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THE AREA IS A GOOD ENOUGH METRIC

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Martin MÖLLER & Jonathan ZACHHUBER

Abstract. — In the first part we extend the construction of the smooth normal-crossing divisors compactification of projectivized strata of abelian differentials given by Bainbridge, Chen, Gendron, Grushevsky and Möller to the case of k-differentials. Since the generalized construction is closely related to the original one, we mainly survey their results and justify the details that need to be adapted in the more general context.

In the second part we show that the flat area provides a canonical hermitian metric on the tautological bundle over the projectivized strata of finite area k-differentials whose curvature form represents the first Chern class. This result is useful in order to apply Chern–Weil theory tools. It has already been used as an assumption in the work of Sauvaget for abelian differentials and is also used in a paper of Chen, Möller and Sauvaget for quadratic differentials.

Résumé. — Dans la première partie de cet article nous étendons aux k-différentielles la construction d'une compactification lisse avec un bord a croisements normaux des strates projectivisées de différentielles abéliennes introduite par Bainbridge, Chen, Gendron, Grushevsky et Moeller. Comme cette construction est très liée à la construction originale, nous ne présentons qu'un survol de celle-ci en soulignant les points qui nécessitent des modifications dans ce contexte plus général.

Dans la deuxième partie nous démontrons que l'aire fournit une métrique canonique hermitienne sur le fibré tautologique au-dessus des strates projectivisées dont la courbure représente la première classe de Chern. Ce résultat est utile pour pouvoir appliquer les outils de la théorie de Chern–Weyl. Ce résultat a déjà été utilisé comme une hypothèse dans les travaux de Sauvaget sur les différentielles abéliennes et dans les travaux de Chen, Moeller et Sauvaget sur les différentielles quadratiques.

1. Introduction

A flat surface (X, ω) is a Riemann surface together with a non-zero holomorphic one-form. Interest in flat surfaces stems from dynamics of polygonal billiards and this paper contributes to justifying the foundations for

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the efficient computation of invariants like Siegel–Veech constants of these billiards. A natural invariant of a flat surface is the flat area $\mathrm{vol}(X,\omega)$, the area taken with respect to the form $|\omega|$. As such, it defines a hermitian metric h on the tautological line bundle $\mathcal{O}(-1)$ over the projectivized strata $\mathbb{P}\Omega\mathcal{M}_g(\mu)$, the moduli space parameterizing flat surfaces whose zeros and poles are of a fixed type $\mu = (m_1, \dots, m_n)$. This metric does not extend smoothly over the boundary, as the area of a flat surface tends to ∞ when X acquires an infinite flat cylinder, i.e. when ω acquires a simple pole. In Chern–Weil theory applications, it suffices to show that the curvature form of the metric connection associated to the metric h represents the first Chern class of $\mathcal{O}(-1)$ on a suitable compactification. This has been used as assumption by Sauvaget in [9] for Masur–Veech volumes of the minimal strata of abelian differentials. While a workaround for this has been given in [6], the computation of the volume of individual spin components in loc. cit. is still based on that assumption. Moreover, the paper [5] extends this line of thought to quadratic differentials. There, too, the volume of the canonical double cover (see Section 2) provides a natural hermitian metric. Even for principal strata, where the Hodge bundle provides a smooth compactification, we do not see an easy route to prove the claim in the title, see the subtleties explained below. This paper consequently makes full use of the smooth compactification of strata of abelian differentials constructed in [4]. Yet another application is a growth justification in the recent computation of the volume of moduli spaces of flat surfaces (in the sense of Veech [11] by Sauvaget [10]).

Given the applications in mind, the first part of this paper is a survey about the construction of the smooth compactification and the formal justification of the tempting claim that the construction extends to k-differentials, if the notions are appropriately adapted in the same way as [3] adapts [2].

1.1. The compactification

Let $\mu = (m_1, \dots, m_n) \in \mathbb{Z}^n$ be a type of a meromorphic k-differential, i.e. m_i are integers such that $\sum m_i = k(2g-2)$. Let $\Omega^k\mathcal{M}_{g,n}(\mu)$ be the moduli space of triples $(X, \mathbf{z}, q)$ consisting of a smooth curve X of genus g with marked points $\mathbf{z} = (z_1, \dots, z_n)$ and a k-differential having zeros or poles of order m_i at the points z_i. We summarize the properties of our compactification $\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ of this moduli space of k-differentials. The canonical cover construction and related notions are recalled in Section 2.

