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EQUI-SINGULARITY OF REAL FAMILIES AND LIPSCHITZ–KILLING CURVATURE DENSITIES AT INFINITY

by Nicolas DUTERTRE & Vincent GRANDJEAN (*)

Abstract. — Fix an o-minimal structure expanding the ordered field of real numbers. Let $(W_{\mathbf{y}})_{\mathbf{y}\in\mathbb{R}^s}$ be a definable family of closed subsets of $\mathbb{R}^n$ whose total space $W = \cup_{\mathbf{y}} W_{\mathbf{y}} \times \mathbf{y}$ is a closed connected C^2 definable sub-manifold of $\mathbb{R}^n \times \mathbb{R}^s$. Let $\varphi : W \to \mathbb{R}^s$ be the restriction of the projection to the second factor.

After defining $K(\varphi)$, the set of generalized critical values of φ, showing that they are closed and definable of positive codimension in $\mathbb{R}^s$, contain the bifurcation values of φ and are stable under generic plane sections, we prove that all the Lipschitz–Killing curvature densities at infinity $\mathbf{y} \mapsto \kappa_i^\infty(W_{\mathbf{y}})$ are continuous functions over $\mathbb{R}^s \setminus K(\varphi)$. When W is a C^2 definable hypersurface of $\mathbb{R}^n \times \mathbb{R}^s$, we further obtain that the symmetric principal curvature densities at infinity $\mathbf{y} \mapsto \sigma_i^\infty(W_{\mathbf{y}})$ are continuous functions over $\mathbb{R}^s \setminus K(\varphi)$.

Résumé. — On fixe une structure o-minimale qui étend le corps ordonné des nombres réels. Soit $(W_{\mathbf{y}})_{\mathbf{y}\in\mathbb{R}^s}$ une famille définissable de sous-ensembles fermés de $\mathbb{R}^n$ dont l'espace total $W = \cup_{\mathbf{y}} W_{\mathbf{y}} \times \mathbf{y}$ est une sous-variété définissable connexe et fermée de $\mathbb{R}^n \times \mathbb{R}^s$ de classe C^2. Soit $\varphi : W \to \mathbb{R}^s$ la restriction de la projection sur le second facteur.

Après avoir défini $K(\varphi)$, l'ensemble des valeurs critiques généralisées de φ, montré qu'elles forment un sous-ensemble définissable fermé de codimension non-nulle de $\mathbb{R}^s$, contiennent les valeurs de bifurcations de φ et sont stables par section plane générique, nous montrons que toutes les densités à l'infini des courbures de Lipschitz–Killing $\mathbf{y} \mapsto \kappa_i^\infty(W_{\mathbf{y}})$ sont des fonctions continues sur $\mathbb{R}^s \setminus K(\varphi)$. Quand W est une hypersurface définissable de $\mathbb{R}^n \times \mathbb{R}^s$ de classe C^2, nous obtenons de plus que les densités à l'infini des courbures symétriques principales $\mathbf{y} \mapsto \sigma_i^\infty(W_{\mathbf{y}})$ sont des fonctions continues sur $\mathbb{R}^s \setminus K(\varphi)$.

Keywords: Real equi-singularity, Lipschitz–Killing curvatures, the Malgrange–Rabier condition.
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1. Introduction

Let $(W_{\mathbf{y}})_{\mathbf{y}\in P}$ be a family of subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ with parameter space P. This is the family of the (projections onto $\mathbb{R}^n$ or $\mathbb{C}^n$ of the) levels of the projection φ of the total space of the family $W := \cup_{\mathbf{y}\in P} W_{\mathbf{y}} \times \mathbf{y}$ onto the parameter space

$$\varphi : W \longrightarrow P.$$

Assuming that φ is continuous, this family is *locally trivial at* $\mathbf{c} \in P$ if φ induces a trivial topological fibre bundle structure over a neighbourhood of $\mathbf{c}$, with model fibre $W_{\mathbf{c}}$ for which φ is the projection-onto-the-base mapping. Similarly, a C^k mapping $F : X \to P$, with $k \geqslant 1$, is *locally trivial at* $\mathbf{c} \in P$ if it induces a C^{k-1} trivial fibre bundle structure over a neighbourhood of $\mathbf{c}$, with model fibre $F^{-1}(\mathbf{c})$. A proper and submersive F induces a locally trivial fibre bundle over P, by Ehresmann's Fibration Theorem [18]. A value at which F is not locally trivial is a *bifurcation value.* Critical values are bifurcation values.

Whenever the mapping or the family is rigid/regular in some explicit sense, bifurcations values are rare, e.g. real and complex polynomial functions admit finitely many bifurcation values [44]. Regular bifurcation values are even rarer and are hard to detect. When X and P are affine spaces, sufficient conditions for F or W to be locally trivial at a regular value already exist. Among others are *Malgrange–Rabier condition* [26, 38, 39], $\mathbf{t}$*-regularity* [10, 40, 45, 46], ρ*-regularity* [10, 32, 34], *spherical-ness at infinity* [7, 8, 17]. Any such regularity condition at infinity at a given value compels the behaviour at infinity of the nearby levels to be "tame".

An interesting task, related to equi-singularity theory, is to find numerical criteria characterizing these regularity conditions at infinity. For instance, following Teissier's works on plane sections and polar invariants [41, 42, 43], Tibăr produced, for each parameter t of a complex family of affine hypersurfaces $(\{\mathbf{x} \in \mathbb{C}^n : F(\mathbf{x}, t) = 0\})_{t\in\mathbb{C}}$, where $F : \mathbb{C}^n \times \mathbb{C} \mapsto \mathbb{C}$ is a polynomial, a polar-like invariant vector $(\alpha^*(t))$ with integer values [45], whose local constancy at a value c is equivalent to $\mathbf{t}$-regularity nearby c. (See also [37, 47] for earlier special occurrences of equi-singularity numerical criteria for complex polynomials.)

Possible avatars, in the real setting, of the previously mentioned local (complex) polar invariants are integrals of Lipschitz–Killing curvatures. Since in general such integrals, as functions over the parameters of a given real family W, do not take only isolated values, we should instead look for the continuity of such functions. A first, yet essential, step consists of chasing sufficient conditions: Given $\mathbf{R}$ a regularity condition at infinity, can we produce a vector valued mapping, defined over the parameter space of the family W, which would be continuous in a neighbourhood of a value $\mathbf{c}$ once the family W would be $\mathbf{R}$-regular at $\mathbf{c}$?

Our approach for the results presented here feeds on two facts: (i) The stability of $\mathbf{t}$-regularity of complex polynomial functions by generic plane sections [45]; (ii) The general results about Lipschitz–Killing curvatures/measures of tame sets of the first named author [14, 15, 16] rely heavily on generic plane sections. In this paper, we produce a vector valued mapping, with components built from Lipschitz–Killing curvatures/measures, and show it is continuous at any regular value which is also Malgrange–Rabier regular.

We treat the more general context of definable families of subsets of $\mathbb{R}^n$ whose total space is a closed sub-manifold. It is sufficient to work with a connected total space, which we assume. Let be given an o-minimal structure expanding the ordered field of real numbers. Consider the following C^2 definable mapping:

$$\varphi : W \longrightarrow \mathbb{R}^s,\ (\mathbf{x},\mathbf{y}) \longmapsto \mathbf{y}$$

defined over the definable closed connected C^2 sub-manifold W of $\mathbb{R}^n \times \mathbb{R}^s$ with $\dim W \geqslant s$. Since, by definition $\varphi^{-1}(\mathbf{y}) = W_{\mathbf{y}} \times \mathbf{y}$, we are interested in the local triviality properties of the definable family $(W_{\mathbf{y}})_{\mathbf{y}\in\mathbb{R}^s}$ of closed subsets of $\mathbb{R}^n$ nearby regular values (of φ). Passing to the graph of the mapping F, the looked for properties for F are in a simple and explicit correspondence with those of φ, as checked here. The choice of C^2 is arbitrary. Everything we will do have analogues in C^k-regularity, $2 \leqslant k \leqslant \infty$.

The results of this paper are of two sorts: (I) those about Malgrange–Rabier regularity condition (see Definition 6.1) later on shortened as (MR), and (II) those about Lipschitz–Killing curvature densities at infinity.

(I) It is simpler to describe (MR)-regularity in the case of function (i.e. $s = 1$): *A value c is (MR)-regular for φ if there exists a positive constant M_c such that for any sequence $(\mathbf{w}_k)_k$ of W such that* (i) $|\mathbf{w}_k| \to \infty$, (ii) $\varphi(\mathbf{w}_k) \to c$, then

$$k \gg 1 \implies |\mathbf{w}_k| \cdot |\nabla\varphi(\mathbf{w}_k)| \geqslant M_c.$$

The set of critical values of φ is $K_0(\varphi)$ and that of non (MR)-regular values is $K_\infty(\varphi)$. We shall proof in this most general context the following results:

Theorem I.

(1) *$K_\infty(\varphi)$ is definable;*

(2) *$K(\varphi) := K_0(\varphi) \cup K_\infty(\varphi)$ is closed, definable (Lemma 6.4) and of positive codimension (Theorem 7.5);*
(3) *the family $(W_{\mathbf{y}})_{\mathbf{y} \notin K(\varphi)}$ is C^1 locally trivial at each of its point (Theorem 8.2), from which we deduce $Bif(\varphi) \subset K(\varphi)$ (Corollary 8.6);*
(4) *(MR)-condition at a regular value is stable by generic plane sections (Theorem 11.2).*

If all known occurrences of statements (1), (2) and (3) of Theorem I are special cases of the present setting, point (4) is new in the real setting, and outside the case of complex polynomial functions.

(II) Let Z be a closed connected C^2 sub-manifold of $\mathbb{R}^n$ of dimension d (equipped with the restriction of the Euclidean metric tensor). The other objects we will consider are the Lipschitz–Killing curvature densities at infinity of Z defined as

$$\kappa_i^\infty(Z) := \lim_{R \to +\infty} \frac{1}{R^{d-i}} \int_{Z \cap \mathbf{B}_R^n} K_i(Z, \mathbf{x}) \, \mathrm{d}\mathbf{x}, \quad \text{for} \quad i = 0, \ldots, d,$$

where $K_i(Z, \mathbf{x})$ is the i-th Lipschitz–Killing curvature of Z at $\mathbf{x}$. Note that $K_i(Z, \mathbf{x}) = 0$ if i is odd or $\geqslant d+1$. When Z is the hypersurface $\{f = 0\}$, oriented by $-\nabla f|_Z$, let $\sigma_i^Z(\mathbf{x})$ be the i-th symmetric function of the principal curvatures of Z at $\mathbf{x}$. We further define the symmetric principal curvature densities at infinity of Z as

$$\sigma_i^\infty(Z) := \lim_{R \to +\infty} \frac{1}{R^{n-1-i}} \int_{Z \cap \mathbf{B}_R^n} \sigma_i^Z(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \quad 0 \leqslant i \leqslant n-1.$$

When i is even $2\sigma_i^\infty(Z) = \kappa_i^\infty(Z)$, but when i is odd in general $\sigma_i^\infty(Z)$ is not identically null. Considering Corollary 8.3, we can reduce to the case where $W_{\mathbf{y}}$ is connected for each $\mathbf{y} \notin K(\varphi)$. A concise way to present our main results Theorem 14.3 and Theorem 16.2 is as follows:

Theorem II. — *Assume that $W_{\mathbf{y}}$ is connected at each $\mathbf{y} \notin K(\varphi)$. For each $i = 0, \ldots, \dim W - s$, the function $\mathbf{y} \mapsto \kappa_i^\infty(W_{\mathbf{y}})$ is continuous over $\mathbb{R}^s \setminus K(\varphi)$. When the $W_{\mathbf{y}}$'s are furthermore hypersurfaces, for each $i = 0, \ldots, n-1$, the function $\mathbf{y} \mapsto \sigma_i^\infty(W_{\mathbf{y}})$ is continuous over $\mathbb{R}^s \setminus K(\varphi)$.*

Theorem II is the continuation of the results of the second named author [20], stating that the following total Gauss–Kronecker curvature functions, associated with a C^2 definable function $f : \mathbb{R}^n \to \mathbb{R}$, have at most finitely many discontinuities

$$y \longmapsto GK(y) := \int_{f^{-1}(y)} \kappa_y(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad \text{and} \quad y \longmapsto |GK|(y) := \int_{f^{-1}(y)} |\kappa_y(\mathbf{x})|\,\mathrm{d}\mathbf{x},$$

where $\kappa_y := \sigma_{n-1}^{f^{-1}(y)}$, and of our recent result [17], where continuity of GK and $|GK|$ is proved at any regular value at which the function is also spherically regular at infinity. It is worth mentioning here the recent paper [11] investigating sufficient conditions of equi-singularity at infinity in terms of topological properties of set valued mappings over the levels of the given mapping F. Theorems I and II are to be associated with the results of [3, 4, 36] about the continuity of local Lipschitz–Killing invariants along Whitney and Verdier strata of definable subsets.

The paper is organized as follows: Section 2 introduces notations and a few reminders used in what follows. Section 3 lists a few facts about definability in order to provide some exhaustive-ness. Section 4 to Section 8 present material for Theorem I. We have given detailed proofs of facts (1) to (3). Our treatment uses heavily the Rabier number (Definition 4.1), and Lemma 4.3 explains why dealing with families includes the case of mappings. The proof of Theorem 8.2 following Rabier's point of view [39] is of interest in its own, and so is Remark 8.4. Section 9 will briefly deal with a trivialisation result of the sub-level family associated with a family of hypersurfaces (Proposition 9.1). Section 10 reviews in an explicit fashion properties on the family of links at infinity, which is a central tool to deal with the Lipschitz–Killing measures/curvatures. We give also proofs of Proposition 10.2 and of the lesser known Proposition 10.3. Section 11 takes care of point (4) of Theorem I. Theorem 11.2 is new and is one of the two cornerstones over which will be established Theorem II. Section 12 to Section 16 are devoted to present all the necessary material around Lipschitz–Killing measures/curvatures to obtain Theorem II. The second cornerstone needed to produce Theorem II is the Gauss–Bonnet type Theorem 12.3 (from the first author [15, 16]).

2. Miscellaneous

Let $\mathbb{R}_{\geqslant 0} := [0, +\infty[$ and $\mathbb{R}_{>0} :=]0, +\infty[$.

We denote by $\mathbf{0}$ the origin (or the null vector) of any vector space or subspace of dimension at least two.

The Euclidean unit closed ball of $\mathbb{R}^q$ centred at the origin is $\mathbf{B}^q$. If $\mathbf{w}$ is a point of $\mathbb{R}^q$, the Euclidean closed ball of radius r and centre $\mathbf{w}$ is $\mathbf{B}^q(\mathbf{w}, r)$. When $\mathbf{w}$ is just the origin we write simply $\mathbf{B}^q_r$.

The Euclidean unit sphere of $\mathbb{R}^q$ centred at the origin is $\mathbf{S}^{q-1}$. The Euclidean sphere of radius r centred at the origin is $\mathbf{S}_r^{q-1}$.

The Grassmann manifold of vector k-planes of $\mathbb{R}^q$ is $\mathbf{G}(k,q)$. We will sometimes write $\mathbf{P}^{q-1}$ for $\mathbf{G}(1,q)$. We will use the same notation P to mean either a point of $\mathbf{G}(k,q)$, or the corresponding vector subspace of $\mathbb{R}^q$.

Any real vector space $\mathbb{R}^q$ comes equipped with the Euclidean metric, and associated scalar product $\langle -,- \rangle$ and norm $|-|$. Any vector subspace E of $\mathbb{R}^q$ turns into an Euclidean space when restricting the Euclidean structure. The unit sphere of E is $\mathbf{S}(E)$ and the orthogonal of E in $\mathbb{R}^q$ is $E^\perp$.

Given a sequence $(\mathbf{w}_k)_k$ of $\mathbb{R}^q$, we will write $\mathbf{w}_k \to \infty$ to mean that $|\mathbf{w}_k| \to +\infty$ as k goes to $+\infty$.

Given any C^1 sub-manifold S of the Euclidean space $\mathbb{R}^q$, the tangent bundle TS is a sub-vector bundle of $T\mathbb{R}^q|_S$, therefore is equipped with the restriction to TS of the Riemannian metric tensor over $T\mathbb{R}^q|_S$.

Let X, Y be two connected C^1 sub-manifolds of $\mathbb{R}^q$. The pair (X,Y) is *Whitney (a)-regular at the point $\mathbf{y}$ of Y*, if (i) Y satisfies *the frontier condition $Y \subset \operatorname{clos}(X) \setminus X$*, and (ii) the following condition holds true: *For any sequence $(\mathbf{x}_k)_k$ of X such that $\mathbf{x}_k \to \mathbf{y}$ and $T_{\mathbf{x}_k}X \to P \in \mathbf{G}(\dim X, q)$, we have that $T_{\mathbf{y}}Y \subset P$.* Let $f : X \cup Y \mapsto \mathbb{R}$ be a continuous function such that each restriction $f|_X$ and $f|_Y$ is C^1. The function f satisfies *Thom condition $(\mathbf{a}_{\rm rel})$ at the point $\mathbf{y}$ of Y* (also called *Thom $(\mathbf{a}_{\rm rel})$-regular*) if the pair (X,Y) is Whitney (a)-regular at $\mathbf{y}$ and the following conditions are verified: (i) the functions $f|_X$ and $f|_Y$ have constant ranks, and (ii) *For any sequence $(\mathbf{x}_k)_k$ of X such that $\mathbf{x}_k \to \mathbf{y}$ and $\ker D_{\mathbf{x}_k}(f|_X) \to T \in \mathbf{G}(\dim X - 1, q)$, we have that*

$$\ker D_{\mathbf{y}}(f|_Y) \subset T.$$

Let $a \in \overline{\mathbb{R}} := [-\infty, +\infty]$. Let $f, g : (I, a) \to \mathbb{R}$ be two germs of real functions at a, where I is any interval with non-empty interior whose frontier in $\overline{\mathbb{R}}$ contains a. We use the following notations:

(i) $f \sim g \iff \lim_a \dfrac{f}{g} \in \mathbb{R}^*$,

(ii) $f \simeq g \iff \lim_a \dfrac{f}{g} = 1$,

(iii) $|f| \ll |g| \iff \lim_a \dfrac{|f|}{|g|} = 0$.

