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Yining Hu & Alain Lasjaunias

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PERIOD-DOUBLING CONTINUED FRACTIONS ARE ALGEBRAIC IN CHARACTERISTIC 2

by Yining HU & Alain LASJAUNIAS (*)

ABSTRACT. — For an arbitrary pair of distinct and non constant polynomials, a and b in $\mathbb{F}_2[t]$, we build a continued fraction in $\mathbb{F}_2((1/t))$ whose partial quotients are only equal to a or b. In a previous work of the first author and Han, the authors considered two cases where the sequence of partial quotients represents in each case a famous and basic 2-automatic sequence, both defined in a similar way by morphisms. They could prove the algebraicity of the corresponding continued fractions for several pairs (a, b) in the first case (the Prouhet–Thue–Morse sequence) and gave the proof for a particular pair for the second case (the period-doubling sequence). Recently Bugeaud and Han proved the algebraicity for an arbitrary pair in the first case. Here we give a short proof for an arbitrary pair in the second case.

RÉSUMÉ. — Pour une paire de polynômes non constants et distincts a et b dans $\mathbb{F}_2[t]$, on construit une fraction continue dans $\mathbb{F}_2((1/t))$ dont les quotients partiels sont égales à a ou b. Dans un travail précédent de la première auteure et Han, les auteurs ont considéré deux cas où les quotients partiels forment des suites 2-automatiques bien connues, définies de façon similaire comme point fixe des morphismes. Ils ont pu démontré l'algébricité des fractions continues associées pour plusieures paires (a, b) dans le premier cas (la suite de Prouhet–Thue–Morse) et ont donné la preuve pour une paire dans le deuxième cas (la suite de doublement de période). Récemment, Bugeaud et Han ont démontré l'algébricité pour une paire arbitraire dans le premier cas. Ici, on donne une preuve courte pour une paire arbitraire dans le deuxième cas.

1. Introduction

Let $\mathbb{F}_2((1/t))$ be the field of power series in $1/t$, where t is a formal indeterminate, over the finite field $\mathbb{F}_2$. A non-zero element is $\alpha = t^n + \sum_{i<n} a_i \cdot t^i$ where n belongs to $\mathbb{Z}$ and $a_i = 0$ or 1. An absolute value on

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$\mathbb{F}_2((1/t))$ is defined by $|0| = 0$ and $|\alpha| = |t|^n$ where $|t| > 1$ is a fixed given real number .

So $\mathbb{F}_2((1/t))$ is the completion of $\mathbb{F}_2(t)$ for this absolute value. Every irrational (resp. rational) element in $\mathbb{F}_2((1/t))$ can be expanded in an infinite (resp. finite) continued fraction: $\alpha = [a_0; a_1, \ldots, a_n, \ldots]$ where the a_i are in $\mathbb{F}_2[t]$ and $\deg(a_i) > 0$ for $i > 0$. For a basic introduction on fields of power series and continued fractions, the reader may consult [6].

The origin of the question discussed here is due to G.-N. Han and the first author. In [4], the authors considered two basic sequences, formed by an infinite word with two letters a and b, belonging to the family of 2-automatic sequences (see [1, p. 173 and p. 176]) . Both sequences are obtained in a similar way, as fixed point of a morphism. For the first one, the $(a;b)$-Prouhet–Thue–Morse sequence, denoted by $\mathbf{t}$, the morphism τ is defined by $\tau(a) = ab$ and $\tau(b) = ba$, and we have

$$\mathbf{t} = \tau^{\infty}(a) = abbabaabbaab\ldots$$

Note that this famous sequence was considered very long ago and has been the starting point of various studies. For the second one, the $(a;b)$-period-doubling sequence, denoted by $\mathbf{p}$, the morphism σ is defined by $\sigma(a) = ab$ and $\sigma(b) = aa$, and we have

$$\mathbf{p} = \sigma^{\infty}(a) = abaaabababaaabaa\ldots$$

In both cases, we consider a pair (a, b) of distinct and non-constant polynomials in $\mathbb{F}_2[t]$, and we can associate with each sequence an infinite continued fraction in $\mathbb{F}_2((1/t))$, $\mathrm{CF}(\mathbf{t})$ and $\mathrm{CF}(\mathbf{p})$, where the sequence of partial quotients is derived from the sequences $\mathbf{t}$ and $\mathbf{p}$:

$$\alpha_{\mathbf{t}} = \mathrm{CF}(\mathbf{t}) = [a; b, b, a, b, a, a, \ldots]$$

and

$$\alpha_{\mathbf{p}} = \mathrm{CF}(\mathbf{p}) = [a; b, a, a, a, b, a, \ldots].$$

