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Daniel DELBOURGO

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ON THE IWASAWA μ -INVARIANT AND λ -INVARIANT ASSOCIATED TO TENSOR PRODUCTS OF NEWFORMS

by Daniel DELBOURGO

ABSTRACT. — Fix an odd prime number p , and let $\bar{\rho}_1, \dots, \bar{\rho}_t$ be a collection of two-dimensional ordinary Galois representations defined over a finite field \mathbb{F}_{p^e} . Suppose that we are given newforms f_1, \dots, f_t whose p -adic representations

$$\rho_{f_1} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathcal{K}), \dots, \rho_{f_t} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathcal{K}) \quad \text{with } \mathcal{O}_{\mathcal{K}}/\pi_{\mathcal{K}} \cong \mathbb{F}_{p^e}$$

satisfy $\rho_{f_1} \otimes \dots \otimes \rho_{f_t} \bmod \pi_{\mathcal{K}} \cong \bar{\rho}_1 \otimes \dots \otimes \bar{\rho}_t$, and some other extra hypotheses. We shall determine the cyclotomic λ -invariant for the Selmer group attached to the product $f_1 \otimes \dots \otimes f_t$ under the assumption that the μ -invariant is zero. If $t = 2$ (i.e. the double product case) this allows us to deduce the Iwasawa Main Conjecture for $f_1 \otimes f_2$ if it is already known for a congruent pair $f'_1 \otimes f'_2$, much as Greenberg and Vatsal [19] did for $t = 1$ (i.e. for elliptic cusp forms).

RÉSUMÉ. — On fixe un nombre premier impair p , et soit $\bar{\rho}_1, \dots, \bar{\rho}_t$ une famille de représentations galoisiennes ordinaires à deux dimensions définies sur un corps fini \mathbb{F}_{p^e} . On suppose données des formes modulaires primitives f_1, \dots, f_t dont les représentations p -adiques

$$\rho_{f_1} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathcal{K}), \dots, \rho_{f_t} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathcal{K}) \quad \text{avec } \mathcal{O}_{\mathcal{K}}/\pi_{\mathcal{K}} \cong \mathbb{F}_{p^e}$$

satisfont $\rho_{f_1} \otimes \dots \otimes \rho_{f_t} \bmod \pi_{\mathcal{K}} \cong \bar{\rho}_1 \otimes \dots \otimes \bar{\rho}_t$, et quelques autres hypothèses supplémentaires. On détermine l'invariant λ cyclotomique pour le groupe de Selmer associé au produit $f_1 \otimes \dots \otimes f_t$ sous l'hypothèse que l'invariant μ est nul. Si $t = 2$ (c'est-à-dire dans le cas du double produit), cela nous permet de déduire la conjecture principale d'Iwasawa pour $f_1 \otimes f_2$ si elle est déjà connue pour une paire congruente $f'_1 \otimes f'_2$, tout comme Greenberg et Vatsal [19] l'ont fait pour $t = 1$ (i.e. pour les formes modulaires paraboliques).

1. History and motivation

Let E be an elliptic curve defined over \mathbb{Q} , and p a prime of good ordinary reduction. For an algebraic extension M/\mathbb{Q} , we write $E(M)$ for its group

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of M -rational points. The Selmer group of E over M sits inside the short exact sequence

$$0 \rightarrow E(M) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow \text{Sel}(E/M) \rightarrow \text{III}(E/M) \rightarrow 0$$

where $\text{III}(E/M)$ denotes the Tate–Shafarevich group for E over the extension M . In the case where M is a number field, it is conjectured that $\text{III}(E/M)$ is finite.

Let \mathbb{Q}^{cyc} be the cyclotomic \mathbb{Z}_p -extension of \mathbb{Q} , with $\Gamma^{\text{cyc}} := \text{Gal}(\mathbb{Q}^{\text{cyc}}/\mathbb{Q}) \cong \mathbb{Z}_p$. Then the p -primary part of $\text{Sel}(E/\mathbb{Q}^{\text{cyc}})$ has the structure of a discrete Λ -module where $\Lambda = \Lambda^{\text{cyc}} := \mathbb{Z}_p[[\Gamma^{\text{cyc}}]]$ denotes the completed group algebra for Γ^{cyc} over \mathbb{Z}_p . In the 1990s, Kato [24] proved the deep result that the Pontragin dual module

$$\mathcal{X}_E := \text{Hom}_{\text{cont}} \left(\text{Sel}(E/\mathbb{Q}^{\text{cyc}})[p^\infty], \mathbb{Q}_p/\mathbb{Z}_p \right)$$

has rank zero over the Iwasawa algebra Λ . By the structure theory of compact finitely-generated torsion Λ -modules, there exists a pseudo-isomorphism

$$\mathcal{X}_E \stackrel{\text{pseudo}}{\cong} \bigoplus_{i=1}^n \frac{\Lambda}{F_i^{\lambda_i} \cdot \Lambda} \oplus \bigoplus_{j=1}^m \frac{\Lambda}{p^{\mu_j} \cdot \Lambda}$$

and the $F_i(X)$'s are distinguished polynomials under the isomorphism $\Lambda \xrightarrow{\sim} \mathbb{Z}_p[[X]]$ sending a topological generator $\gamma_0 \in \Gamma^{\text{cyc}}$ to the element $1 + X$. The two integers

$$\mu^{\text{alg}}(E) = \mu_1 + \cdots + \mu_m \quad \text{and} \quad \lambda^{\text{alg}}(E) = \lambda_1 \cdot \deg(F_1) + \cdots + \lambda_n \cdot \deg(F_n)$$

are called the algebraic μ -invariant and algebraic λ -invariant of E , respectively.

Fix a pair of embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \widehat{\overline{\mathbb{Q}}}_p$. On the analytic side Mazur, Tate and Teitelbaum [31] constructed a p -adic L -function $\mathbf{L}_p^{\text{an}}(E) \in \Lambda[1/p]$, interpolating at finite order characters $\chi : \Gamma^{\text{cyc}} \rightarrow \mathbb{C}^\times$ of conductor p^{n_χ} the data

$$\chi(\mathbf{L}_p^{\text{an}}(E)) = \iota_p \circ \iota_\infty^{-1} \left(\left(1 - \frac{\chi(p)}{\alpha_p} \right)^2 \cdot \frac{\mathfrak{G}_{\overline{\chi}}}{\alpha_p^{n_\chi}} \times \frac{L(E, \chi, 1)}{\Omega_\infty^+(E)} \right).$$

Here $\Omega_\infty^+(E)$ equals the real Néron period attached to the curve, α_p is the unit root of $X^2 - a_p(E)X + p$, and $\mathfrak{G}_\chi = \sum_{j=1}^{p^{n_\chi}} \chi(n) e^{2\pi i j / p^{n_\chi}}$ denotes a Gauss sum for χ . The analytic μ -invariant $\mu^{\text{an}}(E) \in \mathbb{Z}$ is defined to be the largest power of p dividing all the coefficients of $\mathbf{L}_p^{\text{an}}(E)$, while the analytic λ -invariant $\lambda^{\text{an}}(E) \geq 0$ counts the number of zeros (with multiplicity) of the p -adic L -function on the open unit disk.

CONJECTURE. — *The Iwasawa Main Conjecture for E then predicts an equality*

$$\mathbf{L}_p^{\text{an}}(E) = \mathbf{u}_E \cdot \text{char}_\Lambda(\mathcal{X}_E)$$

for some element $\mathbf{u}_E \in \Lambda^\times$, where one defines $\text{char}_\Lambda(\mathcal{X}_E) := p^{\mu^{\text{alg}}(E)} \times \prod_{i=1}^n F_i^{\lambda_i}$.

A simple consequence of this statement is that one must have $\mu^{\text{alg}}(E) = \mu^{\text{an}}(E)$ and likewise $\lambda^{\text{alg}}(E) = \lambda^{\text{an}}(E)$, although these equalities by themselves are not enough to prove the conjecture. The same time as Kato's work, Greenberg and Vatsal considered how this conjecture varies within a family of p -congruent curves.

THEOREM ([19, Theorem 1.4]). — *Suppose that E and E' are two modular elliptic curves defined over \mathbb{Q} , and each elliptic curve has good ordinary reduction at p . If their mod p Galois representations are both absolutely irreducible and satisfy $E[p] \cong E'[p]$ as $G_{\mathbb{Q}}$ -modules, then the following implication holds:*

$$\begin{aligned} \mu^{\text{alg}}(E) = \mu^{\text{an}}(E) = 0 &\implies \mu^{\text{alg}}(E') = \mu^{\text{an}}(E') = 0 \\ \text{and } \lambda^{\text{alg}}(E) = \lambda^{\text{an}}(E) &\quad \text{and } \lambda^{\text{alg}}(E') = \lambda^{\text{an}}(E'). \end{aligned}$$

If one combines the above result with the divisibility of the algebraic p -adic L -function into the analytic version, due again to Kato [24], it is clear that the Iwasawa Main Conjecture only really depends on the mod p Galois representation. Of course, the full conjecture has now been proven via the work of Skinner and Urban [38], who established the reverse divisibility for the two p -adic L -functions. However in the early 2000s this reverse divisibility was certainly not available!

Now an elliptic curve E over \mathbb{Q} is modular by the results of Breuil et al [2], which means that there exists a weight two newform $f_E = \sum_{n=1}^{\infty} a_n(f_E) e^{2\pi i n z}$ with a trivial character and the same conductor as E , satisfying $L(E, s) = L(f_E, s)$. Therefore one may interpret the Iwasawa Main Conjecture for E as a statement about weight two newforms, linking their arithmetic and p -adic analytic properties. Strongly motivated by Greenberg and Vatsal's work, in 2005 Emerton, Pollack and Weston studied both the μ -invariant and λ -invariant attached to p -ordinary families of Hecke eigenforms. In fact they proved the following quite remarkable result.

THEOREM ([13, Theorems 1 and 2]). — *If the μ -invariant vanishes for at least one form f in the Hida family with residual Galois representation*

$\bar{\rho} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathbb{F}_{p^e})$ then it vanishes for every eigenform f' with residual representation $\bar{\rho}$, and further

$$\lambda^{\mathrm{alg}}(f) - \lambda^{\mathrm{alg}}(f') = \lambda^{\mathrm{an}}(f) - \lambda^{\mathrm{an}}(f').$$

In other words, the μ -invariant and λ -invariant for a p -ordinary Hecke eigenform f only really “sees” the residual Galois representation $\bar{\rho}$, and the eigenform f can move around the deformation space (from one irreducible component to the next) and yet both the quantities $\mu^*(f)$ and $\lambda^*(f)$ are still kept tightly in check.

In this article, we extend Greenberg and Vatsal’s ideas in a different direction. Suppose that we are given primitive Hecke eigenforms f_1, \dots, f_t each of weight ≥ 2 . Then by the work of Deligne [12], one can associate a p -adic Galois representation

$$\rho_{f_i} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{Aut}_{\mathcal{O}_{\mathcal{K}}}(\mathbf{T}_{f_i}) \quad \text{with } \mathrm{rk}_{\mathcal{O}_{\mathcal{K}}}(\mathbf{T}_{f_i}) = 2 \text{ for } i \in \{1, \dots, t\},$$

which is unramified at primes $l \nmid pN_{f_i}$ and in addition satisfies $\mathrm{Tr} \rho_{f_i}(\mathrm{Frob}_l) = a_l(f_i)$. Here \mathcal{K} is a finite extension of \mathbb{Q}_p containing all the Fourier coefficients of the f_i ’s, and $a_n(f_i)$ denotes the n -th Fourier coefficient of f_i , i.e. $f_i = \sum_{n=1}^{\infty} a_n(f_i) e^{2\pi i n z}$. Tensoring together these ρ_{f_i} over \mathcal{K} yields a 2^t -dimensional Galois representation

$$\rho_{f_1} \otimes \cdots \otimes \rho_{f_t} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}_{2^t}(\mathcal{O}_{\mathcal{K}})$$

which is unramified outside the primes dividing $p \cdot \prod_{i=1}^t N_{f_i}$ and the infinite place.

Let $\kappa_p : G_{\mathbb{Q}} \rightarrow \mathbb{Z}_p^{\times}$ be the p -th cyclotomic character, so that $\sigma(\zeta_{p^n}) = \zeta_{p^n}^{\kappa_p(\sigma)}$. We also write $\omega : G_{\mathbb{Q}} \rightarrow \mathbb{F}_p^{\times}$ for the Teichmüller character mod p , which associates to each $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ the $(p-1)$ -st root of unit congruent to $\kappa_p(\sigma)$ modulo p .

QUESTION 1.1. — *Can one attach a Selmer group to $\rho_{f_1} \otimes \cdots \otimes \rho_{f_t} \otimes \kappa_p^j$ over $\mathbb{Q}^{\mathrm{cyc}}$ which is Λ -cotorsion, and thereby study its associated μ -invariant and λ -invariant?*

Thanks to the work of several people [1, 17, 18, 39], most notably Greenberg, if the underlying representations are p -ordinary then (conjecturally) the answer is in the affirmative. However there is one important caveat: one needs to restrict the values of j so the points lie within the critical strip for the automorphic L -series. Henceforth a sequence of modular forms f_1, \dots, f_t shall be labelled as *unbalanced* (or more accurately, f_1 -*unbalanced*) if the weight inequality

$$\mathrm{wt}(f_1) \geq 3 - t + \mathrm{wt}(f_2) + \cdots + \mathrm{wt}(f_t) \text{ holds true.}$$

For such an unbalanced sequence, the values of j should be restricted to the strip

$$2 - \text{wt}(f_1) \leq j \leq t - 1 - \text{wt}(f_2) + \cdots - \text{wt}(f_t)$$

otherwise the local $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ -representation $V_{\rho_{f_1}} \otimes \cdots \otimes V_{\rho_{f_t}} \otimes \kappa_p^j$ fails to satisfy a strong Panchishkin condition, and the rest of our arguments will collapse.

QUESTION 1.2. — *Does there exist an analytic p -adic L -function $\mathbf{L}_p^{\text{an}}(f_1 \otimes \cdots \otimes f_t)$ whose values at characters of the form $\chi \kappa_p^j$ interpolate the automorphic L -series?*

If one assumes $t \leq 3$ then such p -adic analytic objects have been constructed. For example, $t = 1$ corresponds to the Mazur, Tate and Teitelbaum L -function [31], for $t = 2$ one has the construction [21, 33] of Hida and Panchishkin, whilst for $t = 3$ one uses Hsieh and Yamana's triple product L -function given in [22, 23]. If $t \geq 4$ there is nothing currently available, and the existence is an open problem.

QUESTION 1.3. — *Is $\mathbf{L}_p^{\text{an}}(f_1 \otimes \cdots \otimes f_t)$ a generator for the characteristic ideal of the (Pontrjagin dual of the) Selmer group associated to $\rho_{f_1} \otimes \cdots \otimes \rho_{f_t}$ over \mathbb{Q}^{cyc} ?*

This is precisely the Iwasawa Main Conjecture for the tensor product motive, and unsurprisingly, this is widely considered to be a difficult problem in the area. For $t = 1$ there is the famous decade-old work of Kato and Skinner–Urban [24, 38], but for $t = 2$ and $t = 3$ the most that has been obtained is a divisibility [6, 27].

Let us examine a less difficult but closely related question. Suppose that we are given another collection of newforms f'_1, \dots, f'_t (again of weight ≥ 2) such that

$$\rho'_{f_1} \otimes \cdots \otimes \rho'_{f_t} \cong \rho_{f_1} \otimes \cdots \otimes \rho_{f_t} \pmod{\pi_{\mathcal{K}}}$$

where $\rho'_{f_i} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathcal{O}_{\mathcal{K}})$ means the Deligne representation attached to each f'_i , and $\pi_{\mathcal{K}}$ is a chosen uniformizer for the discrete valuation field \mathcal{K} .

QUESTION 1.4. — *How do the μ - and λ -invariants (both algebraic and analytic) vary as we pass from $\rho_{f_1} \otimes \cdots \otimes \rho_{f_t}$ to the congruent representation $\rho'_{f_1} \otimes \cdots \otimes \rho'_{f_t}$?*

For $t = 1$ (i.e. for GL_2), the results of Emerton, Pollack and Weston [13] provide a definitive answer in the case where one of the μ -invariants equals zero. For $t = 2$, on the analytic side the author in tandem with Lei and Gilmore [10, 11] derived a transition formula for the λ -invariant, provided

that the weight of the modular forms does not change as we switch between the pairs (f_1, f_2) and (f'_1, f'_2) . For $t \geq 3$ very little is known at present, to the best of the author's knowledge.

One can also ask what happens if the ground field varies in a uniform manner. Let $\mathcal{D}_\infty = \bigcup_{n \geq 1} \mathcal{D}_n$ be a p -adic Lie extension of \mathbb{Q} , containing $\mathbb{Q}(\mu_{p^\infty})$ as a subfield.

QUESTION 1.5. — *Is there a Kida-type formula [26] allowing us to compute the μ - and λ -invariant over $\mathcal{D}_n^{\text{cyc}}$ for $f_1 \otimes \cdots \otimes f_t$ in terms of the invariants over \mathbb{Q}^{cyc} ?*

The main goal of this article is to provide a satisfactory answer to both Question 1.4 and Question 1.5, albeit purely on the algebraic side. If $t = 2$ (the double product case), one can combine these formulae with certain divisibility results of Kings, Loeffler and Zerbes [27], and thereby deduce non-trivial information on Question 1.3 for congruent representations $\rho_{f_1} \otimes \rho_{f_2}$ and $\rho'_{f_1} \otimes \rho'_{f_2}$.

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2. Statement of the main results

Fix an odd prime p , and consider the finite field $\mathbb{F} = \mathbb{F}_{p^e}$ containing p^e elements. Suppose we are given as data a collection two-dimensional Galois representations $\bar{\rho}_1 : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{F}), \dots, \bar{\rho}_t : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{F})$, and let us assume that $\mathbb{F} \cong \mathcal{O}_{\mathcal{K}}/\pi_{\mathcal{K}}$. Without loss of generality, throughout we will suppose that each $\bar{\rho}_i$ has a minimal conductor amongst its twists by characters $\psi : G_{\mathbb{Q}} \rightarrow \bar{\mathbb{F}}^\times$ of conductor coprime to p (if not then replace $\bar{\rho}_i$ with a “smaller” twist $\bar{\rho}_i \otimes \psi_i$, which only changes $\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t$ by at most a ψ -twist). Finally, we impose the following four additional conditions:

HYPOTHESES.

- (H1) *The $G_{\mathbb{Q}}$ -representations $\bar{\rho}_i$ are all absolutely irreducible.*
- (H2) *For every $i \in \{1, \dots, t\}$ the representation $\bar{\rho}_i$ is p -ordinary, namely the restriction $\bar{\rho}_i|_{G_{\mathbb{Q}_p}}$ has an unramified quotient of dimension one over \mathbb{F} .*
- (H3) *Each of the Galois representations $\bar{\rho}_i$ is p -distinguished, so that $\bar{\rho}_i|_{G_{\mathbb{Q}_p}}$ is reducible with a non-scalar semi-simplification for $i \in \{1, \dots, t\}$.*
One of the following two (inequivalent) conditions holds:
- (H4a) *at each prime $l \neq p$, if $l^2 \mid \text{cond}(\bar{\rho}_{i'})$ for some i' then $l^2 \nmid \text{cond}(\bar{\rho}_i)$ if $i \neq i'$;*
- (H4b) *the prime $p > 3$, and for all i one has $\text{Im}(\bar{\rho}_i) \subset \text{GL}_2(\mathbb{F}_p)$ with $\det(\bar{\rho}_i) = \omega$.*

For instance, the conditions (H1)–(H3) are necessary in order to construct a multi-variable universal Hecke algebra associated to $\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t$ in the spirit of [13]. Hypothesis (H4a) is satisfied whenever the intersection of the two ramification sets for any pair of $\bar{\rho}_i$'s equals $\{p, \infty\}$. Hypothesis (H4b) holds when $\bar{\rho}_1, \dots, \bar{\rho}_t$ are the mod p Galois representations arising from a tuple of elliptic curves $E_1/\mathbb{Q}, \dots, E_t/\mathbb{Q}$. Indeed if f_1, \dots, f_t are newforms with trivial character and residual representations $\bar{\rho}_1, \dots, \bar{\rho}_t$, then (H4b) is true whenever $\text{wt}(f_1) \equiv \cdots \equiv \text{wt}(f_t) \equiv 2 \pmod{p-1}$.

For each $\bar{\rho}_i$ as above, we write $\mathcal{H}(\bar{\rho}_i)$ for the set of all p -stabilised newforms f_i of weight ≥ 2 whose residual representation is equivalent to $\bar{\rho}_i$, i.e. $\rho_{f_i} \bmod \pi_K \cong \bar{\rho}_i$. One then defines the *Hida family attached to* $\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t$ to be the set-product

$$\mathcal{H}(\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t) = \mathcal{H}(\bar{\rho}_1) \times \cdots \times \mathcal{H}(\bar{\rho}_t)$$

consisting of t -tuples of Hecke eigenforms $\underline{f} = (f_1, \dots, f_t)$ with associated weights $\underline{k} = (k_1, \dots, k_t) \geq (2, \dots, 2)$ and Nebentypes $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_t)$. In Section 3 we will construct a multi-variable Hecke algebra, $\mathbb{T}_{\underline{\Sigma}}(\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t)$, that is of finite-type over the power series ring $\mathcal{O}[[X_1, \dots, X_t]]$, and which controls the Hida family above. In particular, $\mathbb{T}_{\underline{\Sigma}}(\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t)$ has finitely many “multi-branches” $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \dots, \mathfrak{a}_t)$ where each \mathfrak{a}_i is a minimal prime of the corresponding one-variable Hecke algebra.

2.1. Behaviour of the λ -invariant over \mathbb{Q}^{cyc}

Let us start by taking the ground field to be the rational numbers. For a tuple $\underline{f} \in \mathcal{H}(\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t)$ and $j \in \mathbb{Z}$, its associated Selmer group over

the cyclotomic \mathbb{Z}_p -extension \mathbb{Q}^{cyc} is the subset

$$\text{Sel}_{\mathbb{Q}^{\text{cyc}}}(f, \omega^j) \subset H^1\left(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{\text{cyc}}), \mathbb{Q}/\mathbb{Z} \otimes (\mathbf{T}_{f_1} \otimes_{\mathcal{O}_K} \cdots \otimes_{\mathcal{O}_K} \mathbf{T}_{f_t}) \otimes \omega^j\right)$$

comprised of one-cocycles unramified at the primes of \mathbb{Q}^{cyc} not lying above p , whilst at primes $\mathfrak{p}|p$ they belong to the Bloch and Kato [1] local condition “ $H_g^1(\mathbb{Q}_{\mathfrak{p}}^{\text{cyc}}, -)$ ”. In the case where

$$\text{Sel}_{\mathbb{Q}^{\text{cyc}}}(f, \omega^j)^{\wedge} := \text{Hom}_{\text{cont}}(\text{Sel}_{\mathbb{Q}^{\text{cyc}}}(f, \omega^j), \mathcal{K}/\mathcal{O}_K)$$

is Λ -torsion, the μ - and λ -invariants will then be denoted by $\mu(f, \omega^j)$ and $\lambda(f, \omega^j)$, respectively.

THEOREM 2.1. — *Suppose that Hypotheses (H1)–(H3) hold, at least one of (H4a) or (H4b) holds, and that ω^{-j} is not a sub- $G_{\mathbb{Q}}$ -representation inside $\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t$.*

- (i) *If $\text{Sel}_{\mathbb{Q}^{\text{cyc}}}(f, \omega^j)^{\wedge}$ is Λ -torsion and $\mu(f, \omega^j) = 0$ at some unbalanced tuple $\underline{f} \in \mathcal{H}(\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t)$, then in fact $\text{Sel}_{\mathbb{Q}^{\text{cyc}}}(f, \omega^j)^{\wedge}$ is Λ -torsion and $\mu(f', \omega^j) = 0$ for every unbalanced tuple $\underline{f}' \in \mathcal{H}(\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t)$ of p -stabilised newforms.*
- (ii) *Under the same situation as in (i), if the tuples $\underline{f}^{(1)}, \underline{f}^{(2)} \in \mathcal{H}(\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t)$ are both unbalanced, then*

$$\lambda(\underline{f}^{(1)}, \omega^j) = \lambda(\underline{f}^{(2)}, \omega^j) + \sum_{q|\mathfrak{c}(\underline{f}^{(1)})\mathfrak{c}(\underline{f}^{(2)}), q \neq p} \mathbf{e}_q(\underline{\mathfrak{a}}^{(2)}, \omega^j) - \mathbf{e}_q(\underline{\mathfrak{a}}^{(1)}, \omega^j)$$

where $\mathfrak{c}(\underline{f}^{(*)})$ denotes the conductor of $\rho_{f_1^{(*)}} \otimes \cdots \otimes \rho_{f_t^{(*)}}$ at either choice $\star \in \{1, 2\}$, and $\mathbf{e}_q(\underline{\mathfrak{a}}^{(*)}, \omega^j) \in \mathbb{Z}_{\geq 0}$ depends only on the multi-branch $\underline{\mathfrak{a}}^{(*)}$ which contains $\underline{f}^{(*)}$.

The integer $\mathbf{e}_q(\underline{\mathfrak{a}}, \omega^j)$ is the generic λ -invariant of the function interpolating the Euler factor attached to $\rho_{f_1} \otimes \cdots \otimes \rho_{f_t} \otimes \omega^j$ at the prime q (Definition 5.5). These error terms measure the discrete jumps in the cyclotomic λ -invariant as the eigenforms (f_1, \dots, f_t) switch from one multi-branch of the Hida family to the next. In fact for $t = 1$, the above is an immediate consequence of [13, Theorems 1 and 2]. If $t = 2$ then the Λ -cotorsion of the Selmer group is already known in many cases because of the finiteness results in [27], as we shall now discuss in greater detail.

