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A NEW CLASS OF SOLUTIONS TO THE VAN DANTZIG PROBLEM, THE LEE-YANG PROPERTY, AND THE RIEMANN HYPOTHESIS

by Takis KONSTANTOPOULOS, Pierre PATIE & Rohan SARKAR (*)

ABSTRACT. — The purpose of this paper to carry out an in-depth analysis of the intriguing van Dantzig problem. We start by observing that the celebrated Lee–Yang property and the Riemann hypothesis can be both rephrased in terms of this problem, and, more specifically, in terms of functions in the Laguerre–Pólya class. Motivated by these facts, we proceed by identifying several non-trivial closure properties enjoyed by the set of solutions to this problem. Not only does this revisit but also, by means of probabilistic techniques, deepens the fascinating and intensive studies of functions in the Laguerre–Pólya class. We continue by providing a new class of entire functions that are solutions to the van Dantzig problem. We also characterize the pair of the corresponding van Dantzig random variables. Finally, we investigate the possibility that the Riemann ξ function belongs to this class.

RÉSUMÉ. — Dans cet article, nous effectuons une analyse approfondie du curieux problème de van Dantzig. Nous commençons par observer que la propriété célèbre de Lee—Yang et l'hypothèse de Riemann peuvent être reformulées en termes de ce problème, et, plus spécifiquement, en termes de fonctions dans la classe de Laguerre—Pólya. Motivés par ces faits, nous poursuivons en identifiant plusieurs propriétés non triviales satisfaites par l'ensemble des solutions à ce problème. Cela nous permet non seulement de revisiter mais aussi, au moyen de techniques probabilistes, d'approfondir les nombreuses études fascinantes développées pour les fonctions dans la classe de Laguerre—Pólya. Nous continuons en caractérisant une nouvelle classe de fonctions entières qui sont solutions du problème de van Dantzig. Nous donnons également la paire de variables aléatoires de van Danzig correspondante. Enfin, nous étudions la possibilité que la fonction ξ de Riemann appartienne à cette classe.

Keywords: van Dantzig problem, Characteristic functions, Laguerre–Pólya entire functions, Generalized hypergeometric functions, Lee–Yang property, Riemann hypothesis, Self-similar Markov processes.

²⁰²⁰ Mathematics Subject Classification: 42A38, 33C47, 47D07, 30C15.

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1. Introduction

In his seminal paper [34], E. Lukacs provided comprehensive and fascinating insights towards the characterization of the class of functions

$$(1.1) \mathbb{D} = \{ \mathcal{F} \in \mathbb{P}_+; \ V\mathcal{F} \in \mathbb{P}_+ \}$$

where

$$V\mathcal{F}(t) = \frac{1}{\mathcal{F}(it)}, t \in \mathbb{R},$$

and \mathbb{P}_+ stands for the set of continuous bounded positive-definite functions, normalized to be 1 at the origin. Note that by Bochner's theorem, \mathbb{P}_+ is the set of characteristic functions of real-valued random variables, see e.g. [54]. This question was posed, as a prize-winning problem, by David van Dantzig, a Dutch algebraic topologist, in Nieuw Archief voor Wiskunde.

If $\mathcal{F} \in \mathbb{D}$ then the function $V\mathcal{F}$ is called the (van Dantzig) reciprocal of \mathcal{F} , and also belongs to \mathbb{D} . This last fact entails, see Proposition 3.1, that any $\mathcal{F} \in \mathbb{D}$ admits an analytic extension on a cross section of the complex plane including the imaginary line in its interior. We also let

$$(1.2) \qquad \mathbb{D}_2 = \{ [\mathcal{F}, V\mathcal{F}] \in \mathbb{P}_+ \times \mathbb{P}_+; \ \mathcal{F}(it) \cdot V\mathcal{F}(t) = 1 \text{ for all } t \in \mathbb{R} \}.$$

We say that \mathcal{F} is a van Dantzig function and that $[\mathcal{F}, V\mathcal{F}]$ is a van Dantzig pair. Note that the mapping $V : \mathbb{D} \to \mathbb{D}$ is a multiplicative involution, that is, $V\mathcal{F}_1\mathcal{F}_2 = V\mathcal{F}_1V\mathcal{F}_2$ and $V \circ V\mathcal{F} = \mathcal{F}$, see Proposition 3.2 below.

The first historical instances of non-trivial van Dantzig pairs are

$$\left[\cos t, \frac{1}{\cosh t}\right], \left[\frac{\sin t}{t}, \frac{t}{\sinh t}\right] \text{ and } \left[e^{-\frac{t^2}{2}}, e^{-\frac{t^2}{2}}\right].$$

We notice that, in the last example,

$$\mathcal{F}_{N}(t) = e^{-\frac{t^2}{2}}$$

is the characteristic function of a standard normal random variable for which it is immediate that $V\mathcal{F}_{N}(t) = 1/\mathcal{F}_{N}(it) = \mathcal{F}_{N}(t)$ and hence $\mathcal{F}_{N} \in \mathbb{D}_{S}$, where

$$\mathbb{D}_S = \{ \mathcal{F} \in \mathbb{P}_+; \ \mathcal{F} = V \mathcal{F} \}$$

is the invariant set of V or self-reciprocal elements of \mathbb{D} . It is easy to see that, for any $\mathcal{F} \in \mathbb{D}$, the mapping $t \mapsto \mathcal{F}(t)V\mathcal{F}(t)$ is in \mathbb{D}_S , meaning that the set \mathbb{D}_S contains many more elements than \mathcal{F}_N . However, the following fact, due to Lukacs, offers an original characterization of the characteristic function \mathcal{F}_N

$$\{[\mathcal{F}_1, \mathcal{F}_2] \in \mathbb{D}_2; -\log \mathcal{F}_j \in \mathbb{N}(\mathbb{R}), j = 1, 2\} = \{[\mathcal{F}_N, \mathcal{F}_N]\}$$

where $\mathbb{N}(\mathbb{R})$ is the set of negative definite functions on \mathbb{R} , see [54, Chapter 4] for more information on this set. In other words, if \mathcal{F} and $V\mathcal{F}$ are infinitely divisible then they are identical and both equal to \mathcal{F}_{N} .

Regarding the first example in (1.3), it is immediate that the mapping $t \mapsto \cos t$ is the characteristic function of a random variable taking values ± 1 with equal probability, whereas $t \mapsto (\cosh t)^{-1}$ is the characteristic function of a random variable with density $(2\cosh(\pi x/2))^{-1}$, $x \in \mathbb{R}$. For the second example, the mapping $t \mapsto \sin t/t$ corresponds to the characteristic function of a uniformly distributed random variable on the interval [-1,1] and $t \mapsto t(\sinh t)^{-1}$ to the one of an absolutely continuous probability measure whose density is $\frac{\pi}{4}(1-\tanh(\pi x/2))^2$, $x \in \mathbb{R}$.

Another classical example, discussed at the end of Lukacs's paper, and presented in a more general form in [18, 21], is expressed in terms of the entire functions

(1.4)
$$\mathcal{J}_{\nu}(t) = \Gamma(\nu+1)t^{-\nu}J_{\nu}(t), \quad \mathcal{I}_{\nu}(t) = \mathcal{J}_{\nu}(it),$$

where J_{ν} is the Bessel function of the first kind of index ν , and given by the pair

(1.5)
$$\left[\mathcal{J}_{\nu}, \frac{1}{\mathcal{I}_{\nu}}\right] \in \mathbb{D}_{2}, \quad \nu > -\frac{1}{2}.$$

The corresponding pair of random variables are described in details in [21], see also (5.24) and Lemma 5.7. This set of examples turns out to be the only canonical solutions to the van Dantzig problem available in the literature. Note that they all belong to the subclass $\mathbb{D}_L \subset \mathbb{D}$, originally identified by Lukacs [34], which stands for the set of even entire characteristic functions in the Laguerre–Pólya class. That is, entire functions which are locally the limit of a series of polynomials whose roots are all real, see Section 2 for further discussion on this class. Throughout, we shall provide several ways of generating new instances in \mathbb{D}_L and also present new subclasses of \mathbb{D} .

In this paper, we start by identifying a connection between two fundamental problems in mathematics and the subclass $\mathbb{D}_L \subseteq \mathbb{D}$. First, we explain how the Riemann hypothesis can be equivalently formulated by the membership of the Riemann ξ function to the class \mathbb{D}_L , something which, in a different context, was observed by Roynette and Yor [51]. Similarly, the celebrated Lee–Yang property in statistical physics, discovered first by Lee and Yang in connection to the Ising model on a finite lattice, is also equivalent to the requirement that the partition function, viewed as

a function on the imaginary line, belongs to \mathbb{D}_L , see [20] for a thorough account on this topic. This connection relies on the theory of Pólya frequency functions developed by Schoenberg [55], see Section 2.

In view of its importance, we first aim to develop an in-depth analysis of the subclass \mathbb{D}_L . On the one hand, we adapt several substantial results that one can find in the number theory and statistical mechanics literature to provide new information about the class \mathbb{D}_L . Conversely, by means of probabilistic techniques combined with the theory of entire functions, we also provide original closure properties of the class \mathbb{D}_L which give new insights to the two aforementioned problems. For instance, we find necessary conditions for the product of two independent random variables to belong or to remain in \mathbb{D}_L , see Theorem 3.13. In the same vein, in Theorem 3.14, we revisit, improve and extend to the class \mathbb{D} , a recent result due to Newman and Wu [37] regarding the closure property of this class under locally uniform convergence.

Another objective of this paper is to identify a new subclass, denoted by \mathbb{D}_P , of analytic characteristic functions that belong to the class \mathbb{D} , that is, are solutions to the van Dantzig problem. More specifically, to each Laplace exponent Ψ of a (possibly killed) spectrally negative Lévy process which is non-negative on the interval $[1/2, \infty)$, we associate the function $\mathcal{J}_{\Psi} \in \mathbb{D}_P$ which is defined by

$$\mathcal{J}_{\Psi}(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{\Psi(1) \cdots \Psi(n)} t^{2n}.$$

Since $\lim_{u\to\infty} \Psi(u) = \infty$, we deduce easily that \mathcal{J}_{Ψ} defines an entire function. We refer to Sections 4 and 5.2.2 for more information about the objects introduced here and this class of functions, whose analysis is intimately related to the Wiener-Hopf factorization of the Laplace exponent Ψ . We also point out that the entire function $\mathcal{I}_{\Psi}(t) = \mathcal{J}_{\Psi}(i\sqrt{t})$ was introduced by the second author in [40], and has appeared in various mathematical contexts recently, in probability theory [41], in the spectral theory of some non-self-adjoint operators [42, 45], in the study of special functions [59] and a chapter is devoted to this class of function in the monograph [27]. Note that when $\Psi(n) = n(\nu + n)$ is the Laplace exponent of a (scaled) Brownian motion with drift $\nu \geqslant -\frac{1}{2}$, then \mathcal{J}_{Ψ} boils down to the Bessel-Clifford function \mathcal{J}_{ν} as defined in (1.4). The class \mathbb{D}_{P} includes a wide range of other well-known special functions such as several hypergeometric functions, the Mittag-Leffler functions, and the Wright functions, among others, see Section 4.2.

Theorem 4.1 states that $\mathcal{J}_{\Psi} \in \mathbb{D}$. Its proof relies on identifying the two random variables whose characteristic functions are \mathcal{J}_{Ψ} and $1/\mathcal{I}_{\Psi}, \mathcal{I}_{\Psi}(t) =$ $\mathcal{J}_{\Psi}(it)$, see Lemmas 5.8 and 5.7, respectively. This is achieved by introducing a Markov operator, see (5.19), that serves on the one hand to show that \mathcal{J}_{Ψ} is the characteristic function of the product of two independent random variables (one having as characteristic function the Bessel function \mathcal{J}_0). On the other hand, it also turns out to be an intertwining operator between two Markov semigroups, see e.g. [45] for a review of this concept. From this fact, we deduce first that $t \mapsto \mathcal{I}_{\Psi}(\sqrt{t})$ is an invariant function for one of these semigroups and then its reciprocal is the Laplace transform of a positive random variable associated to the Markov process, from where we conclude by invoking an argument involving the Bochner subordination of a Brownian motion. Moreover, since $\mathcal{J}_{\nu} \in \mathbb{D}_{L} \cap \mathbb{D}_{P}$, it is natural to wonder whether $\mathbb{D}_P \subset \mathbb{D}_L$. This is a very delicate question as it is difficult, in general, to identify the location of zeros of a power series. However, we manage to provide necessary conditions on Ψ for $\mathcal{J}_{\Psi} \in \mathbb{D}_{L}$, that is the entire function has only real zeros. We also identify instances of entire functions in \mathbb{D}_P which have non real zeros, revealing that $\mathbb{D}_L \subsetneq \mathbb{D}$, see Theorem 4.4.

The remaining part of the paper is organized as follows. In Section 2, we introduce the Lukacs class \mathbb{D}_L and present its connection with the Riemann hypothesis and the Lee–Yang property. Section 3 is devoted to a thorough analysis of the sets \mathbb{D} and \mathbb{D}_L , including some closure properties of \mathbb{D}_L under various mappings, and to the identification of original ways to generate new elements in \mathbb{D}_L . In Section 4 we introduce the new subclass \mathbb{D}_P and provide some interesting properties. Finally, in Section 5, we collect the proofs of the results presented in the two previous sections.

2. The Lukacs class \mathbb{D}_L

We start by recalling that an entire function is in the Laguerre–Pólya class \mathbb{LP} if it is the local uniform limit of a sequence of polynomials with real coefficients and real zeros only. In fact, based on ideas of Laguerre, see [17, pp. 167–177] and [6, Chapter 2], Pólya and Schur [48] showed that φ is in \mathbb{LP} if and only if

(2.1)
$$\varphi(z) = Kz^{m}e^{-c^{2}z^{2} + az} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_{k}}\right) e^{z/z_{k}}, \ z \in \mathbb{C},$$

for some $m \in \mathbb{Z}_+$, $K, c, a \in \mathbb{R}$, $z_k \in \mathbb{R} \setminus \{0\}$ such that $\sum 1/z_k^2 < \infty$, and, where z_k , $k \in \mathbb{N}$, are the nonzero zeros of the entire function F, arranged in order of nondecreasing modulus.

Lukacs was interested in even characteristic functions in the class \mathbb{LP} and made the observation that such functions are automatically in \mathbb{D} , something that we explain in the sequel. Let now

(2.2)
$$\mathbb{LP}_e = \{ \varphi \in \mathbb{LP}; \ \varphi \text{ is even and } \varphi(0) = 1 \}.$$

The representation (2.1) immediately gives that $\varphi \in \mathbb{LP}_e$ must be of the form

(2.3)
$$\varphi(z) = e^{-c^2 z^2} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{z_k^2} \right), \quad c \in \mathbb{R}, \quad z_k > 0, \quad \sum_k 1/z_k^2 < \infty.$$

Indeed, the even functions in \mathbb{LP} are the functions of the form (2.1) with m=0, K=1 and zeros that are symmetrically placed around 0 on the real axis, resulting in precisely the form (2.3). Next, we mention that the order ρ and the exponent of convergence ϱ of an entire function φ in the Laguerre-Pólya class are such that

$$0 \le \rho \le \rho \le 2$$

where we recall that

$$\rho = \overline{\lim_{r \to \infty}} \frac{\log \log \max_{|z| = r} |\varphi(z)|}{\log r} \text{ and } \varrho = \inf \left\{ \alpha > 0; \sum_{k \geqslant 1} |z_k|^{-\alpha} < \infty \right\}.$$

Here and throughout, we refer to the monograph of Levin [31] for information related to entire functions.

The functions in the examples (1.3) and (1.5) are all of the form (2.3). However, we emphasize that if an entire function φ is of the form (2.3) then it needs not be the case that the mapping $t \mapsto \varphi(t)$ is positive-definite on \mathbb{R} . Nevertheless, Schoenberg, following Hadamard, proved that the reciprocal of a normalized Laguerre–Pólya entire function (non-necessarily even) is in \mathbb{P}_+ . In particular, we have the following.

Theorem 2.1 ([55, Theorem 1]). — If φ is of the form (2.3) then

(2.4)
$$t \mapsto \frac{1}{\varphi(it)} = e^{-ct^2} \prod_{k=1}^{\infty} \left(1 + \frac{t^2}{z_k^2} \right)^{-1}$$

is the characteristic function of a symmetric Pólya frequency density f_P , namely,

$$\mathcal{F}_{f_P}(t) = \frac{1}{\varphi(it)} = \int_{\mathbb{R}} e^{itx} f_P(x) dx$$

where f_P is a symmetric probability density function on \mathbb{R} such that, for all $n \in \mathbb{N}$, $x_1 < \cdots < x_n$, $y_1 < \cdots < y_n$, the determinant of the matrix $[f_P(x_j - y_k)]_{j,k=1}^n$ is non-negative.

