adic gamma function $\Gamma_p \colon \mathbb{Z}_p \to \mathbb{Z}_p^{\times}$ is the unique continuous function satisfying

(3.1)
$$\Gamma_p(0) = 1, \quad \frac{\Gamma_p(z+1)}{\Gamma_p(z)} = \begin{cases} -z & (z \in \mathbb{Z}_p^{\times}), \\ -1 & (z \in p\mathbb{Z}_p). \end{cases}$$

In this section, we study other functional equations characterizing Γ_p and provide an alternative proof of Coleman's formula in the case $z \in \mathbb{Z}_{(p)}$. Strictly speaking, we only "assume" that the right-hand sides of Theorem 2.4(1), (2) are continuous on z, $(z, \tau(z))$ respectively (of course, this is correct). Then we can recover a "large part" (Corollary 3.6) of Theorem 2.4(1). We assume that p is an odd prime.

3.1. A characterization of Morita's p-adic gamma function

 $\Gamma_p(z)$ satisfies the following *p*-adic analogues of multiplication formulas, which we consider only up to roots of unity in this paper. For the detailed formulation and its proof, see [11, "Basic properties of Γ_p " in Section 2 of Chapter IV].

PROPOSITION 3.1. — Let $d \in \mathbb{N}$ with $p \nmid d$. Then we have for $z \in \mathbb{Z}_p$

(3.2)
$$\prod_{k=0}^{d-1} \Gamma_p\left(z + \frac{k}{d}\right) \equiv d^{1-dz + (dz)_1} \Gamma_p(dz) \mod \mu_{\infty}.$$

Note that if $p \mid d$, then $z + \frac{k}{d}$ is not in the domain of definition of Morita's Γ_p . In the rest of this subsection, we show that multiplication formulas (3.2) and some conditions characterize Morita's p-adic gamma function (at least up to μ_{∞}).

PROPOSITION 3.2. — Assume a continuous function $f(z) \colon \mathbb{Z}_p \to \mathbb{C}_p^{\times}$ satisfies

(3.3)
$$\prod_{k=0}^{d-1} f\left(z + \frac{k}{d}\right) \equiv f(dz) \mod \mu_{\infty} \quad (p \nmid d).$$

Then the following holds.

- (1) $\frac{f(z+1)}{f(z)} \mod \mu_{\infty}$ depends only on $\operatorname{ord}_{p} z$.
- (2) The values

$$c_k \coloneqq \left(\frac{f(p^k+1)}{f(p^k)}\right)^{\flat}$$

characterize the function f(z) up to μ_{∞} . More precisely, for $z \in \mathbb{Z}_p$, we write the p-adic expansion of z-1 as

$$z-1 = \sum_{k=0}^{\infty} x_k p^k \quad (x_k \in \{0, 1, \dots, p-1\}).$$

Then we have

$$f(z) \equiv \prod_{k=0}^{\infty} \alpha_k^{x_k - \frac{p-1}{2}} \mod \mu_{\infty} \quad \text{with} \quad \alpha_k \coloneqq c_k \prod_{i=0}^{k-1} c_i^{p^{k-1-i}(p-1)}.$$

Conversely, assume that

(3.4)
$$f\left(1 + \sum_{k=0}^{\infty} x_k p^k\right) \equiv \prod_{k=0}^{\infty} \alpha_k^{x_k - \frac{p-1}{2}} \mod \mu_{\infty} \quad (x_k \in \{0, 1, \dots, p-1\})$$

for constants $\alpha_k \in \mathbb{C}_p^{\times}$ satisfying $\alpha_k \to 1$ $(k \to \infty)$. Then f(z) satisfies the functional equations (3.3).

Proof. — We suppress $\operatorname{mod} \mu_{\infty}$. Assume (3.3). Replacing z with $z + \frac{1}{d}$, we obtain $\prod_{k=1}^{d} f(z + \frac{k}{d}) \equiv f(dz + 1)$. It follows that $\frac{f(z+1)}{f(z)} \equiv \frac{f(dz+1)}{f(dz)}$. That is,

$$g(z) := \frac{f(z+1)}{f(z)} \equiv g(dz) \quad (p \nmid d \in \mathbb{N}).$$

Then the assertion (1) is clear. Let $c_k := (g(p^k))^{\flat}$, $a_n := x_0 + x_1 p + \dots + x_n p^n$ $(0 \le x_i \le p-1)$. We easily see that

$$\#\{y=1,2,\ldots,a_n\mid \operatorname{ord}_p y=k\}=x_k+\sum_{i=k+1}^n x_i p^{i-k-1}(p-1) \quad (0\leqslant k\leqslant n).$$

