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Steven Dale Cutkosky

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CORRIGENDUM TO “A SIMPLER PROOF OF TOROIDALIZATION OF MORPHISMS FROM 3-FOLDS TO SURFACES”

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by Steven Dale CUTKOSKY (*)

Abstract. — We correct mistakes in the paper [3]. There is a missing case in the analysis of [3]. We give the analysis of this additional case.

Résumé. — Nous corrigeons des erreurs dans l’article [3]. Il manque un cas dans l’analyse de [3]. Nous donnons l’analyse de ce cas supplémentaire.

1. Introduction

A simpler and more conceptual proof of toroidalization of morphisms of 3-folds to surfaces, over an algebraically closed field of characteristic zero, is given in [3]. A toroidalization is obtained by performing sequences of blow ups of nonsingular subvarieties above the domain and range, to make a morphism toroidal. The original proof of toroidalization of morphisms of 3-folds to surfaces, given in [1], is much more complicated. In [2], toroidalization of morphisms from 3-folds to 3-folds is proven (over an algebraically closed field of characteristic zero). This proof uses the result that toroidalization of morphisms from 3-folds to surfaces is possible.

A case was missed in Lemma 3.6 of [3]. In this errata we give the changes that must be made in [3] to include the analysis of this additional case.

The consideration of this new case does not lead to a change of any of the statements of the main results of [3]. It also does not introduce any

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significant change in the proofs. All of the changes which need to be made in [3] to incorporate this new case are written out in detail in this errata. Numbers indexing equations, theorems, definitions, etc. in this errata are as in [3]. New equations and statements of theorems introduced in this errata are indexed by letters. The new case which we must consider is labeled as (A) in the revised Definition 3.3 below.

The author thanks the reviewer for their careful reading and helpful comments to improve the readability of the proofs. The author thanks Andre Belotto and Ed Bierstone for pointing out that a case was missed in the analysis of the original publication; this fault is corrected in this errata.

2. The corrections

Page 869, line 8: "$g^{-1}(D_R)$" should be $g^{-1}(D_T)$".

Page 877, lines 22 - 23: "natural numbers $r_2,\ldots,r_{m-2}$ and a positive integer r_{m-1}" should be "natural numbers $r_2,\ldots,r_{m-1}$".

Page 877: insert after "form of ω after Theorem 4.1." on line 23: Let $\omega'(m,r_2,\ldots,r_m)$ be a function which associates a positive integer to a positive integer m and natural numbers $r_2,\ldots,r_m$. We will give a precise form of ω' after Theorem 4.2.

Page 877: Definition 3.3 should be modified as follows.

Definition 3.3. — *X is 3-prepared (with respect to $f : X \to S$) at a point $p \in D$ if $\sigma_D(p) = 0$ or if $\sigma_D(p) > 0$, f is 2-prepared with respect to D at p and there are permissible parameters x,y,z at p such that x,y,z are uniformizing parameters on an étale cover of an affine neighborhood of p and we have one of the following forms, with $m = \sigma_D(p)+1$:*

(1) *p is a 2-point, and we have an expression (2.2) with*

$$F = \tau_0 z^m + \tau_2 x^{r_2} y^{s_2} z^{m-2} + \cdots + \tau_{m-1} x^{r_{m-1}} y^{s_{m-1}} z + \tau_m x^{r_m} y^{s_m} \tag{3.5}$$

where $\tau_0 \in \widehat{\mathcal{O}}_{X,p}$ is a unit, $\tau_i \in \widehat{\mathcal{O}}_{X,p}$ are units (or zero), $r_i + s_i > 0$ whenever $\tau_i \neq 0$ and $(r_m+c)b - (s_m+d)a \neq 0$. Further, $\tau_{m-1} \neq 0$ or $\tau_m \neq 0$.

(2) *p is a 1-point, and we have an expression (2.1) with*

$$F = \tau_0 z^m + \tau_2 x^{r_2} z^{m-2} + \cdots + \tau_{m-1} x^{r_{m-1}} z + \tau_m x^{r_m} \tag{3.6}$$

where $\tau_0 \in \widehat{\mathcal{O}}_{X,p}$ is a unit, $\tau_i \in \widehat{\mathcal{O}}_{X,p}$ are units (or zero) for $2 \leqslant i \leqslant m-1$, $\tau_m \in \widehat{\mathcal{O}}_{X,p}$ and $\mathrm{ord}(\tau_m(0,y,0)) = 1$ (or $\tau_m = 0$). Further, $r_i > 0$ if $\tau_i \neq 0$, and $\tau_{m-1} \neq 0$ or $\tau_m \neq 0$.

(3) *p is a 1-point, and we have an expression (2.1) with*

$$F=\tau_0 z^m+\tau_2 x^{r_2} z^{m-2}+\cdots+\tau_{m-1} x^{r_{m-1}} z+x^t \Omega \tag{3.7}$$

where $\tau_0 \in \widehat{\mathcal{O}}_{X,p}$ is a unit, $\tau_i \in \widehat{\mathcal{O}}_{X,p}$ are units (or zero) for $2 \leqslant i \leqslant m-1$, $\Omega \in \widehat{\mathcal{O}}_{X,p}$, $\tau_i \neq 0$ for some i with $2 \leqslant i \leqslant m-1$ and $t > \omega(m, r_2, \ldots, r_{m-1})$ (where we set $r_i = 0$ if $\tau_i = 0$). Further, $r_i > 0$ if $\tau_i \neq 0$.

(4) *p is a 1-point, and we have an expression (2.1) with*

$$F=\tau_0 z^m+\tau_2 x^{r_2} y z^{m-2}+\cdots+\tau_{m-1} x^{r_{m-1}} y z+\tau_m x^{r_m} y+x^t \Omega \tag{A}$$

where $\tau_0 \in \widehat{\mathcal{O}}_{X,p}$ is a unit, $\tau_i \in \widehat{\mathcal{O}}_{X,p}$ are units (or zero) for $2 \leqslant i \leqslant m$, $\Omega \in \widehat{\mathcal{O}}_{X,p}$, $\tau_i \neq 0$ for some i with $2 \leqslant i \leqslant m$ and $t > \omega'(m, r_2, \ldots, r_m)$ (where we set $r_i = 0$ if $\tau_i = 0$). Further, $r_i > 0$ if $\tau_i \neq 0$.

X is 3-prepared if X is 3-prepared for all $p \in X$.

Page 881, line 5: “Lemma” 3.5.

Pages 880–882: Lemma 3.6 and its proof should be changed as follows.

Lemma 3.6. — *Suppose that X is 2-prepared with respect to $f: X \to S$. Suppose that $p \in D$ is a 1-point with $m = \sigma_D(p) + 1 > 1$. Let u, v be permissible parameters for $f(p)$ and x, y, z be permissible parameters for D at p such that a form (3.1) holds at p. Let U be an étale cover of an affine neighborhood of p such that x, y, z are uniformizing parameters on U. Let C be the curve in U which has local equations $x = y = 0$ at p.*

Let $T_0 = \operatorname{Spec}(\mathfrak{k}[x, y])$, $\Lambda_0 : U \to T_0$. Then there exists a sequence of quadratic transforms $T_1 \to T_0$ such that if $U_1 = U \times_{T_0} T_1$ and $\psi_1 : U_1 \to U$ is the induced sequence of blow ups of sections over C, $\Lambda_1 : U_1 \to T_1$ is the projection, then U_1 is 2-prepared with respect to $f \circ \psi_1$ at all $p_1 \in \psi_1^{-1}(p)$. Further, for every point $p_1 \in \psi_1^{-1}(p)$, there exist regular parameters x_1, y_1 in $\widehat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$ such that x_1, y_1, z are permissible parameters at p_1, and there exist regular parameters $\widetilde{x}_1, \widetilde{y}_1$ in $\mathcal{O}_{T_1,\Lambda_1(p_1)}$ such that if p_1 is a 1-point, $x_1 = \alpha(\widetilde{x}_1, \widetilde{y}_1)\widetilde{x}_1$ where $\alpha(\widetilde{x}_1, \widetilde{y}_1) \in \widehat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$ is a unit series and $y_1 = \beta(\widetilde{x}_1, \widetilde{y}_1)$ with $\beta(\widetilde{x}_1, \widetilde{y}_1) \in \widehat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$, and if p_1 is a 2-point, then $x_1 = \alpha(\widetilde{x}_1, \widetilde{y}_1)\widetilde{x}_1$ and $y_1 = \beta(\widetilde{x}_1, \widetilde{y}_1)\widetilde{y}_1$, where $\alpha(\widetilde{x}_1, \widetilde{y}_1), \beta(\widetilde{x}_1, \widetilde{y}_1) \in \widehat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$ are unit series. We have one of the following forms:

(1) *p_1 is a 2-point, and we have an expression (2.2) with*

$$F=\tau z^m+\bar a_2(x_1,y_1)x_1^{r_2}y_1^{s_2}z^{m-2}+\cdots+\bar a_{m-1}(x_1,y_1)x_1^{r_{m-1}}y_1^{s_{m-1}}z+\bar a_m x_1^{r_m}y_1^{s_m} \tag{3.10}$$

where $\tau\in\widehat{\mathcal{O}}_{U_1,p_1}$ is a unit, $\bar a_i(x_1,y_1)\in\mathfrak{k}[[x_1,y_1]]$ are units (or zero) for $2\leqslant i\leqslant m-1$, $\bar a_m=0$ or 1 and if $\bar a_m=0$, then $\bar a_{m-1}\neq 0$. Further, $r_i+s_i>0$ whenever $\bar a_i\neq 0$ and $(r_m+c)b-(s_m+d)a\neq 0$.

(2) *p_1 is a 1-point, and we have an expression (2.1) with*

$$F=\tau z^m+\bar a_2(x_1,y_1)x_1^{r_2}z^{m-2}+\cdots+\bar a_{m-1}(x_1,y_1)x_1^{r_{m-1}}z+x_1^{r_m}y_1 \tag{3.11}$$

where $\tau\in\widehat{\mathcal{O}}_{U_1,p_1}$ is a unit, $\bar a_i(x_1,y_1)\in\mathfrak{k}[[x_1,y_1]]$ are units (or zero) for $2\leqslant i\leqslant m-1$. Further, $r_i>0$ (whenever $\bar a_i\neq 0$).

(3) *p_1 is a 1-point, and we have an expression (2.1) with*

$$F=\tau z^m+\bar a_2(x_1,y_1)x_1^{r_2}z^{m-2}+\cdots+\bar a_{m-1}(x_1,y_1)x_1^{r_{m-1}}z+x_1^{t}y_1\Omega \tag{3.12}$$

where $\tau\in\widehat{\mathcal{O}}_{U_1,p_1}$ is a unit, $\bar a_i(x_1,y_1)\in\mathfrak{k}[[x_1,y_1]]$ are units (or zero) for $2\leqslant i\leqslant m-1$ and $r_i>0$ whenever $\bar a_i\neq 0$. We also have $t>\omega(m,r_2,\ldots,r_{m-1})$. Further, $\bar a_i\neq 0$ for some $2\leqslant i\leqslant m-1$ and $\Omega\in\widehat{\mathcal{O}}_{U_1,p_1}$.

(4) *p_1 is a 1-point, and we have an expression (2.1) with*

$$F=\tau z^m+\bar a_2(x_1,y_1)x_1^{r_2}y_1z^{m-2}+\cdots+\bar a_{m-1}(x_1,y_1)x_1^{r_{m-1}}y_1z+x_1^{r_m}y_1+x_1^{t}y_1^2\Omega \tag{B}$$

where $\tau\in\widehat{\mathcal{O}}_{U_1,p_1}$ is a unit, $\bar a_i(x_1,y_1)\in\mathfrak{k}[[x_1,y_1]]$ are units (or zero) for $2\leqslant i\leqslant m$ and $r_i>0$ whenever $\bar a_i\neq 0$. We also have $t>\omega'(m,r_2,\ldots,r_m)$. Further, $\bar a_i\neq 0$ for some $2\leqslant i\leqslant m$ and $\Omega\in\widehat{\mathcal{O}}_{U_1,p_1}$.

Proof. — Let $\bar p=\Lambda_0(p)$. Let $T=\{i\mid a_i(x,y)\neq 0 \text{ and } 2\leqslant i<m\}$. There exists a sequence of blow ups $\varphi_1:T_1\to T_0$ of points over $\bar p$ such that at all points $q\in\psi_1^{-1}(p)$, we have permissible parameters x_1,y_1,z such that x_1,y_1 are regular parameters in $\widehat{\mathcal{O}}_{T_1,\Lambda_1(q)}$ and we have that ug is a monomial in x_1 and y_1 times a unit in $\widehat{\mathcal{O}}_{T_1,\Lambda_1(q)}$, where $g=\prod_{i\in T}a_i(x,y)$.

Suppose that $a_m(x,y)\neq 0$. Let $\bar v=x^ba_m(x,y)$ if (2.1) holds and $\bar v=x^cy^da_m(x,y)$ if (2.2) holds. We have $\bar v\notin\mathfrak{k}[[x]]$ (respectively $\bar v\notin\mathfrak{k}[[x^ay^b]]$). Then by Lemma 3.5 applied to $u,\bar v$, we have that there exists a further sequence of blow ups $\varphi_2:T_2\to T_1$ of points over $\bar p$ such that at all points $q\in(\psi_1\circ\psi_2)^{-1}(p)$, we have permissible parameters x_2,y_2,z such that $x_2,\bar y_2$

are regular parameters in $\widehat{\mathcal{O}}_{T_2,\Lambda_2(q)}$ such that $ug=0$ is a SNC divisor and either

$$u = x_2^{\bar{a}}, \bar{v} = \overline{P}(x_2) + x_2^{\bar{b}}\bar{y}_2^{\bar{c}} \tag{c1}$$

with $\bar{c} > 0$ or

$$u = (x_2^{\bar{a}}\bar{y}_2^{\bar{b}})^t, \bar{v} = \overline{P}(x_2^{\bar{a}}\bar{y}_2^{\bar{b}}) + x_2^{\bar{c}}\bar{y}_2^{\bar{d}} \tag{c2}$$

where $\bar{a}\bar{d} - \bar{b}\bar{c} \neq 0$.

At a 1-point or 2-point $p_2 \in U_2$ above p, we ask if there is an expression (2.1) or (2.2) such that F has an expression

$$\begin{aligned} F = \tau z^m + \bar{a}_2(x_2, y_2)x_2^{r_2}y_2^{s_2}z^{m-2} + \cdots \\ + \bar{a}_{m-1}(x_2, y_2)x_2^{r_{m-1}}y_2^{s_{m-1}}z + \bar{a}_m x_2^{r_m}y_2^{s_m} \end{aligned} \tag{3.13}$$

where τ is a unit series, $\bar{a}_m = 0$ or 1 and $\bar{a}_i$ are unit series (or zero) for $2 \leqslant i < m$.

At a 2-point $p_2 \in U_2$ above p we always have an expression (2.2) such that (3.13) holds, since $ug = 0$ is a SNC divisor and by (c2).

