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DEFORMATIONS OF SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

by Mercedes HAIECH

Abstract. — As in algebraic geometry where the formal neighborhood of a point of a scheme contains informations about the singularities of the object, we extend this study to schemes where a point represents a solution of an algebraic differential equation. The obtained geometric object living naturally in a space of infinite dimension, a first step is to show that the formal neighborhood of a point not canceling the separant is noetherian, using considerations on the embedding dimension. We show that, in the neighborhood of points making the separant invertible, the embedding dimension is exactly the order of the considered differential equation. In a second step, we relate, for a certain type of differential equations of order two, the existence of essential singular components to the decrease of the embedding dimension, in the neighborhood of certain points.

Résumé. — À l'instar de la géométrie algébrique où le voisinage formel d'un point d'un schéma contient des informations sur les singularités de l'objet, nous étendons cette étude à des schémas dont un point représente une solution d'une équation différentielle algébrique. L'objet géométrique obtenu vivant naturellement dans un espace de dimension infinie, une première étape consiste à montrer que le voisinage formel d'un point n'annulant pas le séparant est noethérien, à l'aide de considérations sur la dimension de plongement. Nous montrons également, qu'au voisinage de points rendant le séparant inversible, la dimension de plongement est exactement l'ordre de l'équation différentielle considérée. Dans un second temps, nous relions, pour un certain type d'équations différentielles d'ordre deux, l'existence de composantes singulières essentielles à la décroissance de la dimension de plongement, au voisinage de certains points.

Introduction

The study of the deformations of solutions of differentials equations can be seen as a natural generalization of the local study of the arc scheme. The arc scheme is an object introduced by J. Nash in the 60's in an article published later. It is constructed as the arcs drawn on a given scheme. More

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precisely, if X is a scheme over a field K -in this article K is assumed to be of characteristic zero-, and if A is a K-algebra, then an A-point of the arc scheme of X corresponds to an $A[\![T]\!]$-point of X.

On the other hand the arc scheme has a natural definition thanks to differential algebra. Let $K\{y_1,\ldots,y_n\} \coloneqq K[y_{i,j} \mid 1 \leqslant i \leqslant n, j \in \mathbf{N}]$ be the ring of differential polynomials endowed with the derivation Δ such that $\Delta(y_{i,j}) = y_{i,j+1}$. Then if $X = \mathrm{Spec}(K[y_1,\ldots,y_n]/I)$, where I is some ideal of $K[y_1,\ldots,y_n]$, then the arc scheme of X, denoted by $\mathscr{L}_\infty(X)$, can be described by the K-scheme $\mathrm{Spec}(K\{y_1,\ldots,y_n\}/[I])$, where $[I]$ stands for the differential ideal generated by I.

In the case of $X = \mathrm{Spec}(K[y_1,\ldots,y_n]/I)$, the ideal that defines the arc scheme is -differentially- generated by elements of order 0. This idea can be generalized by studying geometrical objects like $\mathrm{Spec}(K\{y_1,\ldots,y_n\}/J)$ with J a differential ideal of $K\{y_1,\ldots,y_n\}$ (not necessarily generated by differentials polynomials of order 0).

In addition to be some natural generalization of the arc scheme, it has also a meaning regarding differential algebra. The scheme $\mathrm{Spec}(K\{y_1,\ldots,y_n\}/J)$ can be understood as the solutions -which are formal series- of the elements of J seen as differential equations. In order to underline the link between this object and the arc scheme, and since it is related to differential equations, we are going to refer to schemes like $\mathrm{Spec}(K\{y_1,\ldots,y_n\}/J)$ as *differential arc schemes.*

Motivated by a result about the local structure of the arc scheme at the neighborhood of a rational point by M. Grinberg & D. Kazhdan and V. Drinfeld (see [3] and [5]) D. Bourqui & J. Sebag began the study of the local structure of differential arc scheme.

Theorem 1 (Drinfeld, Grinberg–Kazhdan). — *Let X be a scheme of finite type over a field K, let $\gamma\colon \mathrm{Spec}\ K[\![T]\!] \to X$ be a non-degenerated rational arc,* i.e *an element of $(\mathscr{L}_\infty(X) \setminus \mathscr{L}_\infty(\mathrm{nSm}(X)))(k)$. We denote by $\mathscr{L}_\infty(X)_\gamma$ the formal neighborhood of γ in $\mathscr{L}_\infty(X)$. Assume that $\dim_{\gamma(0)}(X) \geqslant 1$. There exists a K-scheme S of finite type, a point $s \in S(k)$ and an isomorphism of formal K-schemes*

$$\mathscr{L}_\infty(X)_\gamma \cong S_s \times_k \mathrm{Spf}(K[\![(T_i)_{i\in\mathbf{N}}]\!]).$$

The local structure theorem of the arc scheme states that all the information about the formal neighborhood of a rational arc is encoded by a scheme of finite type although the arc scheme is, in general, not of finite type. The result of D. Bourqui & J. Sebag (see [1]) states that for differential arc schemes defined by a differential equation F of order 1 the formal neighborhood of non degenerated points is a formal disk of dimension 1. More precisely:

Theorem 2 (Bourqui–Sebag). — *Let K be a field of characteristic zero. Let $F \in K\{y\}$ be a nonconstant differential polynomial of order 1. Let $X^{\partial} = \mathrm{Spec}(K\{y\}/[F])$. Let $\gamma \in X^{\partial}(k)$ be a nonconstant differential arc. Assume that the separant of F does not vanish at $\gamma(T)$. Then the formal neighborhood of γ, denoted X^{∂}_{γ}, is isomorphic to $\mathrm{Spf}(K[\![T]\!])$.*

A natural question is to wonder if this theorem remains true for differential polynomials of order greater than 1. By looking at several examples, it turns out that it is not the case (see Section 6.2). However, we can show that, as in Theorem 2, the formal neighborhood of a solution of $F = 0$ remains Noetherian.

This article is organized as follow : the first section recalls some facts and theorems about differential algebra. In particular we will state the Low power theorem due to J. F. Ritt with gives a effective way, given F a differential polynomial, to decide if a differential polynomial A gives rise to an essential component of F. In the second and third sections we recall the definitions of the functor of deformations (Definition 2.10) and embedding dimension (Definition 3.1) and some useful properties to deal with it. Along the way, we prove the following theorem

Theorem 1. — *Let $(A, \mathfrak{M}_A)$ be a local ring. The following assertions are equivalent:*

(1) *The embedding dimension of the local ring A, denoted* $\mathrm{emb.dim}(A)$ *is finite.*
(2) *The completion $\widehat{A} := \varprojlim_n A/\mathfrak{M}_A^n$ of A is Noetherian.*
(3) *The embedding dimension of the local ring $\widehat{A}$, denoted* $\mathrm{emb.dim}(\widehat{A})$ *is finite.*

Furthermore, if one of this equivalent conditions is verified, then $\mathrm{emb.dim}(A) = \mathrm{emb.dim}(\widehat{A})$ *and the ring $\widehat{A}$ is complete for the $\widehat{\mathfrak{M}_A}$ pre-adic topology.*

Note that the equivalence (1) $\iff$ (2) can already be found in [4, Lemma 10.12]. The rest is, up to my knowledge, new. Moreover, even if it is well known that for A a Noetherian ring the topology on $\widehat{A}$ is the $\widehat{\mathfrak{M}_A}$ pre-adic topology, this statement is, in general, false when A is not Noetherian (see for example [6] or [12]).

In the 4th part, given F a differential polynomial and $\gamma(T)$ a solution of $F = 0$ such that the separant of F does not vanish at $\gamma(T)$, we show that the problem of deciding if the formal neighborhood of F at γ is Noetherian can be reduced to a problem of linear algebra, and we state that the embedding dimension of the formal neighborhood is smaller than the order of F.

Theorem 2. — *Let K be a field of characteristic zero. Let $F \in K\{y\}$ be a non constant differential polynomial of order n. Let $\gamma(T) \in K[\![T]\!]$ be a solution of the ODE $F = 0$ such that the separant of F does not vanish at $\gamma(T)$. Let $X^\partial = \operatorname{Spec}(K\{y\}/[F])$.*

So the formal neighborhood $\widehat{X^\partial_\gamma}$ is Noetherian and its embedding dimension is less or equal to n.

This result gives a upper bound for the embedding dimension, but we can have a more precise result when the embedding dimension is at most one.

Theorem 3. — *Let K be a field of characteritic zero. Let $F \in K\{y\}$ be a non constant irreducible differential polynomial and $X^\partial := \operatorname{Spec}(K\{y\}/[F])$ the associated differential scheme. Let $\gamma(T) \in K[\![T]\!]$ be a non constant solution of the differential equation $F = 0$. Assume that the embedding dimension of X^∂ at γ is at most 1. Then the formal neighborhood $\widehat{X^\partial}_\gamma$ is isomorphic, as formal K-scheme, to the formal disk $\mathbb{D} = \operatorname{Spf}(K[\![T]\!])$ of dimension 1.*

This statement applies for non constant F of any order, in particular it allows us to recover the statement of Theorem 2.

Last section investigates the case of particular differentials equations of order two. For equations $F \in K\{y\}$ of the form $y_i^a - \alpha y_j^b y_k^c$ where $\{i, j, k\} = \{0, 1, 2\}$, called binomial, the embedding dimension seems to be linked to the existence of essential singular components (see Definition 1.4 for a definition). More precisely we have:

Theorem 4. — *Let K be a field of characteristic zero. Let $F \in K[y_0, y_1, y_2]$ an irreducible polynomial. Assume that the ODE $F(y, y', y'') = 0$ is a* binomial *equation of order two with constant coefficients. Let $d \geqslant 2$ be an integer such that $\gamma(T) = T^d$ is a solution of the $F = 0$ and $X^\partial = \operatorname{Spec}(K\{y\}/[F])$. If the perfect differential ideal $\{F\}$[1] has a essential singular component, then* $\operatorname{emb.dim}(\widehat{\mathcal{O}_{X^\partial,\gamma}}) = 1$.

However the converse is false, there exist binomial equations without essential singular components such that the embedding dimension in the neighborhood of a solution is 1.

[1] See beginning of Section 1 for the definition of perfect differential ideal. This terminology is standard in differential algebra but can be confusing compared to the one used in modern algebra and algebraic geometry. In the context of differential algebra, perfect ideal has the same meaning as radical ideal.

To conclude, we present, at the end of the last section, various examples of equations and computations of embedding dimensions.

1. Recollection on the low power theorem

Let K be a field of characteristic 0. Recall that if $(R, +, \cdot)$ is a ring, a derivation Δ on R is a linear map for $+$ and which satisfies the Leibniz Rule $\Delta(ab) = a\Delta(b) + \Delta(a)b$. A ring endowed with one, or many derivations is called a differential ring. An ideal of a differential ring is called differential if it is stable under the action of the derivations. A differential ideal I of a differential ring R is say to be perfect of for every $a \in R$, the fact that $a^n \in I$ implies $a \in I$.

The notation $K\{y\}$ will stand for the differential ring $(K[y_i \mid i \in \mathbf{N}], \Delta)$, where the derivation Δ sends an element of K to 0 and satisfies $\Delta(y_i) = y_{i+1}$. An element of this ring will be called a differential polynomial. If I is an ideal of $K\{y\}$, then $\{I\}$ will denote the intersection of all the perfect differential ideals of $K\{y\}$ containing I. If F is an element of $K\{y\}$ of order ℓ, the separant S_F of F is $\frac{\partial F}{\partial y_\ell}$.

Remark 1.1. — Let $F \in K\{y\}$. In what follows there will be three ideals generated by F.

- The differential ideal $[F]$ generated, as ideal, by F and all its derivative. Its structure is compatible with the action of the derivation.
- The perfect (or radical) differential ideal $\{F\}$, defined before as the intersection of all the perfect differential ideals of $K\{y\}$ containing F. Its structure is also compatible with the one induced by the derivation.
- The ideal $\langle F\rangle$ generated by F. Even if this ideal has no differential structure, it becomes useful when F is seen as a standard polynomial and not as a differential one.

To emphases the difference, let's consider an example. Let $(y')^2 - 4y = 0$ be a differential equation. This equation can be seen as a differential polynomial and as an element of the differential ring $K\{y\}$, where it is denoted by $F = y_1^2 - 4y_0$. The differential ideal $[F]$ is generated by F and all its formal derivative, for example the first formal derivative is $\Delta(F) = 2y_1y_2 - 4y_1$. However, F can also be seen as a polynomial of $K[y_i \mid i \in \mathbf{N}]$, where the differential structure is forgotten, and then $\langle F\rangle$ is the ideal generated by F in $K[y_i \mid i \in \mathbf{N}]$. Sometimes, it may be useful to see F as a polynomial of the K-algebra $K[y_0, y_1]$. In this context, $\langle F\rangle$ will also denote the ideal generated by F but in $K[y_0, y_1]$.

Definition 1.2. — *Let F be a differential polynomial and κ a K-algebra. A formal series $\gamma(T) \in \kappa[T]$ is a solution of F, if*

$$F(\gamma(T), \gamma'(T), \ldots) = 0.$$

For simplicity, we will often denote $F(\gamma(T)) = 0$ when $\gamma(T)$ is a solution of F.

If I in a differential ideal of $K\{y\}$, we denote by $\mathrm{Sol}_\kappa(I)$ the solution of I with coefficients in κ. More precisely

$$\mathrm{Sol}_\kappa(I) = \{\gamma(T) \in \kappa[\![T]\!] \mid \forall F \in I,\ F(\gamma(T)) = 0\}.$$

If $I = [F]$, we sometimes denote $\mathrm{Sol}_\kappa(F)$ instead of $\mathrm{Sol}_\kappa([F])$.

1.1. Decomposition of perfect differential ideals

Let $F \in K\{y\}$ be a non-constant differential polynomial. We will study the decomposition of the ideal $\{F\}$ as intersection of prime differential ideals.

We know that any perfect differential ideal I decomposes into a finite intersection of prime differential ideals (see [10, Chapter 1, 16] or [7, Chapter VII, Theorem 7.5 & Theorem 7.6]). Since $\{F, S_F\}$ is a perfect ideal of $K\{y\}$ there exists an irredundant family of differential prime ideals $\mathfrak{P}_1, \ldots, \mathfrak{P}_s$ such that

$$\{F, S_F\} = \bigcap_{i=1}^{s} \mathfrak{P}_i.$$

Furthermore, the $\mathfrak{P}_i$ are unique up to reordering.

We define $(\{F\} : S_F)$ as the set of elements A in $K\{y\}$ such that $AS_F \in \{F\}$. A classical result of differential algebra can be stated as follows (see [7, Theorem 7.10]):

Proposition 1.3. — *Let K be a field of characteristic 0. Let $F \in K\{y\}$ be an irreducible differential polynomial. Then $\{F\}$ has the following decomposition as intersection of perfect ideals*

$$\{F\} = (\{F\} : S_F) \cap \{F, S_F\}.$$

Moreover $(\{F\} : S_F)$ is a prime differential ideal. If we denote $\{F, S_F\} = \bigcap_{i=1}^{s} \mathfrak{P}_i$, there exists a subset $J \subset \{1, \ldots, s\}$ such that the irredundant irreducible decomposition of the perfect differential ideal $J \subset \{1, \ldots, s\}$ is given by

$$\{F\} = (\{F\} : S_F) \cap \left(\bigcap_{j \in J} \mathfrak{P}_j \right).$$

DEFINITION 1.4. — *The prime differential ideal* $(\{F\} : S_F)$ *is called* component of the general solution of F [(2)] *; the* $\mathfrak{P}_j$ *intervening in the formula, the* components of the essential singular solutions of F [(3)].

Remark 1.5. — Thanks to Proposition 1.3 we can deal with cases where the polynomial F is not irreducible. The irreducible decomposition of the differential ideal $\{F\}$ will, indeed, be obtained from this proposition applied to the irreducible factors of F.

Let's illustrate Proposition 1.3 by an example from [7, Chapter VII, 31].