Then we can write

$$f(a_n+1)^{\flat} = (f(1)g(1)g(2)\cdots g(a_n))^{\flat} = f(1)^{\flat}\alpha_0^{x_0}\alpha_1^{x_1}\cdots\alpha_n^{x_n}$$

with $\alpha_k = c_k \prod_{i=0}^{k-1} c_i^{p^{k-1-i}(p-1)}$. Since $\lim_{n\to\infty} f(a_n+1)$ converges, so do $\lim_{n\to\infty} f(a_n+1)^{\flat}$ and $\prod_{k=0}^{\infty} \alpha_k^{x_k}$. Moreover we can write

$$f(z) \equiv f(1) \prod_{k=0}^{\infty} \alpha_k^{x_k}.$$

Consider the case of d=2, $z=\frac{1}{2}$ of (3.3): $f(\frac{1}{2})f(1)\equiv f(1)$. Therefore, noting that $-\frac{1}{2}=\sum_{k=0}^{\infty}\frac{p-1}{2}p^k$, we obtain

$$1 \equiv f\left(\frac{1}{2}\right) \equiv f(1) \prod_{k=0}^{\infty} \alpha_k^{\frac{p-1}{2}}, \quad \text{that is,} \quad f(1) \equiv \prod_{k=0}^{\infty} \alpha_k^{-\frac{p-1}{2}}.$$

Then the assertion (2) is also clear.

Next, assume (3.4). When $\operatorname{ord}_p z = k$, we see that $\frac{f(z+1)}{f(z)} \equiv \frac{\alpha_k}{\alpha_{k-1}^{p-1}}$ (resp. α_0) if k > 0 (resp. k = 0). In particular, $g(z) \coloneqq \frac{f(z+1)}{f(z)} \mod \mu_{\infty}$ depends only on $\operatorname{ord}_p z$. When z + z' = 1, the p-adic expansions $z - 1 = \sum_{k=0}^{\infty} x_k p^k$, $z' - 1 = \sum_{k=0}^{\infty} x'_k p^k$ satisfy $x_k + x'_k = p - 1$ for any k. Then we have

$$f(z)f(z') \equiv \prod_{k=0}^{\infty} \alpha_k^0 = 1.$$

Therefore the case z=0 of (3.3) holds true since we have $\left(\prod_{k=1}^{d-1} f(\frac{k}{d})\right)^2 = \prod_{k=1}^{d-1} f(\frac{k}{d}) f(1-\frac{k}{d}) \equiv 1$. Then (3.3) for $z \in \mathbb{N}$ follows by mathematical

induction on z noting that

$$\prod_{k=0}^{d-1} f\left(z+1+\frac{k}{d}\right) \equiv \prod_{k=0}^{d-1} f\left(z+\frac{k}{d}\right) g\left(z+\frac{k}{d}\right),$$
$$f(dz+d) \equiv f(dz)g(dz)\cdots g(dz+d-1),$$
$$\operatorname{ord}_{p}(dz+k) = \operatorname{ord}_{p}\left(z+\frac{k}{d}\right).$$

Since \mathbb{N} is dense in \mathbb{Z}_p , we see that (3.3) holds for any $z \in \mathbb{Z}_p$.

The following corollary provides a nice characterization of $\Gamma_p(z)$ mod μ_{∞} in terms of functional equations and one or two special values.

COROLLARY 3.3. — Assume a continuous function $f(z): \mathbb{Z}_p \to \mathbb{C}_p^{\times}$ satisfies

$$\prod_{k=0}^{d-1} f\left(z + \frac{k}{d}\right) \equiv f(dz) \mod \mu_{\infty} \quad (p \nmid d)$$

and put

$$c_n := \left(\frac{f(p^n+1)}{f(p^n)}\right)^{\flat}.$$

Then the following equivalences hold:

(1)
$$c_0 = c_1 = \cdots \Leftrightarrow f(z) \equiv c_0^{z - \frac{1}{2}} \mod \mu_{\infty}$$
.

(1)
$$c_0 = c_1 = \cdots \Leftrightarrow f(z) \equiv c_0^{z - \frac{1}{2}} \mod \mu_{\infty}.$$

(2) $c_1 = c_2 = \cdots \Leftrightarrow f(z) \equiv c_0^{z - \frac{1}{2}} (c_1/c_0)^{z_1 + \frac{1}{2}} \mod \mu_{\infty}.$

Proof. — We suppress $\text{mod}\mu_{\infty}$. For (1), assume that $c_0 = c_1 = \cdots$. Then

$$\alpha_k \coloneqq c_k \prod_{i=0}^{k-1} c_i^{p^{k-1-i}(p-1)} = c_0^{p^k}.$$

Hence we have by Proposition 3.2

$$f\left(1 + \sum_{k=0}^{\infty} x_k p^k\right) \equiv \prod_{k=0}^{\infty} \alpha_k^{x_k - \frac{p-1}{2}} = c_0^{\sum_{k=0}^{\infty} x_k p^k - \frac{p-1}{2} p^k} = c_0^{z-1 + \frac{1}{2}} = c_0^{z - \frac{1}{2}}.$$

The opposite direction is trivial by definition $c_n := (\frac{f(p^n+1)}{f(p^n)})^{\flat}$. For (2), the assumption $c_1 = c_2 = \cdots$ implies $\alpha_0 = c_0$, $\alpha_k = c_0^{p^k} (c_1/c_0)^{p^{k-1}}$ $(k \ge 1)$. In this case we have

$$f\left(1 + \sum_{k=0}^{\infty} x_k p^k\right) \equiv c_0^{\sum_{k=0}^{\infty} x_k p^k - \frac{p-1}{2} p^k} (c_1/c_0)^{\sum_{k=1}^{\infty} x_k p^{k-1} - \frac{p-1}{2} p^{k-1}}$$
$$= c_0^{z - \frac{1}{2}} (c_1/c_0)^{z_1 + \frac{1}{2}}$$

since
$$\sum_{k=1}^{\infty} x_k p^{k-1} = \frac{z-1-x_0}{p} = z_1$$
.

