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A NOTE ON THE SEMICLASSICAL MEASURE AT SINGULAR POINTS OF THE BOUNDARY OF THE BUNIMOVICH STADIUM

by Dan MANGOUBI & Adi WELLER WEISER (*)

Abstract. — An argument by Hassell proving the existence of a Bunimovich stadium for which there are semiclassical measures giving positive mass to the submanifold of bouncing ball trajectories uses a notion of non-gliding points. However, this notion is defined only for domains with C^2-boundaries. The purpose of this note is to clarify the argument.

Résumé. — Un argument de Hassell montrant l'existence d'un stade de Bunimovich sur lequel on trouve une mesure semiclassique chargeant la sous-variété des trajectoires de boules rebondissantes fait appel à la notion de points non-gliding. Pourtant, cette notion suppose un domaine lisse C^2. Le but de cette note est de clarifier ce point.

1. Introduction

In a celebrated paper [4] Hassell proves the existence of quantum ergodic manifolds which are not quantum uniquely ergodic. Furthermore, Hassell shows that there is a Bunimovich stadium, M, with a semiclassical measure giving positive mass to the submanifold of bouncing ball trajectories. For this refinement, in order to rule out formations of certain non-uniform semiclassical measures, Hassell applies a theorem of Burq and Gérard [2] showing that a boundary semiclassical measure cannot put any mass on the set of non-gliding points, a special subset of $S^*\partial M$ where the classical trajectories passing through are not affected by the boundary (see Appendix). However, the definition of non-gliding points requires

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at least C^2-smoothness of the boundary, while the boundary of the Bunimovich stadium is only $C^{1,1}$-smooth. Moreover, in a neighborhood of a singular point p of the boundary, one has points in $S\,\partial M$ which are far from being non-gliding, since the curvature jumps there from zero to a positive constant, and it is not a-priori clear from Burq and Gérard's argument whether one should regard the points of $S_p\partial M$ as gliding or non-gliding. The aim of this note is to clarify Hassell's argument, by showing (in Theorem 3.1) that indeed the semiclassical measure on the boundary gives zero mass to the portion of $S\,\partial M$ lying above the *closure* of the straight part of ∂M. Our result is an instance where one can relax the smoothness assumption in [2] and hints that in two dimensions the $C^{1,1}$ assumption may be sufficient to conclude that the semiclassical measure of the boundary vanishes on the closure of the subset of $S\,\partial M$ where the classical trajectories passing through are not reflected as a replacement for non-gliding points in the smooth case. In this paper, we treat the Bunimovich stadium case. The proof, like in [2], uses Gérard–Leichtnam's transport equation from [3], recalled here in Section 2, and follows similar lines of microlocalization on non-gliding points. However, unlike in [2], we do not make any change of variables, allowing us to treat the singular points of the boundary as well.

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2. Background: Gérard–Leichtnam's transport equation

By Egorov's theorem, in the case of a closed manifold M a semiclassical measure μ is invariant under the geodesic flow; or equivalently $\xi \cdot \partial_x \mu = 0$, where (x, ξ) are canonical coordinates on T^*M. In the presence of a $C^{1,1}$-boundary a semiclassical measure μ evolves under the geodesic flow according to Gérard–Leichtnam's equation. It expresses the distribution $\xi \cdot \partial_x \mu$ in terms of a corresponding semiclassical measure ν on $T^*\partial M$, and Snell's law is manifested in the symmetry of this equation. Here we explain the notions of a corresponding semiclassical measure on $T^*\partial M$, hyperbolic point and glancing point and we are then able to present the equation.

Let $M \subset \mathbb{R}^2$ be a bounded domain with $C^{1,1}$-boundary, and $\{u_j\}$ an orthonormal basis of Laplace eigenfunctions of M with Dirichlet's boundary conditions and with increasing positive eigenvalues $\{E_j\}$ respectively. For the definition below we consider $\{u_j\}$ as a sequence in $L^2(\mathbb{R}^2)$.

Definition 2.1. — *M-Semiclassical measures on $T^*\mathbb{R}^2$.*
Let $\{u_{k_j}\}$ be a subsequence such that for all $a \in C_c^\infty\left(T^*\mathbb{R}^2\right)$ there exists the limit

$$\lim_{j\to\infty} \left\langle Op_{h_{k_j}}(a)\, u_{k_j}, u_{k_j} \right\rangle$$

where $h_{k_j} = E_{k_j}^{-1/2}$ and $Op_h(a)\, u(x) = \frac{1}{(2\pi)^2} \int a(x, h\xi)\, e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi$ is the standard quantization. Then we define the distribution μ by

$$\langle \mu, a \rangle = \lim_{j\to\infty} \left\langle Op_{h_{k_j}}(a)\, u_{k_j}, u_{k_j} \right\rangle .$$

*We call μ an M-*semiclassical measure *associated to the subsequence $\{u_{k_j}\}$.*

Remark 2.2. — The distribution in the above definition is actually a positive measure supported on $\overline{T^*M}$ (see [3, p. 565]).