Example 1.6. — Let $F = y_1^2 - 4y_0 \in K\{y\}$. Thanks to Proposition 1.3, we can decompose the perfect differential ideal $\{F\}$ under the form

$$\{F\} = (\{F\} : S_F) \cap \{F, S_F\}.$$

But $S_F = 2y_1$, hence $\{F, S_F\} = \{y_0\}$. Since the ideal $\{y_0\}$ is prime, we have the decomposition of $\{F, S_F\}$ as an intersection of prime differential ideals. It remains to identify the ideal $(\{F\} : S_F)$, but we already know that it is prime.

The derivative of F factors: $\Delta(F) = 2y_1(y_2 - 2)$. We will prove that $y_1 \notin (\{F\} : S_F)$. Assume that $y_1 \in (\{F\} : S_F)$. Since $y_1 \in \{F, S_F\}$ and that $\{F\}$ is the intersection of these two ideals we should have $y_1 \in \{F\}$. In particular, this implies $\mathrm{Sol}_K(F) \subset \mathrm{Sol}_K(y_1) = K$. But the set

$$\mathrm{Sol}_K(F) = \{x(t) \in K[\![t]\!] \mid F(x(t)) = 0\}$$

contains $x(t) = t^2$ which is not an element of $\mathrm{Sol}_K(y_1)$. Hence $y_1 \notin (\{F\} : S_F)$, but since this ideal is prime and contains $\Delta(F)$, we deduce that $(y_2 - 2) \in (\{F\} : S_F)$. We denote $Q = \{y_1^2 - 4y_0, y_2 - 2\}$. This ideal is contained in $(\{F\} : S_F)$ and is prime because the morphism

$$\begin{aligned} K\{y\}/\{F, y_2 - 2\} &\longrightarrow K[y_0, y_1]/\langle F\rangle \\ y_0 &\longmapsto y_0 \\ y_1 &\longmapsto y_1 \\ y_2 &\longmapsto 2 \\ y_i &\longmapsto 0 \qquad \text{for } i \geqslant 3 \end{aligned}$$

is a ring isomorphism. But $\langle y_1^2 - 4y_0\rangle$ is a prime ideal of $K[y_0, y_1]$, hence the ideal Q is prime. Since $\{F\} \subset Q$, we have

$$\{F\} = (\{F\} : S_F) \cap \{F, S_F\} \cap Q.$$

(2) Sometimes we also say general component.
(3) We sometimes also say essential singular components.

But $Q \subset (\{F\} : S_F)$, then the decomposition of F as prime ideals is the following:

$$\{F\} = \{y_1^2 - 4y_0, y_2 - 2\} \cap \{y_0\}$$

and we deduce that $\{y_1^2 - 4y_0, y_2 - 2\} = (\{F\} : S_F)$.

1.2. Low power theorem

Ritt's low power theorem is one of the great success of the Rittian approach to differential algebra. This theorem is a simple algorithmic statement which fits into the problem of determining the irreducible decomposition of a perfect ideal $\{F\}$ of $K\{y\}$. Introduced in [10, II] in the case of one-derivative differential fields, this theorem also finds a presentation in [8, 13,15] in the case of arbitrary finite sets of derivations. In this section, we will limit ourselves to the case of a differential field K, equipped with a derivation Δ possibly trivial as proposed by Ritt.

This theorem exploits an algorithmic preparation procedure, which can be interpreted as a kind of pseudo-division. Here is how the algorithmic preparation procedure is expressed (see [10, Chapter III, part II, 17]).

Proposition 1.7 (Preparation process). — *Let A and F be two differential polynomials of $K\{y\}$. We note l the order of A and m the order of F. Let's denote by S_A the separant of A. There exists two integers $t \in \mathbf{N}$ and $r \in \setminus\{0\}$ such that $S_A^t F$ is of the form*

$$S_A^t F = \sum_{j=1}^{r} C_j A^{p_i} \Delta(A)^{i_{1,j}} \Delta^2(A)^{i_{2,j}} \cdots \Delta^{m-l}(A)^{i_{m-l,j}}$$

with

1. *the p_j and $i_{k,j}$ are positive integers;*
2. *the $(m-l)$-uplets $(i_{1,j}, \ldots, i_{m-l,j})$ are all distinct*
3. *the order of the C_j is smaller than l and they are not divisible by A.*

Examples 1.8.

1. Let $F = y_1^2 - 4y_0$ and $A = y_0$, then F is already in "prepared" form with $t = 0$, $r = 2$, $C_1 = 1$, $p_1 = 0$, $i_{1,1} = 2$, $C_2 = -4$, $p_2 = 1$, and all other integers are 0.
2. Similarly, if $F = y_1^2 - 4y_0$ and $A = y_1$, then F is already in "prepared" form with $t = 0$, $r = 2$, $C_1 = 1$, $p_1 = 2$ and $C_2 = -4y_0$, and all other integers are zero.

The low power theorem can then be stated as follows (see [10, Chapter III, part II, 20]).

Theorem 1.9 (Low power theorem). — *Let K be a field of characteristic zero. Let $A, F \in K\{y\}$ two irreducible differential polynomials.*

Let

$$S_A^t F = \sum_{j=1}^{r} C_j A^{p_i} \Delta(A)^{i_{1,j}} \Delta^2(A)^{i_{2,j}} \cdots \Delta^{m-l}(A)^{i_{m-l,j}} \tag{1.1}$$

be a preparation of F with respect to A. So the prime ideal differential $(\{A\} : S_A)$ is an irreducible component of $\{F\}$ if and only if

(1) *the right part of* (1.1) *contain a term $C_k A^{p_k}$ free of proper derivations of A*
(2) *for every integer $j \neq k$, we have $p_k < p_j + i_{1,j} + \cdots + i_{m-l,j}$.*[4]

Examples 1.10. — Let's go back to the previous Examples 1.8.

(1) Let $F = y_1^2 - 4y_0$ and $A = y_0$, then there is a term of the form $-4y_0$, where the derivative of A does not appear. And $p_2 = 1 < p_1 + i_{1,1} = 2$. So y_0 is an irreducible component of $\{F\}$.
(2) If $F = y_1^2 - 4y_0$ and $A = y_1$, then the first hypothesis of the lower power theorem is verified, but not the second, so y_1 isn't an irreducible component of $\{F\}$.

Let $F \in K\{y\}$. If we want to test if $\{y_0\}$ is an essential singular component of $\{F\}$, the low power theorem takes the following simpler form: the differential ideal $\{y_0\}$ is a component of $\{F\}$ if and only if the expression of F contains a term of the form αy_0, with $\alpha \in K$, of -total- degree strictly inferior to the degree of any other (non-zero) monomial appearing in the expression of F.

2. Deformations of a point

This section is a recollection about the space of deformations of a point. Notations and definitions are introduced in order to recall the proof that the functor that describes the space of deformation (Definition 2.10) is representable.

[4] If $m = l$ and (1.1) contains a single term of the form $C_k A^{p_k}$, the condition will also be regarded as fulfilled.

2.1. Notations

In the following K will refer to an arbitrary field. We will define three categories:

(1) The category $\mathbf{Algloc}_K$ whose objects are local K-algebras whose residue field is K-isomorphic to K and whose morphisms are morphisms of local rings.
(2) The category $\mathbf{AlgLC}_K$ whose objects are topological and local K-algebras which are the completion of an object of $\mathbf{Algloc}_K$ and whose morphisms are the continuous morphisms of K-algebras.
(3) The category $\mathbf{QArt}_K$, which is a full subcategory of $\mathbf{AlgLC}_K$, whose the objects are the objects of $\mathbf{AlgLC}_K$ whose maximal ideal is nilpotent.

Definition 2.1. — *An object of the category* $\mathbf{QArt}_K$ *will be called a* quasi-artinian *ring.*

Remark 2.2. — Let $(A, \mathfrak{m}_A)$ be an object of $\mathbf{Algloc}_K$ which is Noetherian. Then it is known (see [9, 23.L, Corollary 4]) that the completion of A with respect to its maximal ideal is complete for the $\mathfrak{m}_A$ (pre)adic topology. However this turn out to be false if A is not Noetherian (see [6]).

Remark 2.3. — Let $(A, \mathfrak{M}_A)$ an object of $\mathbf{Algloc}_K$ and $B = \varprojlim_n A/\mathfrak{M}_A^n$ its completion. In addition to the previous remark let us insist on the fact that the maximal ideal of B is described by $\mathfrak{M}_B = \widehat{\mathfrak{M}_A}$. The natural topology on B (resulting from its construction as projective limit) is described by the family $(\widehat{\mathfrak{M}_A^n})_{n\in\mathbf{N}}$. In general, the family $(\widehat{\mathfrak{M}_A^n})_{n\in\mathbf{N}}$ does not coincide with the family of powers of the maximal ideal of B (unless A is Noetherian). If A is not Noetherian, usually $\widehat{\mathfrak{M}_A^n} \neq \widehat{\mathfrak{M}_A}^n$. So we'll take special care to distinguish these two topologies.

Remark 2.4. — The category of quasi-Artinian rings is referred to in some references as *test rings* (see for example [2, 3]).

Proposition 2.5. — *Let* $(A, \mathfrak{M}_A), (B, \mathfrak{M}_B)$ *be two objects of* $\mathbf{Algloc}_K$. *Let* $\widehat{A} = \varprojlim_n A/\mathfrak{M}_A^n$, *and* $\widehat{B} = \varprojlim_n B/\mathfrak{M}_B^n$. *By construction* $\widehat{A}$ *and* $\widehat{B}$ *are objects of* $\mathbf{AlgLC}_K$.

Let $\varphi\colon \widehat{A} \to \widehat{B}$ *be a ring morphism.*

(1) *If the morphism* φ *is continuous, then it's local;*
(2) *If the morphism* φ *is local and if the family of ideals* $(\widehat{\mathfrak{M}_A}^n)_{n\in\mathbf{N}}$ *forms a base for the topology of* $\widehat{A}$ *then* φ *is continuous.*

Proof.

(1). — Let's assume that φ is continuous, and let's show that $\varphi^{-1}(\widehat{\mathfrak{M}_B}) = \widehat{\mathfrak{M}_A}$. Since φ is continuous, then $\varphi^{-1}(\widehat{\mathfrak{M}_B})$ is an open subset of $\widehat{\mathfrak{M}_A}$. In particular, since $0 \in \varphi^{-1}(\widehat{\mathfrak{M}_B})$, there is an integer j such as

$$\widehat{\mathfrak{M}_A}^j \subset \widehat{\mathfrak{M}_A^j} \subset \varphi^{-1}(\widehat{\mathfrak{M}_B}).$$

Now since $\varphi^{-1}(\widehat{\mathfrak{M}_B})$ is a prime ideal of $\widehat{A}$, the previous inclusions imply that $\widehat{\mathfrak{M}_A} \subset \varphi^{-1}(\widehat{\mathfrak{M}_B})$. So φ is local.

(2). — Assume that φ is local. Let U be an open in $\widehat{B}$. Let $x \in \varphi^{-1}(U)$. Since $\varphi(x) \in U$, there exists an integer j such that $\varphi(x) + \widehat{\mathfrak{M}_B^j} \subset U$. Furthermore $\widehat{\mathfrak{M}_B}^j \subset \widehat{\mathfrak{M}_B^j}$. So $\varphi(x + \widehat{\mathfrak{M}_A}^j) \subset \varphi(x) + \widehat{\mathfrak{M}_B}^j \subset U$. So $\varphi^{-1}(U)$ is an open subset, thus φ is continuous. $\square$

Proposition 2.6. — *Let $(A, \mathfrak{M}_A), (B, \mathfrak{M}_B)$ be two objects of $\mathbf{Algloc}_K$. Let $\widehat{A} = \varprojlim_n A/\mathfrak{M}_A^n$, and $\widehat{B} = \varprojlim_n B/\mathfrak{M}_B^n$. By construction $\widehat{A}$ and $\widehat{B}$ are objects of $\mathbf{AlgLC}_K$.*

Let $\varphi\colon A \to B$ be a local ring morphism. Then the morphism induced by $\widehat{\varphi}\colon \widehat{A} \to \widehat{B}$ is continuous.

Proof. — Since the morphism φ is local then, for any integer $n \in \mathbf{N}$, the morphism $\widehat{\varphi}$ satisfies $\widehat{\varphi}(\widehat{\mathfrak{M}_A^n}) \subset \widehat{\mathfrak{M}_B^n}$.

Let U be an open subset of $\widehat{B}$. Let $x \in \widehat{\varphi}^{-1}(U)$. Since $\widehat{\varphi}(x) \in U$, there exists an integer j such that $\widehat{\varphi}(x) + \widehat{\mathfrak{M}_B^j} \subset U$. Thus $\widehat{\varphi}(x + \widehat{\mathfrak{M}_A^j}) \subset \widehat{\varphi}(x) + \widehat{\mathfrak{M}_B^j} \subset U$. So $\widehat{\varphi}^{-1}(U)$ is an open subset, hence $\widehat{\varphi}$ is continuous. $\square$

Proposition 2.7. — *Let $(A, \mathfrak{M}_A), (B, \mathfrak{M}_B)$ be two objets of $\mathbf{Algloc}_K$. Let $\widehat{A} = \varprojlim_n A/\mathfrak{M}_A^n$, and $\widehat{B} = \varprojlim_n B/\mathfrak{M}_B^n$. Let $\varphi\colon A \to \widehat{B}$ be a morphism of local rings. Then there exists an unique continuous morphism $\widehat{\varphi}$ of local rings such that the following diagram is commutative.*

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & \widehat{B} \\ \widehat{A} \arrow[ur, "\widehat{\varphi}"'] & \end{tikzcd}$$

Proof. — Let j be an integer. Since the morphism φ is local and that $\widehat{\mathfrak{M}_B}^j \subset \widehat{\mathfrak{M}_B^j}$, then φ induces a morphism of local rings.

$$\varphi_j\colon A/\mathfrak{M}_A^j \longrightarrow \widehat{B}/\widehat{\mathfrak{M}_B^j} = B/\mathfrak{M}_B^j.$$

Let $\widehat{a} = (a_n)_{n\in\mathbf{N}} \in \widehat{A}$. We denote $\widehat{\varphi}(\widehat{a}) = (\varphi_n(a_n))_{n\in\mathbf{N}}$. Since the morphism φ is local, then $\widehat{\varphi}(\widehat{a}) \in \widehat{B}$ is well defined. And since every φ_j is local, then $\widehat{\varphi}$ is local.

By constructing of the morphism $\widehat{\varphi}$, we also have $\widehat{\varphi}(\widehat{\mathfrak{M}_A^n}) \subset \widehat{\mathfrak{M}_B^n}$, which implies that the morphism $\widehat{\varphi}$ is continuous (see previous proof for details).

Uniqueness is verified because A is dense in $\widehat{A}$ and $\widehat{\varphi}$ is continuous. □

Remark 2.8. — In particular, the Proposition 2.7 proves that, if $(A, \mathfrak{M}_A)$, $(B, \mathfrak{M}_B)$ are two objects of $\mathbf{Algloc}_K$, then

$$\mathrm{Hom}_{\mathbf{Algloc}_K}(A, \widehat{B}) = \mathrm{Hom}_{\mathbf{AlgLC}_K}(\widehat{A}, \widehat{B}).$$

In particular, if B is a quasi-Artinian ring (an object of $\mathbf{QArt}_K$), then we have

$$\mathrm{Hom}_{\mathbf{Algloc}_K}(A, B) = \mathrm{Hom}_{\mathbf{AlgLC}_K}(\widehat{A}, B)$$

because, in this case, $B = \widehat{B}$.