3.2. Alternative proof of a part of Coleman's formula

We fix $\tau \in W_p$ with deg $\tau = 1$ and put

$$(3.5) \quad G_{1}(z) := \left(p^{\frac{1}{2} - \tau^{-1}(z)} \frac{P(z)}{\Phi_{\tau}(P(\tau^{-1}(z)))}\right)^{\flat} \quad (z \in \mathbb{Z}_{(p)} \cap (0, 1)),$$

$$G_{2}(z) := \left(\frac{p^{(\tau^{-1}(z) - z)\operatorname{ord}_{p} z} P(z)}{\Phi_{\tau}(P(\tau^{-1}(z)))}\right)^{\flat} \quad (z \in (\mathbb{Q} - \mathbb{Z}_{(p)}) \cap (0, 1)).$$

Here we added ()^b to the right-hand sides of Coleman's formulas (Theorem 2.4), in order to resolve a root of unity ambiguity, only superficially. Note that G_2 corresponds to Theorem 2.4(2) replaced z with $\tau^{-1}(z)$.

By Theorem 2.4(1), we see that G_1 is continuous for the p-adic topology. G_2 is not p-adically continuous in the usual sense, on the whole of $(\mathbb{Q} - \mathbb{Z}_{(p)}) \cap (0,1)$ (for details, see Remark 3.8). Theorem 2.4(1) only implies the following "continuity":

(3.6) $G_1(z)$ is continuous for the relative topology

induced by
$$z \in (\mathbb{Q} - \mathbb{Z}_{(p)}) \cap (0,1) \hookrightarrow \mathbb{Q}_p \times \mathbb{Q}_p, z \mapsto (z,\tau^{-1}(z)).$$

In Corollary 3.6, oppositely, we show that the p-adic continuity of G_1, G_2 implies a "large part"

$$G_1(z) \equiv a^{z-\frac{1}{2}} b^{z_1+\frac{1}{2}} \Gamma_p(z) \mod \mu_\infty \quad (a,b \in \mathbb{C}_p^\times)$$

of Theorem 2.4(1):

$$G_1(z) \equiv \Gamma_p(z) \mod \mu_{\infty}.$$

Besides we shall show the continuity of $G_1(z)$ in Section 4, independently of Theorem 2.4.

Hereinafter in this section, we forget Theorem 2.4. We assume the following Assumption instead.

Assumption 3.4. — $G_1(z)$ is p-adically continuous and $G_2(z)$ is continuous in the sense of (3.6). In particular, we regard G_1 as a p-adic continuous function:

$$G_1(z)\colon \mathbb{Z}_p\to \mathbb{C}_p.$$

First we derive "multiplication formula":

(3.7)
$$\prod_{k=0}^{d-1} G_1\left(z + \frac{k}{d}\right) \equiv d^{1-dz + (dz)_1} G_1(dz) \mod \mu_\infty \quad (p \nmid d \in \mathbb{N})$$

independently of Theorem 2.4.

Proof of (3.7). — We suppress $\operatorname{mod} \mu_{\infty}$. Let $z \in \mathbb{Z}_{(p)} \cap (0, \frac{1}{d})$. By Definition 2.2(8) and (3.5) we can write

$$\begin{split} &\frac{\prod_{k=0}^{d-1} G_1(z+\frac{k}{d})}{G_1(dz)} \\ &\equiv \frac{\prod_{k=0}^{d-1} \Gamma_{\infty}(z+\frac{k}{d})}{\Gamma_{\infty}(dz)} \Phi_{\tau} \left(\frac{\Gamma_{\infty}(\tau^{-1}(dz))}{\prod_{k=0}^{d-1} \Gamma_{\infty}(\tau^{-1}(z+\frac{k}{d}))} \right) \frac{\prod_{k=0}^{d-1} p^{\frac{1}{2}-\tau^{-1}(z+\frac{k}{d})}}{p^{\frac{1}{2}-\tau^{-1}(dz)}} \end{split}$$

 \times "products of classical or *p*-adic periods",

where the "products of classical or p-adic periods" become trivial by (1.6), as we saw in the proof of Proposition 1.5. Besides we see that

$$\left\{ \tau^{-1} \left(z + \frac{k}{d} \right) \mid k = 0, \dots, d - 1 \right\} = \left\{ \frac{\tau^{-1} (dz)}{d} + \frac{k}{d} \mid k = 0, \dots, d - 1 \right\}.$$