2.2. The double product Main Conjecture

We will study the case $t = 2$. Consider a pair $(f_1, f_2) \in \mathcal{H}(\bar{\rho}_1 \otimes \bar{\rho}_2)$ of eigenforms of weight (k_1, k_2) with $k_1 > k_2$. For a chosen branch ω^j , the

constructions in [21, 33] imply that there is an element $\mathbf{L}_p^{\text{an}}(f_1 \otimes f_2, \omega^j) \in \mathcal{O}_{\mathcal{K}}[[\Gamma^{\text{cyc}}]][1/\pi_{\mathcal{K}}]$, interpolating the double product L -values

$$\begin{aligned} & \chi \kappa_p^r \left(\mathbf{L}_p^{\text{an}}(f_1 \otimes f_2, \omega^j) \right) \\ &= \iota_p \circ \iota_{\infty}^{-1} \left(\mathfrak{M}_p(f_1 \otimes f_2, \bar{\chi} \omega^{-j}, r) \cdot \frac{L(f_1 \otimes f_2, \chi \omega^j, k_2 + r)}{(2\pi i)^{1-k_2} \cdot \langle \tilde{f}_1, \tilde{f}_1 \rangle_{\text{Pet}}} \right) \end{aligned}$$

at critical twists $r \in \{0, \dots, k_1 - k_2 - 1\}$ and finite-order characters $\chi : \Gamma^{\text{cyc}} \rightarrow \bar{\mathbb{Q}}_p^{\times}$. The p -adic multiplier $\mathfrak{M}_p(f_1 \otimes f_2, -, -)$ is described in Section 5.4, and $\langle \tilde{f}_1, \tilde{f}_1 \rangle_{\text{Pet}} = \|\tilde{f}_1\|^2$ is the square of the Petersson norm of the newform, \tilde{f}_1 , whose p -stabilisation is f_1 .

CONJECTURE. — *The Iwasawa Main Conjecture for $f_1 \otimes f_2$ with weights $k_1 > k_2$ and at a fixed ω^j -branch, predicts that $\text{Sel}_{\mathbb{Q}^{\text{cyc}}}(\underline{f}, \omega^j)$ is a Λ^{cyc} -cotorsion module, and secondly that there exists an element $\mathbf{u}_j \in \mathcal{O}_{\mathcal{K}}[[\Gamma^{\text{cyc}}]]^{\times}$ satisfying the equality*

$$\mathbf{h}_{f_1} \cdot \mathbf{L}_p^{\text{an}}(f_1 \otimes f_2, \omega^j) = \mathbf{u}_j \times \text{char}_{\Lambda^{\text{cyc}}} \left(\text{Sel}_{\mathbb{Q}^{\text{cyc}}}(\underline{f}, \omega^j)^{\wedge} \otimes \kappa_p^{-1} \right)$$

where \mathbf{h}_{f_1} generates the congruence ideal [21] attached to

$$\lambda_{f_1} : \mathfrak{h}_{k_1}(\Gamma_1(N_{f_1})) \rightarrow \mathbb{C}.$$

Notation. — For such a pair (f_1, f_2) we abbreviate this conjecture by $\text{IMC}(f_1 \otimes f_2, j)$.

Since we want to use the work in [27] one now assumes that the prime $p > 3$. To apply this Euler system machinery, we also cut down the pairs being considered.

DEFINITION. — *We say that a pair $(f_1, f_2) \in \mathcal{H}(\bar{\rho}_1 \otimes \bar{\rho}_2)$ of weight $(k_1, k_2) \geq (2, 2)$ is admissible if the following list of conditions holds:*

- *the pair of eigenforms (f_1, f_2) is unbalanced, i.e. $k_1 > k_2$;*
- *if the non-dominant weight $k_2 = 2$, then f_2 is not Steinberg at p ;*
- *both f_1 and f_2 are non-Eisenstein modulo the uniformiser $\pi_{\mathcal{K}}$;*
- *there exists $\tau \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_{p^\infty}))$ so that $\frac{\mathbf{T}_{f_1} \otimes \mathbf{T}_{f_2}}{(\tau-1) \cdot \mathbf{T}_{f_1} \otimes \mathbf{T}_{f_2}}$ is free of rank one;*
- *there exists $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_{p^\infty}))$ which acts on $\mathbf{T}_{f_1} \otimes \mathbf{T}_{f_2}$ through $-\text{id}_4$.*

Henceforth assume $\mathbf{L}_p^{\text{an}}(f_1 \otimes f_2, \omega^j) \neq 0$, which is certainly true if $k_1 > k_2 + 1$ by the non-vanishing results of Shahidi [36, Theorem 5.2] (see also [27, Proposition 2.7.6]). For any given branch ω^j , we shall denote by $\mu^{\text{an}}(f_1 \otimes$

$f_2, \omega^j)$ and $\lambda^{\text{an}}(f_1 \otimes f_2, \omega^j)$ the cyclotomic μ - and λ -invariant attached to $\mathbf{h}_{f_1} \cdot \mathbf{L}_p^{\text{an}}(f_1 \otimes f_2, \omega^j)$, respectively.

THEOREM 2.2. — *Suppose that $t = 2$, each of (H1)–(H3) hold, at least one of (H4a) or (H4b) holds, and that ω^{-j} is not a sub- $G_{\mathbb{Q}}$ -representation inside $\bar{\rho}_1 \otimes \bar{\rho}_2$. If the pair $\underline{f} = (f_1, f_2) \in \mathcal{H}(\bar{\rho}_1 \otimes \bar{\rho}_2)$ is admissible, then one has the implication*

$$\text{IMC}(f_1 \otimes f_2, j) \text{ is true} \implies \text{IMC}(f_1 \otimes f'_2, j) \text{ is true for any newform } f'_2 \\ \text{and } \mu^{\text{an}}(f_1 \otimes f_2, \omega^j) = 0 \quad \text{with } \text{wt}(f'_2) = \text{wt}(f_2) \text{ and } \rho_{f'_2} \bmod \pi_K \cong \bar{\rho}_2.$$

The demonstration of this theorem has three main components. The first of these are the aforementioned Euler system results, in particular the divisibility of the algebraic p -adic L -function into its analytic counterpart from [27, Theorem 11.6.4]. The second ingredient is Theorem 2.1, with the specific choice of $t = 2$ naturally. The final component is a (weaker) analytic version of this same theorem, proved by the author and Gilmore in [10, Theorem 1.2]. The restrictions that $f'_1 = f_1$ and $\text{wt}(f'_2) = \text{wt}(f_2)$ are caused unfortunately by technical issues on the analytic side (see the remark at the end of Section 5.4 for a more detailed discussion).

CONJECTURE. — Under the same conditions as Theorem 2.2, and if the analytic μ -invariant $\mu^{\text{an}}(f_1 \otimes f_2, \omega^j) = 0$ for some unbalanced $(f_1, f_2) \in \mathcal{H}(\bar{\rho}_1 \otimes \bar{\rho}_2)$, then

$$\mu^{\text{an}}(f'_1 \otimes f'_2, \omega^j) = 0 \quad \text{for every unbalanced } (f'_1, f'_2) \in \mathcal{H}(\bar{\rho}_1 \otimes \bar{\rho}_2).$$

Further, if $\underline{f} = (f_1, f_2)$ and $\underline{f}' = (f'_1, f'_2)$ lie in $\mathcal{H}(\bar{\rho}_1 \otimes \bar{\rho}_2)$ and are unbalanced, then

$$\lambda^{\text{an}}(f_1 \otimes f_2, \omega^j) = \lambda^{\text{an}}(f'_1 \otimes f'_2, \omega^j) + \sum_{q|\mathfrak{c}(\underline{f})\mathfrak{c}(\underline{f}'), q \neq p} \mathbf{e}_q(\underline{\mathfrak{a}}', \omega^j) - \mathbf{e}_q(\underline{\mathfrak{a}}, \omega^j)$$

where the pair \underline{f} (resp. \underline{f}') lies on the multi-branch $\underline{\mathfrak{a}}$ (resp. the multi-branch $\underline{\mathfrak{a}}'$).

In fact, the very same argument that enables us to prove Theorem 2.2 actually allows us to deduce the stronger implication

$$\text{IMC}(f_1 \otimes f_2, j) \text{ is true} \implies \text{IMC}(f'_1 \otimes f'_2, j) \text{ is true for every} \\ \text{and } \mu^{\text{an}}(f_1 \otimes f_2, \omega^j) = 0 \quad \text{admissible } (f'_1, f'_2) \in \mathcal{H}(\bar{\rho}_1 \otimes \bar{\rho}_2),$$

of course, provided that the above conjecture on the analytic λ -invariant holds. This analytic conjecture is proven in many cases, and forms a work in progress [7].

2.3. Non-commutative Iwasawa theory

Let $\mathcal{D}_\infty/\mathbb{Q}$ denote a p -adic Lie extension of the rational numbers, which satisfies the following three conditions:

- the infinite algebraic extension \mathcal{D}_∞ contains $\mathbb{Q}(\mu_{p^\infty})$ as a subfield;
- the set of rational primes, $S_{\mathcal{D}_\infty}^{\text{ram}}$, that ramify inside $\mathcal{D}_\infty/\mathbb{Q}$ is finite;
- $\text{Gal}(\mathcal{D}_\infty/\mathbb{Q}(\mu_p))$ is a non-abelian pro- p -group without any p -torsion.

If $\mathcal{G}_\infty := \text{Gal}(\mathcal{D}_\infty/\mathbb{Q}(\mu_p))$ and $n > 0$, then we will write \mathcal{D}_n for the fixed field of \mathcal{D}_∞ under the action of the group $\mathcal{G}_\infty^{p^{n-1}}$. In particular, $\mathcal{D}_1 = \mathbb{Q}(\mu_p)$ and $\mathcal{D}_\infty = \bigcup_{n>0} \mathcal{D}_n$. For a tuple $\underline{f} \in \mathcal{H}(\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t)$, its associated Selmer group over $\mathcal{D}_n^{\text{cyc}}$ is the subset

$$\text{Sel}_{\mathcal{D}_n^{\text{cyc}}}(\underline{f}) \subset H^1\left(\text{Gal}(\bar{\mathbb{Q}}/\mathcal{D}_n^{\text{cyc}}), \mathbb{Q}/\mathbb{Z} \otimes (\mathbf{T}_{f_1} \otimes_{\mathcal{O}_K} \cdots \otimes_{\mathcal{O}_K} \mathbf{T}_{f_t})\right)$$

comprised of one-cocycles unramified at the primes of $\mathcal{D}_n^{\text{cyc}}$ not lying above $S_{\mathcal{D}_\infty}^{\text{ram}}$, whilst at the primes $\mathfrak{p}|p$ they belong to the Bloch–Kato local condition $H_g^1(\mathcal{D}_{n,\mathfrak{p}}^{\text{cyc}}, -)$. The μ - and λ -invariants will then be denoted by $\mu_{\mathcal{D}_n}(\underline{f})$ and $\lambda_{\mathcal{D}_n}(\underline{f})$, respectively.

THEOREM 2.3. — Suppose that Hypotheses (H1)–(H3) hold, at least one of (H4a) or (H4b) holds, and that the cohomology group $H^0(G_{\mathbb{Q}(\mu_p)}, V_{\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t})$ is trivial. Furthermore, we shall assume that either (a) $\mathcal{D}_\infty = \mathbb{Q}(\mu_{p^\infty}, m_1^{1/p^\infty}, \dots, m_d^{1/p^\infty})$, or that (b) $\dim(\mathcal{G}_\infty) < 4$ and \mathcal{G}_∞ is not an open subgroup of $\text{SL}_2(\mathbb{Z}_p)$ nor of $\text{SL}_1(\mathbb{D}_p)$.

- If $\text{Sel}_{\mathbb{Q}^{\text{cyc}}}(\underline{f}, \omega^j)^\wedge$ is Λ -torsion and $\mu(\underline{f}, \omega^j) = 0$ at each $j \in \{0, \dots, p-2\}$ for some unbalanced tuple $\underline{f} \in \mathcal{H}(\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t)$, then $\text{Sel}_{\mathcal{D}_n^{\text{cyc}}}(\underline{f}')^\wedge$ is Λ -torsion and $\mu_{\mathcal{D}_n}(\underline{f}') = 0$ for every unbalanced tuple $\underline{f}' \in \mathcal{H}(\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t)$ and integer $n > 0$.
- Under the same situation as in (i), at each unbalanced $\underline{f}' \in \mathcal{H}(\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t)$ and integer $n > 0$, the λ -invariant over $\mathcal{D}_n^{\text{cyc}}$ satisfies the growth formula

$$\begin{aligned} \lambda_{\mathcal{D}_n}(\underline{f}') &= [\mathcal{D}_n : \mathbb{Q}(\mu_p)] \cdot \left(\sum_{j=0}^{p-2} \lambda(\underline{f}', \omega^j) + \sum_{\substack{q|\mathfrak{c}(\underline{f}'), q \neq p \\ \text{or } q \in S_{\mathcal{D}_\infty}^{\text{ram}} - \{p\}}} \mathbf{e}_q(\mathbf{a}', \omega^j) \right) \\ &\quad - \sum_{\substack{q|\mathfrak{c}(\underline{f}'), \\ q \notin S_{\mathcal{D}_\infty}^{\text{ram}}}} \mathbf{e}_{\mathcal{D}_n, q}(\mathbf{a}') \end{aligned}$$

where $\mathbf{e}_{\mathcal{D}_n, q}(\mathbf{a}') \in \mathbb{Z}_{\geq 0}$ depends only on \mathcal{D}_n and the multi-branch \mathbf{a}' containing \underline{f}' .

- (iii) Under the same situation as in (i), if the tuples $\underline{f}^{(1)}, \underline{f}^{(2)} \in \mathcal{H}(\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t)$ are both unbalanced then for each integer $n > 0$, one has the transition formula

$$\lambda_{\mathcal{D}_n}(\underline{f}^{(1)}) = \lambda_{\mathcal{D}_n}(\underline{f}^{(2)}) + \sum_{q | \mathfrak{c}(\underline{f}^{(1)})\mathfrak{c}(\underline{f}^{(2)}), q \neq p} \mathbf{e}_{\mathcal{D}_n, q}(\underline{\mathfrak{a}}^{(2)}) - \mathbf{e}_{\mathcal{D}_n, q}(\underline{\mathfrak{a}}^{(1)})$$

where the eigenform $\underline{f}^{(1)}$ (resp. $\underline{f}^{(2)}$) belongs to the multi-branch $\underline{\mathfrak{a}}^{(1)}$ (resp. $\underline{\mathfrak{a}}^{(2)}$).

Over the n -th layer \mathcal{D}_n in the Lie extension \mathcal{D}_∞ , the non-negative integer $\mathbf{e}_{\mathcal{D}_n, q}(\underline{\mathfrak{a}})$ is the generic λ -invariant of the function interpolating the Euler factor associated to $\rho_{f_1} \otimes \cdots \otimes \rho_{f_t} \otimes \text{reg}_{\mathcal{D}_n/\mathbb{Q}}$ at the prime q (again, see Definition 5.5). The vanishing of $H^0(G_{\mathbb{Q}(\mu_p)}, V_{\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t})$ occurs when $\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t|_{G_{\mathbb{Q}(\mu_p)}}$ is irreducible. However this can fail: if $t > 1$ and $\sigma = \rho_{f_1} \cong \cdots \cong \rho_{f_t}$ then $\sigma^{\otimes t} \cong \rho_{f_1} \otimes \cdots \otimes \rho_{f_t}$ contains both $\text{Sym}^t(\sigma)$ and $\bigwedge^t(\sigma)$, yet the latter may often have trivial reduction as a $G_{\mathbb{Q}(\mu_p)}$ -representation, so the vanishing of $H^0(G_{\mathbb{Q}(\mu_p)}, \bar{\sigma}^{\otimes t})$ is not guaranteed (this case is avoided anyway by the inequality $\text{wt}(f_1) \geq 3 - t + \text{wt}(f_2) + \cdots + \text{wt}(f_t)$).

The growth formula in (ii) is actually predicted by certain congruence relations arising from K -theory, provided the Selmer group attached to \underline{f} over \mathcal{D}_∞ satisfies the $\mathfrak{M}_H(\mathcal{G}_\infty)$ -condition in [5]; the vanishing of the μ -invariant is then equivalent to $\text{Sel}_{\mathcal{D}_\infty}(\underline{f})^\wedge$ having trivial image inside $K_0(\mathbb{F}[\mathcal{G}_\infty])$ – for details, see [3, Proposition 3.4]. There are also results of Lim [28, Theorem 4.2.1] and [29, Theorem 4.1.2], which compare the elementary composition for the π -primary parts of the Selmer group over \mathcal{D}_∞ .

2.4. A brief outline of the strategy

The foundation of almost all the work on this topic [4, 8, 9, 13, 19, 39] exploits a very delicate relationship between the λ -invariant for the dual of the Selmer group (cyclotomic or anti-cyclotomic), and the \mathbb{F} -dimension of the “minimal Selmer group” attached to the residual representation. The key insight of Emerton, Pollack and Weston [13] was to revisit the construction of the p -adic L -function for GL_2 , but to construct it over the universal deformation space which parameterises nearly-ordinary deformations of each representation $\bar{\rho}_i$. Since we are working here with a t -fold product $\text{GL}_2 \times \cdots \times \text{GL}_2$, in Sections 3.1–3.2 we glue together their Hecke algebras into a multi-variable version “ $\mathbb{T}_{\underline{\Sigma}}(\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t)$ ”.

In Section 3.3 we address the existence (or otherwise) of a minimal tuple $(f_1^\dagger, \dots, f_t^\dagger)$ whose coprime-to- p conductor is equal to the geometric conductor of $\bar{\rho}_1 \otimes \dots \otimes \bar{\rho}_t$. For $t = 1$ this is not an issue, as the work of Khare and Wintenberger [25] guarantees the existence of these minimal newforms. However if $t > 1$ we have failed to find a general proof of the existence of such a minimal tuple, which is the main reason why we assume at least one of Hypotheses (H4a) or (H4b) holds to make progress.

In Section 4 we associate Ω -imprimitive Selmer groups to these tensor products where Ω denotes a finite set of rational primes. The characteristic zero Selmer groups have the structure of discrete cofinitely-generated Λ -modules, which are also easily seen to be π -divisible groups. This helps greatly when trying to calculate their λ -invariants as it is enough to compute the dimension of the π -torsion subgroup. We point out that it is only for a small subset of Tate twists that our Selmer groups coincide with their Bloch–Kato cousins, and we undertake this comparison in Section 4.2.

Finally in Section 5, we combine these disparate strands together and give proofs for the three main theorems. Central to our arguments is the nice behaviour of the minimal tuples $(f_1^\dagger, \dots, f_t^\dagger)$ under the reduction modulo π mapping, thereby providing a link between the arithmetic of $\rho_{f_1^\dagger} \otimes \dots \otimes \rho_{f_t^\dagger}$ and its residual counterpart $\bar{\rho}_1 \otimes \dots \otimes \bar{\rho}_t$. Without this minimal tuple one can still establish the vanishing of the μ -invariants, but we cannot obtain explicit formulae for the λ -invariants.

It is important to mention that the proof of our Kida-type formula we found in the non-commutative setting utilises Greenberg’s notion of *Selmer atoms* [18], which is essentially an Iwasawa-theoretic analogue of explicit Brauer induction. Whilst we limited our treatment to p -adic Lie groups with an underlying metabelian structure for this paper, in principle there is nothing preventing us from extending Theorem 2.3 to $\mathrm{GL}_2(\mathbb{Z}_p)$ -extensions, or to even more exotic insoluble Lie groups.

3. The control theory for $\bar{\rho}_1 \otimes \dots \otimes \bar{\rho}_t$

Our principal goal in this section is to associate a multi-variable Hecke algebra “ $\mathbb{T}_{\underline{\Sigma}}(\bar{\rho}_{1,\dots,t})$ ” to the residual representation $\bar{\rho}_{1,\dots,t} := \bar{\rho}_1 \otimes \dots \otimes \bar{\rho}_t : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_{2t}(\mathbb{F})$ and to finite subsets $\underline{\Sigma} = (\Sigma_1, \dots, \Sigma_t) \subset \mathrm{Spec}(\mathbb{Z})^t$. However we must first recall the key properties of the reduced Hecke algebra which controls each individual ρ_i , and the background reference is [13] where the original construction was made.

Let $\mathcal{O} = \mathbf{W}(\mathbb{F})$ be the ring of Witt vectors for \mathbb{F} , and define $\Lambda^{\text{wt}} := \mathcal{O}[[1 + p\mathbb{Z}_p]]$ to be the weight algebra. If $N \geq 1$ with $p \nmid N$, then one writes \mathbb{T}_N for the image of $\mathfrak{h}(Np^\infty; \mathcal{O}) = \varprojlim_r \mathfrak{h}(Np^r; \mathcal{O})$ inside the endomorphism ring of the p -ordinary cusp forms of tame level N , and with \mathcal{O} -coefficients. From [20, Theorems 1.1-1.2], the commutative algebra \mathbb{T}_N is free of finite rank over Λ^{wt} . In terms of specialisations, if $\wp \in \text{Spec}(\mathbb{T}_N)$ is arithmetic of weight k then $\mathbb{T}_N/\wp\mathbb{T}_N$ can be naturally identified with the quotient of $\mathfrak{h}^{\text{ord}}(Np^\infty; \mathcal{O})$ that acts faithfully on $\mathcal{S}_k^{\text{ord}}(Np^\infty; \mathbb{T}_N/\wp)[\varepsilon_\wp]$, where ε_\wp denotes the character on $1 + p\mathbb{Z}_p$ induced by \wp .

We shall write $\mathbb{T}_N^{\text{new}}$ for the quotient of \mathbb{T}_N acting on the subspace of newforms. Let \mathbb{T}'_N denote the Λ^{wt} -subalgebra of \mathbb{T}_N generated by the Hecke operators T_l for $l \nmid Np$, together with the operator U_p . There is a canonical map [13, Proposition 2.3.2]

$$\mathbb{T}'_N \longrightarrow \prod_{M|N} \mathbb{T}_M^{\text{new}}$$

which then becomes an isomorphism after tensoring over the fraction field of Λ^{wt} . Gluing the product of the ρ_M^{new} 's over the divisors M of N yields a representation $\rho'_N : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{T}'_N \otimes_{\Lambda^{\text{wt}}} \text{Frac}(\Lambda^{\text{wt}}))$. At each maximal ideal \mathfrak{m} of \mathbb{T}'_N , taking the localisation of ρ'_N at \mathfrak{m} produces another $(\mathbb{T}'_N)_{\mathfrak{m}}$ -valued Galois representation

$$\rho_{\mathfrak{m}} : G_{\mathbb{Q}} \rightarrow \text{GL}_2((\mathbb{T}'_N)_{\mathfrak{m}} \otimes_{\Lambda^{\text{wt}}} \text{Frac}(\Lambda^{\text{wt}})).$$

If its residual representation $\bar{\rho}_{\mathfrak{m}}$ is irreducible then $\rho_{\mathfrak{m}}$ has a Λ^{wt} -integral model, hence $\rho_{\mathfrak{m}} : G_{\mathbb{Q}} \rightarrow \text{GL}_2((\mathbb{T}'_N)_{\mathfrak{m}})$ will be uniquely determined (up to isomorphism).

3.1. Reduced Hecke algebras associated to $\bar{\rho}_i$

Choose any $i \in \{1, \dots, t\}$. As detailed in Section 2, suppose (H1)–(H3) hold so that $\bar{\rho}_i : G_{\mathbb{Q}} \rightarrow \text{Aut}_{\mathbb{F}}(V_{\bar{\rho}_i})$ is an absolutely irreducible representation which is p -ordinary and p -distinguished. We shall also select a finite set of primes $\Sigma_i = \Sigma_{\mathbb{Q}, i} \subset \text{Spec}(\mathbb{Z})$ such that $p \notin \Sigma_i$.

DEFINITION 3.1. — *If we set $N_i(\Sigma_i) := \text{cond}(\bar{\rho}_i) \cdot \prod_{l \in \Sigma_i} l^{\dim_{\mathbb{F}}(V_{\bar{\rho}_i})_{I_l}}$, then*

$$\mathbb{T}_{\Sigma_i} = \mathbb{T}_{\Sigma_i}(\bar{\rho}_i) := (\mathbb{T}'_{N_i(\Sigma_i)})_{\mathfrak{m}_i}$$

where \mathfrak{m}_i is the unique maximal ideal of $\mathbb{T}'_{N_i(\Sigma_i)}$ at which the residual representation $\bar{\rho}_{\mathfrak{m}_i} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\bar{\mathbb{F}}_p)$, with its canonical p -stabilisation, is isomorphic to $\bar{\rho}_i$.

One calls $\mathrm{Spec}(\mathbb{T}_{\Sigma_i}(\bar{\rho}_i))$ the universal p -ordinary Hida family attached to $\bar{\rho}_i$, minimally ramified outside of Σ_i . The irreducible components (branches) of this Hida family, $\mathcal{H}(\bar{\rho}_i)$, are indexed by the subset of minimal primes in $\mathrm{Spec}(\mathbb{T}_{\Sigma_i}(\bar{\rho}_i))$. If \mathfrak{a}_i is a minimal prime of \mathbb{T}_{Σ_i} , one can take the quotient algebra $\mathbb{T}(\mathfrak{a}_i) = \mathbb{T}_{\Sigma_i}/\mathfrak{a}_i$. In particular, for each branch \mathfrak{a}_i there exists a unique divisor $N_i(\mathfrak{a}_i)$ of $N_i(\Sigma_i)$ and a unique minimal prime $\mathfrak{a}'_i \triangleleft \mathbb{T}_{N_i(\mathfrak{a}_i)}^{\mathrm{new}}$ which lies over \mathfrak{a}_i , such that the diagram

$$\begin{array}{ccccc} \mathbb{T}_{\Sigma_i}(\bar{\rho}_i) & \longrightarrow & \mathbb{T}'_{N_i(\Sigma_i)} & \longrightarrow & \prod_{M|N_i(\Sigma_i)} \mathbb{T}_M^{\mathrm{new}} \\ \downarrow & & & & \downarrow \\ \mathbb{T}_{\Sigma_i}(\bar{\rho}_i)/\mathfrak{a}_i & \xrightarrow{=} & \mathbb{T}(\mathfrak{a}_i) & \longrightarrow & \mathbb{T}_{N_i(\mathfrak{a}_i)}^{\mathrm{new}}/\mathfrak{a}'_i \end{array}$$

is commutative [13, Proposition 2.5.2].