If $f, g : (I, a) \to \mathbb{R}^s$, for $s \geqslant 2$, we will write

(iv) $f \sim g \iff |f| \sim |g|$,

(v) $f \simeq g \iff |f| \simeq |g|$.

3. On o-minimal structures

We collect here a very few definitions and facts about o-minimal structures and definability we will use in the sequel of this paper (see [12, 13] for a proper treatment). We adopt a point of view close to those of [6, 25].

An *o-minimal structure* $\mathcal{M}$ *expanding the ordered field of real numbers* $(\mathbb{R}, +, \cdot, >)$ is a collection $(\mathcal{M}_q)_{q\in\mathbb{N}}$, where each $\mathcal{M}_q$ is a family of subsets of $\mathbb{R}^q$ satisfying the following axioms:

(1) For each $q \in \mathbb{N}$, the family $\mathcal{M}_q$ is a boolean sub-algebra of subsets of $\mathbb{R}^q$.
(2) For any pair of subsets $A \in \mathcal{M}_p$ and $B \in \mathcal{M}_q$, then $A \times B \in \mathcal{M}_{p+q}$.
(3) Let $\pi : \mathbb{R}^{q+1} \to \mathbb{R}^q$ be the projection on the first q factors. Given any subset A of $\mathcal{M}_{q+1}$, its projection $\pi(A)$ is a subset lying in $\mathcal{M}_q$.
(4) The algebraic subsets of $\mathbb{R}^q$ belong to $\mathcal{M}_q$.
(5) The family $\mathcal{M}_1$ consists exactly of the finite unions of points and intervals.

Those points imply that the smallest o-minimal structure is the structure of the semi-algebraic subsets, thus contained in any other one.

Assume that such an o-minimal structure $\mathcal{M}$ is given for the rest of this paper.

A subset A of $\mathbb{R}^q$ is a *definable subset* shortened as *definable* (*in the given o-minimal structure*), if $A \in \mathcal{M}_q$.

Let A be a subset of $\mathbb{R}^q$, a mapping $A \mapsto \mathbb{R}^r$ is *definable* if its graph is a definable subset of $\mathbb{R}^q \times \mathbb{R}^r$.

Let A be a definable subset of $\mathbb{R}^q$ and let B be a definable subset of A. The pair (A, B) admits a *definable Whitney* (a)*-regular* C^k *stratification*, namely the following result holds true:

Theorem (see for instance [9, 13, 33, 35, 48]). — *For each positive integer k, there exists a finite partition of A into connected C^k sub-manifolds, called strata with the following properties:* (i) *Each stratum is definable;* (ii) *B is a union of strata;* (iii) *Each pair (X, Y) such that* $\dim X > \dim Y$ *either verifies the frontier condition $Y \subset \operatorname{clos}(X) \setminus X$, or the intersection $Y \cap \operatorname{clos}(X)$ is empty;* (iv) *Each pair of strata (X, Y) satisfying the frontier condition is Whitney (a)-regular at each point of Y.*

Let $f : A \to \mathbb{R}$ be a definable mapping. Let B be a definable subset of A. The function f admits a *definable* $(\mathbf{a}_{\mathrm{rel}})$*-regular* C^k *stratification*, that is the following statement holds true:

Theorem (see for instance [1, 21, 27, 28, 30, 31]). — *For each positive integer k, there exists a Whitney (a)-regular C^k stratification of the pair (A, B) satisfying the following additional property:* (v) *For each pair of strata (X, Y), the restriction $f|_{X \cup Y}$ is $(\mathbf{a}_{\mathrm{rel}})$-regular at every point of Y.*

The notions of definability of subsets or mappings germ-ify along any definable subset. In particular, the germ of a mapping $(\mathbb{R}, +\infty) \to \mathbb{R}^s$ is *definable at* $+\infty$ if it admits a representative which is definable.

Let I, a as above. Let I^+ be $(I \cap]a, +\infty[, a)$ if $a < +\infty$, or $(\mathbb{R}, +\infty)$ when $a = +\infty$. Similarly let I^- be $(I \cap] - \infty, a[, a)$ when $a > -\infty$, or $(\mathbb{R}, -\infty)$ if $a = -\infty$.

We would like to recall the following well known two elementary facts: *Let $\varepsilon \in \{+, -\}$ be such that I^ε is not empty. Let $f : (I^\varepsilon, a) \to \mathbb{R}$ be definable.*

(i) *For every non-negative integer number k, there exists $\mathcal{U}_k$ an open neighbourhood of a such that f admits a C^k representative $I^\varepsilon \cap \mathcal{U}_k \mapsto \mathbb{R}$.*

(ii) *There exists $\mathcal{U}$ an open neighbourhood of a such that f admits a monotonic representative $I^\varepsilon \cap \mathcal{U} \mapsto \mathbb{R}$.*

Let $t \mapsto \gamma(t) \in \mathbb{R}^n$ be a mapping definable at $+\infty$. The differential mapping $t \mapsto \gamma'(t) \in \mathbb{R}^n$ is also definable at $+\infty$. We define

$$\mathbf{u}(t) := \frac{\gamma(t)}{|\gamma(t)|} \in \mathbf{S}^{n-1}, \text{ and } L(t) := \mathbb{R}\gamma'(t) \in \mathbf{P}^{n-1}$$

to be respectively the secant mapping and tangent direction mapping associated with γ, both definable at $+\infty$. Definability at $+\infty$ guarantees that the following limits exist

$$\mathbf{u}_\gamma := \lim_{t \to +\infty} \mathbf{u}(t), \text{ and } L_\gamma := \lim_{t \to +\infty} L(t).$$

Moreover the following very useful result holds true.

Lemma 3.1 ([6, Lemmas 2.5-7]).

(1) $L_\gamma = \mathbb{R}\mathbf{u}_\gamma$;

(2) *Assume furthermore that $|\gamma(t)| = t$ for t large enough. Then*

$$\lim_{t \to +\infty} \gamma'(t) = \mathbf{u}_\gamma.$$

In particular $|\gamma'|$ is bounded.

We also recall the following:

Lemma 3.2. — *Let $f : (\mathbb{R}_{\geqslant 0}, +\infty) \to \mathbb{R}_{>0}$ be the germ of a function definable at $+\infty$ such that $f \to 0$ as $t \to +\infty$. The following limits hold true*

(i) $\lim_{t\to+\infty} t \cdot f'(t) = 0$, *and*

(ii) $\lim_{t\to+\infty} t^{1+\tau} \cdot \dfrac{f'(t)}{f(t)} = -\infty$, $\forall\, \tau > 0$.

Proof. — We recall that ultimately a function definable at $+\infty$ is monotonic. In both cases an integrability argument near $+\infty$ combined with contradicting the desired statement yields the proof. □

Our Setting

Let W be a C^2 connected and closed sub-manifold of $\mathbb{R}^n \times \mathbb{R}^s$, definable in $\mathcal{M}$. The restriction of the Euclidean metric onto TW induces a definable C^1 Riemannian metric over W. We further require that

$$\dim W \geqslant s.$$

Let $\varphi : W \mapsto \mathbb{R}^s$ be the restriction to W of the projection on $\mathbb{R}^s$, thus φ is a C^2 and definable mapping.

4. Some elementary linear algebra

Let V be a real vector space of dimension q.

Let $\mathbf{G}$ be the union $\bigcup_k \mathbf{G}(k, V)$, the total Grassmann space of V.

We assume that V is equipped with a scalar product $\langle -, - \rangle$. Let $|-|$ be the associated norm. Any vector subspace E of V is equipped with the restriction of the scalar product, to yield a scalar product on E. Let $\mathbf{S}(E)$ be the unit sphere of E.

Let E be a vector subspace of V and let $\mathbf{u}$ be a unit vector of V. *The distance of* $\mathbf{u}$ *to* E is defined as

$$\delta_q(\mathbf{u}, E) := \max\{|\langle \mathbf{u}, \mathbf{v}\rangle| \,:\, \mathbf{v} \in \mathbf{S}(E^\perp)\}.$$

This function $(\mathbf{u}, E) \mapsto \delta_q(\mathbf{u}, E)$ is semi-algebraic over $\mathbf{S}(V) \times \mathbf{G}$.

Let E, F be vector subspaces of V. *The distance from E to F* is defined as

$$\delta_q(E, F) := \max\{\delta_q(\mathbf{u}, F) : \mathbf{u} \in \mathbf{S}(E)\}.$$

The function $(E,F) \to \delta_q(E,F)$ is also semi-algebraic in its entries in $\mathbf{G} \times \mathbf{G}$. It is not symmetric in E and F. Yet its restriction to $\mathbf{G}(k,V) \times \mathbf{G}(k,V)$ yields a distance on $\mathbf{G}(k,V)$.

Let $\mathcal{L}(p,q)$ be the space of linear mappings from $\mathbb{R}^p$ to $\mathbb{R}^q$. It is equipped with its Euclidean norm.

Let $\Sigma(p,q)$ be the algebraic subset of $\mathcal{L}(p,q)$ consisting of all linear maps that are not surjective.

For A a linear operator of $\mathcal{L}(p,q)$, let $A^* \in \mathcal{L}(q,p)$ be the adjoint operator of A.

Definition 4.1. — *Let $A \in \mathcal{L}(p,q)$. The* Rabier number of the operator *A is defined as*

$$\nu(A) := \inf_{|\varphi|=1} |A^*(\varphi)|. \tag{4.1}$$

As can be seen in [22, 26, 39], we know the following result:

Proposition 4.2. — *Let $A \in \mathcal{L}(p,q)$. We have*

$$\inf_{|\varphi|=1} |A^*(\varphi)| = \sup\{r > 0 \,:\, \mathbf{B}^q(\mathbf{0},r) \subset A \cdot \mathbf{B}^p(\mathbf{0},1)\} = \operatorname{dist}(A, \Sigma(p,q)).$$

Let A be any linear map of $\mathcal{L}(p,q)$. Whenever the operator A is not surjective, we get $\nu(A) = 0$. For any real number λ we also get

$$\nu(\lambda A) = |\lambda| \nu(A).$$

Let $N(A)$ be the orthogonal space of $\ker A$ in $\mathbb{R}^p$. Although obvious it is worth singling out the following

$$\nu(A) = \nu(A|_{N(A)}) \geqslant \nu(A|_P) \tag{4.2}$$

for every subspace P of $\mathbb{R}^p$.
In order to work with families rather than mappings, the following elementary result is key to this approach.

Lemma 4.3. — *Let $A \in \mathcal{L}(p,q)$ and let V be the graph of A, subspace of $\mathbb{R}^p \times \mathbb{R}^q$. Let $\varphi : V \to \mathbb{R}^q$ be the restriction to V of the canonical projection $\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$.*

(1) *If $\nu(A) > 0$, then*

$$\nu(A) := \min\{|A \cdot \mathbf{u}| : \mathbf{u} \in \mathbf{S}(N(A))\}. \tag{4.3}$$

(2) *The following is true*

$$\nu(\varphi) = \frac{\nu(A)}{\sqrt{1 + \nu(A)^2}}. \tag{4.4}$$

Proof. — Point (1) is straightforward.

For point (2), observe that $\nu(\varphi) = 0$ if and only if $\nu(A) = 0$. We thus assume that $\nu(A)$ is positive. Working with $N(A)$ instead of $\mathbb{R}^p$ does not change the value of $\nu(A)$. We can thus assume that V is the graph of $A|_{N(A)}$.

Let us consider the following real algebraic subset of $N(A)$

$$S_A := \{\mathbf{u} \in N(A) : |\mathbf{u}|^2 + |A \cdot \mathbf{u}|^2 = 1\}.$$

The subset S_A is compact and does not contain $\mathbf{0}$. Consider the following continuous semi-algebraic mapping

$$a : S_A \longrightarrow \mathbb{R}, \ \ \mathbf{u} \longmapsto \frac{|A \cdot \mathbf{u}|}{|\mathbf{u}|}.$$

Since the radial projection of S_A over $\mathbf{S}(N(A))$ is a homeomorphism, the image of a is the interval $[\nu(A), \|A\|]$. For $\mathbf{u} \in S_A$ we deduce that

$$|\mathbf{u}| = \frac{1}{\sqrt{1 + a(\mathbf{u})^2}}, \ \ \text{and} \ \ |A\mathbf{u}| = \frac{a(\mathbf{u})}{\sqrt{1 + a(\mathbf{u})^2}}.$$

By definitions of φ and of $\nu(\varphi)$ we get

$$\nu(\varphi) = \min\{|\mathbf{v}| : (\mathbf{u}, \mathbf{v}) \in \mathbf{S}(V)\} = \min\{|A \cdot \mathbf{u}| : \mathbf{u} \in S_A\}.$$

Since the function $t \mapsto \frac{t}{\sqrt{1+t^2}}$ is increasing, we obtain

$$\nu(\varphi) = \frac{\nu(A)}{\sqrt{1 + \nu(A)^2}}. \qquad \square$$

5. Blowing-up at infinity

We need to describe conveniently in our setting (and for our purposes) a neighbourhood of infinity.

Let $\mathbf{I} :=]0, +\infty[$. Let $\mathbf{M} := \mathbf{I} \times \mathbf{S}^{n-1}$. Given the point $\mathbf{m} = (\mathbf{r}, \mathbf{u}) \in \mathbf{I} \times \mathbf{S}^{n-1}$, we observe that $T_\mathbf{m}\mathbf{M} = \mathbb{R} \times T_\mathbf{u}\mathbf{S}^{n-1}$.

The spherical blowing-up of $\mathbb{R}^n$ at infinity is the following Nash diffeomorphism, defined as

$$\beta \, : \, \mathbf{M} \longrightarrow \mathbb{R}^n \setminus \mathbf{0}, \ \ \mathbf{m} \longmapsto \mathbf{x} = \frac{\mathbf{u}}{\mathbf{r}}.$$

The mapping β provides a single chart to investigate the behaviour of any mapping, with source a non-bounded domain of $\mathbb{R}^n$, nearby the boundary part of the domain lying at infinity.

Let $\overline{\mathbf{M}} := [0,+\infty[\times\mathbf{S}^{n-1}$. It is a semi-algebraic subset of $\mathbb{R}\times\mathbb{R}^n$ and is a sub-manifold, with smooth and algebraic boundary

$$\mathbf{M}^\infty := 0\times\mathbf{S}^{n-1}.$$

We compactify $\mathbb{R}^n$ as the Nash manifold with boundary $\overline{\mathbb{R}^n} := \overline{\mathbf{M}}\sqcup\mathbf{0} = \mathbb{R}^n\sqcup\mathbf{S}^{n-1}$ identifying $\mathbb{R}^n\setminus\mathbf{0}$ with $\mathbf{M}$ via the mapping β. We furthermore define

$$\mathbf{M}_s := \mathbf{M}\times\mathbb{R}^s,\ \ \overline{\mathbf{M}}_s := \overline{\mathbf{M}}\times\mathbb{R}^s,\ \text{ and }\ \mathbf{M}_s^\infty := \mathbf{M}^\infty\times\mathbb{R}^s.$$

Let $\mathbf{G}(k,\overline{\mathbf{M}}_s)$ be the Grassmann bundle of subspaces of rank k of $T\overline{\mathbf{M}}_s$. Let $\operatorname{clos}(Z)$ be the closure in $\overline{\mathbf{M}}_s$ of the C^1 connected definable sub-manifold Z of $\mathbf{M}_s$. Let

$$Z^\infty := \operatorname{clos}(Z)\cap\mathbf{M}_s^\infty.$$

Let $g: Z\to\mathbb{R}$ be a definable C^1 function. We define the *relative tangent bundle of g over* $\operatorname{clos}(Z)$ as follows

$$\mathcal{T}_g := \operatorname{clos}\{(\mathbf{z},T_\mathbf{z}g)\in\mathbf{G}(\dim Z-1,\overline{\mathbf{M}}_s)|_{Z\setminus\operatorname{crit}(g)}\}\subset\mathbf{G}(\dim Z-1,\overline{\mathbf{M}}_s)|_{\operatorname{clos}(Z)},$$

where $T_\mathbf{z}g$ is the tangent space to the level of g through the point $\mathbf{z}$. Note that $\mathcal{T}_g$ is closed and definable in $\mathbf{G}(\dim Z-1,\overline{\mathbf{M}}_s)$. Let $\pi:\mathbf{G}(\dim Z-1,\overline{\mathbf{M}}_s)\to\overline{\mathbf{M}}_s$ be the projection $(\mathbf{z},T)\mapsto\mathbf{z}$ on the base of the Grassmann bundle. The fibre $\mathcal{T}_{g,\mathbf{z}}$ of $\mathcal{T}_g$ over $\mathbf{z}\in\overline{\mathbf{M}}_s$ is the subset of $\mathbf{G}(\dim Z-1,T_\mathbf{z}\overline{\mathbf{M}}_s)$ defined as

$$\mathbf{z}\times\mathcal{T}_{g,\mathbf{z}} := \mathcal{T}_g\cap\pi^{-1}(\mathbf{z}).$$

Definition 5.1. — *The* relative tangent bundle of g over $\operatorname{clos}(Z)$ at infinity *is defined as*

$$\mathcal{T}_g^\infty := \pi^{-1}(Z^\infty)\cap\mathcal{T}_g.$$

For any $\mathbf{z}\in Z^\infty$, let $(\mathcal{T}_g^\infty)_\mathbf{z}$ be the fibre of $\mathcal{T}_g$ above $\mathbf{z}$.