To see this, it suffices to show that $\{\tau^{-1}(\zeta_N^a\zeta_d^k)\mid k=0,\ldots,d-1\}$ and $\{\tau^{-1}(\zeta_N^{da})^{\frac{1}{d}}\zeta_d^k\mid k=0,\ldots,d-1\}$ coincide with each other. We easily see that both of them are the inverse image of $\tau^{-1}(\zeta_N^{da})$ under the dth power map $\mu_\infty\to\mu_\infty$, $x\mapsto x^d$. Hence we obtain

$$\begin{split} &\frac{\prod_{k=0}^{d-1} G_1(z+\frac{k}{d})}{G_1(dz)} \\ &\equiv \frac{\prod_{k=0}^{d-1} \Gamma_{\infty}(z+\frac{k}{d})}{\Gamma_{\infty}(dz)} \cdot \Phi_{\tau} \left(\frac{\Gamma_{\infty}(\tau^{-1}(dz))}{\prod_{k=0}^{d-1} \Gamma_{\infty}(\frac{\tau^{-1}(dz)}{d}+\frac{k}{d})} \right) \cdot \frac{\prod_{k=0}^{d-1} p^{\frac{1}{2}-(\frac{\tau^{-1}(dz)}{d}+\frac{k}{d})}}{p^{\frac{1}{2}-\tau^{-1}(dz)}} \\ &= d^{\frac{1}{2}-dz} \cdot \Phi_{\tau}(d^{\tau^{-1}(dz)-\frac{1}{2}}) \cdot 1 \equiv d^{\frac{1}{2}-dz} \cdot d^{\tau^{-1}(dz)-\frac{1}{2}} \end{split}$$

by (1.2), (1.6). For the last " \equiv ", we note that Φ_{τ} acts on $\overline{\mathbb{Q}}_{\scriptscriptstyle \parallel} \ni d^{\tau^{-1}(dz)-\frac{1}{2}}$ as τ . By Remark 2.3(2), we have $\tau^{-1}(dz)=(dz)_1+1$. Then the assertion is clear.

Furthermore we can show that $c_n = \left(\frac{f(p^n+1)}{f(p^n)}\right)^{\flat}$ for $f(z) := \frac{G_1(z)}{\Gamma_p(z)}$ is constant, at least for $n \ge 1$.

THEOREM 3.5. — We assume Assumption 3.4 and put $f(z) := \frac{G_1(z)}{\Gamma_p(z)}$.

(1) The following functional equations hold.

$$\prod_{k=0}^{d-1} f\left(z + \frac{k}{d}\right) \equiv f(dz) \mod \mu_{\infty} \quad (p \nmid d).$$

(2) We have
$$c_1 = c_2 = \cdots$$
 for $c_n := \left(\frac{f(p^n + 1)}{f(p^n)}\right)^{\flat}$.

Proof. — We suppress $\operatorname{mod} \mu_{\infty}$. (1) follows from (3.2), (3.7). For (2), we need for $z \in p\mathbb{Z}_p$

$$\frac{G_1(pz)G_1(z+1)}{G_1(pz+1)G_1(z)} \equiv \frac{\Gamma_p(pz)\Gamma_p(z+1)}{\Gamma_p(pz+1)\Gamma_p(z)}.$$

Since the right-hand side is equal to $\begin{cases} l & (p|z) \\ z & (p\nmid z) \end{cases}$ by (3.1), it suffices to show that

$$\frac{G_1(pz)G_1(z+1)}{G_1(pz+1)G_1(z)} \equiv 1 \quad (z \in p\mathbb{Z}_p).$$

Note that we can not use the definition (3.5) directly since z, z+1, pz, pz+1 are not contained in (0,1) simultaneously. Therefore a little complicated argument is needed as follows. Let $z \in \mathbb{Z}_{(p)} \cap (0, \frac{1}{p})$. By Remark 2.3(2), we have

$$\tau(z) = \langle pz \rangle = pz$$
, hence $\tau^{-1}(pz) = z$.

We can write

$$H_1(z) := \frac{G_1(z)G_2(z + \frac{1}{p}) \cdots G_2(z + \frac{p-1}{p})}{G_1(pz)}$$

$$\equiv p^{z + (z + \frac{1}{p}) + \dots + (z + \frac{p-1}{p}) - \tau^{-1}(z) - \tau^{-1}(z + \frac{1}{p}) - \dots - \tau^{-1}(z + \frac{p-1}{p})}$$

$$\times \frac{P(z)P(z + \frac{1}{p}) \cdots P(z + \frac{p-1}{p})}{P(pz)}$$

$$\times \Phi_{\tau} \left(\frac{P(z)}{P(\tau^{-1}(z))P(\tau^{-1}(z + \frac{1}{p})) \cdots P(\tau^{-1}(z + \frac{p-1}{p}))} \right).$$

Here we note that $\operatorname{ord}_p(z+\frac{k}{p})=-1$ for $k=1,\ldots,p-1$. We have

(3.8)
$$\left\{ \tau^{-1} \left(z + \frac{k}{p} \right) \middle| k = 0, \dots, p - 1 \right\} = \left\{ \frac{z+k}{p} \middle| k = 0, \dots, p - 1 \right\}$$

since both of $\{\tau^{-1}(\zeta_N^a\zeta_p^k)\mid k=0,\ldots,p-1\}$, $\{\zeta_{pN}^{a+Nk}\mid k=0,\ldots,p-1\}$ are the set of the pth roots of ζ_N^a when $z=\frac{a}{N}$. Therefore the p-power parts of H_1 become

$$p^{z+(z+\frac{1}{p})+\dots+(z+\frac{p-1}{p})-\frac{z}{p}-\frac{z+1}{p}-\dots-\frac{z+p-1}{p}}=p^{(p-1)z}.$$