DEFINITION 3.2. — *At a minimal prime \mathfrak{a}_i of $\mathbb{T}_{\Sigma_i}(\bar{\rho}_i)$, we call the divisor $N_i(\mathfrak{a}_i)$ above the “tame conductor attached to \mathfrak{a}_i ”, we then define $\mathbb{T}(\mathfrak{a}_i)^0 := \mathbb{T}_{N_i(\mathfrak{a}_i)}^{\mathrm{new}}/\mathfrak{a}'_i$ and write $\rho_{\mathbb{T}(\mathfrak{a}_i)^0} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathbb{T}_{N_i(\mathfrak{a}_i)}^{\mathrm{new}}/\mathfrak{a}'_i \otimes_{\Lambda^{\mathrm{wt}}} \mathrm{Frac}(\Lambda^{\mathrm{wt}}))$ for its Galois representation.*

Let us fix a branch \mathfrak{a}_i of $\mathcal{H}(\bar{\rho}_i)$. For an integer $n \geq 1$, denote by $\mathfrak{a}_n^{(i)} \in \mathbb{T}(\mathfrak{a}_i)^0$ the image of T_n under the natural projection map

$$\mathfrak{h}^{\mathrm{ord}}(N_i(\Sigma_i)p^{\infty}; \mathbb{Z}_p) \twoheadrightarrow \mathbb{T}(\mathfrak{a}_i)^0.$$

One can then associate the formal q -expansion $\mathbf{f}(\mathfrak{a}_i) := \sum_{n=1}^{\infty} \mathfrak{a}_n^{(i)} \cdot q^n \in \mathbb{T}(\mathfrak{a}_i)^0[[q]]$. If \wp_i is an arithmetic height one prime of $\mathbb{T}_{\Sigma_i}(\bar{\rho}_i)$ of weight $k_i \geq 2$, by the control theory in [20, Theorem 1.2] and [13, Proposition 2.5.6(2)], there exists $\wp'_i \in \mathrm{Spec}(\mathbb{T}(\mathfrak{a}_i)^0)$ which pulls back to \wp_i under the mapping $\mathbb{T}(\mathfrak{a}_i) \rightarrow \mathbb{T}(\mathfrak{a}_i)^0$; each \wp_i -specialisation

$$\mathbf{f}_{\wp_i}(\mathfrak{a}_i) = \sum_{n=1}^{\infty} (\mathfrak{a}_n^{(i)} \bmod \wp'_i) \cdot q^n \in \mathcal{S}_{k_i}^{\mathrm{ord}}(N_i(\mathfrak{a}_i)p^{\infty}; \mathbb{T}(\mathfrak{a}_i)^0/\wp'_i)$$

yields a classical p -ordinary Hecke eigenform of weight k_i and tame level $N_i(\mathfrak{a}_i)$.

3.2. A generalisation to several variables

We now explain how to extend these ideas to deal with the 2^t -dimensional representation $\bar{\rho}_{1,\dots,t} = \bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t$. An obvious strategy is to construct the deformation ring directly from the residual representation, but we prefer to use the in-built tensor product structure on $\bar{\rho}_{1,\dots,t}$. At each $i \in \{1, \dots, t\}$, let $\mathcal{B}_i(\bar{\rho}_i; \Sigma_i) \subset \mathrm{Spec}(\mathbb{T}_{\Sigma_i}(\bar{\rho}_i))$ denote the branches of $\mathcal{H}(\bar{\rho}_i)$.

DEFINITION 3.3. — For a finite subset $\underline{\Sigma} = (\Sigma_1, \dots, \Sigma_t) \subset \text{Spec}(\mathbb{Z})^t$, one defines:

- (a) the multi-variable Hecke algebra to be the t -fold completed tensor product

$$\mathbb{T}_{\underline{\Sigma}}(\bar{\rho}_1, \dots, \bar{\rho}_t) := (\mathbb{T}'_{N_1(\Sigma_1)})_{\mathfrak{m}_1} \widehat{\otimes}_{\mathcal{O}} \cdots \widehat{\otimes}_{\mathcal{O}} (\mathbb{T}'_{N_t(\Sigma_t)})_{\mathfrak{m}_t}$$

which is of finite-type over $(\Lambda^{\text{wt}})^{\widehat{\otimes}^t} = \Lambda^{\text{wt}} \widehat{\otimes}_{\mathcal{O}} \cdots \widehat{\otimes}_{\mathcal{O}} \Lambda^{\text{wt}}$;

- (b) the Hida family attached to $\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t$ is then by definition equal to

$$\mathcal{H}(\bar{\rho}_1, \dots, \bar{\rho}_t) := \mathcal{H}(\bar{\rho}_1) \times \cdots \times \mathcal{H}(\bar{\rho}_t)$$

and consists of t -tuples of p -stabilised newforms $\underline{f} = (f_1, \dots, f_t)$ of weight ≥ 2 ;

- (c) the set of multi-branches for $\mathcal{H}(\bar{\rho}_1, \dots, \bar{\rho}_t)$ is given by the t -fold product

$$\underline{\mathcal{B}}(\bar{\rho}_1, \dots, \bar{\rho}_t; \underline{\Sigma}) := \mathcal{B}_1(\bar{\rho}_1; \Sigma_1) \times \cdots \times \mathcal{B}_t(\bar{\rho}_t; \Sigma_t).$$

Because each i -th component has only finitely many minimal associated primes i.e. $\#\mathcal{B}_i(\bar{\rho}_i; \Sigma_i) < \infty$ for all $i \in \{1, \dots, t\}$, in fact $\underline{\mathcal{B}}(\bar{\rho}_1, \dots, \bar{\rho}_t; \underline{\Sigma})$ is always a finite set. We will show stability of the cyclotomic λ -invariant over this set of multi-branches.

DEFINITION 3.4. — Fix a choice of multi-branch $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \dots, \mathfrak{a}_t) \in \underline{\mathcal{B}}(\bar{\rho}_1, \dots, \bar{\rho}_t; \underline{\Sigma})$. Then one defines the big Galois representation

$$\rho_{\mathbb{T}(\underline{\mathfrak{a}})^0} : G_{\mathbb{Q}} \rightarrow \text{GL}_{2^t}(\mathbb{T}(\underline{\mathfrak{a}})^0 \otimes_{\Lambda^{\text{wt}}} \text{Frac}(\Lambda^{\text{wt}}))$$

as the completed tensor product of the Λ^{wt} -adic representations $\rho_{\mathbb{T}(\mathfrak{a}_1)^0}, \dots, \rho_{\mathbb{T}(\mathfrak{a}_t)^0}$ where $\mathbb{T}(\underline{\mathfrak{a}})^0 := \mathbb{T}_{N_1(\mathfrak{a}_1)}^{\text{new}} / \mathfrak{a}'_1 \widehat{\otimes}_{\mathcal{O}} \cdots \widehat{\otimes}_{\mathcal{O}} \mathbb{T}_{N_t(\mathfrak{a}_t)}^{\text{new}} / \mathfrak{a}'_t$ is of finite-type over $(\Lambda^{\text{wt}})^{\widehat{\otimes}^t}$.

For instance, if one considers the tuple $\underline{\wp} = (\wp_1, \dots, \wp_t)$ where each $\wp_i \triangleleft \mathbb{T}(\mathfrak{a}_i)$ is an arithmetic prime of height one and weight $k_i \geq 2$, and if $\underline{\wp}' = (\wp'_1, \dots, \wp'_t)$ is the element of the fiber product $\prod_{i=1}^t \text{Spec}(\mathbb{T}(\mathfrak{a}_i)^0)$ which pulls back to $\underline{\wp}$, then

$$\rho_{\mathbb{T}(\underline{\mathfrak{a}})^0, \underline{\wp}} := \underline{\wp}' \circ \rho_{\mathbb{T}(\underline{\mathfrak{a}})^0} : G_{\mathbb{Q}} \rightarrow \text{GL}_{2^t}(\overline{\mathbb{Q}}_p)$$

takes values in a totally ramified extension $\mathcal{K}_{\underline{\wp}}$ of $\text{Frac } \mathbf{W}(\mathbb{F})$, and is equivalent to the p -adic representation attached to $\mathbf{f}_{\wp_1}(\mathfrak{a}_1) \otimes \cdots \otimes \mathbf{f}_{\wp_t}(\mathfrak{a}_t)$ with $(\mathbf{f}_{\wp_i}(\mathfrak{a}_i))_i \in \mathcal{H}(\bar{\rho}_1, \dots, \bar{\rho}_t)$. Furthermore, if $\pi_{\underline{\wp}}$ is a uniformiser for $\mathcal{K}_{\underline{\wp}}$ then $\rho_{\mathbb{T}(\underline{\mathfrak{a}})^0, \underline{\wp}} \bmod \pi_{\underline{\wp}} \cong \bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t$. (This representation $\rho_{\mathbb{T}(\underline{\mathfrak{a}})^0}$ is used later to define the error terms from Section 2.)

Let \mathcal{D}/\mathbb{Q} be a finite normal extension of fields, and write $\text{reg}_{\mathcal{D}/\mathbb{Q}}$ to denote the regular representation attached to $\text{Gal}(\mathcal{D}/\mathbb{Q})$. Pick a topological generator γ_0 of $\Gamma^{\text{cyc}} = \text{Gal}(\mathbb{Q}^{\text{cyc}}/\mathbb{Q}) \cong \mathbb{Z}_p$, so that sending $[1+p] \mapsto \gamma_0$ induces a (non-canonical) isomorphism $\vartheta : \mathcal{O}[[1+p\mathbb{Z}_p]] \xrightarrow{\sim} \mathcal{O}[[\Gamma^{\text{cyc}}]]$ of Iwasawa algebras. If $x \in \mathbb{Z}_p^\times$ then we write $[x] = x\omega(x)^{-1}$ for its projection to $1+p\mathbb{Z}_p$, so that $\vartheta([x]) = \gamma_0^{\log_p(x)/\log_p(1+p)}$.

LEMMA 3.5. — *Choosing a prime $q \neq p$ and $j \in \mathbb{Z}$, the cyclotomic λ -invariant of*

$$\det\left(1 - \text{Frob}_q \cdot X \mid (\rho_{f_1} \otimes \cdots \otimes \rho_{f_t} \otimes \text{reg}_{\mathcal{D}/\mathbb{Q}} \otimes \omega^j)_{I_q}\right) \Big|_{X=\vartheta([q]^{-1})}$$

for $\underline{f} = (f_1, \dots, f_t) \in \mathcal{H}(\bar{\rho}_1, \dots, t)$ depends only on the multi-branch on which \underline{f} lies.

Proof. — Assume that \underline{f} lies on the multi-branch $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \dots, \mathfrak{a}_t) \in \mathcal{B}(\bar{\rho}_1, \dots, t; \underline{\Sigma})$ in which case each $f_i = \mathbf{f}_{\wp_i}(\mathfrak{a}_i)$ for some arithmetic prime $\wp_i \in \text{Spec}(\mathbb{T}_{\Sigma_i}(\bar{\rho}_i)/\mathfrak{a}_i)$. If we view the eigenforms f_i as belonging to a common ring of q -expansions $\mathcal{K}_{\underline{f}}[[q]]$, then an important observation is that the $\mathcal{O}_{\mathcal{K}_{\underline{f}}}[[X]]$ -ideal generated by

$$E_{q,\underline{f},\mathcal{D},j}(X) = \det\left(1 - \text{Frob}_q \cdot X \mid (\rho_{f_1} \otimes \cdots \otimes \rho_{f_t} \otimes \text{reg}_{\mathcal{D}/\mathbb{Q}} \otimes \omega^j)_{I_q}\right)$$

has unit content, and therefore has a trivial μ -invariant. As a direct consequence,

$$\begin{aligned} \lambda\left(E_{q,\underline{f},\mathcal{D},j}(\vartheta([q]^{-1}))\right) &= \text{rank}_{\mathcal{O}_{\mathcal{K}_{\underline{f}}}/\pi_{\mathcal{K}_{\underline{f}}}}[[\Gamma^{\text{cyc}}]] \left(\frac{\mathcal{O}_{\mathcal{K}_{\underline{f}}}}{\langle \pi_{\mathcal{K}_{\underline{f}}}, E_{q,\underline{f},\mathcal{D},j}(\vartheta([q]^{-1})) \rangle} \right) \\ &= \text{rank}_{\mathcal{O}_{\mathcal{K}_{\underline{\varrho}}}/\pi_{\mathcal{K}_{\underline{\varrho}}}}[[\Gamma^{\text{cyc}}]] \left(\frac{\mathcal{O}_{\mathcal{K}_{\underline{\varrho}}}}{\langle \pi_{\mathcal{K}_{\underline{\varrho}}}, \underline{\varrho}' \circ E_{q,\underline{\mathfrak{a}},\mathcal{D},j}(\vartheta([q]^{-1})) \rangle} \right) \end{aligned}$$

where $E_{q,\underline{\mathfrak{a}},\mathcal{D},j}(X) := \det(1 - \text{Frob}_q \cdot X \mid (\rho_{\mathbb{T}(\underline{\mathfrak{a}}}^0 \otimes \text{reg}_{\mathcal{D}/\mathbb{Q}} \otimes \omega^j)_{I_q}) \in \mathbb{T}(\underline{\mathfrak{a}})^0[[X]]$. However $\underline{\varrho}' \circ E_{q,\underline{\mathfrak{a}},\mathcal{D},j}(X) \bmod \pi_{\underline{\varrho}} \in \mathbb{F}[[X]]$ is independent of the choice of height one primes (\wp_1, \dots, \wp_t) , hence $\lambda(E_{q,\underline{f},\mathcal{D},j}(\vartheta([q]^{-1})))$ depends on $\underline{\mathfrak{a}}$ but not $\underline{\varrho}$. \square

3.3. Existence of minimally ramified t -tuples

Subsequent calculations will require us to compare the cyclotomic Selmer group attached to $f_1 \otimes \cdots \otimes f_t$ with the residual version arising from $\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t$. It is therefore natural to ask: does there exist a tuple $\underline{f} = (f_1, \dots, f_t) \in$

$\mathcal{H}(\bar{\rho}_{1,\dots,t})$ of p -stabilised newforms such that the (tame) conductor of $f_1 \otimes \dots \otimes f_t$ coincides with the conductor of $\bar{\rho}_{1,\dots,t}$? Note that if $t = 1$, this is closely connected Serre's modularity conjecture [25, 35].

DEFINITION 3.6. — *A representation $\bar{\sigma} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_d(\mathbb{F})$ is called “twist-minimal” if $\mathrm{cond}(\bar{\sigma}) \leq \mathrm{cond}(\bar{\sigma} \otimes \psi)$ at all finite characters $\psi : G_{\mathbb{Q}} \rightarrow \mathbb{F}^{\times}$ satisfying $p \nmid \mathrm{cond}(\psi)$.*

The following technical result will prove to be a useful tool once we compare the π -torsion part of the cyclotomic Selmer group with the residual Selmer group.

THEOREM 3.7. — *Suppose that $\bar{\rho}_i$ is twist-minimal for each index $i \in \{1, \dots, t\}$, and assume at least one of the following conditions (see (H4a) and (H4b)) holds:*

- (i) *at each prime $l \neq p$, if $l^2 \mid \mathrm{cond}(\bar{\rho}_{i'})$ for some i' then $l^2 \nmid \mathrm{cond}(\bar{\rho}_i)$ if $i \neq i'$;*
- (ii) *the prime $p \geq 5$, and for all i one has $\mathrm{Im}(\bar{\rho}_i) \subset \mathrm{GL}_2(\mathbb{F}_p)$ with $\det(\bar{\rho}_i) = \omega$.*

Then there is a minimally ramified tuple $\underline{f}^{\dagger} = (f_1^{\dagger}, \dots, f_t^{\dagger}) \in \mathcal{H}(\bar{\rho}_{1,\dots,t})$ such that

$$\mathrm{cond}(\rho_{f_1^{\dagger}} \otimes \dots \otimes \rho_{f_t^{\dagger}})_{(p)} = \mathrm{cond}(\bar{\rho}_1 \otimes \dots \otimes \bar{\rho}_t).$$

Proof. — Let us first recall that for a residual representation $\bar{\sigma} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_d(\mathbb{F})$, the Artin conductor is defined as a product

$$\mathrm{cond}(\bar{\sigma}) = \prod_{\text{primes } l \neq p} l^{n(\bar{\sigma}, l)}$$

where

$$n(\bar{\sigma}, l) = \dim_{\mathbb{F}}(V_{\bar{\sigma}}) - \dim_{\mathbb{F}}(V_{\bar{\sigma}}^{I_l}) + \sum_{m=1}^{\infty} [\Upsilon : \Upsilon_m]^{-1} \cdot \dim_{\mathbb{F}}(V_{\bar{\sigma}} / V_{\bar{\sigma}}^{\Upsilon_m})$$

with $\Upsilon = \bar{\sigma}(I_l)$, and Υ_m denotes its higher inertia subgroups for integers $m \geq 1$. Similarly, if $\sigma : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_d(\mathcal{K})$ is some characteristic zero representation whose reduction is $\bar{\sigma}$, the non- p -part of its conductor is $\mathrm{cond}(\sigma)_{(p)} = \prod_{\text{primes } l \neq p} l^{n(\sigma, l)}$ where the exponent at l is given by

$$n(\sigma, l) = \dim_{\mathcal{K}}(V_{\sigma}) - \dim_{\mathcal{K}}(V_{\sigma}^{I_l}) + \sum_{m=1}^{\infty} [\Xi : \Xi_m]^{-1} \cdot \dim_{\mathcal{K}}(\mathrm{Gr} V_{\sigma} / \mathrm{Gr} V_{\sigma}^{\Xi_m}).$$

Note that $\sigma^{\mathrm{ss}} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(\mathrm{Gr} V_{\sigma})$ is the semi-simplification of σ with respect to Grothendieck's nilpotent operator, while $\Xi = \sigma^{\mathrm{ss}}(I_l)$, and $\Xi_m \subset \Xi$ are its higher inertia subgroups for $m \geq 1$.

It is well known that the Swan conductors of σ^{ss} and $\bar{\sigma}$ at $l \neq p$ are equal to each other (e.g. see [30, Section 1]), implying that

$$\begin{aligned} n(\sigma, l) - n(\bar{\sigma}, l) &= d - \dim_{\mathcal{K}}(V_{\sigma}^{I_l}) - \left(d - \dim_{\mathbb{F}}(V_{\bar{\sigma}}^{I_l}) \right) \\ &= \dim_{\mathbb{F}}(V_{\bar{\sigma}}^{I_l}) - \dim_{\mathcal{K}}(V_{\sigma}^{I_l}) \end{aligned}$$

must be a non-negative integer; hence $\text{cond}(\bar{\sigma})$ will always divide into $\text{cond}(\sigma)_{(p)}$. In particular if $\bar{\sigma} = \bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_t$, the proof of our theorem reduces to showing that:

“There exists $\underline{f}^{\dagger} = (f_1^{\dagger}, \dots, f_t^{\dagger}) \in \mathcal{H}(\bar{\rho}_1, \dots, \bar{\rho}_t)$ such that if either (i) or (ii) holds, then

$$(3.1) \quad \dim_{\mathbb{F}}(H^0(I_l, V_{\bar{\rho}_1, \dots, \bar{\rho}_t})) = \dim_{\mathcal{K}^{\dagger}}(H^0(I_l, V_{\rho_{f_1^{\dagger}} \otimes \cdots \otimes \rho_{f_t^{\dagger}}}))$$

at each prime $l \neq p$.”

To establish this assertion, we analyse each constituent in the tensor product. For $i \in \{1, \dots, t\}$ there exists a p -stabilised newform $f_i^{\dagger} \in \mathcal{K}_i^{\dagger}[[q]]$ of weight ≥ 2 whose tame level $N_{f_i^{\dagger}}^{\text{tame}}$ coincides with $\text{cond}(\bar{\rho}_i)$, and such that $\rho_{f_i^{\dagger}} \bmod \pi_{\mathcal{K}_i^{\dagger}} \cong \bar{\rho}_i$. Indeed if one allows p to divide its level where necessary, then one can even assume that this eigenform f_i^{\dagger} has weight 2.

Note \mathcal{K}_i^{\dagger} is a totally ramified extension of $\text{Frac} \mathbf{W}(\mathbb{F})$, so we define \mathcal{K}^{\dagger} to be the compositum of these \mathcal{K}_i^{\dagger} 's, which is again a totally ramified extension of $\text{Frac} \mathbf{W}(\mathbb{F})$. Without loss of generality, we will assume that each $\rho_{f_i^{\dagger}}$ takes values in $\text{GL}_2(\mathcal{K}^{\dagger})$. Choosing a uniformiser π^{\dagger} of $\mathcal{O}_{\mathcal{K}^{\dagger}}$, it follows that $\rho_{f_1^{\dagger}} \otimes \cdots \otimes \rho_{f_t^{\dagger}} \bmod \pi^{\dagger} \cong \bar{\rho}_1, \dots, \bar{\rho}_t$ as $G_{\mathbb{Q}}$ -representations, and let us therefore set $\sigma := \rho_{f_1^{\dagger}} \otimes \cdots \otimes \rho_{f_t^{\dagger}}$.

At each prime $l \neq p$, the decomposition group $G_{\mathbb{Q}_l} = \text{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$ contains the filtration $G_{\mathbb{Q}_l} \supset I_l \supset \mathcal{Q}_l \supset \mathcal{P}_l$, whose respective quotients satisfy $G_{\mathbb{Q}_l}/I_l \cong \text{Gal}(\bar{\mathbb{F}}_l/\mathbb{F}_l) \cong \hat{\mathbb{Z}}$, $I_l/\mathcal{P}_l \cong \prod_{q \neq l} \mathbb{Z}_q(1)$ and $\mathcal{Q}_l/\mathcal{P}_l \cong \prod_{q \neq l, p} \mathbb{Z}_q(1)$.

Here \mathcal{P}_l denotes the wild inertia subgroup (which is the pro- l -Sylow subgroup of I_l). We make the following claim: *if l divides $\text{cond}(\bar{\rho}_i)$ then $\bar{\rho}_i(\mathcal{Q}_l)$ is non-trivial.*

Suppose the claim were false, i.e. there exists an $l \mid \text{cond}(\bar{\rho}_i)$ with $\bar{\rho}_i(\mathcal{Q}_l) = \{1\}$. By the minimality conditions on f_i^{\dagger} and $\bar{\rho}_i$, clearly $n(\rho_{f_i^{\dagger}}, l) \leq n(\rho_{f_i^{\dagger}} \otimes \psi, l)$ at all twists by characters $\psi : \text{Gal}(\mathbb{Q}_l(\mu_l)/\mathbb{Q}_l) \rightarrow \mathcal{O}_{\mathcal{K}^{\dagger}}^{\times}$ having trivial reduction mod π^{\dagger} . Since $\rho_{f_i^{\dagger}}$ is also ramified at l the conditions (a,b,c) of [30, Proposition 2.3] are satisfied, which means that $\rho_{f_i^{\dagger}}$ belongs (under the local Langlands correspondence for GL_2) to one of the following:

- $l \equiv 1 \pmod{p}$, f_i^\dagger is principal series at l with $n(\rho_{f_i^\dagger}, l) = 1 > 0 = n(\bar{\rho}_i, l)$;
- f_i^\dagger is special series at the prime l with again $n(\rho_{f_i^\dagger}, l) = 1 > 0 = n(\bar{\rho}_i, l)$;
- $l \equiv -1 \pmod{p}$, f_i^\dagger is supercuspidal at l with $n(\rho_{f_i^\dagger}, l) = 2 > n(\bar{\rho}_i, l)$.

In all three cases $n(\rho_{f_i^\dagger}, l) > n(\bar{\rho}_i, l)$ implying that $N_{f_i^\dagger}^{\text{tame}} \neq \text{cond}(\bar{\rho}_i)$, contradiction!

Now suppose $l \mid \text{cond}(\bar{\rho}_i)$ so that $\bar{\rho}_i(\mathcal{Q}_l) \neq \{1\}$. As $\text{Ker}(\text{GL}_2(\mathcal{O}_{\mathcal{K}^\dagger}) \rightarrow \text{GL}_2(\mathbb{F}))$ is a pro- l -group, and the dimension of the invariant spaces for the corresponding inertia subgroups $\Upsilon_m \subset \bar{\rho}_i(I_l)$ and $\Xi_m \subset \rho_{f_i^\dagger}^{\text{ss}}(I_l)$ agree for $m \geq 1$, it follows that $\rho_{f_i^\dagger}(\mathcal{Q}_l)$ is isomorphic to $\bar{\rho}_i(\mathcal{Q}_l)$. Consequently, there are two possible scenarios:

- (a) $H^0(\mathcal{Q}_l, V_{\bar{\rho}_i})$ and $H^0(\mathcal{Q}_l, V_{f_i^\dagger})$ are both zero, and $n(\bar{\rho}_i, l) \geq 2$; or
- (b) $\dim_{\mathbb{F}}(H^0(\mathcal{Q}_l, V_{\bar{\rho}_i})) = \dim_{\mathcal{K}^\dagger}(H^0(\mathcal{Q}_l, V_{f_i^\dagger})) = 1$,
and $\rho_{f_i^\dagger}|_{G_{\mathcal{Q}_l}} \sim \begin{pmatrix} \psi_{1,i} & 0 \\ 0 & \psi_{2,i} \end{pmatrix}$ where $\psi_{1,i}, \psi_{2,i} : G_{\mathcal{Q}_l} \rightarrow \mathcal{O}_{\mathcal{K}^\dagger}^\times$ satisfy $\psi_{1,i}(\mathcal{Q}_l) = \{1\}$ and $\psi_{2,i}(\mathcal{Q}_l) \neq \{1\}$.

In case (b) we must have $\psi_{1,i}(I_l) = \{1\}$ otherwise the level of f_i^\dagger can be further reduced via twisting, i.e. for some ψ' factoring through $\text{Gal}(\mathbb{Q}_l(\mu_l)/\mathbb{Q}_l)$ with trivial reduction we would then find $n(\rho_{f_i^\dagger} \otimes \psi', l) < n(\rho_{f_i^\dagger}, l) = n(\bar{\rho}_i, l)$, which is nonsense.

We proceed by verifying that at each individual prime l dividing $\prod_{i=1}^t N_{f_i^\dagger}^{\text{tame}}$, the desired equality

$$(3.1) \quad \dim_{\mathbb{F}}(H^0(I_l, V_{\bar{\rho}_{1,\dots,t}})) = \dim_{\mathcal{K}^\dagger}(H^0(I_l, V_{\rho_{f_1^\dagger} \otimes \dots \otimes \rho_{f_t^\dagger}}))$$

holds under either of the hypotheses (i) or (ii) from the statement of the theorem. Suppose that $\mathfrak{S}_l \subset \{1, \dots, t\}$ denotes the subset of indices satisfying $n(\bar{\rho}_i, l) > 0$ if and only if $i \in \mathfrak{S}_l$. Let us begin by assuming that the first condition is in place.