2.2. Functor of points

According to the Yoneda's lemma, we have a fully faithful functor defined on the objects by :

$$\begin{aligned} \alpha\colon \mathbf{AlgLC}_K &\longrightarrow \mathrm{Func}(\mathbf{AlgLC}_K, \mathbf{Set}) \\ \widehat{B} &\longmapsto h_{\widehat{B}}\colon \mathbf{AlgLC}_K \longrightarrow \mathbf{Set} \\ &\qquad\qquad \widehat{A} \longmapsto \mathrm{Hom}_{\mathbf{AlgLC}_K}(\widehat{B}, \widehat{A}). \end{aligned}$$

Let $\widehat{B}$ and $\widehat{C}$ be two objects of $\mathbf{AlgLC}_K$. This fully faithful functor is described by the existence of an isomorphism between $\mathrm{Hom}_{\mathbf{AlgLC}_K}(\widehat{C}, \widehat{B})$ and $\mathrm{Hom}(h_{\widehat{C}}, h_{\widehat{B}})$. In other words, a morphism $\varphi \in \mathrm{Hom}_{\mathbf{AlgLC}_K}(\widehat{C}, \widehat{B})$ is the same as considering a collection of morphisms

$$\left\{\mathrm{Hom}_{\mathbf{AlgLC}_K}(\widehat{C}, \widehat{A}) \longrightarrow \mathrm{Hom}_{\mathbf{AlgLC}_K}(\widehat{B}, \widehat{A})\right\}_{\widehat{A}\in\mathbf{AlgLC}_K}$$

functorial in $\widehat{A}$.

This functor α can be restricted to the sub-category $\mathbf{QArt}_K$.

$$\begin{tikzcd} \mathbf{AlgLC}_K \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & \mathrm{Func}(\mathbf{AlgLC}_K, \mathbf{Set}) \arrow[d] \\ & \mathrm{Func}(\mathbf{QArt}_K, \mathbf{Set}) \end{tikzcd}$$

where $\alpha'(\widehat{B}) = h_{\widehat{B}} : \widehat{A} \in \mathbf{QArt}_K \mapsto \mathrm{Hom}_{\mathbf{AlgLC}_K}(\widehat{B}, \widehat{A})$.

The following proposition was used in [3], but the reader can refer to [2, Section 2.1] for more details.

Proposition 2.9. — *The functor* $\alpha' \colon \mathbf{AlgLC}_K \to \mathrm{Func}(\mathbf{QArt}_K, \mathbf{Set})$ *is fully faithful.*

2.3. Infinitesimal deformations of a rational point

This section is a recollection of an important description of the deformation space of a rational point, which is the core of Proposition 2.12. The same considerations can be found in [2, Section 2.2].

If X is a K-scheme and x be a K-point of X, we denote $\mathcal{O}_{X,x}$ the stalk of $\mathcal{O}_X$ at x.

Definition 2.10. — *Let X be a K-scheme and x be a K-point of X. The space of deformations of X at x is the functor from the category* $\mathbf{QArt}_K$ *to* $\mathbf{Set}$

$$A \longmapsto \{x_A \in X(A) \mid \overline{x_A} = x\}$$

where $\overline{x_A}$ is the reduction of x_A via the map $X(A) \to X(K)$.

Lemma 2.11. — *Let K be a field and R a K-algebra. Given two morphisms of K-algebras $f, g \colon R \to K$ such that* $\mathrm{Ker}(f) = \mathrm{Ker}(g)$, *then $f = g$.*

Proof. — Since f and g are morphisms of K-algebras, note that f and g are necessarily surjectives. Thus, the morphism deduced from the universal property of the quotient $R/\mathrm{Ker}(f) \to K$ is bijective. Since $\mathrm{Ker}(f) = \mathrm{Ker}(g)$, we deduce the following commutative diagram by the universal property of the quotient.

$$\begin{tikzcd} R \arrow[r, two heads, "g"] \arrow[d, "\pi_f"'] & K \\ R/\mathrm{Ker}(f) \arrow[ur, equal, "\exists ! \widetilde{g}"'] & \end{tikzcd}$$

Now the only morphism of K-algebra between K and K is the identity. So $\widetilde{g} = \mathrm{id}$, and $g = \pi_f$. In the same way we show that $f = \pi_f$, hence $f = g$. ☐

Proposition 2.12. — *If X is a K-affine scheme, $x \in X(k)$ a rational point and A a quasi-Artinian ring, then*

$$\mathrm{Hom}_{\mathbf{AlgLC}_K}(\widehat{\mathcal{O}_{X,x}}, A) = \mathrm{Hom}_{\mathbf{Algloc}_K}(\mathcal{O}_{X,x}, A) = \{x_A \in X(A) \mid \overline{x_A} = x\}$$

where $\overline{x_A}$ is the reduction of x_A via the map $X(A) \to X(k)$.

Remark 2.13. — If X is a general scheme a proposition like the previous is also true, up to working with an affine open neighborhood of x.

Proof. — Remark 2.8 already proves the equality

$$\mathrm{Hom}_{\mathbf{AlgLC}_K}(\widehat{\mathcal{O}_{X,x}}, A) = \mathrm{Hom}_{\mathbf{Algloc}_K}(\mathcal{O}_{X,x}, A).$$

We aim to define a bijective map from $\{x_A \in X(A) \mid \overline{x_A} = x\}$ to $\mathrm{Hom}_{\mathbf{Algloc}_K}(\mathcal{O}_{X,x}, A)$. Let's consider $x_A \in X(A)$, which corresponds to a morphism of K-algebras $x_A\colon \mathcal{O}_X \to A$ such that $\overline{x_A} = x$. Let's denote $\mathrm{Ker}(x) = \mathfrak{p}_x$.

By the universal property of the localization, x_A factorizes by $\mathcal{O}_{X,x}$ if and only if for $y \notin \mathfrak{p}_x$ then $x_A(y)$ is invertible in A. Since $y \notin \mathfrak{p}_x$, then $\overline{x_A}(y) = x(y) \neq 0$. So x_A factorizes uniquely :

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "x_A"] \arrow[d] & A \\
\mathcal{O}_{X,x} \arrow[ur, "\exists ! \varphi_{x_A}"'] &
\end{tikzcd}$$

This defines an injective morphism

$$\begin{aligned} \alpha\colon \{x_A \in X(A) \mid \overline{x_A} = x\} &\hookrightarrow \mathrm{Hom}_{\mathbf{Algloc}_K}(\mathcal{O}_{X,x}, A) \\ x_A &\longmapsto \varphi_{x_A}. \end{aligned}$$

It remains to show that α is surjective. Let $\varphi \in \mathrm{Hom}_{\mathbf{Algloc}_K}(\mathcal{O}_{X,x}, A)$ be a morphism. We denote $l_x\colon \mathcal{O}_X \to \mathcal{O}_{X,x}$ the canonical morphism of localization. We denote again $\mathrm{Ker}(x) = \mathfrak{p}_x$.

We set $x_A = \varphi \circ l_x$. By definition of φ, if $y \notin \mathfrak{p}_x$, then $x_A(y)$ is invertible in A, in other words $\overline{x_A}(y) \neq 0$ (in K). We deduce that $y \notin \mathrm{Ker}(x)$ implies that $y \notin \mathrm{Ker}(\overline{x_A})$. In other words $\mathrm{Ker}(\overline{x_A}) \subset \mathrm{Ker}(x)$.

But, we also know that $\mathrm{Ker}(x)$ and $\mathrm{Ker}(\overline{x_A})$ are maximal ideals of $\mathcal{O}_X$ since $\mathcal{O}_X/\mathrm{Ker}(x)$ and $\mathcal{O}_X/\mathrm{Ker}(\overline{x_A})$ are K-isomorphic to K. Hence

$$\mathrm{Ker}(\overline{x_A}) = \mathrm{Ker}(x).$$

Thanks to Lemma 2.11, this implies that $\overline{x_A} = x$. This proves that α is surjective and thus the equality

$$\mathrm{Hom}_{\mathbf{Algloc}_K}(\mathcal{O}_{X,x}, A) = \{x_A \in X(A) \mid \overline{x_A} = x\}. \qquad \square$$

3. Tangent space and embedding dimension

In this section, we will present and gather essentially known results on the notion of tangent space and embedding dimension for general schemes.

The material in this chapter constitutes the key ingredients of the results we will obtain in the next chapters. Given the role that the statements will play in the following, we have chosen to present the most important proofs for the ease of reading. Even if most of the results presented in this chapter are known however, to our knowledge, Corollary 3.9 and Proposition 3.10 are new.

3.1. Definition

DEFINITION 3.1. — *Let* $(A, \mathfrak{M}_A)$ *be a local ring. The* embedding dimension *of* A *is the dimension of the* $A/\mathfrak{M}_A$*-vector space* $\mathfrak{M}_A/\mathfrak{M}_A^2$ *and is denoted* $\mathrm{emb.dim}(A)$.

The tangent space of A *is the dual of the* K*-vector space* $\mathfrak{M}_A/\mathfrak{M}_A^2$ *that is denoted* $(\mathfrak{M}_A/\mathfrak{M}_A^2)^\vee$.

Remark 3.2. — Let $(A, \mathfrak{M}_A)$ be a local ring and $n \in \mathbf{N}$ an integer. The A-module $\mathfrak{M}_A^n/\mathfrak{M}_A^{n+1}$ is endowed with a structure of $A/\mathfrak{M}_A$-vector space. If $a \in A$ and $b \in \mathfrak{M}_A^n$ then $(a + \mathfrak{M}_A)(b + \mathfrak{M}_A^{n+1}) = ab + \mathfrak{M}_A^{n+1}$.

Let $(A, \mathfrak{M}_A)$ be a local K-algebra which residual field is K-isomorphic to K. We denote $A/\mathfrak{M}_A = K$ to mean that the structural morphism of K-algebra $K \to A/\mathfrak{M}_A$ is an isomorphism. Let $\pi\colon A \to A/\mathfrak{M}_A = K$ be the quotient morphism. Let $a \in A$. We denote $a_0 = \pi(a) \in K$. Then $\pi(a - a_0) = \pi(a) - a_0 = 0$. Thus $a - a_0 \in \mathfrak{M}_A$. In other words, for every $a \in A$, there exists $a_0 \in K$ and $a_1 \in \mathfrak{M}_A$ such that $a = a_0 + a_1$. We also observe that this decomposition is unique.

Hence the set $\mathrm{Hom}_{\mathbf{Algloc}_K}(A, K[\epsilon]/(\epsilon)^2)$ can be endowed with a structure of K-vector space defined by:

(1) If φ_1 and φ_2 are two elements of $\mathrm{Hom}_{\mathbf{Algloc}_K}(A, K[\epsilon]/(\epsilon)^2)$ and $\lambda \in K$, we define

$$\begin{aligned}\varphi_1 + \lambda\varphi_2\colon A &\longrightarrow K[\epsilon]/(\epsilon)^2\\ a = a_0 + a_1 &\longmapsto a_0 + \varphi_1(a_1) + \lambda\varphi_2(a_1).\end{aligned}$$

(2) The identity element is given by

$$\begin{aligned}0_{\mathrm{Hom}}\colon A &\longrightarrow K[\epsilon]/(\epsilon)^2\\ a = a_0 + a_1 &\longmapsto a_0.\end{aligned}$$

Moreover, this structure of K-vector space is functorial in A, i.e. if $\psi\colon A \to B$ is a morphism of K-algebras, then the induced morphism

$$\begin{aligned}\mathrm{Hom}(\psi)\colon \mathrm{Hom}_{\mathbf{Algloc}_K}(A, K[\epsilon]/(\epsilon)^2) &\longrightarrow \mathrm{Hom}_{\mathbf{Algloc}_K}(B, K[\epsilon]/(\epsilon)^2)\\ \varphi &\longmapsto \psi\circ\varphi\end{aligned}$$

is a morphism of K-vector spaces.

Within this framework, we can therefore propose an equivalent definition of the embedding dimension.

Proposition 3.3. — *Let $(A, \mathfrak{M}_A)$ be a local K-algebra whose residual field is K-isomorphic to K. Then the vector space $\mathrm{Hom}_{\mathbf{Algloc}_K}(A, K[\epsilon]/(\epsilon)^2)$ is isomorphic, as K-vector space, to $(\mathfrak{M}_A/\mathfrak{M}_A^2)^\vee$.*

Proof. — Let $\varphi \in \mathrm{Hom}_{\mathbf{Algloc}_K}(A, K[\epsilon]/(\epsilon)^2)$ and $a \in A$. Then a can be written in a unique way as $a = a_0 + a_1$ with $a_0 \in K$ and $a_1 \in \mathfrak{M}_A$. Then $\varphi(a) = a_0 + \varphi(a_1)$, where $\varphi(a_1) = \epsilon\varphi_1(a_1)$. The map $\varphi_1\colon \mathfrak{M}_A \to K$ sends $\mathfrak{M}_A^2$ to zero, so we can take quotients. We still denote $\varphi_1\colon \mathfrak{M}_A/\mathfrak{M}_A^2 \to K$ the quotient map. This map is K-linear. So it is a morphism of K-vector spaces.

Let's define

$$\begin{aligned}\psi\colon \mathrm{Hom}_{\mathbf{Algloc}_K}(A, K[\epsilon]/(\epsilon)^2) &\longrightarrow (\mathfrak{M}_A/\mathfrak{M}_A^2)^\vee\\ \varphi &\longmapsto \varphi_1.\end{aligned}$$

Let's check that ψ is bijective.

Let $\phi, \varphi \in \mathrm{Hom}_{\mathbf{Algloc}_K}(A, K[\epsilon]/(\epsilon)^2)$, verifying $\psi(\phi) = \psi(\varphi)$. Let $a \in A$. We write, as before, a in the form $a = a_0 + a_1$. Then

$$\phi(a) = a_0 + \epsilon\phi_1(a_1) = a_0 + \epsilon\varphi_1(a_1) = \varphi(a).$$

Hence ψ is injective.

Let $\theta \in (\mathfrak{M}_A/\mathfrak{M}_A^2)^\vee$ and $a = a_0 + a_1 \in A$. We define $\varphi(a) = a_0 + \epsilon\theta(a_1)$. Let $a, b \in A$. Since θ is K-linear, we verify that $\varphi(a+b) = \varphi(a) + \varphi(b)$. Furthermore, with the notations $a = a_0 + a_1$ and $b = b_0 + b_1$, since θ sends $\mathfrak{M}_A^2$ to zero, we have

$$\varphi(ab) = \varphi(a_0b_0 + a_0b_1 + b_0a_1 + a_1b_1) = a_0b_0 + \epsilon(a_0\theta(b_1) + b_0\theta(a_1)) = \varphi(a)\varphi(b).$$

Hence ψ is surjective.

We also check that ψ is a K-linear map. □

Proposition 3.4. — *Let $(A, \mathfrak{M}_A)$ be a local K-algebra which residual field K-isomorphic to K. Then $\mathrm{emb.dim}(A)$ the embedding dimension of A is finite if and only if the dimension of the K-vector space $\mathrm{Hom}_{\mathbf{Algloc}_K}(A, K[\epsilon]/(\epsilon)^2)$ is finite. In this case, these two dimensions are the same.*

Proof. — According to Proposition 3.3, there exists an isomorphism of K-vector space between $\mathrm{Hom}_{\mathbf{Algloc}_K}(A, K[\epsilon]/(\epsilon)^2)$ and the dual of $\mathfrak{M}_A/\mathfrak{M}_A^2$. We conclude that $\mathfrak{M}_A/\mathfrak{M}_A^2$ is of finite dimension if and only if its dual is of finite dimension. □

3.2. Properties and Characterizations

Lemma 3.5. — *Let $(A, \mathfrak{M}_A)$ be a local ring and $\widehat{A} = \varprojlim_n A/\mathfrak{M}_A^n$ its completion. Let $n \in \mathbf{N}^*$ be an integer. Assume that there exists an integer d such that $\dim_{A/\mathfrak{M}_A}(\mathfrak{M}_A^n/\mathfrak{M}_A^{n+1}) = d$, then $\widehat{\mathfrak{M}_A^n}$ is an $\widehat{A}$-module of finite type generated by at most d elements.*

In particular, with $n = 1$, if $\mathrm{emb.dim}(A) = d$ then $\widehat{\mathfrak{M}_A}$ is a $\widehat{A}$-module of finite type generated by at most d elements.