THEOREM 1.1. — *There exists a complex orbifold $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$, the moduli space of multi-scale k-differentials, with the following properties.*

(i) *The space $\Omega^k\mathcal{M}_{g,n}(\mu)$ is dense in $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$.*
(ii) *The boundary $D = \Xi^k\overline{\mathcal{M}}_{g,n}(\mu) \smallsetminus \Omega^k\mathcal{M}_{g,n}(\mu)$ is a normal crossing divisor.*
(iii) *The rescaling action of $\mathbb{C}^*$ on $\Omega^k\mathcal{M}_{g,n}$ extends to $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ and the resulting projectivization $\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ is a compactification of $\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)$.*
(iv) *Via the canonical cover construction, the space $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ has a map to the compactification $\Xi\overline{\mathcal{M}}_{\hat{g},\{\hat{n}\}}(\widehat{\mu})$ of the corresponding stratum of abelian differentials with partially labeled points. In the interior, this map is a closed immersion.*

Here we only prove that $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ is a "moduli space" in a very weak form, namely by exhibiting what its complex points correspond to, the multi-scale k-differentialsintroduced below. We leave it to the interested reader to adapt the functor from [4] to the context of k-differentials. Recall that a k-differential is called *primitive* if it cannot be written as d-th power of a k/d-differential for any $d > 1$. In general our notion of k-differentials does not imply primitivity. This is convenient for defining twisted k-differentials below, but as a consequence the spaces in Theorem 1.1 have many connected components, those consisting of k-th powers of abelian differentials having dimension one more than the other components.

Besides the normal crossing boundary, the most relevant property for us is the existence of a convenient coordinate system, given by *perturbed period coordinates*. To introduce this, we first have to explain how to parameterize boundary points of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$.

Let $\Gamma = (V, H, E, g)$ be a stable graph, as in [1], where V is the set of vertices, H is the set of half-edges, E is the set of edges and g is the genus assignment. A *level graph* is a stable graph together with a weak total order on the set of vertices, which is determined by a level function normalized to take values in $0, -1, \ldots, -L$, with zero being the top level and the rest often referred to as *lower levels*. An edge of a level graph is called *horizontal* if it is adjacent vertices are on the same level and *vertical otherwise*. This leads to a partition of the edges $E = E^h \cup E^v$ and we use the adjectives horizontal and vertical for the corresponding nodes accordingly. An *enhanced level graph* is a level graph together with an enhancement $\kappa\colon H \to \mathbb{Z}$ on the half-edges that specifies the number of prongs of the

differential at the corresponding marked point, see Section 2 for the full definition.

Each of the levels of Γ thus specifies a moduli space of k-differentials, the type being given by the enhancement and the restrction of μ to the legs at that level. A collection of these differentials, one for each level, on a given pointed stable curve $(X, \mathbf{z})$ is called *twisted differential* and we call a twisted differential *compatible with the enhanced level graph* Γ if the underlying graph of Γ is the stable graph of $(X, \mathbf{z})$ and if the collection moreover satisfies the global k-residue-condition (GRC) from [3]. A *multi-scale k-differential* is a twisted differential compatible with Γ up to projectivization of the levels below zero, together with the choice of an equivalence class of prong-matchings. The details are given in Section 3 using the notion of level rotation torus. Leaving them aside, we can now describe the coordinates.

Proposition 1.2. — *In a neighborhood $U \subset \Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ of every point in the boundary stratum corresponding to an enhanced level graph Γ with $L+1$ levels and h horizontal edges there is an orbifold chart given by the perturbed period map*

$$\mathrm{PPer} : U \to \mathbb{C}^{|E^h|} \times \mathbb{C}^{L+1} \times \prod_{i=0}^{L} \mathbb{C}^{\dim E^{\mathrm{grc}}_{(-i)} - 1},$$

where $E^{\mathrm{grc}}_{(-i)}$ is some eigenspace in homology constrained by the GRC and where the corresponding coordinates are obtained by integrating perturbations of the twisted differential against these homology classes.