Suppose that $p_2 \in U_2$ is a 1-point above p. Let $x_2, \bar{y}_2$ be the regular parameters of (c1) in $\widehat{\mathcal{O}}_{T_2,\Lambda_2(p_2)}$. There exists $\widetilde{y} \in \widehat{\mathcal{O}}_{T_2,\Lambda_2(p_2)}$ such that $x_2, \widetilde{y}$ are regular parameters and $ug = \bar{\alpha}x_2^{\bar{a}}\widetilde{y}^{\bar{b}}$ where $\bar{\alpha}$ is a unit. Since $x_2, \widetilde{y}_2$ is a regular system of parameters in $\widehat{\mathcal{O}}_{T_2,\Lambda_2(p_2)} \cong k[[x_2, \bar{y}_2]]$, we have that $\operatorname{ord} \widetilde{y}(0, \bar{y}_2) = 1$. Thus by the Weierstrass Preparation Theorem, there exists a unit $\bar{\tau} \in k[[x_2, \bar{y}_2]]$ and nonunit series $\varphi(x_2) \in k[[x_2]]$ such that $\widetilde{y} = \bar{\tau}(\bar{y}_2 + \varphi(x_2))$. Substituting into F, we obtain an expression (2.1) with

$$\begin{aligned} F = \tau z^m + \bar{a}_2(x_2, \bar{y}_2)x_2^{r_2}(\bar{y}_2 + \varphi(x_2))^{s_2}z^{m-2} + \cdots \\ + \bar{a}_{m-1}(x_2, \bar{y}_2)x_2^{r_{m-1}}(\bar{y}_2 + \varphi(x_2))^{s_{m-1}}z + \bar{a}_m x_2^{r_m}\bar{y}_2^{s_m} \end{aligned} \tag{C}$$

where τ is a unit series, $\bar{a}_m = 0$ or 1, $s_m \geqslant 1$ and $\bar{a}_i$ are unit series (or zero) for $2 \leqslant i \leqslant m$, $\varphi \in k[[x_2]]$ is a series with $0 \leqslant r := \operatorname{ord} \varphi \leqslant \infty$, and $x_2, \bar{y}_2, z$ are permissible parameters with $\bar{y}_2 \in \widehat{\mathcal{O}}_{T_2}$.

We will show that after a finite number of blow ups of points $T_3 \to T_2$, we have an expression (2.2) with (3.13) for all 2-points $p_3 \in U_3$ above p and an expression (2.1) with (3.13) for all 1-points $p_3 \in U_3$ above p.

Suppose that $p_2 \in U_2$ is a 1-point. If $r = 0$ or ∞, $\bar{a}_m = 0$ or $s_m = 1$, then after a permissible change of variables we have a form (3.13). In fact, if a form (C) holds at p_2 with $s_m = 1$, then we set $\widetilde{y}_2 = \bar{y}_2 + \varphi(x_2)$ and write

$$v = P(x_2) + x_2^b F = P(x_2) + \bar{a}_m x_2^{b+r_m}\varphi(x_2) + x_2^b F'$$

where

$$F' = \tau z^m + \bar{a}_2 x_2^{r_2} \widetilde{y}_2^{s_2} z^{m-2} + \cdots + \bar{a}_{m-1} x_2^{r_{m-1}} \widetilde{y}_2^{s_{m-1}} z + \bar{a}_m x_2^{r_m} \widetilde{y}_2$$

to obtain the form (3.13).

Suppose that a form (3.13) does not hold at p_2. Then $0 < r = \operatorname{ord} \varphi < \infty$, $\bar{a}_m = 1$ and $s_m > 1$ in (C). Let $T_3 \to T_2$ be the blow up of $\Lambda_2(p_2) \in T_2$ with induced blow up $\psi_3 : U_3 \to U_2$. Suppose $p_3 \in \psi_3^{-1}(p_2)$. We have permissible parameters x_3, y_3, z in $\widehat{\mathcal{O}}_{U_3,p_3}$ such that one of the following cases holds:

(a) $x_2 = x_3, \bar{y}_2 = x_3(y_3 + \alpha)$ with $0 \neq \alpha \in \mathfrak{k}$.

(b) $x_2 = x_3$, $\bar{y}_2 = x_3 y_3$.

(c) $x_2 = x_3 y_3$, $\bar{y}_2 = y_3$.

Suppose that (a) holds at p_3. We have

$$\bar{y}_2^{s_m} = x_3^{s_m}(y_3 + \alpha)^{s_m} = x_3^{s_m}(\alpha^{s_m} + \mu(y_3) y_3)$$

where $\mu(y_3)$ is a unit series.

If $\alpha + \frac{1}{x_3}\varphi(x_3)$ is a unit series, then setting $\widetilde{y}_3 = \mu(y_3) y_3$, we have

$$v = P(x_2) + x_2^b F = P(x_3) + \bar{a}_m \alpha^{s_m} x_3^{b + r_m + s_m} F'$$

where

$$F' = \tau z^m + a'_2 x_2^{r_2 + s_2} + \cdots + a'_{m-1} x_3^{r_{m-1} + s_{m-1}} + \bar{a}_m x_3^{r_m + s_m} \widetilde{y}_3,$$

of the form (3.13).

If $\alpha + \frac{1}{x_3}\varphi(x_3)$ is not a unit series, then setting $\bar{y}_3 = \mu(y_3) y_3$, we obtain an expression (C) at p_3 with $s_m = 1$,

$$\text{(c3)} \quad F = \tau z^m + a'_2 x_3^{r'_2} (\bar{y}_3 + \varphi'(x_3))^{s'_2} z^{m-2} + \cdots + a'_{m-1} x_3^{r'_{m-1}} (\bar{y}_3 + \varphi'(x_3))^{s'_{m-1}} z + \bar{a}_m x_3^{r'_m} \bar{y}_3).$$

As observed above, making the change of variables $\widetilde{y}_3 = \bar{y}_3 + \varphi'(x_3)$ now gives a form (3.13).

If (b) holds at p_3, then an expression (2.1) holds at p_3 with a form (C) with $\operatorname{ord} \varphi < r$. If $\operatorname{ord} \varphi = 0$ we then have an expression (3.13) (with $s_i = 0$ for $2 \leqslant i \leqslant m - 1$). If (c) holds then

$$\bar{y}_2 + \varphi(x_2) = y_3 + \varphi(x_3 y_3) = y_3 \gamma(x_3, y_3)$$

where γ is a unit series. Thus we have an expression (2.2) with a form (3.13).

By induction on r, we must obtain an expression (2.1) or (2.2) with a form (3.13) at all points above p. We may assume that this already holds in U_2.

Suppose that p_2 is a 1-point above p.

Let

$$J = I_2\begin{pmatrix} \frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial y_2} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x_2} & \frac{\partial v}{\partial y_2} & \frac{\partial v}{\partial z}\end{pmatrix} = x^n\left(\frac{\partial F}{\partial y_2}, \frac{\partial F}{\partial z}\right)$$

for some positive integer n. Since D contains the locus where f is not smooth, we have that the localization $J_{\mathfrak{p}} = (\widehat{\mathcal{O}}_{U_2,q})_{\mathfrak{p}}$, where $\mathfrak{p}$ is the prime ideal (y_2, z) in $\widehat{\mathcal{O}}_{U_2,q}$.

We compute

$$\frac{\partial F}{\partial z} = \bar{a}_{m-1}x_2^{r_{m-1}}y_2^{s_{m-1}} + \Lambda_1 z$$

and

$$\frac{\partial F}{\partial y_2} = s_m\bar{a}_m x_2^{r_m}y_2^{s_m-1} + \Lambda_2 z$$

for some $\Lambda_1, \Lambda_2 \in \widehat{\mathcal{O}}_{U_2,q}$, to see that

$$\text{(D)} \qquad \bar{a}_{m-1} \neq 0 \text{ and } s_{m-1} = 0, \text{ or } \bar{a}_m \neq 0 \text{ and } s_m = 1.$$

Suppose $p_2 \in U_2$ is a 2-point above p. Deforming p_2 to a 1-point, we see from (D) (and since (3.13) holds) that (3.10) holds at p_2.