Proof. — Since $\dim_{A/\mathfrak{M}_A}(\mathfrak{M}_A^n/\mathfrak{M}_A^{n+1}) = d$, there exists elements

$$a_1, \ldots, a_d \in \mathfrak{M}_A^n$$

which form a basis of the $A/\mathfrak{M}_A$-vector space $\mathfrak{M}_A^n/\mathfrak{M}_A^{n+1}$. Let's consider the morphism of A-modules $\psi\colon A^d \to \mathfrak{M}_A^n, e_i \mapsto a_i$, where $e_i = (0, \ldots, 1, \ldots, 0)$ with the unique 1 is at the i-th position.

According to [11, Tag 0315 (1)] if R is a ring, $I \subset R$ an ideal and $\varphi\colon M \to N$ a morphism of R-modules then, if $M/IM \to N/IN$ is surjective, then $\widehat{M} \to \widehat{N}$ is also surjective.

By applying this Proposition with $R = A$, $I = \mathfrak{M}_A$, $M = A^d$, $N = \mathfrak{M}_A^n$ and $\varphi = \psi$, we deduce that the morphism

$$\widehat{\psi}\colon \widehat{A}^d \longrightarrow \widehat{\mathfrak{M}_A^n}$$

is surjective (because $\varprojlim_m A^d/\mathfrak{M}_A^{n+m} = \varprojlim_m A^d/\mathfrak{M}_A^m$). □

The following lemma is an adaptation of the Lemma [11, Tag 05GH], which does not apply directly in our context because, in general, if the local ring $(A, \mathfrak{M}_A)$ is not Noetherian, then, $\widehat{A} = \varprojlim_n A/\mathfrak{M}_A^n$ and $\widehat{\widehat{A}} = \varprojlim_n \widehat{A}/\widehat{\mathfrak{M}_A}^n$ are not isomorphic (see Remark 2.2). However the proof of the lemma can be adjusted for our statement by noticing that if a sequence is Cauchy for the $\widehat{\mathfrak{M}_A}$ pre-adic topology on $\widehat{A}$, then this sequence is also Cauchy for the topology that makes $\widehat{A}$ complete.

Lemma 3.6. — *Let $(A, \mathfrak{M}_A)$ be a local ring and $\widehat{A} = \varprojlim_n A/\mathfrak{M}_A^n$ its completion. Assume that $\widehat{\mathfrak{M}_A}$ is an ideal of finite type. Then $\widehat{A}$ is Noetherian.*

Proof. — Let's denote $f_1, \dots, f_t \in \widehat{\mathfrak{M}_A}$ a generating family of $\widehat{\mathfrak{M}_A}$. Consider the direct sum $B = \bigoplus_{n \geqslant 0} \widehat{\mathfrak{M}_A}^n / \widehat{\mathfrak{M}_A}^{n+1}$ in the category of $\widehat{A}$-modules. This direct sum has an additional ring structure. Let's consider the morphism of rings

$$\widehat{A}/\widehat{\mathfrak{M}_A}[T_1, \dots, T_t] \longrightarrow B$$

which send T_j on $\overline{f_j}$. This morphism is surjective, so B is a Noetherian ring.

Let J be an ideal of $\widehat{A}$. Consider the ideal $J_B = \bigoplus_{n \geqslant 0} J \cap \widehat{\mathfrak{M}_A}^n / J \cap \widehat{\mathfrak{M}_A}^{n+1}$ of B. Since B is Noetherian, there exists a finite family $\overline{g_1}, \dots, \overline{g_m}$ of elements of B generating J_B. Up to increasing the family size, we can assume that, for every j, we have $\overline{g_j} \in J \cap \widehat{\mathfrak{M}_A}^{d_j} / J \cap \widehat{\mathfrak{M}_A}^{d_j+1}$ for some integer d_j. There exists $g_j \in J \cap \widehat{\mathfrak{M}_A}^{d_j}$ which maps to $\overline{g_j}$ in the quotient. We will show that the family $g_1, \dots, g_m$ generates J.

Let $x \in J$. There exists an integer n such that $x \in J \cap \widehat{\mathfrak{M}_A}^n$ and $x \notin J \cap \widehat{\mathfrak{M}_A}^{n+1}$. If we consider $\bar{x}$ the image of x in $J \cap \widehat{\mathfrak{M}_A}^n / J \cap \widehat{\mathfrak{M}_A}^{n+1}$, we deduce that there exists a family $a_j \in \widehat{\mathfrak{M}_A}^{\max(0, n-d_j)}$ such that $\bar{x} = \sum_{j=1}^m \overline{a_j} \overline{g_j}$ since the family of the g_j generate J_B. Thus $x - \sum_{j=1}^m a_j g_j \in J \cap \widehat{\mathfrak{M}_A}^{n+1}$. By iterating this process, for every integer N, we can find a family $(a_{1,\ell}, \dots, a_{m,\ell})_{n \leqslant \ell \leqslant N}$ with $a_{j,\ell} \in \widehat{\mathfrak{M}_A}^{\max(0, \ell - d_j)}$ such that

$$x = \sum_{\ell=0}^{N} \sum_{j=1}^{m} a_{j,\ell} g_j \mod \widehat{\mathfrak{M}_A}^N.$$

We set $A_j = \sum_{\ell \geqslant 0} a_{j,\ell}$. The sequence $(\sum_{\ell=0}^N a_{j,\ell})_{N \in \mathbf{N}}$ is a Cauchy sequence (for the topology for which $\widehat{A}$ is complete) because, for every integer $\ell \in \mathbf{N}$, we have $\widehat{\mathfrak{M}_A}^\ell \subset \widehat{\mathfrak{M}_A^\ell}$ hence A_j is well defined in $\widehat{A}$. Thus we can write $x = \sum_{j=0}^m A_j g_j$, since $\bigcap_{N \in \mathbf{N}} \widehat{\mathfrak{M}_A^N} = 0$ (because $\widehat{A}$ is complete -and separate- for the filtration $(\widehat{\mathfrak{M}_A^N})_{N \in \mathbf{N}}$) which proves that J is of finite type. □

Lemma 3.7. — *Let $(A, \mathfrak{M}_A)$ be a local ring. The following assertions are equivalent:*

(1) *The embedding dimension of the local ring $(A, \mathfrak{M}_A)$ is finite.*
(2) *The completion $\widehat{A}$ of A, with respect to its maximal ideal, is Noetherian.*

Proof. — Assume that $\dim_{A/\mathfrak{M}_A}(\mathfrak{M}_A / \mathfrak{M}_A^2) < \infty$, the by applying Lemma 3.5 we deduce that $\widehat{\mathfrak{M}_A}$ is an ideal of finite type, and then, by appliying Lemma 3.6, we deduce that the ring $\widehat{A}$ is Noetherian.

Conversely, if the ring $\widehat{A}$ is Noetherian, the inclusion $\widehat{\mathfrak{M}_A}^2 \subset \widehat{\mathfrak{M}_A^2}$ implies that the morphism $\widehat{\mathfrak{M}_A}/\widehat{\mathfrak{M}_A}^2 \to \mathfrak{M}_A/\mathfrak{M}_A^2$ is surjective. Hence the embedding dimension of A is finite and $\mathrm{emb.dim}(A) \leqslant \mathrm{emb.dim}(\widehat{A})$. $\square$

The previous lemma is a result that can be found in [4, Lemma 10.12] in the following form.

Remark 3.8. — The previous statement can be reformulate as follows from the point of view of schemes. Let K be a field. Let X be a K-scheme. Let $x \in X$. The following assertions are equivalent:

(1) The embedding dimension of the local ring $\mathcal{O}_{X,x}$ is finite.
(2) The completion $\widehat{\mathcal{O}_{X,x}}$ of the local ring at x is Noetherian.

COROLLARY 3.9. — *Let $(A, \mathfrak{M}_A)$ be a local ring. We denote $\widehat{A} = \varprojlim_n A/\mathfrak{M}_A^n$ the completion of A. Then* $\mathrm{emb.dim}(A)$ *is finite if and only if* $\mathrm{emb.dim}(\widehat{A})$ *is finite, and in this case* $\mathrm{emb.dim}(A) = \mathrm{emb.dim}(\widehat{A})$.

Proof. — The fact that $\mathrm{emb.dim}(A)$ is finite if and only if $\mathrm{emb.dim}(\widehat{A})$ is finite, can be deduced from Lemma 3.7.

Let's assume that these embedding dimensions are finite. Thanks to the inclusion $\widehat{\mathfrak{M}_A}^2 \subset \widehat{\mathfrak{M}_A^2}$, we have a surjective morphism of $A/\mathfrak{M}_A$-vector spaces

$$\widehat{\mathfrak{M}_A}/\widehat{\mathfrak{M}_A}^2 \longrightarrow \widehat{\mathfrak{M}_A}/\widehat{\mathfrak{M}_A^2} = \mathfrak{M}_A/\mathfrak{M}_A^2.$$

Hence $\mathrm{emb.dim}(A) \leqslant \mathrm{emb.dim}(\widehat{A})$.

Furthermore, we know thanks to Lemma 3.5 that $\widehat{\mathfrak{M}_A}$ is generated as $\widehat{A}$-module by at most $d := \mathrm{emb.dim}(A)$ elements. Denote by $u_1, \ldots, u_d \in \widehat{\mathfrak{M}_A}$ a generating family of $\widehat{\mathfrak{M}_A}$. Let $x \in \widehat{\mathfrak{M}_A}$, then there exists $(a_1, \ldots, a_d) \in \widehat{A}^d$ such that $x = \sum_{i=1}^d a_i u_i$. Let $\pi \colon \widehat{A} \to \widehat{A}/\widehat{\mathfrak{M}_A} = A/\mathfrak{M}_A$ the canonical morphism of projection. Since $\widehat{\mathfrak{M}_A}/\widehat{\mathfrak{M}_A}^2$ is endowed with a structure of $A/\mathfrak{M}_A$-vector space, we can consider the element $y = \sum_{i=0}^d \pi(a_i)\overline{u_i}$ of $\widehat{\mathfrak{M}_A}/\widehat{\mathfrak{M}_A}^2$. Hence $\overline{x} - y = 0$ in $\widehat{\mathfrak{M}_A}/\widehat{\mathfrak{M}_A}^2$. Thus the $A/\mathfrak{M}_A$-vector space $\widehat{\mathfrak{M}_A}/\widehat{\mathfrak{M}_A}^2$ is of dimension at most d. Hence

$$\mathrm{emb.dim}(\widehat{A}) \leqslant \mathrm{emb.dim}(A). \qquad \square$$

The above propositions allow us to prove the following statement about ring completion:

PROPOSITION 3.10. — *Let $(A, \mathfrak{M}_A)$ be a local ring. Let's denote $\widehat{A} = \varprojlim_n A/\mathfrak{M}_A^n$ the completion of A. If $\widehat{A}$ is a Noetherian ring then $\widehat{A}$ is complete for the $\widehat{\mathfrak{M}_A}$ pre-adic topology.*

Proof. — We are going to show by induction, that for every $n \in \mathbf{N}$, we have $\widehat{\mathfrak{M}_A}^n = \widehat{\mathfrak{M}_A^n}$.

For $n = 1$ there is nothing to prove.

Assume now that $\widehat{\mathfrak{M}_A}^n = \widehat{\mathfrak{M}_A^n}$ for some integer n. Thanks to the inclusion $\widehat{\mathfrak{M}_A}^{n+1} \subset \widehat{\mathfrak{M}_A^{n+1}}$, we deduce a surjective morphism

$$\varphi_n \colon \widehat{\mathfrak{M}_A}^n / \widehat{\mathfrak{M}_A}^{n+1} \longrightarrow \widehat{\mathfrak{M}_A^n} / \widehat{\mathfrak{M}_A^{n+1}} = \mathfrak{M}_A^n / \mathfrak{M}_A^{n+1}.$$

Since $\widehat{A}$ is Noetherian, then the $A/\mathfrak{M}_A$-vector space $\widehat{\mathfrak{M}_A}^n / \widehat{\mathfrak{M}_A}^{n+1}$ is of finite dimension and so $\widehat{\mathfrak{M}_A^n} / \widehat{\mathfrak{M}_A^{n+1}}$ is also of finite dimension.

We set $d = \dim_{A/\mathfrak{M}_A}(\mathfrak{M}_A^n / \mathfrak{M}_A^{n+1})$, then, thanks to Lemma 3.5, we deduce that $\widehat{\mathfrak{M}_A^n}$ is an $\widehat{A}$-module generated by at most d elements. But, by hypothesis $\widehat{\mathfrak{M}_A^n} = \widehat{\mathfrak{M}_A}^n$. Denote $u_1, \ldots, u_d \in \widehat{\mathfrak{M}_A}^n$ a generating family of $\widehat{\mathfrak{M}_A}^n$. Let $x \in \widehat{\mathfrak{M}_A}^n$, then there exists $(a_1, \ldots, a_d) \in \widehat{A}^d$ such that $x = \sum_{i=1}^d a_i u_i$. Let $\pi \colon \widehat{A} \to \widehat{A}/\widehat{\mathfrak{M}_A} = A/\mathfrak{M}_A$ be the canonical morphism of projection. Since $\widehat{\mathfrak{M}_A}^n / \widehat{\mathfrak{M}_A}^{n+1}$ is endowed with a structure of $A/\mathfrak{M}_A$-vector space, we consider the element $y = \sum_{i=0}^d \pi(a_i) u_i$ of $\widehat{\mathfrak{M}_A}^n / \widehat{\mathfrak{M}_A}^{n+1}$. Hence $x - y = 0$ in $\widehat{\mathfrak{M}_A}^n / \widehat{\mathfrak{M}_A}^{n+1}$. Thus the $A/\mathfrak{M}_A$-vector space $\widehat{\mathfrak{M}_A}^n / \widehat{\mathfrak{M}_A}^{n+1}$ is of dimension at most d.

We deduce that φ_n is a surjective morphism between two $A/\mathfrak{M}_A$-vector spaces of same dimension, so it is bijective. Hence $\widehat{\mathfrak{M}_A}^{n+1} = \widehat{\mathfrak{M}_A^{n+1}}$. This concludes the induction. □

Remark 3.11. — Proposition 3.10 is false in general if the ring $\widehat{A}$ is not Noetherian as underlined in Remark 2.2.

PROPOSITION 3.12. — *Let $(A, \mathfrak{M}_A)$ be a local K-algebra which residue field is K-isomorphic to K such that* emb.dim(A)*, the embedding dimension of A, is finite and equal to d. Then there exists a surjective morphism of local K-algebras $K[\![x_1, \ldots, x_d]\!] \to \widehat{A}$.*

Proof. — According to Corollary 3.9, we know that emb.dim$(\widehat{A}) = d$. Let's consider $(u_1, \ldots, u_d) \in \widehat{\mathfrak{M}_A}$ a basis of the K-vector space $\widehat{\mathfrak{M}_A} / \widehat{\mathfrak{M}_A}^2$. Let $u \in \widehat{A}$, we will show by induction that, for every $i \in \mathbf{N}$, there exists an homogeneous polynomial $P_{u,i} \in K[x_1, \ldots, x_d]$ of total degree i such that, for every $n \in \mathbf{N}$, we have

$$u - \sum_{i=1}^{n} P_{u,i}(u_1, \ldots, u_d) \in \widehat{\mathfrak{M}_A}^{n+1}.$$

For $n=0$, we note that $\widehat{A}/\widehat{\mathfrak{M}_A}=K$, then if $P_{u,0}\in K$ denotes the image of u through the morphism $\widehat{A}\to\widehat{A}/\widehat{\mathfrak{M}_A}$, we have $u-P_{u,0}\in\mathfrak{M}_A$.

If $u\in\widehat{\mathfrak{M}_A}$, then, since the K-vector space $\widehat{\mathfrak{M}_A}/\widehat{\mathfrak{M}_A}^2$ is generated by $(u_1,\dots,u_d)$, there exists $P_{u,1}$ an homogeneous polynomial of total degree 1 such that $u-P_{u,1}(u_1,\dots,u_d)\in\widehat{\mathfrak{M}_A}^2$.