Let $\mathbf{r}_Z: Z\to\mathbb{R}$ be the C^1 definable function given by the restriction of $\mathbf{r}$ to Z, that is for $\mathbf{z}=(\mathbf{r},\mathbf{u},\mathbf{y})\in\overline{\mathbf{M}}_s$ we find $\mathbf{r}_Z(\mathbf{z})=\mathbf{r}$.

The behaviour nearby Z^∞ of the function $\mathbf{r}_Z$ contains some information about the accumulation of Z onto Z^∞. The notion of $\mathbf{t}$-regularity [40, 46] is about the nature of the relative conormal space at the divisor at infinity of a projective compactification of the graph of a complex or real polynomial function. In our present and most general real and definable setting, the analogue of the divisor at infinity would be the "boundary subset" Z^∞. Using the Rabier number, it is much clearer and easier to present our avatar of the notion of $\mathbf{t}$-regularity in terms of limits of tangent spaces (see Definition 5.2). We introduce the following mapping

$$\beta_s := \beta\times\mathrm{Id}_{\mathbb{R}^s}:\mathbf{M}_s\longrightarrow\mathbb{R}^n\times\mathbb{R}^s.$$

For any limit of tangent spaces T of $(\mathcal{T}^\infty_{\mathbf{r}_Z})_{\mathbf{z}}$, with $\mathbf{z} = (0, \mathbf{u}, \mathbf{c})$, the projection of T onto $\mathbb{R}^s$ means the restriction to T of the projection of $T_{\mathbf{z}}\overline{\mathbf{M}}_s = T_{(0,\mathbf{u})}\overline{\mathbf{M}} \times T_{\mathbf{c}}\mathbb{R}^s$ onto the second factor $T_{\mathbf{c}}\mathbb{R}^s$ of this product.

Definition 5.2. — *Let W be a closed connected definable C^1 sub-manifold of $\mathbb{R}^n \times \mathbb{R}^s$ of dimension $d \geqslant s$. Let $Z := \beta_s^{-1}(W)$.*

The subset W is **t**-regular at the value **c** *if for any $\mathbf{z} \in Z^\infty \cap \overline{\mathbf{M}} \times \mathbf{c}$ and any $T \in \mathcal{T}^\infty_{\mathbf{r}_Z,\mathbf{z}}$ the projection mapping $T \to \mathbb{R}^s$ is surjective.*

If W is the graph of $F : X \to \mathbb{R}^s$, the mapping F is **t**-regular at the value **c** *if W is.*

When working with a polynomial or semi-algebraic C^1 mapping $\mathbb{R}^n \mapsto \mathbb{R}^s$, observe that [46, Definition 2.10] and [10, Definition 2.5] are somehow equivalent to our Definition 5.2.

Equipping $[0, +\infty[$ and $\mathbf{S}^{n-1}$ with their respective Euclidean metric, the space $\overline{\mathbf{M}}$ inherits this metric product structure, and so does $\overline{\mathbf{M}}_s$.
If $\mathbf{m} = (\mathbf{r}, \mathbf{u}, \mathbf{c})$ is any point of $\overline{\mathbf{M}}_s$ then $T_{\mathbf{m}}\overline{\mathbf{M}}_s = \mathbb{R} \times T_{\mathbf{u}}\mathbf{S}^{n-1} \times T_{\mathbf{c}}\mathbb{R}^s$. For any given subspace T of $T_{\mathbf{m}}\overline{\mathbf{M}}$, let $L : T \mapsto \mathbb{R}^s = T_{\mathbf{c}}\mathbb{R}^s$ be the projection of T onto $\mathbb{R}^s$. The Rabier number $\nu_s(L)$ of L is defined w.r.t the product metric on $\overline{\mathbf{M}}_s$ and the canonical one on $\mathbb{R}^s$. Let $\pi_s(\mathbf{m})$ be the projection of $T_{\mathbf{m}}\overline{\mathbf{M}}_s$ onto $T_{\mathbf{c}}\mathbb{R}^s$.

We get the following useful and quantified characterization of **t**-regularity.

Lemma 5.3. — *The sub-manifold $W = \beta_s(Z)$ is **t**-regular at the regular value **c** if and only if there exists $\nu_{\mathbf{c}} > 0$ such for every $\mathbf{z} \in Z^\infty \cap \mathbf{M}^\infty \times \mathbf{c}$ and every $T \in \mathcal{T}^\infty_{r_Z,\mathbf{z}}$ we have*

$$\nu_s\left(\pi_s(\mathbf{z})|_T\right) \geqslant \nu_{\mathbf{c}}.$$

Proof. — It is a straightforward consequence of the following ingredients: $Z^\infty \cap \mathbf{M}^\infty \times \mathbf{c}$ is definable and compact, the mapping $\mathbf{z} \mapsto \nu_s(\pi_s(\mathbf{z})|_{T_{\mathbf{z}}\mathbf{r}_Z})$ is definable and continuous, the projection of $(\dim W - 1)$-planes of $\mathbb{R}^n \times \mathbb{R}^s$ onto $\mathbb{R}^s$ is continuous, and the definition of **t**-regularity. ☐

6. Rabier number, Malgrange–Rabier condition

Let $\mathbf{w} := (\mathbf{x}, \mathbf{y})$ be the Euclidean coordinates of $\mathbb{R}^n \times \mathbb{R}^s$. Let W be a closed connected definable C^2 definable sub-manifold of $\mathbb{R}^n \times \mathbb{R}^s$ of dimension $d \geqslant s$ and let $\varphi : W \to \mathbb{R}^s$ be the projection $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{y}$. We introduce the following (see [26, 39]):

Definition 6.1. — *The mapping φ satisfies the* Malgrange–Rabier condition (MR) at the value $\mathbf{c} \in \mathbb{R}^s$ *if there exists a positive constant L such that*

$$|\varphi(\mathbf{w}) - \mathbf{c}| \ll 1 \text{ and } |\mathbf{w}| \gg 1 \Longrightarrow |\mathbf{w}| \cdot \nu(D_{\mathbf{w}}\varphi) \geqslant L. \tag{6.1}$$

We will also say that φ is (MR)-regular at $\mathbf{c}$.

In Definition 6.1, the mapping φ tends to a finite value $\mathbf{c}$, therefore the asymptotic phenomenon at infinity only occurs in the directions of the $\mathbb{R}^n$ component. In other words, since $\mathbf{w} = (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^s$, instead of requiring $|\mathbf{w}| \gg 1$ and $|\mathbf{w}| \cdot |D_{\mathbf{w}}\varphi| \geqslant L$, we can equivalently require $|\mathbf{x}| \gg 1$ and $|\mathbf{x}| \cdot |D_{\mathbf{w}}\varphi| \geqslant L$.

As counterpart to (MR)-regular values we explicitly introduce the next

Definition 6.2. — *An* asymptotic critical value $\mathbf{c}$ of the mapping φ is simply *a Malgrange–Rabier critical value of φ, namely there exists a sequence $\mathbf{w}_k := (\mathbf{x}_k, \mathbf{y}_k)_k$ of points of W such that:* (i) $\mathbf{w}_k \to \infty$, (ii) $\mathbf{y}_k \to \mathbf{c}$ *and*

$$|\mathbf{w}_k| \cdot \nu(D_{\mathbf{w}_k}\varphi) \longrightarrow 0.$$

It is customary to write $K_\infty(\varphi)$ for the set of ACVs of the mapping φ.

Definition 6.3. — *A* generalized critical value of φ *is any value of the subset*

$$K(\varphi) := K_0(\varphi) \cup K_\infty(\varphi) \subset \mathbb{R}^s.$$

Notation. — Let $\nu^\varphi(\mathbf{w}) := \nu(D_{\mathbf{w}}\varphi)$.

We start with the following expected result:

Lemma 6.4. — *The subset $K(\varphi)$ is definable and closed.*

Proof. — Since the Malgrange–Rabier condition proceeds from a first order formula in the language of the structure $\mathcal{M}$, the subset $K_\infty(\varphi)$ is definable in the structure. Therefore the subset $K(\varphi)$ is too.
Compactify $\mathbb{R}^q$ as $\mathbf{S}^q$, and $\mathbb{R}_{\geqslant 0}$ as $[0, +\infty]$, by adding a single point at infinity. The compactification is semi-algebraic. Consider the following compact definable subset

$$G := \operatorname{clos}\{(\mathbf{w}, (1 + |\mathbf{w}|) \cdot \nu^\varphi(\mathbf{w})) \in W \times \mathbb{R}_{\geqslant 0}\} \subset \mathbf{S}^n \times \mathbf{S}^s \times [0, +\infty].$$

Denoting by $g : G \to \mathbf{S}^s$ the projection on $\mathbf{S}^s = \mathbb{R}^s \sqcup \infty$, the following identity

$$K(\varphi) = \mathbb{R}^s \cap g\,(G \cap \mathbf{S}^n \times \mathbf{S}^s \times 0)$$

concludes the proof. □

The next result tells us about the behaviour at infinity nearby a value $\mathbf{c} \notin K(\varphi)$ of some feature of the mapping φ.

Lemma 6.5. — *Let* $\mathbf{c}$ *be a regular value of* φ *which is also (MR)-regular. Let* $(\mathbf{w}_k)_k$ *be any sequence of* W *such that* (i) $\mathbf{w}_k \to \infty$, (ii) $\mathbf{y}_k \to \mathbf{c}$. *The following limit holds true*

$$\lim_{k \to +\infty} \delta_{n+s}\left(\frac{\mathbf{w}_k}{|\mathbf{w}_k|}, T_{\mathbf{w}_k}\varphi\right) = 0. \tag{6.2}$$

Proof. — By hypothesis, there exists a positive constant L such that along any sequence $(\mathbf{w}_k)_k$ as in the statement we have

$$|\mathbf{w}_k| \cdot \nu^{\varphi}(\mathbf{w}_k) \geqslant L, \text{ for } k \gg 1.$$

It is enough to work with paths definable at $+\infty$ instead of sequences. Let

$$t \longmapsto \mathbf{w}(t) = (\mathbf{x}(t), \mathbf{y}(t)) \in W$$

be a germ of definable mapping at $+\infty$, such that as t goes to $+\infty$

$$\mathbf{w}(t) := t\mathbf{u}(t), \text{ with } \mathbf{u}(t) \in \mathbf{S}^{n+s-1};\ \mathbf{y}(t) \longrightarrow \mathbf{c}.$$

In particular $\mathbf{u}(t) \to \mathbf{u} \in \mathbf{S}^{n-1} \times \mathbf{0}$. Since φ is definable, the tangent-to-the-fibre mapping

$$\mathbf{w} \longmapsto T_{\mathbf{w}}\varphi \in \mathbf{G}(\dim W - s, n+s)$$

is definable. Thus the following mappings are definable at $+\infty$

$$t \longmapsto T_{\mathbf{w}(t)}\varphi, \quad \text{and} \quad T_{\mathbf{w}(t)}\varphi \longrightarrow T$$

as t goes to $+\infty$. We want to show that

$$\lim\nolimits_{\infty} \delta_{n+s}(\mathbf{w}', T_{\mathbf{w}}\varphi) = 0,$$

since Lemma 3.1 gives that $\left|\mathbf{w}' - \frac{\mathbf{w}}{|\mathbf{w}|}\right| \to 0$ at infinity. The mapping

$$t \longmapsto f(t) := \varphi(\mathbf{w}(t)) - \mathbf{c}$$

is definable at $+\infty$ and tends to $\mathbf{0}$ as t goes to $+\infty$. Working with each component of f, Lemma 3.2 yields

$$t \cdot f'(t) \longrightarrow \mathbf{0}.$$

Let $N_{\mathbf{w}}\varphi \subset T_{\mathbf{w}}W$ be the orthogonal complement of $T_{\mathbf{w}}\varphi$. Let $\mathbf{w}' = \mathbf{w}'_T + \mathbf{w}'_N$ be the decomposition of $\mathbf{w}'$ in the orthogonal direct sum $T_{\mathbf{w}(t)}\varphi \oplus N_{\mathbf{w}(t)}\varphi = T_{\mathbf{w}(t)}W$. Of course the germ of each of the following mappings

$$t \longmapsto T_{\mathbf{w}(t)}W,\ \ t \longmapsto N_{\mathbf{w}(t)}\varphi,\ \ t \longmapsto \mathbf{w}'_T, \ \text{and} \ \ t \longmapsto \mathbf{w}'_N$$

is definable at $+\infty$. By definition we have

$$\delta_{n+s}(\mathbf{w}', T_{\mathbf{w}}\varphi) = |\mathbf{w}'_N|.$$

Combining the previous equality with the hypothesis, we find

$$|D_{\mathbf{w}}\varphi \cdot \mathbf{w}'| = |D_{\mathbf{w}}\varphi \cdot \mathbf{w}'_N| \geqslant |\mathbf{w}'_N| \cdot \nu^{\varphi}(\mathbf{w}(t)),$$

from which we get

$$|\mathbf{w}'_N(t)| \leqslant \frac{t \cdot |f'(t)|}{L} \longrightarrow 0. \qquad \square$$

We introduce the following:

Definition 6.6. — *A germ of a continuous, positive function* $\mu : (\mathbb{R}_{\geqslant 0}, +\infty) \to \mathbb{R}_{\geqslant 0}$ *is* increasing fast at $+\infty$ *if*

$$\lim_{t \to +\infty} t^{-1} \cdot \mu(t) = +\infty.$$

For a function germ μ *increasing fast at* $+\infty$*, we define*

$$K^{\mu}_{\infty}(\varphi) = \left\{ \mathbf{c} \in \mathbb{R}^s \, : \, \begin{array}{l} \exists (\mathbf{w}_k)_k \in W \text{ with } \mathbf{w}_k \longrightarrow +\infty, \, \mathbf{y}_k \longrightarrow \mathbf{c} \\ \text{such that } \mu(|\mathbf{w}_k|) \cdot \nu^{\varphi}(\mathbf{w}_k) \longrightarrow 0 \end{array} \right\}.$$

In Definition 6.6 we can, equivalently, require that $\mu(|\mathbf{x}_k|) \cdot \nu^{\varphi}(\mathbf{w}_k) \to \infty$. When the function germ μ is definable at $+\infty$, the subset $K^{\mu}_{\infty}(\varphi)$ is definable, contained in $K_{\infty}(\varphi)$. As in [6, 26], we have

Proposition 6.7. — *There exists a function germ at infinity* $\mu : (\mathbb{R}_{\geqslant 0}, +\infty) \to \mathbb{R}_{\geqslant 0}$ *increasing fast and definable at* $+\infty$ *such that*

$$K_{\infty}(\varphi) = K^{\mu}_{\infty}(\varphi).$$

Proof. — The principle of the proof is the same as that of [6, Lemma 3.3] or [26, Lemma 3.1]. Using the inverse of the stereographic projection, any Euclidean space $\mathbb{R}^q$ compactifies smoothly and semi-algebraically as $\mathbf{S}^q = \mathbb{R}^q \sqcup \infty$. Let $\operatorname{clos}(W)$ be the closure of W in $\mathbf{S}^{n+s}$. We consider the following definable mapping

$$G : W \longrightarrow \mathbb{R}^s \times \mathbb{R}, \; \mathbf{w} \longmapsto (\varphi(\mathbf{w}), |\mathbf{w}| \cdot \nu^{\varphi}(\mathbf{w})).$$

Let $Y := \operatorname{clos}(W) \times \mathbf{S}^s \times \mathbf{S}^1$. Define

$$\Gamma := \operatorname{clos}(\operatorname{graph}(G)) \subset Y \text{ and } A := \infty \times \operatorname{clos}(K_{\infty}(\varphi)) \times 0 \subset Y.$$

Observe that A is closed. The definable Wing Lemma ([48], [29, Lemma 2.7]) states that there exists a closed definable subset B of Γ such that

$$B \cap A = \emptyset, \quad \text{and} \quad \operatorname{clos}(B) \cap \Gamma \setminus \operatorname{graph}(G) = A.$$

The point $\infty \times \mathbf{u} \times 0$ belongs to A if there exists a sequence $(\mathbf{w}_k)_k \in \pi_W(B)$ such that $\mathbf{w}_k \to \infty$, $\mathbf{y}_k \to \mathbf{c}$ and $|\mathbf{w}| \cdot \nu^{\varphi}(\mathbf{w}) \to 0$, where π_W :

$\Gamma \mapsto \mathrm{clos}(W)$ is the projection onto the first factor. Consider the following definable function

$$b(r) := r \cdot \max\{\nu^\varphi(\mathbf{w}) \text{ for } \mathbf{w} \in \pi_W(B) \text{ and } |\mathbf{w}| = r\}.$$

We check as in [6, Lemme 3.3] that the function defined as

$$r \longmapsto \mu(r) := \frac{r}{\sqrt{b(r)}}$$

satisfies the announced statement. □

To conclude this section, we would like to add a few words about the special case of graphs.

Let X be a closed connected definable C^2-sub-manifold of $\mathbb{R}^n$. Let $F : X \to \mathbb{R}^s$ be a C^2 and definable mapping. We suppose that $\dim X \geqslant s$. The corresponding Rabier number function to consider is

$$\mathbf{x} \longmapsto \nu^F(\mathbf{x}) := \nu(D_\mathbf{x} F).$$

We recall the following complementary definitions:

A value $\mathbf{c}$ *is* a Malgrange–Rabier regular value of the mapping F if there exists a positive constant L such that for any sequence $(\mathbf{x}_k)_k$ of X such that (i) $\mathbf{x}_k \to +\infty$, (ii) $F(\mathbf{x}_k) \to \mathbf{c}$, we have

$$|\mathbf{x}_k| \cdot \nu^F(\mathbf{x}_k) \geqslant L.$$

A value $\mathbf{c} \in \mathbb{R}^s$ is *an ACV of the mapping* F if there exists a sequence $(\mathbf{x}_k)_k$ of X such that (i) $\mathbf{x}_k \to +\infty$, (ii) $F(\mathbf{x}_k) \to \mathbf{c}$ and (iii) $|\mathbf{x}_k| \cdot \nu^F(\mathbf{x}_k) \to 0$.