In this proposition, the first set of coordinates in $\mathbb{C}^{|E^h|}$ measures the opening of horizontal nodes and the second set in $\mathbb{C}^L$ measures the rescaling of the differentials on each level. Neither of them is a period, in fact they are exponentials, respectively roots, of periods. The statement about integration is intentionally vague, since we are not exactly integrating the (roots of) k-differentials parameterized by U, but its sum with a modification differential, as constructed in Section 3. Moreover, the path of integration is not between the zeros of those differentials but between neighboring points, thus the name "perturbed". Technically important is that these perturbations go to zero faster than the rescaling of the k-differential. The map PPer depends on many choices (see Section 3.6 for more details), however they are irrelevant for many local computations.

1.2. Boundary divisors

To a first approximation the boundary divisors, i.e., the irreducible components of the boundary $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)\smallsetminus\Omega^k\mathcal{M}_{g,n}(\mu)$, are given by graphs with one level and a single horizontal edge, and by graphs with two levels and no horizontal edge. However, in the setting of k-differentials the level graph does not specify the boundary divisor uniquely. In Section 2 we recall the notion of canonical k-cover, which is unique for k-differentials on smooth curves, but not in the stable case. An example for two different covers that give rise to different components of the boundary is given by [3, Figure 2]. In fact, the residue conditions are different in the two cases. Consequently, as second approximation the choice of a cyclic k-cover $\pi\colon\widehat{\Gamma}\to\Gamma$ compatible with the canonical covers of the components (see Section 2 for the definition of both notions) characterizes boundary components.

Proposition 1.3. — *For each k-cyclic cover $\pi\colon\widehat{\Gamma}\to\Gamma$ of enhanced level graphs with Γ of type (g,n) there is a boundary stratum $D_{\hat{\Gamma}}$ of the compactification $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$. Each $D_{\hat{\Gamma}}$ is commensurable to the product of the level-wise projectivised moduli space of twisted differentials on $\widehat{\Gamma}$.*

Here "commensurable" is a shorthand for the existence of a complex space with a finite map to the two spaces in question.

We will not address the subtle question of connectivity of those $D_{\hat{\Gamma}}$. The details of the construction of a space that admits a finite covering to both $D_{\hat{\Gamma}}$ and the product level-wise projectivised moduli spaces is given in [7, Section 4.2]. There, the construction is given for Abelian differentials, but it can verbatim be applied for k-differentials, too.

1.3. The metric

We now return to our primary goal. The statement is about flat surfaces of finite area, so we suppose from now on that $m_i>-k$. If $\pi\colon\widehat{X}\to X$ denotes the canonical covering associated with $(X,q)\in\Omega^k\mathcal{M}_{g,n}(\mu)$ such that $\pi^*q=\omega^k$ is a k-th power, then the definition

$$h(X,q)^{1/k}=\operatorname{area}_{\widehat{X}}(\omega)=\frac{i}{2}\int_{\widehat{X}}\omega\wedge\overline{\omega} \tag{1.1}$$

provides the tautological bundle $\mathcal{O}(-1)$ on $\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)$ with a hermitian metric h. The moduli space $\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)$ has, besides the nice compactification $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ discussed above, a highly singular compactification, the

incidence variety compactification $\mathbb{P}\overline{\Omega^k\mathcal{M}}_{g,n}(\mu)$ that has been studied in [2] and [3]. It is the closure of $\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)$ inside the projectivized bundle of k-fold stable differentials twisted by the polar part of μ. This projectivized bundle provides an extension of the tautological bundle $\mathcal{O}(-1)$, whose restriction to the incidence variety compactification we denote by the same symbol.

There is a natural forgetful map $\varphi\colon \mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu) \to \mathbb{P}\overline{\Omega^k\mathcal{M}}_{g,n}(\mu)$, which is an isomorphism restricted to $\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)$. The pullback of $\mathcal{O}(-1)$ thus provides an extension of the tautological bundle on $\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ that we still denote by the same symbol. It is this bundle whose Chern classes are relevant ([6, 9]) for computation of Masur–Veech volumes and Siegel–Veech constants. Our main theorem is:

Theorem 1.4. — *The curvature form* $\frac{i}{2\pi}[F_h]$ *of the metric* h *is a closed current on* $\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ *that represents the first Chern class* $c_1(\mathcal{O}(-1))$. *More generally, the* d*-th wedge power of the curvature form represents* $c_1(\mathcal{O}(-1))^d$ *for any* $d \geqslant 1$.