(i). — At each prime $l \neq p$, if $l^2 \mid \text{cond}(\bar{\rho}_{i'})$ for some i' then $l^2 \nmid \text{cond}(\bar{\rho}_i)$ if $i \neq i'$. Clearly either $n(\bar{\rho}_i, l) = 1$ for all $i \in \mathfrak{S}_l$, or instead $n(\bar{\rho}_{i'}, l) \geq 2$ for some $i' \in \mathfrak{S}_l$ with $n(\bar{\rho}_i, l) = 1$ at each $i \in \mathfrak{S}_l - \{i'\}$.

If we are in the former scenario then upon choosing suitable bases, our previous discussion for (b) implies both $\rho_{f_i^\dagger}|_{G_{\mathcal{Q}_l}}$ and $\bar{\rho}_i|_{G_{\mathcal{Q}_l}}$ are equivalent to $\begin{pmatrix} \psi_{1,i} & 0 \\ 0 & \psi_{2,i} \end{pmatrix}$ where $\psi_{1,i}$ is unramified and $\psi_{2,i}(\mathcal{Q}_l) \neq \{1\}$. A straightforward

calculation shows

$$\dim_{\mathbb{F}} \left(H^0(I_l, V_{\bar{\rho}_1, \dots, t}) \right) = \dim_{\mathcal{K}^\dagger} \left(H^0(I_l, V_{\rho_{f_1^\dagger} \otimes \dots \otimes \rho_{f_t^\dagger}}) \right) = 2^{t - \#\mathfrak{S}_l} \times m_l$$

where the positive integer

$$m_l := \# \left\{ (\delta_i)_i \in \mathbb{F}_2^{\#\mathfrak{S}_l} \mid \prod_{i \in \mathfrak{S}_l} \psi_{2,i}^{\delta_i} \text{ is unramified at } l \right\}.$$

Alternatively, suppose instead we are in the latter situation, so that $n(\bar{\rho}_{i'}, l) \geq 2$ while $n(\bar{\rho}_i, l) \leq 1$ at each $i \neq i'$. The action of inertia on $V_{f_{i'}^\dagger}$ cannot factor through $I_l^{\text{ab}} \cong \frac{I_l}{[I_l, I_l]}$ otherwise we could find a twist $f_{i'}^\dagger \otimes \psi''$ with $n(\rho_{f_{i'}^\dagger} \otimes \psi'', l) < n(\rho_{f_{i'}^\dagger}, l)$, which contradicts the equality $n(\rho_{f_{i'}^\dagger}, l) = n(\bar{\rho}_{i'}, l)$ and the twist-minimality of $\bar{\rho}_{i'}$. Consequently I_l acts on both $V_{f_{i'}^\dagger}$ and $V_{\bar{\rho}_{i'}}$ through a finite non-abelian group, whilst its action on each $V_{f_i^\dagger}$ and $V_{\bar{\rho}_i}$ for $i \neq i'$ is through I_l^{ab} . One concludes that

- $H^0(I_l, V_{\bar{\rho}_1, \dots, t}) = H^0(I_l^{\text{ab}}, V_{\bar{\rho}_{i'}^{[I_l, I_l]} \otimes_{\mathbb{F}} \bigotimes_{i \neq i'} V_{\bar{\rho}_i}}) = \{0\}$
- $H^0(I_l, V_{\rho_{f_1^\dagger} \otimes \dots \otimes \rho_{f_t^\dagger}}) = H^0(I_l^{\text{ab}}, V_{f_{i'}^\dagger^{[I_l, I_l]} \otimes_{\mathcal{K}^\dagger} \bigotimes_{i \neq i'} V_{f_i^\dagger}}) = \{0\}.$

(ii). — The prime $p \geq 5$, and for all i one has $\text{Im}(\bar{\rho}_i) \subset \text{GL}_2(\mathbb{F}_p)$ with $\det(\bar{\rho}_i) = \omega$. Under these circumstances, for each $i \in \{1, \dots, t\}$ there exists an elliptic curve E_i defined over \mathbb{Q} such that f_i^\dagger is the p -stabilisation of the weight two newform associated to E_i by modularity (note that each E_i has either good ordinary or bad multiplicative reduction at p). Let $\rho_{E_i, p^\infty} : G_{\mathbb{Q}} \rightarrow \text{Aut}_{\mathbb{Z}_p}(T_p(E_i))$ denote the Galois representation arising from the action on its Tate module $T_p(E_i) := \varprojlim_m E_i[p^m]$.

Choose a prime $l \neq p$ with l dividing $\prod_{i=1}^t \text{cond}(E_i)$. Let \mathcal{J}_l consist of the indices i with $\text{ord}_l(j_{E_i}) \geq 0$, where j_{E_i} is the j -invariant of the elliptic curve E_i . If $i \in \{1, \dots, t\} - \mathcal{J}_l$ so that $\text{ord}_l(j_{E_i}) < 0$, then E_i does not have bad additive reduction over \mathbb{Q}_l , otherwise twisting E_i over $\mathbb{Q}(\sqrt{(-1)^{(l-1)/2} \cdot l})/\mathbb{Q}$ would imply

$$n \left(\rho_{E_i, p^\infty} \otimes \left(\frac{(-1)^{(l-1)/2} \cdot l}{-} \right), l \right) = 1 < n(\rho_{E_i, p^\infty}, l) = n(\bar{\rho}_i, l)$$

violating the equality $|\text{cond}(E_i)|_l^{-1} = |\text{cond}(\bar{\rho}_i)|_l^{-1}$ and the twist-minimality of $\bar{\rho}_i$.

Alternately if $i \in \mathcal{J}_l$ so that $\text{ord}_l(j_{E_i}) \geq 0$, then by the fundamental work of Serre [34, Section 5.6] it is known that the action of I_l on $T_p(E_i)$ factors through a finite group, $\Phi_l^{(i)}$, which has one of the following possible structures listed below:

- if $l \nmid \text{cond}(E_i)$ then $\Phi_l^{(i)}$ is the trivial group;
- if $l \mid \text{cond}(E_i)$ and $l \geq 5$, then $\Phi_l^{(i)}$ is cyclic with $\#\Phi_l^{(i)} \in \{2, 3, 4, 6\}$;
- if $l \mid \text{cond}(E_i)$ and $l = 3$, then either $\Phi_l^{(i)}$ is cyclic with $\#\Phi_l^{(i)} \in \{2, 3, 4, 6\}$, or instead $\Phi_l^{(i)} \cong C_4 \rtimes C_3$;
- if $l \mid \text{cond}(E_i)$ and $l = 2$, then either $\Phi_l^{(i)}$ is cyclic with $\#\Phi_l^{(i)} \in \{2, 3, 4, 6\}$, or $\Phi_l^{(i)} \cong \text{SL}_2(\mathbb{F}_3)$, or lastly $\Phi_l^{(i)}$ is the quaternion group of order 8.

We write $\Phi_l^{(i)} = (\Phi_l^{(i)})_p \rtimes \rho_{E_i, p^\infty}(\mathcal{Q}_l)$ where $(\Phi_l^{(i)})_p$ is a quotient of $I_l/\mathcal{Q}_l \cong \mathbb{Z}_p(1)$; since $p \geq 5$ clearly $\gcd(\#\Phi_l^{(i)}, p) = 1$, and so $(\Phi_l^{(i)})_p$ acts trivially on $\rho_{E_i, p^\infty}(\mathcal{Q}_l)$. Exploiting the isomorphisms $\rho_{E_i, p^\infty}(\mathcal{Q}_l) \cong \bar{\rho}_i(\mathcal{Q}_l)$ from earlier and using induction, $\bigotimes_{i \in \mathcal{J}_l} \rho_{E_i, p^\infty}(I_l) \cong \bigotimes_{i \in \mathcal{J}_l} \bar{\rho}_i(I_l)$. Passing to the algebraic closures of \mathbb{F}_p and \mathcal{K}^\dagger , one obtains a decomposition of I_l -modules

$$\bar{\mathbb{F}} \otimes_{\mathbb{F}_p} \left(\bigotimes_{i \in \mathcal{J}_l} V_{\bar{\rho}_i} \right)^{[I_l, I_l]} \cong \bigoplus_{c \in \mathfrak{C}} \bar{\mathbb{F}} \cdot \eta_c \quad \text{and} \quad \bar{\mathbb{Q}}_p \otimes_{\mathcal{K}^\dagger} \left(\bigotimes_{i \in \mathcal{J}_l} V_{f_i^\dagger} \right)^{[I_l, I_l]} \cong \bigoplus_{c \in \mathfrak{C}} \bar{\mathbb{Q}}_p \cdot \eta_c$$

where the inertial characters $\eta_c : I_l \twoheadrightarrow I_l^{\text{ab}} \rightarrow \bar{\mathbb{Q}}_p^\times$ are indexed by the finite set \mathfrak{C} .

For those indices $i \in \{1, \dots, t\} - \mathcal{J}_l$ we have already seen that E_i must have bad multiplicative reduction at l with $l \parallel \text{cond}(E_i)$, and as discussed in (b) earlier

- $\dim_{\mathbb{F}_p} (H^0(I_l, V_{\bar{\rho}_i})) = \dim_{\mathcal{K}^\dagger} (H^0(I_l, V_{f_i^\dagger})) = 1$, and

$$\rho_{f_i^\dagger}|_{G_{\mathbb{Q}_l}} \sim \begin{pmatrix} \psi_{1,i} & 0 \\ 0 & \psi_{2,i} \end{pmatrix}$$

where $\psi_{1,i}, \psi_{2,i} : G_{\mathbb{Q}_l} \rightarrow \mathcal{O}_{\mathcal{K}^\dagger}^\times$ satisfy $\psi_{1,i}(I_l) = \{1\}$ and $\psi_{2,i}(\mathcal{Q}_l) \neq \{1\}$.

Combining these strands together, one first deduces that

$$\begin{aligned} \dim_{\mathcal{K}^\dagger} (H^0(I_l, V_{f_1^\dagger} \otimes \cdots \otimes V_{f_t^\dagger})) \\ = \dim_{\mathcal{K}^\dagger} H^0 \left(I_l^{\text{ab}}, \bigotimes_{i \in \mathcal{J}_l} V_{f_i^\dagger}^{[I_l, I_l]} \otimes_{\mathcal{K}^\dagger} \bigotimes_{i \in \{1, \dots, t\} - \mathcal{J}_l} V_{f_i^\dagger} \right) \end{aligned}$$

and secondly,

$$\begin{aligned} \dim_{\mathbb{F}_p} (H^0(I_l, V_{\bar{\rho}_1} \otimes \cdots \otimes V_{\bar{\rho}_t})) \\ = \dim_{\mathbb{F}_p} H^0 \left(I_l^{\text{ab}}, \bigotimes_{i \in \mathcal{J}_l} V_{\bar{\rho}_i}^{[I_l, I_l]} \otimes_{\mathbb{F}_p} \bigotimes_{i \in \{1, \dots, t\} - \mathcal{J}_l} V_{\bar{\rho}_i} \right). \end{aligned}$$

A basic counting argument then reveals that both of these dimensions are equal to the non-negative integer

$$m'_l := \sum_{c \in \mathfrak{C}} \# \left\{ (\delta_i)_i \in \mathbb{F}_2^{t - \#\mathcal{I}_l} \left| \prod_{i \in \{1, \dots, t\} - \mathcal{I}_l} \psi_{2,i}^{\delta_i} \cdot \eta_c \text{ is unramified at } l \right. \right\}$$

thereby completing the demonstration of the theorem. \square

Remark. — One strongly suspects the existence of a minimally-ramified tuple $\underline{f}^\dagger = (f_1^\dagger, \dots, f_t^\dagger) \in \mathcal{H}(\bar{\rho}_{1, \dots, t})$ satisfying $\text{cond}(\rho_{f_1^\dagger} \otimes \dots \otimes \rho_{f_t^\dagger})_{(p)} = \text{cond}(\bar{\rho}_{1, \dots, t})$ without needing to assume either of the conditions (i) or (ii) in Theorem 3.7 hold, in other words that Equation (3.1) should be true unconditionally for some \underline{f}^\dagger . Unfortunately an argument that could establish this claim has so far proved elusive.

4. The Λ^{cyc} -structure of Selmer groups

We now outline how to choose an appropriate set of local conditions (at finite places) which allow us to associate a Selmer group to a tuple of eigenforms $\underline{f} \in \mathcal{H}(\bar{\rho}_{1, \dots, t})$. For $t = 2$ these local conditions cut out the double product Selmer group in [27], whilst for $t = 3$ they cut out the “unbalanced” triple product version from [23].

4.1. Local conditions and the f_1 -filtration

Given a tuple $\underline{f} = (f_1, \dots, f_t)$ lying in $\mathcal{H}(\bar{\rho}_{1, \dots, t})$ of level $\underline{N}_{\underline{f}} = (N_{f_1}, \dots, N_{f_t})$ and character $\underline{\varepsilon}_{\underline{f}} = (\varepsilon_{f_1}, \dots, \varepsilon_{f_t})$, one again considers Deligne’s p -adic Galois representations constructed in [12]:

$$\rho_{f_1} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathcal{K}_1), \dots, \rho_{f_t} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathcal{K}_t).$$

For each i , these are unramified at primes $l \nmid pN_{f_i}$ and satisfy

$$\text{Tr } \rho_{f_i}(\text{Frob}_l) = a_l(f_i).$$

By taking the compositum of the local fields \mathcal{K}_i we may assume every ρ_{f_i} takes values in a common $\text{GL}_2(\mathcal{K})$, where \mathcal{K} is a totally ramified extension of $\text{Frac } \mathbf{W}(\mathbb{F})$.

As mentioned before, for $i \in \{1, \dots, t\}$ let us denote by V_{f_i} the \mathcal{K} -vector space afforded by ρ_{f_i} . Henceforth we will fix a rank two $G_{\mathbb{Q}}$ -stable lattice $\mathbf{T}_{f_i} \subset V_{f_i}$, and the cofree $\mathcal{O}_{\mathcal{K}}$ -modules $A_{f_i} := V_{f_i}/\mathbf{T}_{f_i}$ naturally inherit

their $G_{\mathbb{Q}}$ -action from ρ_{f_i} . If $\pi = \pi_{\underline{f}, \mathcal{K}}$ denotes a uniformiser for $\mathcal{O}_{\mathcal{K}}$ then as local $G_{\mathbb{Q}_p}$ -modules, there exists a short exact sequence of π -divisible groups

$$0 \rightarrow \mathcal{K}/\mathcal{O}_{\mathcal{K}} \otimes \kappa_p^{k-1} \varepsilon_{f_i} \varphi_i^{-1} \rightarrow A_{f_i} \rightarrow \mathcal{K}/\mathcal{O}_{\mathcal{K}} \otimes \varphi_i \rightarrow 0$$

where the unramified character $\varphi_i : G_{\mathbb{Q}_p} \twoheadrightarrow G_{\mathbb{Q}_p}/I_p \rightarrow \mathcal{O}_{\mathcal{K}}^{\times}$ sends $\text{Frob}_p \mapsto a_p(f_i)$. We label the left-hand module as A'_{f_i} , and $A''_{f_i} = A_{f_i}/A'_{f_i}$ denotes the étale quotient. Similarly, for each of the lattices we have the exact sequence of $\mathcal{O}_{\mathcal{K}}[G_{\mathbb{Q}_p}]$ -modules

$$(4.1) \quad 0 \rightarrow \text{Fil}^+ \mathbf{T}_{f_i} \rightarrow \mathbf{T}_{f_i} \rightarrow \mathbf{T}_{f_i}^{\text{ét}} \rightarrow 0$$

where $\text{Fil}^+ \mathbf{T}_{f_i} \cong \mathcal{O}_{\mathcal{K}}(\kappa_p^{k-1} \varepsilon_{f_i} \varphi_i^{-1})$ and $\mathbf{T}_{f_i}^{\text{ét}} = \mathbf{T}_{f_i} / \text{Fil}^+ \mathbf{T}_{f_i} \cong \mathcal{O}_{\mathcal{K}}(\varphi_i)$.

DEFINITION 4.1. — Let $\underline{f} = (f_1, \dots, f_t) \in \mathcal{H}(\bar{\rho}_1, \dots, t)$ be a t -tuple of eigenforms.

- (i) We define $\mathbf{T}_{\underline{f}} := \mathbf{T}_{f_1} \otimes_{\mathcal{O}_{\mathcal{K}}} \cdots \otimes_{\mathcal{O}_{\mathcal{K}}} \mathbf{T}_{f_t}$ which is $\mathcal{O}_{\mathcal{K}}$ -free of rank 2^t , and set

$$A_{\underline{f}} := \mathbf{T}_{\underline{f}} \otimes \mathbb{Q}/\mathbb{Z} \cong (V_{f_1} \otimes_{\mathcal{K}} \cdots \otimes_{\mathcal{K}} V_{f_t}) / \mathbf{T}_{f_1} \otimes_{\mathcal{O}_{\mathcal{K}}} \cdots \otimes_{\mathcal{O}_{\mathcal{K}}} \mathbf{T}_{f_t}$$

which is the associated π -divisible group.

- (ii) We shall construct a pair of rank 2^{t-1} local $G_{\mathbb{Q}_p}$ -modules $\mathcal{F}^{\pm} \mathbf{T}_{\underline{f}}$ by taking

$$\mathcal{F}^+ \mathbf{T}_{\underline{f}} := (\text{Fil}^+ \mathbf{T}_{f_1}) \otimes_{\mathcal{O}_{\mathcal{K}}} \mathbf{T}_{f_2} \otimes_{\mathcal{O}_{\mathcal{K}}} \cdots \otimes_{\mathcal{O}_{\mathcal{K}}} \mathbf{T}_{f_t} \text{ and } \mathcal{F}^- \mathbf{T}_{\underline{f}} := \mathbf{T}_{\underline{f}} / \mathcal{F}^+ \mathbf{T}_{\underline{f}}$$

with the π -divisible versions, $A_{\underline{f}}^+ := \mathcal{F}^+ \mathbf{T}_{\underline{f}} \otimes \mathbb{Q}/\mathbb{Z}$ and $A_{\underline{f}}^- := (\mathbf{T}_{\underline{f}} / \mathcal{F}^+ \mathbf{T}_{\underline{f}}) \otimes \mathbb{Q}/\mathbb{Z}$. Both $\mathcal{F}^{\pm} \mathbf{T}_{\underline{f}}$ and $A_{\underline{f}}^{\pm}$ depend on making a distinguished choice of the eigenform f_1 (or more precisely, $\text{Fil}^+ \mathbf{T}_{f_1}$ and $\mathbf{T}_{f_1}^{\text{ét}}$) over the other forms f_2, \dots, f_t in this tuple.

For any profinite group G and topological G -module M , one writes $H^i(G, M)$ for its i -th continuous cohomology group. If L/K is an algebraic extension of fields and $G = \text{Gal}(L/K)$, then we abbreviate this group by $H^i(L/K, M)$. In particular for $L = \bar{K}$ a fixed algebraic closure of K , we shall set $H^i(K, M) := H^i(\bar{K}/K, M)$.

Let F be a number field, and suppose $\chi : G_F \rightarrow \mu_{p^\infty}$ is a finite order character. Adjoining the image of χ to \mathcal{K} yields yet another (possibly larger) totally ramified extension of $\text{Frac } \mathbf{W}(\mathbb{F})$, so without loss of generality we assume that $\text{Im}(\chi) \subset \mathcal{K}$. For every $j \in \mathbb{Z}$ and χ as above, one constructs $\chi \kappa_p^j$ -twisted $\mathcal{O}_{\mathcal{K}}[G_F]$ -modules by

$$\mathbf{T}_{\underline{f}}[\chi, j] := \mathbf{T}_{\underline{f}} \otimes \chi \kappa_p^j \quad \text{and} \quad A_{\underline{f}}[\chi, j] := A_{\underline{f}} \otimes \chi \kappa_p^j,$$

and makes an analogous definition for the local modules $\mathcal{F}^\pm \mathbf{T}_f[\chi, j]$ and $A_f^\pm[\chi, j]$. If G is a closed normal subgroup of $G_F = \text{Gal}(\bar{F}/F)$ then the short exact sequence $0 \rightarrow V_{\bar{\rho}_1, \dots, t} \otimes \omega^j \rightarrow A_f[\chi, j] \xrightarrow{\times \pi} A_f[\chi, j] \rightarrow 0$ induces a truncated exact sequence⁽¹⁾

$$(4.2) \quad 0 \rightarrow \frac{H^i(G, A_f[\chi, j])}{\pi} \xrightarrow{\partial_i} H^{i+1}(G, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j) \xrightarrow{\beta_{i+1}} H^{i+1}(G, A_f[\chi, j])[\pi] \rightarrow 0$$

in G -cohomology, for all indices $i \geq 0$.

DEFINITION 4.2. — If $\Omega \subset \text{Spec}(\mathbb{Z})$ is a finite set of primes with $p \notin \Omega$, we define the “ Ω -imprimitive $\chi \kappa_p^j$ -twisted Selmer group of f over F^{cyc} ” to be the kernel of the global-to-local restriction maps

$$H^1(F^{\text{cyc}}, A_f[\chi, j]) \xrightarrow{\prod_{\text{res}_\nu} \prod_{\nu \nmid p, \nu \notin \Omega^{\text{cyc}}} H^1(F_\nu^{\text{cyc}}, A_f[\chi, j]) \times \prod_{\mathfrak{p}|p} H_{\text{ord}}^1(F_{\mathfrak{p}}^{\text{cyc}}, A_f[\chi, j])}$$

where the first product is over the places ν of F^{cyc} not lying above $\Omega \cup \{p\}$, and

$$H_{\text{ord}}^1(F_{\mathfrak{p}}^{\text{cyc}}, A_f[\chi, j]) := \text{Im} \left(H^1(F_{\mathfrak{p}}^{\text{cyc}}, A_f[\chi, j]) \rightarrow H^1(I_{F_{\mathfrak{p}}^{\text{cyc}}}, A_f^+[\chi, j]) \right).$$

Notation. — We shall label this cyclotomic Selmer group as $\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_f)$.

PROPOSITION 4.3. — Suppose $H^0(F, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j) = \{0\}$ for some $j \in \mathbb{Z}$. Then

(a) $\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_f)$ is cofinitely-generated over the algebra

$$\Lambda_F^{\text{cyc}} := \mathcal{O}_{\mathcal{K}}[[\Gamma_F^{\text{cyc}}]];$$

(b) $\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_f)$ has finite π -torsion if and only if its Pontrjagin dual

$$\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_f)^\wedge := \text{Hom}_{\text{cont}} \left(\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_f), \mathcal{K}/\mathcal{O}_{\mathcal{K}} \right)$$

is a Λ_F^{cyc} -torsion module with a trivial μ -invariant;

(c) if $\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_f)[\pi]$ is finite then $\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_f)$ is π -divisible, in which case

$$\lambda \left(\text{char}_{\Lambda_F^{\text{cyc}}} \left(\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_f)^\wedge \right) \right) = \dim_{\mathbb{F}} \left(\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_f)[\pi] \right).$$

⁽¹⁾ Whenever $\mu_p \subset F$, writing the ω^j -twist in $H^{i+1}(G, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j)$ is completely redundant.

The proof of this result is identical to [9, Proposition 2.2] therefore we omit it. We should point out that the triviality of the $H^0(F, V_{\bar{\rho}_1, \dots, \bar{\rho}_t} \otimes \omega^j)$ is needed in (c) to establish the vanishing of both $H^0(F^{\text{cyc}}, A_{\underline{f}} \otimes \chi \kappa_p^s)$ and $H^0(F^{\text{cyc}}, (A_{\underline{f}} \otimes \chi \kappa_p^s)^*)$ for all $s \in \mathbb{Z}$. Indeed it is straightforward to show that as a discrete Λ_F^{cyc} -module,

$$\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_{\underline{f}}) \cong \text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j']}(\underline{A}_{\underline{f}}) \otimes [\kappa_p]^{j-j'}$$

if $j \equiv j' \pmod{p-1}$, so $\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_{\underline{f}})$ only depends on the class of j modulo $p-1$.

4.2. Comparison with the Bloch–Kato version

Throughout this section our goal is to compare $\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_{\underline{f}})$ to the objects Bloch and Kato defined in [1]. We begin by recalling some background material in [15] on p -adic representations. Fontaine has defined topological rings $\mathbf{B}_{\text{cris}} \subset \mathbf{B}_{\text{dR}}$ with a continuous $G_{\mathbb{Q}_p}$ -action. The discrete valuation field \mathbf{B}_{dR} has the ring of integers \mathbf{B}_{dR}^+ , the residue field \mathbb{C}_p , and a filtration $\text{Fil}^i \mathbf{B}_{\text{dR}} = \mathfrak{t}^i \cdot \mathbf{B}_{\text{dR}}^+$ given by powers of the uniformiser \mathfrak{t} for \mathbf{B}_{dR}^+ . Its subring \mathbf{B}_{cris} is equipped with a Frobenius operator φ , and a filtration induced from that on \mathbf{B}_{dR} , i.e. $\text{Fil}^i \mathbf{B}_{\text{cris}} = \mathfrak{t}^i \cdot \mathbf{B}_{\text{dR}}^+ \cap \mathbf{B}_{\text{cris}}$.

Let K be a finite extension of \mathbb{Q}_p . If we are given a vector space V with a continuous G_K -action, Bloch and Kato construct three subgroups of $H^1(K, V)$ via

$$\begin{aligned} H_e^1(K, V) &:= \text{Ker} \left(H^1(K, V) \rightarrow H^1(K, V \otimes \mathbf{B}_{\text{cris}}^{\varphi=1}) \right), \\ H_f^1(K, V) &:= \text{Ker} \left(H^1(K, V) \rightarrow H^1(K, V \otimes \mathbf{B}_{\text{cris}}) \right) \end{aligned}$$

and

$$H_g^1(K, V) := \text{Ker} \left(H^1(K, V) \rightarrow H^1(K, V \otimes \mathbf{B}_{\text{dR}}) \right).$$

For a G_K -stable \mathbb{Z}_p -lattice $\mathbf{T} \hookrightarrow V$ and $\star \in \{e, f, g\}$, one defines $H_{\star}^1(K, \mathbf{T})$ to be the preimage of each $H_{\star}^1(K, V)$, and secondly

$$H_{\star}^1(K, V/\mathbf{T}) = H_{\star}^1(K, V) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p/\mathbb{Z}_p.$$

Finally for infinite extensions L/\mathbb{Q}_p , one simply takes the direct limit of the relevant subgroups $H_{\star}^1(L', -)$ over the finite extensions L'/\mathbb{Q}_p contained inside L .

Now fix a number field F , and a Hecke character $\chi : G_F \rightarrow \mu_{p^\infty}$ of finite order.