Let $p_2 \in U_2$ be a 1-point above p where the conclusions of the lemma do not hold. Let $T_3 \to T_2$ be the blow up of $\Lambda_2(p_2) \in T_2$ with induced blow up $\psi_3 : U_3 \to U_2$. Suppose $p_3 \in \psi_3^{-1}(p_2)$. We have that the conclusions of the lemma hold in the form (3.10) if p_3 is the 2-point which has permissible parameters x_3, y_3, z defined by $x_2 = x_3y_3$ and $y_2 = y_3$. At a 1-point which has permissible parameters x_3, y_3, z defined by $x_2 = x_3$, $y_2 = x_3(y_3 + \alpha)$ with $\alpha \neq 0$, we have that a form (3.11) or (3.12) holds. Thus the only case where we may possibly have not achieved the conclusions of the lemma is at the 1-point which has permissible parameters x_3, y_3, z defined by $x_2 = x_3$ and $y_2 = x_3y_3$. We continue to blow up, so that there is at most one point p_3 above p_2 where the conclusions of the lemma do not hold. This point is a 1-point which has permissible parameters x_3, y_3, z with $x_2 = x_3$ and $y_2 = x_3^n y_3$ where we can take n as large as we like. Substituting into (3.13), we have an expression (2.1) at p_3 with

$$F = \tau z^m + \bar{a}_2 x_3^{r_2+s_2n}y_3^{s_2}z^{m-2} + \cdots + \bar{a}_{m-1}x_3^{r_{m-1}+s_{m-1}n}y_3^{s_{m-1}}z + \bar{a}_m x_3^{r_m+s_mn}y_3^{s_m}.$$

Since we must have some $\bar{a}_i \neq 0$ with $s_i \leqslant 1$ by (D) for $n \gg 0$, we obtain a form (3.12) or (B). $\square$

Pages 882–883: The statement of Lemma 3.7 should be changed to:

LEMMA 3.7. — *Suppose that X is 2-prepared with respect to $f : X \to S$. Suppose that $p \in D$ is a 1-point with $\sigma_D(p) > 0$. Let $m = \sigma_D(p) + 1$. Let*

x, y, z be permissible parameters for D at p such that a form (3.1) *holds at p.*

Let notation be as in Lemma 3.6. For $p_1 \in \psi_1^{-1}(p)$ let $\bar{r}(p_1) = m+1+r_m$, if a form (3.11) *holds at p_1, and*

$$\bar{r}(p_1) = \begin{cases} \max\{m+1+r_m, m+1+s_m\} & \text{if } \bar{a}_m = 1 \\ \max\{m+1+r_{m-1}, m+1+s_{m-1}\} & \text{if } \bar{a}_m = 0 \end{cases}$$

if a form (3.10) *holds at p_1. Let $\bar{r}(p_1) = \omega(m, r_2, \ldots, r_{m-1}) + m + 1$ if a form* (3.12) *holds at p_1, $\bar{r}(p_1) = \omega'(m, r_2, \ldots, r_m) + m + 1$ if a form* (B) *holds at p_1. Let*

$$r = \max\{\bar{r}(p_1) \mid p_1 \in \psi_1^{-1}(p)\}. \tag{3.15}$$

Suppose that $x^* \in \mathcal{O}_{X,p}$ is such that $x = \overline{\gamma} x^*$ for some unit $\overline{\gamma} \in \widehat{\mathcal{O}}_{X,p}$ with $\overline{\gamma} \equiv 1 \bmod m_p^r \widehat{\mathcal{O}}_{X,p}$.

Let V be an affine neighborhood of p such that $x^*, y \in \Gamma(V, \mathcal{O}_X)$, and let C^* be the curve in V which has local equations $x^* = y = 0$ at p.

Let $T_0^* = \mathrm{Spec}(\mathfrak{k}[x^*, y])$. Then there exists a sequence of blow ups of points $T_1^* \to T_0^*$ above (x^*, y) such that if $V_1 = V \times_{T_0^*} T_1^*$ and $\psi_1^* : V_1 \to V$ is the induced sequence of blow ups of sections over C^*, $\Lambda_1^* : V_1 \to T_1^*$ is the projection, then V_1 is 2-prepared at all $p_1^* \in (\psi_1^*)^{-1}(p)$. Further, for every point $p_1^* \in (\psi_1^*)^{-1}(p)$, there exist $\widehat{x}_1, \overline{y}_1 \in \widehat{\mathcal{O}}_{V_1, p_1^*}$ such that $\widehat{x}_1, \overline{y}_1, z$ are permissible parameters at p_1^* and we have one of the following forms:

(1) *p_1^* is a 2-point, and we have an expression (2.2) with*

$$F = \overline{\tau}_0 z^m + \overline{\tau}_2 \widehat{x}_1^{r_2} \overline{y}_1^{s_2} z^{m-2} + \cdots + \overline{\tau}_{m-1} \widehat{x}_1^{r_{m-1}} \overline{y}_1^{s_{m-1}} z + \overline{\tau}_m \widehat{x}_1^{r_m} \overline{y}_1^{s_m} \tag{3.16}$$

where $\overline{\tau}_0 \in \widehat{\mathcal{O}}_{V_1, p_1^*}$ is a unit, $\overline{\tau}_i \in \widehat{\mathcal{O}}_{V_1, p_1^*}$ are units (or zero) for $0 \leqslant i \leqslant m-1$, $\overline{\tau}_m$ is zero or 1, $\overline{\tau}_{m-1} \neq 0$ if $\overline{\tau}_m = 0$, $r_i + s_i > 0$ if $\overline{\tau}_i \neq 0$, and

$$(r_m + c)b - (s_m + d)a \neq 0.$$

(2) *p_1^* is a 1-point, and we have an expression (2.1) with*

$$F = \overline{\tau}_0 z^m + \overline{\tau}_2 \widehat{x}_1^{r_2} z^{m-2} + \cdots + \overline{\tau}_{m-1} \widehat{x}_1^{r_{m-1}} z + \overline{\tau}_m \widehat{x}_1^{r_m} \tag{3.17}$$

where $\overline{\tau}_0 \in \widehat{\mathcal{O}}_{V_1, p_1^*}$ is a unit, $\overline{\tau}_i \in \widehat{\mathcal{O}}_{V_1, p_1^*}$ are units (or zero), and $\mathrm{ord}\, \overline{\tau}_m(0, \overline{y}_1, 0) = 1$. Further, $r_i > 0$ if $\overline{\tau}_i \neq 0$.

(3) *p_1^* is a 1-point, and we have an expression (2.1) with*

$$F = \overline{\tau}_0 z^m + \overline{\tau}_2 \widehat{x}_1^{r_2} z^{m-2} + \cdots + \overline{\tau}_{m-1} \widehat{x}_1^{r_{m-1}} z + \widehat{x}_1^t \overline{\Omega} \tag{3.18}$$

where $\overline{\tau}_0 \in \widehat{\mathcal{O}}_{V_1, p_1^*}$ is a unit, $\overline{\tau}_i \in \widehat{\mathcal{O}}_{V_1, p_1^*}$ are units (or zero), $\overline{\Omega} \in \widehat{\mathcal{O}}_{V_1, p_1^*}$, $\overline{\tau}_i \neq 0$ for some $2 \leqslant i \leqslant m-1$ and $t > \omega(m, r_2, \ldots, r_{m-1})$. Further, $r_i > 0$ if $\overline{\tau}_i \neq 0$.

(4) *p_1^* is a 1-point, and we have an expression (2.1) with*

$$F = \overline{\tau}_0 z^m + \overline{\tau}_2 \widehat{x}_1^{r_2} \overline{y}_1 z^{m-2} + \cdots + \overline{\tau}_{m-1} \widehat{x}_1^{r_{m-1}} \overline{y}_1 z + x_1^{r_m} y_1 + \widehat{x}_1^t \overline{\Omega} \tag{F}$$

where $\overline{\tau}_0 \in \widehat{\mathcal{O}}_{V_1,p_1^*}$ is a unit, $\overline{\tau}_i \in \widehat{\mathcal{O}}_{V_1,p_1^*}$ are units (or zero) for $2 \leqslant i \leqslant m$, $\overline{\Omega} \in \widehat{\mathcal{O}}_{V_1,p_1^*}$, $\overline{\tau}_i \neq 0$ for some $2 \leqslant i \leqslant m$ and $t > \omega'(m, r_2, \ldots, r_m)$. Further, $r_i > 0$ if $\overline{\tau}_i \neq 0$.