In an earlier version of the paper we had claimed that the metric h is good in the sense of Mumford. This is not true at boundary points where there are both horizontal and vertical edges, as explained in Section 4. We thank Duc-Manh Nguyen for bringing this to our attention.

More precisely, in the case of only horizontal nodes the metric diverges as we approach the boundary. However in perturbed period coordinates coordinates the local calculation is essentially the calculation of Mumford for the special case of elliptic curves (times the number of horizontal nodes).

In the absence of horizontal nodes, the metric extends continuously over the boundary. This fits with the intuition that the area of the lower level surfaces goes to zero. The area is not a C^2-function near vertical notes, but the second derivative is integrable, which turns out to be good enough. In the presence of both horizontal and vertical edges we estimate directly the growth of the curvature form to justify Theorem 1.4. This is a delicate computation that makes full use of the coordinate system that we have near the boundary, which in turn is the main reason for working with the compactification we constructed rather than say simple with the full Hodge bundle $\mathbb{P}\overline{\Omega^k\mathcal{M}}_{g,n}$.

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2. Period coordinates and canonical covers of k-differentials

In this section we summarize well-known results about period coordinates, but also recall the period coordinates along the boundary strata of the incidence variety compactification from [3]. We start by recalling properties of the canonical k-cover.

Let X be a Riemann surface and let q be a meromorphic k-differential of type μ. This datum defines (see e.g. [3, Section 2.1]) a k-fold cover $\pi\colon \widehat{X} \to X$ such that $\pi^* q = \omega^k$ is the k-power of an abelian differential. Note that $\widehat{X}$ is disconnected, if q is a d-th power of a k/d-differential for some $d > 1$. This differential ω is of type

$$\widehat{\mu} := \Big(\underbrace{\widehat{m}_1, \dots, \widehat{m}_1}_{\gcd(k,m_1)}, \underbrace{\widehat{m}_2, \dots, \widehat{m}_2}_{\gcd(k,m_2)}, \dots, \underbrace{\widehat{m}_n, \dots, \widehat{m}_n}_{\gcd(k,m_n)}\Big),$$

where $\widehat{m}_i := \frac{k+m_i}{\gcd(k,m_i)} - 1$. We let $\widehat{g} = g(\widehat{X})$ and $\widehat{n} = \sum_i \gcd(k, m_i)$. The type of the covering determines a natural subgroup $S_{\widehat{\mu}} \subset S_{\widehat{n}}$ of the symmetric group that allows only the permutations of each the $\gcd(k, m_i)$ points corresponding to a preimage of the i-th point.

We fix *once and for all a primitive k-th root of unity* ζ. The Deck group of π contains a unique element τ such that $\tau^*\omega = \zeta\omega$. We fix this automorphism as well. We denote by $\mathbf{z} = (z_1, \dots, z_n)$ the tuple of marked points in X. The preimages in $\widehat{X}$ of these marked points give a tuple that is labeled up to the action of $S_{\widehat{\mu}}$ and which we denote by $\widehat{\mathbf{z}}$. By the canonical cover construction there is an isomorphism of orbifolds between the moduli space $\Omega^k\mathcal{M}_{g,n}(\mu)$ and the space of

$$\{(\widehat{X}, \widehat{\mathbf{z}}, \omega, \langle\tau\rangle) : \tau \in \mathrm{Aut}(\widehat{X}), \quad \mathrm{ord}(\tau) = k, \quad \tau^*\omega = \zeta_k\omega\} \tag{2.1}$$

which has a natural closed immersion to $\Omega\mathcal{M}_{\widehat{g},\widehat{n}}(\widehat{\mu})/S_{\widehat{\mu}}$.

For the analogous statements about coverings in the stable case we first need to define twisted k-differentials and further preparation. An *enhanced level graph* for k-differentials is a level graph together with an enhancement map $\kappa : H \to \mathbb{Z}$ on the half-edges, satisfying the following properties:

(i) If h and h' are paired to an edge, then $\kappa(h) + \kappa(h') = 0$.
(ii) At a leg $h \in H \smallsetminus E$ with order m_i, we impose that $\kappa(h) = m_i + k$.