DEFINITION 4.4. — For $\Omega \subset \text{Spec}(\mathbb{Z})$ a finite set such that $p \notin \Omega$ and $\star \in \{e, f, g\}$, the Bloch–Kato Selmer group is the kernel of the global-to-local restriction maps

$$H^1(F^{\text{cyc}}, A_{\underline{f}}[\chi, j]) \xrightarrow{\prod_{\nu} \text{res}_{\nu}} \prod_{\nu \nmid p, \nu \notin \Omega^{\text{cyc}}} H^1(I_{F_{\nu}^{\text{cyc}}}, A_{\underline{f}}[\chi, j]) \times \prod_{\mathfrak{p} \mid p} \frac{H^1(F_{\mathfrak{p}}^{\text{cyc}}, A_{\underline{f}}[\chi, j])}{H_{\star}^1(F_{\mathfrak{p}}^{\text{cyc}}, A_{\underline{f}}[\chi, j])}$$

where the first product is over the places ν of F^{cyc} not lying above $\Omega \cup \{p\}$, and

$$H_{\star}^1(F_{\mathfrak{p}}^{\text{cyc}}, A_{\underline{f}}[\chi, j]) = \varinjlim_n H_{\star}^1(F_{n, \mathfrak{p}_n}, A_{\underline{f}}[\chi, j]) \text{ at each choice of } \star \in \{e, f, g\}$$

with F_n/F denoting the finite layer of degree p^n in the cyclotomic \mathbb{Z}_p -extension.

Notation. — These Ω -imprimitive versions will be written as $H_{\star, \Omega}^1(F^{\text{cyc}}, A_{\underline{f}} \otimes \chi \kappa_p^j)$.

PROPOSITION 4.5. — Suppose that the eigenforms $\underline{f} = (f_1, \dots, f_t) \in \mathcal{H}(\bar{\rho}_{1, \dots, t})$ have weight $\underline{k} = (k_1, \dots, k_t) \geq \underline{2}$, and assume that $\chi = \mathbf{1}_F$ is the trivial character. If the inequalities $k_1 \geq 3 - t + \sum_{i=2}^t k_i$ and $2 - k_1 \leq j \leq t - 1 - \sum_{i=2}^t k_i$ both hold, and for $j = 0$ if $H^0(F_{\mathfrak{P}}(\mu_p), \mathcal{F}^{-\mathbf{T}}_{\underline{f}})$ vanishes at primes \mathfrak{P} of F dividing p , then

$$\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\mathbf{1}_F, j]}(A_{\underline{f}}) = H_{g, \Omega}^1(F^{\text{cyc}}, A_{\underline{f}} \otimes \kappa_p^j).$$

Furthermore, if the twisted representation $\rho_{f_1} \otimes \dots \otimes \rho_{f_t} \otimes \kappa_p^j$ does not satisfy an exceptional p -adic zero condition⁽²⁾ at those primes \mathfrak{P} of F dividing p , then

$$\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\mathbf{1}_F, j]}(A_{\underline{f}}) = H_{g, \Omega}^1(F^{\text{cyc}}, A_{\underline{f}} \otimes \kappa_p^j) = H_{f, \Omega}^1(F^{\text{cyc}}, A_{\underline{f}} \otimes \kappa_p^j).$$

The first inequality cuts out a subset of \mathbb{Z}^t in which the weight of f_1 dominates the other weights of f_2, \dots, f_t , therefore we call this the f_1 -unbalanced region. Note outside of this unbalanced region the Selmer groups above might not be equal.

Proof. — If ν is a place of F^{cyc} lying above a prime $\varpi \nmid p$ of F then because ϖ does not split completely in the cyclotomic \mathbb{Z}_p -extension, the quotient group $G_{F_{\nu}^{\text{cyc}}}/I_{F_{\nu}^{\text{cyc}}}$ has profinite degree prime to the residue characteristic of the field F_{ϖ} . Consequently, the restriction mapping $H^1(F_{\nu}^{\text{cyc}}, A_{\underline{f}} \otimes$

⁽²⁾ A p -adic representation V is said to satisfy an “exceptional p -adic zero condition” if either of 1 or p^{-1} occurs an eigenvalue of the Frobenius φ acting on a suitable subquotient of $D_{\text{cris}}(V)$.

$\kappa_p^j) \rightarrow H^1(I_{F_\nu^{\text{cyc}}}, A_{\underline{f}} \otimes \kappa_p^j)$ is injective, hence the local conditions at $\nu \mid \varpi$ agree for all of these Selmer groups. We thus focus attention on the local conditions at primes \mathfrak{p} of F^{cyc} dividing p .

Let \mathfrak{P} be a place of F above p and below \mathfrak{p} , and consider the field $K = F_{\mathfrak{P}}$. Recalling the sequence in (4.1), one knows the Hodge–Tate weights for each V_{f_i} are 0 and $1 - k_i$, in which case $\mathbf{V}_{\underline{f}} := V_{f_1} \otimes_K \cdots \otimes_K V_{f_t}$ has Hodge–Tate weights in

$$\underline{\text{HT}}(\mathbf{V}_{\underline{f}}) = \left\{ \sum_{i=1}^t \delta_i \cdot (1 - k_i) \mid \delta_i \in \{0, 1\} \right\}.$$

Setting $\mathcal{F}^\pm \mathbf{V}_{\underline{f}} = \mathcal{F}^\pm \mathbf{T}_{\underline{f}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, it is then easy to determine that for all $j \in \mathbb{Z}$:

- $\underline{\text{HT}}(\mathcal{F}^+ \mathbf{V}_{\underline{f}}(j)) = \left\{ 1 - k_1 - j + \sum_{i=2}^t \delta_i \cdot (1 - k_i) \mid \delta_i \in \{0, 1\} \right\}$
- $\underline{\text{HT}}(\mathcal{F}^- \mathbf{V}_{\underline{f}}(j)) = \left\{ -j + \sum_{i=2}^t \delta_i \cdot (1 - k_i) \mid \delta_i \in \{0, 1\} \right\}.$

As an immediate corollary, one has $\underline{\text{HT}}(\mathcal{F}^+ \mathbf{V}_{\underline{f}}(j)) \subset [1 - k_1 - j - r_{\underline{f}}, 1 - k_1 - j]$ and $\underline{\text{HT}}(\mathcal{F}^- \mathbf{V}_{\underline{f}}(j)) \subset [-j - r_{\underline{f}}, -j]$ where the integer $r_{\underline{f}} = 1 - t + \sum_{i=2}^t k_i \geq 0$.

Now the assumption $2 - k_1 \leq j \leq t - 1 - \sum_{i=2}^t k_i$ implies $j \in [2 - k_1, -r_{\underline{f}}] \cap \mathbb{Z}$, hence $\mathcal{F}^+ \mathbf{V}_{\underline{f}}(j)$ has strictly negative HT-weights, whilst $\mathcal{F}^- \mathbf{V}_{\underline{f}}(j)$ has only positive or zero HT-weights. One thereby obtains the following two equalities:

$$\text{Fil}^0 D_{\text{dR}, K}(\mathcal{F}^+ \mathbf{V}_{\underline{f}}(j)) = \{0\}$$

and

$$\text{Fil}^0 D_{\text{dR}, K}(\mathcal{F}^- \mathbf{V}_{\underline{f}}(j)) = D_{\text{dR}, K}(\mathcal{F}^- \mathbf{V}_{\underline{f}}(j))$$

where, as usual, $D_{\text{dR}, K}(V) := (V \otimes \mathbf{B}_{\text{dR}})^{G_K}$ and $\text{Fil}^0 D_{\text{dR}, K}(V) := (V \otimes \mathbf{B}_{\text{dR}}^+)^{G_K}$. In other words, if the integer $j \in [2 - k_1, -r_{\underline{f}}]$ then the representation $\mathbf{V}_{\underline{f}}(j)$ satisfies a “strong Panchishkin condition” as a local G_K -module, at each $K = F_{\mathfrak{P}}$. Thus for every index $n \geq 0$, the G_{K_n} -cohomology sequence

$$(4.3) \quad H^1(K_n, \mathcal{F}^+ \mathbf{V}_{\underline{f}}(j)) \rightarrow H_g^1(K_n, \mathbf{V}_{\underline{f}}(j)) \rightarrow H_g^1(K_n, \mathcal{F}^- \mathbf{V}_{\underline{f}}(j))$$

must be exact, upon applying the same reasoning as Nekovář in [32, §6.7].

For simplicity, let us first assume that $\mu_p \subset F$ so that $K^{\text{cyc}} = K(\mu_{p^\infty})$. As a local G_K -module, each V_{f_i} sits inside the short exact sequence

$$0 \rightarrow \text{Fil}^+ V_{f_i} \rightarrow V_{f_i} \rightarrow V_{f_i}^{\text{ét}} \rightarrow 0$$

where $\mathrm{Fil}^+ V_{f_i} \cong \mathcal{K}(\kappa_p^{k-1} \varepsilon_{f_i} \varphi_i^{-1})$ and $V_{f_i}^{\mathrm{ét}} \cong \mathcal{K}(\varphi_i)$, with $\varepsilon_{f_i}, \varphi_i$ described in (4.1). Now choosing any integer $n \geq \max\{\mathrm{ord}_p(\mathrm{cond}(\varepsilon_i)) \mid i = 1, \dots, t\}$, the restriction

$$\rho_{f_i}|_{G_{K(\mu_{p^n})}} \in \mathrm{Ext}_{\mathcal{K}[G_{K(\mu_{p^n})}]}^1 \left(\mathcal{K}(\varphi_i), \mathcal{K}(\kappa_p^{k-1}(\varepsilon_{f_i})_{(p)} \varphi_i^{-1}) \right)$$

corresponds to an extension class of two semistable $\mathrm{Gal}(\bar{K}/K(\mu_{p^n}))$ -representations, and V_{f_i} is therefore a semistable representation over this larger extension $K(\mu_{p^n})$. Applying [14, Théorème 5.1.7(i)], the tensor product $\mathbf{V}_{\underline{f}}$ of the individual V_{f_i} 's is semistable too, and then Théorème 5.1.7(iii) of *op. cit.* implies both the quotient $\mathcal{F}^- \mathbf{V}_{\underline{f}}$ and subrepresentation $\mathcal{F}^+ \mathbf{V}_{\underline{f}} \subset \mathbf{V}_{\underline{f}}$ are semistable $\mathcal{K}[G_{K(\mu_{p^n})}]$ -modules. Twisting the above by κ_p^j , one deduces the semistability of $\mathcal{F}^+ \mathbf{V}_{\underline{f}}(j)$ and $\mathcal{F}^- \mathbf{V}_{\underline{f}}(j)$ over these larger field extensions $K(\mu_{p^n})$. As a consequence, we may establish that

$$\begin{aligned} H_g^1(K(\mu_{p^n}), \mathcal{F}^- \mathbf{V}_{\underline{f}}(j)) &= H_f^1(K(\mu_{p^n}), \mathcal{F}^- \mathbf{V}_{\underline{f}}(j)) \\ &\cong \frac{D_{\mathrm{cris}, K(\mu_{p^n})}(\mathcal{F}^- \mathbf{V}_{\underline{f}}(j))}{(\varphi - 1) \cdot D_{\mathrm{cris}, K(\mu_{p^n})}(\mathcal{F}^- \mathbf{V}_{\underline{f}}(j))} \end{aligned}$$

by utilising weak admissibility and Bloch–Kato [1, Corollary 3.8.4].

Let us write $\tilde{K} = K \cap \mathbb{Q}_p^{\mathrm{unr}}$ to denote the maximal unramified subfield in K . A straightforward calculation shows that

$$\begin{aligned} \dim_{\tilde{K}} \left(\frac{D_{\mathrm{cris}, K(\mu_{p^n})}(\mathcal{F}^- \mathbf{V}_{\underline{f}}(j))}{(\varphi - 1)} \right) \\ &= \dim_{\tilde{K}} \left(D_{\mathrm{cris}, K(\mu_{p^n})}(\mathcal{F}^- \mathbf{V}_{\underline{f}}(j))^{\varphi=1} \right) \\ &= \dim_{\tilde{K}} \left(D_{\mathrm{cris}, K(\mu_{p^n})}(\mathcal{F}^- \mathbf{V}_{\underline{f}}(j)) \cap \mathrm{Fil}^0 D_{\mathrm{dR}, K(\mu_{p^n})}(\mathcal{F}^- \mathbf{V}_{\underline{f}}(j)) \right) \end{aligned}$$

which coincides with the dimension of $H^0(K(\mu_{p^n}), \mathcal{F}^- \mathbf{V}_{\underline{f}}(j))$. One concludes that:

- if $j < 0$ then this latter group vanishes by HT-weight considerations
- if $j = 0$ then $H^0(K(\mu_p), \mathcal{F}^- \mathbf{V}_{\underline{f}}) = \{0\} \implies H^0(K(\mu_{p^n}), \mathcal{F}^- \mathbf{V}_{\underline{f}}) = \{0\}$.

In summary, we have shown that $H_{\star}^1(K(\mu_{p^n}), \mathcal{F}^- \mathbf{V}_{\underline{f}}(j))$ is zero for $\star \in \{f, g\}$. From the exactness of (4.3) and noting $K_n = K(\mu_{p^{n+n_0}})$ for some $n_0 \in \mathbb{N}$ if $\mu_p \subset F$,

$$H_g^1(K_n, \mathbf{V}_{\underline{f}}(j)) = \mathrm{Im} \left(H^1(K_n, \mathcal{F}^+ \mathbf{V}_{\underline{f}}(j)) \rightarrow H^1(K_n, \mathbf{V}_{\underline{f}}(j)) \right).$$

Taking the direct limit over n , the first assertion in our proposition will follow.

Alternatively, let us now assume $\mu_p \not\subset F$ so that $[K(\mu_{p^\infty}) : K^{\text{cyc}}] = p-1$. Repeating the above argument once more identifies $H_g^1(K(\mu_{p^n}), \mathbf{V}_{\underline{f}}(j))$ with the image of $H^1(K(\mu_{p^n}), \mathcal{F}^+ \mathbf{V}_{\underline{f}}(j))$ inside $H^1(K(\mu_{p^n}), \mathbf{V}_{\underline{f}}(j))$ provided that $n \gg 0$. One can then decompose these H^1 -cohomology groups into their ω^i -eigenspaces, i.e. $H^1(K(\mu_{p^n}), -) \cong \bigoplus_{i=0}^{p-2} H^1(K(\mu_{p^n}), -)^{(\omega^i)} \cong \bigoplus_{i=0}^{p-2} H^1(K_{n-1}, - \otimes \omega^{-i})$, whence

$$H_g^1(K_{n-1}, \mathbf{V}_{\underline{f}}(j)) = \text{Im} \left(H^1(K_{n-1}, \mathcal{F}^+ \mathbf{V}_{\underline{f}}(j)) \rightarrow H^1(K_{n-1}, \mathbf{V}_{\underline{f}}(j)) \right).$$

As we already saw, the first assertion follows from taking the direct limit over n .

It remains to establish the second assertion which was stated in our proposition. In fact, one can exploit the perfect duality (for de Rham representations) between

$$\frac{H_g^1(K_n, \mathbf{V}_{\underline{f}}(j))}{H_f^1(K_n, \mathbf{V}_{\underline{f}}(j))} \quad \text{and} \quad \frac{D_{\text{cris}, K_n}(\mathbf{V}_{\underline{f}}^*(1-j))}{(\varphi-1) \cdot D_{\text{cris}, K_n}(\mathbf{V}_{\underline{f}}^*(1-j))}.$$

The latter space has the same dimension (as a \tilde{K} -vector space) as the φ -invariant subspace inside $D_{\text{cris}, K_n}(\mathbf{V}_{\underline{f}}^*(1-j))$, and $D_{\text{cris}, K}(\mathbf{V}_{\underline{f}}^*(1-j))^{\varphi=1}$ clearly vanishes whenever $\mathbf{V}_{\underline{f}}(j)$ does *not* satisfy an exceptional zero condition at the prime p . \square

4.3. Residual f_1 -induced Selmer structures

The next problem we face is to construct a residual avatar of the cyclotomic Selmer group defined in Section 4.1. The local conditions away from p are easy, but we need to tread carefully at primes dividing p , as working with the maximal étale quotient of $V_{\bar{\rho}_1, \dots, t}$ is the wrong choice. For each index $i \in \{1, \dots, t\}$, there exists an exact sequence of local $G_{\mathbb{Q}_p}$ -modules

$$0 \rightarrow V'_{\bar{\rho}_i} \rightarrow V_{\bar{\rho}_i} \rightarrow V''_{\bar{\rho}_i} \rightarrow 0$$

where $\dim_{\mathbb{F}}(V'_{\bar{\rho}_i}) = \dim_{\mathbb{F}}(V''_{\bar{\rho}_i}) = 1$, and the quotient $V''_{\bar{\rho}_i} = V_{\bar{\rho}_i}/V'_{\bar{\rho}_i}$ is unramified. We define $G_{\mathbb{Q}_p}$ -modules $V'_{\bar{\rho}_1, \dots, t}$ and $V''_{\bar{\rho}_1, \dots, t}$ by

$$V'_{\bar{\rho}_1, \dots, t} := V'_{\bar{\rho}_1} \otimes_{\mathbb{F}} (V_{\bar{\rho}_2} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} V_{\bar{\rho}_t})$$

and

$$V''_{\bar{\rho}_1, \dots, t} := V''_{\bar{\rho}_1} \otimes_{\mathbb{F}} (V_{\bar{\rho}_2} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} V_{\bar{\rho}_t})$$

which closely mirrors our earlier construction of $A_{\underline{f}}^+$ and $A_{\underline{f}}^-$, respectively.

Recall from Section 4.1 that for $\underline{f} \in \mathcal{H}(\bar{\rho}_1, \dots, t)$ satisfying $f_1, \dots, f_t \in \mathcal{K}[[q]]$ with $\text{Im}(\chi) \subset \mathcal{K}$ and associated uniformiser $\pi = \pi_{\underline{f}, \mathcal{K}}$, there is the exact sequence (4.2): for every $i \geq 0$ and closed normal subgroup G of the absolute Galois group of F ,

$$0 \rightarrow \frac{H^i(G, A_{\underline{f}}[\chi, j])}{\pi} \xrightarrow{\partial_i} H^{i+1}(G, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j) \xrightarrow{\beta_{i+1}^+} H^{i+1}(G, A_{\underline{f}}[\chi, j])[\pi] \rightarrow 0.$$

Indeed if $G \triangleleft \text{Gal}(\bar{F}_{\mathfrak{P}}/F_{\mathfrak{P}})$ for some prime $\mathfrak{P}|p$, there are short exact sequences

$$(4.4) \quad 0 \rightarrow \frac{H^i(G, A_{\underline{f}}^+[\chi, j])}{\pi} \xrightarrow{\partial_i^+} H^{i+1}(G, V'_{\bar{\rho}_1, \dots, t} \otimes \omega^j) \xrightarrow{\beta_{i+1}^+} H^{i+1}(G, A_{\underline{f}}^+[\chi, j])[\pi] \rightarrow 0,$$

$$(4.5) \quad 0 \rightarrow \frac{H^i(G, A_{\underline{f}}^-[\chi, j])}{\pi} \xrightarrow{\partial_i^-} H^{i+1}(G, V''_{\bar{\rho}_1, \dots, t} \otimes \omega^j) \xrightarrow{\beta_{i+1}^-} H^{i+1}(G, A_{\underline{f}}^-[\chi, j])[\pi] \rightarrow 0.$$

DEFINITION 4.6. — Let $\Omega \subset \text{Spec}(\mathbb{Z})$ be a set of primes with $p \notin \Omega$, and $j \in \mathbb{Z}$.

- (i) We define the group $\text{Sel}_{F^{\text{cyc}}, \Omega}^{\min, [j]}(\bar{\rho}_1, \dots, t)$ to be the kernel of the restriction maps

$$H^1(F^{\text{cyc}}, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j) \xrightarrow{\prod_{\text{res}_\nu} \prod_{\nu \nmid p, \nu \notin \Omega^{\text{cyc}}} H^1(F_\nu^{\text{cyc}}, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j) \times \prod_{\mathfrak{p}|p} H^1(I_{F_{\mathfrak{p}}^{\text{cyc}}}, V''_{\bar{\rho}_1, \dots, t} \otimes \omega^j)}$$

with the first product ranging over the places ν of F^{cyc} not lying above $\Omega \cup \{p\}$.

- (ii) If $\underline{f} \in \mathcal{H}(\bar{\rho}_1, \dots, t)$ then $\text{Sel}_{F^{\text{cyc}}, \Omega}^{\underline{f}, [\chi, j]}(\bar{\rho}_1, \dots, t)$ is the kernel of the restriction maps

$$H^1(F^{\text{cyc}}, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j) \xrightarrow{\prod_{\text{res}_\nu} \prod_{\nu \nmid p, \nu \notin \Omega^{\text{cyc}}} \frac{H^1(F_\nu^{\text{cyc}}, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j)}{H_{\underline{f}\text{-ind}}^{1, [\chi, j]}(F_\nu^{\text{cyc}})} \times \prod_{\mathfrak{p}|p} H^1(I_{F_{\mathfrak{p}}^{\text{cyc}}}, V''_{\bar{\rho}_1, \dots, t} \otimes \omega^j)}$$

where at each place $\nu \nmid p$ of F^{cyc} , one selects the specific \underline{f} -induced local condition

$$H_{\underline{f}\text{-ind}}^{1, [\chi, j]}(F_\nu^{\text{cyc}}) := \text{Ker} \left(H^1(F_\nu^{\text{cyc}}, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j) \xrightarrow{\beta_1} H^1(F_\nu^{\text{cyc}}, A_{\underline{f}}[\chi, j])[\pi] \right).$$

For $i = 1$, the homomorphism β_i occurring in the short exact sequence (4.2) produces a mapping

$$\beta_{1, \star} : \text{Sel}_{F^{\text{cyc}}, \Omega}^{f, [\chi, j]}(\bar{\rho}_1, \dots, t) \rightarrow \text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_{\underline{f}})[\pi]$$

of \mathbb{F} -vector spaces. The next result confirms (amongst other things) this mapping is an isomorphism, thereby providing a mechanism to compare $\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_{\underline{f}})[\pi]$ and $\text{Sel}_{F^{\text{cyc}}, \Omega}^{\min, [j]}(\bar{\rho}_1, \dots, t)$.

PROPOSITION 4.7. — Assume that $H^0(F, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j) = \{0\}$ for some $j \in \mathbb{Z}$.

(a) If $\nu \nmid p$ is a finite place of F^{cyc} , then

$$H_{\underline{f}\text{-ind}}^{1, [\chi, j]}(F_\nu^{\text{cyc}}) = \text{Im} \left(H^0(F_\nu^{\text{cyc}}, A_{\underline{f}}[\chi, j]) / \pi \xrightarrow{\partial_0} H^1(F_\nu^{\text{cyc}}, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j) \right).$$

(b) If $\nu \nmid p$ lies over the prime l such that $l \nmid \text{disc}(F) \cdot N_{F/\mathbb{Q}}(\text{cond}(\chi))$ and $|\text{cond}(\rho_{f_1} \otimes \dots \otimes \rho_{f_t})|_l^{-1} = |\text{cond}(\bar{\rho}_1, \dots, t)|_l^{-1}$ then the cohomology group $H^0(F_\nu^{\text{cyc}}, A_{\underline{f}}[\chi, j])$ is π -divisible,

$$\beta_1 : H^1(F_\nu^{\text{cyc}}, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j) \xrightarrow{\sim} H^1(F_\nu^{\text{cyc}}, A_{\underline{f}}[\chi, j])[\pi]$$

and the local condition $H_{\underline{f}\text{-ind}}^{1, [\chi, j]}(F_\nu^{\text{cyc}})$ is identically zero.

(c) The induced mapping $\beta_{1, \star} : \text{Sel}_{F^{\text{cyc}}, \Omega}^{f, [\chi, j]}(\bar{\rho}_1, \dots, t) \rightarrow \text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_{\underline{f}})[\pi]$ gives an isomorphism of \mathbb{F} -vector spaces, for every tuple $\underline{f} \in \mathcal{H}(\bar{\rho}_1, \dots, t)$ in the Hida family.

Proof. — Statement (a) follows from (4.2), taking $i = 0$ and

$$G = \text{Gal}(\bar{F}_\nu / F_\nu^{\text{cyc}}).$$

The proof for (b) is entirely identical to that of [9, Lemma 2.3 and Corollary 2.8]. Lastly to prove (c), consider the commutative diagram below with

exact columns

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 H^0(F^{\text{cyc}}, A_{\underline{f}}[\chi, j]) / \pi & \xrightarrow{\prod \text{loc}_{\mathfrak{p}}^{(0)}} & \prod_{\nu \notin \Omega^{\text{disc}}, \nu \nmid p} \{0\} \times \prod_{\mathfrak{p}|p} H^0(I_{F_{\mathfrak{p}}^{\text{cyc}}}, A_{\underline{f}}^{-}[\chi, j]) / \pi \\
 \downarrow \partial_0 & & \downarrow (0, \partial_0^{-}) \\
 H^1(F^{\text{cyc}}, V_{\bar{\rho}_1, \dots, t}) & \xrightarrow{\prod \text{res}_{\nu}} & \prod_{\nu \notin \Omega^{\text{cyc}}, \nu \nmid p} H^1(F_{\nu}^{\text{cyc}}, A_{\underline{f}}[\chi, j])[\pi] \times \prod_{\mathfrak{p}|p} H^1(I_{F_{\mathfrak{p}}^{\text{cyc}}}, V_{\bar{\rho}_1, \dots, t}'' \otimes \omega^j) \\
 \downarrow \beta_1 & & \downarrow (\text{id}, \beta_1^{-}) \\
 H^1(F^{\text{cyc}}, A_{\underline{f}}[\chi, j])[\pi] & \xrightarrow{\prod \text{res}_{\nu}} & \prod_{\nu \notin \Omega^{\text{cyc}}, \nu \nmid p} H^1(F_{\nu}^{\text{cyc}}, A_{\underline{f}}[\chi, j])[\pi] \times \prod_{\mathfrak{p}|p} H^1(I_{F_{\mathfrak{p}}^{\text{cyc}}}, A_{\underline{f}}^{-}[\chi, j])[\pi] \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

where at each place $\mathfrak{p}|p$, the map $\text{loc}_{\mathfrak{p}}^{(0)}$ is induced modulo π from the composition

$$\begin{aligned}
 A_{\underline{f}}[\chi, j]^{G_{F^{\text{cyc}}}} &\hookrightarrow A_{\underline{f}}[\chi, j]^{G_{F_{\mathfrak{p}}^{\text{cyc}}}} \\
 &\xrightarrow{\text{mod } \mathcal{F}^+} H^0(F_{\mathfrak{p}}^{\text{cyc}}, A_{\underline{f}}^{-}[\chi, j]) \subset H^0(I_{F_{\mathfrak{p}}^{\text{cyc}}}, A_{\underline{f}}^{-}[\chi, j]).
 \end{aligned}$$

Applying the Snake Lemma to this diagram yields an exact sequence

$$\begin{aligned}
 0 \rightarrow \prod_{\mathfrak{p}|p} \text{Ker} \left(\text{loc}_{\mathfrak{p}}^{(0)} \right) &\xrightarrow{\partial_0, *}} \text{Sel}_{F^{\text{cyc}}, \Omega}^{f, [\chi, j]}(\bar{\rho}_1, \dots, t) \\
 &\xrightarrow{\beta_1, *}} \text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_{\underline{f}})[\pi] \rightarrow \prod_{\mathfrak{p}|p} \text{Coker} \left(\text{loc}_{\mathfrak{p}}^{(0)} \right).
 \end{aligned}$$

The proof will therefore be complete, provided that we can establish the triviality of both the kernel and cokernel of $\text{loc}_{\mathfrak{p}}^{(0)}(-)$ at every prime \mathfrak{p} of F^{cyc} lying over p .