Page 885, line 6: "$\overline{\tau}_m \in \mathfrak{k}[[x_1, \overline{y}_1]]$ with $\operatorname{ord}(\overline{\tau}_m(0, \overline{y}_1)) = 1$" should be "$\overline{\tau}_m \in \mathfrak{k}[[x_1, \overline{y}_1, z]]$ with $\operatorname{ord}(\overline{\tau}_m(0, \overline{y}_1, 0)) = 1$".

Page 885, line 11: after "form (3.18)" insert: "and in the case when p_1 has a form (B) a similar argument shows that p_1^* has a form (F)." The following is a detailed proof of this statement.

Suppose that p_1 has a form (B). With the notation of Lemma 3.6, we have polynomials φ, ψ such that

$$x = \varphi(\widetilde{x}_1, \widetilde{y}_1), \quad y = \psi(\widetilde{x}_1, \widetilde{y}_1)$$

determines the birational extension $\mathcal{O}_{T_0,p_0} \to \mathcal{O}_{T_1,\Lambda_1(p_1)}$, and we have a formal change of variables

$$x_1 = \alpha(\widetilde{x}_1, \widetilde{y}_1)\widetilde{x}_1, \quad y_1 = \beta(\widetilde{x}_1, \widetilde{y}_1)$$

for some unit series α and series β. In this case, p_1 maps to a 1-point with an expression (3.13). Let $W \subset \{2, \ldots, m-1\}$ be the set of i such that $s_i \geqslant 2$. Then $W \neq \emptyset$, $i \notin W$ implies, with the notation of (B),

$$a_i(x, y) = \overline{a}_i(x_1, y_1) x_1^{r_i} y_1,$$
$$a_m(x, y) = \overline{a}_m(x_1, y_1) x_1^{r_m} y_1$$

and

$$\sum_{i \in W} a_i(x, y) z^i = x_1^t y_1^2 \Omega$$

with $t > \omega'(m, r_1, \ldots, r_m)$.

We have $x = \overline{\gamma} x^*$ with $\overline{\gamma} \equiv 1 \bmod m_p^r \widehat{\mathcal{O}}_{X,p}$. Set $y^* = y$. At p_1^*, we have regular parameters x_1^*, y_1^* in $\mathcal{O}_{T_1^*,\Lambda_1^*(p_1^*)}$ such that

$$x^* = \varphi(x_1^*, y_1^*), \quad y^* = \psi(x_1^*, y_1^*),$$

and $x_1^*, y_1^*, \widetilde{z}$ are regular parameters in $\mathcal{O}_{V_1,\overline{p}_1^*}$ (recall that $z = \sigma \widetilde{z}$ in Lemma 3.1). We have regular parameters $\overline{x}_1, \overline{y}_1, \in \widehat{\mathcal{O}}_{T_1^*,\Lambda_1^*(p_1^*)}$ defined by

$$\overline{x}_1 = \alpha(x_1^*, y_1^*) x_1^*, \quad \overline{y}_1 = \beta(x_1^*, y_1^*).$$

We have $u = x^a = x_1^{a_1}$ where $a_1 = ad$ for some $d \in \mathbb{Z}_+$. Since $[\alpha(\widetilde{x}_1, \widetilde{y}_1)\widetilde{x}_1]^d = x$, we have that $[\alpha(x_1^*, y_1^*) x_1^*]^d = x^*$. Set $\widehat{x}_1 = \overline{\gamma}^{\frac{1}{d}} \overline{x}_1 = \overline{\gamma}^{\frac{1}{d}} \alpha(x_1^*, y_1^*) x_1^*$. We have that $\overline{\gamma}^{\frac{1}{d}} \alpha(x_1^*, y_1^*)$ is a unit in $\widehat{\mathcal{O}}_{V_1,p_1^*}$, and $x = \widehat{x}_1^d$.

Thus $x_1 = \widehat{x}_1$ (with an appropriate choice of root $\overline{\gamma}^{\frac{1}{d}}$). We have $u = \widehat{x}_1^{ad}$, so that $\widehat{x}_1, \overline{y}_1, z$ are permissible parameters at p_1^*.

For $i \notin W$, we have

$$a_i(x,y) = a_i(\overline{\gamma}x^*, y^*) \equiv a_i(x^*, y^*) \bmod m_p^r \widehat{\mathcal{O}}_{V,p}$$

and

$$\begin{aligned} a_i(x^*, y^*) &= a_i(\varphi(x_1^*, y_1^*), \psi(x_1^*, y_1^*)) \\ &= \overline{x}_1^{r_i} \overline{a}_i(\overline{x}_1, \overline{y}_1) \\ &\equiv x_1^{r_i} \overline{a}_i(x_1, \overline{y}_1) \bmod m_p^r \mathcal{O}_{V_1,p_1^*}. \end{aligned}$$

We further have

$$a_m(x^*, y^*) \equiv x_1^{r_m} \overline{y}_1 \bmod m_p^r \widehat{\mathcal{O}}_{V_1,p_1^*}$$

and

$$x_1^t y_1^2 \Omega \equiv x_1^t \overline{y}_1^2 \Omega(x_1, \overline{y}_1, z) \bmod m_p^r \mathcal{O}_{V_1,p_1^*}.$$

Thus we have expressions (2.1) with

$$\begin{aligned} &u = x_1^{da} \\ \text{(c4)}\quad &v = P(x_1^d) + x_1^{bd} P_1(x_1) + x_1^{bd}(\overline{\tau} z^m + x_1^{r_2} \overline{a}_2(x_1, \overline{y}_1) \overline{y}_1 z^{m-2} + \cdots \\ &\qquad + x_1^{r_m} \overline{y}_1 + x_1^t \overline{y}_1^2 \overline{\Omega} + h) \end{aligned}$$

where $\overline{\tau} \in \widehat{\mathcal{O}}_{V_1,p_1^*}$ is a unit series and

$$h \in m_p^r \widehat{\mathcal{O}}_{V_1,p_1^*} \subset (x_1, z)^r.$$

Set $s = r - m$, and write

$$\begin{aligned} h = z^m \Lambda_0(x_1, \overline{y}_1, z) + z^{m-1} x_1^{1+s} \Lambda_1(x_1, \overline{y}_1) + z^{m-2} x_1^{2+s} \Lambda_2(x_1, \overline{y}_1) + \cdots \\ + z x_1^{(m-1)+s} \Lambda_{m-1}(x_1, \overline{y}_1) + x_1^{m+s} \Lambda_m(x_1, \overline{y}_1) \end{aligned}$$

with $\Lambda_0 \in m_{p_1^*} \widehat{\mathcal{O}}_{V_1,p_1^*}$ and $\Lambda_i \in \mathfrak{k}[[x_1, \overline{y}_1]]$ for $1 \leqslant i \leqslant m$.

Substituting into (c4), we obtain an expression

$$\begin{aligned} u &= x_1^{da} \\ v &= P(x_1^d) + x_1^{bd} P_1(x_1) + x_1^{bd}(\overline{\tau}_0 z^m + x_1^{r_2} \overline{\tau}_2 \overline{y}_1 z^{m-2} + \cdots \\ &\qquad + x_1^{r_{m-1}} \overline{\tau}_{m-1} \overline{y}_1 z + x_1^{r_m} \overline{y}_1 + x_1^{t'} \overline{\Omega}) \end{aligned}$$

where $\overline{\tau}_0 \in \widehat{\mathcal{O}}_{V_1,p_1^*}$ is a unit, $\overline{\tau}_i \in \widehat{\mathcal{O}}_{V_1,p_1^*}$ are units (or zero), for $1 \leqslant i \leqslant m-1$.