Now, let $u\in\widehat{\mathfrak{M}_A}^{n+1}$. We will prove that there exists an homogeneous polynomial $P_{u,n+1}\in K[x_1,\dots,x_d]$ of total degree $n+1$ such that $u-P_{u,n+1}\in\widehat{\mathfrak{M}_A}^{n+2}$. Since $u\in\widehat{\mathfrak{M}_A}^{n+1}$, there exists a finite set J, a family $\alpha_j\in\widehat{A}$ and $(b_{j,i})_{j\in J,1\leqslant i\leqslant n+1}$ a family of elements of $\widehat{\mathfrak{M}_A}$ such that

$$u=\sum_{j\in J}\alpha_j\prod_{i=1}^{n+1}b_{j,i}.$$

Without loss of generality, we can assume that, for every $j\in J$, we have $\alpha_j\notin\widehat{\mathfrak{M}_A}$. Then there exists $P_{\alpha_j,0}$ an homogeneous polynomial of degree 0 (i.e. an element of K) such that $\alpha_j-P_{\alpha_j,0}\in\widehat{\mathfrak{M}_A}$. Furthermore, there exists $P_{b_{j,i},1}\in K[x_1,\dots,x_d]$ homogeneous polynomials of total degree 1 such that, for every $j\in J$ and $1\leqslant i\leqslant n+1$, we have

$$b_{j,i}-P_{b_{j,i},1}(u_1,\dots,u_d)\in\widehat{\mathfrak{M}_A}^2.$$

We check that u and

$$P_{u,n+1}(u_1,\dots,u_d)\coloneqq\sum_{j\in J}P_{\alpha_j,0}\prod_{i=1}^{n+1}P_{b_{j,i},1}(u_1,\dots,u_d)$$

have the same image in $\widehat{\mathfrak{M}_A}^{n+1}/\widehat{\mathfrak{M}_A}^{n+2}$. To do so, it is sufficient to note that

$$\sum_{j\in J}(\alpha_j-P_{\alpha_j,0})\prod_{i=1}^{n+1}(b_{j,i}-P_{b_{j,i},1}(u_1,\dots,u_d))=0\mod\widehat{\mathfrak{M}_A}^{n+2}$$

and that by developing the left member, we obtain

$$0=u-P_{u,n+1}(u_1,\dots,u_d)\mod\widehat{\mathfrak{M}_A}^{n+2}.$$

If $u\in\widehat{A}$, this shows the existence, for every $i\in\mathbf{N}$, of an homogeneous polynomial $P_{u,i}\in K[x_1,\dots,x_d]$ of total degree i such that, for every integer $n\in\mathbf{N}$, we have

$$u-\sum_{i=1}^{n}P_{u,i}(u_1,\dots,u_d)\in\widehat{\mathfrak{M}_A}^{n+1}.$$

If we set $P \coloneqq \sum_{i\geqslant 0} P_{u,i}$, then $P \in K[\![x_1,\dots,x_d]\!]$ because all of the polynomials $P_{u,i}$ are homogeneous of total degree i. Furthermore

$$u - P(u_1,\dots,u_d) \in \bigcap_{N\in\mathbf{N}} \widehat{\mathfrak{M}_A}^N,$$

and the intersection $\bigcap_{N\in\mathbf{N}} \widehat{\mathfrak{M}_A}^N$ is zero because

$$\bigcap_{N\in\mathbf{N}} \widehat{\mathfrak{M}_A}^N \subset \bigcap_{N\in\mathbf{N}} \widehat{\mathfrak{M}_A^N} = \{0\}.$$

Finally this shows that there is a sujection of $K[\![x_1,\dots,x_d]\!] \to \widehat{A}$, defined by $x_i \mapsto u_i$. $\square$

4. A property of noetherianity of the formal neighborhoods of the differential arc scheme.

Let K be a field of characteristic zero. Let $F \in K\{y\}$ be a differential polynomial and $X^\partial = \operatorname{Spec}(K\{y\}/[F])$ be the associated differential arc scheme.

When we consider $\operatorname{Spec}(K\{y\}/[F])$ we forget the differential structure on the K-algebra $K\{y\}/[F]$. Moreover, a K-point of X^∂ is the data of a morphism of K-algebras $\gamma\colon K\{y\}/[F] \to K$. This data also corresponds to an unique morphism of differential K-algebras,

$$\begin{aligned}\gamma\colon K\{y\}/[F] &\longrightarrow K[\![T]\!]\\ G &\longmapsto \sum_{i\geqslant 0} \frac{\gamma(\Delta^i(G))}{i!} T^i\end{aligned}$$

where $K[\![T]\!]$ is considered as a differential K-algebra endowed with the derivation ∂_T.

More generally, if A is a K-algebra, we have a bijection

$$\operatorname{Hom}_{\mathbf{Alg}_K}(K\{y\}/[F], A) = \operatorname{Hom}_{\mathbf{Algdiff}_K}(K\{y\}/[F], A[\![T]\!])$$

functorial in A. In the following, when we are going to consider a solution $\gamma(T) \in K[\![T]\!]$ of the differential polynomial F. Note that it will be the same as considering a K-point γ of $\operatorname{Spec}(K\{y\}/[F])$.

Remark 4.1. — If $F \in K\{y\}$ is a differential polynomial and $\gamma(T) = \sum_{i\geqslant 0} \frac{a_i}{i!} T^i \in K[\![T]\!]$ a solution of F, then the morphism of K-algebras $\gamma\colon K\{y\}/[F] \to K$ associated to $\gamma(T)$ is defined by

$$\begin{aligned}\gamma\colon K\{y\}/[F] &\longrightarrow K\\ y_i &\longmapsto a_i.\end{aligned}$$

Note that $\mathrm{Ker}(\gamma)$ is, in general, not a differential ideal of $K\{y\}/[F]$. In particular the stalk of X^∂ at γ, denoted by $\mathcal{O}_{X^\partial,\gamma}$, will refer to the local ring $(K\{y\}/[F])_{\mathrm{Ker}(\gamma)}$. However, if $\mathfrak{p}$ refers to the ideal $\gamma(T)^{-1}(\langle T\rangle)$, the stalk of X^∂ at γ this is the same as considering the ring $(K\{y\}/[F])_{\mathfrak{p}}$.

We denote by $\widehat{X^\partial_\gamma}$ the formal neighborhood of γ, which is defined as $\mathrm{Spf}(\widehat{\mathcal{O}_{X^\partial,\gamma}})$.

Definition 4.2. — *Let K be a field of characteristic zero. Let $F \in K\{y\}$ a differential polynomial of order n. Let $\gamma(T) \in K[\![T]\!]$ be a solution of the ODE $F = 0$. The solution $\gamma(T)$ is said to be a* non-degenerated arc *if the separant of F does not vanish at $\gamma(T)$,* i.e *if $S_F(\gamma(T)) \neq 0$.*

We recall that S_F is the differential polynomial defined as $\partial F/\partial y_n$, where n is the order of F (*i.e* the highest derivative appearing in the writing of F).

The main point of this chapter is to prove the following result:

Theorem 4.3. — *Let K be a field of characteristic zero. Let $F \in K\{y\}$ a differential polynomial of order n. Let $\gamma(T) \in K[\![T]\!]$ be a solution of the ODE $F = 0$ which is a non-degenerated arc. Then the formal neighborhood $\widehat{X^\partial_\gamma}$ is Noetherian and its embedding dimension is less or equal to n.*

Remark 4.4. — If $\gamma(T)$ is a solution of $F = 0$. The condition "is a non-degenerated arc" can be easily replaced by "there exists $i \in [0, n]$, such that $\partial_{y_i} F(\gamma(T)) \neq 0$".

If, for every integer $i \in [0, n]$, we have $\partial_{y_i} F(\gamma(T)) = 0$, then the embedding dimension is infinite since the linearization (see Definition 4.14) is zero.

The proof will follow the strategy of [5, Theorem 1] and [3], by using the functor of point of theses objects. Our differential problem has the particularity that it can be "linearized" (see Section 4.2), which will reduce the noetherianity problem to a question of linear algebra.

The property of noetherianity can be compared to the statement in [5] and [3]. Even if in the algebraic framework the formal neighborhood is, in general, non Noetherian, in the differential case the objects can be understood as generalization of arc spaces $\mathscr{L}_\infty(X)$ where X is 0-dimensional. And when X is 0-dimensional, formal neighborhood are Noetherian.

4.1. Deformation functor

In the Section 2 we have developed the notion of deformation in the algebraic framework. In this section, we are going to introduce the differential deformation functor on the model of the construction of the Section 2. We will also establish similar properties.

Let K be a field of characteristic zero. Recall that a local K-algebra $(A, \mathfrak{M}_A)$ is called *quasi-Artinian* if its residue field is K-isomorphic to K and if its maximal ideal is nilpotent. Let $F \in K\{y\}$. We denote $X^\partial = \operatorname{Spec}(K\{y\}/[F])$. Let $\gamma \in X^\partial(K)$.

Definition 4.5. — *The* functor of differential deformations *of X^∂ at γ is the functor*

$$A \longmapsto \operatorname{Def}_\gamma(X^\partial, A) := \left\{\gamma_A \in X^\partial(A) \mid \gamma_A(T) - \gamma(T) \in \mathfrak{M}_A[\![T]\!]\right\}$$

for every quasi-Artinian K-algebra A. Every element $\gamma_A \in \operatorname{Def}_\gamma(X^\partial, A)$ is called differential deformation of γ.

Notation 4.6. — If $X^\partial = \operatorname{Spec}(K\{y\}/[F])$, we will denote sometimes $\operatorname{Def}_\gamma(X^\partial, A) = \operatorname{Def}_\gamma([F], A)$. And to study the deformations of ${X^\partial}_{\mathrm{red}} = \operatorname{Spec}(K\{y\}/\{F\})$, we will denote

$$\operatorname{Def}_\gamma(\{F\}, A) := \left\{\gamma_A \in X^\partial_{\mathrm{red}}(A) \mid \gamma_A(T) - \gamma(T) \in \mathfrak{M}_A[\![T]\!]\right\}.$$

As in the algebraic framework, we have the following result:

Proposition 4.7. — *Let K be a field of characteristic zero. Let $F \in K\{y\}$ a non zero polynomial and $\gamma \in X^\partial(K)$ a solution of the associated ODE. For every quasi-Artinian K-algebra A we have a natural bijection*

$$\operatorname{Hom}_{\mathbf{AlgLC}_k}(\widehat{\mathcal{O}_{X^\partial,\gamma}}, A) \cong \operatorname{Def}_\gamma(X^\partial, A).$$

In other words, the completion of the local ring of X^∂ at γ represent the functor of the differential deformations of γ.

Proof. — This proposition follows from Proposition 2.12. □

Remark 4.8. — By combining this proposition with Proposition 3.3 and Remark 2.8, note that the study of the deformations for $A = K[\epsilon]/(\epsilon^2)$ is equivalent to the study of the tangent space of X^∂.

Example 4.9. — Let $X^\partial = \operatorname{Spec}(K\{y\}/[y_1 - y_0])$. In other words X^∂ is defined by the differential equation $y' = y$. A solution of this equation is of the form $\gamma(T) = Ce^T$ with $C \in K$. Such a solution corresponds to a K-point of X^∂. Let $C \in K$ and $\gamma(T) = Ce^T$ be a solution. We will compute $\operatorname{Def}_\gamma([y_1 - y_0], A)$.

$$\operatorname{Def}_\gamma([y_1 - y_0], A) := \left\{\gamma_A \in X^\partial(A) \mid \gamma_A(T) - \gamma(T) \in \mathfrak{M}_A[\![T]\!]\right\}.$$

Let's denote $\gamma_A(T) = \gamma(T) + \sum_{i\geqslant 0} a_i T^i$. By definition, $\gamma_A(T)$ must verify the equation $\gamma_A(T)' - \gamma_A(T) = 0$. We deduce that

$$\forall i \geqslant 0, \quad (i+1)a_{i+1} - a_i = 0.$$

In particular, the data of $a_0 \in \mathfrak{M}_A$ sets all the a_i. We deduce that there is a bijection between $\mathrm{Def}_\gamma([y_1 - y_0], A)$ and $\mathfrak{M}_A$ which is functional in A, furthermore there is also a functorial bijection in A between $\mathfrak{M}_A$ and $\mathrm{Hom}_{\mathbf{AlgLC}_k}(K[\![T]\!], A)$, and hence, by Yoneda's Lemma, we have that $\widehat{\mathcal{O}_{X^\partial,\gamma}}$ is isomorphic to $K[\![T]\!]$ in the category of $\mathbf{AlgLC}_k$.

Note however that this is a toy example since $K\{y\}/[y_1 - y_0]$ is K-isomorphic to $K[y_0]$. In particular $X^\partial = \mathrm{Spec}(K[y_0])$ is a line, which gives an example of easily computed differential arc scheme of finite dimension.

Let $F \in K\{y\}$ be an irreducible polynomial. We have seen (Proposition 1.3) that $\{F\} = (\{F\} : S_F) \cap \{F, S_F\}$.

Lemma 4.10. — *Let $F \in K\{y\}$ be an irreducible polynomial, then*

$$\{F\} \subset ([F] : S_F^\infty) \subset (\{F\} : S_F).$$

In particular, we deduce that $([F] : S_F^\infty) = (\{F\} : S_F)$.

Proof. — The inclusion $([F] : S_F^\infty) \subset (\{F\} : S_F)$ is clear.

For the inclusion $\{F\} \subset ([F] : S_F^\infty)$ we have that, for every element $G \in K\{y\}$, there exists an element $G_1 \in K\{y\}$ of order less or equal that the order of F and $m \in \mathbf{N}$ such that $S_F^m G = G_1 \mod [F]$ (for this result see [10, Chapter I, Section 6 Reduction]).

If $G \in \{F\}$, then the element G_1 is also in $\{F\}$. Hence there exists $n \in \mathbf{N}$ such that $G_1^n \in [F]$ (since $\mathbf{Q} \subset K$, see [7, Lemma 1.8]). Since the order of G_1 is less than or equal to the order of F, then F divides G_1^n (see [7, Lemma 7.8]). Finally, since F is assumed to be irreducible, then F divides G_1 and hence $G_1 \in [F]$. Thus $S_F^m G \in [F]$. □

Proposition 4.11. — *Let $F \in K\{y\}$ be a irreducible differential polynomial. Let $\gamma \in X^\partial(K)$ be a solution of the associated ODE, which is a non-degenerated arc. Then $\gamma(T)$ is a solution of $([F] : S_F^\infty)$. Furthermore, if A is a quasi-Artinian ring, then*

$$\mathrm{Def}_\gamma([F], A) = \mathrm{Def}_\gamma(([F] : S_F^\infty), A) = \mathrm{Def}_\gamma(\{F\}, A).$$

Proof. — The following inclusion are clear (thanks to Lemma 4.10):

$$\mathrm{Def}_\gamma(([F] : S_F^\infty), A) \subset \mathrm{Def}_\gamma(\{F\}, A) \subset \mathrm{Def}_\gamma([F], A).$$

Let's consider $\gamma_A \in \mathrm{Def}_\gamma([F], A)$. Let $G \in ([F] : S_F^\infty)$, we will prove that $G(\gamma(T) + \gamma_A(T)) = 0$. There exists $m \in \mathbf{N}$ such that $S_F^m G \in [F]$. In particular $S_F^m(\gamma(T)+\gamma_A(T))G(\gamma(T)+\gamma_A(T))=0$. Since $S_F^m(\gamma(T))\neq 0$ then $S_F^m(\gamma(T)+\gamma_A(T))$ is not a zero divisor in A (since it is non zero modulo $\mathfrak{M}_A$), hence $G(\gamma(T)+\gamma_A(T))=0$.