By assumption $H^0(F, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j)$ is zero, and as $\Gamma_F^{\text{cyc}} \cong \mathbb{Z}_p$ is a pro- p -group it follows that $H^0(F^{\text{cyc}}, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j) \cong H^0(F^{\text{cyc}}, A_{\underline{f}}[\chi, j])[\pi]$ must also be trivial. Consequently the kernel of $\text{loc}_{\mathfrak{p}}^{(0)}$ vanishes at each finite place $\mathfrak{p}|p$, as required.

We now show the vanishing of $\text{Coker}(\text{loc}_{\mathfrak{p}}^{(0)})$. In fact, it is enough to establish π -divisibility for $H^0(I_{F_{\mathfrak{p}}^{\text{cyc}}}, A_{\underline{f}}^{-}[\chi, j])$ at $\mathfrak{p}|p$. We should point out that this module is equal to $\mathbf{T}_{f_1}^{\text{ét}} \otimes_{\mathcal{O}_{\mathcal{K}}} H^0(I_{F_{\mathfrak{p}}^{\text{cyc}}}, A_{(f_2, \dots, f_t)}[\chi, j])$ since $\mathbf{T}_{f_1}^{\text{ét}}$ has a trivial $I_{F_{\mathfrak{p}}^{\text{cyc}}}$ -action, and we will show that the latter H^0 -group is π -divisible.

Suppose initially that we have $\mu_p \subset F$. Here one knows that $F_{\mathfrak{p}}^{\text{cyc}} = F(\mu_{p^\infty})_{\mathfrak{p}}$, in which case $0 \rightarrow \mathcal{K}/\mathcal{O}_{\mathcal{K}} \otimes \varphi_i^{-1} \rightarrow A_{f_i} \rightarrow \mathcal{K}/\mathcal{O}_{\mathcal{K}} \otimes \varphi_i \rightarrow 0$ is a short exact sequence of divisible $\text{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}}^{\text{cyc}})$ -modules for $i \in \{2, \dots, t\}$. By an inductive argument, one obtains a long exact sequence of $G_{F_{\mathfrak{p}}^{\text{cyc}}}$ -modules

$$\begin{aligned} 0 \rightarrow A_{(f_2, \dots, f_t)}^{(1)} \otimes \chi \rightarrow \cdots \rightarrow A_{(f_2, \dots, f_t)}^{(2^{t-2})} \otimes \chi \rightarrow A_{(f_2, \dots, f_t)}[\chi, j] \\ \rightarrow A_{(f_2, \dots, f_t)}^{(2^{t-2}+1)} \otimes \chi \rightarrow \cdots \rightarrow A_{(f_2, \dots, f_t)}^{(2^{t-1})} \otimes \chi \rightarrow 0 \end{aligned}$$

where $A_{(f_2, \dots, f_t)}^{(n)}$ is unramified, and

$$\text{ck}_{\mathcal{O}_{\mathcal{K}}}\left(A_{(f_2, \dots, f_t)}^{(n)}\right) = \begin{cases} n & \text{if } n \leq 2^{t-2}, \\ 2^{t-1} - n + 1 & \text{if } n > 2^{t-2}. \end{cases}$$

We may therefore conclude that:

- if $\chi(I_{F_{\mathfrak{p}}^{\text{cyc}}}) \neq \{1\}$ then $H^0(I_{F_{\mathfrak{p}}^{\text{cyc}}}, A_{(f_2, \dots, f_t)}[\chi, j])$ is equal to zero;
- if $\chi(I_{F_{\mathfrak{p}}^{\text{cyc}}}) = \{1\}$ then $H^0(I_{F_{\mathfrak{p}}^{\text{cyc}}}, A_{(f_2, \dots, f_t)}[\chi, j]) \cong (\mathcal{K}/\mathcal{O}_{\mathcal{K}})^{\oplus 2^{t-1}}$.

In both situations the π -divisibility of these H^0 -cohomology groups is clear.

Alternatively, suppose instead that $\mu_p \not\subset F$. From the standard properties of continuous cohomology, there is a short exact sequence

$$\begin{aligned} 0 \rightarrow \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} H^0(I_{F_{\mathfrak{p}}^{\text{cyc}}}, \mathbf{T}_{(f_2, \dots, f_t)}[\chi, j]) \rightarrow H^0(I_{F_{\mathfrak{p}}^{\text{cyc}}}, A_{(f_2, \dots, f_t)}[\chi, j]) \\ \rightarrow H^1(I_{F_{\mathfrak{p}}^{\text{cyc}}}, \mathbf{T}_{(f_2, \dots, f_t)}[\chi, j])[p^\infty] \rightarrow 0 \end{aligned}$$

and the proof will be complete if the right-hand p -primary torsion group vanishes. If $\Delta_{\mathfrak{p}} = \text{Gal}(F_{\mathfrak{p}}^{\text{cyc}}(\mu_p)/F_{\mathfrak{p}}^{\text{cyc}})$, then under the inflation-restriction exact sequence

$$\begin{aligned} 0 \rightarrow H^1(\Delta_{\mathfrak{p}}, \mathbf{T}_{(f_2, \dots, f_t)}[\chi, j]^{I_{F_{\mathfrak{p}}^{\text{cyc}}(\mu_p)}}) \xrightarrow{\text{inf}} H^1(I_{F_{\mathfrak{p}}^{\text{cyc}}}, \mathbf{T}_{(f_2, \dots, f_t)}[\chi, j]) \\ \xrightarrow{\text{res}} H^1(I_{F_{\mathfrak{p}}^{\text{cyc}}(\mu_p)}, \mathbf{T}_{(f_2, \dots, f_t)}[\chi, j]) \end{aligned}$$

the p^∞ -part of $H^1(I_{F_{\mathfrak{p}}^{\text{cyc}}}, \mathbf{T}_{(f_2, \dots, f_t)}[\chi, j])$ injects into

$$H^1(I_{F_{\mathfrak{p}}^{\text{cyc}}(\mu_p)}, \mathbf{T}_{(f_2, \dots, f_t)}[\chi, j])$$

because $\#\Delta_{\mathfrak{p}}$ is coprime to p . Now over the larger cyclotomic extension $F_{\mathfrak{p}}^{\text{cyc}}(\mu_p)$ we have already shown $H^0(I_{F_{\mathfrak{p}}^{\text{cyc}}(\mu_p)}, A_{(f_2, \dots, f_t)}[\chi, j])$ is π -divisible,

implying that

$$\begin{aligned} 0 \rightarrow \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} H^0(I_{F_p^{\text{cyc}}(\mu_p)}, \mathbf{T}_{(f_2, \dots, f_t)}[\chi, j]) \\ \xrightarrow{\sim} H^0(I_{F_p^{\text{cyc}}(\mu_p)}, A_{(f_2, \dots, f_t)}[\chi, j]) \\ \rightarrow H^1(I_{F_p^{\text{cyc}}(\mu_p)}, \mathbf{T}_{(f_2, \dots, f_t)}[\chi, j])[p^\infty] = 0. \end{aligned}$$

This latter sequence forces the p^∞ -part of $H^1(I_{F_p^{\text{cyc}}}, \mathbf{T}_{(f_2, \dots, f_t)}[\chi, j])$ to vanish too.

The triviality of the cokernel of $\text{loc}_p^{(0)}$ at those $\mathfrak{p}|p$ now follows in both cases. \square

5. Variation of the algebraic Iwasawa invariants

The technical portion of the paper is complete, and we apply these ideas to study the μ - and λ -invariant of the Iwasawa module $\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_f)^\wedge$ in various situations. *Throughout assume there is a minimal tuple $\underline{f}^\dagger = (f_1^\dagger, \dots, f_t^\dagger) \in \mathcal{H}(\bar{\rho}_{1, \dots, t})$ as in Theorem 3.7, such that $\text{cond}(\rho_{f_1^\dagger} \otimes \dots \otimes \rho_{f_t^\dagger})_{(p)} = \text{cond}(\bar{\rho}_{1, \dots, t})$ is true.* We shall not continually restate this assumption in the forthcoming results in Section 5.

5.1. Minimality and the finiteness condition

We begin by relating the arithmetic of the minimal tuple \underline{f}^\dagger over F^{cyc} to the \mathbb{F} -vector space $\text{Sel}_{F^{\text{cyc}}, \Omega}^{\text{min}, [j]}(\bar{\rho}_{1, \dots, t})$. As usual F denotes a number field, and $\chi : G_F \rightarrow \mu_{p^\infty}$ is a finite order character.

LEMMA 5.1. — *Suppose that $\text{supp}(\text{disc}(F) \cdot N_{F/\mathbb{Q}}(\text{cond}(\chi))) \subset \Omega$ with $p \notin \Omega$, and that $H^0(F, V_{\bar{\rho}_{1, \dots, t}} \otimes \omega^j) = \{0\}$ for a particular twist $j \in \{0, \dots, p-2\}$.*

- (i) *At every minimal tuple $\underline{f}^\dagger \in \mathcal{H}(\bar{\rho}_{1, \dots, t}) \cap \mathcal{K}^\dagger[[q]]$, there are isomorphisms*

$$\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_{\underline{f}^\dagger})[\pi^\dagger] \cong \text{Sel}_{F^{\text{cyc}}, \Omega}^{\underline{f}^\dagger, [\chi, j]}(\bar{\rho}_{1, \dots, t}) \cong \text{Sel}_{F^{\text{cyc}}, \Omega}^{\text{min}, [j]}(\bar{\rho}_{1, \dots, t})$$

where the uniformiser $\pi^\dagger = \pi_{\underline{f}^\dagger, \mathcal{K}^\dagger}$.

- (ii) *$\text{Sel}_{F^{\text{cyc}}, \Omega}^{\text{min}, [j]}(\bar{\rho}_{1, \dots, t})$ is finite if and only if for some (every) tuple $\underline{f} \in \mathcal{H}(\bar{\rho}_{1, \dots, t})$ and (every) Hecke character $\chi : G_F \rightarrow \mu_{p^\infty}$ of finite order, the Pontrjagin dual $\text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_{\underline{f}})^\wedge$ is a finitely-generated Λ_F^{cyc} -torsion module with a trivial μ -invariant.*

Proof. — To show assertion (i), recall that Proposition 4.7(c) states the mapping

$$\beta_{1,\star} : \text{Sel}_{F^{\text{cyc}},\Omega}^{f^\dagger, [\chi, j]}(\bar{\rho}_{1,\dots,t}) \longrightarrow \text{Sel}_{F^{\text{cyc}},\Omega}^{[\chi, j]}(A_{f^\dagger})[\pi^\dagger]$$

is an isomorphism, so we need to compare $\text{Sel}_{F^{\text{cyc}},\Omega}^{f^\dagger, [\chi, j]}(\bar{\rho}_{1,\dots,t})$ and $\text{Sel}_{F^{\text{cyc}},\Omega}^{\min, [j]}(\bar{\rho}_{1,\dots,t})$. The local conditions at $\mathfrak{p}|p$ are identical for both Selmer groups by Definition 4.6. Alternatively, if $\nu \nmid p$ does not lie above the primes in Ω then the requirements of Proposition 4.7(b) are clearly satisfied, hence $H_{f^\dagger\text{-ind}}^{1, [\chi, j]}(F_\nu^{\text{cyc}})$ must be equal to zero.

In order to establish part (ii), if $\dim_{\mathbb{F}}(\text{Sel}_{F^{\text{cyc}},\Omega}^{\min, [j]}(\bar{\rho}_{1,\dots,t})) < \infty$ then because $\frac{\text{Sel}_{F^{\text{cyc}},\Omega}^{f, [\chi, j]}(\bar{\rho}_{1,\dots,t})}{\text{Sel}_{F^{\text{cyc}},\Omega}^{\min, [j]}(\bar{\rho}_{1,\dots,t})} \cong \prod_{\nu \nmid p, \nu \notin \Omega^{\text{cyc}}} H_{f\text{-ind}}^{1, [\chi, j]}(F_\nu^{\text{cyc}})$ is a finite group by Proposition 4.7(b), it therefore follows that $\dim_{\mathbb{F}}(\text{Sel}_{F^{\text{cyc}},\Omega}^{f, [\chi, j]}(\bar{\rho}_{1,\dots,t})) < \infty$. As a consequence of 4.7(c) the π -torsion subgroup in $\text{Sel}_{F^{\text{cyc}},\Omega}^{[\chi, j]}(A_f)$ must be finite as well, in which case both the Λ_F^{cyc} -rank and μ -invariant of $\text{Sel}_{F^{\text{cyc}},\Omega}^{[\chi, j]}(A_f)^\wedge$ are zero using Proposition 4.3(b). Note that the “ \Leftarrow ” statement follows by carefully reversing this argument. \square

DEFINITION 5.2. — *For every finite set of primes $\Omega \subset \text{Spec}(\mathbb{Z})$ not containing p , we write $\mu_{F,\Omega}^{[\chi, j]}(f)$ and $\lambda_{F,\Omega}^{[\chi, j]}(f)$ for (respectively) the μ -invariant and λ -invariant of*

$$\text{char}_{\Lambda_F^{\text{cyc}}}(\text{Sel}_{F^{\text{cyc}},\Omega}^{[\chi, j]}(A_f)) \in \Lambda_F^{\text{cyc}}/(\Lambda_F^{\text{cyc}})^\times \cong \mathcal{O}_K[[X]]/\mathcal{O}_K[[X]]^\times,$$

at each tuple $f \in \mathcal{H}(\bar{\rho}_{1,\dots,t})$ in the Hida family attached to $\bar{\rho}_{1,\dots,t} = \bar{\rho}_1 \otimes \dots \otimes \bar{\rho}_t$.

As we already observed, $\mu_{F,\Omega}^{[\chi, j]}(f)$ and $\lambda_{F,\Omega}^{[\chi, j]}(f)$ only depend on j modulo $p-1$. The following result provides an explicit formula used to compute these invariants.

THEOREM 5.3. — *Assume that each of the following three conditions hold:*

- $\text{supp}(\text{disc}(F) \cdot N_{F/\mathbb{Q}}(\text{cond}(\chi))) \subset \Omega \cup \{p\}$
- $H^0(F, V_{\bar{\rho}_{1,\dots,t}} \otimes \omega^j) = \{0\}$ for some integer j
- $\text{Sel}_{F^{\text{cyc}},\Omega}^{\min, [j]}(\bar{\rho}_{1,\dots,t})$ is finite dimensional over \mathbb{F} .

Then at every tuple $\underline{f} \in \mathcal{H}(\bar{\rho}_1, \dots, t)$ one finds that $\mu_{F, \Omega}^{[\chi, j]}(\underline{f})$ equals zero, and moreover

$$\begin{aligned} \lambda_{F, \Omega}^{[\chi, j]}(\underline{f}) &= \dim_{\mathbb{F}} \left(\text{Sel}_{F^{\text{cyc}}, \Omega}^{\min, [j]}(\bar{\rho}_1, \dots, t) \right) \\ &\quad + \sum_{\substack{\nu | \text{cond}(\mathbf{V}_{\underline{f}}), \\ \nu \nmid p, \nu \notin \Omega^{\text{cyc}}}} \dim_{\mathbb{F}} \left(H^1(F_{\nu}^{\text{cyc}}, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j) \right) - \delta_{F, \nu}^{[\chi, j]}(\underline{f}) \end{aligned}$$

where $\delta_{F, \nu}^{[\chi, j]}(\underline{f}) \in \mathbb{Z}_{\geq 0}$ denotes the λ -invariant of

$$\text{char}_{\Lambda_{F^{\text{cyc}}}} \left(H^1(F_{\nu}^{\text{cyc}}, A_{\underline{f}}[\chi, j])^{\wedge} \right).$$

Proof. — Firstly the vanishing of $\mu_{F, \Omega}^{[\chi, j]}(\underline{f})$ follows directly from Lemma 5.1(ii). Let us fix a larger set of primes $\tilde{\Omega} \supset \Omega \cup \text{supp}(N_{f_1}^{\text{tame}} \times \dots \times N_{f_t}^{\text{tame}})$ with $p \notin \tilde{\Omega}$. Now

$$0 \rightarrow \text{Sel}_{F^{\text{cyc}}, \Omega}^{[\chi, j]}(A_{\underline{f}}) \rightarrow \text{Sel}_{F^{\text{cyc}}, \tilde{\Omega}}^{[\chi, j]}(A_{\underline{f}}) \rightarrow \prod_{\nu \in \tilde{\Omega}^{\text{cyc}} - \Omega^{\text{cyc}}} H^1(F_{\nu}^{\text{cyc}}, A_{\underline{f}}[\chi, j]) \rightarrow 0$$

is an exact sequence of π -divisible groups, thus by the additivity of the λ -invariant

$$(5.1) \quad \lambda_{F, \tilde{\Omega}}^{[\chi, j]}(\underline{f}) = \lambda_{F, \Omega}^{[\chi, j]}(\underline{f}) + \sum_{\nu \in \tilde{\Omega}^{\text{cyc}} - \Omega^{\text{cyc}}} \delta_{F, \nu}^{[\chi, j]}(\underline{f}).$$

Furthermore, our preceding work establishes that

$$(5.2) \quad \begin{aligned} \lambda_{F, \tilde{\Omega}}^{[\chi, j]}(\underline{f}) &\stackrel{\text{by 4.3(c)}}{=} \dim_{\mathbb{F}} \left(\text{Sel}_{F^{\text{cyc}}, \tilde{\Omega}}^{[\chi, j]}(A_{\underline{f}})[\pi] \right) \\ &\stackrel{\text{by 4.7(c)}}{=} \dim_{\mathbb{F}} \left(\text{Sel}_{F^{\text{cyc}}, \tilde{\Omega}}^{f, [\chi, j]}(\bar{\rho}_1, \dots, t) \right). \end{aligned}$$

We next make the important comment that

$$\text{Sel}_{F^{\text{cyc}}, \tilde{\Omega}}^{f, [\chi, j]}(\bar{\rho}_1, \dots, t) = \text{Sel}_{F^{\text{cyc}}, \tilde{\Omega}}^{f^{\dagger}, [\chi, j]}(\bar{\rho}_1, \dots, t)$$

for any minimally ramified tuple $\underline{f}^{\dagger} = (f_1^{\dagger}, \dots, f_t^{\dagger}) \in \mathcal{H}(\bar{\rho}_1, \dots, t)$ (see Theorem 3.7) – this assertion follows because $H_{\underline{f}^{\dagger}\text{-ind}}^{1, [\chi, j]}(F_{\nu}^{\text{cyc}}) = H_{\underline{f}\text{-ind}}^{1, [\chi, j]}(F_{\nu}^{\text{cyc}}) = \{0\}$ for all primes $\nu \notin \tilde{\Omega}^{\text{cyc}}$ with $\nu \nmid p$, upon utilising Proposition 4.7(b).

As an immediate corollary,

$$\begin{aligned}
 \dim_{\mathbb{F}} \left(\text{Sel}_{F^{\text{cyc}}, \tilde{\Omega}}^{\underline{f}, [\chi, j]} (\bar{\rho}_{1, \dots, t}) \right) \\
 &= \dim_{\mathbb{F}} \left(\text{Sel}_{F^{\text{cyc}}, \tilde{\Omega}}^{\underline{f}^{\dagger}, [\chi, j]} (\bar{\rho}_{1, \dots, t}) \right) \\
 &\stackrel{\text{by 5.1(i)}}{=} \dim_{\mathbb{F}} \left(\text{Sel}_{F^{\text{cyc}}, \tilde{\Omega}}^{\min, [j]} (\bar{\rho}_{1, \dots, t}) \right) \\
 &= \dim_{\mathbb{F}} \left(\text{Sel}_{F^{\text{cyc}}, \tilde{\Omega}}^{\min, [j]} (\bar{\rho}_{1, \dots, t}) \right) \\
 &\quad + \sum_{\nu \in \tilde{\Omega}^{\text{cyc}} - \Omega^{\text{cyc}}} \dim_{\mathbb{F}} (H^1(F_{\nu}^{\text{cyc}}, V_{\bar{\rho}_{1, \dots, t}} \otimes \omega^j))
 \end{aligned}$$

and combining these identities above with Equations (5.1) and (5.2):

$$\begin{aligned}
 \lambda_{F, \Omega}^{[\chi, j]}(\underline{f}) &= \dim_{\mathbb{F}} \left(\text{Sel}_{F^{\text{cyc}}, \tilde{\Omega}}^{\min, [j]} (\bar{\rho}_{1, \dots, t}) \right) \\
 &\quad + \sum_{\nu \in \tilde{\Omega}^{\text{cyc}} - \Omega^{\text{cyc}}} \dim_{\mathbb{F}} (H^1(F_{\nu}^{\text{cyc}}, V_{\bar{\rho}_{1, \dots, t}} \otimes \omega^j)) - \delta_{F, \nu}^{[\chi, j]}(\underline{f}).
 \end{aligned}$$

Finally at those places $\nu \nmid \text{cond}(\mathbf{V}_{\underline{f}})$ lying outside the finite set Ω^{cyc} , one finds that

$$\begin{aligned}
 \delta_{F, \nu}^{[\chi, j]}(\underline{f}) &= \dim_{\mathbb{F}} (H^1(F_{\nu}^{\text{cyc}}, A_{\underline{f}}[\chi, j])[\pi]) \\
 &\stackrel{\text{by 4.7(b)}}{=} \dim_{\mathbb{F}} (H^1(F_{\nu}^{\text{cyc}}, V_{\bar{\rho}_{1, \dots, t}} \otimes \omega^j))
 \end{aligned}$$

and the formula stated in the theorem follows at once. \square

5.2. Controlling the error terms $e_q(-)$

We next focus on the λ -invariant associated to the terms $H^1(F_{\nu}^{\text{cyc}}, A_{\underline{f}}[\chi, j])$, and study the variation over $\mathbb{T}_{\Sigma}(\bar{\rho}_{1, \dots, t})$. Let $\mathbb{Q} \subset F_n := \mathbb{Q}(\mu_{p^n}) \subset \mathcal{D}$ be an increasing chain of finite normal extensions of fields, and we write $\text{reg}_{\mathcal{D}/\mathbb{Q}}$ for the regular representation attached to $\text{Gal}(\mathcal{D}/\mathbb{Q})$. In addition, we shall suppose that $\text{Gal}(\mathcal{D}/F_n)$ is an abelian group of p -power order. Given a character $\chi : \text{Gal}(\mathcal{D}/F_w) \rightarrow \mu_{p^\infty}$ for some $w \in \{0, \dots, n\}$, let us define $F_{\chi} := \mathbb{Q}(\mu_{p^\infty})^{\text{Stab}(\chi)}$ where $\text{Stab}(\chi) \subset \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ is the stabilizer of χ .

Now choosing appropriate subsets $\widehat{R}_{\mathcal{D}/F}^{(w)} \subset \text{Hom}(\text{Gal}(\mathcal{D}/\mathbb{Q}(\mu_{p^w})), \mu_{p^\infty})$ and $\widehat{R}_{F/\mathbb{Q}}^{(w, \chi)} \subset \text{Hom}(\text{Gal}(F_n/\mathbb{Q}), \overline{\mathbb{Q}}_p^\times)$, the regular representation neatly

decomposes into

(5.3)

$$\mathrm{reg}_{\mathcal{D}/\mathbb{Q}} \cong \bigoplus_{w=0}^n \bigoplus_{\chi \in \widehat{R}_{\mathcal{D}/F}^{(w)}} \bigoplus_{\psi \in \widehat{R}_{F/\mathbb{Q}}^{(w,\chi)}} \varrho_{\chi,\psi}^{\dim(\varrho_{\chi,\psi})} \text{ where } \dim(\varrho_{\chi,\psi}) = [F_\chi : \mathbb{Q}],$$

with each $\varrho_{\chi,\psi} := \psi \otimes \mathrm{Ind}_{\mathrm{Stab}(\chi) \ltimes \mathrm{Gal}(\mathcal{D}/F_n)}^{\mathrm{Gal}(\mathcal{D}/\mathbb{Q})} (\chi_{F_n})$ an irreducible $G_{\mathbb{Q}}$ -representation. The following result generalises work of Greenberg and Vatsal [19, Proposition 2.4].

LEMMA 5.4. — *For all rational primes $q \neq p$ and $j \in \mathbb{Z}$, the non-negative integer*

$$\delta_{\mathcal{D},q}^{[j]}(\underline{f}) := \sum_{\nu|q} \sum_{w=0}^n \sum_{\chi \in \widehat{R}_{\mathcal{D}/F}^{(w)}} \sum_{\psi \in \widehat{R}_{F/\mathbb{Q}}^{(w,\chi)}} \dim(\varrho_{\chi,\psi}) \cdot [F_\chi : \mathbb{Q}] \times \delta_{F_w,\nu}^{[\chi\psi_F,j]}(\underline{f})$$

equals the λ -invariant of the $(\mathrm{reg}_{\mathcal{D}/\mathbb{Q}} \otimes \omega^j)$ -twisted Euler factor (cf. Lemma 3.5)

$$\det \left(1 - \mathrm{Frob}_q \cdot X \left| (\rho_{f_1} \otimes \cdots \otimes \rho_{f_t} \otimes \mathrm{reg}_{\mathcal{D}/\mathbb{Q}} \otimes \omega^j)_{I_q} \right. \right) \Big|_{X=\vartheta([q]^{-1})}$$

at every tuple $\underline{f} = (f_1, \dots, f_t) \in \mathcal{H}(\bar{\rho}_1, \dots, t)$ of eigenforms in the Hida family.