We have $\overline{\tau}_0 = \overline{\tau} + \Lambda_0$, $\tau_i = \overline{a}_i(x_1, \overline{y}_1)$ for $2 \leqslant i \leqslant m-1$, and

$$\overline{\Omega} = x_1^{t-t'}\overline{y}_1^2\Omega(x, \overline{y}_1, z) + z^{m-1}x_1^{1+s-r_m}\Lambda_1(x_1, \overline{y}_1) + \cdots \\ + x_1^{m+s-r_m}\Lambda_m(x_1, \overline{y}_1)).$$

with $t' = \omega'(m, r_2, \ldots, r_m) + 1$

We thus have the desired form (F).

Page 894, line 16: After "in a neighborhood of q" insert "Further, if $I\mathcal{O}_{U',q}$ is principal, then $\sigma_D(q) = 0$".

Page 895, line 3: "$\sigma_{D'}(q)$" should be "$\sigma_D(q)$".

Page 895, line 10: "$\sigma_D(q) < m - h - 1$" should be "$\sigma_D(q) \leqslant m - h - 1$"

Page 896, line 17: "r_m if $\tau_i \neq 0$" should be "r_m if $\tau_m \neq 0$"

Page 896, line 17:

$$s = \min\{b_1 m, r_i + b_1(m-i) \text{ with } \tau_i \neq 0 \text{ for } 2 \leqslant i \leqslant m-1,\ r_m \text{ if } \tau_m \neq 0\}.$$

Page 896, line −8: "Then x^{r_m} generates $I\widehat{\mathcal{O}}_{U',q}$" should be "If $\tau_m \neq 0$, then x^{r_m} generates $I\widehat{\mathcal{O}}_{U',q}$".

Page 896, line −6: After "Thus U' is prepared at q" insert: "If $\tau_m = 0$ then $\tau_{m-1} \neq 0$ so $x^{r_{m-1}}z$ generates $I\widehat{\mathcal{O}}_{U',q}$. Then $G' = x_1^{r_{m-1}+b_1}\Lambda$ with $\operatorname{ord}\Lambda(0,0,z) = 1$ so U' is prepared at q."

Page 896, line −4: Remove "and $r_{m-1} > 0$"

Page 896, line −2 and page 897, line 8: "$r_i > 0$" should be $\tau_i \neq 0$".

Page 897, line 7: Should be "$z = x_1^{b_1}z_1$ for some $b_1 \in \mathbb{Z}_+$"

Page 897, lines −10, −7, −5: "$r_i > 0$" should be $\tau_i \neq 0$".

Page 898, line 10, Replace "The proof is similar to that of the Theorem 4.1" with the following detailed proof of Theorem 4.2.

Proof. — Let I be the ideal in $\Gamma(U, \mathcal{O}_X)$ generated by

$$z^{r_m} \text{ and } \{x^{r_i}z^{m-i} \mid 2 \leqslant i \leqslant m-1 \text{ and } \tau_i \neq 0\}.$$

Suppose that $\psi : U' \to U$ is toroidal for $\overline{D}$ and U' is nonsingular. Let $\overline{D}' = \psi^{-1}(\overline{D})$.

The set of 2-curves of $\overline{D}'$ is the disjoint union of the 2-curves of $D_{U'}$ and the 2-curve which is the intersection of the strict transform of the surface

$z=0$ on U' with $D_{U'}$. ψ factors as a sequence of blow ups of 2-curves of (the preimage of) $\overline{D}$. We will verify the following three statements, from which the conclusions of the theorem follow.

(c5) If $q\in\psi^{-1}(p)$ and $I\mathcal{O}_{U',q}$ is principal, then $\sigma_D(q)<\sigma_D(p)$. In particular, $\sigma_D(q)<\sigma_D(p)$ if q is a 1-point of $\overline{D}'$.

(c6) If C' is a 2-curve of $D_{U'}$, then U' is prepared at $q=C'\cap\psi^{-1}(p)$ if and only if $\sigma_D(q)<\infty$
if and only if $I\mathcal{O}_{U',q}$ is principal
if and only if U' is prepared at all $q'\in C'$ in a neighborhood of q.

(c7) If C' is the 2-curve of $\overline{D}'$ which is the intersection of $D_{U'}$ with the strict transform of $\widetilde{z}=0$ in U', then $\sigma_D(q)\leqslant\sigma_D(p)$ if $q=C'\cap\psi^{-1}(p)$, and $\sigma_D(q')=\sigma_D(q)$ for $q'\in C'$ in a neighborhood of q.

Suppose that $q\in\psi^{-1}(p)$ is a 1-point for $\overline{D}'$. Then $I\widehat{\mathcal{O}}_{U',q}$ is principal. At q, we have permissible parameters x_1,y,z_1 defined by

$$x=x_1^{a_1},\quad z=x_1^{b_1}(z_1+\alpha)\tag{c8}$$

for some $a_1,b_1\in\mathbb{Z}_+$ and $0\neq\alpha\in\mathfrak{k}$. Substituting into (3.7), we have

$$u=x_1^{aa_1},\quad v=P(x_1^{a_1})+x_1^{ba_1}G$$

where

$$G=\tau_0x_1^{b_1m}(z_1+\alpha)^m+\tau_2x_1^{a_1r_2+b_1(m-2)}(z_1+\alpha)^{m-2}+\cdots\\ +\tau_{m-1}x_1^{a_1r_{m-1}+b_1}(z_1+\alpha)+x_1^{a_1t}\Omega.$$

Let x_1^s be a local generator of $I\widehat{\mathcal{O}}_{U',q}$. We have that $a_1t>s$ by our construction of $\omega(m,r_2,\ldots,r_{m-1})$ before the statement of Theorem 4.2. Let $G'=\frac{G}{x_1^s}$.

If z^m is a local generator of $I\widehat{\mathcal{O}}_{U',q}$, then G' has an expansion

$$G'=\tau'(z_1+\alpha)^m+g_2(z_1+\alpha)^{m-2}+\cdots+g_{m-1}(z_1+\alpha)++x_1\Omega_1+y\Omega_2$$

where $0\neq\tau'=\tau(0,0,0)\in\mathfrak{k}$, $g_2,\ldots,g_m\in\mathfrak{k}$ and $\Omega_1,\Omega_2\in\widehat{\mathcal{O}}_{U',q}$. We have $\operatorname{ord}(G'(0,0,z_1))\leqslant m-1$. Setting $F'=G'-G'(x_1,0,0)$ and $P'(x_1)=P(x_1^{a_1})+x_1^{ba_1+b_1m}G'(x_1,0,0)$, we have an expression

$$u=x_1^{aa_1},v=P'(x_1)+x_1^{ba_1+b_1m}F'$$

of the form of (2.1). Thus U' is 2-prepared at q with $\sigma_{D'}(q)<m-1=\sigma_D(p)$.

Suppose that z^m is not a local generator of $I\widehat{\mathcal{O}}_{U',q}$, but there exists some i with $2 \leqslant i \leqslant m-1$ such that $x^{r_i} z^{m-i}$ is a local generator of $I\widehat{\mathcal{O}}_{U',q}$. Let h be the smallest i with this property. Then G' has an expression

$$G' = g_h(z_1+\alpha)^{m-h} + \cdots + g_m + x_1\Omega_1 + y_1\Omega_2$$

for some $g_i \in \mathfrak{k}$ with $g_h \neq 0$ and $\Omega_1, \Omega_2 \in \widehat{\mathcal{O}}_{U',q}$. As in the previous case, we have that U' is 2-prepared at q with $\sigma_D(q) \leqslant m-h-1 < m-1 = \sigma_D(p)$.