(iii) At each vertex $v \in V(\Gamma)$

$$k(2g(v)-2) = \sum_{h \vdash v} (\kappa(h) - k)\,.$$

The next notion is a combinatorial model of canonical curves, as they occur in the limit when tending to stable curves. Let $\widehat{\Gamma}$ be an enhanced level graph for abelian differentials of type $(\widehat{g}, \widehat{n})$ and let Γ be an enhanced level graph for k-differentials of type (g, n). A *(cyclic) k-cover of nhanced level graphs* $\pi\colon \widehat{\Gamma} \to \Gamma$ is a morphism f graphs (with legs), given as the quotient map by a graph automorphism $\tau \in \mathrm{Aut}(\widehat{\Gamma})$ of order k that preserves levels, the orders $\widehat{m}_i$ and enhancements, and with the following two properties: An edge e has $\gcd(\kappa_e, k)$ preimages and a marked point of type m_i has $\gcd(m_i, k)$ preimages.

We next give a the definition of a twisted differential. The case of k-differentials is reduced to the case of abelian differentials. Let $\widehat{\Gamma}$ be an enhanced level graph for the stable curve $(\widehat{X}, \widehat{\mathbf{z}})$. A *twisted* 1*-differential* $\boldsymbol{\omega} = (\omega_v)$ *compatible with* $\widehat{\Gamma}$ is a collection of differentials on $\widehat{X}_v$ for each vertex $v \in V(\widehat{\Gamma})$ of type given by the markings and enhancements, i.e., $\mathrm{ord}_e(\omega_v) = \kappa_e - 1$ for each edge e adjacent to v. This collection is required to satisfy the usual residue condition at horizontal nodes and moreover the global residue condition (GRC), see [3] for details on this. Later it will be convenient to group together the differentials on all vertices of the same level and we thus write $\boldsymbol{\omega} = (\omega_i)_{i \in L(\widehat{\Gamma})}$.

A collection of k-differentials on X naturally defines a k-cover for each component of X, but it does not uniquely define how to glue them to a stable curve $\widehat{X}$. A k-cover of level graphs contains this gluing information. Let Γ be an enhanced level graph for the stable curve $(X, \mathbf{z})$. A *twisted k-differential compatible with* Γ is a collection $\mathbf{q} = (q_v)$ of differentials on X_v for each vertex $v \in V(\Gamma)$ such that the pullback to the stable curve $\widehat{X}$ given by the canonical covers induced by q_v and some k-cover $\pi\colon \widehat{\Gamma} \to \Gamma$ is a k-th power of a twisted abelian differential. (A formulation of this condition directly on X is given in [3]. This includes however a quite complicated formulation of the GRC.) As above we group these differentials according to levels and write $\mathbf{q} = (q_i)_{i \in L(\Gamma)}$. We say that $\mathbf{q}$ is *compatible with* $\widehat{\Gamma}$, if the above condition holds for a chosen k-cover $\pi\colon \widehat{\Gamma} \to \Gamma$. By definition we can specify a twisted k-differential either by $(X, \mathbf{z}, \mathbf{q}, \pi)$ or by $(\widehat{X}, \widehat{\mathbf{z}}, \boldsymbol{\omega}, \tau)$, where τ is an automorphism of order k.

The starting point for the construction of the compactification is the moduli space of twisted k-differentials compatible with $\widehat{\Gamma}$, which we denote by $\mathfrak{W}^k(\widehat{\Gamma})$ suppressing the dependence on the initial type μ. By definition

this is a subspace defined by the GRC inside a product of strata. The main result of [3] implies that points in $\mathfrak{W}^k(\widehat{\Gamma})$ are smoothable: they arise as limits of k-differentials in $\Omega^k\mathcal{M}_{g,n}(\mu)$.

Next we define the subspaces of homology that we use for period coordinates. We fix some reference smooth surface Σ of genus g with n marked points that we partition as $P \cup Z$ according to the poles of order $\leqslant -k$ among μ and the "zeros", i.e. points with order $> -k$. We let $\widehat{\Sigma}$ be a model for the canonical covering surface, which is of genus $\widehat{g}$, and which comes with a map $\pi\colon \widehat{\Sigma} \to \Sigma$. We let $\widehat{P}$ and $\widehat{Z}$ be the preimages of P and Z. They now correspond indeed to the zeros and poles of the type $\widehat{\mu}$.

$$\begin{array}{ccc} \widehat{P}, \widehat{Z} & \widehat{\Sigma} & \widehat{\Lambda}^\circ \supset \widehat{\Lambda} \\ \downarrow \pi & \downarrow \pi & \downarrow \pi \\ P, Z & \Sigma & \Lambda^\circ \supset \Lambda \end{array}$$

If X is a stable curve and π a covering as above, we may find a multicurve $\widehat{\Lambda}$ in $\widehat{\Sigma}$ mapping under π to the multicurve Λ in Σ, such that $\widehat{X}$ and X are obtained by pinching $\widehat{\Sigma}$ and Σ along $\widehat{\Lambda}$ and Λ respectively.