Moreover, the probability measure with density f_P is infinitely divisible, that is, equivalently, $t \mapsto -\log \mathcal{F}_P(it) \in \mathbb{N}(\mathbb{R})$, see Kwaśnicki [26, Proposition 5.3].

We point out that an easy way to see why (2.4) is a characteristic function is by a probabilistic argument. Indeed, let N, Z_1, Z_2, \ldots be independent random variables where N is a standard normal and each Z_j a standard Laplace random variable, that is, it has density $\frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$. Then, one observes that the random variable

(2.5)
$$X = c\sqrt{2}N + \sum_{j=1}^{\infty} \frac{Z_j}{z_j}$$

has characteristic function (2.4). Then, relying on Theorem 2.1, Lukacs observed the very useful fact that if an even characteristic function \mathcal{F} is in the Laguerre–Pólya class then it is necessarily solution to the van Dantzig problem. This leads us to introduce the following class.

Definition 2.2. — Let

$$\mathbb{D}_L = \mathbb{L}\mathbb{P}_e \cap \mathbb{P}_+ \subset \mathbb{D}$$

be the Lukacs class of solutions to the van Dantzig problem.

The examples in (1.3) and (1.5) are not just in \mathbb{D} but also in \mathbb{D}_L . Further elements of \mathbb{D}_L can be generated by means of the mappings described in Theorem 3.5 below, and, also from the subclass of \mathbb{D} that we introduce and study in Section 4, see Theorem 4.4. We already mention that in Section 3 we provide some (partial) characterizations of the class \mathbb{D}_L .

The first appearance of this class of Laguerre–Pólya characteristic functions traces back to Pólya [48] who, motivated by Riemann hypothesis, was interested in characterizing all (complex valued) functions f on \mathbb{R} such that the analytic extension of

(2.6)
$$t \mapsto \int_{\mathbb{R}} e^{itx} f(x) dx$$
 is an entire function with only real zeros.

We refer to the recent paper by Newman and Wu [38] for an excellent account on Pólya's approach and to de Bruijn's fascinating contributions to this problem. Theorems 3.10, 3.11 and 3.12 below form the adaptation of these results in the context of the van Dantzig problem. From that time onwards, the Laguerre-Pólya class of characteristic functions has become ubiquitous and plays a central role in various fields of mathematics. In what follows, we describe the connection between the class \mathbb{D}_L and both the Riemann hypothesis, and, the Lee-Yang property that appears in some statistical mechanics models and in Euclidean quantum field theory.

2.1. The Lukacs class and the Riemann hypothesis

There is a fascinating literature describing the role played by the Riemann ζ function in probability theory, see the excellent papers [3, 4] and the references therein. In the spirit of the work of Roynette and Yor [51, Théorème V.3.2], we now explain how the Riemann hypothesis can be formulated in terms of the van Dantzig problem. Consider the Riemann ζ function

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}, \Re z > 1,$$

which, when extended meromorphically to the whole complex plane, has a single simple pole at 1 with residue 1 and the following zeros: the trivial ones located at the negative even integers and the nontrivial ones lying in the critical strip $0 < \Re z < 1$. The Riemann hypothesis states that all nontrivial zeros are located on the critical line $\Re z = 1/2$. It is well-known that the function

$$\eta(z) = \frac{1}{2}z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z), \quad z \in \mathbb{C},$$

is an entire function (because 1-z cancels the pole of ζ , whereas the trivial zeros of ζ cancel the poles of the gamma function $\Gamma(z/2)$, located at the same places). Hence the zeros of η are the nontrivial zeros of ζ . Moreover, it satisfies

$$\eta(z) = \eta(1-z), \quad z \in \mathbb{C}.$$

Performing an affine transformation on \mathbb{C} so that the critical line maps onto the real line, we get the Landau function

$$\xi(z) = \eta(\frac{1}{2} + iz), \quad z \in \mathbb{C}.$$

The functional equation above then becomes

$$\xi(z) = \xi(-z), \quad z \in \mathbb{C}.$$

Using the standard integral expression for the gamma function and the definition of the ζ function one can show that

(2.7)
$$\xi(t) = \int_{-\infty}^{\infty} e^{itx} \Phi(x) \, \mathrm{d}x, \ t \in \mathbb{R},$$

where

$$\Phi(x) = \sum_{n=1}^{\infty} \left(4\pi^2 n^4 e^{9x/2} - 6\pi n^2 e^{5x/2} \right) \, e^{-\pi n^2 e^{2x}}.$$

It is plain that $\Phi(x) > 0$ for all $x \ge 0$. Since ξ is even, one gets that $\Phi(x) > 0$ for all $x \in \mathbb{R}$. Also, Φ is integrable as $\int_{-\infty}^{\infty} \Phi(x) dx = \xi(0) < \infty$, we have

that the function $\Phi/\xi(0)$ is the density of a symmetric real-valued random variable and $\xi/\xi(0)$ is its characteristic function. We have the following.

THEOREM 2.3. — The function $\xi/\xi(0)$ is in \mathbb{D}_L if and only if the Riemann hypothesis holds.

Proof. — If the function $\xi/\xi(0)$ is in \mathbb{D}_L then it has only real zeros which means that η has all its zeros on the critical line. Hence the Riemann hypothesis holds. If the Riemann hypothesis holds then $\xi/\xi(0)$ is an even entire characteristic function with real zeros only. Using the result of Proposition 3.3 below, we have that $\xi/\xi(0)$ is in \mathbb{D}_L .

The previous result combined with Theorem 2.1 yields this reformulation.

COROLLARY 2.4. — The function $t \mapsto \xi(0)/\xi(it)$ is the characteristic function of a symmetric Pólya frequency function if and only if the Riemann hypothesis holds.

Further connections between the Riemann ξ and the van Dantzig problem will be discussed in Section 4.1.

2.2. The Lukacs class and the Lee-Yang property

Entire characteristic functions with only real zeros appear naturally in various models of statistical mechanics and quantum field theory. We briefly state the examples taken from the excellent paper [38]. In their pioneering works, Lee and Yang [29, 30] considered the Ising model in the presence of an external magnetic field and discovered that the zeros, in the magnetic field variable, of the partition function are purely imaginary. This is equivalent to the following. Let μ denote the probability measure on $\{-1,1\}^N, N \in \mathbb{N}$, defined by

$$\mu(x) = Ke^{\sum_{j,k=1}^{N} J_{j,k}x_{j}x_{k}}, \quad x = (x_{1}, \dots, x_{N}),$$

where, here and below, $J_{j,k} \ge 0$ and $J_{j,k} = J_{k,j}$ for all j,k and K is a positive normalizing constant. Then the partition function

$$\mathcal{P}_{\boldsymbol{\lambda},\mu}(z) = \sum_{\boldsymbol{x} \in \{-1,1\}^N} e^{z \boldsymbol{\lambda} \cdot \boldsymbol{x}} \boldsymbol{\mu}(\boldsymbol{x}), \quad \boldsymbol{\lambda} = (\lambda_1, \dots \lambda_N) \in \mathbb{R}_+^N,$$

is an entire function whose zeros are all purely imaginary. This was discovered by Lee and Yang and refer to it as the Lee-Yang property. We emphasize that the location and distribution of the zeros of the partition function are useful to determine substantial properties of the underlying

physical system such as phase transitions, the infinite volume limit and existence of a mass gap under an external magnetic field. In relation to the van Dantzig problem, one observes that the mapping $t \mapsto \mathcal{P}_{\lambda,\mu}(it)$ is the characteristic function of the random variable

$$\lambda \cdot x = \sum_{j=1}^{N} \lambda_j x_j$$

under the measure μ . By Theorem 2.1, the Lee–Yang property is then equivalent to the statement that this characteristic function is in \mathbb{D}_L . Generalizing the Lee–Yang result, Simon and Griffiths [56] showed that if μ_0 is a symmetric probability measure⁽¹⁾ on \mathbb{R} such that

$$\int_{\mathbb{R}} e^{bx^2} \mu_0(\mathrm{d}x) < \infty \text{ for all } b \in \mathbb{R} \text{ and } \mathcal{F}_{\mu_0}(z) \neq 0 \text{ for all } \Im(z) < 0$$

then, to the probability measure

$$\boldsymbol{\mu}_{\beta}(\mathrm{d}\boldsymbol{x}) = Ke^{\beta \sum_{j,k} J_{j,k} x_j x_k} \prod_{k=1}^{N} \mu_0(\mathrm{d}x_k) \quad \boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N,$$

there corresponds the partition function

$$\mathcal{P}_{\boldsymbol{\lambda},\boldsymbol{\mu}_{\beta}}(z) = \int_{\mathbb{R}^{N}} e^{z\boldsymbol{\lambda}\cdot\boldsymbol{x}} \boldsymbol{\mu}_{\beta}(\mathrm{d}\boldsymbol{x}), \quad \boldsymbol{\lambda} = (\lambda_{1},\ldots,\lambda_{N}) \in \mathbb{R}_{+}^{N},$$

with the property that, for all $\beta \geq 0$, the zeros of the entire function $z \mapsto \mathcal{P}_{\lambda,\mu_{\beta}}(z)$ are all purely imaginary. Hence, the mapping $t \mapsto \mathcal{P}_{\lambda,\mu_{\beta}}(it)$ is in \mathbb{D}_L . We also point out that these Lee–Yang type theorems arise in quantum field theory, and refer again to [38] for a more detailed description. We simply point out that an important measure in this context is the measure $\mu_0(\mathrm{d}x) = e^{-ax^4 - bx^2} \mathrm{d}x$, where a > 0 and $x, b \in \mathbb{R}$. From Theorem 2.1, we deduce the following.

THEOREM 2.5. — A partition function \mathcal{P} has the Lee–Yang property if and only if $t \mapsto \mathcal{P}(it) \in \mathbb{D}_L$.

3. Properties of the van Dantzig and Lukacs classes

We collect here several properties of both the classes \mathbb{D} and \mathbb{D}_L . We start by presenting some basic properties that both classes share. Some of them were stated in Lukacs' paper [34] without proof; for sake of conciseness, we indicate the main lines of proof. Then, we move to the properties to the

⁽¹⁾ In fact, a signed measure is also allowed in [56].

Lukacs class \mathbb{D}_L which are of different types. Some results can be found in the number theory or statistical mechanics literature in some form that we adapt, revisit or extend to identify new properties for the set \mathbb{D}_L . Moreover, based on ideas coming from probability theory, we also present original and substantial results about this set, see Theorem 3.13, and, we improve, in Theorem 3.14, a very interesting closure property due to Newman and Wu. Let us start with the following simple but useful result which is a reformulation of [32, Theorems A.2.1 and A.2.2, p. 335].

PROPOSITION 3.1. — If $\mathcal{F} \in \mathbb{D}$ then \mathcal{F} is a real and even function, meaning that the associated random variables are symmetric. Moreover, \mathcal{F} admits an analytic extension to some cross $\{z \in \mathbb{C}; |\Im(z)| < z_1\}$, $\{z \in \mathbb{C}; |\Re(z)| < z_2\}$ where $z_1 > 0, z_2 > 0$. The same claims hold for $V\mathcal{F}$.

We also point out that since \mathcal{F} is even, a theorem of Schoenberg on positive-definite radial functions entails that the mapping $t \mapsto \mathcal{F}(\sqrt{t})$ is completely monotone on \mathbb{R}^+ , that is, it is the Laplace transform of a nonnegative Radon measure on $[0, \infty)$, see e.g. [10].

We proceed with the following the simple observation which follows readily from the previous Proposition since $V \circ V\mathcal{F}(t) = 1/V\mathcal{F}(it) = \mathcal{F}(-t) = \mathcal{F}(t)$.

PROPOSITION 3.2. — The mapping V defined by (1.2) is an involution on \mathbb{D} .

In other words, the set \mathbb{D}_2 is closed under commutation, that is, if $[\mathcal{F}, V\mathcal{F}] \in \mathbb{D}_2$ then $[V\mathcal{F}, \mathcal{F}] \in \mathbb{D}_2$.

Next, from the Definition 2.2 of \mathbb{D}_L , a first natural question is to understand whether the Lukacs class \mathbb{D}_L contains all possible entire characteristic functions with only real zeros, that is, whether there exists such an entire function of order $\rho > 2$. Here is the definitive answer to this issue, which is, in fact, a direct consequence of a very nice result due to Gol'dberg and Ostrovs'ki [15], see also [35, Theorem 4.4.1], regarding entire characteristic functions having only real zeros. Note that, in these references, the statement is proven for a more general class of entire functions, namely the ones possessing the so-called ridge property, that is, $|\varphi(z)| \leq |\varphi(\Im(z))|$.

PROPOSITION 3.3. — Every entire characteristic function with only real zeros belongs to the class \mathbb{D}_L .

We proceed by deriving some closure properties of the sets \mathbb{D} and \mathbb{D}_L . First, since multiplication of characteristic functions remain characteristic function as they correspond to addition of independent random variables, we get that \mathbb{D} is stable by multiplication and having the constant function 1 as identity element, we obtain that it is a monoid. Similarly, since any reciprocal of a function in \mathbb{D}_L being the moment generating function, at least on an imaginary strip, of a Pólya frequency function, it remains to identify transforms that preserve the positive definiteness property of a Laguerre-Pólya function. It is the program that we develop in the remaining part of this section. In this spirit, there is this first closure property which follows readily since the product of two characteristic functions in \mathbb{LP} , that is of the form (2.3), remain an even entire characteristic function in \mathbb{LP} , see the item (3.4) above for the same property for the set \mathbb{D} .

PROPOSITION 3.4. — The sets \mathbb{D} and \mathbb{D}_L constitute a monoid under the operation of function multiplication.

Note that the previous claim could also be interpreted as the set \mathbb{D}_L being invariant by the convolution of probability distributions or equivalently by taking the sum of independent random variables. The following mappings and results were proposed by Lukacs

(3.1)
$$L^{(p)}f(t) = t^{p-2} \frac{f^{(p)}(t)}{f^{(2)}(0)}, \quad t \in \mathbb{R}, \quad p = 1, 2,$$

where, for a sufficiently smooth function f, we write $f^{(p)}(t) = \frac{d^p}{dt^p} f(t)$.

THEOREM 3.5 ([34, Theorem 4]). — We have, for $p = 1, 2, L^{(p)}(\mathbb{LP}) \subset \mathbb{LP}$ and $L^{(p)}(\mathbb{D}_L) \subset \mathbb{D}_L$.

The proof of this theorem relies on Laguerre's theory, see e.g. Borel [6, Chapter 2], which provides a closure property of the class \mathbb{LP} by differentiation. Then Lukacs showed, by analytical means, that his mappings leave the set \mathbb{P}_+ invariant. In Section 5 we shall provide an alternative proof of this last fact based on probabilistic arguments.

Remark 3.6.

- (i) Since $L^{(p)}$, p=1,2, are differential operators, we easily get that the unique invariant of $L^{(1)}$ (resp. $L^{(2)}$) in \mathbb{P}_+ , i.e. $L^{(1)}\mathcal{F}=\mathcal{F}$, is $\mathcal{F}(t)=e^{-\sigma^2t^2/2}$ (resp. $\mathcal{F}(t)=\cos(\sigma t)$, $\sigma\in\mathbb{R}$).
- (ii) For any finite sequence (p_1, \ldots, p_n) whose elements take values in $\{1, 2\}$ we have that $L^{(p_n)} \circ \cdots \circ L^{(p_1)}(\mathbb{D}_L) \subseteq \mathbb{D}_L$. For example, one observes that

$$L^{(1)}\cos(t) = \frac{\sin t}{t} \ \text{ and } \ L^{(1)}\circ L^{(1)}\cos(t) = 3\frac{\sin t - t\cos t}{t^3}.$$

More generally, define $\mathcal{F}_{n+1}(t) = L^{(1)}\mathcal{F}_n(t), n \geq 0$, with $\mathcal{F}_0(t) = \cos t$. Thus, \mathcal{F}_n is obtained by the *n*-fold application of the operator $L^{(1)}$ to the cosine function. We obtain the expression

$$\mathcal{F}_{n+1}(t) = \frac{-(2n+1)!!}{t^{2n+1}} \Re(\overline{P}_n(-it)ie^{-it})$$

where $(2n+1)!! = (2n+1)(2n-1)(2n-3)\cdots 1$ and \overline{P}_n is the socalled reverse Bessel polynomial, expressed in terms of the degreen Bessel polynomial P_n via $\overline{P}_n(x) = x^n P_n(1/x)$, see [24]. Note that $\mathcal{F}_n \to \mathcal{F}_0$ pointwise as $n \to \infty$ and this is natural from the probabilistic interpretation of the operator $L^{(1)}$, see Lemma 5.1.