Let W be the graph of F, and let $\mathbf{w} = (\mathbf{x}, F(\mathbf{x}))$ be a point of W. Since $T_\mathbf{w} W$ is just the graph of the linear mapping $D_\mathbf{x} F : T_\mathbf{x} X \to \mathbb{R}^s$, Equation (4.4) of Lemma 4.3 guarantees that the value $\mathbf{c} \in \mathbb{R}^s$ lies in $K_\infty(F)$ if and only if it lies in $K_\infty(\varphi)$.

Any statement and notion for φ presented in this section admits a version for the mapping F. The modifications to be done are clear, and we can check that each demonstration for a statement about F adapts easily from the corresponding statement for φ, following the same steps.

7. Malgrange–Rabier condition and geometry at infinity

We are going to combine here the local point of view at infinity of Section 5 and the affine one of Section 6.

The blowing-up at infinity mapping $\beta : \mathbf{M} \to \mathbb{R}^n \setminus \mathbf{0}$, although being a diffeomorphism is by no means an isometry. Nevertheless spheres $r \times \mathbf{S}^{n-1}$ are mapped onto spheres $\mathbf{S}^{n-1}_{\frac{1}{r}}$, angles of pairs of vectors tangent to a sphere at a same point are preserved and orthogonality to the spheres as well, as we can check below: Let $\mathbf{m} = (\mathbf{r}, \mathbf{u})$ be a point of $\mathbf{M}$. We find

$$D_{\mathbf{m}}\beta|_{T_{\mathbf{u}}\mathbf{S}^{n-1}} = \mathbf{r}^{-1}\,\mathrm{Id}_{T_{\mathbf{u}}\mathbf{S}^{n-1}}, \ \text{ and } \ D_{\mathbf{m}}\beta \cdot \partial_{\mathbf{r}} = -r^{-2}\mathbf{u} = -\mathbf{r}^{-1}\beta(\mathbf{m}).$$

Working with Z nearby $\mathbf{r}_Z = 0$ is working in a neighbourhood at infinity of W. We find the following:

Lemma 7.1. — *If φ is (MR)-regular at the regular value $\mathbf{c}$, then W is $\mathbf{t}$-regular at $\mathbf{c}$.*

Proof. — It is sufficient to show the result along a path at infinity.

Let $t \mapsto \mathbf{w}(t) \in W$ be the germ of a mapping definable at $+\infty$ such that: $\mathbf{z}(t) := \beta_s^{-1}(\mathbf{w}(t)) \to \mathbf{z}_\infty = (0, \mathbf{u}, \mathbf{c}) \in \mathbf{M}_s^\infty$, for $\mathbf{c}$ a (MR)-regular value of φ, and

$$T(t) := T_{\mathbf{z}(t)}Z \longrightarrow T, \ \text{ and } \ R(t) := T_{\mathbf{z}(t)}\mathbf{r}_Z \longrightarrow R.$$

Without loss of generality we further assume that $\mathbf{w}(t) = t\mathbf{u} + o(t)$, and set

$$\mathbf{u}(t) = \frac{\mathbf{x}(t)}{|\mathbf{x}(t)|} = \frac{\mathbf{w}(t)}{|\mathbf{w}(t)|} + o(1).$$

We recall that

$$R(t) \subset T(t) \cap 0 \times T_{\mathbf{u}(t)}\mathbf{S}^{n-1} \times \mathbb{R}^s.$$

Let $D(t) := D_{\mathbf{w}(t)}\varphi$. Let $\nu(t) = \nu^\varphi(\mathbf{w}(t))$. By hypothesis we have

$$t \cdot \nu(t) \geqslant L, \ \text{for some constant } L > 0.$$

Let us still denote by $D(t)$ the extension of $D(t)$ to $\mathbb{R}^n \times \mathbb{R}^s$ as the null mapping over $(T_{\mathbf{w}(t)}W)^\perp$. In other words, we have

$$t \gg 1 \Longrightarrow \frac{1}{\nu(t)} D(t) \cdot \mathbf{B}^{n+s}(\mathbf{0}, 1) \supset \mathbf{B}^s(\mathbf{0}, 1).$$

Let ξ_s be any vector of $\mathbf{S}^{s-1}$. There exists a germ of mapping definable at infinity

$$t \longmapsto \chi(t) \in \mathbf{B}^{n+s}(\mathbf{0}, 1)$$

tangent to X along $\mathbf{w}$, such that for t large enough

$$\xi_s = \frac{1}{\nu(t)} D(t) \cdot \chi(t) \in \mathbf{S}^{s-1}.$$

In the orthogonal direct sum

$$T_{\mathbf{w}(t)}(\mathbb{R}^n \times \mathbb{R}^s) = \mathbb{R}\mathbf{u}(t) \oplus T_{\mathbf{u}(t)}\mathbf{S}^{n-1} \oplus \mathbb{R}^s,$$

we decompose $\chi(t)$ as the definable orthogonal sum

$$\chi(t) = \psi(t)\mathbf{u}(t) + \chi_{\mathbf{S}}(t) + \nu(t)\xi_s.$$

We find

$$\xi(t) := D_{\mathbf{w}(t)}\beta_s^{-1} \cdot \frac{1}{\nu(t)}\chi(t) = \left(\frac{-\psi(t)}{t^2\nu(t)}, \frac{1}{t\nu(t)}\chi_{\mathbf{S}}(t), \xi_s\right).$$

Since $|\chi(t)| \leqslant 1$ and $t \cdot \nu(t) \geqslant L$ once t is large enough, we deduce there exists $\xi_{\mathbf{S}} \in T_{\mathbf{u}}\mathbf{S}^{n-1}$ such that

$$\xi(t) \longrightarrow (0, \xi_{\mathbf{S}}, \xi_s) \in \mathbb{R} \times T_{\mathbf{u}}\mathbf{S}^{n-1} \times \mathbb{R}^s,$$

proving the announced result. Indeed the vector $(0, \xi_{\mathbf{S}}, \xi_s)$ lies necessarily in R, since R is a subspace of $T \cap 0 \times T_{\mathbf{u}}\mathbf{S}^{n-1} \times \mathbb{R}^s$. □

The next lemma is a converse of Lemma 7.1.

LEMMA 7.2. — *If W is $\mathbf{t}$-regular at the regular value $\mathbf{c}$ of φ, then φ is (MR)-regular at $\mathbf{c}$.*

Proof. — Again, it is enough to check that the property holds true along a definable path at infinity.

Let $t \mapsto \mathbf{w}(t) := (\mathbf{x}(t), \mathbf{y}(t))$ be a germ of a mapping definable at $+\infty$, and let $\mathbf{z} = \beta_s^{-1} \circ \mathbf{w}$. Let us consider the following limits as $t \to +\infty$

$$\mathbf{z}(t) \longrightarrow \mathbf{z}_\infty := (0, \mathbf{u}, \mathbf{c}) \in \overline{\mathbf{M}}_s^\infty, \quad \text{and} \quad R(t) := T_{\mathbf{z}(t)}\mathbf{r}_Z \longrightarrow R,$$

where $Z := \beta_s^{-1}(W)$. We can assume that $\mathbf{z}(t) = (t^{-1}, \mathbf{u}(t), \mathbf{y}(t))$ with $\mathbf{y}(t) \to \mathbf{c} \notin K(\varphi)$. Let

$$\pi_s(t) := \pi_s(\mathbf{z}(t)) \quad \text{and} \quad \nu_s(t) := \nu_s\left(\pi_s(t)|_{T_{\mathbf{z}(t)}\mathbf{r}_Z}\right).$$

The hypothesis implies that

$$\lim_{t \to +\infty} \nu_s(t) = \nu \geqslant \nu_{\mathbf{c}}.$$

Let ξ be a unit vector of $\mathbb{R}^s$. Let $\xi(t)$ be a definable path with values in $T\mathbf{r}_Z$, along the path $t \mapsto \mathbf{z}(t)$ such that for t large enough we have

$$\pi_s(t) \cdot \xi(t) = \xi.$$

Therefore we can write the following orthogonal decomposition

$$\xi(t) = r(t)\partial_{\mathbf{r}} \oplus \zeta(t) \oplus \xi \in T_{\mathbf{z}(t)}\mathbf{M}_s = \mathbb{R} \oplus T_{\mathbf{u}(t)}\mathbf{S}^{n-1} \oplus \mathbb{R}^s,$$

so that the hypotheses yield

$$r(t) = 0, \quad \text{and} \quad |\xi(t)| \leqslant \frac{1}{\nu_s(t)} \leqslant \frac{2}{\nu}.$$

Let $t \mapsto \chi(t)$ be the definable path at infinity

$$\chi(t) := D_{\mathbf{z}(t)}\beta_s(t) \cdot \xi(t) \in T_{\mathbf{w}(t)}W.$$

Decomposing $\chi(t)$ in the orthogonal direct sum $\mathbb{R}\mathbf{u}(t) \oplus T_{\mathbf{u}(t)}\mathbf{S}^{n-1} \oplus \mathbb{R}^s$, we get

$$\chi(t) = 0 \oplus \mathbf{v}(t) \oplus \xi.$$

In particular we obviously find

$$D_{\mathbf{w}(t)}\varphi \cdot \chi(t) = \xi.$$

By the choice of parametrization of $\mathbf{z}$ we have $|\mathbf{x}(t)| = t$. Since

$$\xi(t) = 0 \oplus \frac{1}{t}\mathbf{v}(t) \oplus \xi,$$

we observe furthermore that

$$\frac{2}{\nu} \cdot t \ \geqslant \ t \cdot |\xi(t)| \ \geqslant \ |\chi(t)|.$$

Since $|\mathbf{w}(t)| \geqslant t$, the previous inequalities imply the following inclusion

$$\mathbf{B}_1^s \subset D(t) \cdot \mathbf{B}^{n+s}_{\frac{2}{\nu}|\mathbf{w}(t)|},$$

where $D(t) \in \mathcal{L}(n+s, s)$ is the linear extension of $D_{\mathbf{w}(t)}\varphi$ to $\mathbb{R}^n \times \mathbb{R}^s$, as the null mapping over the normal space $N_{\mathbf{w}(t)}W$ of $T_{\mathbf{w}(t)}W$ in $\mathbb{R}^n \times \mathbb{R}^s$. In other words we have proved that

$$\nu^\varphi(\mathbf{w}(t)) = \nu(D(t)) \geqslant \frac{\nu_{\mathbf{c}}}{2} \cdot \frac{1}{|\mathbf{w}(t)|},$$

which ends the proof. $\square$

Combining Lemma 7.1, Lemma 7.2 we have proved in our most general context the following

Proposition 7.3. — *A regular value* $\mathbf{c}$ *of* φ *is (MR)-regular if and only if the sub-manifold* W *is* $\mathbf{t}$*-regular at the value* $\mathbf{c}$.

In other words, we have rigorously showed, in this most general context, that (MR)-regularity and (the current avatar of) $\mathbf{t}$-regularity are equivalent (see [10, 17, 46] for special cases).

Remark 7.4. — When $d = s$, the condition of Malgrange–Rabier regularity at $\mathbf{c}$ is equivalent to the local properness of the mapping φ at the value $\mathbf{c}$, which means that φ is proper over some neighbourhood of $\mathbf{c}$. Indeed the ACVs coincide with the non-properness values of the mapping φ (see [26, pages 86 to 88] and adapt almost readily their arguments to our context). In particular Lemma 7.1 and Lemma 7.2 are trivial.

Using Proposition 6.7 we are now in position to show the following Morse–Sard type result, whose proof is different from those given in [24, 26] and gives a more precise insight of the geometric phenomenon at hand than that of [10].

THEOREM 7.5. — *The subset $K_\infty(\varphi)$ of $\mathbb{R}^s$ (is definable) and has positive codimension.*

Proof. — By Proposition 6.7 we know that $K_\infty(\varphi) = K_\infty^\mu(\varphi)$, for some fast decreasing function μ definable at $+\infty$. The following subset

$$W_1 := \{\mathbf{w} \in W \,:\, \mu(|\mathbf{w}|) \cdot \nu^\varphi(\mathbf{w}) \leqslant 1\}$$

is a closed and definable subset of $\mathbb{R}^n \times \mathbb{R}^s$. Let

$$Z_1 := \beta_s^{-1}(W_1), \;\; \text{and} \;\; Z_1^\infty := \operatorname{clos}(Z_1) \cap \mathbf{M}_s^\infty.$$

By Lemma 7.1 and Lemma 7.2, any point $\mathbf{z}_\infty = (0, \mathbf{u}, \mathbf{c})$ of Z_1^∞ is a limit of a sequence $(\mathbf{z}_k)_k$ of Z_1 such that $T_{\mathbf{z}_k}\mathbf{r}_Z \to R \subset T_{\mathbf{z}_\infty}\overline{M}_s$ which does not projects surjectively onto $\mathbb{R}^s$. Let τ_Z be the projection of $\operatorname{clos}(Z)$ onto $\mathbb{R}^s$. Thus

$$\tau_Z(Z_1^\infty) \;=\; K_\infty(\varphi).$$

Let $\mathbf{r}_1$ be the restriction of $\mathbf{r}_Z$ to $\operatorname{clos}(Z_1)$. It is a continuous definable mapping. The pair (Z_1, Z_1^∞) of germs at $\mathbf{M}_s^\infty$ can be definably stratified to be $(\mathbf{a}_{rel})$-regular w.r.t. the function $\mathbf{r}_1$ [31]. Let $\mathbf{z}_\infty$ be a point of S^∞, a strata contained in Z_1^∞. Therefore $T_{\mathbf{z}_\infty}S^\infty$ is contained in the limit T at $\mathbf{z}_\infty$ of the relative tangent spaces $T_{\mathbf{z}_k}\mathbf{r}_Z = T_{\mathbf{z}_k}\mathbf{r}_1$, given any sequence $(\mathbf{z}_k)_k$ of Z_1 converging to $\mathbf{z}_\infty$. By Proposition 7.3 and the definition of Z_1, the limit T cannot project surjectively onto $\mathbb{R}^s$. Therefore we deduce

$$\text{rank } D_{z_\infty}(\tau_Z|_{S^\infty}) \leqslant s - 1,$$

and thus we find

$$\dim \tau_Z(Z_1^\infty) \leqslant s - 1. \qquad \square$$

8. Bifurcation values and triviality results

We recall here in this most general setting the notions and results about equi-singularity nearby a (MR)-regular value $\mathbf{c}$ in the context of the previous sections.

Let X be a connected, definable C^a sub-manifold of $\mathbb{R}^n$ which is also a closed subset of $\mathbb{R}^n$, where $a \in \mathbb{N}_{\geqslant 2} \cup \infty$. Let $F : X \to \mathbb{R}^s$ be a C^k definable mapping where $2 \leqslant k \leqslant a$. In this definition C^k is understood as the maximal possible regularity of F on X. In order to have an interesting statement we assume that $\dim X \geqslant s$.

Definition 8.1. — *A value* $\mathbf{c}$ *of* $\mathbb{R}^s$ *is* typical for the mapping F *if there exists a neighbourhood* $\mathcal{V}$ *of* $\mathbf{c}$ *in* $\mathbb{R}^s$ *such that the restriction*

$$F|_{\mathcal{U}} : \mathcal{U} := F^{-1}(\mathcal{V}) \longrightarrow \mathcal{V}$$

is a trivial C^{k-1}*-fibre bundle over* $\mathcal{V}$ *with model fibre* $F^{-1}(\mathbf{c})$ *and* $F|_{\mathcal{U}}$ *is the projection mapping onto the base.*

We can now see the relation of (MR)-regularity condition previously introduced with equi-singularity in the following expected

Theorem 8.2. — *If a regular value* $\mathbf{c}$ *of* $\mathbb{R}^s$ *is (MR)-regular for the mapping* F*, then* $\mathbf{c}$ *is a typical value for* F*.*

There are many occurrences of this result over the last forty years, mostly for functions though [6, 10, 17, 22, 23, 24, 26, 32, 37, 39, 40, 46, 47] (this list is far from being exhaustive). The proof we present here is explicit and does not get into the messy task of composing s flows (which is rarely explicitly done once $s \geqslant 2$). We follow the simple and beautiful idea of Rabier [39] (see also Jelonek's variation in [22, 23] using Gaffney's characterization of (MR)-regularity [19]).