It was proved in [4] that, for pairs (a, b) with $\deg a + \deg b \leqslant 7$, $\alpha_{\mathbf{t}}$ was a root of a polynomial of degree 4, with five coefficients in $\mathbb{F}_2[t]$ depending on (a, b). In a recent work, extending the case of $\mathrm{CF}(\mathbf{t})$, Bugeaud and Han [3] obtained the same result for all pairs (a, b) and they gave the explicit formulas for the 5 coefficients as polynomials in a and b.

Also in [4], it was proved that for $(a, b) = (t^3, t^2 + t + 1)$ we have

$$(t^5 + t^3 + t^2) \cdot \alpha_{\mathbf{p}}^4 + (t^8 + t^6 + t^5 + t^3) \cdot \alpha_{\mathbf{p}}^3 + (t^5 + t^4 + t^3) \cdot \alpha_{\mathbf{p}}^2 + 1 = 0.$$

In this note, we give a short proof of the general case for the sequence $\mathbf{p}$ with the following theorem, thus confirming [4, Conjecture 1.5].

THEOREM 1.1. — *Let a, b be two distinct non constant elements in $\mathbb{F}_2[t]$. Let $\alpha_{\mathbf{p}} = [\sigma^\infty(a)] \in \mathbb{F}_2((1/t))$. Define $P(x) \in \mathbb{F}_2(t)[x]$ to be*

$$P(x) = Ax^4 + Bx^3 + Cx^2 + 1 \tag{1.1}$$

with

$$A = ab + b^2 + 1, \quad B = ab(a+b), \quad C = ab.$$

Then $P(\alpha_{\mathbf{p}}) = 0$.

Remark 1.2. — An elementary proof shows that $P(x)$ has no solution in $\mathbb{F}_2(t)$. Consequently, since $\alpha_{\mathbf{p}}$ is not quadratic, $P(x)$ is irreducible and $\alpha_{\mathbf{p}}$ has degree 4 over $\mathbb{F}_2(t)$.

Remark 1.3. — The theorem remains true when we replace $\mathbb{F}_2$ by any other field K of characteristic 2.

Remark 1.4. — In fact we have proven that the continued fraction $\alpha_{\mathbf{p}}$ as a series in the ring $\mathbb{F}_2((1/a, 1/b))$ is algebraic over $\mathbb{F}_2(a, b)$.

In Section 2 we recall notation and formulas for continued fractions. In Section 3 we give the proof of Theorem 1.1 and in a last section we make some comments about the link with Riccati differential equations.

2. Notation and basic formulas for continued fractions

We use the same notation as in [6], which we recall in this section.

Let $W = w_1, w_2, \ldots, w_n$ be a sequence of variables over a ring $\mathbb{A}$. We set $|W| = n$ for the length of the word W. We define the following operators for the word W.

$$\begin{aligned} W' &= w_2, w_3, \ldots, w_n && \text{or } W' = \emptyset \text{ if } |W| = 1. \\ W'' &= w_1, w_2, \ldots, w_{n-1} && \text{or } W'' = \emptyset \text{ if } |W| = 1. \\ W^* &= w_n, w_{n-1}, \ldots, w_1. \end{aligned}$$

We consider the finite continued fraction associated with W to be

$$[W] = [w_1, \ldots, w_n] = w_1 + \cfrac{1}{w_2 + \cfrac{1}{w_3 + \cfrac{1}{\ddots + \cfrac{1}{w_n}}}} \tag{2.1}$$