We proceed with the following two results that give a characterization or a partial characterization of the set \mathbb{D}_L . First, we have the following Lévy–Khintchine type representation which follows readily from Theorem 2.1 combined with the characterization result due to Kwaśnicki of the Laplace transform of Pólya frequency densities.

THEOREM 3.7 ([26, Proposition 5.3]). — If $\mathcal{F} \in \mathbb{D}_L$ then, for all $t \geq 0$,

(3.2)
$$\mathcal{F}(t) = e^{\Psi_L(t)}$$

where $\Psi_L(t) = -ct^2 - \int_{-\infty}^{\infty} \left(\frac{1}{t+r} - \frac{1}{r} + \frac{t}{r^2}\right) \rho(r) dr, c \ge 0$ and $\rho : \mathbb{R} \to \mathbb{Z}$ is an even non-decreasing integer-valued function such that $\int_{-\infty}^{\infty} \frac{|\rho(r)|}{|r|^3} dr < \infty$.

The next theorem, proved by Newman [36], who was, as discussed in Section 2.2, motivated by problems arising in statistical physics and Euclidean field theory, characterizes a subclass of \mathbb{D}_L . For a random variable X and a real number λ such that $Z_{\lambda} := \int_{-\infty}^{\infty} e^{\lambda x^2} F_{\rm X}(\mathrm{d}x) < \infty$, let ${\rm X}_{\lambda}$ denote a random variable whose distribution $F_{\rm X}_{\lambda}$ is given by

(3.3)
$$F_{X_{\lambda}}(\mathrm{d}x) = \frac{1}{Z_{\lambda}} e^{\lambda x^2} F_{X}(\mathrm{d}x), \ x \in \mathbb{R}.$$

THEOREM 3.8 ([36, Theorem 1]). — Let X be a symmetric random variable. Then, $\mathcal{F}_{X_{\lambda}} \in \mathbb{D}_{L}$ for all $\lambda \in D_{X} = \{\lambda \in \mathbb{R}; \mathcal{F}_{X^{2}}(-i\lambda) < \infty\} \supseteq (-\infty, 0]$ if and only if either, for some x_{0} ,

$$F_{\mathbf{X}} = \frac{1}{2} \left(\delta_{x_0} + \delta_{-x_0} \right),$$

or F_X is absolutely continuous with respect to the Lebesgue measure with a density f_X which takes the form

(3.4)
$$f_{X}(x) = Kx^{2m}e^{-\alpha x^{4} - \beta x^{2}} \prod_{k=1}^{N} \left(\left(1 + \frac{x^{2}}{a_{k}^{2}} \right) e^{-\frac{x^{2}}{a_{k}^{2}}} \right)$$

where K > 0 is a normalizing constant, m a nonnegative integer, α, β real numbers, N a nonnegative integer or ∞ , with the a_k positive, and either $\alpha = 0$, $\sum a_k^{-4} < \infty$, or $\alpha > 0$, $\beta + \sum a_k^{-2} > 0$ (the case $\sum a_k^{-2} = \infty$ is allowed).

Remark 3.9. — This theorem does not fully characterize the set \mathbb{D}_L as, for instance, \mathcal{J}_0 , the Bessel function of order 0, see (1.5), is in \mathbb{D}_L , but it is the characteristic function of the arc-sine law whose density, see (5.24), does not have the form (3.4). Another well-known function that does not belong to Newman characterization is the function

$$t\mapsto \mathcal{F}_{\lambda,\Phi}(t) = \frac{1}{\int_{\mathbb{R}} e^{\lambda x^2} \Phi(x) \, \mathrm{d}x} \int_{\mathbb{R}} e^{itx} e^{\lambda x^2} \Phi(x) \, \mathrm{d}x,$$

introduced by Pólya, where Φ is the inverse Fourier transform of the Landau function ξ ; see (2.7). It is well-known that

$$\mathcal{F}_{\lambda,\Phi} \in \mathbb{D}_L$$
 if and only if $\lambda \geqslant \Lambda_{DN}$

where Λ_{DN} is the celebrated de Bruijn–Newman constant. Since, by Theorem 2.3, the Riemann hypothesis is equivalent to $\Lambda_{DN} \leq 0$, this observation has motivated an intensive research activity on the computation of Λ_{DN} . The current state of art is $0 \leq \Lambda_{DN} \leq 0.22$ and was obtained by Tao and collaborators [50, 57].

The following three claims are adaptation to our setting of some deep results due to Pólya [47] (the first two) and to de Bruijn [8], see also [38, Sections 2.2 and 2.3]. We omit their proofs as they follow readily from the aforementioned results combined with Theorem 2.1.

THEOREM 3.10 ([47]). — Let $f : \mathbb{R} \to \mathbb{R}_+$ be the density of a symmetric random variable such that, for some $A, \alpha > 0$,

$$(3.5) f(x) \leqslant Ae^{-x^{2+\alpha}}, \ x \geqslant 0.$$

Let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be even, real analytic, and such that $\int_{\mathbb{R}} f(x)\varphi(x) dx = 1$. Assume further that $t \mapsto \mathcal{F}_f(t) = \int_{\mathbb{R}} e^{itx} f(x) dx \in \mathbb{D}_L$. Then,

$$t \mapsto \mathcal{F}_{f \cdot \varphi}(t) = \int_{\mathbb{R}} e^{itx} f(x) \varphi(x) \, \mathrm{d}x \in \mathbb{D}_L$$

if and only if the analytic extension of φ is such that $t \mapsto \varphi(it) \in \mathbb{LP}$.

Note that the function φ in the above theorem is called a universal factor by Pólya. An interesting and simple application of this theorem is for the entire function $\varphi(x) = e^{\lambda x^2}$ where $\lambda > 0$. Indeed, using the notation of Theorem 3.8, if X is a symmetric random variable with an absolutely continuous distribution whose density satisfies the bound (3.5) and for some

 $\underline{\lambda} \in \mathbb{R}$, $\mathcal{F}_{X_{\underline{\lambda}}} \in \mathbb{D}_L$, then $\mathcal{F}_{X_{\lambda}} \in \mathbb{D}_L$ for all $\lambda \geqslant \underline{\lambda}$ as plainly here $D_X = \mathbb{R}$. For instance, from the expression (5.24), we deduce that, for any $\nu < \frac{1}{2}$ and $\lambda > 0$, the mapping

(3.6)
$$t \mapsto K_{\lambda} \int_{|x| < 2} e^{itx} e^{\lambda x^2} (4 - x^2)^{-\nu - \frac{1}{2}} dx \in \mathbb{D}_L$$

where $K_{\lambda} > 0$ is a normalizing constant. However, note that since the density above when $\lambda = 0$ is not of the form (3.4), by Theorem 3.8, there exists $\underline{\lambda} < 0$ such that the entire function

(3.7)
$$t \mapsto K_{\underline{\lambda}} \int_{|x| < 2} e^{itx} e^{\underline{\lambda}x^2} (4 - x^2)^{-\nu - \frac{1}{2}} dx$$

has non-real zeros.

THEOREM 3.11 ([8]). — Let $f:[0,\infty)\to\mathbb{R}_+$ be the density of a positive random variable, and, for all $y\geqslant 0$,

$$f(y) \leqslant Be^{-y^{\frac{1}{2}+\beta}}$$

for some $B, \beta > 0$. Suppose that f has an analytic extension in a neighborhood of the origin. Then the function

$$\mathcal{M}_f(t) = \int_0^\infty y^{t-1} f(y) \, \mathrm{d}y$$

has an extension on \mathbb{C} as a meromorphic function. If the function \mathcal{M}_f has only negative zeros and n is a positive integer, then, writing $\frac{1}{C_n} = \int_{\mathbb{R}} f(x^{2n}) dx$,

$$t \mapsto C_n \int_{\mathbb{R}} e^{itx} f(x^{2n}) \, \mathrm{d}x \in \mathbb{D}_L.$$

THEOREM 3.12 ([38]). — Let f be an entire function such that its derivative $f^{(1)}$ is the limit (uniform in any bounded domain) of a sequence of polynomials, all of whose roots lie on the imaginary axis. Suppose further that f is not a constant, f(x) = f(-x), and $f(x) \ge 0$ for $x \in \mathbb{R}$ and $\int_{\mathbb{R}} e^{-f(x)} dx = 1$. Then

$$t \mapsto \int_{\mathbb{R}} e^{itx} e^{-f(x)} \, \mathrm{d}x \in \mathbb{D}_L.$$

As instances illustrating these results, there are the following entire functions that were derived by Pólya

(3.8)
$$t \mapsto K \int_{\mathbb{R}} e^{itx} \cosh(ax) e^{-a \cosh x} dx$$
, $t \mapsto K \int_{\mathbb{R}} e^{itx} e^{-x^{2n}} dx$
and $t \mapsto K \int_{\mathbb{R}} e^{itx} e^{-ax^{4n} + bx^{2n} + cx^{2}} dx \in \mathbb{D}_{L}$

where K is, in each expression, a normalizing constant, and $n \in \mathbb{N}, a > 0$.

We continue Pólya's and de Bruijn's line of research by presenting original additional closure properties of the set \mathbb{D}_L . More specifically, we investigate its stability under product of independent variables. We have already mentioned that this property is intimately connected to the concept of intertwining relationship between Markov semigroups, as we will discuss later in the paper. To state it, we say that a positive linear operator Λ on the space of bounded borelian functions is Markov multiplicative if there exists a random variable I with distribution function $F_{\rm I}$, such that, for any bounded borelian function f, writing $\Lambda = \Lambda_{\rm I}$,

(3.9)
$$\Lambda_{\rm I} f(t) = \int_{\mathbb{R}} f(xt) F_{\rm I}(\mathrm{d}x).$$

In the following, we identify a mapping from the set of even entire functions in \mathbb{P}_+ into \mathbb{D}_L . In other words, we provide a way of creating entire characteristic functions with only real zeros from any even characteristic functions. We also find necessary conditions on a Markov multiplicative operator to leave invariant the set \mathbb{D}_L . Let us now denote by \mathbb{LP}_+ the class of functions in \mathbb{LP} with only strictly negative zeros and of the form

$$\varphi(z) = Ke^{az} \prod_{k=1}^{\infty} \left(1 + \frac{z}{z_k} \right) e^{-z/z_k}$$

with $K, a \in \mathbb{R}$ and $z_k > 0$ for all k with $\sum_{k \geqslant 1} z_k^{-2} < \infty$.

THEOREM 3.13.

(1) Let $\mathcal{F}_D \in \mathbb{P}_+$ be entire and even, and, assume that there exist a random variable I such that for any non-negative integer n

$$\mathcal{M}_{\mathrm{I}}(2n)\mathcal{M}_{\mathrm{D}}(2n) = a_{\varphi}(n)G(n)\Gamma(2n+1)$$

where $\varphi(z) = \sum_{n=0}^{\infty} a_{\varphi}(n) z^n \in \mathbb{LP}$ and $G \in \mathbb{LP}_+$. Then, $\Lambda_I \mathcal{F}_D \in \mathbb{D}_L$.

(2) Let us now assume that there exist $\varphi \in \mathbb{LP}_+$ and a random variable I_L such that for any non-negative integer n,

$$\varphi(n) = \mathcal{M}_{\mathrm{Ir}}(2n).$$

Then,
$$\Lambda_{\mathrm{I}_L}(\mathbb{D}_L) \subset \mathbb{D}_L$$
.

Before continuing further, let us first illustrate part (1) of this theorem by an example. Let us take $\varphi(z) = e^z \in \mathbb{LP}$ and then $a_{\varphi}(n) = \frac{1}{n!}, n \geqslant 0$. For instance, let $D = J_0$ where J_0 is the symmetric random variable whose distribution is the arc-sine law and is recalled in (5.24) below. Then, from (5.28), $\mathcal{M}_{J_0}(2n) = \frac{(2n)!}{n!n!}$ and $\mathcal{F}_{J_0} = \mathcal{J}_0 \in \mathbb{D}_L$, the Bessel function of order 0, see (1.5) above.

Next fix b>0 and define I_b as the positive random variable with distribution $F_{I_b}(\mathrm{d}x)=\frac{2b}{b+1}e^{-x^2}x\left(\frac{x^2}{b}+1\right)\mathrm{d}x,\ x>0$. Simple algebra yields that $\mathcal{M}_{I_b}(2n)=\frac{(n+b)}{b}n!$, and, with the previous choice of φ , the equation $\mathcal{M}_{I_b}(2n)\mathcal{M}_{J_0}(2n)=a_{\varphi}(n)G(n)\Gamma(2n+1)$ gives

$$G(n) = \frac{(2n)!n!(n+b)n!}{bn!n!(2n)!} = \frac{(n+b)}{b}.$$

Since $G(z) = \frac{z+b}{b}$ is in \mathbb{LP}_+ , the theorem above shows that

$$\Lambda_{\mathbf{I}_{b}}\mathcal{F}_{\mathbf{J}_{0}}(t) = \sum_{n=0}^{\infty} (-1)^{n} \frac{n+b}{n!b} t^{2n} = \frac{b-t^{2}}{b} e^{-t^{2}} \in \mathbb{D}_{L},$$

for all b > 0. Note that this characteristic function already appeared in [32, Example III.8] where the authors showed that it is not decomposable with respect to the additive convolution of probability measures. However, our approach reveals that it is decomposable with respect to the multiplicative one. We point out that Theorem 3.13 enables one to generate many new examples of elements in \mathbb{D}_L , and, we refer to Theorem 5.6 where this idea is exploited and to Section 4.2 below, where some additional examples are provided. Another interesting aspect of the set \mathbb{D}_L is the following additional closure property.

THEOREM 3.14. — The sets \mathbb{D} and \mathbb{D}_L are closed under locally uniform convergence.

Note that the locally uniform convergence of characteristic functions is equivalent to the pointwise convergence to a continuous function at 0, see [5, Theorem 3.2.1], and, by the Lévy continuity theorem, see [7, Theorem 8.28], this implies the weak convergence of the corresponding sequence of random variables to a random variable. In other words, Theorem 2.5 entails that Theorem 3.14 is a generalization to [37, Theorem 7] regarding the closure under weak convergence of the set \mathcal{X} , which is the set of symmetric probability measures F whose characteristic function is in \mathbb{LP}_e and such that $\int_{\mathbb{R}} e^{bx^2} F(dx) < \infty$ for some b > 0. We do not need this last condition for our closure result. However, the result from Newman and Wu [37, Theorem 7] ensures that the gaussian tail property for elements in \mathbb{D}_L is preserved under weak convergence.

4. The new class \mathbb{D}_P

To present the main results of this Section, we start by introducing some objects and notation. First, with $\mathbb{M}_{+}(\mathbb{R}^{+})$ denoting the set of non-negative

Radon measures on $(0, \infty)$, we define the mapping $\Psi : \mathbb{R}^+ \to \mathbb{R}$ by

(4.1)
$$\Psi(u) = -\kappa + au + \frac{1}{2}\sigma^2 u^2 - \int_0^\infty (1 - e^{-ur} - ur \mathbb{I}_{\{r < 1\}}) \mu(dr)$$

where $\kappa \geqslant 0$, $a \in \mathbb{R}$, $\sigma^2 \geqslant 0$ and $\mu \in \mathbb{M}_+(\mathbb{R}^+)$ is such that $\int_0^\infty \inf(1,r^2)\mu(\mathrm{d}r) < \infty$. We exclude the case when $\sigma = 0$, $a + \int_0^1 r\mu(\mathrm{d}r) \leqslant 0$ and $\int_0^\infty \inf(1,r)\mu(\mathrm{d}r) < \infty$, which is seen as degenerate in our context. This function has a nice probabilistic interpretation. Indeed, it is the so-called Laplace exponent of a possibly killed real-valued spectrally negative Lévy process $Y = (Y_t)_{t\geqslant 0}$, i.e. a stochastic process without positive jumps and with stationary and independent increments starting from 0 and, when $\kappa > 0$, it is killed at an independent exponential time of parameter κ . Moreover, we have, for any $u, t \geqslant 0$,

(4.2)
$$\mathcal{F}_{Y_{\star}}(-iu) = e^{\Psi(u)t}.$$

Note that under the three conditions we excluded above, Y is negative-valued and has non-increasing sample paths. We refer to the monograph [27] for a thorough study of these processes. It is a well established (and an easy to check) fact that the Laplace exponent Ψ is strictly convex on $[0, \infty)$ with

(4.3)
$$\lim_{u \to \infty} \Psi(u) = +\infty \text{ and } \Psi \text{ is increasing on } [\theta, \infty)$$

where $\theta = \sup\{u \ge 0; \Psi(u) = 0\}$. In fact, by convexity, we have $\theta > 0$ if and only if

(4.4) (i)
$$\kappa > 0$$
 or (ii) $\kappa = 0$ and $\Psi^{(1)}(0^+) = \lim_{u \downarrow 0} \Psi^{(1)}(u) < 0$

and a monotone convergence argument yields that $\Psi^{(1)}(0^+) = a - \int_1^\infty r\mu(\mathrm{d}r) \in [-\infty, +\infty)$. We are now ready to define the sets

(4.5)
$$\mathbb{N} = \{ \Psi : \mathbb{R}^+ \to \mathbb{R} \text{ of the form (4.1)} \}$$

and

$$\mathbb{N}_{\mathbb{D}} = \left\{ \Psi \in \mathbb{N}; \ 0 \leqslant \theta \leqslant \frac{1}{2} \right\}.$$

Note that (4.4) entails that

(4.7)
$$\Psi \in \mathbb{N}_{\mathbb{D}} \text{ if } \kappa = 0 \text{ and } \Psi^{(1)}(0^+) \geqslant 0$$

as, in this case, $\theta = 0$. Moreover, from (4.3), one easily gets that a necessary and sufficient condition for a function Ψ to be in $\mathbb{N}_{\mathbb{D}}$ is that $\Psi \in \mathbb{N}$ with $\Psi(\frac{1}{2}) \geqslant 0$. The notation of the set $\mathbb{N}_{\mathbb{D}}$ is motivated by the following facts. On the one hand, due to the infinite divisibility of Y_1 , we have, for any $\Psi \in \mathbb{N}$, that $z \mapsto -\Psi(iz) \in \mathbb{N}(\mathbb{R})$, the set of continuous and negative-definite functions on \mathbb{R} . On the other hand, we now define a class of entire

functions that are generated by the set $\mathbb{N}_{\mathbb{D}}$, that will be shown to belong to \mathbb{D} .