To define the corresponding boundary semiclassical measures recall that in [3] Gérard and Leichtnam showed that the standard quantization procedure may be extended to the following class of symbols defined on any closed C^1-manifold N

$$\Sigma_c(T^*N) = \{b \in C_c(T^*N) \mid b \text{ has continuous vertical derivatives up to order } n+1\}$$

where n is the dimension of N, while keeping the resulting operators $Op_h(b)$ bounded on $L^2(N)$.

In our case, this extension allows to define below a notion of semiclassical measure on $T^*\partial M$. For the sake of this definition we also recall that due to our assumption that ∂M is $C^{1,1}$-smooth, the normal derivatives on the boundary $\partial_n u_k$ exist in $L^2(\partial M)$, and moreover, the sequence $h_k \partial_n u_k$ is bounded in $L^2(\partial M)$ (see e.g. [3, Lemma 2.1]).

Definition 2.3 (Corresponding semiclassical measures on $T^*\partial M$ for a Dirichlet problem). — *For an M-semiclassical measure μ associated to the subsequence $\{u_{k_j}\}$, assume that for all $b \in \Sigma_c(T^*\partial M)$ there exists the limit*

$$\lim_{j\to\infty} \left\langle Op_{h_{k_j}}(b)\, h_{k_j}\partial_n u_{k_j}, h_{k_j}\partial_n u_{k_j} \right\rangle$$

Then, define a positive measure ν on $T^*\partial M$ by

$$\langle \nu, b\rangle = \lim_{j\to\infty} \left\langle Op_{h_{k_j}}(b)\, h_{k_j}\partial_n u_{k_j}, h_{k_j}\partial_n u_{k_j} \right\rangle .$$

We call ν a boundary semiclassical measure corresponding to *μ.*

Remark 2.4. — For any given subsequence $\{u_{k_j}\}$ there is a subsequence $\{u_{k_{j_l}}\}$ for which the limits in Definitions 2.1 and 2.3 exist (see [3, p. 565, 576]).

Next, points in $T^*\partial M$ are classified according to their dynamical behaviour [5] (see also [1]). The definitions below are only for the case of the billiard dynamics.

Definition 2.5. — *Glancing points*

$$G = \{(x,\xi) \in T^*\partial M \mid |\xi| = 1\} = S^*\partial M$$

Definition 2.6. — *Hyperbolic points*

$$H = \{(x,\xi) \in T^*\partial M \mid |\xi| < 1\}$$

The Gérard–Leichtnam transport equation is

Theorem 2.7 ([3]). — *For ν a boundary semiclassical measure corresponding to an M-semiclassical measure μ we have for all $a \in C_c^\infty(T^*\mathbb{R}^2)$*

$$(2.1)\quad -\int_{T^*\mathbb{R}^2} \xi\cdot\partial_x a\, d\mu = \int_{H\cup G} \frac{a(x(\rho),\xi^+(\rho)) - a(x(\rho),\xi^-(\rho))}{\langle \xi^+(\rho)-\xi^-(\rho), n(x(\rho))\rangle}\, d\nu(\rho)$$

where $n(x(\rho))$ is the inward pointing normal at $x(\rho)$, and $\xi^+(\rho)$ is the co-vector in $S^*_{x(\rho)}\mathbb{R}^2$ such that its orthogonal projection on $T^*_{x(\rho)}\partial M$ is equal to ρ's co-vector and so that $\langle \xi^+(\rho), n(x(\rho))\rangle > 0$. The co-vector $\xi^-(\rho)$ is similarly defined with $\langle \xi^-(\rho), n(x(\rho))\rangle \leq 0$.

3. Zero mass on the set of glancing points of the straight part of the boundary

It is shown in [2] that in the case that the boundary is smooth the set of non-gliding points is of boundary semiclassical measure zero. We extend the proof to the case of the Bunimovich stadium and show that the set of glancing points lying above the closure of the straight part of the boundary is of boundary semiclassical measure zero. The main new point is the avoidance of change of variables which allows us to handle non-smooth points of ∂M.