The continued fraction $[W]$ is a quotient of multivariate polynomials in the variables $w_1, w_2, \dots, w_n$. These polynomials are called continuants built on W. They are defined inductively as follows:

Set $\langle \emptyset \rangle = 1$. If the sequence W has only one element, then we have $\langle W \rangle = W$. Hence, with the above notation, the continuants can be computed, recursively on the length $|W|$, by the following formula

$$\langle W \rangle = w_1 \langle W' \rangle + \langle (W')' \rangle \text{ for } |W| \geqslant 2. \tag{2.2}$$

Thus, with this notation, for any finite word W, the finite continued fraction $[W]$ satisfies

$$[W] = \frac{\langle W \rangle}{\langle W' \rangle}. \tag{2.3}$$

It is easy to prove by induction that $\langle W^* \rangle = \langle W \rangle$. For any finite sequences A and B of variables over $\mathbb{A}$, defining A, B as the concatenation of the sequences, by induction on $|A|$, we also have the following generalization of (2.2)

$$\langle A, B \rangle = \langle A \rangle \langle B \rangle + \langle A'' \rangle \langle B' \rangle. \tag{2.4}$$

Using induction of $|W|$, we have the following classical identity

$$\langle W \rangle \langle (W')'' \rangle - \langle W' \rangle \langle W'' \rangle = (-1)^{|W|} \text{ for } |W| \geqslant 2. \tag{2.5}$$

3. Proof of Theorem 1.1

For $n \geqslant 0$, set

$$W_n = (\sigma^n(a))''.$$

We have $W_0 = \emptyset$, $W_1 = a$, $W_2 = aba$, etc., and $|W_n| = 2^n - 1$.

We will prove that $P([W_n])$ converges to 0 . For this, we need to following two lemmas.

Lemma 3.1. — *For all $n \geqslant 0$,*

$$W_{n+1} = W_n, \varepsilon_n, W_n \tag{3.1}$$

where $\varepsilon_n = a$ if n is even and $\varepsilon_n = b$ if n is odd. In consequence, for $n \geqslant 0$, we have

$$W_n = W_n^*.$$

Proof. — We prove by induction. Suppose identity (3.1) holds for n, then

$$\begin{aligned} W_{n+2} &= \sigma(\sigma^{n+1}(a))'' \\ &= \sigma(W_{n+1})a \\ &= \sigma(W_n), \sigma(\varepsilon_n), \sigma(W_n)a \\ &= W_{n+1}, \varepsilon_{n+1}, W_{n+1}. \end{aligned}$$

In the last equality, we use the fact that $\sigma(\varepsilon_n) = a\varepsilon_{n+1}$. □

For $n \geqslant 1$, define $u_n = \langle W_n \rangle$ and $v_n = \langle W_n' \rangle$, so that

$$[W_n] = \frac{u_n}{v_n}.$$

LEMMA 3.2. — *For all $n \geqslant 1$, we have*

$$\begin{aligned} u_{n+1} &= \varepsilon_n u_n^2, \\ v_{n+1} &= \varepsilon_n u_n v_n + 1. \end{aligned}$$

Proof. — We use Lemma 3.1, identity (2.4), and the fact that we are in characteristic 2.

$$\begin{aligned} u_{n+1} &= \langle W_{n+1} \rangle \\ &= \langle W_n, \varepsilon_n, W_n \rangle \\ &= \langle W_n \rangle \langle \varepsilon_n, W_n \rangle + \langle W_n'' \rangle \langle W_n \rangle \\ &= \langle W_n \rangle (\varepsilon_n \langle W_n \rangle + \langle W_n' \rangle) + \langle W_n'' \rangle \langle W_n \rangle \\ &= \langle W_n \rangle (\varepsilon_n \langle W_n \rangle + \langle W_n' \rangle) + \langle W_n' \rangle \langle W_n \rangle \\ &= \langle W_n \rangle \varepsilon_n \langle W_n \rangle \\ &= \varepsilon_n u_n^2. \end{aligned}$$