We obtain the equalities

$$\mathrm{Def}_\gamma([F],A)=\mathrm{Def}_\gamma(([F]:S_F^\infty),A).$$ □

Example 4.12. — Let K be a field of characteristic zero. Let $F=(y_2-y_1)^2-(y_1-y_0)\in K\{y\}$ be a differential polynomial. Its separant is $S_F=\partial F/\partial y_2=2(y_2-y_1)$. In particular $\{F,S_F\}=\{y_1-y_0\}$. Thanks to the low power theorem, it can be checked that $A=y_1-y_0$ is a singular essential component of F. Since $\Delta(F)=(y_2-y_1)(2(y_3-y_2)-1)$, then $2(y_3-y_2)-1\in([F]:S_F^\infty)$. However $\gamma(T)=\mathrm{e}^T$ is a solution of $[F]$ but not a solution of $([F]:S_F^\infty)$, so Proposition 4.11 does not apply if the solution canceals the separant.

Note that in general, similarly to the algebraic framework, the differential deformations of a differential polynomial do not correspond to those of the associated reduced differential polynomial: the functors $\mathrm{Def}_\gamma([F],A)$ and $\mathrm{Def}_\gamma(\{F\},A)$ do not usually coincide. If we consider, for example, $F=x_2^m$ and A a quasi-Artinian K-algebra which nilpotency index of its maximal ideal is m, then, for every $a\in\mathfrak{M}_A$, the deformation $\gamma_A(T)=T+aT^2$ is in $\mathrm{Def}_T([x_2^m],A)$ but not in $\mathrm{Def}_T(\{x_2^m\},A)$ since x_2 does not vanish at $\gamma_A(T)$.

Next proposition will prove that in the neighborhood of a solution that makes the separant invertible, the deformation space is described as a formal disk. Thus, in this case, the formal neighborhood in well described.

Proposition 4.13. — *Let K be a field of characteristic zero. Let $F\in K\{y\}$ be a differential polynomial of order n. Let $\gamma(T)\in K[\![T]\!]$ a solution of F such that $S_F(\gamma(T))$ is a unit in $K[\![T]\!]$ (or equivalently that $S_F(\gamma)\neq 0$). Let $X^\partial=\mathrm{Spec}(K\{y\}/[F])$ Then the neighborhood $\mathcal{O}_{X^\partial,\gamma}$ is a Noetherian ring and the formal neighborhood $\widehat{\mathcal{O}_{X^\partial,\gamma}}$ is a formal disk of dimension n.*

Proof. — Let's denote $\gamma(T)=\sum_{i\geqslant 0}a_i/i!\,T^i$. We consider the associated morphism

$$\begin{aligned}\gamma\colon K\{y\}/[F]&\longrightarrow K\\ y_i&\longmapsto a_i.\end{aligned}$$

Since $S_F(\gamma(T))$ is invertible, then $\gamma(S_F)$ is non zero. In particular, $S_F\notin\mathrm{Ker}(\gamma)$.

The Leibiz's formula applied to F in the differential K-algebra $(K\{y\},\Delta)$ produces the following formula:

$$\Delta(F)=\sum_{i=0}^{n}\partial_{y_i}(F)y_{i+1}.$$

Consequently, since $\partial_{y_n}F=S_F$, the differential ideal $[F]$ seen in the localization $K\{y\}_{\mathrm{Ker}(\gamma)}$ contains the element

$$y_{n+1}=\sum_{i=0}^{n-1}\frac{\partial_{y_i}(F)}{S_F}y_{i+1}$$

and all of its derivatives. Let's denote

$$\begin{aligned}\widetilde{\gamma}\colon K[y_0,\dots,y_n]/\langle F(y_0,\dots,y_n)\rangle &\longrightarrow K\\ y_i &\longmapsto a_i.\end{aligned}$$

The result is an isomorphism of K-algebras

$$\mathcal{O}_{X^\partial,\gamma}\cong (K[y_0,\dots,y_n]/\langle F\rangle)_{\mathrm{Ker}(\widetilde{\gamma})}.$$

Now since $(a_0,\dots,a_n)\in K^n$ is a smooth point of the variety $K[y_0,\dots,y_n]/\langle F\rangle$ (since $\partial_{y_n}(F)(\gamma(T))\neq 0$), then $\widehat{\mathcal{O}_{X^\partial,\gamma}}=K[\![T_1,\dots,T_n]\!]$. $\square$

4.2. The linearized differential equation

Let $F\in K\{y\}$ be a differential polynomial of order $n\geqslant 1$, $\gamma(T)\in K[\![T]\!]$ a solution of F and $X^\partial=\mathrm{Spec}(K\{y\}/[F])$. Let $K[\epsilon]=K[X]/(X^2)$, where the symbol ϵ refer to the class of the element X in the quotient. Note that there is an obvious isomorphism of K-algebras $K[\epsilon][\![T]\!]\cong K[\![T]\!][\epsilon]$. As a result, any deformation of γ in $K[\epsilon]$, i.e. every solution $\gamma(T)+\gamma_\epsilon(T)\in K[\epsilon][\![T]\!]$ of F, can uniquely be rewriten as an element of $K[\![T]\!][\epsilon]$. More precisely

$$\begin{aligned}\mathrm{Def}_\gamma(X^\partial,K[\epsilon]))\coloneqq&\ \{\gamma_\epsilon(T)\in(\epsilon)[\![T]\!]\mid F(\gamma(T)+\gamma_\epsilon(T))=0\}\\ =&\ \{\eta(T)\in K[\![T]\!]\mid F(\gamma(T)+\epsilon\eta(T))=0\}.\end{aligned}$$

Let $\gamma(T)+\epsilon\eta(T)\in\mathrm{Def}_\gamma(X,K[\epsilon]))$. In the sequel we will denote by $\eta^{(i)}$ the i-th derivative of η with respect to T, which can be also denoted $\partial_T^i\eta$.

By the Taylor's formula and the property $\epsilon^2 = 0$ in the ring $K[\epsilon]$, we deduce

$$\begin{aligned} F(\gamma(T) + \epsilon\eta(T)) &= F(\gamma(T)) + \epsilon \left(\sum_{i=0}^{n} \eta^{(i)}(T)(\partial_{y_i} F)(\gamma(T)) \right) \\ &= \epsilon \left(\sum_{i=0}^{n} \eta^{(i)}(T)(\partial_{y_i} F)(\gamma(T)) \right). \end{aligned} \tag{4.1}$$

Thanks to the formula (4.1), we deduce that $F(\gamma(T)+\epsilon\eta(T)) = 0$ is equivalent to the condition

$$\sum_{i=0}^{n} \eta^{(i)}(T)(\partial_{y_i} F)(\gamma(T)) = 0. \tag{4.2}$$

Definition 4.14. — *Let K be a field of characteristic zero. Let $F \in K\{y\}$ be a differential polynomial of order $n \geqslant 1$, $\gamma(T) \in K[\![T]\!]$ a solution of F. We call* linearized differential equation *of F at $\gamma(T)$ the linear differential polynomial $\sum_{i=0}^{n} (\partial_{y_i} F(\gamma(T))) y_i \in K\{y\}$. We will denote it $\mathcal{L}(F, \gamma)$.*

Note that the linearized differential equation will not have constant coefficients in general.

Proposition 4.15. — *The embedding dimension of the completion of the local ring of X^∂ at γ is finite if and only if the dimension of the K-vector space*

$$\mathrm{Sol}(\mathcal{L}(F,\gamma))(K[\![T]\!]) = \{\eta \in K[\![T]\!] \mid \mathcal{L}(F,\gamma)(\eta) = 0\}$$

is finite. In this case, these two dimension coincide.

Proof. — According to the above considerations, this proposition is a consequence of Proposition 3.4 applied to $A = \widehat{K\{y\}/[F]_\gamma}$. □

4.3. Proof of Theorem 4.3

The key argument of the proof of Theorem 4.3 lies in the computation of this embedding dimension. By using Proposition 4.15 and the following lemma due to Ritt, we will be able to provide an upper bound of this dimension, which will be sufficient for our purpose.

Definition 4.16. — *An set of elements $\eta_1, \ldots, \eta_s \in K[\![T]\!]$ is said* linearly dependent *if there exists elements $c_1, \ldots, c_s \in K$, not all zero, such that*

$$\sum_{i=1}^{s} c_i \eta_i = 0.$$

LEMMA 4.17. — *An set of elements $\eta_1, \dots, \eta_s \in K[\![T]\!]$ is linearly dependent if and only if the Wronskian*

$$\begin{vmatrix} \eta_1 & \cdots & \eta_s \\ \eta_1' & \cdots & \eta_s' \\ \dots & \cdots & \dots \\ \eta_1^{(s-1)} & \cdots & \eta_s^{(s-1)} \end{vmatrix} = 0.$$

Proof. — See [10, Chapter 2, The resolvant, p. 34]. □

COROLLARY 4.18. — *Let K be a field of characteristic zero. Let $F \in K\{y\}$ be a differential polynomial of order $n \geqslant 1$, $\gamma(T) \in K[\![T]\!]$ a solution of F, which is a non-degenerated arc, and $X^\partial = \operatorname{Spec}(K\{y\}/[F])$. Then, we have*

$$\operatorname{emb.dim}(\widehat{\mathcal{O}_{X^\partial,\gamma}}) \leqslant n.$$

For the definition of the embedding dimension and its properties see Section 3.

Proof. — For simplicity, we will denote $(\partial_{y_i} f)(\gamma(T)) = a_i(T)$ and

$$\sum_{i=0}^{n} a_i(T) y_i = 0 \tag{4.3}$$

the linearized equation $\mathcal{L}(f, \gamma)$. Let $\eta_1(T), \dots, \eta_s(T)$ be an independent family of solutions in $\operatorname{Def}_\gamma(X^\partial, K[\epsilon]/(\epsilon^2))$, i.e. a independent familly of solutions in $K[\![T]\!]$ of the differential polynomial (4.3). In particular the $\eta_i(T)$ are linearly independent. Thanks to Lemma 4.17, we deduce that the Wronskian

$$\begin{vmatrix} \eta_1 & \cdots & \eta_s \\ \eta_1' & \cdots & \eta_s' \\ \dots & \cdots & \dots \\ \eta_1^{(s-1)} & \cdots & \eta_s^{(s-1)} \end{vmatrix}$$

must not be equal to zero. Assume that $s > n$, then the n-th row of the Wronskian is equal to $(\eta_1^{(n)} \ \cdots \ \eta_s^{(n)})$. We can multiply the Wronskian by $a_n(T)$ which is non zero (since it is the separant of F evaluated $\gamma(T)$), and thus the row $a_n(T) y_1^{(n)} \ \cdots \ a_n(T) y_s^{(n)}$ is a linear combination of the other's. Thus the Wronskian is zero, which is the desired contradiction. So $s \leqslant n$. □

The statement of Theorem 4.3 is then directly deduced from Corollary 4.18 and Lemma 3.7.

5. Differential arc scheme which embedding dimension is at most 1

Let K be a field of characteritic zero. In this chapter we will clarify Theorem 4.3 in the particular case of a differential polynomial which embedding dimension at most 1. In particular, this covers the case of differential polynomials of order 1.

In this particular case, we will show the following result:

Theorem 5.1. — *Let K be a field of characteritic zero. Let $F := F(y, \dots, y^{(n)}) \in K\{y\}$ be an irreducible differential polynomial and $X^\partial := \mathrm{Spec}(K\{y\}/[F])$. Let $\gamma(T) \in K[\![T]\!]$ be a non constant solution of the equation $F = 0$. Assume that the embedding dimension of X^∂ at γ is at most 1. Then the formal neighborhood $\widehat{X^\partial}_\gamma$ is isomorphic, as formal K-scheme, to a formal disk $\mathbb{D}_K = \mathrm{Spf}(K[\![T]\!])$ of dimension 1.*

According to Theorem 4.3, if F is a differential equation of order 1 and $\gamma(T)$ a non-constant solution of $F = 0$, then the embedding dimension of X^∂ at γ is exactly 1. We deduce the following corollary:

Corollary 5.2. — *Let K be a field of characteritic zero. Let $F := F(y_0, y_1) \in K\{y\}$ be an irreducible differential polynomial and $X^\partial := \mathrm{Spec}(K\{y\}/[F])$. Let $\gamma(T) \in K[\![T]\!]$ be a non constant solution of the equation $F = 0$. Then the formal neighborhood $\widehat{X^\partial}_\gamma$ is isomorphic, as formal K-scheme, to a formal disk $\mathbb{D} = \mathrm{Spf}(K[\![T]\!])$ of dimension 1.*

Remark 5.3. — This corollary is also a result of Bourqui & Sebag in [1, Theorem 1.2].

5.1. Preliminary propositions

Proposition 5.4. — *Let K be a field of characteritic zero. Let $F \in K\{y\}$ be an irreducible polynomial of order n and $X^\partial := \mathrm{Spec}(K\{y\}/[F])$. Let $\gamma(T) \in K[\![T]\!]$ be a non constant solution of the equation $F = 0$. Let A be an object of the category $\mathbf{AlgLC}_K$ of the local K-algebras which are completion of local K-algebras. Then there is a map*

$$\iota_A \colon \mathrm{Hom}_{\mathbf{AlgLC}_K}(K[\![T]\!], A) \hookrightarrow \mathrm{Hom}_{\mathbf{AlgLC}_K}(\widehat{\mathcal{O}_{X^\partial,\gamma}}, A)$$

injective and functorial in A.

Proof. — We know from Proposition 2.12 that

$$\mathrm{Hom}_{\mathbf{AlgLC}_K}(\widehat{\mathcal{O}_{X^\partial,\gamma}}, A) = \{\gamma_A(T) \in \mathfrak{M}_A[\![T]\!] \mid F(\gamma(T) + \gamma_A(T)) = 0\}.$$

But, for every $a \in \mathfrak{M}_A$, we have that $\gamma(T+a)$ is solution of F, since F is autonomous, *i.e* has no dependency in T. In particular, if $\gamma(T)$ is not constant, we get a map

$$\begin{aligned} \mathfrak{M}_A &\longrightarrow \{\gamma_A(T) \in \mathfrak{M}_A[\![T]\!] \mid f(\gamma(T) + \gamma_A(T)) = 0\} \\ a &\longmapsto \gamma(T+a) - \gamma(T) \end{aligned}$$

injective and functorial in A. Yet we have a functorial bijection in A between $\mathfrak{M}_A$ and $\mathrm{Hom}_{\mathbf{AlgLC}_K}(K[\![T]\!], A)$. This gives the required result. □

Remark 5.5. — In particular, the statement also shows that the embedding dimension of X^∂ at a non-constant γ cannot be 0 (by considering $A = K[\epsilon]/(\epsilon)^2$).

Lemma 5.6. — *Let $\mathcal{C}$ be a category and B and C be objects of this category. Assume that, for every object A of the category $\mathcal{C}$, we have a map*

$$\iota_A \colon \mathrm{Hom}_{\mathcal{C}}(B, A) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(C, A)$$

injective and functorial in A. Then, there exists an epimorphism $\varphi\colon C \to B$.

Proof. — Let's take $A = B$, and denote $\varphi = \iota_B(\mathrm{id})$. We have to check that $\varphi\colon C \to B$ is an epimorphism.