Recall ([3, Section 2]) that the moduli space $\Omega^k\mathcal{M}_{g,n}(\mu)$ of k-differentials is locally modeled on the ω-periods of the eigenspace

$$E(\widehat{\Sigma} \setminus \widehat{P}, \widehat{Z}) = H_1(\widehat{\Sigma} \setminus \widehat{P}, \widehat{Z}, \mathbb{C})_{\tau=\zeta}\,.$$

Similarly, we can describe local coordinates for the components of a twisted k-differential on a stable curve X with enhanced level graph Γ (not yet imposing full compatibility, i.e. the GRC). Let Λ° be an open thickening of Λ. We let $\Lambda^\pm$ be the upper and lower boundaries of Λ°. The level structure on Γ organizes $\Sigma \smallsetminus \Lambda$ into levels $\Sigma_{(i)}$ and we denote the adjacent poles, zeros and boundaries $\Lambda^\pm$ with the subscript (i). All the notation is applied with a hat to the corresponding objects on the k-cover. The level-i component of the twisted differential is thus modeled on

$$E_{(i)} = H_1(\widehat{\Sigma}_{(i)} \setminus \{\widehat{P}_{(i)} \cup \widehat{\Lambda}^\circ_{(i)}\}, \widehat{\Lambda}^+_{(i)} \cup \widehat{Z}_{(i)}, \mathbb{C})_{\tau=\zeta}\,. \tag{2.2}$$

We can now restate the main dimension estimate in the proof of [3, Theorem 6.2].

Proposition 2.1. — *The moduli space of twisted k-differentials compatible with an enhanced level graph Γ is locally modeled on the $\omega_{(-i)}$-periods of $\prod_{i=0}^{L} E^{\mathrm{grc}}_{(-i)}$, where $E^{\mathrm{grc}}_{(i)} \subseteq E_{(i)}$ is the subspace at level $i \in L(\Gamma)$ cut out by the global residue condition. The dimensions of these subspaces*

is given by

$$\sum_{i=0}^{L} \dim_{\mathbb{C}} E^{\mathrm{grc}}_{(-i)} = \dim_{\mathbb{C}} \Omega^k \mathcal{M}_{g,n} \; - \; |E^h| \,,$$

where E^h is the set of horizontal edges of Γ.

3. Construction of $\Xi^k \overline{\mathcal{M}}_{g,n}(\mu)$

In this section we recall the main technical tools from [4], construct the compactification and eventually prove Theorem 1.1. The definitions in Sections 3.1–3.4 are direct adaptations of the abelian case by working on the canonical k-covers. Avoiding the discussion of Teichmüller spaces means omission of that aspect but also a simplification of notations. In the remaining sections we have to ensure at some places that constructions can be performed τ-equivariantly.

3.1. Degeneration, undegeneration

We describe here two types of maps between level graphs Γ that encode the degeneration of curves, together with the compatible maps between the coverings graphs $\widehat{\Gamma}$ that form part of the degeneration datum. In fact, it is easier to first describe the inverse process of undegeneration that encodes all the k-differentials in a neighborhood of a given degenerate k-differential.

Let $\pi\colon \widehat{\Gamma} \to \Gamma$ by a cyclic k-covering of enhanced level graphs with $L+1$ normalized levels. For any subset $I \subset \{1, \dots, L\}$, to be memorized as the *the level passages that remain*, we define the *vertical undegeneration* δ_I as the following contraction of certain vertical edges. An edge e is contracted by δ_I if and only it crosses the level passages indexed by the complement of I. Vertices are merged if an edge connecting them has been contracted. This edge contraction is performed simultaneously on the domain and range of π and induces a cyclic k-covering $\delta_I(\widehat{\Gamma}) \to \delta_I(\Gamma)$ that we abbreviate as $\delta_I(\pi)$. We write $\delta_I(j)$ for the image of the j-th level under δ_I.