Proof. — Examining the decomposition in (5.3) and applying Shapiro's Lemma,

$$\prod_{\nu|q} \prod_{w=0}^n \prod_{\chi} \prod_{\psi} H^1(F_{\chi,\nu}^{\mathrm{cyc}}, A_{\underline{f}}[\chi\psi_F, j])^{\dim(\varrho_{\chi,\psi})} \\ \cong \prod_{\eta|q} H^1(\mathbb{Q}_{\eta}^{\mathrm{cyc}}, A_{\underline{f}} \otimes \kappa_p^j \otimes \mathrm{reg}_{\mathcal{D}/\mathbb{Q}})$$

where η ranges over the primes of $\mathbb{Q}^{\mathrm{cyc}}$ lying above q (see also [9, Lemma 2.4]). The λ -invariant of the Pontrjagin dual of the left-hand side is by definition $\delta_{\mathcal{D},q}^{[j]}(\underline{f})$. It thus remains to show that the λ -invariant for the (dual of the) right-hand side coincides with the λ -invariant of the $(\mathrm{reg}_{\mathcal{D}/\mathbb{Q}} \otimes \omega^j)$ -twisted Euler factor above.

To begin with $H^1(\mathbb{Q}_{\eta}^{\mathrm{cyc}}, A_{\underline{f}} \otimes \kappa_p^j \otimes \mathrm{reg}_{\mathcal{D}/\mathbb{Q}}) \cong H^1(\mathbb{Q}_{\eta}^{\mathrm{cyc}}, A_{\underline{f}} \otimes \omega^j \otimes \mathrm{reg}_{\mathcal{D}/\mathbb{Q}}) \otimes [\kappa_p^j]$ so it is enough to compute the λ -invariant of the Pontrjagin dual of the latter group. Applying [17, Proposition 2], a local Euler characteristic calculation implies that

$$\mathrm{ck}_{\mathcal{O}_{\mathcal{K}}} \left(\prod_{\eta|q} H^1(\mathbb{Q}_{\eta}^{\mathrm{cyc}}, A_{\underline{f}} \otimes \omega^j \otimes \mathrm{reg}_{\mathcal{D}/\mathbb{Q}}) \right) = \mathfrak{s}_q \times \dim_{\mathcal{K}} \left(H^0(\mathbb{Q}_{\eta}^{\mathrm{cyc}}, \mathcal{V}_{\mathcal{D}/\mathbb{Q}}^{*,[j]}) \right)$$

for any such η lying over q ; here \mathfrak{s}_q is the largest power of p with $q^{p-1} \equiv 1 \pmod{\mathfrak{s}_q p}$, whilst $\mathcal{V}_{\mathcal{D}/\mathbb{Q}}^{*,[j]} := \text{Hom}_{\mathcal{K}}(\mathbf{V}_{\underline{f}} \otimes \omega^j \otimes \text{reg}_{\mathcal{D}/\mathbb{Q}}, \mathcal{K}(1))$ denotes the Kummer dual module.

Let \mathcal{I}_q be the inertia subgroup inside $\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q^{\text{unr}})$, and write $(\mathcal{V}_{\mathcal{D}/\mathbb{Q}}^{[j]})_{\mathcal{I}_q}$ for the maximal quotient of the vector space $\mathbf{V}_{\underline{f}} \otimes \omega^j \otimes \text{reg}_{\mathcal{D}/\mathbb{Q}}$ on which \mathcal{I}_q acts trivially. The eigenvalues of $\text{Frob}_q \in \text{Gal}(\mathbb{Q}_q^{\text{unr}}/\mathbb{Q}_q)$ acting on the \mathcal{I}_q -coinvariants $(\mathcal{V}_{\mathcal{D}/\mathbb{Q}}^{[j]})_{\mathcal{I}_q}$ will henceforth be labelled as $\alpha_1, \dots, \alpha_e$, and counted with multiplicity.

Because \mathcal{I}_q acts trivially on $\mathcal{K}(1)$ and $\mathbb{Q}_{\eta}^{\text{cyc}} \subset \mathbb{Q}_q^{\text{unr}}$ for $\eta|q$, there is an injection $H^0(\mathbb{Q}_{\eta}^{\text{cyc}}, \mathcal{V}_{\mathcal{D}/\mathbb{Q}}^{*,[j]}) \hookrightarrow \text{Hom}_{\mathcal{K}}((\mathcal{V}_{\mathcal{D}/\mathbb{Q}}^{[j]})_{\mathcal{I}_q}, \mathcal{K}(1))$ which is a Galois equivariant mapping. The eigenvalues of Frob_q on the right-hand group are clearly $q^{-1}\alpha_1, \dots, q^{-1}\alpha_e$. Moreover $\text{Gal}(\mathbb{Q}_q^{\text{unr}}/\mathbb{Q}_{\eta}^{\text{cyc}})$ operates through a finite quotient of size coprime to p , hence the eigenvalues of Frob_q acting on $H^0(\mathbb{Q}_{\eta}^{\text{cyc}}, \mathcal{V}_{\mathcal{D}/\mathbb{Q}}^{*,[j]})$ must belong to the set

$$\left\{ q^{-1}\alpha_s \mid s = 1, \dots, e \text{ and } q^{-1}\alpha_s \text{ is a principal local unit of } \mathcal{K}(\alpha_s) \right\}.$$

The rest of the argument follows very similar lines to [19, Proposition 2.4].

For each $\eta|q$ let $\Gamma_{\eta} \subset \Gamma^{\text{cyc}}$ denote the corresponding decomposition subgroup, in particular $\Gamma_{\eta} \cong \mathbb{Z}_p$ and $[\Gamma^{\text{cyc}} : \Gamma_{\eta}] = \mathfrak{s}_q$. We will then write γ_{η} for the image of Frob_q inside Γ_{η} , so that $\Gamma_{\eta} = \overline{\langle \gamma_{\eta} \rangle}$ and also $\Gamma^{\text{cyc}} \ni \vartheta([q]) \mapsto (\gamma_{\eta})_{\eta} \in \prod_{\eta|q} \Gamma_{\eta}$. One next defines the power series element

$$\mathcal{P}_{\eta}(\gamma_{\eta} - 1) := \prod_{s=1}^e (1 - q^{-1}\alpha_s \gamma_{\eta}) \in \mathcal{O}_{\mathcal{K}}[\Gamma_{\eta}] \subset \mathcal{O}_{\mathcal{K}}[\Gamma^{\text{cyc}}] \cong \mathcal{O}_{\mathcal{K}}[X].$$

The μ -invariant of \mathcal{P}_q is trivial, hence using the Weierstrass Preparation Theorem, $\mathcal{P}_{\eta}(X) = u_{\eta}(X) \cdot h_{\eta}(X)$ where $u_{\eta}(X) \in \mathcal{O}_{\mathcal{K}}[[X]]^{\times}$ and $h_{\eta}(X) \in \mathcal{O}_{\mathcal{K}}[X]$ is a product of distinguished polynomials. By the above discussion and the construction of \mathcal{P}_{η} :

$$\begin{aligned} \lambda \left(\prod_{\eta|q} H^1(\mathbb{Q}_{\eta}^{\text{cyc}}, A_{\underline{f}} \otimes \omega^j \otimes \text{reg}_{\mathcal{D}/\mathbb{Q}})^{\wedge} \right) \\ = \text{ck}_{\mathcal{O}_{\mathcal{K}}} \left(\prod_{\eta|q} H^1(\mathbb{Q}_{\eta}^{\text{cyc}}, A_{\underline{f}} \otimes \omega^j \otimes \text{reg}_{\mathcal{D}/\mathbb{Q}}) \right) \\ = [\Gamma^{\text{cyc}} : \Gamma_{\eta}] \times \dim_{\mathcal{K}} \left(H^0(\mathbb{Q}_{\eta}^{\text{cyc}}, \mathcal{V}_{\mathcal{D}/\mathbb{Q}}^{*,[j]}) \right) = \deg(h_{\eta}(X)). \end{aligned}$$

Let us set $\mathcal{J}_\eta = \text{Gal}(\overline{\mathbb{Q}}_q^{\text{cyc}}/\mathbb{Q}_\eta^{\text{cyc}})$, $\bar{I}_q = I_q/\mathcal{Q}_q$ and $\mathcal{A} = A_{\underline{f}} \otimes \omega^j \otimes \text{reg}_{\mathcal{D}/\mathbb{Q}}$, where as before \mathcal{Q}_q denotes the unique normal subgroup of I_q such that $I_q/\mathcal{Q}_q \cong \mathbb{Z}_p(1)$. The proof in *op. cit.* implies $H^1(\mathbb{Q}_\eta^{\text{cyc}}, \mathcal{A}) = H^1(\mathcal{J}_\eta, \mathcal{A}_{\mathcal{Q}_q}) \cong H^1(\bar{I}_q, \mathcal{A}_{\mathcal{Q}_q})^{\mathcal{J}_\eta/\bar{I}_q}$, and secondly that there exist $\text{Gal}(\mathbb{Q}_q^{\text{unr}}/\mathbb{Q}_q)$ -equivariant isomorphisms

$$H^1(\bar{I}_q, \mathcal{A}_{\mathcal{Q}_q}) \cong \text{Hom}(\mathbb{Z}_p(1), \mathcal{A}_{\mathcal{Q}_q}) \cong \mathcal{A}_{\mathcal{Q}_q}(-1).$$

The eigenvalues of Frob_q on $\mathcal{A}_{\mathcal{Q}_q}(-1)^\wedge$ are $q^{-1}\alpha_1, \dots, q^{-1}\alpha_e$, and the eigenvalues of γ_η acting on $(\mathcal{A}_{\mathcal{Q}_q}(-1)^{\mathcal{J}_\eta/\bar{I}_q})^\wedge$ are the subset of these which are principal units.

In conclusion, we have just shown that \mathcal{P}_η generates the characteristic power series of the compact torsion $\mathcal{O}_K[[\Gamma^{\text{cyc}}]]$ -module $\prod_{\eta|q} H^1(\mathbb{Q}_\eta^{\text{cyc}}, A_{\underline{f}} \otimes \omega^j \otimes \text{reg}_{\mathcal{D}/\mathbb{Q}})^\wedge$. This element has trivial μ -invariant, and its zeroes are in one-to-one correspondence with those of the twisted Euler factor from the statement of the lemma. \square

Recalling Lemma 3.5, the λ -invariant associated to the $(\text{reg}_{\mathcal{D}/\mathbb{Q}} \otimes \omega^j)$ -twisted Euler factor at a rational prime q depends exclusively on the multi-branch \underline{a} on which $\underline{f} = (f_1, \dots, f_t)$ lies. One then interprets $\delta_{\mathcal{D},q}^{[j]}(\underline{f})$ as the generic λ -invariant for the determinant of Frob_q acting on the I_q -coinvariants of $\rho_{\mathbb{T}(\underline{a})^0} \otimes \text{reg}_{\mathcal{D}/\mathbb{Q}} \otimes \omega^j$.

DEFINITION 5.5. — *Let \mathcal{D} be a number field which is a normal extension of \mathbb{Q} . For each multi-branch $\underline{a} = (\mathbf{a}_1, \dots, \mathbf{a}_t) \in \underline{\mathcal{B}}(\bar{\rho}_1, \dots, t; \underline{\Sigma})$ and prime $q \neq p$, we define*

$$\mathbf{e}_{\mathcal{D},q}^{[j]}(\underline{a}) := \lambda \left(\det \left(1 - \text{Frob}_q \cdot X \middle| (\rho_{f_1} \otimes \dots \otimes \rho_{f_t} \otimes \text{reg}_{\mathcal{D}/\mathbb{Q}} \otimes \omega^j)_{I_q} \right) \middle|_{X=\vartheta([q]^{-1})} \right)$$

for any (all) tuples $\underline{f} = (f_1, \dots, f_t) \in \mathcal{H}(\bar{\rho}_1, \dots, t)$ lying on the same multi-branch \underline{a} .

These error terms $\mathbf{e}_{\mathcal{D},q}^{[j]}(-)$ measure the (discrete) jumps in the λ -invariant for the cyclotomic Selmer group attached to \underline{f} as we switch between multi-branches. In the particular situation where $\mu_p \subset \mathcal{D}$, one already knows $\text{reg}_{\mathcal{D}/\mathbb{Q}} \otimes \omega^j \cong \text{reg}_{\mathcal{D}/\mathbb{Q}}$ in which case we drop the superscript “[j]” entirely from the notation for $\mathbf{e}_{\mathcal{D},q}^{[j]}$.

5.3. Proof of Theorem 2.1

We now prove the three main results from Section 2, starting with the first. We shall set $\mathcal{D} = F = \mathbb{Q}$, $\Omega = \emptyset$ and the character $\chi = \mathbf{1}_{\mathbb{Q}}$. As

one of Hypotheses (H4a) or (H4b) holds, the existence of a minimally ramified tuple $\underline{f}^\dagger \in \mathcal{H}(\bar{\rho}_{1,\dots,t})$ is guaranteed by Theorem 3.7. In fact, because ω^{-j} is not a sub- $G_{\mathbb{Q}}$ -representation inside $\bar{\rho}_{1,\dots,t}$, it also follows that $H^0(\mathbb{Q}, V_{\bar{\rho}_{1,\dots,t}} \otimes \omega^j)$ is zero. Applying Lemma 5.1(ii), if one knows $\text{Sel}_{\mathbb{Q}^{\text{cyc}}}(\underline{f}, \omega^j)^\wedge$ is Λ -torsion and $\mu(\underline{f}, \omega^j) = 0$ at some tuple of forms $\underline{f} \in \mathcal{H}(\bar{\rho}_1 \otimes \dots \otimes \bar{\rho}_t)$, then $\text{Sel}_{F^{\text{cyc}}, \Omega}^{\text{min}, [j]}(\bar{\rho}_{1,\dots,t})$ must be finite. Moreover this same lemma implies both the Λ -torsion property of $\text{Sel}_{\mathbb{Q}^{\text{cyc}}}(\underline{f}', \omega^j)^\wedge$ and the triviality of the μ -invariant extends to every $\underline{f}' \in \mathcal{H}(\bar{\rho}_1 \otimes \dots \otimes \bar{\rho}_t)$.

To establish the second statement in Theorem 2.1, if $\underline{f}^{(1)}, \underline{f}^{(2)} \in \mathcal{H}(\bar{\rho}_1 \otimes \dots \otimes \bar{\rho}_t)$ then recall we have written $\mathfrak{c}(\underline{f}^{(*)})$ to indicate the conductor of $\mathbf{V}_{\underline{f}^{(*)}}$ over \mathbb{Q} . Writing $\mathbf{d}^{\text{min}, [j]}$ for the \mathbb{F} -dimension of $\text{Sel}_{\mathbb{Q}^{\text{cyc}}, \emptyset}^{\text{min}, [j]}(\bar{\rho}_{1,\dots,t})$, and utilising Theorem 5.3:

$$(5.4) \quad \lambda_{\mathbb{Q}, \emptyset}^{[j]}(\underline{f}^{(1)}) = \mathbf{d}^{\text{min}, [j]} + \sum_{\eta | \mathfrak{c}(\underline{f}^{(1)}) \mathfrak{c}(\underline{f}^{(2)}), \eta \nmid p} \dim_{\mathbb{F}}(H^1(\mathbb{Q}_{\eta}^{\text{cyc}}, V_{\bar{\rho}_{1,\dots,t}} \otimes \omega^j)) - \delta_{\mathbb{Q}, \eta}^{[j]}(\underline{f}^{(1)}),$$

$$(5.5) \quad \lambda_{\mathbb{Q}, \emptyset}^{[j]}(\underline{f}^{(2)}) = \mathbf{d}^{\text{min}, [j]} + \sum_{\eta | \mathfrak{c}(\underline{f}^{(1)}) \mathfrak{c}(\underline{f}^{(2)}), \eta \nmid p} \dim_{\mathbb{F}}(H^1(\mathbb{Q}_{\eta}^{\text{cyc}}, V_{\bar{\rho}_{1,\dots,t}} \otimes \omega^j)) - \delta_{\mathbb{Q}, \eta}^{[j]}(\underline{f}^{(2)})$$

since $\delta_{\mathbb{Q}, \eta}^{[j]}(\underline{f}^{(1)})$ equals $\dim_{\mathbb{F}}(H^1(\mathbb{Q}_{\eta}^{\text{cyc}}, V_{\bar{\rho}_{1,\dots,t}} \otimes \omega^j))$ if $\eta | \mathfrak{c}(\underline{f}^{(2)})$ but $\eta \nmid \mathfrak{c}(\underline{f}^{(1)})$, while $\delta_{\mathbb{Q}, \eta}^{[j]}(\underline{f}^{(2)})$ equals $\dim_{\mathbb{F}}(H^1(\mathbb{Q}_{\eta}^{\text{cyc}}, V_{\bar{\rho}_{1,\dots,t}} \otimes \omega^j))$ if $\eta | \mathfrak{c}(\underline{f}^{(1)})$ but $\eta \nmid \mathfrak{c}(\underline{f}^{(2)})$. Subtracting Equation (5.5) from (5.4), one readily deduces that

$$\lambda_{\mathbb{Q}, \emptyset}^{[j]}(\underline{f}^{(1)}) = \lambda_{\mathbb{Q}, \emptyset}^{[j]}(\underline{f}^{(2)}) + \sum_{q | \mathfrak{c}(\underline{f}^{(1)}) \mathfrak{c}(\underline{f}^{(2)}), q \neq p} \sum_{\eta | q} \delta_{\mathbb{Q}, \eta}^{[j]}(\underline{f}^{(2)}) - \delta_{\mathbb{Q}, \eta}^{[j]}(\underline{f}^{(1)})$$

and using Lemma 5.4 and Definition 5.5, the inner sum is $\mathbf{e}_{\mathbb{Q}, q}^{[j]}(\underline{\mathfrak{a}}^{(2)}) - \mathbf{e}_{\mathbb{Q}, q}^{[j]}(\underline{\mathfrak{a}}^{(1)})$. Lastly, $\lambda_{\mathbb{Q}, \emptyset}^{[j]}(\underline{f}^{(*)}) = \lambda(\underline{f}^{(*)}, \omega^j)$ at unbalanced tuples $\underline{f}^{(*)}$ by Proposition 4.5.

5.4. Proof of Theorem 2.2

In this section we take $t = 2$, $\Omega = \emptyset$ and $p > 3$. In particular, the condition that a pair $(f_1, f_2) \in \mathcal{H}(\bar{\rho}_1 \otimes \bar{\rho}_2)$ is unbalanced just means that

the weight of f_1 is strictly greater than the weight of f_2 , i.e. $k_1 > k_2$. For a Dirichlet character χ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) \gg 0$, the Rankin L -function is

$$L(f_1 \otimes f_2, \chi, s) = \zeta(2s+2-k_1-k_2, \varepsilon_{f_1} \varepsilon_{f_2} \chi^2) \times \sum_{n=1}^{\infty} a_n(f_1) a_n(f_2) \chi(n) \cdot n^{-s}.$$

Also, the completed L -function $\Psi_{\infty}(f_1 \otimes f_2, \chi, s) = \frac{\Gamma(s)\Gamma(s+1-k_2)}{(2\pi)^{2s}} \cdot L(f_1 \otimes f_2, \chi, s)$ admits an analytic continuation to the whole complex plane (see [37] for details).

Before we give the proof of Theorem 2.2, let us first recall some of the p -adic theory. From the work of Hida and Panchishkin [21, 33], for each $j \in \{0, \dots, p-2\}$ there exists a unique analytic element $\mathbf{L}_p^{\text{an}}(f_1 \otimes f_2, \omega^j) \in \mathcal{O}_{\mathcal{K}}[[\Gamma^{\text{cyc}}]][1/\pi_{\mathcal{K}}]$ interpolating

$$\begin{aligned} & \chi \kappa_p^r \left(\mathbf{L}_p^{\text{an}}(f_1 \otimes f_2, \omega^j) \right) \\ &= \frac{\mathfrak{G}_{\bar{\chi}\omega^{-j}}^2 p^{n_{\chi\omega^j}(k_2+2r-1)} \mathcal{E}_p(r, \bar{\chi}\omega^{-j})}{(-1)^r \cdot a_p(f_1)^{2n_{\chi\omega^j}}} \cdot \frac{L_{\{p\}}(f_1 \otimes f_2, \chi\omega^j, k_2+r)}{(2\pi i)^{1-k_2} \cdot \langle \tilde{f}_1, \tilde{f}_1 \rangle_{\text{Pet}}} \end{aligned}$$

at points $r \in \{0, \dots, k_1-k_2-1\}$, and characters $\chi : \Gamma^{\text{cyc}} \rightarrow \bar{\mathbb{Q}}_p^{\times}$ of conductor $p^{n_{\chi}}$. Here $\mathfrak{G}_{\chi} = \sum_{j=1}^{p^{n_{\chi}}} \chi(n) \exp(2\pi\sqrt{-1}j/p^{n_{\chi}})$ indicates the Gauss sum associated to χ , and the Euler factor $\mathcal{E}_p(s, \bar{\chi}\omega^{-j})$ equals 1 if the character χ is non-trivial, while

$$\mathcal{E}_p(s, \omega^j) = (1-\omega^{-j}(p)a_p(f_1)^{-1}a_p(f_2)p^s) \cdot (1-\omega^j(p)a_p(f_1)a_p(f_2)p^{-s-k_2})^{-1}.$$

We next utilise the Euler system of Beilinson–Flach elements, which requires us to check some necessary conditions from [27, §11] hold in order to use these results. By the definition of admissibility for a pair (f_1, f_2) in Section 2.2, the conditions

- the pair of eigenforms (f_1, f_2) is unbalanced
- if the weight $k_2 = 2$ then f_2 is not Steinberg at p
- both f_1 and f_2 are non-Eisenstein modulo $\pi_{\mathcal{K}}$

imply that [27, Hypothesis 1.11] is true (note the prime p divides the level of the newform, \tilde{f}_i say, whose associated p -stabilisation is f_i , only if $k_i = 2$ and $p \parallel N_{\tilde{f}_i}$). One also checks that [27, Hypothesis 1.12] holds whenever the following are true:

- there exists $\tau \in \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_{p^\infty}))$ so that $\frac{\mathbf{T}_{f_1} \otimes \mathbf{T}_{f_2}}{(\tau-1) \cdot \mathbf{T}_{f_1} \otimes \mathbf{T}_{f_2}}$ is free of rank one
- there exists $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_{p^\infty}))$ which acts on $\mathbf{T}_{f_1} \otimes \mathbf{T}_{f_2}$ through $-\operatorname{id}_4$.

Using deep arithmetic properties of Rankin–Eisenstein classes, the fundamental results of Kings, Loeffler and Zerbes [27, Theorem 11.6.4(ii)] imply⁽³⁾ that $\mathrm{Sel}_{\mathbb{Q}^{\mathrm{cyc}}, \emptyset}^{[1_{\mathbb{Q}}, j]}(A_{\underline{f}})^{\wedge}$ is Λ^{cyc} -torsion for all $\underline{f} = (f_1, f_2) \in \mathcal{H}(\bar{\rho}_1 \otimes \bar{\rho}_2)$ such that $\mathbf{L}_p^{\mathrm{an}}(f_1 \otimes f_2, \omega^{1-j}) \neq 0$. Furthermore, applying [27, Theorem 11.6.4(iii)] produces a divisibility of power series

$$\mathrm{char}_{\Lambda^{\mathrm{cyc}}} \left(\mathrm{Sel}_{\mathbb{Q}^{\mathrm{cyc}}, \emptyset}^{[1_{\mathbb{Q}}, j]}(A_{\underline{f}})^{\wedge} \right) \text{ divides } \mathbf{h}_{f_1} \cdot \mathrm{Tw}_{1-j} \left(\mathbf{L}_p^{\mathrm{an}}(f_1 \otimes f_2, \omega^j) \right)$$

for each integer j , where the isomorphism $\mathrm{Tw}_r : \Lambda^{\mathrm{cyc}} \xrightarrow{\sim} \Lambda^{\mathrm{cyc}}$ sends $\gamma_0 \mapsto \kappa_p^r(\gamma_0)\gamma_0$.

As an immediate consequence, one may deduce some useful relations between the algebraic Iwasawa invariants and the analytic Iwasawa invariants, namely that:

$$(5.6) \quad \left\{ \text{zeros of } \mathrm{char}_{\Lambda^{\mathrm{cyc}}} \left(\mathrm{Sel}_{\mathbb{Q}^{\mathrm{cyc}}, \emptyset}^{[1_{\mathbb{Q}}, j]}(A_{\underline{f}})^{\wedge} \right) \right\} \\ \subset \left\{ \text{zeros of } \mathrm{Tw}_{1-j} \left(\mathbf{L}_p^{\mathrm{an}}(f_1 \otimes f_2, \omega^j) \right) \right\}$$

$$(5.7) \quad \mu(f_1 \otimes f_2, \omega^j) \leq \mu^{\mathrm{an}}(f_1 \otimes f_2, \omega^j) \\ \text{and } \lambda(f_1 \otimes f_2, \omega^j) \leq \lambda^{\mathrm{an}}(f_1 \otimes f_2, \omega^j).$$

Now by assumption $\mu^{\mathrm{an}}(f_1 \otimes f_2, \omega^j) = 0$, therefore $\mu(f_1 \otimes f_2, \omega^j) = 0$ from the first part of (5.7) because it is sandwiched between zero from both below and above. Theorem 2.1(i) tells us that $\mathrm{Sel}_{\mathbb{Q}^{\mathrm{cyc}}}(\underline{f}', \omega^j)$ is Λ^{cyc} -cotorsion and $\mu(f'_1 \otimes f'_2, \omega^j) = 0$ for every unbalanced tuple $\underline{f}' = (f'_1, f'_2)$. Applying Theorem 2.1(ii), one finds that

$$(5.8) \quad \lambda(f'_1 \otimes f'_2, \omega^j) - \lambda(f_1 \otimes f_2, \omega^j) \\ = \sum_{q | \mathfrak{c}(f'_1 \otimes f'_2) \mathfrak{c}(f_1 \otimes f_2), q \neq p} \mathbf{e}_q(\underline{a}, \omega^j) - \mathbf{e}_q(\underline{a}', \omega^j).$$

The analytic version of this formula is evidently given by

$$(5.9) \quad \lambda^{\mathrm{an}}(f'_1 \otimes f'_2, \omega^j) - \lambda^{\mathrm{an}}(f_1 \otimes f_2, \omega^j) \\ = \sum_{q | \mathfrak{c}(f'_1 \otimes f'_2) \mathfrak{c}(f_1 \otimes f_2), q \neq p} \mathbf{e}_q(\underline{a}, \omega^j) - \mathbf{e}_q(\underline{a}', \omega^j)$$

⁽³⁾The authors work with $\widetilde{H}^2(\mathbb{Z}[1/S], T_p(f_1 \otimes f_2)^* \otimes \Lambda_{\Gamma}(-\mathbf{j}); \Delta^{(f_1)})$ instead of $\mathrm{Sel}_{\mathbb{Q}^{\mathrm{cyc}}, \emptyset}^{[1_{\mathbb{Q}}, j]}(A_{\underline{f}})^{\wedge}$ (using the specific notation of *op. cit.*), which is also a compact finitely-generated Λ^{cyc} -module; however the former object is isomorphic to the latter by [27, Proposition 11.2.9], and these both coincide with the Bloch–Kato version within the critical region $j \in \{2 - k_1, \dots, 1 - k_2\}$ by Proposition 4.5.

which was proven if $f'_1 = f_1$ and $k'_2 = k_2$ in [10, Theorem 1.2(ii)], and has subsequently been proven by the author in more generality (see the preprint [7]).