Let $M = ([-a,a]\times(0,1)) \cup \left\{(x\pm a)^2 + \left(y-\frac{1}{2}\right)^2 < \frac{1}{4}\right\}$. Decompose ∂M as $S \cup C$ where S is the straight part of the boundary and C is its curved part. To be precise, we let $S = S_0 \cup S_1$ where $S_i = [-a,a]\times\{i\}$ and put $C = (\partial M)\setminus S$. Let $\pi : T^*\partial M \to \partial M$ and $\pi_{\mathbb{R}^2} : T^*\mathbb{R}^2 \to \mathbb{R}^2$ denote the projection maps.

Theorem 3.1. — *Let ν be a semiclassical measure on $T^*\partial M$ corresponding to an M-semiclassical measure. Let G be the set of glancing points of $T^*\partial M$. Then $\nu\left(G\cap\pi^{-1}(S)\right) = 0$.*

Proof. — Let μ be an M-semiclassical measure to which ν corresponds. Fix a point $\rho_0 \in G\cap\pi^{-1}(S_0)$ and consider

$$a_\varepsilon(x,\xi) = \xi_2 b\left(x_1, \frac{x_2}{\varepsilon}, \frac{\xi_2^2}{\varepsilon}\right)\phi(\xi_1)$$

where $b \in C_c^\infty\left(\mathbb{R}^3\right)$ is nonnegative with $b(x_1(\rho_0), 0, 0) > 0$ and where the cutoff function ϕ is in $C_c^\infty(\mathbb{R})$ with $|\phi| \leqslant 1$ and $\phi(\xi_1) = 1$ for $|\xi_1| \leqslant 1$. In equation (2.1) we make the substitution $a = a_\varepsilon$ and take the limit as ε tends to zero.

For calculating the left hand side of (2.1) we note that

$$\xi\cdot\partial_x a_\varepsilon = \xi_1\xi_2\phi(\xi_1)(\partial_1 b)\left(x_1, \frac{x_2}{\varepsilon}, \frac{\xi_2^2}{\varepsilon}\right) + \xi_2^2\phi(\xi_1)\frac{1}{\varepsilon}(\partial_2 b)\left(x_1, \frac{x_2}{\varepsilon}, \frac{\xi_2^2}{\varepsilon}\right)$$

We claim that $\xi\cdot\partial_x a_\varepsilon \longrightarrow 0$ as $\varepsilon\to 0$. Indeed, if $\xi_2 = 0$ then $\xi\cdot\partial_x a_\varepsilon = 0$. If $\xi_2 \neq 0$, then for all ε small enough we have that $\partial_i b\left(x_1, \frac{x_2}{\varepsilon}, \frac{\xi_2^2}{\varepsilon}\right) \equiv 0$ since b is compactly supported. Furthermore, $\xi\cdot\partial_x a_\varepsilon$ is dominated by a constant independent of $\varepsilon \leqslant 1$. In fact, we have

$$\begin{aligned}\left|\xi_1\xi_2\phi(\xi_1)(\partial_1 b)\left(x_1, \frac{x_2}{\varepsilon}, \frac{\xi_2^2}{\varepsilon}\right)\right| &= \left|\xi_1\phi(\xi_1)\frac{\xi_2}{\sqrt{\varepsilon}}(\partial_1 b)\left(x_1, \frac{x_2}{\varepsilon}, \frac{\xi_2^2}{\varepsilon}\right)\sqrt{\varepsilon}\right| \\ &\leqslant \sup_{x_1,x_2,y,\xi_1}\left|y\partial_1 b\left(x_1,x_2,y^2\right)\right| |\xi_1\phi(\xi_1)|\end{aligned}$$

while

$$\left|\frac{\xi_2^2}{\varepsilon}\phi(\xi_1)(\partial_2 b)\left(x_1, \frac{x_2}{\varepsilon}, \frac{\xi_2^2}{\varepsilon}\right)\right| \leqslant \sup_{x_1,x_2,y}|y\partial_2 b(x_1,x_2,y)|\,.$$

Since μ has compact support ($\operatorname{supp}\mu \subset S^*\mathbb{R}^2 \cap \pi^{-1}_{\mathbb{R}^2}\left(\overline{M}\right)$, see [3]) we may conclude by the Lebesgue Dominated Convergence Theorem that

$$\int_{T^*\mathbb{R}^2} \xi \cdot \partial_x a_\varepsilon \, d\mu \xrightarrow[\varepsilon \to 0]{} 0 \,. \tag{3.1}$$

Next, we calculate the right hand side of (2.1). We set

$$b_\varepsilon (x_1, x_2, \xi_2) := b\left(x_1, \frac{x_2}{\varepsilon}, \frac{\xi_2^2}{\varepsilon}\right) .$$