Now suppose that $q \in \psi^{-1}(p)$ is a 2-point for $D_{U'}$. We have permissible parameters x_1, y, z_1 in $\widehat{\mathcal{O}}_{U',q}$ such that

$$x = x_1^{a_1} z_1^{b_1}, z = x_1^{c_1} z_1^{d_1} \tag{c9}$$

with $a_1, b_1 > 0$ and $a_1 d_1 - b_1 c_1 = \pm 1$. Substituting into (3.7), we have

$$u = x_1^{a_1 a} z_1^{b_1 a}, v = P(x_1^{a_1} z_1^{b_1}) + x_1^{a_1 b} z_1^{b_1 b} G$$

where

$$\begin{aligned} G = \tau_0 x_1^{c_1 m} z_1^{d_1 m} + \tau_2 x_1^{r_2 a_1 + c_1(m-2)} z_1^{r_2 b_1 + d_1(m-2)} + \cdots \\ + \tau_{m-1} x_1^{a_1 r_{m-1} + c_1} z_1^{b_1 r_{m-1} + d_1} + x_1^{a_1 t} z_1^{b_1 t} \Omega. \end{aligned}$$

Let C' be the 2-curve of $D_{U'}$ containing q. The three statements $\sigma_D(q) < \infty$, $\sigma_D(q) = 0$ and $I\mathcal{O}_{U',q}$ is principal are equivalent. Further, we have that $\sigma_D(q') = \sigma_D(q)$ for $q' \in C'$ in a neighborhood of q.

Suppose that $I\mathcal{O}_{U',q}$ is principal and let $x_1^s z_1^{\bar{t}}$ be a local generator of $I\widehat{\mathcal{O}}_{U',q}$. We have that $a_1 t > s$ and $b_1 t > \bar{t}$ by our construction of $\omega(m, r_2, \ldots, r_{m-1})$ before the statement of Theorem 4.2. Let $G' = G/x_1^s z_1^{\bar{t}}$. We have that

$$u = (x_1^{a_1} z_1^{b_1})^a, \quad v = P(x_1^{a_1} z_1^{b_1}) + x_1^{a_1 b + s} z_1^{b b_1 + \bar{t}} G'$$

has the form (2.2), since we have made a monomial substitution in x and z. Since z^m or $x^{r_i} z^{m-i}$ for some $i < m$ is a local generator of $I\widehat{\mathcal{O}}_{U',q}$, we have that G' is a unit in $\widehat{\mathcal{O}}_{U',q}$. We thus have that U' is prepared at q.

The final case is when $q \in \psi^{-1}(p)$ is on the 2-curve C' of $\overline{D}'$ which is the intersection of $D_{U'}$ with the strict transform of $z = 0$ in U'. Then there exist permissible parameters x_1, y, z_1 at q such that

$$x = x_1, \quad z = x_1^{b_1} z_1 \tag{c10}$$

for some $b_1 \in \mathbb{Z}_+$. The equations $x_1 = z_1 = 0$ are local equations of C' at q. Let

$$s = \min\{b_1 m, r_i + b_1(m-i) \text{ with } \tau_i \neq 0 \text{ for } 2 \leqslant i \leqslant m-1\}.$$

We have that $t > s$ by our construction of $\omega(m, r_2, \dots, r_{m-1})$ before the statement of Theorem 4.1. We have an expression of the form (2.1) at q,

$$u = x_1^a$$
$$v = P(x_1^a) + x_1^{ab+s} G'$$

with

$$G' = \tau_0 x_1^{b_1 m - s} z_1^m + \tau_2 x_1^{r_2 + b_1(m-2) - s} z_1^{m-2} + \cdots + \tau_{m-1} x_1^{r_{m-1} + b_1 - s} z_1 + x_1^{t-s} \Omega.$$

We see that $\sigma_D(q) \leqslant \sigma_D(p)$ (with $\sigma_D(q) < \sigma_D(p)$ if $s = r_i + b_1(m-i)$ for some i with $2 \leqslant i \leqslant m-1$) and $\sigma_D(q') = \sigma_D(q)$ for q' in a neighborhood of q on C'.

Suppose that $I\mathcal{O}_{U',q}$ is principal. Let h be the largest i such that $\tau_i \neq 0$ in (3.7). Then $x^{r_h} z^{m-h}$ is the local generator of $I\widehat{\mathcal{O}}_{U',q}$. Since $2 \leqslant h \leqslant m-1$, we have that

$$\sigma_D(q) = m - h - 1 < m - 1 = \sigma_D(p). \qquad \square$$

Before the statement of Theorem 4.3 on page 898, add the following:

We construct the function $\omega'(m, r_2, \dots, r_m)$ in a similar way to the construction of ω. Let I be the ideal in $k[x, z]$ generated by z^m and $x^{r_i} z^{m-i}$ for all i such that $2 \leqslant i \leqslant m$ and $\tau_i \neq 0$. We define $\omega'(m, r_2, \dots, r_m)$ as we define ω, except we allow i to range within $2 \leqslant i \leqslant m$.

Theorem G. — *Suppose that $p \in \mathrm{Sing}_1(X)$ is a 1-point and X is 3-prepared at p. Let x, y, z be permissible parameters at p giving a form* (A) *at p. Let U be an étale cover of an affine neighborhood of p in which x, y, z are uniformizing parameters. Then $xz = 0$ gives a toroidal structure $\overline{D}$ on U.*

There is (after possibly replacing U with a smaller neighborhood of p) a unique, minimal toroidal morphism $\psi : U' \to U$ with respect to $\overline{D}$ which has the property that U' is nonsingular, 2-prepared and $\Gamma_D(U') < \sigma_D(p)$. This map ψ factors as a sequence of permissible blowups $\pi_i : U_i \to U_{i-1}$ of sections C_i over the two curve C of $\overline{D}$. U_i is 1-prepared for $U_i \to S$. We have that the curve C_i blown up in $U_{i+1} \to U_i$ is in $\mathrm{Sing}_{\sigma_D(p)}(U_i)$ if C_i is not a 2-curve of D_{U_i}, and that C_i is in $\mathrm{Sing}_1(U_i)$ if C_i is a 2-curve of D_{U_i}.

Proof. — Let I be the ideal in $\Gamma(U, \mathcal{O}_X)$ generated by

$$z^{r_m} \text{ and } \{x^{r_i} z^{m-i} \mid 2 \leqslant i \leqslant m \text{ and } \tau_i \neq 0\}.$$

Suppose that $\psi : U' \to U$ is toroidal for $\overline{D}$ and U' is nonsingular. Let $\overline{D}' = \psi^{-1}(\overline{D})$.

The set of 2-curves of $\overline{D}'$ is the disjoint union of the 2-curves of $D_{U'}$ and the 2-curve which is the intersection of the strict transform of the surface $z=0$ on U' with $D_{U'}$. ψ factors as a sequence of blow ups of 2-curves of (the preimage of) $\overline{D}$. We will verify the following three statements, from which the conclusions of the theorem follow.

$$\text{If } q \in \psi^{-1}(p) \text{ and } I\mathcal{O}_{U',q} \text{ is principal, then } \sigma_D(q) < \sigma_D(p). \text{ In particular, } \sigma_D(q) < \sigma_D(p) \text{ if } q \text{ is a 1-point of } \overline{D}'. \tag{c11}$$

(c12) If C' is a 2-curve of $D_{U'}$, then U' is prepared at $q = C' \cap \psi^{-1}(p)$
if and only if $\sigma_D(q) < \infty$
if and only if $I\mathcal{O}_{U',q}$ is principal
if and only if U' is prepared at all $q' \in C'$ in a neighborhood of q.

(c13) If C' is the 2-curve of $\overline{D}'$ which is the intersection of $D_{U'}$ with the strict transform of $\tilde{z} = 0$ in U', then $\sigma_D(q) \leqslant \sigma_D(p)$ if $q = C' \cap \psi^{-1}(p)$, and $\sigma_D(q') = \sigma_D(q)$ for $q' \in C'$ in a neighborhood of q.