$$\begin{aligned} v_{n+1} &= \langle W_{n+1}' \rangle \\ &= \langle W_n', \varepsilon_n, W_n \rangle \\ &= \langle W_n' \rangle \langle \varepsilon_n, W_n \rangle + \langle (W_n')'' \rangle \langle W_n \rangle \\ &= \langle W_n' \rangle (\varepsilon_n \langle W_n \rangle + \langle W_n' \rangle) + \langle (W_n')'' \rangle \langle W_n \rangle \\ &= \varepsilon_n u_n v_n + \langle W_n' \rangle \langle W_n' \rangle + \langle (W_n')'' \rangle \langle W_n \rangle \\ &= \varepsilon_n u_n v_n + 1. \end{aligned}$$

In the last step, we also use (2.5), and the symmetry of W_n. □

Proof of Theorem 1.1. — For $n \geqslant 1$, set

$$X_n = Au_n^4 + Bu_n^3 v_n + Cu_n^2 v_n^2 + v_n^4,$$

so that $P(u_n/v_n) = X_n/v_n^4$.

Using Lemma 3.2 and noticing that $\varepsilon_n^2 + ab = \varepsilon_n(a+b)$, we obtain

$$X_{n+1} + 1 = \varepsilon_n^4 u_n^4 (X_n + 1) + \varepsilon_n^3 u_n^4 (a+b)(1 + abu_n^2). \tag{3.2}$$

We observe that

$$X_1 + 1 = a^3(a+b) = \varepsilon_0 u_1^2 (a+b),$$

this allows us, using (3.2) and the fact that $\varepsilon_n \varepsilon_{n-1} = ab$, to get by induction that for all $n \geqslant 1$

$$X_n + 1 = \varepsilon_{n-1} u_n^2 (a+b). \tag{3.3}$$

Therefore

$$P\left(\frac{u_n}{v_n}\right) = \frac{\varepsilon_{n-1} u_n^2 (a+b) + 1}{v_n^4}$$

This is a quotient of polynomials in $\mathbb{F}_2[t]$, and it can be easily seen from Lemma 3.2 that the degree in t of the denominator minus that of the numerator tends to plus infinity as n goes to infinity. And therefore the sequence converges to 0 in $\mathbb{F}_2((1/t))$. □

4. Link with Riccati differential equations

In both cases, the Prouhet–Thue–Morse and the period-doubling continued fraction, a particular choice of the pair (a, b) in $\mathbb{F}_2[t]^2$, brings us back to an investigation undertaken 45 years ago. Indeed, in these two cases, taking $(a, b) = (t, t+1)$, all the partial quotients of these continued fractions are t or $t+1$. In 1977, Baum and Sweet [2] considered all the infinite continued fractions in $\mathbb{F}_2((1/t))$ whose partial quotients have all degree one.

They considered the subset D of irrational continued fractions α in $\mathbb{F}_2((1/t))$ such that

$$\alpha = [0; a_1, a_2, \ldots, a_n, \ldots]$$

where $\deg(a_i) = 1$ (i.e., $a_i = t$ or $t+1$) for $i \geqslant 1$. Let P the subset of $\mathbb{F}_2((1/t))$ containing all α such that $|\alpha| < 1$. Hence P contains D. We have:

THEOREM 4.1 (Baum–Sweet, 1977). — *An element $\alpha \in P$ is in D if and only if α satisfies*

$$\alpha^2 + t\alpha + 1 = (1+t)\beta^2$$

for some $\beta \in P$.

In order to obtain another formulation of Theorem 4.1, we will now use formal differentiation in $\mathbb{F}_2((1/t))$. Continued fractions and formal differentiation in power series fields are related, these are the main tools in Diophantine approximation (see [5, p. 221-225]). We recall that in characteristic 2, which is the case considered here, an element has derivative zero if and only if this element is a square. Moreover the derivative of an element is a square (hence the second derivative is zero). We have the following theorem.