Proof. — Assume that $\mathbf{c}$ is a (MR)-regular value of F. There exists a positive number ε such that the closed ball $\mathbf{B} := \mathbf{B}^p(\mathbf{c}, \varepsilon)$ does not intersect with $K(F)$. Let $\mathbf{F}$ be the closed definable C^k sub-manifold with boundary

$$\mathbf{F} := F^{-1}(\mathbf{B}) \subset X.$$

Observe that a sequence $(\mathbf{x}_k)_k$ of $\mathbf{F}$ escaping $\mathbf{F}$ as k goes to infinity either goes to the boundary $\partial\mathbf{F}$ or goes to infinity (see Condition (4.1) in [39, Lemma 4.1]). Up to a translation we can assume that the origin of $\mathbb{R}^n$ does not lie in $\mathbf{F}$. By definition of Malgrange–Rabier regularity, we can assume that there exists a positive constant M such that the following estimate holds true in $\mathbf{F}$:

$$|\mathbf{x}| \cdot \nu(D_{\mathbf{x}}F) \geqslant M. \tag{8.1}$$

For $\mathbf{x} \in \mathbf{F}$, let $A(\mathbf{x}) := D_{\mathbf{x}}F$. We consider the following linear mapping

$$\mathbf{V}(\mathbf{x}) := A(\mathbf{x})^* \cdot (A(\mathbf{x}) \cdot A(\mathbf{x})^*)^{-1}.$$

Following Rabier's terminology, $\mathbf{V}(\mathbf{x})$ is a right inverse of $A(\mathbf{x})$:

$$A(\mathbf{x}) \cdot \mathbf{V}(\mathbf{x}) = \mathrm{Id}_{T_{F(\mathbf{x})}}\, \mathbb{R}^s.$$

The mapping $\mathbf{x} \mapsto \mathbf{V}(\mathbf{x})$ is a C^{k-1} definable section of the vector bundle $F^*(T\mathbb{R}^s)|_{\mathbf{F}}$ and takes values in $NF|_{\mathbf{F}} \subset T\mathbb{R}^n|_{\mathbf{F}}$, the normal bundle of F

over $\mathbf{F}$. We follow now Rabier's proof [39, Section 4]. Consider the following differential equation with parameters:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{V}(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{c}) \\ \mathbf{x}(0) \in \mathbf{F} \end{cases} \tag{8.2}$$

where $\mathbf{y} \in \mathbf{B}$. The vector field with parameters

$$\xi(\mathbf{x}, \mathbf{y}) := \mathbf{V}(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{c})$$

is definable and C^{k-1}. If Ψ denotes the flow of ODE (8.2), observe that

$$F(\Psi(t, \mathbf{x}, \mathbf{y})) = F(\mathbf{x}) + t(\mathbf{y} - \mathbf{c}).$$

Let $\mathbf{z} = (\mathbf{x}, \mathbf{y})$ be in $\mathbf{F} \times \mathbf{B}$ and let $I_{\mathbf{z}}$ be the domain over which the solution

$$t \longmapsto \varphi(t; \mathbf{z})$$

of the ODE (8.2) exists and where $\varphi(0; \mathbf{z}) = \mathbf{x}$. Let

$$\ell(t; \mathbf{z}) := \left| \int_0^t |\mathbf{V}(\varphi(\tau; \mathbf{z}))| \mathrm{d}\tau \right|,$$

be the length of the trajectory $t \mapsto \varphi(t; \mathbf{z})$ between the time 0 and t. Using Estimate (8.1) combined with a Grönwall argument (see [8]), we find that

$$\ell(t; \mathbf{z}) \leqslant \frac{1}{M} \cdot |\mathbf{y} - \mathbf{c}| \cdot |\mathbf{x}| \cdot e^{\frac{t}{M}}.$$

From here we conclude as in the proof of [39, Lemma 4.2], that for each $\mathbf{x} \in F^{-1}(\mathbf{c})$ the interval $I_{\mathbf{z}}$ contains $[-1, 1]$. The trivialization is achieved as the C^{k-1} diffeomorphism

$$F^{-1}(\mathbf{c}) \times \mathbf{B} \longrightarrow \mathbf{F}, \quad (\mathbf{x}, \mathbf{y}) \longmapsto \Psi(1, \mathbf{x}, \mathbf{y}). \tag{8.3}$$ □

Let $\mathcal{V}$ be any connected component of $\mathbb{R}^s \setminus K(F)$. Let $\mathcal{U}_1, \ldots, \mathcal{U}_\alpha$ be the connected components of $F^{-1}(\mathcal{V})$ and F^a be the restriction of F to $\mathcal{U}_a$. Theorem 8.2 implies the following:

Corollary 8.3. — *For each $a = 1, \ldots, \alpha$, the mapping $F^a : \mathcal{U}_a \mapsto \mathcal{V}$ induces a locally trivial C^{k-1} fibre bundle structure over $\mathcal{V}$, with connected model fibre $F^{-1}(\mathbf{c}_a) \cap \mathcal{U}_a$ for some $\mathbf{c}_a \in \mathcal{V}$.*

Of interest is the following

Remark 8.4. — Assume that F is real analytic and globally sub-analytic. The proof of Theorem 8.2 shows that nearby (MR)-regular values the produced flow of Equation (8.3) is real analytic, since ξ is a real analytic mapping.

Definition 8.5. — *A value* $\mathbf{c}$ *of* $\mathbb{R}^s$ *is a* bifurcation value of the mapping F *if it is not typical. Let* $\mathrm{Bif}(F)$ *be the set of bifurcation values of* F*.*

The (expected) corollary of Theorem 8.2 and Definition 8.5 is the following

Corollary 8.6. — $\mathrm{Bif}(F) \subset K(F)$.

9. Sub-level sets family associated with a family of hypersurfaces

This is the setting that will be used in Section 16.

Let $F : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$ be a C^2 definable function such that 0 is a regular value of F. Define the following C^2 closed definable subsets of $\mathbb{R}^n \times \mathbb{R}^s$,

$$\mathcal{W} := \{\mathbf{p} \in \mathbb{R}^n \times \mathbb{R}^s : F(\mathbf{p}) \leqslant 0\}, \text{ and } W := \{\mathbf{p} \in \mathbb{R}^n \times \mathbb{R}^s : F(\mathbf{p}) = 0\}.$$

We assume that $\mathcal{W}$ and W are both not empty. Observe that $\mathcal{W}$ is a definable C^2 sub-manifold with definable C^2 boundary W. Let $\varphi : W \to \mathbb{R}^s$, $\omega : \mathcal{W} \to \mathbb{R}^s$ be respectively the projection mappings $(\mathbf{x}, \mathbf{y}) \to \mathbf{y}$. For each parameter $\mathbf{y} \in \mathbb{R}^s$, we define the function

$$f_{\mathbf{y}} : \mathbb{R}^n \longrightarrow \mathbb{R}, \text{ as } \mathbf{x} \longmapsto F(\mathbf{x}, \mathbf{y}),$$

yielding the definable family of C^2 functions $(f_{\mathbf{y}})_{\mathbf{y}\in\mathbb{R}^s}$. For $\mathbf{y} \in \mathbb{R}^s$ we have

$$\varphi^{-1}(\mathbf{y}) = f_{\mathbf{y}}^{-1}(0) \times \mathbf{y} \subset \mathbb{R}^n \times \mathbb{R}^s \text{ and } \omega^{-1}(\mathbf{y}) = \{f_{\mathbf{y}} \leqslant 0\} \times \mathbf{y} \subset \mathbb{R}^n \times \mathbb{R}^s.$$

Let us write

$$\mathcal{W}_{\mathbf{y}} := \{\mathbf{x} \in \mathbb{R}^n : f_{\mathbf{y}}(\mathbf{x}) \leqslant 0\}, \text{ and } W_{\mathbf{y}} := \{\mathbf{x} \in \mathbb{R}^n : f_{\mathbf{y}}(\mathbf{x}) = 0\}.$$

Whenever $\mathbf{y}$ is a regular value of φ, the level $W_{\mathbf{y}}$ is a C^2 closed definable hypersurface of $\mathbb{R}^n$ bounding the C^2 closed definable sub-manifold with boundary $\mathcal{W}_{\mathbf{y}}$.

We need some equi-singularity results about the family $(\mathcal{W}_{\mathbf{y}})_{\mathbf{y}\notin K(\varphi)}$ similar to those presented in Section 8, but for the mapping ω instead of simply φ.

Proposition 9.1. — *For every* $\mathbf{c} \notin K(\varphi)$*, there exists a closed ball* $\mathbf{B}$ *of* $\mathbb{R}^s \setminus K(\varphi)$ *centred at* $\mathbf{c}$ *such that* $\omega^{-1}(\mathbf{B})$ *is* C^1 *trivial fibre bundle over* $\mathbf{B}$ *with model fibre* $\mathcal{W}_{\mathbf{c}} \times \mathbf{c}$*.*

Proof. — Following [5, Theorem 6.11], the hypersurface W admits a definable neighbourhood in $\mathbb{R}^n \times \mathbb{R}^s$ which is C^1 definably diffeomorphic to

$$NW(\rho) \coloneqq \{(\mathbf{w},\xi) \in NW : |\xi| \leqslant \rho(\mathbf{w})\},$$

where ρ is C^2 positive definable function on W and NW is the normal bundle of W in $T(\mathbb{R}^n \times \mathbb{R}^s)$. The diffeomorphism is obtained by the restriction of "the end point mapping"

$$E : T(\mathbb{R}^n \times \mathbb{R}^s) \longrightarrow \mathbb{R}^n \times \mathbb{R}^s, \ \ (\mathbf{p},\xi) \longmapsto \mathbf{p}+\xi$$

to $NW(\rho)$. The aforementioned neighbourhood of W is defined as

$$\mathscr{T}_\rho \coloneqq E(NW(\rho)).$$

Let $N\varphi$ be the normal bundle of $T\varphi$ taken into $T(W \setminus \mathrm{crit}(\varphi))$. It is C^1, definable and of rank s. Let $N_\rho\varphi$ be the definable C^1 vector bundle of $T(\mathbb{R}^n \times \mathbb{R}^s)|_{\mathscr{T}_\rho}$ of rank s obtained by parallel transport of $N\varphi$: any $\mathbf{p} \in \mathscr{T}_\rho$ writes uniquely as $\mathbf{p} = \mathbf{w} + \xi$ with $(\mathbf{w},\xi)$ in $NW(\rho)$, yielding, when $\mathbf{w}$ is not a critical point of φ,

$$(N_\rho\varphi)_\mathbf{p} \coloneqq N_\mathbf{w}\varphi.$$

Assume that $\mathbf{B} = \mathbf{B}^s(\mathbf{c},\varepsilon)$ for some positive radius ε. For $\mathbf{p} = (\mathbf{x},\mathbf{y}) \in \omega^{-1}(\mathbf{B})$, we define

$$A(\mathbf{p}) \coloneqq D_\mathbf{p}\omega, \ \text{ and } \ V(\mathbf{p}) \coloneqq A(\mathbf{p})^* \cdot (A(\mathbf{p}) \cdot A(\mathbf{p})^*)^{-1}.$$

Observe that $V(\mathbf{p}) : T_\mathbf{y}\mathbb{R}^s \mapsto \mathbf{0} \times T_\mathbf{y}\mathbb{R}^s \subset T_\mathbf{p}\mathcal{W}$ is the "identity" mapping of $T_\mathbf{y}\mathbb{R}^s$. Similarly we define for $\mathbf{p} \in \mathscr{T}_\rho \cap \omega^{-1}(\mathbf{B})$ the following linear operators:

$$A_\rho(\mathbf{p}) \coloneqq D_\mathbf{p}\omega|_{N_\rho\varphi}, \ \text{ and } \ V_\rho(\mathbf{p}) \coloneqq A_\rho(\mathbf{p})^* \cdot (A_\rho(\mathbf{p}) \cdot A_\rho(\mathbf{p})^*)^{-1}.$$

For any $\mathbf{p}$ in $\mathscr{T}_\rho$ we have $A_\rho(\mathbf{p}) \cdot V_\rho(\mathbf{p}) = \mathrm{Id}_{T_\mathbf{y}\mathbb{R}^s}$. Note that the mapping $\mathbf{p} \to V_\rho(\mathbf{p})$ is C^1 and definable. Since any $\mathbf{p} \in \mathscr{T}_\rho$ writes uniquely as $\mathbf{p} = \mathbf{w} + \xi$ we find

$$A_\rho(\mathbf{p}) = D_\mathbf{w}\varphi|_{N_\mathbf{w}\varphi}.$$

There exist C^1 definable functions $a_0, b_0 : \mathbb{R}_{\geqslant 0} \mapsto [0,1]$ such that: (i) $a_0 + b_0 \equiv 1$; (ii) $a_0^{-1}(0) = [1,+\infty[$; (iii) $b_0^{-1}(0) = [0,\frac{1}{2}]$. Any $\mathbf{p} \in \mathscr{T}_\rho$ has a unique form $\mathbf{p} = \mathbf{w} + \xi$. Define the following mapping over $\omega^{-1}(\mathbf{B})$:

$$\mathbf{p} \longmapsto \mathbf{V}(\mathbf{p}) \coloneqq \begin{cases} V(\mathbf{p}) & \text{if } \mathbf{t} \notin \mathscr{T}_\rho, \\ V_\rho(\mathbf{p}) & \text{if } \mathbf{t} \in \mathscr{T}_{\frac{\rho}{2}}, \\ a_0\left(\frac{|\xi|}{\rho(\mathbf{p})}\right) V_\rho(\mathbf{p}) + b_0\left(\frac{|\xi|}{\rho(\mathbf{p})}\right) V(\mathbf{p}) & \text{if } \mathbf{t} \in \mathscr{T}_\rho \setminus \mathscr{T}_{\frac{\rho}{2}}. \end{cases}$$

It is definable and C^1 and satisfies

$$D_{\mathbf{p}}\omega \cdot \mathbf{V}(\mathbf{p}) = \mathrm{Id}_{T_{\mathbf{y}}\mathbb{R}^s} .$$

Observe that whenever $\mathbf{p}$ lies in $\mathcal{W} \setminus W$, we have $\nu(D_{\mathbf{p}}\omega) = 1$. By definition of $\mathbf{c}$ and $\mathbf{B}$ there exists a positive constant M (depending on $\mathbf{B}$) such that

$$\mathbf{w} \in \varphi^{-1}(\mathbf{B}) \Longrightarrow (1 + |\mathbf{w}|) \cdot \nu^{\varphi}(\mathbf{w}) \geqslant M.$$

Observe that for any $\mathbf{p} = \mathbf{w} + \xi \in \mathscr{T}_\rho \cap \omega^{-1}(\mathbf{B})$, we have

$$\nu(A_\rho(\mathbf{p})) = \nu^{\varphi}(\mathbf{w}).$$

We consider again the ODE (8.2). The remarks about the estimates of $\nu(D_{\mathbf{p}}\omega)$ and $\nu^{\varphi}(\mathbf{w})$ are just to guarantee that Grönwall arguments of the proof of Theorem 8.2 will go through so that the rest of this proof adapts readily to our current $\mathbf{V}$, and produces a C^1 flow that gives the announced trivialisation. $\square$

Let $\mathcal{V}$ be any connected component of $\mathbb{R}^s \setminus K(\varphi)$. Let $\mathcal{U}_1, \ldots, \mathcal{U}_\alpha$ be the connected components of $\omega^{-1}(\mathcal{V})$ and ω^a be the restriction of ω to $\mathcal{U}_a$. Proposition 9.1 implies the following

Corollary 9.2. — *For each $a = 1, \ldots, \alpha$, the mapping $\omega^a : \mathcal{U}_a \to \mathcal{V}$ induces a locally trivial fibre bundle structure over $\mathcal{V}$, with connected model fibre $\omega^{-1}(\mathbf{c}_a) \cap \mathcal{U}_a$ for some $\mathbf{c}_a \in \mathcal{V}$.*

10. On the link at infinity

Let S be a closed definable subset of $\mathbb{R}^p$. For any positive R, let

$$S_R := S \cap \mathbf{S}_R^{p-1}.$$

The family $(S_R)_{R>0}$ is a definable family, thus there exists a positive radius r_S such that the topological type of S_R is constant once $R > r_S$ (see for instance [5, Theorem 5.22]). In order to use the techniques developed in [15, 16], we recall the following key notion:

Definition 10.1. — *The* link at infinity $\mathrm{Lk}^\infty(S)$ of S *is any subset S_R for $R > r_S$.*

Let W be a C^2 connected closed definable sub-manifold of $\mathbb{R}^n \times \mathbb{R}^s$ and let $\varphi : W \to \mathbb{R}^s$ be the restriction to W of the canonical projection $\mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^s$. For any value $\mathbf{y}$ of $\mathbb{R}^s$, we recall that $\varphi^{-1}(\mathbf{y}) = W_{\mathbf{y}} \times \mathbf{y}$.

Let $\mathcal{V}$ be any connected component of $\mathbb{R}^s \setminus K(\varphi)$. Let $\mathcal{U}^1, \ldots, \mathcal{U}^\alpha$ be the connected components of the germ at infinity of $\varphi^{-1}(\mathcal{V})$.

Let us write $(\varphi|_{\mathcal{U}^a})^{-1}(\mathbf{y}) = W^a_{\mathbf{y}} \times \mathbf{y}$.

The following result is probably known folklore, yet we will provide a proof.

Proposition 10.2. — *Let $\mathbf{c}$ be a regular and (MR)-regular value of φ lying in $\mathcal{V}$. For each $a = 1, \ldots, \alpha$, there exists a closed ball $\mathbf{B}$ of $\mathbb{R}^s$ centred at $\mathbf{c}$ and not intersecting with $K(\varphi)$ and there exists $R_{\mathbf{B}} > 0$ such that for each $\mathbf{y} \in \mathbf{B}^\circ$ and each $R \geqslant R_{\mathbf{B}}$ the links $(\varphi|_{\mathcal{U}^a})^{-1}(\mathbf{y})_R$ and $(\varphi|_{\mathcal{U}^a})^{-1}(\mathbf{c})_{R_{\mathbf{B}}}$ are C^1 diffeomorphic, and the links $(W^a_{\mathbf{y}})_R$ and $(W^a_{\mathbf{c}})_{R_{\mathbf{B}}}$ are C^1 diffeomorphic as well.*

Proof. — We can assume that $\alpha = 1$. Let $\psi : W \to \mathbb{R}_{\geqslant 0} \times \mathbb{R}^s$ be defined as $\mathbf{w} \mapsto (|\mathbf{x}|^2, \mathbf{y})$ when $\mathbf{w} = (\mathbf{x}, \mathbf{y})$. Lemma 6.5 implies that there exist a closed ball $\mathbf{B}$ centred at $\mathbf{c}$ and a positive radius R_1 such that ψ is a proper submersion over $\varphi^{-1}(\mathbf{B}) \cap \{(\mathbf{x}, \mathbf{y}) : |\mathbf{x}| \geqslant R_1\}$. Observe that the following closed definable subset

$$\mathbf{F} := \varphi^{-1}(\mathbf{B}) \cap \{(\mathbf{x}, \mathbf{y}) : |\mathbf{x}| > R_1\}.$$

is C^2 sub-manifold with boundary. The restriction of φ to the definable C^2 sub-manifold with boundary $(\mathbf{F}, \partial\mathbf{F})$ is a C^2 submersion. Ehresmann Fibration Theorem over $\mathbf{B}^\circ$ yields the result once $R_{\mathbf{B}} > R_1$.