For $y \in \partial M$, let $n^*(y) \in S^*_y\mathbb{R}^2$ be the co-vector such that $\langle n^*(y), n(y)\rangle = 1$ and $\langle n^*(y), \dot{y}\rangle = 0$ where $\dot{y} \in S_y\partial M$. Note that for $\rho \in H \cup G$

$$\xi^\pm(\rho) = \xi(\rho) \pm \sqrt{1 - |\xi(\rho)|^2}\, n^*(x(\rho)) \,.$$

It will be convenient to write

$$\xi^\pm(\rho) = \xi_1^\pm(\rho)\, dx^1 + \xi_2^\pm(\rho)\, dx^2 \,.$$

The numerator of the integrand in the right hand side of (2.1) is

$$\begin{aligned} & a_\varepsilon\left(x(\rho), \xi^+(\rho)\right) - a_\varepsilon\left(x(\rho), \xi^-(\rho)\right) \\ &\qquad \overset{|\xi_1^\pm| \leqslant 1}{=} \xi_2^+(\rho)\, b_\varepsilon\left(x(\rho), \xi_2^+(\rho)\right) - \xi_2^-(\rho)\, b_\varepsilon\left(x(\rho), \xi_2^-(\rho)\right) \\ &\qquad\qquad\qquad\qquad = f_\varepsilon\left(x(\rho), \xi_2^+(\rho)\right) - f_\varepsilon\left(x(\rho), \xi_2^-(\rho)\right) \end{aligned}$$

where $f_\varepsilon(x, \xi_2) := \xi_2 b_\varepsilon(x, \xi_2)$. Hence we can write the integrand in the right hand side of (2.1) as

$$\begin{aligned} A_\varepsilon(\rho) &= \frac{f_\varepsilon\left(x(\rho), \xi_2^+(\rho)\right) - f_\varepsilon\left(x(\rho), \xi_2^-(\rho)\right)}{\xi^+(\rho) - \xi^-(\rho)} \\ &= \frac{\xi_2^+(\rho) - \xi_2^-(\rho)}{\xi^+(\rho) - \xi^-(\rho)} \cdot \frac{1}{\xi_2^+(\rho) - \xi_2^-(\rho)} \int_{\xi_2^-(\rho)}^{\xi_2^+(\rho)} \partial_{\xi_2} f_\varepsilon(x(\rho), y)\, dy \\ &= \frac{\xi_2^+(\rho) - \xi_2^-(\rho)}{\xi^+(\rho) - \xi^-(\rho)} \int_0^1 \partial_{\xi_2} f_\varepsilon\left(x(\rho), t\xi_2^+(\rho) + (1-t)\,\xi_2^-(\rho)\right) dt \,. \end{aligned} \tag{3.2}$$

To take the limit notice first that

$$\begin{aligned} |\partial_{\xi_2} f_\varepsilon(x_1, x_2, \xi_2)| &= \left| b_\varepsilon(x_1, x_2, \xi_2) + 2\frac{\xi_2^2}{\varepsilon}(\partial_3 b)\left(x_1, \frac{x_2}{\varepsilon}, \frac{\xi_2^2}{\varepsilon}\right)\right| \\ &\leqslant \sup_y \left| b\left(x_1, \frac{x_2}{\varepsilon}, y\right)\right| + 2\sup_y \left| y\partial_3 b\left(x_1, \frac{x_2}{\varepsilon}, y\right)\right| \end{aligned} \tag{3.3}$$

and

$$\frac{\left|\xi_2^+(\rho) - \xi_2^-(\rho)\right|}{\xi^+(\rho) - \xi^-(\rho)} \leqslant 1 \,. \tag{3.4}$$

As $x_2(\rho) = 0$ when $\rho \in \pi^{-1}(C \setminus S_1)$ and b is with compact support we can infer that $\lim_{\varepsilon\to 0} A_\varepsilon(\rho) = 0$ in this case. For $\rho \in \pi^{-1}(S_0)$ notice that we have $\xi_2^+(\rho) = -\xi_2^-(\rho) = \sqrt{1-|\xi(\rho)|^2}$, and since $b_\varepsilon(x,\xi_2) = b_\varepsilon(x,-\xi_2)$ we get

$$A_\varepsilon(\rho) = \frac{\xi_2^+(\rho)\, b_\varepsilon\left(x(\rho), \xi_2^+(\rho)\right) - \xi_2^-(\rho)\, b_\varepsilon\left(x(\rho), \xi_2^-(\rho)\right)}{\xi_2^+(\rho) - \xi_2^-(\rho)}$$
$$= b_\varepsilon\left(x_1(\rho), 0, \sqrt{1-|\xi(\rho)|^2}\right) = b\left(x_1(\rho), 0, \frac{\sqrt{1-|\xi(\rho)|^2}}{\varepsilon}\right).$$