Moreover the "period parts" of H_1 become trivial by (1.6), (3.8). Namely we can write

$$H_1(z) \equiv p^{(p-1)z} \frac{\Gamma_{\infty}(z) \Gamma_{\infty}(z + \frac{1}{p}) \cdots \Gamma_{\infty}(z + \frac{p-1}{p})}{\Gamma_{\infty}(pz)} \times \Phi_{\tau} \left(\frac{\Gamma_{\infty}(z)}{\Gamma_{\infty}(\frac{z}{p}) \Gamma_{\infty}(\frac{z+1}{p}) \cdots \Gamma_{\infty}(\frac{z+p-1}{p})} \right).$$

By using the original Multiplication formula (1.2) for Γ_{∞} , we obtain

$$H_1(z) \equiv p^{(p-1)z} p^{\frac{1}{2} - pz} p^{z - \frac{1}{2}} = 1.$$

Next, let $z = \frac{a}{N} \in \mathbb{Z}_{(p)} \cap (-\frac{1}{n}, 0)$. Then we have

- $\tau(z+1) = pz + 1$. Hence $\tau^{-1}(pz+1) = z + 1$.
- $\{\tau^{-1}(\zeta_N^a \zeta_p^k) \mid k = 1, \dots, p\} = \{\zeta \mid \zeta^p = \zeta_N^a\} = \{\zeta_{pN}^{a+Nk} \mid k = 1, \dots, p\}$. Hence $\{\tau^{-1}(z + \frac{k}{p}) \mid k = 1, \dots, p\} = \{\frac{z+k}{p} \mid k = 1, \dots, p\}$.

Then we can prove similarly that

$$\begin{split} H_2(z) &\coloneqq \frac{G_2(z+\frac{1}{p})\cdots G_2(z+\frac{p-1}{p})G_1(z+1)}{G_1(pz+1)} \\ &\equiv p^{(z+\frac{1}{p})+\cdots+(z+\frac{p-1}{p})+(z+1)-\tau^{-1}(z+\frac{1}{p})-\cdots-\tau^{-1}(z+\frac{p-1}{p})-\tau^{-1}(z+1)} \\ &\times \frac{P(z+\frac{1}{p})\cdots P(z+\frac{p-1}{p})P(z+1)}{P(pz+1)} \\ &\times \Phi_\tau \left(\frac{P(z+1)}{P(\tau^{-1}(z+\frac{1}{p}))\cdots P(\tau^{-1}(z+\frac{p-1}{p}))P(\tau^{-1}(z+1))}\right) \\ &\equiv p^{(z+\frac{1}{p})+\cdots+(z+\frac{p-1}{p})+(z+1)-\frac{z+1}{p}-\cdots-\frac{z+p-1}{p}-\frac{z+p}{p}} \, p^{\frac{1}{2}-(pz+1)} p^{z+1-\frac{1}{2}} = 1. \end{split}$$

Here $H_i(z) \equiv 1 \mod \mu_{\infty}$ implies $H_i(z) = 1$ (i = 1, 2) since we have $x^{\flat} = \exp_p(\log_p x) = \exp_p(0) = 1$ for $x \in \mu_{\infty}$. $(G_1(z), G_2(z))$ are in the image under $(x)^{\flat}$ by definition, so are $H_i(z)$.) In particular, we have

$$\frac{G_1(pz)}{G_1(z)} = G_2\left(z + \frac{1}{p}\right) \cdots G_2\left(z + \frac{p-1}{p}\right) \quad \left(z \in \mathbb{Z}_{(p)} \cap \left(0, \frac{1}{p}\right)\right),$$

$$\frac{G_1(pz+1)}{G_1(z+1)} = G_2\left(z + \frac{1}{p}\right) \cdots G_2\left(z + \frac{p-1}{p}\right) \quad \left(z \in \mathbb{Z}_{(p)} \cap \left(-\frac{1}{p}, 0\right)\right).$$

Let $z \in p\mathbb{Z}_{(p)}$. Then there exist $z_n^+ \in p\mathbb{Z}_{(p)} \cap (0, \frac{1}{p}), z_n^- \in p\mathbb{Z}_{(p)} \cap (-\frac{1}{p}, 0)$ which converge to z when $n \to \infty$ respectively. Then we can write

$$\frac{G_1(pz)}{G_1(z)} = \lim_{n \to \infty} \frac{G_1(pz_n^+)}{G_1(z_n^+)} = \lim_{n \to \infty} G_2\left(z_n^+ + \frac{1}{p}\right) \cdots G_2\left(z_n^+ + \frac{p-1}{p}\right),$$

$$\frac{G_1(pz+1)}{G_1(z+1)} = \lim_{n \to \infty} \frac{G_1(pz_n^- + 1)}{G_1(z_n^- + 1)} = \lim_{n \to \infty} G_2\left(z_n^- + \frac{1}{p}\right) \cdots G_2\left(z_n^- + \frac{p-1}{p}\right).$$