To get the result for the levels of φ instead of the family $(W_{\mathbf{y}})_{\mathbf{y}}$, we follow the same scheme of proof as the previous one, only changing ψ into the mapping $\mathbf{w} \mapsto (|\mathbf{w}|^2, \mathbf{y})$. □

We complete this section dealing also with the links at infinity of the "family of the sub-level sets" of the hypersurface family context presented in Section 9. We keep the exact same notations.

Let $\mathcal{V}$ be any connected component of $\mathbb{R}^s \setminus K(\varphi)$. Let $\mathcal{U}^1, \ldots, \mathcal{U}^\alpha$ be the connected components of the germ at infinity of $\omega^{-1}(\mathcal{V})$. Let us write $(\omega|_{\mathcal{U}^a})^{-1}(\mathbf{y}) = \mathcal{W}^a_{\mathbf{y}} \times \mathbf{y}$. We have the following result.

Proposition 10.3. — *Let $\mathbf{c}$ be a regular and (MR)-regular value of $\mathcal{V}$. For every $a = 1, \ldots, \alpha$, there exists a closed ball $\mathbf{B}$ of $\mathbb{R}^s$ centred at $\mathbf{c}$ and not intersecting with $K(\varphi)$ and there exists $R_{\mathbf{B}} > 0$ such that for each $\mathbf{y} \in \mathbf{B}^\circ$ and each $R \geqslant R_{\mathbf{B}}$ the links $(\omega|_{\mathcal{U}^a})^{-1}(\mathbf{y})_R$ and $(\omega|_{\mathcal{U}^a})^{-1}(\mathbf{c})_{R_{\mathbf{B}}}$ are C^1 diffeomorphic, and the links $(\mathcal{W}^a_{\mathbf{y}})_R$ and $(\mathcal{W}^a_{\mathbf{c}})_{R_{\mathbf{B}}}$ are C^1 diffeomorphic as well.*

Proof. — We can assume that $\mathcal{V} = \mathbb{R}^s \setminus K(\varphi)$ and the germ at infinity of $\omega^{-1}(\mathcal{V})$ are both connected.

Consider the mapping $\psi : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}_{\geqslant 0} \times \mathbb{R}^s$ defined as $(\mathbf{x}, \mathbf{y}) \mapsto (|\mathbf{x}|^2, \mathbf{y})$. For $\mathbf{c} \in \mathbb{R}^s \setminus K(\varphi)$, Proposition 10.2 implies that there exist a small neighbourhood U of $\mathbf{c}$ in $\mathbb{R}^s$ and a compact subset K_0 of $\mathbb{R}^n$ such that

$$(\varphi^{-1}(U) \setminus K_0 \times U) \cap \operatorname{crit}(\psi|_W) = \emptyset.$$

Observe also the obvious fact that $\operatorname{crit}(\psi) = \mathbf{0} \times \mathbb{R}^s$. Let $\mathbf{B}$ be a closed ball of $\mathbb{R}^s$ centred at $\mathbf{c}$ of radius $r > 0$. The subset $K_1 := \mathbf{B}_1^n \times \mathbf{B}$ is compact in $\mathbb{R}^n \times \mathbb{R}^s$.

Claim 10.4. — $(\varphi^{-1}(\mathbf{B}) \setminus K_1) \cap \operatorname{crit}(\psi) = \emptyset$.

Proof. — If $(\mathbf{x}, \mathbf{y}) \in \varphi^{-1}(\mathbf{B})$ then $\mathbf{y} \in \mathbf{B}$. If $(\mathbf{x}, \mathbf{y})$ lies in $\operatorname{crit}(\psi)$, then $\mathbf{x} = \mathbf{0}$. □

Up to reducing r, we can assume that the closed ball $\mathbf{B}$ is contained in U. Let $R_1 > 0$ be such that $\mathbf{B}_{R_1}^n$ contains $K_0 \cup \mathbf{B}_1^n$. Define

$$K := \mathbf{B}_{R_1}^n \times \mathbf{B}$$

so that $(K_0 \times \mathbf{B}) \cup K_1$ is a subset of K. Claim 10.4 implies that the mapping $\psi|_{\mathcal{W}}$ is a proper submersion over $\varphi^{-1}(\mathbf{B}) \setminus K$. Ehresmann Fibration Theorem for a manifold with boundary applies. Therefore we have obtained the announced result of constancy of the C^1 type. Since

$$\psi^{-1}(\mathbb{R} \times \mathbf{y}) \cap \mathcal{W} = \mathcal{W}_{\mathbf{y}} \times \mathbf{y}$$

we conclude that for all $\mathbf{y} \in \mathbf{B}^\circ$, the links $\operatorname{Lk}^\infty(\mathcal{W}_{\mathbf{y}})$ and $\operatorname{Lk}^\infty(\mathcal{W}_{\mathbf{c}})$ are C^1-diffeomorphic.

To get the result for the levels of ω instead of the family $(\mathcal{W}_{\mathbf{y}})_{\mathbf{y}}$, there is just to follow the same scheme of demonstration as the previous one, only modifying ψ to become $\mathbf{p} \mapsto (|\mathbf{p}|^2, \mathbf{y})$. □

11. Hyperplane sections and Malgrange–Rabier Condition

We start with the following simple lemma:

Lemma 11.1 (see also [15]). — *Let S be a connected definable C^2 sub-manifold of $\mathbb{R}^q$ of positive codimension. For each $k \geqslant \dim S$, there exists an open dense and definable subset Ω_S^k of $\mathbf{G}(k, q)$ such that $S \cap P$ is transverse (but possibly at the origin) for any k-plane P of Ω_S^k.*

Proof. — Let $\mathbf{0}$ be the origin of $\mathbb{R}^q$ and let $S^* := S \setminus \mathbf{0}$. Let $k \geqslant \dim S$. Let us consider the following subset

$$B_k := \{(\mathbf{x}, P) \in S^* \times \mathbf{G}(k, q) : \mathbf{x} \in P\}.$$

It is a closed definable C^2 sub-manifold of $S^* \times \mathbf{G}(k, q)$ of dimension $\dim S + (k-1)(q-k)$ since it is a fibre bundle over S^* with model fibre $\mathbf{G}(k-1, q-1)$.

Let π_k be the projection $B_k \to \mathbf{G}(k,q)$. It is obviously a C^2 definable mapping. The subset $\Delta = \pi_k(\mathrm{crit}(\pi_k))$ of its critical values is definable in $\mathbf{G}(k,q)$ and of positive codimension. The complement $\mathbf{G}(k,q) \setminus \mathrm{clos}(\Delta)$ is the desired looked for Ω_S^k. $\square$

Of course the proof works for any k, but for $q - k \geqslant \dim S$, we have $\pi(B_k) = \Delta$, so that the fibre of π_k over $P \notin \Delta$ is empty.

Since any linear subspace P of dimension $k - s$ of $\mathbb{R}^n$ gives rise to a unique linear subspace $P \times \mathbb{R}^s$ of $\mathbb{R}^n \times \mathbb{R}^s$ of dimension k, we check that the following subset

$$\Pi(k) := \{P \times \mathbb{R}^s \in \mathbf{G}(k, n+s) \,:\, P \in \mathbf{G}(k-s, n)\}$$

is a non-singular projective sub-variety of $\mathbf{G}(k, n+s)$ isomorphic to $\mathbf{G}(k-s, n)$.

The first main new result of the paper is the following:

Theorem 11.2. — *Let $\mathbf{c} \notin K(\varphi)$. For every $k \geqslant n + s - \dim W$, there exists a definable open dense subset $\Omega_{\mathbf{c}}^k$ of $\Pi(k)$ consisting of k-vector planes $P \times \mathbb{R}^s$ such that the value $\mathbf{c}$ does not lie in $K(\varphi|_{P\times\mathbb{R}^s})$.*

Proof. — Observe that for any vector p-plane P of $\mathbb{R}^n$ with $p \geqslant 1$, we have

$$\beta^{-1}(P) =]0, +\infty[\times \mathbf{S}(P) \subset \overline{\mathbf{M}}.$$

Let us define the "p-plane" induced from P in $\overline{\mathbf{M}}$

$$P^+ := [0, +\infty[\times \mathbf{S}(P) \subset \overline{\mathbf{M}}.$$

Let $\mathbf{c}$ be a (regular) value of φ. The set of accumulation points at infinity of the value $\mathbf{c}$ of the mapping φ is the closed, definable subset of $\mathbf{S}^{n-1} \times \mathbf{0} \subset \mathbf{S}^{n+s-1}$ defined as

$$W_{\mathbf{c}}^\infty := \left\{ \mathbf{u} \in \mathbf{S}^{n+s-1} \,;\, \begin{array}{c} \exists\,(\mathbf{w}_k)_{k\in\mathbb{N}} \subset W \,: \\ \mathbf{w}_k \longrightarrow \infty,\ \mathbf{y}_k \longrightarrow \mathbf{c},\quad \dfrac{\mathbf{w}_k}{|\mathbf{w}_k|} \longrightarrow \mathbf{u} \end{array} \right\}. \tag{11.1}$$

We write $W_{\mathbf{c}}^\infty = A_{\mathbf{c}}^\infty \times \mathbf{0}$. We recall that $Z = \beta_s^{-1}(W)$ and that $Z^\infty = \mathrm{clos}(Z) \cap \overline{\mathbf{M}}_s^\infty$. Observe that

$$Z_{\mathbf{c}}^\infty := Z^\infty \cap 0 \times \mathbf{S}^{n-1} \times \mathbf{c} = 0 \times A_{\mathbf{c}}^\infty \times \mathbf{c}.$$

We recall that $\mathbf{z} = (r, \mathbf{u}, \mathbf{y}) \in \overline{\mathbf{M}}_s$, and $\mathbf{r}_Z(\mathbf{z}) = r$ for $\mathbf{z} \in Z$. We need the following intermediary result:

Claim 11.3. — *Let* $\mathbf{c} \in \mathbb{R}^s \setminus K(\varphi)$. *There exists a definable stratification of* $Z^\infty_{\mathbf{c}}$ *such that for any point* $\mathbf{u}$ *of* $A^\infty_{\mathbf{c}}$ *and for any sequence* $(\mathbf{z}_k)_{k\in\mathbb{N}}$ *of* Z *such that* (i) $r_k \to 0$, (ii) $\mathbf{y}_k \to \mathbf{c}$, (iii) $\mathbf{u}_k \to \mathbf{u}$, *and* (iv) $T_{\mathbf{z}_k}\mathbf{r}_Z \to R$, *we have*

$$T_{\mathbf{u}}S \subset R,$$

where S *is the stratum of* $Z^\infty_{\mathbf{c}}$ *containing* $0 \times \mathbf{u} \times \mathbf{c}$.

Proof of the claim. — The function $\mathbf{r}_Z$ extends continuously and definably to 0 on Z^∞. Following [30, 31], it can be definably stratified $(\mathbf{a}_{\text{rel}})$. We can further ask that $0 \times A^\infty_{\mathbf{c}} \times \mathbf{c}$ is a union of strata. Any stratum of $0 \times A^\infty_{\mathbf{c}} \times \mathbf{c}$ is of the form $0 \times S \times \mathbf{c}$, for some sub-manifold S of $\mathbf{S}^{n-1}$. □

Let us write again $W_{\mathbf{c}} \times \mathbf{c} := \varphi^{-1}(\mathbf{c})$. It is more convenient to work in a neighbourhood of $\overline{\mathbf{M}}^\infty_s$, more precisely nearby $Z^\infty_{\mathbf{c}}$.

Let $S_1, \ldots, S_l$ be the strata of $A^\infty_{\mathbf{c}}$ obtained from Claim 11.3.

Let $k \leqslant n+s$ be any integer such that $k + \dim W - s \geqslant n$. Let us consider the following definable open dense subset of $\mathbf{G}(k-s, n)$

$$\Omega^k_{\mathbf{c}} := \cap_{j=0}^{l} \Omega^{k-s}_{S_j},$$

where $\Omega^{k-s}_{S_j}$ is the definable open dense subset of $\mathbf{G}(k-s,n)$ of Lemma 11.1 corresponding to S_j.

Assume that $\dim W - s \geqslant n - k + 1$.

Let P be a $(k-s)$-plane of $\Omega^k_{\mathbf{c}}$. Let $(\mathbf{z}_m)_m$ be a sequence of $P^+ \times \mathbb{R}^s \setminus Z^\infty$ such that $\mathbf{z}_m \to \mathbf{z}_\infty = (0, \mathbf{u}, \mathbf{c}) \in Z^\infty_{\mathbf{c}}$. We can assume that

$$T_{\mathbf{z}_m}\mathbf{r}_Z \longrightarrow R, \text{ and } \mathbf{u}_m \longrightarrow \mathbf{u} \in (0 \times A^\infty_{\mathbf{c}} \cap P^+) \times \mathbf{0}.$$

Since $\mathbf{u} \in P^+$, we deduce that $P = \mathbb{R}\mathbf{u} \oplus T_{\mathbf{u}}\mathbf{S}(P)$.
All the computations which will follow are done in

$$T_{\mathbf{z}_\infty}\overline{\mathbf{M}}_s = \mathbb{R} \times T_{\mathbf{u}}\mathbf{S}^{n-1} \times \mathbb{R}^s = \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}^s.$$

Therefore we write P for $\mathbb{R} \times T_{\mathbf{u}}\mathbf{S}(P) = T_{(0,\mathbf{u})}P^+ \subset \mathbb{R} \times \mathbb{R}^{n-1}$.
By hypothesis $R = 0 \times R_1 \subset 0 \times \mathbb{R}^{n-1} \times \mathbb{R}^s$ and the space R projects surjectively onto $\mathbb{R}^s$ (see Definition 5.2 and Lemma 7.1). We want to show that $R \cap P \times \mathbb{R}^s$ projects surjectively onto $\mathbb{R}^s$. Let

$$0 \times K_R \times \mathbf{0} := R \cap (\mathbb{R} \times \mathbb{R}^{n-1}) \times \mathbf{0} = \ker(R \longmapsto \mathbb{R}^s).$$

We can assume that $K_R = \mathbb{R}^p \times \mathbf{0} \subset \mathbb{R}^{n-1}$ where $p = \dim W - s - 1$. By hypothesis, denoting by S the stratum of $A^\infty_{\mathbf{c}}$ containing $\mathbf{u}$, we have

$$T_{\mathbf{u}}S \subset K_R, \text{ and } P + T_{\mathbf{u}}S = \mathbb{R} \times \mathbb{R}^{n-1}.$$

Since $T_{\mathbf{u}}S$ is contained in $0 \times \mathbb{R}^{n-1}$, the k-plane P is not. Let $0 \times P_1 := P \cap 0 \times \mathbb{R}^{n-1}$. Since we find

$$K_R + P_1 = \mathbb{R}^{n-1},$$

we deduce the following key fact

$$\mathbf{0} \times \mathbb{R}^{n-1-p} \subset P_1.$$

Let $N_R := K_R^{\perp} \cap R$ be the orthogonal complement of K_R in R, thus the projection to $\mathbb{R}^s$ when restricted to N_R is an isomorphism. Since the space N_R is contained in $\mathbf{0} \times \mathbb{R}^{n-1-p} \times \mathbb{R}^s$, we deduce that

$$N_R = (0 \times P_1 \times \mathbb{R}^s) \cap N_R \subset (P \times \mathbb{R}^s) \cap R.$$

In other words $R \cap (P \times \mathbb{R}^s)$ projects surjectively onto $\mathbb{R}^s$.

Assume $\dim W - s = n - k$. For each generic k-plane P of $\Omega_{\mathbf{c}}^k$, the fibre $(\varphi|_{P\times\mathbb{R}^s})^{-1}(\mathbf{c})$ is finite. The definition of $W_{\mathbf{c}}^{\infty}$ prohibits, when it is not empty, that $\mathbf{c}$ be a properness value of $\varphi|_{P\times\mathbb{R}^s}$, therefore $\mathbf{c}$ lies in $K(\varphi|_{P\times\mathbb{R}^s})$ if $W_{\mathbf{c}}^{\infty}$ is not empty. In such a case, the arguments used in the previous case show the existence of a surjective mapping from a space of dimension $s-1$ onto $\mathbb{R}^s$. □

Let $F : X \to \mathbb{R}^s$ be a C^2 definable mapping over a closed connected C^2 definable sub-manifold X of $\mathbb{R}^n$ with $\dim X \geqslant s$. The original goal of this section, consequence of Theorem 11.2, is the following result:

Corollary 11.4. — *Let $\mathbf{c}$ be a value not in $K(F)$. For every $k \geqslant n-(\dim X - s)$, there exists a definable and dense open subset $\mathcal{V}_{\mathbf{c}}^k$ of $\mathbf{G}(k,n)$ such that for every plane P of $\mathcal{V}_{\mathbf{c}}^k$ the value $\mathbf{c}$ does not lie in $K(F|_P)$.*

Proof. — Let W be the graph of F and let $\varphi : W \to \mathbb{R}^s$ be the restriction to W of the projection $\mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^s$. The regular mapping

$$\Pi(k+s) \xmapsto{\iota_n} \mathbf{G}(k,n),\ P \longmapsto P_n := P \cap \mathbb{R}^n \times 0$$

is an isomorphism. Let $\mathbf{c} \in \mathbb{R}^s$ which does not lie in $K(\varphi) = K(F)$. For any $l \geqslant (n+s)-(\dim W - s)$, let $\mathcal{U}_{\mathbf{c}}^{l+s}$ be the open dense subset of Theorem 11.2. The image

$$\mathcal{V}_{\mathbf{c}}^k := \iota_n(\mathcal{U}_{\mathbf{c}}^{k+s})$$

is definable, open and dense in $\mathbf{G}(k,n)$. □

12. Lipschitz–Killing Measures

We present very briefly in this section the Lipschitz–Killing measures of a definable set in an o-minimal structure following Bröcker and Kuppe's approach [2]. They are the essential ingredients to define the functions $\Lambda_k^\infty(-)$ (Definition 12.2), bricks of the general Gauss–Bonnet Formula presented in Theorem 12.3.

We start with a few reminders about the extrinsic geometry of sub-manifolds of the Euclidean space $\mathbb{R}^n$.