For any $\Psi \in \mathbb{N}$, we introduce the power series

(4.8)
$$\mathcal{J}_{\Psi}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{W_{\Psi}(n+1)} t^{2n}$$

where, here and below, for a function φ defined on \mathbb{R}^+ , we set

$$W_{\varphi}(1) = 1, \quad W_{\varphi}(n+1) = \prod_{k=1}^{n} \varphi(k), \quad n \geqslant 1.$$

Since, from (4.3), $\lim_{n\to\infty} \frac{W_{\Psi}(n+2)}{W_{\Psi}(n+1)} = \lim_{n\to\infty} \Psi(n+1) = \infty$, \mathcal{J}_{Ψ} defines an entire function.

We also write

(4.9)
$$\mathcal{I}_{\Psi}(t) = \mathcal{J}_{\Psi}(it) = \sum_{n=0}^{\infty} \frac{1}{W_{\Psi}(n+1)} t^{2n}$$

and point out that the entire function $\mathcal{I}_{\Psi}(\sqrt{t})$ was introduced by the second author in [40] where it was shown that it is an invariant function of some self-similar integro-differential operator. Therein, the complete monotonicity property was identified for several of its transformations, and, we also refer to [1] for a more recent and refined studies of this class of functions. We are now ready to introduce the following class of entire functions

$$(4.10) \mathbb{D}_P = \{ \mathcal{J}_{\Psi} \text{ of the form } (4.8) \text{ with } \Psi \in \mathbb{N}_{\mathbb{D}} \}.$$

THEOREM 4.1. — We have $\mathbb{D}_P \subseteq \mathbb{D}$, and, for all $\Psi \in \mathbb{N}_{\mathbb{D}}$,

$$\left[\mathcal{J}_{\Psi}, \frac{1}{\mathcal{I}_{\Psi}}\right] \in \mathbb{D}_{2}$$

with

$$(4.12) t \mapsto -\log \mathcal{J}_{\Psi}(t) \notin \mathbb{N}(\mathbb{R}) but t \mapsto \log \mathcal{I}_{\Psi}(t) = \phi_{\Psi}(t^2) \in \mathbb{N}(\mathbb{R})$$

where the function ϕ_{Ψ} is a Bernstein function that belongs to the class \mathbb{B}_{J} , that is, it is of the form (5.4) below with the additional property that $r \mapsto r\bar{\mu}(r)$ is non-increasing on \mathbb{R}^{+} . Finally, writing $\mathcal{F}_{\Psi}(t) = \frac{\mathcal{J}_{\Psi}(t)}{\mathcal{I}_{\Psi}(t)}, t \in \mathbb{R}$, we have $\mathcal{F}_{\Psi} \in \mathbb{D}_{S}$, that is $\mathcal{F}_{\Psi}(t)\mathcal{F}_{\Psi}(it) = 1$ for all $t \in \mathbb{R}$.

Remark 4.2. — The random variables whose characteristic functions appear above shall be explicitly described in Section 5.2.4.

Remark 4.3. — Note that in [25], it is proved that the entire function $\mathcal{J}_{\Psi}(\sqrt{t})$ has its smallest (in modulus) zero, say z_1 , which is simple and located on the positive real line. On the one hand, this shows that \mathcal{J}_{Ψ} is

not the characteristic function of an infinitely divisible variable as their characteristic functions are zero-free, see [53]. On the other hand, $z_1 < 0$ corresponds to a singularity of the Bernstein function ϕ_{Ψ} . However, from Theorem 4.1, we get the identity, for all $t \in \mathbb{R}$,

(4.13)
$$\mathcal{I}_{\Psi}(t)\mathcal{J}_{\Psi}(t) = e^{-\phi_{\Psi}(t^2)}e^{\phi_{\Psi}(-t^2)} = 1,$$

which entails that the Bernstein function ϕ_{Ψ} admits a meromorphic extension on \mathbb{C} , but it is not necessarily a Pick function (see the definition below). This reveals that such an identity cannot be possible for Bernstein functions, especially for those having an essential singularity, e.g. $\phi(u) = u^a, 0 < a < 1$.

The purpose of the next result is to explain how the classes \mathbb{D}_P and \mathbb{D}_L are related, which consists on investigating the difficult issue of locating the zeros of the entire function $\mathcal{J}_{\Psi} \in \mathbb{D}_P$. To state it, we recall that a Bernstein Pick function [54, p. 56] is a Bernstein function which admits an holomorphic extension which maps the upper half-plane into its closure. We say that an entire function (resp. Pick meromorphic function) has the 1-separation property if its sequence of zeros $(z_k)_{k\geqslant 1}$ (resp. and poles $(\rho_k)_{k\geqslant 1}$) satisfies, for all k, $z_{k+1} < z_k - 1$ (resp. $\rho_k = z_k - 1 > z_{k+1}$). We now introduce the set (4.14)

 $\mathbb{B}_{P_1} = \{ \phi \in \mathbb{B}; \phi \text{ is a Pick function having the 1 separation property} \}.$

THEOREM 4.4. — Let $\phi \in \mathbb{B}_{P_1}$. Then, $u \mapsto \Psi(u) = u\phi(u) \in \mathbb{N}_{\mathbb{D}}$ and $\mathcal{J}_{\Psi} \in \mathbb{D}_{L} \cap \mathbb{D}_{P}$. However, $\mathbb{D}_{P} \nsubseteq \mathbb{D}_{L}$ as there are $\Psi's \in \mathbb{N}_{\mathbb{D}}$ such that \mathcal{J}_{Ψ} has at least a non-real zero.

Remark 4.5. — In Section 4.2, we provide instances of the two situations presented in this theorem, see e.g. the Bernstein functions that define the Bessel functions and the Fox–Wright functions, see 4.2.3, which both belong to \mathbb{D}_L . On the other hand there are the examples involving hypergeometric functions and the Mittag–Leffler functions which have non-real zeros.

We proceed with the following result that shows that the Lukacs mappings, introduced in (3.1), also leave our class \mathbb{D}_P invariant.

PROPOSITION 4.6. — Let $L^{(p)}$, p=1,2, be the operators that were defined in (3.1) above. Then, we have $L^{(1)}(\mathbb{D}_P) \subset \mathbb{D}_P$. Moreover, $L^{(2)}(\mathbb{D}_P) \subset \mathbb{D}$ and the same remains true for their iterates.

4.1. The Riemann ξ function and the class \mathbb{D}_P

A natural and important question that arises at this stage is to understand whether the Riemann ξ function defined in (2.7) belongs to the class \mathbb{D}_P . Indeed, this would yield a power series representation of this function whose coefficients would be expressed in terms of negative definite functions offering new tools to study the location of its zeros, using for instance Theorem 4.4. To this end, let us recall that the Riemann ξ function, defined in (2.7), can be expressed in terms of the following power series

(4.15)
$$\Theta(z) = \xi(\sqrt{z}) = \sum_{n=0}^{\infty} \frac{\gamma(n)}{n!} z^n, \ z \in \mathbb{C},$$

where, with $F(n) = \int_1^\infty (\log x)^n x^{-3/4} \theta_0(x) dx$ and $\theta_0(x) = \sum_{n=1}^\infty e^{-\pi n^2 x}$ the theta series, we have set

(4.16)
$$\gamma(n) = \frac{n!}{(2n)!} \frac{32\binom{2n}{2}F(2n-2) - F(2n)}{2^{2n-1}},$$

see e.g. [16]. The question whether there exists $\Psi \in \mathbb{N}_{\mathbb{D}}$ such that $\mathcal{J}_{\Psi} = \xi$ boils down to the existence of $\phi \in \mathbb{B}_{\mathbb{D}}$ such that, for all $n \in \mathbb{N}$,

(4.17)
$$\frac{1}{W_{\phi}(n+1)} = \frac{n!}{(2n)!} \cdot \frac{32\binom{2n}{2}F(2n-2) - F(2n)}{2^{2n-1}},$$

which, after some easy algebra, is equivalent to show that

(4.18)
$$\phi(n+1) = -\frac{G(2n)}{8(n+1)G(2n+2)},$$

where we have set $G(2n) = 64n(2n-1)\frac{F(2n-2)}{F(2n)} - 1$. Since this question does not seem straightforward, we investigate, instead here, whether this possibility could be excluded from the properties that we know about elements of the class \mathbb{D}_P and the Riemann ξ function. Recalling that the latter is an entire function of order 1 with infinite type and it is the characteristic function of the density of a probability measure whose support is \mathbb{R} , we have the following.

PROPOSITION 4.7. — Let $\Psi \in \mathbb{N}_{\mathbb{D}}$ such that $\Psi(u) = \frac{u^2}{\ell(u)}, u \in \mathbb{R}$, with $\lim_{u \to \infty} \ell(u) = \infty$, ℓ being a slowly varying function at infinity, i.e. for every u > 0, $\lim_{t \to \infty} \frac{\ell(ut)}{\ell(t)} = 1$. Then \mathcal{J}_{Ψ} is an entire function of order 1 and infinite type. This condition holds when $\Psi(u) = (u-\theta)\phi(u), \theta \geqslant 0$, with ϕ a special Bernstein function, i.e. $\phi(u)\widehat{\phi}(u) = u$, such that its conjugate Bernstein function $\widehat{\phi}(u) = \ell(u)$. Moreover, under this condition, \mathcal{J}_{Ψ} is the characteristic function of a probability density function whose support is \mathbb{R} .

The first part of the Proposition follows readily from [1, Proposition 2.1], see also Proposition 5.4 below, where we notice that when the order is 1, with the notation of Proposition 5.4, $\underline{\Psi}=2$ and thus the type $\tau_{\Psi} \geqslant \left(\overline{\lim}_{n\to\infty}\frac{n}{\phi(n)}\right)^{\frac{1}{2}} = \left(\overline{\lim}_{n\to\infty}\ell(n)\right)^{\frac{1}{2}} = \infty$ under the condition of the Proposition. The last claim is a specific instance of Lemma 5.8.

Remark 4.8. — One instance when $\Psi(u) = \frac{u^2}{\ell(u)}, u \in \mathbb{R}$, with ℓ as in the Proposition is when $\Psi(u) = ue^{W(u)}$, where W is the Lambert function. Indeed, it is well known that for all $u \geq 0$, $W(u)e^{W(u)} = u$ and W is a complete Bernstein function that is a Bernstein function whose Lévy measure is absolutely continuous with a completely monotone density, and $\lim_{u\to\infty} \frac{W(u)}{\ln u} = 1$, and hence it is a special Bernstein function with $e^{W(u)}$ as conjugate, see [39]. Therefore $\Psi(u) = ue^{W(u)} \sim \frac{u^2}{\ln u}$.

4.2. Some examples in the class \mathbb{D}_P

In this section, we give several specific examples of the function \mathcal{J}_{Ψ} including the modified Bessel functions, the Mittag–Leffler functions and several type of hypergeometric functions, and refer to [14, 23] as classical references on these functions. The interested reader can also consult the monograph [54] for several examples of Bernstein functions from which one can provide additional interesting instances of \mathcal{J}_{Ψ} .

4.2.1. The Bessel functions

Let $\Psi(u) = u(u+\nu)$, $\nu \geqslant -\frac{1}{2}$. We get that $W_{\Psi}(n+1) = n! \frac{\Gamma(n+\nu+1)}{\Gamma(\nu+1)}$ and thus

(4.19)
$$\mathcal{J}_{\Psi}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\nu+1)}{\Gamma(n+\nu+1)} \frac{t^{2n}}{n!} = \Gamma(\nu+1)t^{-\nu}J_{\nu}(2t)$$

where J_{ν} stands for the Bessel function of order ν . It is well-known that $\mathcal{J}_{\Psi} \in \mathbb{D}_{L}$.

4.2.2. The confluent hypergeometric function

Let 0 < a < 1 < a + b and $\Psi(u) = u \frac{u+1-a}{u+b}$. Note that, in this case $\sigma = 0$, and hence the support of the variable $D_{\phi,\theta}$ is the real line. Moreover, simple algebra yields

$$\Psi(u) = u \frac{1-a}{b} + u \int_0^\infty (1 - e^{-ur}) (a+b-1) e^{-br} dr.$$

We have $W_{\Psi}(n+1) = n! \frac{\Gamma(b+1)\Gamma(n+2-a)}{\Gamma(n+b+1)\Gamma(2-a)}$ and thus

$$\mathcal{J}_{\Psi}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+b+1)\Gamma(2-a)}{\Gamma(b+1)\Gamma(n+2-a)} \frac{t^{2n}}{n!} = {}_{1}F_{1}(b+1;2-a;-t^2)$$

where ${}_{1}F_{1}$ is the confluent hypergeometric function. If a+b=2,3,..., then $\mathcal{J}_{\Psi} \in \mathbb{D}_{L}$, see e.g. [22, Theorem 4]. Therein, the authors conjecture, in particular, that $\mathcal{J}_{\Psi} \in \mathbb{D}_{L}$ for all a+b>1.

4.2.3. The Fox-Wright function

Let $\alpha \in (0,1)$ and $\beta \geqslant \alpha$, then

$$\Psi(u) = u \frac{\Gamma(\alpha u + \beta)}{\Gamma(\alpha(u - 1) + \beta)}$$

$$= u \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} + \frac{u}{\Gamma(1 - \alpha)} \int_0^\infty (1 - e^{-ur}) \frac{e^{-\frac{\beta}{\alpha}r}}{(1 - e^{-r/\alpha})^{\alpha + 1}} dr$$

yielding $W_{\Psi}(n+1) = n! \frac{\Gamma(\alpha n + \beta)}{\Gamma(\beta)}$ and thus

$$\mathcal{J}_{\Psi}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\beta)}{\Gamma(\alpha n + \beta)} \frac{t^{2n}}{n!} = {}_{0}\Psi_1^* \left((\beta, \alpha) ; -t^2 \right)$$

where ${}_{0}\Psi_{1}^{*}$ is the normalized Fox–Wright function, which is in \mathbb{D}_{L} by Laguerre Theorem, see e.g. [52, Theorem 4] as the mapping $z \mapsto \frac{1}{\Gamma(\alpha z + \beta)} \in \mathbb{LP}_{+}$.