Note the congruence in Theorem 1.2(i) of *op. cit.* implies $\mathbf{L}_p^{\text{an}}(f'_1 \otimes f'_2, \omega^{1-j}) \neq 0$, and that $\mu^{\text{an}}(f'_1 \otimes f'_2, \omega^j) = \mu^{\text{an}}(f_1 \otimes f_2, \omega^j)$ which we already know to be zero. Combining together Equations (5.8) and (5.9), it follows readily that

$$\lambda^{\text{an}}(f'_1 \otimes f'_2, \omega^j) = \lambda(f'_1 \otimes f'_2, \omega^j) + \left(\lambda^{\text{an}}(f_1 \otimes f_2, \omega^j) - \lambda(f_1 \otimes f_2, \omega^j) \right).$$

However $\lambda^{\text{an}}(f_1 \otimes f_2, \omega^j) = \lambda(f_1 \otimes f_2, \omega^j)$ because $\text{IMC}(f_1 \otimes f_2, j)$ is true, so that

$$(5.10) \quad \begin{aligned} \mu^{\text{an}}(f'_1 \otimes f'_2, \omega^j) &= \mu(f'_1 \otimes f'_2, \omega^j) = 0 \\ \text{and } \lambda^{\text{an}}(f'_1 \otimes f'_2, \omega^j) &= \lambda(f'_1 \otimes f'_2, \omega^j). \end{aligned}$$

Exploiting the containment in (5.6) for the admissible pair $\underline{f}' = (f'_1, f'_2)$ now yields

$$(5.11) \quad \left\{ \text{zeros of } \text{char}_{\Lambda^{\text{cyc}}} \left(\text{Sel}_{\mathbb{Q}^{\text{cyc}}, \emptyset}^{[\mathbf{1}_{\mathbb{Q}}, j]}(A_{\underline{f}'}^\wedge) \right) \right\} \\ \subset \left\{ \text{zeros of } \text{Tw}_{1-j} \left(\mathbf{L}_p^{\text{an}}(f'_1 \otimes f'_2, \omega^j) \right) \right\}$$

and the second part of (5.10) turns this containment into an actual equality of sets. Since any two elements $F, G \in \Lambda^{\text{cyc}}$ satisfying: (i) $\mu(F) = \mu(G)$, (ii) $\lambda(F) = \lambda(G)$ and (iii) $\{\text{zeros of } F\} = \{\text{zeros of } G\}$ must be equal up to a unit, clearly one has

$$\text{char}_{\Lambda^{\text{cyc}}} \left(\text{Sel}_{\mathbb{Q}^{\text{cyc}}, \emptyset}^{[\mathbf{1}_{\mathbb{Q}}, j]}(A_{\underline{f}'}^\wedge) \right) \cdot \Lambda^{\text{cyc}} = \mathbf{h}_{f'_1} \cdot \text{Tw}_{1-j} \left(\mathbf{L}_p^{\text{an}}(f'_1 \otimes f'_2, \omega^j) \right) \cdot \Lambda^{\text{cyc}}$$

in which case $\text{IMC}(f'_1 \otimes f'_2, j)$ must be true as well, and the theorem is established.

Remark. — The restrictions $f'_1 = f_1$ and $k'_2 = k_2$ are an unwanted by-product of the low-brow proof of [10, Theorem 1.2], which prevents the weights of (f'_1, f'_2) from moving around. To restore variation in (f'_1, f'_2) one uses a Hida theory approach: first allow f'_1 and f'_2 to vary over weight-space, producing families \mathbf{F}'_1 and \mathbf{F}'_2 , and then compute the \mathbf{F}'_1 -isotypic projection of $\mathbf{F}'_2 \cdot \delta^{(-)}(\mathbf{Eis})$ where \mathbf{Eis} is an appropriate Λ^{wt} -adic Eisenstein series, and $\delta^{(-)}$ denotes the Maass–Shimura differential operator.

5.5. Proof of Theorem 2.3

We begin by recalling from Section 2.3 that \mathcal{D}_∞ is a p -adic Lie extension containing $\mathbb{Q}(\mu_{p^\infty})$, in which only finitely many primes ramify. We

shall write \mathcal{D}_n for the n -th layer, so that $\mathcal{D}_\infty = \bigcup_{n \geq 0} \mathcal{D}_n$ and $\mathcal{D}_1 = \mathbb{Q}(\mu_p)$. Furthermore, $\mathcal{G}_\infty = \text{Gal}(\mathcal{D}_\infty/\mathbb{Q}(\mu_p))$ is endowed with the additional structure of a compact, non-abelian, finite-dimensional pro- p -group without any p -torsion.

It is worthwhile to list which types of non-abelian group can occur for \mathcal{G}_∞ . Firstly, if $\mathcal{D}_\infty = \mathbb{Q}(\mu_{p^\infty}, m_1^{1/p^\infty}, \dots, m_d^{1/p^\infty})$ is a d -fold false Tate extension, then

- $\mathcal{G}_\infty \cong \langle \gamma \rangle \ltimes \mathbb{Z}_p^d$ where γ acts on \mathbb{Z}_p^d through a diagonal matrix in $\text{GL}_d(\mathbb{Z}_p)$.

Secondly, if \mathcal{G}_∞ is two-dimensional then it is automatically solvable, in which case

- $\mathcal{G}_\infty \cong \langle \gamma \rangle \ltimes \mathbb{Z}_p$ where γ acts on the second factor by scalar multiplication.

Thirdly, if \mathcal{G}_∞ is three-dimensional and solvable⁽⁴⁾ then by the work of Klopsch and González-Sánchez [16, Theorem 7.4], \mathcal{G}_∞ must be isomorphic to one of the following:

- an open subgroup of the Heisenberg group, i.e. a group presented in the form $\langle \gamma, h_1, h_2 : [h_1, h_2] = 1, [h_1, \gamma] = 1, [h_2, \gamma] = h_1^{p^s} \rangle$ for some $s \in \mathbb{Z}_{\geq 0}$;
- the group $\langle \gamma, h_1, h_2 : [h_1, h_2] = 1, [h_1, \gamma] = h_1^{p^s}, [h_2, \gamma] = h_2^{p^s} \rangle$ where $s \in \mathbb{N}$;
- the group $\langle \gamma, h_1, h_2 : [h_1, h_2] = 1, [h_1, \gamma] = h_1^{p^s} h_2^{p^{s+r}c}, [h_2, \gamma] = h_1^{p^{s+r}} h_2^{p^s} \rangle$ for some $s, r \in \mathbb{N}$ with $c \in \mathbb{Z}_p$;
- $\langle \gamma, h_1, h_2 : [h_1, h_2] = 1, [h_1, \gamma] = h_2^{p^s c}, [h_2, \gamma] = h_1^{p^s} h_2^{p^{s+r}} \rangle$ where $s, r \in \mathbb{N}_0$ and $c \in \mathbb{Z}_p$, such that either $s \geq 1$, or instead $r \geq 1$ and $c \in p\mathbb{Z}_p$;
- either one of:
 - (a) $\langle \gamma, h_1, h_2 : [h_1, h_2] = 1, [h_1, \gamma] = h_2^{p^{s+r}}, [h_2, \gamma] = h_1^{p^s} \rangle$ or
 - (b) $\langle \gamma, h_1, h_2 : [h_1, h_2] = 1, [h_1, \gamma] = h_2^{p^{s+r}t}, [h_2, \gamma] = h_1^{p^s} \rangle$
 where $s, r \in \mathbb{Z}_{\geq 0}$ so that $s + r \geq 1$, and $t \in \mathbb{Z}_p^\times$ is not a square modulo p .

In all seven of the cases listed above, $\mathcal{G}_\infty \cong \langle \gamma \rangle \ltimes \mathcal{H}_\infty$ where \mathcal{H}_∞ is a free \mathbb{Z}_p -module of rank $= \dim(\mathcal{G}_\infty) - 1$. We also identify γ with the topological generator $\gamma_0 \in \Gamma^{\text{cyc}}$.

⁽⁴⁾ If \mathcal{G}_∞ is isomorphic to an open subgroup of $\text{SL}_2(\mathbb{Z}_p)$ or $\text{SL}_1(\mathbb{D}_p)$ then it is insolvable, and so does not occur on this list – we can thus infer nothing about the λ -invariant in these two cases.

Let us now choose the finite set Ω^{ram} to contain all primes that ramify in the infinite extension $\mathcal{D}_\infty/\mathbb{Q}$ with the exception of the prime p , i.e. $\Omega^{\text{ram}} = S_{\mathcal{D}_\infty}^{\text{ram}} - \{p\}$. We also recall that $F_\chi = \mathbb{Q}(\mu_{p^\infty})^{\text{Stab}(\chi)}$ and $\mathcal{D}_n = H^0(\mathcal{G}_\infty^{p^{n-1}}, \mathcal{D}_\infty)$ is the n -th layer. Combining Equation (5.3) with [9, Corollary 2.5], for every tuple $\underline{f}' \in \mathcal{H}(\bar{\rho}_{1,\dots,t})$ there is a natural decomposition of compact finitely-generated Λ^{cyc} -modules

$$(5.12) \quad \text{Sel}_{\mathcal{D}_n^{\text{cyc}}, \Omega^{\text{ram}}}^{[j]}(A_{\underline{f}'})^\wedge \cong \bigoplus_{w=0}^n \bigoplus_{\chi \in \widehat{R}_{\mathcal{D}/F}^{(w)}} \bigoplus_{\psi \in \widehat{R}_{F/\mathbb{Q}}^{(w, \chi)}} \left(\text{Sel}_{F_\chi^{\text{cyc}}, \Omega^{\text{ram}}}^{[\chi\psi_F, j]}(A_{\underline{f}'})^\wedge \right)^{\dim(\varrho_{\chi, \psi})}.$$

Furthermore, the additivity of the Λ^{cyc} -rank, the μ - and the λ -invariant implies

$$(5.13) \quad \beta \left(\text{Sel}_{\mathcal{D}_n^{\text{cyc}}, \Omega^{\text{ram}}}^{[j]}(A_{\underline{f}'})^\wedge \right) = \sum_{w=0}^n \sum_{\chi \in \widehat{R}_{\mathcal{D}/F}^{(w)}} \sum_{\psi \in \widehat{R}_{F/\mathbb{Q}}^{(w, \chi)}} \dim(\varrho_{\chi, \psi}) \cdot \beta \left(\text{Sel}_{F_\chi^{\text{cyc}}, \Omega^{\text{ram}}}^{[\chi\psi_F, j]}(A_{\underline{f}'})^\wedge \right)$$

for $\beta \in \{\text{rk}_{\Lambda^{\text{cyc}}}, \mu, \lambda\}$, where we take all these invariants over Λ^{cyc} rather than $\Lambda_{F_\chi}^{\text{cyc}}$.

We shall proceed by establishing the three statements of Theorem 2.3 in order. Starting with assertion (i), the condition that $H^0(G_{\mathbb{Q}(\mu_p)}, V_{\bar{\rho}_{1,\dots,t}})$ is zero implies that ω^{-j} cannot occur as a sub- $G_{\mathbb{Q}}$ -representation inside of $\bar{\rho}_{1,\dots,t}$ for all $j \in \mathbb{Z}$. Applying Lemma 5.1(ii), one has twin vanishing identities

$$\beta \left(\text{Sel}_{F_\chi^{\text{cyc}}, \Omega^{\text{ram}}}^{[\chi\psi_F, j]}(A_{\underline{f}'})^\wedge \right) = \beta \left(\text{Sel}_{F_\chi^{\text{cyc}}, \emptyset}^{[\chi\psi_F, j]}(A_{\underline{f}'})^\wedge \right) = 0 \quad \text{where } \beta \in \{\text{rk}_{\Lambda^{\text{cyc}}}, \mu\}$$

at all unbalanced tuples \underline{f}' , since the characteristic ideal for the quotient of the two Selmer groups has unit content. Consequently the Λ^{cyc} -rank and μ -invariant of $\text{Sel}_{\mathcal{D}_n^{\text{cyc}}, \Omega^{\text{ram}}}^{[j]}(A_{\underline{f}'})^\wedge$ vanish too by the previous summation formula, and (i) follows.

To establish (ii), we note for all $w \in \{0, \dots, n\}$ and characters $\chi \in \widehat{R}_{\mathcal{D}/F}^{(w)}$ that $\text{supp}(\text{disc}(F_\chi) \cdot N_{F_\chi/\mathbb{Q}}(\text{cond}(\chi))) \subset \Omega^{\text{ram}} \cup \{p\}$, that $H^0(F_\chi, V_{\bar{\rho}_{1,\dots,t}} \otimes \omega^j) = \{0\}$ at every twist $j \in \{0, \dots, p-2\}$, and lastly that $\text{Sel}_{F_\chi^{\text{cyc}}, \Omega^{\text{ram}}}^{\min, [j]}(\bar{\rho}_{1,\dots,t})$ is a finite group. For each ψ on $\text{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$, one uses Theorem 5.3 (with

$\Omega = \Omega^{\text{ram}}$) to show that

$$(5.14) \quad \lambda_{F_\chi, \Omega^{\text{ram}}}^{[\chi\psi_{F_\chi}, j]}(\underline{f}') = \mathbf{d}_{F_\chi, \Omega^{\text{ram}}}^{\min, [j]} + \sum_{\substack{\nu | \text{cond}(\mathbf{V}_{f'}), \\ \nu \nmid p, \nu \notin \Omega_{F_\chi}^{\text{cyc}}}} \mathbf{d}_{F_\chi, \nu}^{[\chi, j]} - \delta_{F_\chi, \nu}^{[\chi\psi_{F_\chi}, j]}(\underline{f}')$$

where $\mathbf{d}_{F_\chi, \Omega^{\text{ram}}}^{\min, [j]} := \dim_{\mathbb{F}}(\text{Sel}_{F_\chi^{\text{cyc}}, \Omega^{\text{ram}}}^{\min, [j]}(\bar{\rho}_1, \dots, t))$, and at places $\nu \nmid p$ of F_χ^{cyc} we define

$$\mathbf{d}_{F_\chi, \nu}^{[\chi, j]} := \dim_{\mathbb{F}}(H^1(F_{\chi, \nu}^{\text{cyc}}, V_{\bar{\rho}_1, \dots, t} \otimes \omega^j)).$$

If $F_\chi = \mathbb{Q}$ then clearly both

$$F_\chi^{\text{cyc}} = \mathbb{Q}^{\text{cyc}} \quad \text{and} \quad \lambda_{F_\chi, \Omega^{\text{ram}}}^{[\chi\psi_{F_\chi}, j]}(\underline{f}) = \lambda\left(\text{Sel}_{\mathbb{Q}(\mu_{p^\infty}), \Omega^{\text{ram}}}^{[\chi\psi_F, j]}(A_{\underline{f}'}^\wedge)\right),$$

whilst if $F_\chi \neq \mathbb{Q}$ then

$$F_\chi^{\text{cyc}} = \mathbb{Q}(\mu_{p^\infty}) \quad \text{and} \quad \lambda_{F_\chi, \Omega^{\text{ram}}}^{[\chi\psi_{F_\chi}, j]}(\underline{f}) = \frac{\lambda(\text{Sel}_{\mathbb{Q}(\mu_{p^\infty}), \Omega^{\text{ram}}}^{[\chi\psi_F, j]}(A_{\underline{f}'}^\wedge))}{[F_\chi : \mathbb{Q}(\mu_p)]}.$$

Substituting this scaling information into our earlier formula

$$\begin{aligned} & \lambda\left(\text{Sel}_{\mathcal{D}_n^{\text{cyc}}, \Omega^{\text{ram}}}^{[j]}(A_{\underline{f}'}^\wedge)\right) \\ &= \sum_{w=0}^n \sum_{\chi \in \widehat{R}_{\mathcal{D}/F}^{(w)}} \sum_{\psi \in \widehat{R}_{F/\mathbb{Q}}^{(w, \chi)}} \dim(\varrho_{\chi, \psi}) \cdot \lambda\left(\text{Sel}_{F_\chi^{\text{cyc}}, \Omega^{\text{ram}}}^{[\chi\psi_F, j]}(A_{\underline{f}'}^\wedge)\right) \end{aligned}$$

and then rearranging, we deduce directly from Equation (5.14) that

$$\begin{aligned} & \lambda\left(\text{Sel}_{\mathcal{D}_n^{\text{cyc}}, \Omega^{\text{ram}}}^{[j]}(A_{\underline{f}'}^\wedge)\right) + \sum_{\substack{q | \text{cond}(\mathbf{V}_{f'}), \\ q \notin \Omega^{\text{ram}} \cup \{p\}}} \delta_{\mathcal{D}_n, q}^{[j]}(\underline{f}') \\ &= \sum_{\psi \in \widehat{R}_{F/\mathbb{Q}}^{(0, 1_{\mathbb{Q}})}} \left(\mathbf{d}_{\mathbb{Q}, \Omega^{\text{ram}}}^{\min, [j]} + \sum_{\nu} \mathbf{d}_{\mathbb{Q}, \nu}^{[\chi, j]} \right) \\ & \quad + \sum_{w=1}^n \sum_{\chi \in \widehat{R}_{\mathcal{D}/F}^{(w)}} \sum_{\psi \in \widehat{R}_{F/\mathbb{Q}}^{(w, \chi)}} \dim(\varrho_{\chi, \psi}) \cdot \frac{[F_\chi : \mathbb{Q}]}{p-1} \\ & \quad \times \left(\mathbf{d}_{F_\chi, \Omega^{\text{ram}}}^{\min, [j]} + \sum_{\substack{\nu | \text{cond}(\mathbf{V}_{f'}), \\ \nu \nmid p, \nu \notin \Omega_{F_\chi}^{\text{cyc}}}} \mathbf{d}_{F_\chi, \nu}^{[\chi, j]} \right). \end{aligned}$$

The left-hand side is independent of j as $\mu_p \subset \mathcal{D}_n$, hence so is the right-hand side. In fact as $[F_\chi : \mathbb{Q}] = \dim(\varrho_{\chi, \psi})$, one can reinterpret the right-hand

side as being

$$\sum_{w=0}^n \sum_{\chi \in \widehat{R}_{\mathcal{D}/F}^{(w)}} \sum_{\psi \in \widehat{R}_{F/\mathbb{Q}}^{(w,\chi)}} \frac{\dim(\varrho_{\chi,\psi})^2}{p-1} \times \left(\mathbf{d}_{\mathbb{Q}(\mu_p), \Omega^{\text{ram}}}^{\min, [-]} + \sum_{\substack{\nu | \text{cond}(\mathbf{V}_{f'}), \\ \nu \nmid p, \nu \notin \Omega_{\mathbb{Q}(\mu_p)}^{\text{cyc}}}} \mathbf{d}_{\mathbb{Q}(\mu_p), \nu}^{[\chi, -]} \right).$$

We next observe that this triple summation for $\dim(\varrho_{\chi,\psi})^2$ is equal to the sum of $\dim(\varrho)^2$ as ϱ ranges over the irreducible representations factoring through \mathcal{D}_n/\mathbb{Q} ; in other words $\sum_{w,\chi,\psi} \dim(\varrho_{\chi,\psi})^2 = \sum_{\text{irr } \varrho} \dim(\varrho)^2 = [\mathcal{D}_n : \mathbb{Q}]$. As a nice corollary,

$$(5.15) \quad \lambda \left(\text{Sel}_{\mathcal{D}_n^{\text{cyc}}, \Omega^{\text{ram}}}^{[j]} (A_{\underline{f}'}^\wedge) \right) + \sum_{\substack{q | \text{cond}(\mathbf{V}_{f'}), \\ q \notin \Omega^{\text{ram}} \cup \{p\}}} \delta_{\mathcal{D}_n, q}^{[j]}(\underline{f}') = \frac{[\mathcal{D}_n : \mathbb{Q}]}{p-1} \times \mathfrak{J}(\underline{f}')$$

where the non-negative constant $\mathfrak{J}(\underline{f}') = \mathfrak{J}(\underline{f}'; \bar{\rho}_1, \dots, t, \Omega^{\text{ram}})$ is given by

$$\begin{aligned} \mathfrak{J}(\underline{f}') &= \dim_{\mathbb{F}} \left(\text{Sel}_{\mathbb{Q}(\mu_{p^\infty}), \Omega^{\text{ram}}}^{\min, [-]} (\bar{\rho}_1, \dots, t) \right) \\ &\quad + \sum_{\substack{\nu | \text{cond}(\mathbf{V}_{f'}), \\ \nu \nmid p, \nu \notin \Omega_{\mathbb{Q}(\mu_p)}^{\text{cyc}}}} \dim_{\mathbb{F}} \left(H^1(\mathbb{Q}(\mu_{p^\infty})_\nu, V_{\bar{\rho}_1, \dots, t}) \right). \end{aligned}$$

However $\delta_{\mathcal{D}_n, q}^{[j]}(\underline{f}')$ coincides with $\mathbf{e}_{\mathcal{D}_n, q}^{[j]}(\underline{\mathbf{a}}')$ for all \underline{f}' lying on the multi-branch \mathbf{a}' , and one may then use Equation (5.15) to conclude that

$$(5.16) \quad [\mathcal{D}_n : \mathcal{D}_1]^{-1} \times \left(\lambda \left(\text{Sel}_{\mathcal{D}_n^{\text{cyc}}, \Omega^{\text{ram}}}^{[j]} (A_{\underline{f}'}^\wedge) \right) + \sum_{\substack{q | \text{cond}(\mathbf{V}_{f'}), \\ q \notin \Omega^{\text{ram}} \cup \{p\}}} \mathbf{e}_{\mathcal{D}_n, q}^{[j]}(\underline{\mathbf{a}}') \right)$$

is a non-negative integer, independent of $n \geq 1$ and j but dependent on $\text{cond}(\mathbf{V}_{f'})$. Note that assertion (ii) now follows inductively from this independence property.

Finally to establish assertion (iii), we first remark that one can rearrange (5.14) into the modified form

$$(5.14)^\dagger \quad \lambda_{F_X, \Omega^{\text{ram}}}^{[\chi\psi_{F_X}, j]}(\underline{f}') = \mathbf{d}_{F_X, \Omega^{\text{ram}}}^{\min, [j]} + \sum_{\substack{\nu | D \cdot \text{cond}(\mathbf{V}_{f'}), \\ \nu \nmid p, \nu \notin \Omega_{F_X}^{\text{cyc}}}} \mathbf{d}_{F_X, \nu}^{[\chi, j]} - \delta_{F_X, \nu}^{[\chi\psi_{F_X}, j]}(\underline{f}')$$

for any $D \in \mathbb{N}$, because $\mathbf{d}_{F_X, \nu}^{[\chi, j]} = \delta_{F_X, \nu}^{[\chi\psi_{F_X}, j]}(\underline{f}')$ at places $\nu \nmid p \cdot \text{cond}(\mathbf{V}_{f'}) \cdot \text{disc}(\mathcal{D}_n)$. We shall study what happens to the cyclotomic λ -invariant when we switch between the two pairs $(\underline{f}', \mathbf{a}') = (\underline{f}^{(1)}, \mathbf{a}^{(1)})$ and $(\underline{f}', \mathbf{a}') = (\underline{f}^{(2)}, \mathbf{a}^{(2)})$ in the Hida family. Let us carefully choose the positive integer

D to equal $\text{cond}(\mathbf{V}_{\underline{f}(1)}) \cdot \text{cond}(\mathbf{V}_{\underline{f}(2)})$. Applying the same reasoning as we employed for the proof of statement (ii) above, and using Equation (5.14)[†] in the rôle of (5.14), one deduces that

$$(5.15)^\dagger \quad \lambda \left(\text{Sel}_{\mathcal{D}_n^{\text{cyc}}, \Omega^{\text{ram}}}^{[j]} (A_{\underline{f}'}^\wedge) \right) + \sum_{\substack{q|D, \\ q \notin \Omega^{\text{ram}} \cup \{p\}}} \mathbf{e}_{\mathcal{D}_n, q}^{[j]}(\mathbf{a}') = \frac{[\mathcal{D}_n : \mathbb{Q}]}{p-1} \times \mathfrak{J}^\dagger(D)$$

where the modified constant $\mathfrak{J}^\dagger(D) = \mathfrak{J}^\dagger(D; \bar{\rho}_1, \dots, t, \Omega^{\text{ram}})$ is explicitly defined as

$$\begin{aligned} \mathfrak{J}^\dagger(D) = & \dim_{\mathbb{F}} \left(\text{Sel}_{\mathbb{Q}(\mu_{p^\infty}), \Omega^{\text{ram}}}^{\min, [-]} (\bar{\rho}_1, \dots, t) \right) \\ & + \sum_{\substack{\nu|D, \\ \nu \nmid p, \nu \notin \Omega_{\mathbb{Q}(\mu_p)}^{\text{cyc}}}} \dim_{\mathbb{F}} \left(H^1(\mathbb{Q}(\mu_{p^\infty})_\nu, V_{\bar{\rho}_1, \dots, t}) \right). \end{aligned}$$

Since $\mathfrak{J}^\dagger(D)$ is unchanged whenever we switch between $\underline{f}' = \underline{f}^{(1)}$ and $\underline{f}' = \underline{f}^{(2)}$ so is the left-hand side of Equation (5.15)[†], and the last part of Theorem 2.3 is proved.

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Daniel DELBOURGO
Department of Mathematics
University of Waikato
Hamilton (New Zealand)
daniel.delbourgo@waikato.ac.nz