Recall that $G_2(z)$ is continuous in the sense of (3.6). Clearly we have for k = 1, ..., p - 1

$$z_n^{\pm} + \frac{k}{p} \to z + \frac{k}{p} \quad (n \to \infty).$$

Additionally we see that

$$\tau^{-1}\left(z_n^{\pm} + \frac{k}{p}\right) = \frac{z_n^{\pm}}{p} + \tau^{-1}\left(\frac{k}{p}\right) \to \frac{z}{p} + \tau^{-1}\left(\frac{k}{p}\right) \quad (n \to \infty)$$

by noting that $\tau^{-1}(z+z') \equiv \tau^{-1}(z) + \tau^{-1}(z') \mod \mathbb{Z} \ (\forall z,z'), \ \tau^{-1}(z) \equiv \frac{z}{p} \mod \mathbb{Z} \ \text{if} \ p \mid z, \frac{z_n^{\pm}}{p} \in (-\frac{1}{p},\frac{1}{p}), \ \tau^{-1}(\frac{k}{p}) \in [\frac{1}{p},\frac{p-1}{p}].$ It follows that

$$\lim_{n \to \infty} G_2\left(z_n^+ + \frac{k}{p}\right) = \lim_{n \to \infty} G_2\left(z_n^- + \frac{k}{p}\right).$$

Then the assertion is clear.

By Corollary 3.3, we obtain the following.

COROLLARY 3.6. — Assume Assumption 3.4. Then there exist constants a, b satisfying

$$G_1(z) \equiv a^{z - \frac{1}{2}} b^{z_1 + \frac{1}{2}} \Gamma_p(z) \mod \mu_{\infty}.$$

Remark 3.7. — In addition to the above results, by computing the absolute Frobenius on only one Fermat curve, we obtain Coleman's formula $G_1(z) \equiv \Gamma_p(z) \mod \mu_{\infty}$. For example, when p=3, we obtain it for $z=\frac{1}{5},\frac{2}{5}$ by the computation on F_5 . It follows that $a^{\frac{-3}{10}}b^{\frac{-1}{10}} \equiv a^{\frac{-1}{10}}b^{\frac{3}{10}} \equiv 1$, hence $a \equiv b \equiv 1$.

Remark 3.8. — We used the assumption $p \mid z$ only in the last paragraph of the proof for Theorem 3.5 because G_2 is not p-adically continuous on the whole of $(\mathbb{Q} - \mathbb{Z}_{(p)}) \cap (0, 1)$. For example, we put

$$z_n := \frac{1}{p^2} + \frac{p^{n+1}}{p^{n+2} + (1-p)^n} \in (\mathbb{Q} - \mathbb{Z}_{(p)}) \cap (0,1) \quad (n \in \mathbb{N})$$

and take $\tau \in W_p$ with deg $\tau = 1$ so that

$$\tau(\zeta_{p^2}) = \zeta_{n^2}^{-1}.$$

In particular we see that

$$z_n \to \frac{1}{p^2}$$
 for the *p*-adic topology.

On the other hand we see that

$$\tau^{-1}(z_n) \equiv \tau^{-1} \left(\frac{1}{p^2}\right) + \tau^{-1} \left(\frac{p^{n+1}}{p^{n+2} + (1-p)^n}\right) \mod \mathbb{Z}$$

$$= \frac{p^2 - 1}{p^2} + \frac{p^n}{p^{n+2} + (1-p)^n} = 1 - \frac{(1-p)^n}{p^2(p^{n+2} + (1-p)^n)},$$

$$1 - \frac{(1-p)^n}{p^2(p^{n+2} + (1-p)^n)} \in \begin{cases} (1,2) & \text{if } n \text{ is odd,} \\ (0,1) & \text{if } n \text{ is even.} \end{cases}$$

Hence we have

$$\tau^{-1}(z_n) = \begin{cases} -\frac{(1-p)^n}{p^2(p^{n+2} + (1-p)^n)} \to -\frac{1}{p^2} & \text{if } n = 2k+1, \ k \to \infty, \\ 1 - \frac{(1-p)^n}{p^2(p^{n+2} + (1-p)^n)} \to 1 - \frac{1}{p^2} & \text{if } n = 2k, \ k \to \infty. \end{cases}$$

Then, by Theorem 2.4(2), we see that $G_2(z_n) = (\Gamma_p(z_n)/\Gamma_p(\tau^{-1}(z_n)))^{\flat}$ does not converge *p*-adically although z_n does.

4. On the p-adic continuity

In the previous section, we showed that the p-adic continuity of the right-hand sides of Theorem 2.4(1), (2) implies a large part of Theorem 2.4(1) itself. In this section, we see that it is relatively easy to show such p-adic continuity properties, without explicit computation. For simplicity, we consider only the case $z \in \mathbb{Z}_p$. Assume that $p \nmid N$.