4.2.4. The Mittag-Leffler function

Let now $\alpha \in (1,2)$ and $\beta \in (\alpha - 1, \alpha)$ and set

$$\begin{split} \Psi(u) &= \frac{\Gamma(\alpha u + \beta)}{\Gamma(\alpha u + \beta - \alpha)} \\ &= \frac{\alpha \Gamma(\beta)}{\Gamma(\beta - \alpha + 1)} u + \frac{(\alpha - 1)u}{\Gamma(2 - \alpha)} \int_0^\infty \left(1 - e^{-ur}\right) \frac{e^{-\frac{\beta}{\alpha}r}}{\left(1 - e^{-r/\alpha}\right)^\alpha} \, \mathrm{d}r. \end{split}$$

Then, $W_{\Psi}(n+1) = \frac{\Gamma(\alpha n + \beta)}{\Gamma(\beta)}$ and

$$\mathcal{J}_{\Psi}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\beta)}{\Gamma(\alpha n + \beta)} t^{2n} = \Gamma(\beta) E_{\alpha,\beta}(-t^2)$$

where $E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + \beta)}$ is the general Mittag–Leffler function. It is well known, see [13, Chapter 3.2], that, for $1 < \alpha < 2$, $E_{\alpha,\beta}$ have nonreal zeros. For the last 3 examples, we do not know whether $\mathcal{J}_{\Psi} \in \mathbb{D}_L$ or not.

4.2.5. The Barnes-Hypergeometric function

Let now $\alpha \in (1,2), \rho \in (0,1/\alpha]$ and set

$$\Psi(u) = \frac{\Gamma(\alpha\rho + \alpha u)u}{\Gamma(\alpha u)}$$
$$= u \int_0^\infty (1 - e^{-ur}) \frac{\rho}{\Gamma(1 - \alpha\rho)} e^{-\rho r} \left(1 - e^{-r/\alpha}\right)^{-\alpha\rho - 1} dr.$$

Observe that

$$(4.20) W_{\Psi}(n+1) = n! \frac{G\left(1, \frac{1}{\alpha}\right)}{G\left(1 + \rho, \frac{1}{\alpha}\right)} \frac{G\left(n + 1 + \rho, \frac{1}{\alpha}\right)}{G\left(n + 1, \frac{1}{\alpha}\right)}$$

where the Barnes G-function $G(n; \frac{1}{\alpha})$ is defined as the unique log-convex solution to recurrence equation $G(n+1; \frac{1}{\alpha}) = \Gamma(\alpha n) G(n; \frac{1}{\alpha})$, and refer to [33] for more details on this example. Then,

$$\mathcal{J}_{\Psi}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{G\left(n+1,\frac{1}{\alpha}\right) G\left(1+\rho,\frac{1}{\alpha}\right)}{G\left(1,\frac{1}{\alpha}\right) G\left(n+1+\rho,\frac{1}{\alpha}\right)} \frac{t^{2n}}{n!}.$$

4.2.6. The hypergeometric function

Let
$$\alpha > 0$$
, and, writing $\alpha_1 = \frac{\alpha + \sqrt{\alpha^2 + 4}}{2}$ and $\alpha_2 = \frac{\alpha - \sqrt{\alpha^2 + 4}}{2}$, consider
$$\Psi(u) = \frac{u}{u + \alpha} (u + \alpha_1) (u + \alpha_2).$$

It follows that $W_{\Psi}(n+1) = n! \frac{\Gamma(\alpha+1)\Gamma(n+\alpha_1+1)\Gamma(n+\alpha_2+1)}{\Gamma(n+\alpha+1)\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}$ and thus

$$\mathcal{J}_{\Psi}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\alpha+1)\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}{\Gamma(\alpha+1)\Gamma(n+\alpha_1+1)\Gamma(n+\alpha_2+1)} \frac{t^{2n}}{n!}$$
$$= {}_{1}F_{2}\left(\alpha+1; \alpha_1+1, \alpha_2+1; -t^{2}\right)$$

where $_1F_2$ is an hypergeometric function.

4.2.7. The power-gamma function

Let $\alpha \in (0,1), \gamma \geqslant 0$ and consider

$$\Psi(u) = u (u + \gamma)^{\alpha} = \gamma^{\alpha} u + u \int_{0}^{\infty} (1 - e^{-ur}) e^{-\gamma r} \frac{r^{-\alpha - 1}}{\Gamma(-\alpha)} dr.$$

It follows that $W_{\Psi}(n+1) = n! \frac{\Gamma^{\alpha}(n+\gamma+1)}{\Gamma^{\alpha}(\gamma+1)}$ and thus

$$\mathcal{J}_{\Psi}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma^{\alpha}(\gamma+1)}{\Gamma^{\alpha}(n+\gamma+1)} \frac{t^{2n}}{n!}.$$

5. Proofs

5.1. Proofs of Section 3

5.1.1. Proof of Theorem 3.5

First, resorting to the Laguerre theory, see e.g. Borel [6, Chapter 2], on the additional number of zeros obtained by differentiating an entire function of finite genus, we get that if $\mathcal{F} \in \mathbb{LP}$ then $t \mapsto \mathcal{F}^{(1)}(t)/t$ and $t \mapsto \mathcal{F}^{(2)}(t)$ are also in \mathbb{LP} . Recall that \mathbb{D}_L contains even and positive definite functions in \mathbb{LP} that take value 1 at 0. Suppose that \mathcal{F} is even with $\mathcal{F}(0) = 1$. it is easy to see that $L^{(p)}\mathcal{F}$, p = 1, 2, are also even and take value 1 at 0. Hence, to show that $L^{(p)}(\mathbb{D}_L) \subset \mathbb{D}_L$ it suffices to show that if \mathcal{F} is a characteristic function then so are the $L^{(p)}\mathcal{F}$, p = 1, 2. One can check the latter by a probabilistic argument. First, recall if X is a positive random variable with finite expectation $m_X(1)$ and distribution F_X then the random variable X(1) with distribution

$$F_{X(1)}(dx) = \frac{x}{m_X(1)} F_X(dx), \ x \geqslant 0,$$

is called the size-biased version of X. Size-biasing appears frequently and naturally in probability theory, most notably in the theory of stationary point processes on the real line and in the theory of branching processes and random walks.

LEMMA 5.1. — Let \mathcal{F}_X be the characteristic function of a real-valued random variable X with finite second moment. Let $X^2(1)$ be the size-biased version of X^2 . Then $L^{(p)}\mathcal{F}_X$ is the characteristic function of the random variable

$$(5.1) U^{(p)} \times \sqrt{X^2(1)}$$

where $U^{(p)}$ is a uniform random variable on the interval [-1,1] if p=1 or a random variable taking values +1 or -1 with probability 1/2 each if p=2, and, in both cases, is taken independent of $X^2(1)$.

Proof. — We first recall that, for a random variable X, we denote its distribution function by F_X . Then, using the definition of size-biasedness and writing $X^{(p)} = U^{(p)} \times \sqrt{X^2(1)}$ we obtain, since the variables are independent, that

$$F_{\mathbf{X}^{(1)}}(\mathrm{d}x) = \frac{\int_{|y|>|x|} |y| F_{\mathbf{X}(1)}(\mathrm{d}y)}{m_{\mathbf{X}}(2)} \mathrm{d}x, \quad F_{\mathbf{X}^{(2)}}(\mathrm{d}x) = \frac{x^2}{m_{\mathbf{X}}(2)} F_{\mathbf{X}(1)}(\mathrm{d}x),$$

where, by assumption, $m_{\rm X}(2) = \int_{-\infty}^{\infty} x^2 F_{\rm X}(\mathrm{d}x) < \infty$. We can then directly verify that $L^{(p)}f(t) = \int_{-\infty}^{\infty} e^{itx} F_{{\rm X}^{(p)}}(\mathrm{d}x), p = 1, 2.$

5.1.2. Proof of Theorem 3.13

First, note that if $\mathcal{F}_X \in \mathbb{P}_+$, then for any random variable I, $\Lambda_I \mathcal{F}_X \in \mathbb{P}_+$ as it is the characteristic function of the random variable XI where the two random variables are considered to be independent. Next, take $\mathcal{F}_D \in \mathbb{P}_+$ even and entire, that is, D is a symmetric real-valued random variable such that, for any $t \in \mathbb{R}$,

$$\mathcal{F}_{\rm D}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{\mathcal{M}_{\rm D}(2n)}{(2n)!} t^{2n}.$$

Moreover, under the conditions of the item (1), that is $\varphi \in \mathbb{LP}$ and $G \in \mathbb{LP}_+$, Laguerre's theorem [52, Theorem 4] entails that the function

$$f(z) = \sum_{n=0}^{\infty} G(n)a_{\varphi}(n)z^{n}$$

is an entire function with only real zeros. We made the assumption that $G(n)a_{\varphi}(n) = \mathcal{M}_{\mathrm{D}}(2n)\mathcal{M}_{\mathrm{I}}(2n)/(2n)!$. Since these are nonnegative numbers, f cannot have nonnegative zeros. On the other hand,

$$\Lambda_{\rm I} \mathcal{F}_{\rm D}(t) = \int_{\mathbb{R}} \mathcal{F}_{\rm D}(xt) F_{\rm I}(dx) = \sum_{n=0}^{\infty} (-1)^n \frac{\mathcal{M}_{\rm D}(2n) \mathcal{M}_{\rm I}(2n)}{(2n)!} t^{2n}
= \sum_{n=0}^{\infty} (-1)^n G(n) a_{\varphi}(n) t^{2n} = f(-t^2)$$

where the interchange of the integral and sum is justified by a classical Fubini argument as the series defines an entire function. Since f has only real negative zeros, it follows that $\Lambda_{\rm I}\mathcal{F}_{\rm D}$ is an even entire function with only real zeros. Since $\Lambda_{\rm I}\mathcal{F}_{\rm D}$ is a characteristic function, it is in \mathbb{P}_+ . Hence $\Lambda_{\rm I}\mathcal{F}_{\rm D} \in \mathbb{D}_L$.

For the second one, let $\mathcal{F} \in \mathbb{D}_L$, and thus, one has from (2.6), that \mathcal{F} is of the form (2.3), and thus $t \mapsto \mathcal{F}(\sqrt{t}) \in \mathbb{LP}$. Since $\mathcal{F} \in \mathbb{D}_L$ there exists a symmetric real-valued random variable D such that $\mathcal{F} = \mathcal{F}_D$. Proceeding as above, we get, for any $t \in \mathbb{R}$,

$$\Lambda_{\mathrm{I}}\mathcal{F}_{\mathrm{D}}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{\mathcal{M}_{\mathrm{D}}(2n)\mathcal{M}_{\mathrm{I}}(2n)}{(2n)!} t^{2n}.$$

We apply again Laguerre's Theorem [52, Theorem 4]. On the one hand, by assumption, there is a function $\varphi \in \mathbb{LP}_+$ such that $\mathcal{M}_{\mathrm{I}}(2n) = \varphi(n)$. On the other hand, we observed above that

$$f(z) = \sum_{n=0}^{\infty} \frac{\mathcal{M}_{\mathrm{D}}(2n)}{(2n)!} z^n = \mathcal{F}(\sqrt{z}) \in \mathbb{LP}.$$

Since $\Lambda_{\rm I}\mathcal{F}_{\rm D}(t)=f(-t^2)$ we conclude, as above, that $\Lambda_{\rm I}\mathcal{F}_{\rm D}\in\mathbb{D}_L$.

5.1.3. Proof of Theorem 3.14

Let first $(\mathcal{F}_n)_{n\geqslant 0}$ be a sequence in \mathbb{D} and assume that for all t in a bounded interval, $\lim_{n\to\infty}\mathcal{F}_n=\mathcal{F}$ uniformly. Then, according to the Lévy continuity theorem, see [7, Theorem 8.28], $\mathcal{F}\in\mathbb{P}_+$ and is continuous on \mathbb{R} . Let us write, for all $t\in\mathbb{R}, n\geqslant 0$, $G_n(t)=\frac{1}{\mathcal{F}_n(t)}$. Since by assumption, for all $n\geqslant 0$, $t\mapsto G_n(it)\in\mathbb{P}_+$, and, \mathcal{F}_n is real, even and non-vanishing around 0, we get that G_n is well defined and even in a neighborhood of 0. Hence, it is the moment generating function of a (unique) symmetric random variable. Moreover, we have, for all $t\in\mathbb{R}$, $\lim_{n\to\infty}G_n(t)=G(t)=\frac{1}{\mathcal{F}(t)}$, and, by continuity of \mathcal{F} and the fact that $\mathcal{F}(0)=1$, there exits $\alpha>0$ such that G is finite-valued on $|t|<\alpha$. According to [12, Theorem 3], G is the moment generating function of a random variable, whose law is uniquely determined by G, see [12, Theorem 1]. Finally, its characteristic function is plainly the mapping $t\mapsto G(it)$ and is such that, for all $t\in\mathbb{R}$, $\mathcal{F}(t)G(it)=1$, that is $\mathcal{F}\in\mathbb{D}$.

Now assume that the sequence $(\mathcal{F}_n)_{n\geqslant 0}$ is in \mathbb{D}_L . We shall show that its locally uniform limit $\mathcal{F}\in\mathbb{D}_L$. Since, for all $n\geqslant 0$, $\mathcal{F}_n\in\mathbb{D}_L$, by Theorem 2.1, we have that $\mathcal{G}_n:t\mapsto\frac{1}{\mathcal{F}_n(it)}$ is the characteristic function of a symmetric Pólya frequency function. As $\mathbb{D}_L\subset\mathbb{D}$, from the previous proof, we deduce that, for all $t\in\mathbb{R}$, $\lim_{n\to\infty}\mathcal{G}_n(t)=\mathcal{G}(t)=\frac{1}{\mathcal{F}(it)}$ with $\mathcal{G}\in\mathbb{P}_+$. However, the set of Pólya frequency functions (probability measures) is closed under weak convergence as it is the closure, under weak convergence, of probability measures whose characteristic functions are the reciprocal of a polynomial having only real roots, see [19, Chapter III, Theorem 4.1]. We obtain, using the fact that \mathcal{G}_n is real and even for all $n\geqslant 0$, that the mapping $t\mapsto \mathcal{F}(t)=\frac{1}{\mathcal{G}(it)}\in\mathbb{LP}_e$ and hence $\mathcal{F}\in\mathbb{P}_+\cap\mathbb{LP}_e=\mathbb{D}_L$ which completes the proof.

5.2. Proofs of Section 4

5.2.1. A Wiener-Hopf type mapping between sets of negative-definite functions

We start the proof by a one-to-one mapping, emanating from the Wiener–Hopf factorization, between the set $\mathbb{N}_{\mathbb{D}}$ and a set of Bernstein functions.

This allows us to provide a representation of the coefficients of the function in terms of a Stieltjes moment sequence that will be helpful in identifying one of the random variables solving the van Dantzig problem.

To this end we introduce some notation. First, let us denote by \mathbb{B} the set of Bernstein functions, i.e. functions $\phi:(0,\infty)\to[0,\infty)$ having derivatives of all orders such that $(-1)^{n+1}\phi^{(n)}\geqslant 0$ for all $n\geqslant 1$; see [54, Chapter 3] These functions are in a one-to-one correspondence with functions that admit the so-called Lévy–Khintchine representation

(5.2)
$$\phi(u) = \nu + \frac{\sigma^2}{2}u + \int_0^\infty (1 - e^{-ur})\mu(dr), \ u \geqslant 0,$$

where $\nu, \sigma \geqslant 0$ and $\mu \in \mathbb{M}^+(\mathbb{R}^+)$ such that $\int_0^\infty \inf(1,r)\mu(\mathrm{d}r) < \infty$. The set \mathbb{B} is invariant under several transformations. In particular, it is a convex cone, and, it is also stable by the action of the semigroup of translations, Indeed, for any $\beta \geqslant 0$, easy algebra yields

(5.3)
$$\phi(u+\beta) = \phi(\beta) + \frac{\sigma^2}{2}u + \int_0^\infty (1 - e^{-ur})e^{-\beta r}\mu(\mathrm{d}r), \ u \geqslant 0,$$

which is plainly a Bernstein function as $\phi(\beta) \ge 0$ and $e^{-\beta r} \mu(dr)$ is a Lévy measure as defined above. We shall also need the subset $\mathbb{B}_J \subset \mathbb{B}$ which is the convex cone of Bernstein functions which take the form

(5.4)
$$\phi(u) = \nu + \frac{\sigma^2}{2}u + \int_0^\infty (1 - e^{-ur})\overline{\mu}(r) dr$$

where $\nu, \sigma \geqslant 0$ and $\overline{\mu}$ is a non-negative and non-increasing function on \mathbb{R}^+ such that $\int_0^\infty \inf(1, r)\overline{\mu}(r) dr < \infty$. We refer to the monograph [54] for an excellent account on all these sets of functions.