Let Z be a C^2 connected orientable sub-manifold of $\mathbb{R}^n$ equipped with the restriction of the Euclidean metric tensor. Let $\mathbf{x}$ be a point of Z. Let $\mathbf{S}(N_\mathbf{x}Z)$ be the unit sphere of the normal space $N_\mathbf{x}Z = T_\mathbf{x}Z^\perp$ in $T_\mathbf{x}\mathbb{R}^n = \mathbb{R}^n$. For $\mathbf{v}$ a given vector of $\mathbf{S}(N_\mathbf{x}Z)$, let $\mathbf{v}^*$ be the linear form over $T_\mathbf{x}\mathbb{R}^n$ defined by the scalar product with $\mathbf{v}$.

Let ∇ be the covariant differentiation in $\mathbb{R}^n$ w.r.t. the Euclidean metric tensor.

Of fundamental importance to define the Lipschitz–Killing curvatures of Z at any of its points is the family of second fundamental forms $(II_\mathbf{n})_{\mathbf{n}\in\mathbf{S}(NZ)}$, where $\mathbf{S}(NZ)$ is the unit sphere bundle of the normal bundle NZ of Z in $T\mathbb{R}^n|_Z$. Given $\mathbf{n} = (\mathbf{x}, \mathbf{v})$ in $\mathbf{S}(NZ)$, we recall that $II_\mathbf{n}$, *the second fundamental form in the direction* $\mathbf{v}$, is the symmetric bilinear form over $T_\mathbf{x}Z$ defined as

$$II_\mathbf{n}(\mathbf{u}_1, \mathbf{u}_2) = -\langle \nabla_{\mathbf{u}_1}\nu, \mathbf{u}_2\rangle,$$

where $\mathbf{u}_1, \mathbf{u}_2$ are vectors of $T_\mathbf{x}Z$ and ν is any C^1 local extension of $\mathbf{v}$ normal to Z at $\mathbf{x}$.

For each $l = 0, \dots, \dim Z$, let $\sigma_l^Z(\mathbf{n})$ be the l-th elementary symmetric function of the eigenvalues of $II_\mathbf{n}$ when considered as a symmetric endomorphism of $T_\mathbf{x}Z$.

Let X be a closed definable subset of $\mathbb{R}^n$ equipped with $\mathcal{S} = \{S_a\}_{a\in A}$, a finite C^2 definable Whitney stratification. Let S be a stratum S_a of dimension d_S. Let $\mathbf{x}$ be a point of S and let $\mathbf{v}$ be a unit vector normal to S at $\mathbf{x}$. We recall the definition of the following index

$$\mathrm{ind}_{\mathrm{nor}}(\mathbf{v}^*, X, \mathbf{x}) = 1 - \chi\left(X \cap N_\mathbf{x} \cap \mathbf{B}^n(\mathbf{x}, \varepsilon) \cap \{\mathbf{v}^* = \mathbf{v}^*(\mathbf{x}) - \delta\}\right),$$

where $0 < \delta \ll \varepsilon \ll 1$ and $N_\mathbf{x}$ is a normal (definable) slice to S at $\mathbf{x}$ in $\mathbb{R}^n$.

For each $k = 0, \ldots, n$, we define the function $\lambda_k^S : S \to \mathbb{R}$ as

$$\lambda_k^S(\mathbf{x}) := \begin{cases} \frac{1}{s_{n-k-1}} \int_{\mathbf{S}(N_\mathbf{x} S)} \mathrm{ind}_{\mathrm{nor}}(v^*, X, \mathbf{x}) \sigma_{d_S - k}^S(\mathbf{x}, \mathbf{v}) \, \mathrm{d}\mathbf{v} & \text{if } 0 \leqslant k \leqslant d_S, \\ 0 & \text{if } d_S + 1 \leqslant k \leqslant n, \end{cases}$$

where s_l is the l-volume of the unit Euclidean sphere $\mathbf{S}^l$ of $\mathbb{R}^{l+1}$.

If S has dimension n then for all $\mathbf{x} \in S$, we find

$$\lambda_0^S = \cdots = \lambda_{n-1}^S \equiv 0 \ \text{ and } \ \lambda_n^S \equiv 1.$$

If S has dimension 0 then

$$\begin{aligned} \mathrm{ind}_{\mathrm{nor}}(\mathbf{v}^*, X, \mathbf{x}) &= \mathrm{ind}(\mathbf{v}^*, X, \mathbf{x}) \\ &:= 1 - \chi\left(X \cap \mathbf{B}^n(\mathbf{x}, \varepsilon) \cap \{\mathbf{v}^* = \mathbf{v}^*(\mathbf{x}) - \delta\}\right), \end{aligned}$$

and we set

$$\lambda_0^S(\mathbf{x}) = \frac{1}{s_{n-1}} \int_{\mathbf{S}^{n-1}} \mathrm{ind}(\mathbf{v}^*, X, \mathbf{x}) \mathrm{d}\mathbf{v}, \ \text{ and } \ \lambda_k^S \equiv 0, \ \text{ once } \ k \geqslant 1.$$

Definition 12.1. — *Let* $k \in \{0, \ldots, n\}$. *The* k-th Lipschitz–Killing measure $\Lambda_k(X, -)$ of X *is defined as follows:*

$$U \longmapsto \Lambda_k(X, U) = \sum_{a \in A} \int_{S_a \cap U} \lambda_k^{S_a}(\mathbf{x}) \, \mathrm{d}\mathbf{x},$$

where U *is any bounded Borel subset of* X*.*

Denoting by d the dimension of X, we obviously have

$$\Lambda_{d+1}(X, -) = \cdots = \Lambda_n(X, -) \equiv 0,$$

and for any bounded Borel subset U of X we get

$$\Lambda_d(X, U) = \mathcal{H}_d(U),$$

where $\mathcal{H}_d$ is the d-th dimensional Hausdorff measure in $\mathbb{R}^n$.

An exhaustive family of compact subsets $(K_R)_{R>0}$ of X is an increasing, for the inclusion, family of compacts of X covering X:

$$\bigcup_{R>0} K_R = X.$$

We introduce some new notations in order to present short formulae. Let g_n^l be the volume of the Grassmann manifold $\mathbf{G}(l, n)$ of vector l-planes of $\mathbb{R}^n$ when equipped with the Euclidean metric (see Section 4). We start with the following two sets of numbers:

Definition 12.2. — *Let X be a closed definable subset of $\mathbb{R}^n$. For each $l = 0, \ldots, n$, let $\chi_l^\infty(X)$ be defined as*

$$\chi_l^\infty(X) := \frac{1}{2g_n^l} \int_{\mathbf{G}(l,n)} \chi\left(\mathrm{Lk}^\infty(X \cap P)\right) \mathrm{d}P, \tag{12.1}$$

and let $\Lambda_l^\infty(X)$ be the l-th Lipschitz–Killing invariant of X at infinity *defined as*

$$\Lambda_l^\infty(X) := \lim_{R \to +\infty} \frac{\Lambda_l(X, X \cap \mathbf{B}_R^n)}{b_l R^l}, \tag{12.2}$$

where b_l is the volume of the Euclidean unit ball $\mathbf{B}_1^l$.

Observe that for $l = n$ we have the obvious equality

$$\chi(\mathrm{Lk}^\infty(X)) = 2\chi_n^\infty(X). \tag{12.3}$$

When X is a closed definable connected sub-manifold, note that $\chi_k^\infty(X) = 0$ whenever $\mathrm{Lk}^\infty(X \cap P)$ has odd dimension, that is when $\dim X - (n-k)$ is even. When $n - k \geqslant \dim X$ we also find that

$$\chi_k^\infty(X) = 0,$$

since the link $\mathrm{Lk}^\infty(X \cap P)$ is empty for P lying in a definable open dense subset of $\mathbf{G}(k,n)$.

We recall now several Gauss–Bonnet type formulas for a closed definable set X subset of $\mathbb{R}^n$ established by the first author in [15, 16] and which relate the numbers $\chi_l^\infty(-)$ and $\Lambda_l^\infty(-)$.

Theorem 12.3 ([15, 16]).

(1) *The limit*

$$\Lambda_0^\infty(X) := \lim_{R \to +\infty} \Lambda_0(X, X \cap K_R)$$

exists and does not depend on the choice of the exhaustive family of compact subsets $(K_R)_{R>0}$. More precisely the following equality holds:

$$\Lambda_0^\infty(X) = \chi(X) - \chi_n^\infty(X) - \chi_{n-1}^\infty(X). \tag{12.4}$$

(2) *For each $k = 1, \ldots, n-2$, we furthermore have:*

$$\Lambda_k^\infty(X) = -\chi_{n-k-1}^\infty(X) + \chi_{n-k+1}^\infty(X), \tag{12.5}$$

and for $l = n-1, n$, we have

$$\Lambda_l^\infty(X) = \chi_{n-l+1}^\infty(X). \tag{12.6}$$

Remark 12.4. — Consider the following two vectors of $\mathbb{R}^{n+1}$

$$\Lambda_*^\infty(X) = (\Lambda_0^\infty(X), \dots, \Lambda_n^\infty(X))$$

and

$$\chi_{-*}^\infty(X) = (\chi(X), \chi_n^\infty(X), \dots, \chi_1^\infty(X)).$$

As seen in Theorem 12.3, both vectors carry the same information about X at infinity. Precisely, there exists a triangular superior matrix L, depending only on n, with coefficients in $\{-1, 1\}$ and only with 1 on the diagonal such that

$$\Lambda_*^\infty(X) = L \cdot \chi_{-*}^\infty(X).$$

13. The case of sub-manifolds with boundary

We describe the Lipschitz–Killing measures when $(X, \partial X)$ is a closed C^2 definable sub-manifold with boundary of $\mathbb{R}^n$ of dimension d. In this case, the partition $X = X^\circ \sqcup \partial X$, where $X^\circ = X \setminus \partial X$, is a C^2 definable Whitney stratification of X, to which we can apply the construction of Section 12.

Let $\mathbf{x} \in X^\circ$. If $d < n$ then for $k = 0, \dots, d$, we have

$$\lambda_k^{X^\circ}(\mathbf{x}) = \frac{1}{s_{n-k-1}} \int_{\mathbf{S}(N_\mathbf{x} X^\circ)} \sigma_{d-k}^{X^\circ}(\mathbf{x}, \mathbf{v}) \mathrm{d}\mathbf{v}.$$

We get the following identities of the extrinsic geometry of Euclidean sub-manifolds

$$\lambda_k^{X^\circ}(\mathbf{x}) = \frac{1}{s_{n-k-1}} K_{d-k}(X^\circ, \mathbf{x}),$$

where $K_{d-k}(X^\circ, -)$ is the $(d-k)$-th Lipschitz–Killing curvature of X°. We recall that

$$d-k \ \text{ odd } \implies K_{d-k}(X^\circ, -) \equiv 0.$$

We recall that if $d = n$, then $\lambda_n^{X^\circ} \equiv 1$ and $\lambda_0^{X^\circ} = \dots = \lambda_{n-1}^{X^\circ} \equiv 0$.

Let $\mathbf{y} \in \partial X$ and let $\nu_\mathbf{y}$ be the unit vector tangent to X at $\mathbf{y}$, normal to the boundary ∂X and pointing inwards. For $\mathbf{v} \in \mathbf{S}^{n-1}$ we recall that the following alternative holds

$$\mathrm{ind}_{\mathrm{nor}}(\mathbf{v}^*, X, y) = \begin{cases} 1 & \text{if } \langle \mathbf{v}, \nu_\mathbf{y} \rangle > 0, \\ 0 & \text{if } \langle \mathbf{v}, \nu_\mathbf{y} \rangle < 0. \end{cases}$$

Let $\mathbf{S}(N_\mathbf{y} \partial X)^+$ be the following open half-sphere:

$$\mathbf{S}(N_\mathbf{y} \partial X)^+ = \{\mathbf{v} \in \mathbf{S}(N_\mathbf{y} \partial X) \ : \ \langle \mathbf{v}, \nu_\mathbf{y} \rangle > 0\}.$$

Therefore, for $k = 0, \ldots, d-1$, we can write

$$\lambda_k^{\partial X}(\mathbf{y}) = \frac{1}{s_{n-k-1}} \int_{\mathbf{S}(N_{\mathbf{y}}\partial X)^+} \sigma_{d-1-k}^{\partial X}(\mathbf{y}, \mathbf{v}) \mathrm{d}\mathbf{v}.$$

The case of a definable sub-manifold with boundary of dimension d yields the following explicit description of the Lipschitz–Killing measures. For any bounded Borel subset U of X we find

$$\Lambda_d(X, U) = \mathcal{H}_d(U). \tag{13.1}$$

When $d < n$ and $k = 0, \ldots, d-1$, we find

$$\Lambda_k(X, U) = \int_{X^\circ \cap U} \lambda_k^{X^\circ}(\mathbf{x}) \mathrm{d}\mathbf{x} + \int_{\partial X \cap U} \lambda_k^{\partial X}(\mathbf{y}) \mathrm{d}\mathbf{y}. \tag{13.2}$$

If $d = n$ and $k = 0, \ldots, n-1$,

$$\Lambda_k(X, U) = \int_{\partial X \cap U} \lambda_k^{\partial X}(\mathbf{y}) \mathrm{d}\mathbf{y}. \tag{13.3}$$

14. Definable families and continuity of Lipschitz–Killing curvature densities at infinity

We return to the setting of the previous sections: let W be a C^2 connected sub-manifold of $\mathbb{R}^n \times \mathbb{R}^s$ which is also a closed subset of $\mathbb{R}^n$, definable in $\mathcal{M}$, and let $\varphi : W \to \mathbb{R}^s$ be the restriction to W of the canonical projection on $\mathbb{R}^s$. Let $d = \dim W$. Hence for $\mathbf{y}$ any regular value of φ either the level $\varphi^{-1}(\mathbf{y})$ is empty or is a C^2 definable sub-manifold of dimension $d - s$ of $\mathbb{R}^n$, of the form

$$\varphi^{-1}(\mathbf{y}) = W_{\mathbf{y}} \times \mathbf{y} \subset \mathbb{R}^n \times \mathbb{R}^s.$$

Instead of working with the family of levels $(\varphi^{-1}(\mathbf{y}))_{\mathbf{y}}$ we will work with the family $(W_{\mathbf{y}})_{\mathbf{y}}$ of the projections onto $\mathbb{R}^n$ of the levels of φ, a definable family of closed subsets of $\mathbb{R}^n$.

Let $\mathcal{V}_1, \ldots, \mathcal{V}_\beta$ be the connected components of $\mathbb{R}^s \setminus K(\varphi)$ for which $\varphi^{-1}(\mathcal{V}_b)$ is not empty. For each $b = 1, \ldots, \beta$, let $\mathcal{U}_{b,1}, \ldots, \mathcal{U}_{b,\alpha_b}$ be the connected components of $\varphi^{-1}(\mathcal{V}_b)$. For each $\mathbf{y} \in \mathcal{V}_b$ and each $a = 1, \ldots, \alpha_b$, let

$$W_{\mathbf{y}}^a \times \mathbf{y} := W_{\mathbf{y}} \times \mathbf{y} \cap \mathcal{U}_{b,a}. \tag{14.1}$$

Each such sub-manifold $W_{\mathbf{y}}^a$ is not empty, closed and connected, by Corollary 8.3.

We present in this section now the looked for continuity results of the functions $\mathbf{y} \mapsto \Lambda_k^\infty(W_{\mathbf{y}})$ nearby a regular value which is also a (MR)-regular value of this definable family.

Proposition 14.1. — *For each $k = 0, \ldots, d-s$, for each $b = 1, \ldots, \beta$, and each $a = 1, \ldots, \alpha_b$, the functions*

$$\mathbf{y} \longmapsto \chi^\infty_{n+1-k}(W^a_\mathbf{y}), \ \ \text{and} \ \ \mathbf{y} \longmapsto \Lambda^\infty_{d-s-k}(W^a_\mathbf{y})$$

are continuous over $\mathcal{V}_b$, (where $\chi^\infty_{n+1}(-)$ is the null function over the subsets of $\mathbb{R}^n$).

Proof. — Theorem 12.3 and Remark 12.4 guarantee that it is sufficient to prove it for the functions $\mathbf{y} \mapsto \chi^\infty_{n+1-k}(W^a_\mathbf{y})$. Observe that the mapping $\mathbf{y} \mapsto \chi(W^a_\mathbf{y})$ is constant on $\mathcal{V}_b$ for each $a = 1, \ldots, \alpha_b$, by Corollary 8.3.

For a closed definable subset X of $\mathbb{R}^n$, let us write

$$\chi^\infty(X) := \chi(\mathrm{Lk}^\infty(X)).$$

- Assume first that $W_\mathbf{y}$ is connected for $\mathbf{y} \notin K(\varphi)$. The case $k = 0$ is obvious by definition, so we treat first the case $k = 1$. Let $\mathbf{c} \in \mathbb{R}^s \setminus K(\varphi)$. Proposition 10.2 implies the existence of an open neighbourhood U of $\mathbf{c}$ in $\mathbb{R}^s$ such that

$$\mathbf{y} \in U \Longrightarrow 2\chi^\infty_n(W_\mathbf{y}) = \chi^\infty(W_\mathbf{y}) = \chi^\infty(W_\mathbf{c}).$$

In other words, the mapping $\mathbf{y} \mapsto \chi^\infty_n(W_\mathbf{y})$ is constant on each connected component of $\mathbb{R}^s \setminus K(\varphi)$.