Let D a object of the category $\mathcal{C}$ and $g_1, g_2 \in \mathrm{Hom}_{\mathcal{C}}(B, D)$ such that $g_1 \circ \varphi = g_2 \circ \varphi$. The functoriality of the inclusion in A gives us the following commutative diagram (for g_1):

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,B) \arrow[r, "\iota_B"] \arrow[d, "g_1\circ -"'] & \mathrm{Hom}_{\mathcal{C}}(C,B) \arrow[d, "g_1\circ -"] \\ \mathrm{Hom}_{\mathcal{C}}(B,D) \arrow[r, "\iota_D"] & \mathrm{Hom}_{\mathcal{C}}(C,D) \end{tikzcd}$$

We consider the image of $\mathrm{id} \in \mathrm{Hom}_{\mathcal{C}}(B,B)$ in the above diagram and we deduce that $g_1 \circ \iota_B(\mathrm{id}) = \iota_D(g_1)$, or, in other words, that $g_1 \circ \varphi = \iota_D(g_1)$. But $g_1 \circ \varphi = g_2 \circ \varphi$, so $\iota_D(g_1) = \iota_D(g_2)$. Since ι_D is injective, we deduce that $g_1 = g_2$, so φ is an epimorphism. □

Proposition 5.7. — *Let A be an object of the category $\mathbf{AlgLC}_K$. Let d be a non negative integer. Assume that there exists an epimorphism φ of the category $\mathbf{AlgLC}_K$ from A to $K[\![T_1, \dots, T_d]\!]$. Then φ is surjective.*

Proof. — Assume that the morphism $\varphi\colon A \to K[\![T_1,\dots,T_d]\!]$ is not surjective. Then, there exists an integer $i_0 \in \{1,\dots,d\}$ such that, for every $a \in A$, we have $\varphi(a) \neq T_{i_0}$. We denote I the ideal generated by $\varphi(\mathfrak{M}_A)$ in $K[\![T_1,\dots,T_d]\!]$ and $\mathfrak{m}$ the maximal ideal of $K[\![T_1,\dots,T_d]\!]$. Then $I \subsetneq \mathfrak{m}$ because $T_{i_0} \notin I$ and $I \subset \mathfrak{m}$.

We consider the K-algebra $B = K[\![T_1,\dots,T_d]\!]/I$. It's a local K-algebra which is complete (in particular B is the completion of B for the pre-adic topology defined by its maximal ideal). We consider $\pi\colon K[\![T_1,\dots,T_d]\!] \to K[\![T_1,\dots,T_d]\!]/I$ the quotient morphism. This morphism is local and continuous. We also consider

$$\mathrm{ev}_{T=0}\colon K[\![T_1,\dots,T_d]\!] \longrightarrow K[\![T_1,\dots,T_d]\!]/I$$

where $\mathrm{ev}_{T=0}$ is the morphism of K-algebra which sends T_i to 0, for every integer $i \in \{1,\dots,d\}$. It is also a local and continuous morphism of K-algebras.

Let $a \in A$, then $\varphi(a) = P(T_1,\dots,T_d)$ for a certain polynomial P. We denote $a_0 = P(0,\dots,0) \in K$. Then $\varphi(a-a_0) = P(T_1,\dots,T_d) - P(0,\dots,0) \in \mathfrak{m}$. Since φ is a local morphism, this implies that $a - a_0 \in \mathfrak{M}_A$ and hence that $\varphi(a-a_0) \in I$. Hence $\pi \circ \varphi(a-a_0) = 0$ et $\mathrm{ev}_{T=0} \circ \varphi(a-a_0) = 0$. We can see that $\pi \circ \varphi = \mathrm{ev}_{T=0} \circ \varphi$.

Since φ is an epimorphism, we should have $\pi = \mathrm{ev}_{T=0}$, which is not the case because $I \subsetneq \mathfrak{m}$. So φ is surjective. □

The following proposition is an immediate consequence of the propositions and Lemmas 5.4, 5.6 and 5.7.

Proposition 5.8. — *Let K be a field of characteristic zero. Let $F \in K\{y\}$ be an irreducible polynomial of order n and $X^\partial \coloneqq \mathrm{Spec}(K\{y\}/[F])$. Let $\gamma(T) \in K[\![T]\!]$ be a non-constant solution of the equation $F = 0$. Hence, there exists a surjective morphism of K-algebras $\mathcal{O}_{X^\partial,\gamma} \to K[\![T]\!]$. In other words, there exists a closed immersion $\mathbb{D}_K \to \mathrm{Spf}(\widehat{\mathcal{O}_{X,\gamma}})$.*

Lemma 5.9. — *Let R be a Noetherian ring and $\phi\colon R \to R$ be a surjective morphism. Then ϕ is bijective.*

Remark 5.10. — Let $d > 0$ be a non negative integer. We will use the previous lemma only in the case where $R = K[\![T_1,\dots,T_d]\!]$. In particular if there exists a surjective morphism of local K-algebras $\phi\colon K[\![T_1,\dots,T_d]\!] \to K[\![T_1,\dots,T_d]\!]$. Then ϕ is a bijection.

Proof. — The statement of this lemma is well-known, we remind here the proof.

The kernels of the iterates of ϕ form an accending chain of ideals of R

$$\mathrm{Ker}(\phi) \subset \mathrm{Ker}(\phi^2) \subset \cdots \subset \mathrm{Ker}(\phi^n) \subset \cdots$$

which becomes stable since R is Noetherian. In particular, there exists $m \in \mathbf{N}$ such that $\mathrm{Ker}(\phi^m) = \mathrm{Ker}(\phi^{m+1})$. Let $x \in \mathrm{Ker}(\phi)$. Since ϕ is surjective, then so is ϕ^m, there exists $y \in R$ such that $x = \phi^m(y)$. Hence $0 = \phi(x) = \phi^{m+1}(y)$. In particular $y \in \mathrm{Ker}(\phi^{m+1}) = \mathrm{Ker}(\phi^m)$. So $x = \phi(y) = 0$. $\square$

5.2. Proof of Theorem 5.1

We now have the elements to prove Theorem 5.1, we only need the following result:

Theorem 5.11. — *Let K be a field of characteristic zero. Let $F \in K\{y\}$ be an irreducible differential polynomial of order n and $X^\partial := \mathrm{Spec}(K\{y\}/[F])$. Let $\gamma(T) \in K[\![T]\!]$ be a non-constant solution of the equation $F = 0$. Let A be an object of the category $\mathbf{AlgLC}_K$. Assume that the embedding dimension of X^∂ at γ is at most d and that there exists a map*

$$\iota_A \colon \mathrm{Hom}_{\mathbf{AlgLC}_K}(K[\![T_1, \dots, T_d]\!], A) \hookrightarrow \mathrm{Hom}_{\mathbf{AlgLC}_K}(\widehat{\mathcal{O}_{X^\partial,\gamma}}, A)$$

injective and functorial in A.

Then the formal neighborhood $\widehat{X^\partial}_\gamma$ is isomorphic, as a formal K-scheme, to a formal disk $\mathbb{D} = \mathrm{Spf}(K[\![T_1, \dots, T_d]\!])$ of dimension d.

Proof. — From ι_A, we deduce from Lemma 5.6 and Proposition 5.7 that there exists a surjective morphism of local K-algebras

$$\varphi \colon \widehat{\mathcal{O}_{X^\partial,\gamma}} \longrightarrow K[\![T_1, \dots, T_d]\!].$$

Furthermore, as an upper bound of the embedding dimension is d, there exists a surjective morphsim $\pi \colon K[\![T_1, \dots, T_d]\!] \to \widehat{\mathcal{O}_{X^\partial,\gamma}}$ (according to Proposition 3.12). In particular, thanks to Lemma 5.9, we deduce that $\varphi \circ \pi$ is bijective. We deduce that π et φ are bijective, and hence that the formal neighborhood of $\widehat{X^\partial}_\gamma$ is isomorphic, as formal K-scheme, to a formal disk $\mathbb{D}^d_K = \mathrm{Spf}(K[\![T_1, \dots, T_d]\!])$ of dimension d. $\square$

Thanks to Theorem 5.11 we deduce Theorem 5.1, using Proposition 5.4.

5.3. Remark

Let $F \in K\{y\}$ a differential polynomial of order 2. Assume that there is no term in y_0 in F. Then $F \in K\{y\}$ is a differential polynomial of order 2 but without term in y_0. Let's define $G := F(0, y_0, y_1)$. Let $\gamma(T)$ be a solution of F which is a non-degenerated arc, then $\gamma'(T)$ is a solution of G such that $S_G(\gamma(T)) \neq 0$. Let A be an object of $\mathbf{AlgLC}_K$.

$$\begin{aligned}\mathrm{Def}([F], A) &= \{\gamma_A(T) \in \mathfrak{M}_A[\![T]\!] \mid f(\gamma(T) + \gamma_A(T)) = 0\} \\ &= \{\gamma_A(T) \in \mathfrak{M}_A[\![T]\!] \mid G(\gamma'(T) + \gamma_A'(T)) = 0\} \\ &= \mathfrak{M}_A \times \{\gamma_A(T) \in \mathfrak{M}_A[\![T]\!] \mid G(\gamma'(T) + \gamma_A(T)) = 0\}.\end{aligned}$$

Since G is of order 1, we have seen in Theorem 5.1 that the ring $(\widehat{K\{y\}/[G]})_{\gamma'}$ is isomorphic to $K[\![T]\!]$ in the category of $\mathbf{AlgLC}_K$. Hence

$$\{\gamma_A(T) \in \mathfrak{M}_A[\![T]\!] \mid G(\gamma'(T) + \gamma_A(T)) = 0\}$$

is naturally in bijection with $\mathfrak{M}_A$, and so the ring $(\widehat{K\{y\}/[F]})_\gamma$ is isomorphic to $K[\![T_1, T_2]\!]$ in the category $\mathbf{AlgLC}_K$.

We can generalize the previous considerations and make the following remark:

Remark 5.12. — Let $F \in K\{y\}$ be a differential polynomial of order n where the only terms that appears effectively in it are y_{n-1} and y_n. Let $\gamma(T)$ be a solution of F which is a non-degenerated arc. Hence the ring $(\widehat{K\{y\}/[F]})_\gamma$ is isomorphic to $K[\![T_1, T_2, \ldots, T_n]\!]$ in the category $\mathbf{AlgLC}_K$.

6. Differential arc scheme of differential equations of order 2

In this chapter, we will link Theorem 4.3 and the notion of essential singularity introduced in Section 1.1 for specific differential polynomials of order 2.

DEFINITION 6.1. — *Let $F \in K\{y\}$ of order n. We say that $F = 0$ is a* binomial equation *if there exists two monomials $M, N \in K\{y\}$ of order n such that $F = M - N$.*

Specifically, in the case of *binomial* differential polynomials of order 2, we will prove the following theorem:

Theorem 6.2. — *Let K be a field of characteristic zero. Let $F \in K\{y\}$ be an irreducible binomial differential polynomial of order 2. Let $d \geqslant 2$ be an integer such that $\gamma(T) = T^d$ is a solution of the differential polynomial F and $X^\partial = \operatorname{Spec}(K\{y\}/[F])$. If the perfect differential ideal $\{F\}$ has a essential singular component, then* $\operatorname{emb.dim}(\widehat{\mathcal{O}_{X^\partial,\gamma}}) = 1$.

6.1. Singular solutions of binomial ordinary differential polynomials of order 2 and tangent space

Assume that $F \in K\{y\}$ is an irreducible binomial differential polynomial of order 2. Such binomial equations are of one of the three following forms:

$$y_0^a - \alpha y_1^b y_2^c$$
$$y_1^b - \alpha y_0^a y_2^c$$
$$y_2^c - \alpha y_0^a y_2^b$$

with $a, b \in \mathbf{N}$, $c \in \mathbf{N}^\times$ and $\alpha \in K^\times$. Note that $\gamma(T) = T^d$ is a non-degenerated arc. Moreover, since $\gamma(T) = T^d$ is a solution of $F = 0$ the coefficient α is forced to be in $\mathbf{Q}_+$.

Assume that there exists $\gamma(T) = T^d$ a solution of the differential equation $F(y_0, y_1, y_2) = 0$. The general idea is to show that, in this particular case, we can calculate independently, and then link, the fact that the perfect differential ideal $\{F\}$ has a essential singular solution and the embedding dimension of the completion of the local ring of the differential arc scheme at γ.

Following the strategy of Section 4, we consider the linear equation associated with F given by :

$$\mathcal{L}(F,\gamma) := \sum_{j=0}^{2} \partial_{y_j}(F)(\gamma(T)) y_j = 0. \tag{6.1}$$

Proposition 4.15 allows us to conclude that the embedding dimension of F at γ is given by the dimension of the solution space of the linearization.

The relation $\partial_T(F(\gamma(T), \gamma'(T), \gamma''(T)) = 0$ implies that $\gamma'(T)/(d-1) = T^{d-1}$ is a solution $\mathcal{L}(F,\gamma)$.

We want to find a criterion to understand when the dimension of the K-vector space $\operatorname{Sol}(\mathcal{L}(F,\gamma))(K[\![T]\!])$ is 1 or 2.

Let us make some preliminary observations. For every $j \in \{0,1,2\}$, there exists $m_j \in \mathbf{N}$ and $\beta_j \in K$ such that $\partial_{y_j}(F)(\gamma(T)) = \beta_j T^{m_j}$. Thus, since T^{d-1} is a solution of $\mathcal{L}(F,\gamma) = 0$ we know that

$$\beta_0 T^{m_0} T^{d-1} + (d-1)\beta_1 T^{m_1} T^{d-2} + \beta_2 (d-1)(d-2) T^{m_2} T^{d-3} = 0. \tag{6.2}$$

For equality (6.2) to be true, it is necessary that

$$m_0 + d - 1 = m_1 + d - 2 = m_2 + d - 3,$$

which is equivalent to have $m_0 + 2 = m_1 + 1 = m_2$. Thus, if $F = 0$ is binomial, to compute the dimension of the solutions' space in $K[\![T]\!]$ of the equation (6.1) is the same as to compute the dimension of the solutions' space in $K[\![T]\!]$ of the equation

$$a_2 T^2 y_2 + a_1 T y_1 + a_0 y_0 = 0 \tag{6.3}$$

(with $a_i \in K$ and $a_2 \neq 0$). We recognize here *Euler's equation* (or Cauchy–Euler's equation), whose resolution is well-known. The solutions are parameterized by the associated *characteristic equation* which is, here, equal to:

$$a_2 x(x-1) + a_1 x + a_0 = a_2 \left(x^2 + \left(\frac{a_1}{a_2} - 1 \right) x + \frac{a_0}{a_2} \right) = 0. \tag{6.4}$$

Since $\gamma(T) = T^d$ is assumed to be a solution of F, then $\gamma'(T)$ is a solution of the differential polynomial (6.1). Then $x = d-1$ is solution of (6.4) and we want to find the second solution, that we denote by s and which can *a priori* be in $\mathbf{C}$. Note that

$$\begin{aligned} 1 - \frac{a_1}{a_2} &= d - 1 + s \\ \frac{a_0}{a_2} &= (d-1)s. \end{aligned} \tag{6.5}$$

Remark 6.3. — If $d = 2$ we have to be a little bit more careful with the previous arguments, but with small adjustments everything works, and we actually get an Euler equation with $a_0 = 0$.

The next step in the strategy is to study each of the three cases.

The case $y_2^c = \alpha y_1^b y_0^a$

Given $a, b \in \mathbf{N}$ and $c \in \mathbf{N}^{\times}$, we consider the differential polynomial $y_2^c - \alpha y_1^b y_0^a \in K\{y\}$, with $\alpha \in \mathbf{Q}_+$. Assume $\gamma(T) = T^d$, with $d \geqslant 3$, is a solution. In particular this implies

$$(d-2)c = b(d-1) + ad. \tag{6.6}$$

Since T^d is a solution of $y_2^c = \alpha y_1^b y_0^a$, we have:

$$\begin{aligned} \partial_{y_2}(F)(\gamma(T)) &= c(d(d-1))^{c-1} T^{(d-2)(c-1)} \\ \partial_{y_1}(F)(\gamma(T)) &= -\alpha b d^{b-1} T^{(d-1)(b-1)+da} \\ \partial_{y_0}(F)(\gamma(T)) &= -\alpha a d^b T^{(d-1)b+d(a-1)}. \end{aligned}$$

Thus $a_2 = c(d(d-1))^{c-1}$, $a_1 = -\alpha b d^{b-1}$ et $a_0 = -\alpha a d^b$ hence, we know that

$$\alpha = \frac{(d(d-1))^c}{d^b}.$$

So $\alpha \geqslant 0$. Furthermore

$$(d-1)s = \frac{-\alpha a d^b}{c(d(d-1))^{c-1}}.$$

In particular

$$s = -\frac{ad}{c}.$$

If $a = 0$, the equation can be written $y_2^c = \alpha y_1^b$, so $s = 0$. In this case, we note thanks to Theorem 1.9 that the differential polynomial has an essential singular solution if and only if $b \geqslant c$, but the condition (6.6), here $(d-2)c = b(d-1)$, is incompatible. Hence the embedding dimension is equal to 2, and there is no essential singular solution.