Suppose that $q \in \psi^{-1}(p)$ is a 1-point for $\overline{D}'$. Then $I\hat{\mathcal{O}}_{U',q}$ is principal. At q, we have permissible parameters x_1, y, z_1 defined by

$$x = x_1^{a_1}, \quad z = x_1^{b_1}(z_1 + \alpha) \tag{c14}$$

for some $a_1, b_1 \in \mathbb{Z}_+$ and $0 \neq \alpha \in \mathfrak{k}$. Substituting into (A), we have

$$u = x_1^{aa_1}, \quad v = P(x_1^{a_1}) + x_1^{ba_1} G$$

where

$$G = \tau_0 x_1^{b_1 m}(z_1+\alpha)^m + \tau_2 x_1^{a_1 r_2 + b_1(m-2)} y (z_1+\alpha)^{m-2} + \cdots + \tau_{m-1} x_1^{a_1 r_{m-1} + b_1} y (z_1 + \alpha) + \tau_m x_1^{a_1 r_m} y + x_1^{a_1 t}\Omega.$$

Let x_1^s be a local generator of $I\hat{\mathcal{O}}_{U',q}$. We have that $a_1 t > s$ by our construction of $\omega'(m, r_2, \ldots, r_m)$. Let $G' = \frac{G}{x_1^s}$.

If z^m is a local generator of $I\hat{\mathcal{O}}_{U',q}$, then G' has an expansion

$$G' = \tau'(z_1+\alpha)^m + x_1\Omega_1 + y\Omega_2$$

where $0 \neq \tau' = \tau(0,0,0) \in \mathfrak{k}$, $g_2, \ldots, g_m \in \mathfrak{k}$ and $\Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U',q}$. We have $\mathrm{ord}(G'(0,0,z_1)) \leqslant m-1$. Setting $F' = G' - G'(x_1, 0, 0)$ and $P'(x_1) = P(x_1^{a_1}) + x_1^{ba_1 + b_1 m} G'(x_1, 0, 0)$, we have an expression

$$u = x_1^{aa_1}, \quad v = P'(x_1) + x_1^{ba_1 + b_1 m} F'$$

of the form of (2.1). Thus U' is 2-prepared at q with $\sigma_{D'}(q) < m - 1 = \sigma_D(p)$.

Suppose that z^m is not a local generator of $I\widehat{\mathcal{O}}_{U',q}$, but there exists some i with $2 \leqslant i \leqslant m$ such that $x^{r_i}z^{m-i}$ is a local generator of $I\widehat{\mathcal{O}}_{U',q}$. Let h be the smallest i with this property. Then G' has an expression

$$G' = g_h(z_1+\alpha)^{m-h}y + \cdots + g_m y + x_1\Omega_1 + y_1^2\Omega_2$$

for some $g_i \in \mathfrak{k}$ with $g_h \neq 0$ and $\Omega_1, \Omega_2 \in \widehat{\mathcal{O}}_{U',q}$. We have that U' is 2-prepared at q with $\sigma_D(q) \leqslant (m-h+1)-1 < m-1 = \sigma_D(p)$ since $h \geqslant 2$.

Now suppose that $q \in \psi^{-1}(p)$ is a 2-point for $D_{U'}$. We have permissible parameters x_1, y, z_1 in $\widehat{\mathcal{O}}_{U',q}$ such that

$$x = x_1^{a_1}z_1^{b_1}, \quad z = x_1^{c_1}z_1^{d_1} \tag{c15}$$

with $a_1, b_1 > 0$ and $a_1d_1 - b_1c_1 = \pm 1$. Substituting into (A), we have

$$u = x_1^{a_1a}z_1^{b_1a}, \quad v = P(x_1^{a_1}z_1^{b_1}) + x_1^{a_1b}z_1^{b_1b}G$$

where

$$\begin{aligned} G = {} & \tau_0 x_1^{c_1m}z_1^{d_1m} + \tau_2 x_1^{r_2a_1+c_1(m-2)}z_1^{r_2b_1+d_1(m-2)}y + \cdots \\ & + \tau_{m-1}x_1^{a_1r_{m-1}+c_1}z_1^{b_1r_{m-1}+d_1}y + \tau_m x_1^{a_1r_m}z_1^{b_1r_m}y + x_1^{a_1t}z_1^{b_1t}\Omega. \end{aligned}$$

Let C' be the 2-curve of $D_{U'}$ containing q. The three statements $\sigma_D(q) < \infty$, $\sigma_D(q) = 0$ and $I\mathcal{O}_{U',q}$ is principal are equivalent. Further, we have that $\sigma_D(q') = \sigma_D(q)$ for $q' \in C'$ in a neighborhood of q.

Suppose that $I\mathcal{O}_{U',q}$ is principal and let $x_1^s z_1^{\bar{t}}$ be a local generator of $I\widehat{\mathcal{O}}_{U',q}$. We have that $a_1t > s$ and $b_1t > \bar{t}$ by our construction of $\omega'(m, r_2, \ldots, r_m)$. Let $G' = G/x_1^s z_1^{\bar{t}}$. We have that

$$u = (x_1^{a_1}z_1^{b_1})^a, \quad v = P(x_1^{a_1}z_1^{b_1}) + x_1^{a_1b+s}z_1^{bb_1+\bar{t}}G'$$

has the form (2.2), since we have made a monomial substitution in x and z. Since z^m or $x^{r_i}z^{m-i}$ for some i is a local generator of $I\widehat{\mathcal{O}}_{U',q}$, we have that $\operatorname{ord} G'(0,y,0) \leqslant 1$. We thus have that U' is prepared at q.

The final case is when $q \in \psi^{-1}(p)$ is on the 2-curve C' of $\overline{D}'$ which is the intersection of $D_{U'}$ with the strict transform of $z = 0$ in U'. Then there exist permissible parameters x_1, y, z_1 at q such that

$$x = x_1, \quad z = x_1^{b_1}z_1 \tag{c16}$$

for some $b_1 \in \mathbb{Z}_+$. The equations $x_1 = z_1 = 0$ are local equations of C' at q. Let

$$s = \min\{b_1m, r_i + b_1(m-i) \text{ with } \tau_i \neq 0 \text{ for } 2 \leqslant i \leqslant m\}.$$

We have that $t > s$ by our construction of $\omega'(m, r_2, \ldots, r_m)$. We have an expression of the form (2.1) at q,

$$\begin{aligned} u &= x_1^a \\ v &= P(x_1^a) + x_1^{ab+s} G' \end{aligned}$$

with

$$\begin{aligned} G' = \tau_0 x_1^{b_1 m - s} z_1^m + \tau_2 x_1^{r_2 + b_1(m-2) - s} z_1^{m-2} y + \cdots \\ + \tau_{m-1} x_1^{r_{m-1} + b_1 - s} z_1 y + \tau_m x_1^{r_m} y + x_1^{t-s} \Omega. \end{aligned}$$

We see that $\sigma_D(q) \leqslant \sigma_D(p)$ (with $\sigma_D(q) < \sigma_D(p)$ if $s = r_i + b_1(m-i)$ for some i with $2 \leqslant i \leqslant m$) and $\sigma_D(q') = \sigma_D(q)$ for q' in a neighborhood of q on C'.

Suppose that $I\mathcal{O}_{U',q}$ is principal. Let h be the largest i such that $\tau_i \neq 0$ in (A). Then $x^{r_h} z^{m-h}$ is the local generator of $I\widehat{\mathcal{O}}_{U',q}$. Since $2 \leqslant h \leqslant m-1$, we have that

$$\sigma_D(q) = (m - h + 1) - 1 < m - 1 = \sigma_D(p). \qquad \square$$

Page 907, lines 19–20 and line 29: should be "(3.6), (3.7) or (A)"

Page 908, line 9: should be "Theorem 4.1, 4.2 or G".

Page 918, lines 16 and −2: should be "Theorem 4.2 and G"

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Steven Dale CUTKOSKY
Department of Mathematics
University of Missouri
Columbia MO 65211 (USA)
cutkoskys@missouri.edu