We treat now the case $k = 2, \ldots d-s$, and $\mathbf{c} \in \mathbb{R}^s \setminus K(\varphi)$. Theorem 11.2 ensures the existence of an open dense definable subset $\mathcal{U}^k_\mathbf{c}$ of $\mathbf{G}(n+1-k, n)$ such that

$$P \in \mathcal{U}^k_\mathbf{c} \Longrightarrow \mathbf{c} \notin K(\varphi|_{P\times\mathbb{R}^s}).$$

For any $P \in \mathbf{G}(n+1-k, n)$ and any $\mathbf{y} \in \mathbb{R}^s$ we find

$$(\varphi|_{P\times\mathbb{R}^s})^{-1}(\mathbf{y}) = (W_\mathbf{y} \cap P) \times \mathbf{y} \subset P \times \mathbb{R}^s.$$

As for $k = 1$, given $P \in \mathcal{U}^k_\mathbf{c}$, there exists an open neighbourhood U_P of $\mathbf{c}$ contained in $\mathbb{R}^s \setminus K(\varphi|_{P\times\mathbb{R}^s})$ such that for all $\mathbf{y} \in U_P$, we find

$$\chi^\infty(W_\mathbf{y} \cap P) = \chi^\infty(W_\mathbf{c} \cap P),$$

which we can write as

$$\lim_{\mathbf{y}\to\mathbf{c}} \chi^\infty(W_\mathbf{y} \cap P) = \chi^\infty(W_\mathbf{c} \cap P).$$

Now Lebesgue Dominated Convergence Theorem provides

$$\lim_{\mathbf{y}\to\mathbf{c}} \int_{\mathbf{G}_n^{n+1-k}} \chi^\infty(W_\mathbf{y}\cap P)\mathrm{d}P = \int_{\mathbf{G}_n^{n+1-k}} \lim_{\mathbf{y}\to\mathbf{c}} \chi^\infty(W_\mathbf{y}\cap P)\mathrm{d}P$$
$$= \int_{\mathbf{G}_n^{n+1-k}} \chi^\infty(W_\mathbf{c}\cap P)\mathrm{d}P,$$

namely the desired continuity result.

- Let $b \in \{1,\ldots,\beta\}$ and let $\mathbf{y} \in \mathcal{V}_b$. For each $a = 1,\ldots,\alpha_b$, the connected case implies that each function $\mathbf{y} \mapsto \chi^\infty_{n+1-k}(W^a_\mathbf{y})$ is continuous over $\mathcal{V}_b$. □

Let X be a closed definable sub-manifold of $\mathbb{R}^q$ of dimension d. For $i = 0,\ldots,d$, let $K_i(X,\mathbf{x})$ be the i-th Lipschitz–Killing curvature of X at $\mathbf{x}$. For $i \geqslant d+1$, we may define $K_i(X,-) \equiv 0$ if need be. The global polar-like invariant avatars we are looking for in our real context are presented in the next definition.

Definition 14.2. — *Let $i \in \{0,\ldots,d\}$. The* i-th Lipschitz–Killing curvature density at infinity of the closed definable sub-manifold X of $\mathbb{R}^q$ of dimension d *is defined as*

$$\kappa_i^\infty(X) := \lim_{R\to+\infty} \frac{1}{R^{d-i}} \int_{X\cap\mathbf{B}_R^n} K_i(X,\mathbf{x})\mathrm{d}\mathbf{x}.$$

The second main result of the paper is the following

Theorem 14.3. — *For each $i = 0,\ldots,d-s$, for each $b = 1,\ldots,\beta$, and each $a = 1,\ldots,\alpha_b$, the functions*

$$\mathbf{y} \longrightarrow \kappa_i^\infty(W^a_\mathbf{y})$$

are continuous over $\mathcal{V}_b$. Thus the functions $\mathbf{y} \mapsto \kappa_i^\infty(W_\mathbf{y})$ are continuous on $\mathbb{R}^s \setminus K(\varphi)$.

Proof. — As we saw in the demonstration of Proposition 14.1, we can work only with connected $W_\mathbf{y}$ as far as $\mathbf{y}$ lies in $\mathbb{R}^s \setminus K(\varphi)$, which we will assume for the rest of the proof.

We treat first the case $i = d-s$. Theorem 12.3 (see also [15, Theorem 5.6]) gives

$$\frac{1}{s_{n-1}}\kappa^\infty_{d-s}(W_\mathbf{y}) = \chi(W_\mathbf{y}) - \chi^\infty_n(W_\mathbf{y}) - \chi^\infty_{n-1}(W_\mathbf{y})$$

for $\mathbf{y}$ any regular value. Applying Proposition 14.1 concludes this case.

For $2 \leqslant i \leqslant d-s-1$ and $\mathbf{y}$ any regular value of φ, Theorem 12.3 (see also [16, Theorem 4.1]) gives

$$\frac{1}{s_{n-(d-s)+i-1}\cdot b_{d-s-i}}\kappa_i^\infty(W_\mathbf{y}) = -\chi^\infty_{n-(d-s)+i-1}(W_\mathbf{y}) + \chi^\infty_{n-(d-s)+i+1}(W_\mathbf{y}).$$

We conclude again by Proposition 14.1.

For $i = 0$ or 1 and $\mathbf{y}$ any regular value of φ, Theorem 12.3 gives again

$$\frac{1}{s_{n-(d-s)+i-1} \cdot b_{d-s-i}} \kappa_i^\infty(W_\mathbf{y}) = \chi_{n-(d-s)+i+1}^\infty(W_\mathbf{y}).$$

We remark that the term

$$\chi_{n-(d-s)}^\infty(W_\mathbf{y})$$

that appears in the equality concerning the curvature K_1 is in fact zero, because generically $W_\mathbf{y} \cap P$ is of dimension 0, hence compact. We conclude with the previous proposition again. □

Remark 14.4. — For odd i, the i-th Lipschitz–Killing curvature function $K_i(X, -)$ of a given sub-manifold X is identically null, therefore the mapping in our context $\mathbf{y} \mapsto \kappa_i^\infty(W_\mathbf{y})$ is the null mapping over $\mathbb{R}^s \setminus K(\varphi)$.

15. Curvatures of hypersurfaces

In this section, we study the special case of regular levels of a C^2 definable function, and express some curvature-like integrals over them as Lipschitz–Killing measures.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 definable function. We assume that 0 is a regular value of f taken by f, so that $Y = f^{-1}(0)$ is a non-empty C^2 hypersurface. We orientate Y by the normal vector $-\nabla f|_Y$, with the convention that $(\xi_1, \ldots, \xi_{n-1})$ is a positive basis of $T_\mathbf{x}Y$ if and only if $(\xi_1, \ldots, \xi_{n-1}, -\nabla f(\mathbf{x}))$ is a positive basis of $\mathbb{R}^n$.

The Gauss mapping ν_Y of Y is the following mapping:

$$\nu_Y : Y \longrightarrow \mathbf{S}^{n-1}, \ \ \mathbf{x} \longmapsto -\frac{\nabla f}{|\nabla f|}(\mathbf{x}).$$

Its derivative

$$D_\mathbf{x}\nu_Y : T_\mathbf{x}Y \longmapsto T_{\nu(\mathbf{x})}\mathbf{S}^{n-1} = T_\mathbf{x}Y$$

is a self-adjoint operator. The principal curvatures $k_1(\mathbf{x}), \ldots, k_{n-1}(\mathbf{x})$ are the opposite of the eigenvalues of $D_\mathbf{x}\nu_Y$. Let σ_i^Y be the i-th elementary symmetric function of the principal curvatures. Note that σ_{n-1}^Y is the Gauss–Kronecker curvature and $\sigma_0^Y \equiv 1$. These elementary symmetric functions are related to the Lipschitz–Killing curvatures of Y and the sub-level set $\mathcal{Y} := \{f \leqslant 0\}$ in the following way: Given $\mathbf{x} \in Y$, since Y is the boundary of $\mathcal{Y}$, we note that

$$i \text{ even } \implies 2\sigma_i^Y(\mathbf{x}) = K_i(Y, \mathbf{x}) = s_i \cdot \lambda_{n-1-i}^Y(\mathbf{x}).$$

Definition 15.1. — *Let $i \in \{0,\dots,n-1\}$. The i-th* symmetric principal curvature density at infinity of the hypersurface Y *is defined as*

$$\sigma_i^\infty(Y) := \lim_{R\to+\infty} \frac{1}{R^{n-1-i}} \int_{Y\cap \mathbf{B}_R^n} \sigma_i^Y(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{15.1}$$

The density at infinity of $\mathcal{Y}$ is defined as

$$\Theta_n^\infty(\mathcal{Y}) := \lim_{R\to+\infty} \frac{1}{R^n} \mathcal{H}_n(\mathcal{Y}\cap \mathbf{B}_R^n). \tag{15.2}$$

In this context, Equations (13.3) give

$$\sigma_{n-1-i}^\infty(Y) = s_{n-1-i} b_i \cdot \Lambda_i^\infty(\mathcal{Y}), \quad i = 0,\dots,n-1.$$

The next corollary gives explicit relations between the numbers $\chi_l^\infty(-)$ of Equation (12.1) and Equation (12.3) and the numbers $\sigma_l^\infty(-)$ of Equation (15.1). It is an application of the Gauss–Bonnet formulas of Theorem 12.3 for the levels of the function f, the description of the Gauss–Bonnet measures for manifolds with boundary and Equations (13.3) for the numbers $\sigma_i^\infty(-)$.

Corollary 15.2.

$$\Theta_n^\infty(\mathcal{Y}) = \chi_1^\infty(\mathcal{Y}), \tag{15.3}$$

and for $i = n-1$

$$\frac{1}{s_{n-1}} \sigma_{n-1}^\infty(Y) = \chi(\mathcal{Y}) - \chi_n^\infty(\mathcal{Y}) - \chi_{n-1}^\infty(\mathcal{Y}), \tag{15.4}$$

and for $i = 0,\dots,n-2$,

$$\frac{1}{s_i b_{n-i-1}} \sigma_i^\infty(Y) = -\chi_i^\infty(\mathcal{Y}) + \chi_{i+2}^\infty(\mathcal{Y}). \tag{15.5}$$

Remark 15.3. — We can orientate Y by the normal vector $\nabla f|_Y$, with the convention that $(\xi_1,\dots,\xi_{n-1})$ is a positive basis of $T_\mathbf{x}Y$ if and only if $(\nabla f(\mathbf{x}), \xi_1, \dots, \xi_{n-1})$ is a positive basis of $\mathbb{R}^n$. In this case, the Gauss mapping ν_H of H is the mapping:

$$\nu_Y : Y \longrightarrow \mathbf{S}^{n-1}, \ \ \mathbf{x} \longmapsto \frac{\nabla f}{|\nabla f|}(\mathbf{x}).$$

and the principal curvatures $k_1(\mathbf{x}),\dots,k_{n-1}(\mathbf{x})$ are the eigenvalues of $D_\mathbf{x}\nu_Y$.

16. Continuity of curvature integrals and families of hypersurfaces

In this last section we expand and strengthen the study initiated in [17], addressing here the special case of families of hypersurfaces.

We work within the context of Section 9. We are given a C^2 definable function $F : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$ for which 0 is a regular value. Let again $\mathcal{W}$ and W be defined as

$$\mathcal{W} := \{\mathbf{p} : F(\mathbf{p}) \leqslant 0\}, \text{ and } W := \{\mathbf{p} : F(\mathbf{p}) = 0\}.$$

The mappings ω, φ are the restrictions of the projection $\mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^s$ to $\mathcal{W}, W$, respectively. Practically we work with the connected components of W, thus we further assume that W is connected. For each $\mathbf{y} \in \mathbb{R}^s$, the function $f_{\mathbf{y}} : \mathbb{R}^n \to \mathbb{R}$ defined as $\mathbf{x} \mapsto F(\mathbf{x}; \mathbf{y})$ is also C^2 and definable. We recall that

$$\mathcal{W} \cap \mathbb{R}^n \times \mathbf{y} = \mathcal{W}_{\mathbf{y}} \times \mathbf{y}, \text{ and } \varphi^{-1}(\mathbf{y}) = W_{\mathbf{y}} \times \mathbf{y}.$$

Whenever $\mathbf{y}$ is a regular value of φ, the hypersurface $W_{\mathbf{y}} = f_{\mathbf{y}}^{-1}(0)$ bounds the C^2 closed definable n-dimensional sub-manifold with boundary $\mathcal{W}_{\mathbf{y}} := \{f_{\mathbf{y}} \leqslant 0\}$.

Let $\mathcal{V}_1, \ldots, \mathcal{V}_\beta$ be the connected components of $\mathbb{R}^s \setminus K(\varphi)$ for which $\varphi^{-1}(\mathcal{V}_b)$ is not empty. For each $b = 1, \ldots, \beta$, let $\mathcal{U}_{b,1}, \ldots, \mathcal{U}_{b,\alpha_b}$ be the connected components of $\omega^{-1}(\mathcal{V}_b)$. For each couple (b, a) and each $\mathbf{y} \in \mathcal{V}_b$, let

$$\mathcal{W}_{\mathbf{y}}^a := \mathcal{W}_{\mathbf{y}} \cap \mathcal{U}_{b,a}.$$

We obtain the following:

Proposition 16.1. — *For each $b = 1, \ldots, \beta$, for each $a = 1, \ldots, \alpha_b$, and for $k = 1, \ldots, n$, the functions $\mathbf{y} \mapsto \chi_k^\infty(\mathcal{W}_{\mathbf{y}}^a)$ are continuous over $\mathbb{R} \setminus K(\varphi)$.*

Proof. — We can assume that $b = 1$ and $\alpha_1 = 1$ as well. Thus $\mathcal{W}_{\mathbf{y}}^a = \mathcal{W}_{\mathbf{y}}$.

We treat first the case $k = n$. Proposition 10.3 shows that the following function is constant on $\mathcal{V}_b$

$$\mathbf{y} \longmapsto \chi_n^\infty(\mathcal{W}_{\mathbf{y}}) = \chi^\infty(\mathcal{W}_{\mathbf{y}}).$$

We assume that $1 \leqslant k \leqslant n$. Let $\mathbf{c} \in \mathbb{R}^s \setminus K(\varphi)$. Theorem 11.2 guarantees that for almost all $P \in \mathbf{G}(k, n)$, the value $\mathbf{c}$ does not lie in $K_\infty(\varphi|_{P \times \mathbb{R}^s})$. Proposition 10.3 gives the existence of an open neighbourhood V_P of $\mathbf{c}$ in $\mathbb{R}^s$ such that for all $\mathbf{y}$ in V_P,

$$\chi^\infty(\mathcal{W}_{\mathbf{y}} \cap P) = \chi^\infty(\mathcal{W}_{\mathbf{c}} \cap P).$$

The proof ends applying Lebesgue Dominated Convergence Theorem, as was done in the proof of Proposition 14.1. □

For $b = 1, \ldots, \beta$, let $\mathcal{U}_{b,1}, \ldots, \mathcal{U}_{b,\alpha_b}$ be the connected components of $\varphi^{-1}(\mathcal{V}_b)$. For $\mathbf{y}$ in $\mathcal{V}_b$ and $a = 1, \ldots, \alpha_b$, let

$$\mathcal{W}_{\mathbf{y}}^a := \mathcal{W}_{\mathbf{y}} \cap \mathcal{U}_{b,a}, \text{ and } W_{\mathbf{y}}^a := W_{\mathbf{y}} \cap \mathcal{U}_{b,a}.$$

The last main result of the paper is the following:

THEOREM 16.2. — *For each $b = 1, \ldots, \beta$, and for each $a = 1, \ldots, \alpha_b$, the following statements hold true:*

(1) *For each $i = 0, \ldots, n-1$, the functions $\mathbf{y} \mapsto \sigma_i^\infty(W_{\mathbf{y}}^a)$ are continuous on $\mathcal{V}_b$. Thus $\mathbf{y} \mapsto \sigma_i^\infty(W_{\mathbf{y}})$ is continuous over $\mathbb{R}^s \setminus K(\varphi)$.*

(2) *The function $\mathbf{y} \mapsto \Theta_n^\infty(\mathcal{W}_{\mathbf{y}}^a)$ is also continuous on $\mathcal{V}_b$. Thus $\mathbf{y} \mapsto \Theta_n^\infty(\mathcal{W}_{\mathbf{y}})$ is continuous over $\mathbb{R}^s \setminus K(\varphi)$.*

Proof. — Let $\mathbf{c} \in \mathbb{R}^s \setminus K(f)$. We can assume that both $\mathcal{W}_{\mathbf{c}}$ and $W_{\mathbf{c}}$ are connected. Let us apply the Gauss–Bonnet formulas of Theorem 12.3 to the n-dimensional closed definable set $\mathcal{W}_{\mathbf{y}}$. We get

$$\Lambda_0^\infty(\mathcal{W}_{\mathbf{y}}) = \chi(\mathcal{W}_{\mathbf{y}}) - \chi_n^\infty(\mathcal{W}_{\mathbf{y}}) - \chi_{n-1}^\infty(\mathcal{W}_{\mathbf{y}}). \tag{16.1}$$

Moreover, for $k = 0, \ldots, n-2$, we have:

$$\Lambda_k^\infty(\mathcal{W}_{\mathbf{y}}) = -\chi_{n-k-1}^\infty(\mathcal{W}_{\mathbf{y}}) + \chi_{n-k+1}^\infty(\mathcal{W}_{\mathbf{y}}), \tag{16.2}$$

and

$$\Lambda_{n-1}^\infty(\mathcal{W}_{\mathbf{y}}) = \chi_2^\infty(\mathcal{W}_{\mathbf{y}}) \tag{16.3}$$

$$\Theta_n^\infty(\mathcal{Y}) = \Lambda_n^\infty(\mathcal{W}_{\mathbf{y}}) = \chi_1^\infty(\mathcal{W}_{\mathbf{y}}). \tag{16.4}$$

Since $\mathcal{W}_{\mathbf{y}}$ is n-dimensional, Equalities (16.1), (16.2), (16.3), and (16.4) are non-trivial.

Applying Proposition 16.1, we obtain that for $k = 0, \ldots, n$, the functions

$$\mathbf{y} \longmapsto \Lambda_k^\infty(\mathcal{W}_{\mathbf{y}})$$

are continuous on $\mathbb{R}^s \setminus K(\varphi)$. It is enough to apply the expressions of Corollary 15.2 to conclude. □

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