Next, for a Radon measure μ on $(0, \infty)$ and $\theta \ge 0$, let

$$\overline{\mu}_{\theta}(r) := \int_{r}^{\infty} e^{\theta(r-s)} \mu(\mathrm{d}s), \quad r > 0.$$

Letting \mathcal{M}_{θ} be the set of all functions $\bar{\mu}_{\theta}$, where μ ranges over Radon measures, such that $\int_{0}^{\infty} \min(1, r) \bar{\mu}_{\theta}(r) dr < \infty$, we define, in relation to the van Dantzig problem, the following set of Bernstein functions:

$$(5.5) \quad \mathbb{B}_{\mathbb{D}} = \left\{ \phi \in \mathbb{B}; \mu(\mathrm{d}r) = \overline{\mu}_{\theta}(r) \,\mathrm{d}r, \text{ with } \overline{\mu}_{\theta}(r) \in \mathcal{M}_{\theta}, 0 \leqslant \theta \leqslant \frac{1}{2} \right\}.$$

Note first that when $\theta > 0$ the function $\bar{\mu}_{\theta}$ may fail to be monotone, for a suitable choice of μ . Note also that when $\theta = 0$ we have $\bar{\mu}_0(r) = \int_r^{\infty} \mu(\mathrm{d}s)$ is simply a non-negative and non-increasing function that satisfies the integrability condition above, meaning that \mathbb{B}_J is a strict subset of $\mathbb{B}_{\mathbb{D}}$.

We have the following.

Proposition 5.2.

(1) There exists a one-to-one mapping between the sets $\mathbb{N}_{\mathbb{D}}$ and $\mathbb{B}_{\mathbb{D}}$. More specifically, for any $\Psi \in \mathbb{N}_{\mathbb{D}}$ of the form (4.1), we have, for any $u \geqslant 0$,

(5.6)
$$\Psi(u) = (u - \theta)\phi(u)$$

where, with $\nu_{\theta} = \frac{\kappa}{\theta} \mathbb{I}_{\{\theta>0\}} + \Psi^{(1)}(0^+) \mathbb{I}_{\{\theta=0\}}$, we have set $\phi(u) = \nu_{\theta} + \frac{1}{2}\sigma^2 u + \int_0^{\infty} (1 - e^{-ur}) \overline{\mu}_{\theta}(r) dr \in \mathbb{B}_{\mathbb{D}}$. Moreover, with such a notation, we have, for all $z \in \mathbb{C}$,

(5.7)
$$\mathcal{J}_{\Psi}(z) = \mathcal{J}_{\phi,\theta}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(1-\theta)}{\Gamma(n+1-\theta) W_{\phi}(n+1)} z^{2n}.$$

- (2) Let us define the mapping $\mathcal{T}_{\beta}f(u) = \frac{u}{u+\beta}f(u+\beta)$. Then, $(\mathcal{T}_{\beta})_{\beta\geqslant 0}$ form a semigroup, i.e., $\mathcal{T}_{\beta}\circ\mathcal{T}_{\gamma} = \mathcal{T}_{\beta+\gamma}$, and, for any $\beta\geqslant 0$, $\mathcal{T}_{\beta}(\mathbb{N})\subset\mathbb{N}$ and $\mathcal{T}_{\beta}(\mathbb{N}_{\mathbb{D}})\subset\mathbb{N}_{\mathbb{D}}$.
- (3) Let us write $\overline{\mathcal{T}}_{\beta}f(u) = \frac{u-\beta}{u+\beta}f(u+\beta)$. Then, for any $\beta \geqslant \theta$, $\overline{\mathcal{T}}_{\beta}(\mathbb{N}) \subset \mathbb{N}$ and for any $\theta \leqslant \beta \leqslant \frac{1}{2}$, $\overline{\mathcal{T}}_{\beta}(\mathbb{N}_{\mathbb{D}}) \subset \mathbb{N}_{\mathbb{D}}$.

Proof. — Let $\Psi \in \mathbb{N}_{\mathbb{D}}$. Then, by means of the Wiener–Hopf factorization of Lévy–Khintchine exponents of spectrally negative Lévy process, see e.g. [27], there exists $\phi \in \mathbb{B}$ such that, for all $u \geqslant 0$,

(5.8)
$$\Psi(u) = (u - \theta)\phi(u)$$

where we recall that $\theta = \sup\{u \geq 0; \Psi(u) = 0\}$, see (4.4). In order to characterize ϕ , we write $\bar{\nu}_{\theta} = a + \sigma^2 \theta + \int_0^1 (1 - e^{-\theta r}) r \mu(dr) - \int_1^\infty e^{-\theta r} \bar{\mu}_{\theta}(r) dr$ and, observe first that

$$\Psi(u+\theta) = (\bar{\nu}_{\theta} + \int_{1}^{\infty} e^{-\theta r} \bar{\mu}_{\theta}(r) dr) u + \frac{1}{2} \sigma^{2} u^{2}$$
$$- \int_{0}^{\infty} (1 - e^{-ur} - ur \mathbb{I}_{\{r < 1\}}) e^{-\theta r} \mu(dr)$$
$$= u \left(\bar{\nu}_{\theta} + \frac{1}{2} \sigma^{2} u + \int_{0}^{\infty} (1 - e^{-ur}) e^{-\theta r} \bar{\mu}_{\theta}(r) dr \right)$$

and, thus

$$\Psi(u) = (u - \theta) \left(\nu_{\theta} + \frac{1}{2} \sigma^2 u + \int_0^{\infty} (1 - e^{-ur}) \overline{\mu}_{\theta}(r) dr \right)$$

which, after recalling that $-\theta\phi(0) = \Psi(0) = -\kappa$ and $\Psi^{(1)}(0^+) = \phi(0)$ if $\theta = 0$, completes the proof of the first item as the rest follows at once. Next, the claim of item (2) in the case $\beta = 0$ is obvious and thus we

assume that $\beta > 0$. In [11], see also [43, Proposition 2.1], it is shown that $(\mathcal{T}_{\beta})_{\beta \geqslant 0}$ is a semigroup and the set N is invariant under the action of the mapping $\mathcal{T}_{\beta}, \beta \geqslant 0$, that is $\mathcal{T}_{\beta}(\mathbb{N}) \subset \mathbb{N}$. Thus, it simply remains to show that $\mathcal{T}_{\beta}(\mathbb{N}_{\mathbb{D}}) \subset \mathbb{N}_{\mathbb{D}}$ for all $\beta \geq 0$. Let $\Psi \in \mathbb{N}_{\mathbb{D}}$. Then, note that $\mathcal{T}_{\beta}\Psi(0)=0$, i.e. $\kappa=0$ in (4.1), and, there exists $0\leqslant\theta\leqslant\frac{1}{2}$ such that $\Psi(\theta) = 0$. Suppose, first, that $\beta \geqslant \theta$, then recalling that Ψ is non-decreasing on $[\theta, \infty)$ with $\Psi(\theta) \ge 0$, we get the claim, from (4.7), by observing that $(\mathcal{T}_{\beta}\Psi)^{(1)}(0) = \frac{\Psi(\beta)}{\beta} \geqslant 0$. Next, assuming that $0 < \beta < \theta$, one easily sees that $\mathcal{T}_{\beta}\Psi(\theta-\beta) = \frac{\theta-\beta}{4}\Psi(\theta) = 0$ and since $0 < \theta - \beta < \frac{1}{2}$, we deduce that also $\mathcal{T}_{\beta}\Psi \in \mathbb{N}_{\mathbb{D}}$, completing the proof of this item as the semigroup property is obvious. For the last claim, first observe that, $\overline{\mathcal{T}}_{\beta}\Psi(u)=\frac{u+\beta-2\beta}{u+\beta}\Psi(u+\beta)$ $(\beta) + \Psi(\beta) - \Psi(\beta) = \mathcal{T}_{2\beta,\beta}\Psi(u) - \Psi(\beta)$ (the last identity serves to fix a notation), and then we know from [11, Proposition 2.2], that $\mathcal{T}_{2\beta,\beta}(\mathbb{N}) \subset \mathbb{N}$ and we obtain the statement since $\Psi(\beta) \ge 0$ as we choose $\beta \ge \theta$. Finally, if $\Psi \in \mathbb{N}_{\mathbb{D}}$, then, from (4.3), Ψ is non-negative on $\left[\frac{1}{2}, \infty\right)$ and, from the preceding discussion, $\overline{\mathcal{T}}_{\beta}\Psi \in \mathbb{N}$ with $\overline{\mathcal{T}}_{\beta}\Psi(\frac{1}{2}) = \frac{\frac{1}{2}-\beta}{\frac{1}{2}+\beta}\Psi(\frac{1}{2}+\beta) \geqslant 0$.

5.2.2. Some analytical properties of the entire functions in the class \mathbb{D}_P

LEMMA 5.3. — Let $\Psi \in \mathbb{N}_{\mathbb{D}}$, and, for p = 1, 2, we set $L_1^{(p)} = L^{(p)}$, where we recall that $L^{(p)}$ was defined in (3.1), and, for k = 1, 2, ..., we define

$$L_{k+1}^{(p)} = L^{(p)} \circ L_k^{(p)}.$$

(1) For any k = 1, 2, ..., we have

$$(5.9) L_{\iota}^{(1)} \mathcal{J}_{\Psi} = \mathcal{J}_{\mathcal{T}_{\iota} \Psi}$$

where we recall that the mapping \mathcal{T}_k was defined in Proposition 5.2(2).

(2) Moreover, for any k = 1, 2, ..., we have

$$(5.10) L_k^{(2)} \mathcal{J}_{\Psi} = \mathcal{J}_{\overline{\mathcal{T}}_{\frac{1}{n}}^{2k} \Psi}$$

where, $\overline{\mathcal{T}}_{\beta}^1 = \overline{\mathcal{T}}_{\beta}$ and $\overline{\mathcal{T}}_{\beta}^k = \overline{\mathcal{T}}_{\beta} \circ \overline{\mathcal{T}}_{\beta}^{k-1}$, and, the mapping $\overline{\mathcal{T}}_{\beta}$ was defined in Proposition 5.2(3).

Proof. — To prove the identity (5.9), we first let k = 1 and observe that

$$(5.11) \quad \mathcal{J}_{\Psi}^{(1)}(t) = \sum_{n=1}^{\infty} \frac{2n(-1)^n t^{2n-1}}{W_{\Psi}(n+1)} = \frac{-2t}{\Psi(1)} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{\prod_{k=1}^n \frac{k}{k+1} \Psi(k+1)} = \frac{-2t}{W_{\Psi}(2)} \mathcal{J}_{\mathcal{T}_1 \Psi}(t).$$

Moreover, differentiating the right-hand side one more time yields

$$\mathcal{J}_{\Psi}^{(2)}(0) = \lim_{t \to 0} \frac{-2}{W_{\Psi}(2)} \mathcal{J}_{\mathcal{T}_1 \Psi}(t) = \frac{-2}{W_{\Psi}(2)}$$

where we used that $\mathcal{J}_{T_1\Psi}^{(1)}(0) = 0$, which, itself, follows by applying (5.11) to $\mathcal{J}_{T_1\Psi}$. Hence $L^{(1)}\mathcal{J}_{\Psi} = \mathcal{J}_{T_1\Psi}$. To complete the proof for p = 1, we resort to an induction argument combined with the semigroup property of the mapping \mathcal{T}_k . Indeed, for any $k = 1, \ldots$, one has

$$L_{k+1}^{(1)} \mathcal{J}_{\Psi} = L^{(1)} \circ L_k^{(1)} \mathcal{J}_{\Psi} = L^{(1)} \mathcal{J}_{\mathcal{T}_k \Psi} = \mathcal{J}_{\mathcal{T}_{k+1} \Psi}.$$

On the other hand, we note that

$$\begin{split} \mathcal{J}_{\Psi}^{(2)}(t) &= \frac{-2}{W_{\Psi}(2)} \frac{\mathrm{d}}{\mathrm{d}t} \sum_{n=0}^{\infty} \frac{(-1)^n}{W_{\mathcal{T}_1 \Psi}(n+1)} t^{2n+1} \\ &= \frac{-2}{W_{\Psi}(2)} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{W_{\mathcal{T}_1 \Psi}(n+1)} t^{2n} \\ &= \frac{-2}{W_{\Psi}(2)} \sum_{n=0}^{\infty} \frac{(-1)^n}{\prod_{k=1}^n \frac{k-\frac{1}{2}}{k+\frac{1}{2}} \frac{k}{k+1} \Psi(k+1)} t^{2n} \\ &= \frac{-2}{W_{\Psi}(2)} \mathcal{J}_{\overline{\mathcal{T}}_{\frac{1}{2}} \Psi}(t) \end{split}$$

which provides the proof of the last claim for k = 1. As above invoking an induction argument yields

$$L_{k+1}^{(2)}\mathcal{J}_{\Psi}=L^{(2)}\circ L_{k}^{(2)}\mathcal{J}_{\Psi}=L^{(2)}\mathcal{J}_{\overline{\mathcal{T}}_{\frac{1}{k}}^{2k}\Psi}=\mathcal{J}_{\overline{\mathcal{T}}_{\frac{1}{k}}^{2k+2}\Psi}$$

which completes the proof.

We proceed with the following result regarding the order and the type of the entire function \mathcal{J}_{Ψ} . It is a slight refinement of a result obtained recently by Bartholmé and Patie [1] in the case $\theta=0$, and, for the entire function $\mathcal{I}_{\Psi}(\sqrt{t})=\mathcal{J}_{\Psi}(i\sqrt{t})$, this later transformation affects simply the order by a factor of 2. To deal with the general case $\theta>0$, it is not difficult to see that both the order and type remain the same when replacing the term n! by $C\Gamma(n+1-\theta), C>0$, in the coefficients of a power series. We left the details of the easy modification to the reader and recall that a measurable function ℓ on \mathbb{R}^+ is said to be slowly varying at infinity if, for every u>0, $\lim_{t\to\infty}\frac{\ell(ut)}{\ell(t)}=1$.

PROPOSITION 5.4 ([1, Proposition 2.1]). — Let $\Psi \in \mathbb{N}_{\mathbb{D}}$. Then, \mathcal{J}_{Ψ} is an entire function of order

(5.12)
$$\rho_{\Psi} = \frac{2}{\Psi} \in [1, 2]$$

where $\underline{\Psi} = \sup\{a > 0; \lim_{u \to \infty} u^{-a}\Psi(u) = \infty\} = \lim\inf_{u \to \infty} \frac{\ln \Psi(u)}{\ln u} \in [1, 2]$, is its so-called Blumenthal–Getoor lower index. Moreover, its type τ_{Ψ} is given by

$$(5.13) \quad \tau_{\Psi} = \underline{\Psi}e^{-\frac{\underline{\Psi}-1}{\underline{\Psi}}} \lim_{n \to \infty} e^{-\frac{1}{\underline{\Psi}} \left(\int_{1}^{n} \frac{\ln \phi(u)}{n} du - (\underline{\Psi}-1) \ln n \right)} \geqslant \left(\overline{\lim}_{n \to \infty} \frac{n^{\underline{\Psi}-1}}{\phi(n)} \right)^{\frac{1}{\underline{\Psi}}}$$

where we recall that $\Psi(u) = (u - \theta)\phi(u)$. In particular, $\rho_{\Psi} = 1$ if $\Psi(u) = \frac{u^2}{\ell(u)}$, and, otherwise, $\rho_{\Psi} = \frac{2}{\mathfrak{a}}$, $\mathfrak{a} \in [1,2)$, if $\Psi(u) = u^{\mathfrak{a}}\ell(u)$ where, in both cases, ℓ is a slowly varying function, and, $\rho_{\Psi} = 2$ also if $\int_0^{\infty} r\mu(\mathrm{d}r) < \infty$.

5.2.3. Useful bounds for the Bernstein-gamma functions

For a Bernstein function $\phi \in \mathbb{B}$, we write $z_{\phi} = \inf\{u > 0; \phi(-u) = 0\} \in [0, \infty]$ and $e_{\phi} = \sup\{u > 0; \int_{1}^{\infty} e^{ur} \mu(\mathrm{d}r) < \infty\} \in [0, \infty]$ and set

$$a_{\phi} = \min(z_{\phi}, e_{\phi}).$$

We also denote by W_{ϕ} the so-called Bernstein-gamma function which admits the following generalized Weierstrass product representation

$$W_{\phi}(z) = \frac{e^{-\gamma_{\phi}z}}{\phi(z)} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)}z}$$

where

$$\gamma_{\phi} = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{\phi'(k)}{\phi(k)} - \log \phi(n) \right).$$

From [44, Theorem 4.1], we know that W_{ϕ} defines a zero-free and analytic function on the right-half plane $\Re(z) > -a_{\phi}$, which admits a meromorphic extension to the right-half plane $\Re(z) > -e_{\phi}$. We mention that when $\phi(z) = z$, then W_{ϕ} boils down to the gamma function as the infinite product above corresponds to its Weierstrass representation and γ_{ϕ} is the Euler–Mascheroni constant. This class of functions has been thoroughly studied in the papers by Patie and Savov [44, 45]. Thereout, it has been shown that $W_{\phi} \in \mathbb{P}_{+}$ and it is the unique element in \mathbb{P}_{+} solution to the functional equation $W_{\phi}(z+1) = \phi(z)W_{\phi}(z), W_{\phi}(1) = 1$. We also find in [44] the following result which will be useful in the sequel.