THEOREM 4.2. — *An element $\alpha \in P$ is in D if and only if α satisfies*

$$(\alpha \cdot t(t+1))' = \alpha^2 + 1.$$

Proof. — If

$$\alpha^2 + t\alpha + 1 = (1+t)\beta^2$$

then by derivation we get

$$(t\alpha)' = \beta^2.$$

Consequently, we have

$$\alpha^2 + 1 = t\alpha + (1+t)(t\alpha)' = (\alpha \cdot t(t+1))'.$$

Conversely, if

$$(\alpha \cdot t(t+1))' = \alpha^2 + 1.$$

then

$$\alpha^2 + t\alpha + 1 = (\alpha \cdot t(t+1))' + t\alpha = (t+1)(\alpha \cdot t)' = (t+1)\beta^2,$$

for some $\beta \in P$ since $(t\alpha)'$ is a square in P. $\square$

Thus the Riccati differential equation

$$(x \cdot t(t+1))' = x^2 + 1 \qquad \text{(R0)}$$

is characteristic of infinite continued fraction expansions in $\mathbb{F}_2((1/t))$, having all partial quotients of degree one. Indeed, there exists a direct proof, without Theorem 4.1, showing that a solution in $\mathbb{F}_2((1/t))$ of (R0) is irrational and has all partial quotients of degree one (see [5, p. 225]).

Now we turn to elements in $\mathbb{F}_2((1/t))$ which are algebraic over $\mathbb{F}_2(t)$ of degree $d \geqslant 1$. If α is such an element, we have (see, e.g., [5, p. 221])

$$\alpha' = a_0 + a_1\alpha + \cdots + a_{d-1}\alpha^{d-1} \tag{4.1}$$

where the coefficients a_i belong to $\mathbb{F}_2(t)$.

Concerning the algebraic continued fraction $\alpha_{\mathbf{t}} = \mathrm{CF}(\mathbf{t}) \in \mathbb{F}_2((1/t))$ associated with the Prouhet–Thue–Morse sequence, we have observed that it satisfies the Riccati differential equation (see [3, Proposition 2.4]):

$$(ab(a+b)x)' = (ab)'(1+x^2). \tag{R}$$

In fact, the continued fraction $\mathrm{CF}(\mathbf{p})$ (also algebraic of degree 4), associated with the period-doubling sequence satisfies the same Riccati differential equation.

Proposition 4.3. — *The continued fraction $\alpha_{\mathbf{p}} = \mathrm{CF}(\mathbf{p})$ satisfies the Riccati differential equation* (R)

Proof. — We know from Theorem 1.1 that

$$A\alpha_{\mathbf{p}}^4 + B\alpha_{\mathbf{p}}^3 + C\alpha_{\mathbf{p}}^2 + 1 = 0 \tag{4.2}$$

with

$$A = ab + b^2 + 1, \quad B = ab(a+b), \quad C = ab.$$

Hence, by straightforword derivation of (4.2), we get

$$B\alpha_{\mathbf{p}}' + B'\alpha_{\mathbf{p}} = A'\alpha_{\mathbf{p}}^2 + C'.$$

Therefore, observing that $A' = C' = (ab)'$, we obtain

$$(ab(a+b)\alpha_{\mathbf{p}})' = (B\alpha_{\mathbf{p}})' = A'\alpha_{\mathbf{p}}^2 + C' = (ab)'(1+\alpha_{\mathbf{p}}^2). \qquad \square$$

Note that for the pair (a, b)=$(t, t+1)$, equation (R) reduces to (R0), the one stated in Theorem 4.2. This was the way, taking in consideration differential aspects for $\mathrm{CF}(\mathbf{t})$ and $\mathrm{CF}(\mathbf{p})$, knowing the equation (R0) in the basic case, and inspired by the particular case published in [4], that allowed us to guess the coefficients for the algebraic equation satisfied by $\mathrm{CF}(\mathbf{p})$.

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Yining HU
School of Mathematics and Statistics,
Huazhong University of Science and Technology,
Wuhan (P. R. China)
Current address:
Institute for Advanced Study in Mathematics,
Harbin Institute of Technology,
Harbin (P. R. China)
huyining@protonmail.com

Alain LASJAUNIAS
Mathematical Institute Bordeaux University,
Bordeaux (France)
lasjauniasalain@gmail.com