We conclude that

$$A_\varepsilon(\rho) \xrightarrow[\varepsilon\to 0]{} \begin{cases} b(x_1(\rho), 0, 0), & \text{if } \pi(\rho) \in S_0 \text{ and } |\xi(\rho)| = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

We verify that A_ε is bounded independently of ε. Indeed, from the integral expression (3.2) and the bounds (3.3) and (3.4) it follows that

$$|A_\varepsilon(\rho)| \leqslant \sup|b| + 2 \sup_{x_1,x_2,y} |y\partial_3 b(x_1,x_2,y)| .$$

From (3.5) we have by the Lebesgue Dominated Convergence Theorem the convergence of the integral

$$\int_{H \setminus G} A_\varepsilon(\rho)\, d\nu(\rho) \xrightarrow[\varepsilon\to 0]{} \int_{G\cap\pi^{-1}(S_0)} b(x_1(\rho), 0, 0)\, d\nu(\rho) . \tag{3.6}$$

Comparing (3.1) and (3.6) we learn that

$$0 = \int_{G\cap\pi^{-1}(S_0)} b(x_1(\rho), 0, 0)\, d\nu(\rho) .$$

Since $b(x_1(\rho), 0, 0) \geq 0$ and $b(x_1(\rho_0), 0, 0) > 0$ we have that $\rho_0 \notin \operatorname{supp}(\nu)$. Because $\rho_0 \in G \cap \pi^{-1}(S_0)$ is arbitrary, we see that $\nu\left(G \cap \pi^{-1}(S_0)\right) = 0$. Similarly $\nu\left(G \cap \pi^{-1}(S_1)\right) = 0$.

3.1. An argument under a vanishing assumption on the curved part

In [4] the zero mass of non-gliding points theorem from [2] is applied in a case where it is known that the boundary semiclassical measure ν on $T^*\partial M$ vanishes on the portion lying above the curved part of ∂M, $\pi^{-1}(C)$. In this circumstance one does not need the full power of Theorem 3.1 and we bring here an ad hoc approach which was explained to us by Gérard.

Since ν vanishes on $\pi^{-1}(C)$, it follows from Gérard–Leichtnam's equation that if ν corresponds to an M-semiclassical measure μ, then μ is supported on the portion of $T\ \mathrm{R}^2$ lying above the closed rectangular part of the billiard table. As a result, one may replace the singular billiard table by an infinite strip, and then apply the analysis in [2] on a smooth domain in order to conclude that ν vanishes on $G \cap \pi^{-1}(S)$.

Appendix. Non-Gliding points in the case of a two dimensional billiard with C^2-boundary

We recall the definition of a special subset $G_{\mathrm{ng}} \quad G \quad T\ \partial M$ which is known as the set of non-gliding points. For a two dimensional C^2-billiard M, one can first positively orient ∂M, and then define

Definition A.1. — *Non-gliding points for $M \quad \mathrm{R}^2$ with billiard dynamics.*

$$G_{\mathrm{ng}} = \{(x,\xi) \quad G \mid \text{The curvature of } \partial M \text{ at } x \text{ is nonpositive}\}$$

Remark A.2. — In [2] these points are called non-strictly-gliding, a term which we avoid due to grammatical ambiguity.

More generally [5], the notion of non-gliding points is defined in the context of a Hamiltonian dynamical system on a manifold with boundary M, and in any case it requires that the normal vector to ∂M be C^1, or, equivalently, that the boundary be C^2. The set of non-gliding points is the union of the set of diffractive points and the set of high order glancing points as defined in [5]. In the case of two dimensional billiards the definition simplifies to the one above.

BIBLIOGRAPHY

[1] K. G. Andersson & R. B. Melrose, "The propagation of singularities along gliding rays", *Invent. Math.* **41** (1977), no. 3, p. 197-232.

[2] N. Burq & P. Gérard, "Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes", *C. R. Math. Acad. Sci. Paris* **325** (1997), no. 7, p. 749-752.

[3] P. Gérard & É. Leichtnam, "Ergodic properties of eigenfunctions for the Dirichlet problem", *Duke Math. J.* **71** (1993), no. 2, p. 559-607.

[4] A. Hassell, "Ergodic billiards that are not quantum unique ergodic", *Ann. Math.* **171** (2010), no. 1, p. 605-618, with an appendix by the author and Luc Hillairet.

[5] R. B. Melrose, "Microlocal parametrices for diffractive boundary value problems", *Duke Math. J.* **42** (1975), no. 4, p. 605-635.

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