PROPOSITION 5.5 ([44, Proposition 6.2]). — Let $\phi \in \mathbb{B}$ and write $\overline{\mu}(r) = \int_r^{\infty} \mu(\mathrm{d}s), \ r > 0$. Then, the following estimates holds.

(1) If $\sigma^2 > 0$ we have, for any $\epsilon, a > 0$ such that $\int_0^\infty e^{-ar} \overline{\mu}(r) dr < 1$, as $|b| \to \infty$,

(5.14)
$$\left| \frac{\Gamma(a+ib)}{W_{\phi}(a+ib)} \right| \leqslant C|b|^{-\frac{2}{\sigma^2}(\overline{\mu}(1/|b|)+\phi(0))+\epsilon}$$

where C > 0.

(2) If $\sigma^2 = 0$ then, for any $u \ge 0$ and a > 0 fixed,

(5.15)
$$\lim_{|b| \to \infty} |b|^u \left| \frac{\Gamma(a+ib)}{W_{\phi}(a+ib)} \right| = 0.$$

5.2.4. Characterization and properties of the van Dantzig pair of random variables

To define the van Dantzig pair of random variables that appear in Theorem 4.1, we need to introduce some notations. Let $\Psi \in \mathbb{N}$ and recall that it is the Laplace exponent of a spectrally negative Lévy process Y, see (4.2). Then, according to Lamperti [28], there exists a 1-self-similar Markov process on $(0, \infty)$ denoted by $X = (X_t)_{t \geq 0}$ such that

(5.16)
$$X_t = e^{Y_{A_t}}, \quad 0 \le t < \xi = \inf\{t > 0; X_t = 0\},$$

where $A_t = \inf \{s > 0; \int_0^s e^{Y_u} du > t \}$. Moreover, if $\theta = 0$, i.e. the conditions (4.4) are not fulfilled, then $\xi = \infty$ almost surely and X is in fact a Feller process on $[0, \infty)$. On the other hand, if $0 < \theta < 1$, then ξ is finite almost surely but there exists an unique 1-self-similar extension of the process X which is also a Feller process on $[0, \infty)$ and with the path property to leave the recurrent boundary point 0 continuously, see [49]. From now on, if $0 \le \theta < 1$, we denote by $X(\Psi) = (X_t(\Psi))_{t \ge 0}$ the realization of the 1-self-similar Feller semigroup $(P_t^{\Psi})_{t \ge 0}$ on $[0, \infty)$ (that is the recurrent extension if $\theta > 0$) associated to Y and let T_{Ψ} be a random variable whose distribution is that of

$$\inf\{t > 0; \ X_t(\Psi) \geqslant 1\},\$$

conditional on $X(\Psi)$ starting from 0. We also need to introduce the exponential functional of a subordinator, a positive random variable which has been intensively studied over the last two decades and refer to [43] for a nice historical account and a thorough study. Let $\phi \in \mathbb{B}$ and let

$$I_{\phi} = \int_0^{\infty} e^{-\mathbf{Z}_t} \, \mathrm{d}t$$

where $(\mathbf{Z}_t)_{t\geq 0}$ is a subordinator such that $\mathcal{F}_{\mathbf{Z}_t}(iu) = e^{-\phi(u)t}, u, t \geq 0$. Next, we recall, from [9], the following expression for the integer moments of \mathbf{I}_{ϕ} ,

(5.18)
$$\mathcal{M}_{I_{\phi}}(n) = \frac{n!}{W_{\phi}(n+1)}, \quad n = 0, 1, \dots,$$

and the notation of the associated Markov operator, see (3.9),

(5.19)
$$\Lambda_{\mathrm{I}_{\phi}} f(t) = \int_{0}^{\infty} f(xt) F_{\mathrm{I}_{\phi}}(\mathrm{d}x).$$

We are now ready to state the following result which can be found in [40]. To emphasize the role played by the concept of intertwining in the van Dantzig problem, we provide another original proof. Intertwining, in our context, refers to the relation (5.22) below. To state the theorem, recall the notation

$$\mathcal{J}_{-\theta}(x) = \Gamma(1-\theta)t^{\theta}J_{-\theta}(2x), \quad \mathcal{I}_{-\theta}(x) = \Gamma(1-\theta)t^{\theta}I_{-\theta}(2x),$$

appearing in (4.19), where $J_{-\theta}$ (respectively $I_{-\theta}$) is the ordinary (respectively modified) Bessel function of the first kind of order $-\theta$.

THEOREM 5.6 ([40, Theorem 2.1]). — Let $\Psi \in \mathbb{N}$ with $0 \leq \theta < 1$. Then, writing $\widetilde{\mathcal{I}}_{\Psi}(x) = \mathcal{I}_{\Psi}(\sqrt{x})$ and $\widetilde{\mathcal{I}}_{-\theta}(x) = \mathcal{I}_{-\theta}(\sqrt{x})$, we have

(5.20)
$$\Lambda_{I_{\phi}} \widetilde{\mathcal{I}}_{-\theta}(x) = \widetilde{\mathcal{I}}_{\Psi}(x), \quad x > 0,$$

where $\phi(u) = \frac{\Psi(u)}{u-\theta} \in \mathbb{B}$. Moreover, T_{Ψ} is a positive self-decomposable random variable, and, for any u > 0, we have

(5.21)
$$\mathcal{F}_{\mathrm{T}_{\Psi}}(iu) = \frac{1}{\mathcal{I}_{\Psi}(\sqrt{u})} = e^{-\phi_{\Psi}(u)}$$

where ϕ_{Ψ} is the Bernstein function defined in Theorem 4.1.

Proof. — Let $\Psi \in \mathbb{N}$ with $0 \le \theta < 1$. We recall that in [45] for the case $\theta = 0$, and in [46] for the case $0 < \theta < 1$, the following intertwining relation has been identified

$$(5.22) P_t^{\Psi} \Lambda_{\mathbf{I}_{\phi}} = \Lambda_{\mathbf{I}_{\phi}} P_t^{\Psi_{\theta}}$$

where $\Psi_{\theta}(u) = u(u - \theta) \in \mathbb{N}$. Then, using successively Tonelli theorem and the identity (5.18), we obtain that, for any x > 0,

$$\begin{split} \Lambda_{\mathrm{I}_{\phi}} \widetilde{\mathcal{I}}_{-\theta}(x) &= \sum_{n=0}^{\infty} \frac{\mathcal{M}_{\mathrm{I}_{\phi}}(n) \Gamma(1-\theta)}{n! \Gamma(n+1-\theta)} x^n \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(1-\theta)}{W_{\phi}(n+1) \Gamma(n+1-\theta)} x^n = \widetilde{\mathcal{I}}_{\Psi}(x). \end{split}$$

It is well-known that the mapping $x\mapsto d_q\widetilde{\mathcal{I}}_{-\theta}(x)=\widetilde{\mathcal{I}}_{-\theta}(qx), q>0$, is a q-invariant for the squared Bessel semigroup P^{Ψ_θ} , that is, for all $t\geqslant 0$, $e^{-qt}P_t^{\Psi_\theta}d_q\widetilde{\mathcal{I}}_{-\theta}(x)=d_q\widetilde{\mathcal{I}}_{-\theta}(x)$. Now observing that $d_q\Lambda_{\mathrm{I}_\phi}=\Lambda_{\mathrm{I}_\phi}d_q$, we deduce from the intertwining relation above that

$$\begin{split} P_t^{\Psi} d_q \widetilde{\mathcal{I}}_{\Psi}(x) &= P_t^{\Psi} d_q \Lambda_{\mathcal{I}_{\phi}} \widetilde{\mathcal{I}}_{-\theta}(x) = P_t^{\Psi} \Lambda_{\mathcal{I}_{\phi}} d_q \widetilde{\mathcal{I}}_{-\theta}(x) \\ &= \Lambda_{\mathcal{I}_{\phi}} P_t^{\Psi\theta} d_q \widetilde{\mathcal{I}}_{-\theta}(x) = e^{qt} \Lambda_{\mathcal{I}_{\phi}} d_q \widetilde{\mathcal{I}}_{-\theta}(x) \\ &= e^{qt} d_q \Lambda_{\mathcal{I}_{\phi}} \widetilde{\mathcal{I}}_{-\theta}(x) = e^{qt} d_q \widetilde{\mathcal{I}}_{\Psi}(x), \end{split}$$

which shows, since it is positive, that $d_q \widetilde{\mathcal{I}}_{\Psi}, q > 0$, is a q-invariant function for the semigroup P^{Ψ} . Then, noting that $\widetilde{\mathcal{I}}_{\Psi}(0) = 1$, an application of Dynkin's formula to the bounded stopping $T_{\Psi}(t) = \inf(T_{\Psi}, t)$, yields, for all $t \geq 0$, that

$$e^{-qt} P_{\mathcal{T}_{\Psi}(t)}^{\Psi} d_q \widetilde{\mathcal{I}}_{\Psi}(0) = 1$$

where $(P_{T_{\Psi}(t)}^{\Psi})_{t\geqslant 0}$ is the semigroup corresponding to the process $(X_{T_{\Psi}(t)})_{t\geqslant 0}$. Differentiating term by term the series $\widetilde{\mathcal{I}}_{\Psi}$, we observe that $x\mapsto d_q\widetilde{\mathcal{I}}_{\Psi}(x)$ is, for all q>0, non-decreasing on \mathbb{R}^+ . This allows us to invoke a dominated convergence argument, while combined with the absence of positive jumps of $X(\Psi)$, which entails that $X_{T_{\Psi}}(\Psi)=1$ almost surely, gives the first identity in (5.21). The rest of the statement, that is, T_{Ψ} is a positive self-decomposable variable is justified in [40].

We are ready to state the following result that identifies the first random variable of our pair of solutions to the van Dantzig problem.

LEMMA 5.7. — Let $\Psi \in \mathbb{N}$ with $0 \le \theta < 1$ and $B = (B_t)_{t \ge 0}$ be a standard Brownian motion independent of $X(\Psi)$. Then, the random variable $\overline{D}_{\Psi} = \sqrt{2}B_{T_{\Psi}}$ is real-valued, symmetric, and infinitely divisible. Moreover, for any $t \in \mathbb{R}$,

(5.23)
$$\mathcal{F}_{\overline{D}_{\Psi}}(t) = \frac{1}{\mathcal{I}_{\Psi}(t)}.$$

Proof. — According to Theorem 5.6, T_{Ψ} is self-decomposable and hence infinitely divisible. Since B is a symmetric Lévy process, we have, by Bochner subordination, see [53, Theorem 30.1], that \overline{D}_{Ψ} is real-valued, symmetric, and infinitely divisible. Next, using the independence of B and $X(\Psi)$ and hence of T_{Ψ} , one gets, for any $t \in \mathbb{R}$,

$$\mathcal{F}_{\overline{\mathrm{D}}_{\Psi}}(t) = \mathcal{F}_{\mathrm{T}_{\Psi}}\left(it^{2}\right) = \frac{1}{\mathcal{I}_{\Psi}(t)}$$

where the last line follows from (5.21).

To characterize the second random variable, we need to introduce the random variable J_{ν} , $\nu \in (-\infty, \frac{1}{2}]$, whose law is, when $\nu < -\frac{1}{2}$, absolutely continuous with a density $f_{J_{\nu}}$ given by

(5.24)
$$f_{J_{\nu}}(x) = \frac{2^{2\nu} \Gamma(1-\nu)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\nu)} (4-x^2)^{-\nu-\frac{1}{2}} \mathbb{I}_{\{|x|\leqslant 2\}}$$

and, otherwise for $\nu = -\frac{1}{2}$, J_{ν} has the Bernoulli distribution $\frac{1}{2}(\delta_2 + \delta_{-2})$. We note that the characteristic function corresponding to the random variable J_{ν} is the function $\mathcal{J}_{-\nu}(t) = \Gamma(1-\nu)t^{\nu}J_{-\nu}(2t)$, see [2, p. 38]. We are now ready to state the following which defines a new class of random variables indexed by the whole set of Bernstein functions, and, when restricted to the subet $\mathbb{B}_{\mathbb{D}}$ gives the other set of van Dantzig variables.

LEMMA 5.8. — Let $\phi \in \mathbb{B}$ and define, for any $\nu \leqslant -\frac{1}{2}$, the random variable

$$D_{\phi,\nu} = \sqrt{I_{\phi}} \times J_{\nu}$$

where J_{ν} is chosen independent of I_{ϕ} . Then, $D_{\phi,\nu}$ is a symmetric random variable taking values in the possibly infinite interval $\left(-\frac{2\sqrt{2}}{\sigma},\frac{2\sqrt{2}}{\sigma}\right)$ (we use the convention $\frac{1}{0}=\infty$). Moreover, all its even moments exist and are given, for any $\nu<-\frac{1}{2}$ (resp. $\nu=-\frac{1}{2}$), by

(5.25)
$$\mathcal{M}_{D_{\phi,\nu}}(2n) = \frac{\Gamma(2n+1)\Gamma(1-\nu)}{W_{\phi}(n+1)\Gamma(n+1-\nu)} \text{ (resp. = 2}^{2n}), \quad n = 0, 1, \dots,$$

and, for any $t \in \mathbb{R}$, we have

(5.26)
$$\mathcal{F}_{D_{\phi,\nu}}(t) = \mathcal{J}_{\phi,\nu}(t)$$

where the entire function $\mathcal{J}_{\phi,\nu}$ is defined as in (5.7). Finally, the law of $D_{\phi,\nu}$ is absolutely continuous with a density $f_{D_{\phi,\nu}}$ which is continuous on \mathbb{R} and $f_{D_{\phi,\nu}} \in C_0^{\infty}(\mathbb{R} \setminus \{0\})$ (resp. $f_{D_{\phi,\nu}} \in C_0^p(\mathbb{R} \setminus \{0\})$), where $p = \lceil \frac{2}{\sigma^2} (\overline{\mu}(0^+) + \phi(0)) - \nu - \frac{1}{2} \rceil$) if $\sigma^2 = 0$ or $\overline{\mu}(0^+) = \infty$ (resp. otherwise), such that $f_{D_{\phi,\nu}}(x) = f_{D_{\phi,\nu}}(-x)$, and, for any $n \in \mathbb{N}$ (resp. $n = 0, \ldots, p$), x > 0 and $a > \frac{1}{2} + n$,

(5.27)
$$f_{D_{\phi,\nu}}^{(n)}(x) = (-1)^n \frac{\Gamma(1-\nu)}{2\pi i} \times \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(z)\Gamma(2z-1-2n)}{\Gamma(z-n)W_{\phi}(z-n)\Gamma(z-\nu-n)} dz.$$

Proof. — First, note that the symmetry property of J_{ν} entails the one of $D_{\phi,\nu}$. Since the support of J_{ν} is [-2,2] (or $\{-2,2\}$ when $\nu=\frac{1}{2}$) and the one of I_{ϕ} is $\left[0,\frac{2}{\sigma^2}\right]$, see e.g. [44, Theorem 2.4], we deduce readily the one of

 $D_{\phi,\nu}$. Being symmetric, only the even moments are non-zero and are given, for any $n=0,1,\ldots$, by

$$\mathcal{M}_{\mathrm{D}_{\phi,\nu}}(2n) = \mathcal{M}_{\mathrm{J}_{\nu}}(2n)\mathcal{M}_{\mathrm{I}_{\phi}}(n)$$

where we used that the random variables are independent. Using the identity (5.18) and, observing, from the duplication formula of the gamma function, that, for $\nu < -\frac{1}{2}$,

(5.28)
$$\int_{-2}^{2} x^{2n} f_{J_{\nu}}(x) dx = \frac{\Gamma(1-\nu)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\nu)} 2^{2n} \int_{0}^{1} y^{n-\frac{1}{2}} (1-y)^{-\nu-\frac{1}{2}} dy$$
$$= 2^{2n} \frac{\Gamma(1-\nu)\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1-\nu)} = \frac{\Gamma(1-\nu)(2n)!}{\Gamma(n+1-\nu)n!}$$

we derive easily the identity (5.25). The case $\nu = -\frac{1}{2}$ follows easily and from the identity before (5.28), the expression of moment is continuous (from below) in ν . Next, since, see e.g. [21, Theorem 1], for any $t \in \mathbb{R}$,