If $a \neq 0$, we deduce that $s < 0$ and the embedding dimension is equal to 1. In this case, the only possible essential singular component is $\{y_1\}$ and it is a component if and only if $b < c$. But, if $b \geqslant c$, the equality $(d-2)c = b(d-1) + ad$ is not possible, so the $\{F\}$ always has an essential singular component, and Theorem 6.2 is true in this case.

The case $y_0^a - \alpha y_1^b y_2^c$

Given $a, b \in \mathbf{N}$ and $c \in \mathbf{N}^\times$, we consider the differential polynomial $y_0^a - \alpha y_1^b y_2^c \in K\{y\}$, with $\alpha \in \mathbf{Q}_+$. Assume that $\gamma(T) = T^d$, with $d \geqslant 3$, is a solution. In particular this implies:

$$\begin{aligned} da &= b(d-1) + c(d-2) \\ \partial_{y_2}(F)(\gamma(T)) &= -\alpha c (d(d-1))^{c-1} d^b T^{(d-2)(c-1)+(d-1)b} \\ \partial_{y_1}(F)(\gamma(T)) &= -\alpha b d^{b-1} T^{(d-1)(b-1)+(d-2)c} \\ \partial_{y_0}(F)(\gamma(T)) &= a T^{d(a-1)}. \end{aligned}$$

Thus $a_2 = -\alpha c (d(d-1))^{c-1} d^b$, $a_1 = -\alpha b d^{b-1}$ and $a_0 = a$. Since T^d is a solution of $y_0^a = \alpha y_1^b y_2^c$, we know that

$$\alpha = \frac{1}{d^b (d(d-1))^c}.$$

Hence $\alpha \geqslant 0$. Furthermore

$$(d-1)s = \frac{a}{-\alpha c (d(d-1))^{c-1} d^b}.$$

In particular

$$s = -\frac{ad}{c}.$$

If $a = 0$, the equation becomes $1 = \alpha y_1^b y_2^c$ and $s = 0$. Hence, the low power theorem ensures that the perfect differential ideal $\{F\}$ has no essential singular component. In this case the embedding dimension is equal to 2 ans $\{F\}$ has no essential singular solution.

If $a \neq 0$, we have $s < 0$ and the embedding dimension of the equation is 1. The only component possible for the singular solution is $\{y_0\}$, and that's the case if and only if $a < b + c$. If we assume that $a \geqslant b + c$, we deduce, thanks to the equation $da = b(d-1) + c(d-2)$, that $-b - 2c \geqslant 0$ which is absurd. Hence $\{y_0\}$ is always an irreducible component.

The case $y_1^b - \alpha y_0^a y_2^c$

This case is the most diverse. Given $a, b \in \mathbf{N}$ and $c \in \mathbf{N}^\times$, we consider the differential polynomial $y_1^b - \alpha y_0^a y_2^c \in K\{y\}$, with $\alpha \in \mathbf{Q}_+$. Assume that $\gamma(T) = T^d$, with $d \geqslant 3$, is a solution. In particular, this implies:

$$(d-1)b = ad + (d-2)c$$

$$\partial_{y_2}(F)(\gamma(T)) = -\alpha c(d(d-1))^{c-1} T^{(d-2)(c-1)+ad}$$

$$\partial_{y_1}(F)(\gamma(T)) = bd^{b-1} T^{(d-1)(b-1)}$$

$$\partial_{y_0}(F)(\gamma(T)) = -\alpha a(d(d-1))^c T^{(d-2)c+d(a-1)}.$$

Hence $a_2 = -\alpha c(d(d-1))^{c-1}$, $a_1 = bd^{b-1}$ and $a_0 = -\alpha a(d(d-1))^c$. Since T^d is a solution of $y_1^b = \alpha y_0^a y_2^c$, we know that

$$\alpha = \frac{d^b}{(d(d-1))^c}.$$

Thus $\alpha \geqslant 0$. Furthermomre $(d-1)s \geqslant 0$ and

$$(d-1)s = \frac{\alpha a(d(d-1))^c}{\alpha c(d(d-1))^{c-1}} = \frac{ad(d-1)}{c}.$$

Hence

$$s = \frac{ad}{c}.$$

We will show that Theorem 4.3 is still true in this case.

Lemma 6.4. — *Let $(a, b, c, d) \in \mathbf{N}^2$ be integers. Assume that $d \geqslant 2$ and that $b < c$. Assume, furthermore, that the equality $(d-1)b = ad + (d-2)c$ holds. Then ad/c is not an integer.*

Proof. — The integer d verifies the equality $(d-1)b = ad + (d-2)c$, which can be written as

$$(d-1)(b-c) = ad - c.$$

If we assume that $b < c$ then $ad - c < 0$ because $d - 1 \geqslant 1$. Thus $ad/c < 1$, so ad/c is not an integer. □

If the perfect differential ideal $\{F\}$ has an essential singular component, then it is $\{y_1\}$. And $\{y_1\}$ is an essential singular component of $\{F\}$ if and only if $b < c$ according to Theorem 1.9.

Proposition 6.5. — *Let $F = y_1^b - \alpha y_0^a y_2^c$ and let $d \geqslant 2$ verifying the equation $(d-1)b = ad + (d-2)c$. Assume that $\gamma(T) = T^d$ is a solution of F. If $\{y_1\}$ is a component of F then the embedding dimension of the completion of the local ring of the differential arc scheme associated to F to $\gamma(T) = T^d$ is 1.*

Proof. — If $\{y_1\}$ is an irreducible component of $\{F\}$ then, according to the low power theorem (see Theorem 1.9), we have that $b < c$ and thanks to Lemma 6.4 we have that ad/c is not an integer. Hence the embedding dimension is equal to 1. □

Remark 6.6. — In general, the converse of Theorem 6.2 is false. In other words, there exists binomial differential polynomials whose embedding dimension is 1, but that have no essential singular component. The following examples highlight this observation:

(1) $F = y_1^5 - 3/16 y_0^2 y_2^4$ and $\gamma(T) = T^3$, then $ad/c = 3/2$,
(2) $F = y_1^7 - 3^2/2^5 y_0^3 y_2^5$ and $\gamma(T) = T^3$ then $ad/c = 9/5$,
(3) $F = y_1^8 - 2^5 y_0^4 y_2^3$ and $\gamma(T) = T^2$ then $ad/c = 8/3$,
(4) $F = y_1^{10} - 2^6 y_0^5 y_2^4$ and $\gamma(T) = T^2$ then $ad/c = 5/2$.

Remark 6.7. — The following code can be used to compute examples of equations for which the embedding dimension is 1 but such that F has no essential singular component.

```
#Compute the list of [a,b,c,d,ad/c] and return it if the
    embedding dimension is 1 but f has no components.
def Liste(n):
    L=[]
    for a in range (1,n):
        for b in range (1,n):
            for c in range (1,n):
                if a+c-b!=0:
                    d=(2*c-b)/(a+c-b)
                    if int(d)==d and d>1 and mod(a*d,c)!=0 and
                        b>=c:
                        f=a*d/c
                        L=L+[[a,b,c,d,f]]
    return L
```

Remark 6.8. — An interesting sub-case of the case $y_1^b - \alpha y_0^a y_2^c$ is this where the equation is homogeneous, that is if it is of the form $y_1^{2a} - \alpha y_0^a y_2^a$. In this case, the deformations can be fully described.

Let $d \geqslant 2$ be an integer. Then $\gamma(T) = T^d$ is solution of the equation $F = y_1^{2a} - \frac{d}{(d(d-1))^a} y_0^a y_2^a$. Let A be a ring in the category $\mathbf{AlgLC}_k$. Let $\beta_1, \beta_2 \in \mathfrak{M}_A$. Then $\gamma_{\beta_1,\beta_2} = (1+\beta_1)(T+\beta_2)^d$ is a deformation of $\operatorname{Def}_\gamma([F], A)$. This implies the existence of a map

$$\iota_A \colon \operatorname{Hom}_{\mathbf{AlgLC}_k}(K[\![T_1, T_2]\!], A) \hookrightarrow \operatorname{Hom}_{\mathbf{AlgLC}_k}(\widehat{K\{y\}/[F]}_\gamma, A)$$

injective and functorial in A. Furthermore, thanks to Theorem 4.3, we know that the embedding dimension of the ring $\widehat{K\{y\}/[F]}_\gamma$ is at most 2.

Hence thanks to Theorem 5.11, we deduce that the ring $\widehat{K\{y\}/[F]}_\gamma$ is isomorphic, in the category $\mathbf{AlgLC}_k$, to $K[\![T_1, T_2]\!]$.

6.2. Examples

Examples 6.9. — Examples in the case $y_2^c = \alpha y_1^b y_0^a$.

(1) Let $F = y_2^3 - 6^3 y_0$. According to the low power Theorem 1.9, This equation has an essential singular component $\{y_0\}$. According to Theorem 6.2, we deduce that in the neighborhood of $\gamma(T) = T^3$, we have $\widehat{\mathcal{O}_{X^\partial,\gamma}} \simeq K[\![T]\!]$.

(2) Let $F = 3y_2^5 - 6^5 y_0 y_1$. This equation has an essential singular component $\{y_1\}$. According to Theorem 6.2, we deduce that in the neighborhood of $\gamma(T) = T^3$, we have $\widehat{\mathcal{O}_{X^\partial,\gamma}} \simeq K[\![T]\!]$.

(3) Let $F = 2^5 y_0^2 - y_1^4 y_2$. This equation has an essential singular component $\{y_0\}$. According to Theorem 6.2, we deduce that in the neighborhood of $\gamma(T) = T^2$, we have $\widehat{\mathcal{O}_{X^\partial,\gamma}} \simeq K[\![T]\!]$.

(4) Let $F = 1600 y_0 y_1 - y_2^3$. This equation has an essential singular component $\{y_1\}$. According to Theorem 6.2, we deduce that in the neighborhood of $\gamma(T) = T^5$, we have $\widehat{\mathcal{O}_{X^\partial,\gamma}} \simeq K[\![T]\!]$.

(5) $F = 3^5 \times 2^6 y_0 y_1^2 - y_2^5$. This equation has an essential singular component $\{y_1\}$. According to Theorem 6.2, we deduce that in the neighborhood of $\gamma(T) = T^4$, we have $\widehat{\mathcal{O}_{X^\partial,\gamma}} \simeq K[\![T]\!]$.

Examples 6.10. — Examples in the case $y_0^a = \alpha y_1^b y_2^c$.

(1) Let $F = 18 y_0 - y_1 y_2$. En vertu du low power Theorem 1.9, This equation has an essential singular component $\{y_0\}$. According to Theorem 6.2, we deduce that in the neighborhood of $\gamma(T) = T^3$, we have $\widehat{\mathcal{O}_{X^\partial,\gamma}} \simeq K[\![T]\!]$.

(2) Let $F = y_2y_1^2 - 4^3 \times 3y_0^2$. This equation has an essential singular component $\{y_0\}$. According to Theorem 6.2, we deduce that in the neighborhood of $\gamma(T) = T^4$, we have $\widehat{\mathcal{O}_{X^\partial,\gamma}} \simeq K[\![T]\!]$.

(3) Let $F = 8y_0 - y_1^2y_2$. This equation has an essential singular component $\{y_0\}$. According to Theorem 6.2, we deduce that in the neighborhood of $\gamma(T) = T^2$, we have $\widehat{\mathcal{O}_{X^\partial,\gamma}} \simeq K[\![T]\!]$.

Examples 6.11. — Examples in the case $y_1^b - \alpha y_0^a y_2^c$.

(1) Let $F = 2y_1^2 - 3y_0y_2$. Thanks to the low power Theorem 1.9, we know that this equation has no essential singular component. We want to study the deformation of F at neighborhood of $\gamma(T) = T^3$.

$$\partial_{y_0}(F) = -3y_2 = -18T$$
$$\partial_{y_1}(F) = 4y_1 = 12T^2$$
$$\partial_{y_2}(F) = -3y_0 = -3T^3.$$

The linearization of F is given by

$$-3T^2y_2 + 12Ty_1 - 18y_0.$$

The associated characteristic equation is

$$-3x^2 + 15x - 18 = 0.$$

This equation has two solutions which are 2 et 3. Furthermore, it can be seen that, given an object A of the category $\mathbf{AlgLC}_k$ then, for every $\beta_1, \beta_2 \in \mathfrak{M}_A$, the deformation $\gamma_{\beta_1,\beta_2} = (1+\beta_1)(T+\beta_2)^d$ is in $\mathrm{Def}_\gamma([F], A)$. We deduce, thanks to the Theorems 4.3 and 5.11 that $\widehat{\mathcal{O}_{X^\partial,\gamma}}$ is isomorphic to $K[\![T_1, T_2]\!]$ in the category $\mathbf{AlgLC}_k$.

(2) Let $F = y_1^4 - 2^3y_0^2y_2$. This equation has no essential singular component. We want to study the deformation of F at neighborhood of $\gamma(T) = T^2$.

$$\partial_{y_0}(F) = -2^4y_0y_2 = -2^5T^2$$
$$\partial_{y_1}(F) = 2^2y_1^3 = 2^5T^3$$
$$\partial_{y_2}(F) = -2^3y_0^2 = -2^3T^4.$$

The linearization of F is given by

$$T^2y_2 - 4Ty_1 + 4y_0.$$

The associated characteristic equation is

$$x^2 - 5x + 4 = 0.$$

This equation has two solutions which are 1 et 4. We deduce that the embedding dimension of the ring $\widehat{\mathcal{O}_{X^\partial,\gamma}}$ is equal to 2. Furthermore, the ring $\widehat{\mathcal{O}_{X^\partial,\gamma}}$ is isomorphic to $K[\![T_1, T_2]\!]/I$, for a certain ideal I, in the category $\mathbf{AlgLC}_k$. Furthermore, a computation can show that $I \neq 0$ since, if we study the solutions of the linearization in $K[\epsilon](\epsilon^8)$, we can show that $T_1^4 T_2^3 \in I/I^8$.

(3) Let $F = 2y_1^5 - 3^4 y_0^3 y_2$. This equation has no essential singular component. We want to study the deformation of F at neighborhood ofe $\gamma(T) = T^3$.

$$\begin{aligned}\partial_{y_0}(F) &= -3^5 y_0^2 y_2 = -3^6 \times 2T^7\\ \partial_{y_1}(F) &= 2 \times 5y_1^4 = 2 \times 3^4 \times 5T^8\\ \partial_{y_2}(F) &= -3^4 y_0^3 = -3^4 T^9.\end{aligned}$$

The linearization of F is given by

$$T^2 y_2 - 10Ty_1 + 18y_0.$$

The associated characteristic equation is

$$x^2 - 11x + 18 = 0.$$

This equation has two solutions which are 2 et 9. We deduce that the embedding dimension of the ring $\widehat{\mathcal{O}_{X^\partial,\gamma}}$ is equal to 2. Furthermore, the ring $\widehat{\mathcal{O}_{X^\partial,\gamma}}$ is isomorphic to $K[\![T_1, T_2]\!]/I$, for a certain ideal I, in the category $\mathbf{AlgLC}_k$. Furthermore, a computation can show that $I \neq 0$ since, if we study the solutions of the linearization $K[\epsilon](\epsilon^9)$, we can show that $T_1^6 T_2^2 \in I/I^9$.

(4) Let $F = 2^{3a} y_1^{3a} - y_0^a y_2^{3a}$. This equation has an essential singular component $\{y_1\}$. According to Theorem 6.2, we deduce that in the neighborhood of $\gamma(T) = T^3$, on a $\widehat{\mathcal{O}_{X^\partial,\gamma}} \simeq K[\![T]\!]$.

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