LEMMA 4.1 ([3, §VI]). — Let $1 \le r, s < N$ with $r + s \ne N$. We consider the formal expansion of the differential form $\eta_{r,s} = x^r y^{s-N} \frac{dx}{x}$ on $F_N : x^N + y^N = 1$ at (x,y) = (0,1):

$$\begin{split} \eta_{r,s} &= \sum_{n=0}^{\infty} b_{r,s}(n) x^n \frac{dx}{x}, \\ b_{r,s}(n) &\coloneqq \begin{cases} (-1)^{\frac{n-r}{N}} \binom{\frac{s}{N}-1}{\frac{n-r}{N}} & (n \equiv r \mod N), \\ 0 & (n \not\equiv r \mod N). \end{cases} \end{split}$$

Let Φ be the absolute Frobenius on $H^1_{dR}(F_N, \mathbb{Q}_p)$. Then there exists $\alpha_{r',s'} \in \mathbb{Q}_p$ satisfying

$$\Phi(\eta_{r,s}) = \alpha_{r',s'} \eta_{r',s'}$$
for r', s' with $1 \le r', s' < N$, $pr \equiv r' \mod N$, $ps \equiv s' \mod N$.

Then we have

(4.1)
$$\alpha_{r',s'} = \lim_{\substack{N \ni n \mapsto 0 \\ n \equiv r \bmod N}} \frac{pb_{r,s}(n)}{b_{r',s'}(pn)}$$

$$= \lim_{\substack{N \ni k \to -\frac{r}{N}}} (-1)^{(p-1)k + \frac{pr-r'}{N}} \frac{p\binom{\frac{s}{N}}{N} - 1}{\binom{\frac{s'}{N}}{N} - 1}.$$

We note that $\alpha_{r',s'}$ depends only on $(\frac{r'}{N},\frac{s'}{N})$. That is $\alpha_{r',s'}$ with $N=N_1$ is equal to $\alpha_{tr',ts'}$ with $N=tN_1$.

PROPOSITION 4.2. — $\alpha_{r',s'}$ is p-adically continuous on $(\frac{r'}{N},\frac{s'}{N}) \in (\mathbb{Z}_{(p)} \cap (0,1))^2$.

Proof. — It suffices to show that $\alpha_{r'_1,s'_1}$ with $N=N_1$ is close to $\alpha_{r'_2,s'_2}$ with $N=N_2$ when $\frac{r'_1}{N_1}$ is close to $\frac{r'_2}{N_2}$ and $\frac{s'_1}{N_1}$ is close to $\frac{s'_2}{N_2}$. We may assume $N:=N_1=N_2$ by considering $N=N_1N_2$. First we fix $r':=r'_1=r'_2$ and assume that s'_1 is close to s'_2 . Then we can take the same k for the limit expressions (4.1) of α_{r',s'_1} , α_{r',s'_2} . We easily see that if $p^l \mid (s'_1-s'_2)$, then $p^{l-1} \mid (s_1-s_2)$. In fact, we can write $s'_i=ps_i-l_iN$ with $l_i=0,1,\ldots,p-1$ since $0 < s_i, s'_i < N$ for i=1,2. If $p \mid (s'_1-s'_2)$, then we have $p \mid (l_1-l_2)$, so $l_1=l_2$. Therefore we obtain $s_1-s_2=\frac{s'_1-s'_2}{p}$. It follows that s_1 also is close to s_2 . Hence the continuity on $\frac{s'}{N}$ is clear since the numerator (resp. the denominator) of the expression (4.1) is a polynomial on $\frac{s}{N}$ (resp. $\frac{s'}{N}$).

For the variable $\frac{r'}{N}$, we replace x with y. In other words, replace the point (x,y)=(0,1) for the expansion with (1,0). Then the continuity on $\frac{r'}{N}$ also follows from the same argument.

COROLLARY 4.3. — $G_1(z)$ defined in (3.5) is p-adically continuous on $z \in \mathbb{Z}_{(p)} \cap (0,1)$. In particular, we may regard $G_1(z)$ as a continuous function on \mathbb{Z}_p .

Proof. — CM-types $\Xi_{r,s}$ of (1.4), corresponding to $\eta_{r,s}$, generate the \mathbb{Q} -vector space $\{\sum_{\sigma} c_{\sigma} \cdot \sigma \mid c_{\sigma} + c_{\rho \circ \sigma} \text{ is a constant}\}$. More explicitly, we claim

that

$$\sum_{(b,N)=1} \left(\frac{1}{2} - \left\langle \frac{ab}{N} \right\rangle \right) \sigma_b = \frac{1}{N} \sum_{1 \leqslant s < N, \ a+s \neq N} \Xi_{a,s} - \frac{N-2}{2N} \sum_{(b,N)=1} \sigma_b,$$

where s runs over $1 \leq s < N$ with $a+s \neq N$ in the first sum of the right-hand side. By the definition (1.4), $\sigma_b \in \Xi_{a,s}$ if and only if $\langle \frac{ab}{N} \rangle + \langle \frac{sb}{N} \rangle < 1$. Namely $\langle \frac{sb}{N} \rangle = \frac{1}{N}, \frac{2}{N}, \dots, 1 - \frac{1}{N} - \langle \frac{ab}{N} \rangle$. The number of such b is congruent to $-1 - ab \mod N$. Hence we have

$$\frac{1}{N} \sum_{\substack{1 \leqslant s < N, \\ a+s \neq N}} \Xi_{a,s} = \sum_{(b,N)=1} \left\langle \frac{-1-ab}{N} \right\rangle \sigma_b = \sum_{(b,N)=1} \left(1 - \frac{1}{N} - \left\langle \frac{ab}{N} \right\rangle \right) \sigma_b.$$