(5.29)
$$\mathcal{F}_{J_{\nu}}(t) = \mathcal{J}_{-\nu}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(1-\nu)}{\Gamma(n+1-\nu)n!} t^{2n}$$

and recalling, from [44], that the law of I_{ϕ} is absolutely continuous, the independence of I_{ϕ} and J_{ν} yields, for any $t \in \mathbb{R}$, that

$$\mathcal{F}_{\mathcal{D}_{\phi,\nu}}(t) = \int_0^\infty \mathcal{F}_{\mathcal{J}_{\nu}}(\sqrt{x}t) f_{\mathcal{I}_{\phi}}(x) \, \mathrm{d}x$$

$$= \int_0^\infty \mathcal{J}_{\nu}(\sqrt{x}t) f_{\mathcal{I}_{\phi}}(x) \, \mathrm{d}x$$

$$= \sum_{n=0}^\infty \frac{(-1)^n \Gamma(1-\nu) \mathcal{M}_{\mathcal{I}_{\phi}}(n)}{\Gamma(n+1-\nu)n!} t^{2n}$$

$$= \mathcal{J}_{\phi,\nu}(t)$$

where to justify the interchange of sums in the third equality, we resort to a classical Fubini's argument whose details are described in [58, Section 1.77], and relies on the fact that the series $\mathcal{J}_{\phi,\nu}$ is absolutely convergent on \mathbb{R} . For the last identity we used (5.18). Next, it is not difficult to see that the computation (5.28) extends to any complex z such that $\Re(z) > -\frac{1}{2}$, and, from [44, Theorem 2.4], the expression (5.18) also extends (at least) to $\Re(z) > -1$, to get, for $\Re(z) > -\frac{1}{2}$,

$$\mathcal{M}_{\mathrm{D}_{\phi,\nu}^2}(z) = \frac{\Gamma(2z+1)\Gamma(1-\nu)}{W_{\phi}(z+1)\Gamma(z+1-\nu)},$$

which proves (5.25). Then, recalling the Stirling formula of the gamma function, for any a > 0 and |b| large,

(5.30)
$$|\Gamma(a+ib)| \sim Ce^{-|b|\frac{\pi}{2}}|b|^{a-\frac{1}{2}}$$

where, here and below, C > 0 is a generic constant, we deduce, that, for any $a > -\frac{1}{2}$ and |b| large,

$$\left| \frac{\Gamma(2a+1+2ib)}{\Gamma(a+1+ib)\Gamma(a+1-\nu+ib)} \right| \sim C|b|^{\nu-\frac{1}{2}}.$$

This combines with (5.14) when $\sigma^2 > 0$ and $\bar{\mu}(0^+) < \infty$, gives for any $\epsilon > 0$ and $a > -\frac{1}{2}$,

(5.31)
$$\left| \mathcal{M}_{\mathcal{D}^{2}_{\phi,\nu}}(a+ib) \right| \leqslant C|b|^{-\frac{2}{\sigma^{2}}\left(\overline{\mu}(0^{+})+\phi(0)\right)+\nu-\frac{1}{2}+\epsilon}.$$

We deduce that the mapping $b \mapsto |b|^p |\mathcal{M}_{\mathrm{D}^2_{\phi,\nu}}(a+ib)|$ is integrable on \mathbb{R} whenever $p < \frac{2}{\sigma^2} (\bar{\mu}(0^+) + \phi(0)) - \nu - \frac{1}{2}$. Then, invoking classical results on Mellin inversion, see e.g. [45, Section 1.7.4], we get the Mellin–Barnes representation of $f_{\mathrm{D}^2_{\phi,\nu}}$ which takes the form for any x > 0 and $a > \frac{1}{2}$,

$$f_{\mathcal{D}^2_{\phi,\nu}}(x) = \frac{\Gamma(1-\nu)}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^{-z} \frac{\Gamma(2z-1)}{W_{\phi}(z)\Gamma(z-\nu)} \,\mathrm{d}z.$$

Next, with $\nu < -\frac{1}{2}$, since $f_{D_{\phi,\nu}}$ is symmetric, we have that $f_{D_{\phi,\nu}}(x) = x f_{D_{\phi,\nu}^2}(x^2), x > 0$, from where we deduce the Mellin–Barnes representation of $f_{D_{\phi,\nu}}$. Next, since the mapping $z \mapsto \frac{\Gamma(2z-1)}{W_{\phi}(z)\Gamma(z-\nu)}$ is meromorphic on $(a', a' + \frac{1}{2}), 0 < a' < \frac{1}{2}$ with a simple pole at $\frac{1}{2}$, an application of the residues theorem yields

$$f_{\mathrm{D}_{\phi,\nu}}(x) = \frac{\Gamma(1-\nu)}{2W_{\phi}(\frac{1}{2})\Gamma(\frac{1}{2}-\nu)} + \frac{\Gamma(1-\nu)}{2\pi i} \int_{a'-i\infty}^{a'+i\infty} x^{1-2z} \frac{\Gamma(2z-1)}{W_{\phi}(z)\Gamma(z-\nu)} \,\mathrm{d}z$$

from where we conclude, as $a' < \frac{1}{2}$, that $\lim_{x\downarrow 0} f_{D_{\phi,\nu}}(x) = \frac{\Gamma(1-\nu)}{2W_{\phi}(\frac{1}{2})\Gamma(\frac{1}{2}-\nu)}$. Moreover, by Mellin inversion, see again [45, Section 1.7.4], and by symmetry, we get that $f \in C_0(\mathbb{R})$. From the same reference and by a similar reasoning, we obtain the expression and the smoothness properties for the successive derivatives for all $n \leq p$. The cases $\sigma = 0$ or $\overline{\mu}(0^+) = \infty$ follow easily by means of a similar reasoning and using (5.15) in place of (5.14).

We continue our program with the following observation which is the key step in proving Proposition 4.6 for the $L^{(2)}$ mapping. To state it, we recall that $X(\gamma)$, $\gamma \in \mathbb{R}$, is the γ -length-biased random variable of a non-negative random variable X, if its γ^{th} moment $m_X(\gamma)$ is finite and

$$F_{\mathbf{X}(\gamma)}(\mathrm{d}x) = \frac{x^{\gamma}}{m_{\mathbf{X}}(\gamma)} F_{\mathbf{X}}(\mathrm{d}x).$$

This notion was used in the proof of the Lukacs mapping defined in (3.5). We are now ready to state the following.

PROPOSITION 5.9. — For any $\phi \in \mathbb{B}$, we have $\mathcal{T}_1\phi(u) = \frac{u}{u+1}\phi(u+1) \in \mathbb{B}$. Moreover, with the notation of (5.18), we have, for any $t \in \mathbb{R}$,

$$\mathcal{F}_{\mathbf{I}_{\mathcal{T}_1\phi}}(t) = \mathcal{F}_{\mathbf{I}_{\phi}(1)}(t)$$

and, for any twice continuously differentiable function $\mathcal{F}_X \in \mathbb{P}_+$,

$$(5.32) L^{(2)}\Lambda_{\mathbf{I}_{\phi}}\mathcal{F}_{X} = \Lambda_{\mathbf{I}_{\phi}(1)}L^{(2)}\mathcal{F}_{X}$$

where $\Lambda_{\mathrm{I}_{\phi}}f(t)=\mathbb{E}\left[f(t\sqrt{\mathrm{I}_{\phi}})\right]$ and I_{ϕ} is chosen independent of X. In particular, for any $k=1,2,\ldots,$ and, any $t\in\mathbb{R}$, with $\nu<\frac{1}{2}$,

(5.33)
$$L_k^{(2)} \mathcal{F}_{D_{\phi,\nu}}(t) = \mathcal{F}_{J_{\nu}(2k)\sqrt{I_{\phi}(k)}}(t) = \mathcal{F}_{D_{\phi,\nu}(2k)}(t).$$

Proof. — The first statement can be found in [11]. Then, observing that $W_{\mathcal{T}_1\phi}(n+1) = \prod_{k=1}^n \frac{k}{k+1}\phi(k+1) = \frac{W_{\phi}(n+2)}{W_{\phi}(1)(n+1)}$, we deduce, from (5.18), that, for any integer n,

(5.34)
$$\mathcal{M}_{I_{\mathcal{T}_{1}\phi}}(n) = \frac{n!}{W_{\mathcal{T}_{1}\phi}(n+1)} = \frac{(n+1)!W_{\phi}(1)}{W_{\phi}(n+2)}$$
$$= \frac{\mathcal{M}_{I_{\phi}}(n+1)}{\mathcal{M}_{I_{\phi}}(1)} = \mathcal{M}_{I_{\phi}(1)}(n)$$

which yields the second claim since these random variables are moment determinate, see [9], and the characteristic function uniquely determines the law of a random variable. Next, as \mathcal{F}_X is twice continuously differentiable, one gets, for any $t \in \mathbb{R}$,

$$\begin{split} L^{(2)} \Lambda_{\mathrm{I}_{\phi}} \mathcal{F}_{X}(t) &= L^{(2)} \mathcal{F}_{X \sqrt{\mathrm{I}_{\phi}}}(t) \\ &= \frac{\mathcal{F}_{X \sqrt{\mathrm{I}_{\phi}}}^{(2)}(t)}{\mathcal{F}_{X \sqrt{\mathrm{I}_{\phi}}}^{(2)}(0)} \\ &= \frac{1}{\mathcal{F}_{\sqrt{\mathrm{I}_{\phi}}}^{(2)}(0)} \int_{0}^{\infty} \frac{\mathcal{F}_{X}^{(2)}(t \sqrt{x})}{\mathcal{F}_{X}^{(2)}(0)} x f_{\mathrm{I}_{\phi}}(x) \, \mathrm{d}x \\ &= \Lambda_{\mathrm{I}_{\phi}(1)} L^{(2)} \mathcal{F}_{X}(t) \end{split}$$

where for the third equality, we used that X and I_{ϕ} are independent and for the last one that $\mathcal{F}^{(2)}_{\sqrt{I_{\phi}}}(0) = \int_{0}^{\infty} x f_{I_{\phi}}(x) dx = \frac{1}{\phi(1)}$. To prove the last relation, we recall that $D_{\phi,\nu} = \sqrt{I_{\phi}} J_{\nu}$, where J_{θ} is chosen independent of I_{ϕ} and thus resorting to the commutation type relation (5.32), one gets, for any $t \in \mathbb{R}$,

$$\begin{split} L^{(2)}\mathcal{F}_{\mathcal{D}_{\phi,\nu}}(t) &= L^{(2)}\Lambda_{\mathcal{I}_{\phi}}\mathcal{F}_{\mathcal{J}_{\nu}}(t) = \Lambda_{\mathcal{I}_{\phi}(1)}L^{(2)}\mathcal{F}_{\mathcal{J}_{\nu}}(t) = \Lambda_{\mathcal{I}_{\phi}(1)}\mathcal{F}_{\mathcal{J}_{\nu}(2)}(t) \\ &= \mathcal{F}_{\mathcal{J}_{\nu}(2)}\sqrt{\mathcal{I}_{\mathcal{I}_{\phi}(1)}}(t) \end{split}$$

which provides the claim for k = 1. Then, an induction argument gives for any k,

$$\begin{split} L_{k+1}^{(2)} \mathcal{F}_{\mathcal{D}_{\phi,\nu}}(t) &= L^{(2)} \circ L_k^{(2)} \mathcal{F}_{\mathcal{D}_{\phi,\nu}}(t) = L^{(2)} \mathcal{F}_{\mathcal{J}_{\nu}(2k)\sqrt{\mathcal{I}_{\mathcal{I}_{\phi}(k)}}} \\ &= \mathcal{F}_{\mathcal{J}_{\nu}(2k+2)\sqrt{\mathcal{I}_{\mathcal{I}_{\phi}(k+1)}}}(t) \end{split}$$

which completes the proof.

5.2.5. End of the proof of Theorem 4.1

To complete the proof of Theorem 4.1, we take $\Psi \in \mathbb{N}_{\mathbb{D}}$, and, recall from Proposition 5.2, that there exists $\phi \in \mathbb{B}_{\mathbb{D}}$ and $0 \leqslant \theta \leqslant \frac{1}{2}$ such that $\Psi(u) = (u - \theta)\phi(u)$. Then combining Lemmas 5.7 and 5.8, we obtain, using the notation of the latter lemma, that is $D_{\phi,\theta} = \sqrt{I_{\phi}}J_{\theta}$,

$$\mathcal{F}_{\overline{D}_{\Psi}}(it)\mathcal{F}_{D_{\phi,\theta}}(t) = 1, \quad t \in \mathbb{R}.$$

Hence $[\mathcal{F}_{\overline{D}_{\Psi}}, \mathcal{F}_{D_{\phi,\theta}}]$ form a van Dantzig pair, which completes the proof of the theorem after invoking Theorem 5.6.

5.2.6. Proof of Theorem 4.4

We simply sketch the proof of the first claim as the detailed arguments can be found in [25]. Let $\phi \in \mathbb{B}_{P_1}$. Then, observing that $W_{\Psi}(n+1) = n!W_{\phi}(n+1), n \geqslant 0$, we write $F_{\phi} = \frac{1}{W_{\phi}}$, that is F_{ϕ} is a function which is analytic and zero-free on the half-plane $\Re(z) > z_1, z_1$ being the largest zero of ϕ which is simple, and, at least on this former half-plane, F_{ϕ} is solution to the recurrence equation

(5.35)
$$F_{\phi}(z+1) = \frac{1}{\phi(z)} F_{\phi}(z).$$

From this recurrence equation, we get that z_1 is also a simple zero of F_{ϕ} , and F_{ϕ} admits an analytic extension to $\Re(z) > \rho_1 > z_2$ due to the interlacing

property. Then, the 1-separation property entails that $\rho_1 = z_1 - 1$ is a simple pole of ϕ , and, hence, ρ_1 is neither a zero nor a pole of F_{ϕ} , which, yields, by the recurrence (5.35), that F_{ϕ} admits an analytic extension to $\Re(z) > z_2 = \rho_2 - 1$. An induction argument gives that F_{ϕ} is indeed an entire function with the sequence of simple zeros $(z_k)_{k\geqslant 1}$, which completes the proof of the first claim by invoking Laguerre theorem [52, Theorem 4] as $\mathcal{J}_{\Psi}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n F_{\phi}(n+1)}{n!} z^{2n}$. For the last claim, we invoke, for instance, the Mittag–Leffler function presented in example 4.2.4 which has non-real zeros.

5.2.7. Proof of Proposition 4.6

The case p=1 is simply the combination of Lemma 5.3, Proposition 5.2 and Theorem 4.1. To prove the case p=2, we take $\Psi \in \mathbb{N}_{\mathbb{D}}$ and recall that $D_{\phi,\theta} = \sqrt{I_{\phi}}J_{\theta}$, where $\Psi(u) = (u-\theta)\phi(u)$. Then, using (5.27), Proposition 5.2 and the identity (5.33), one gets, for any $k=1,2,\ldots$,

(5.36)
$$L_k^{(2)} \mathcal{J}_{\Psi}(t) = L_k^{(2)} \mathcal{F}_{D_{\phi,\theta}}(t) = \mathcal{F}_{J_{\theta}(2k)\sqrt{I_{I_{\phi}(k)}}}(t).$$

On the other hand, using (5.10), and recalling the identity $\mathcal{I}_{\Psi}(t) = \mathcal{J}_{\Psi}(e^{i\frac{\pi}{2}}t)$, one has

$$(5.37) L_k^{(2)} \mathcal{I}_{\Psi} = \mathcal{I}_{\overline{\mathcal{T}}_{\frac{1}{2}}^{2k} \Psi}.$$

However, since $\Psi \in \mathbb{N}_{\mathbb{D}}$, then $\frac{1}{2} \geqslant \theta$ and, from Proposition 5.2(3), the mapping $u \mapsto \overline{\mathcal{T}}_{\frac{1}{2}}^{2k} \Psi(u) = \overline{\mathcal{T}}_{\frac{1}{2}}^{1} \circ \overline{\mathcal{T}}_{\frac{1}{2}}^{2k-1} \Psi(u) = \frac{u-\frac{1}{2}}{u+\frac{1}{2}} \overline{\mathcal{T}}_{\frac{1}{2}}^{2k-1} \Psi(u+\frac{1}{2}) \in \mathbb{N}$ with $\overline{\mathcal{T}}_{\frac{1}{2}}^{2k} \Psi(\frac{1}{2}) = 0$. Then, $\overline{\mathcal{T}}_{\frac{1}{2}}^{2k} \Psi$ fulfills the requirement of Theorem 5.6 and hence

$$\mathcal{F}_{\mathrm{T}_{\mathcal{T}_{\frac{1}{2}}^{2k}\Psi}}(iu) = \frac{1}{\mathcal{I}_{\mathcal{T}_{\frac{1}{2}}^{2k}\Psi}(\sqrt{u})}$$

which concludes the proof after invoking Lemma 5.7.

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