Here we note that $ab \not\equiv 0 \mod N$ since (b, N) = 1, $a \not\equiv 0 \mod N$. Then the above claim follows. By substituting this into Definition 2.2(8), we can write

$$P\left(\frac{a}{N}\right) \equiv \frac{\Gamma_{\infty}(\frac{a}{N})(2\pi i)_p^{\frac{1}{2}-\frac{a}{N}} \prod_{1 \leqslant s < N, \ a+s \neq N} \left((2\pi i)_p^{e_s} \int_{\gamma,p} \eta_{a,s}\right)^{\frac{1}{N}}}{(2\pi i)^{\frac{1}{2}-\frac{a}{N}} \prod_{1 \leqslant s < N, \ a+s \neq N} \left((2\pi i)_p^{e_s} \int_{\gamma} \eta_{a,s}\right)^{\frac{1}{N}}} \mod \mu_{\infty},$$

$$e_s := \begin{cases} -1 & (a+s < N) \\ 0 & (a+s > N) \end{cases}$$

since the part $\sum_{(b,N)=1} \sigma_b$ becomes trivial by Proposition 2.1(2). We can strengthen the congruence relation \equiv of the formula (1.3) into an equality =, by selecting a specific closed path γ_0 (e.g., $\gamma_0 = N\gamma_N$ with γ_N in [12, Proposition 4.9]). Then we have

$$P\left(\frac{a}{N}\right) \equiv c \cdot (2\pi i)_p^{\frac{-1}{2} + \frac{1}{N}} \prod_{1 \le s < N, \ a+s \ne N} \left(\int_{\gamma_0, p} \eta_{a,s}\right)^{\frac{1}{N}} \mod \mu_{\infty},$$

where we put

$$c \coloneqq \frac{\Gamma(\frac{a}{N})}{(2\pi)^{\frac{1}{N}}} \left(\prod_{1 \leqslant s < N, \ a+s \neq N} \frac{\Gamma(\frac{a+s}{N})}{\Gamma(\frac{a}{N})\Gamma(\frac{s}{N})} \right)^{\frac{1}{N}}.$$

Since (1.2) implies that

$$\prod_{1 \leqslant s \leqslant N} \frac{\Gamma(\frac{a+s}{N})}{\Gamma(\frac{a}{N})\Gamma(\frac{s}{N})} = \frac{N^{-a}a!}{\Gamma(\frac{a}{N})^N},$$

we obtain

$$c = \frac{\Gamma(\frac{a}{N})}{(2\pi)^{\frac{1}{N}}} \left(\frac{\Gamma(\frac{a}{N})\Gamma(\frac{N-a}{N})}{\Gamma(1)} \frac{\Gamma(\frac{a}{N})\Gamma(\frac{N}{N})}{\Gamma(\frac{a+N}{N})} \frac{N^{-a}a!}{\Gamma(\frac{a}{N})^N} \right)^{\frac{1}{N}} = \left(\frac{N^{1-a}(a-1)!}{2\sin(\frac{a}{N}\pi)} \right)^{\frac{1}{N}}.$$

For the last equality we used (1.1) and the difference equation $\Gamma(z+1) = z\Gamma(z)$. Take $\tau \in W_p$ with deg $\tau = 1$. Then we have

$$\begin{split} G_1\left(\frac{a'}{N}\right) &\equiv p^{\frac{1}{2}-\frac{a}{N}} \frac{P\left(\frac{a'}{N}\right)}{\Phi_{\tau}(P\left(\frac{a}{N}\right))} \\ &\equiv \left(\frac{N^{a-a'}(a'-1)!}{p^{a-N-1}(a-1)!} \prod_{1\leqslant s < N, \ a+s \neq N} \alpha_{a',s'}^{-1}\right)^{\frac{1}{N}} \mod \mu_{\infty}, \end{split}$$

by noting that $\Phi_{\tau}((2\pi i)_p) = p(2\pi i)_p$ and $\Phi_{\tau}(\sin(\frac{a}{N}\pi)) = \tau(\sin(\frac{a}{N}\pi)) = \pm \sin(\frac{a'}{N}\pi)$. Here a', s' denote integers satisfying $1 \leqslant a', s' < N$, $pa \equiv a' \mod N$, $ps \equiv s' \mod N$ as above. By Proposition 4.2, $\alpha_{a',s'}$ are continuous for a'. When a is in a small open ball, as we saw in the proof of Proposition 4.2, we may write a' = pa - M for a fixed M (M is lN in the proof of Proposition 4.2). Then the remaining part becomes

$$\frac{N^{a-a'}(a'-1)!}{p^{a-N-1}(a-1)!} = \pm \Gamma_p(a'+M+1) \frac{p^N N^{\frac{(1-p)a'+M}{p}}(a'+M)}{a'(a'+1)(a'+2)\cdots(a'+M)},$$

which is also continuous as desired.

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