Moreover, we define for any subset $E_0 \subset E^{\mathrm{h}}$ of the horizontal edges of Γ the *horizontal undegeneration* δ_{E_0} to be the edge contraction that contracts precisely the edges in E_0 in Γ. Contracting simultaneously also on the π-preimages of E_0 in $\widehat{\Gamma}$, we obtain a new cyclic k-covering $\delta_{E_0}(\pi) : \delta_{E_0}(\widehat{\Gamma}) \to \delta_{E_0}(\Gamma)$.

A general undegeneration is a pair $\delta = (\delta_I, \delta_{E_0})$, defined by composing a horizontal and a vertical undegeneration in either order. A *degeneration*

is the inverse of an undegeneration. We write $\widehat{\Gamma}' \rightsquigarrow \widehat{\Gamma}$ for a general degeneration of level graphs and δ^{ver} and δ^{hor} for the two constituents of an undegeneration δ.

For any edge e of $\widehat{\Gamma}$, we will denote by e^- both the point in $\widehat{X}$ at the bottom end of the edge e and its level. The meaning should be clear from the context. Similarly, e^+ refers to the top end.

3.2. Prong-matchings as extra structure on twisted differentials

We start with the definition of prong-matchings and the welded surfaces constructed from these. Given a differential ω on X that has been put in standard form, which is $z^\kappa \mathrm{d}z/z$ if $\kappa > 0$ or $(z^\kappa + r)\mathrm{d}z/z$ if $\kappa < 0$, the *prongs* are the $|\kappa|$ tangent vectors $\mathrm{e}^{2\pi i j/|\kappa|}\frac{\partial}{\partial z}$ for $\kappa > 0$ and $-\mathrm{e}^{2\pi i j/|\kappa|}\frac{\partial}{\partial z}$ for $j = 0, \dots, |\kappa|-1$. At simple poles (i.e. for $\kappa = 0$), prongs are not defined.

We now get back to a twisted 1-differential $(\widehat{X}, \boldsymbol{\omega}, \widehat{\Gamma})$. Define a *local prong-matching* σ_e at the vertical edge e of $\widehat{\Gamma}$ to be a cyclic order-reversing bijection between the κ_e prongs of $\boldsymbol{\omega}$ at the upper and lower end of e. A *global prong-matching* is a collection $\sigma = (\sigma_e)_{e \in E(\widehat{\Gamma})}$ of local prong-matchings. If the twisted 1-differential stems from a twisted k-differential $(X, \mathbf{z}, \mathbf{q}, \pi)$, i.e. if it contains the additional information of the automorphism τ, we require moreover that the global prong-matching is equivariant with respect to the action of τ permuting the edges and multiplying the local coordinates z by ζ.

A global prong-matching σ on $\widehat{X}$ gives an *almost-smooth surface* $\widehat{X}_\sigma$, i.e. a smooth surface except for nodes corresponding to the horizontal nodes of $\widehat{X}$, constructed by the following procedure of *welding*. Take the partial normalization of $\widehat{X}$ separating branches at vertical nodes and perform the real oriented blowup of each pair of preimages. Then identify the boundary circles isometrically so as to identify boundary points that are paired by the prong-matching. More details of the construction can be found in Section 4 of [4], see also [1]. (We only use subscripts σ to denote weldings here and suppress the overline used in loc. cit. to avoid double decorations.) Horizontal nodes remain untouched in the welding procedure.

On almost-smooth surfaces any *good arc* γ, i.e. any arc transversal to the seams created by welding, has a well-defined *turning number* with respect to the flat structure ω, that we denote by $\rho(\gamma)$.

Adding the information of prong-matching to points in the space of twisted differentials $\mathfrak{W}^k(\widehat{\Gamma})$ will give us a finite covering space as follows.

We define the set $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$ to be tuples $(\widehat{X}, \widehat{\mathbf{z}}, \boldsymbol{\omega}, \tau, \widehat{\sigma})$ consisting of a point $(\widehat{X}, \widehat{\mathbf{z}}, \boldsymbol{\omega}, \tau) \in \mathfrak{W}^k(\widehat{\Gamma})$ together with a prong-matching $\widehat{\sigma}$. There is an obvious notion of parallel transport of prong-matchings (since the finitely many tangent vectors $\pm e^{2\pi i j/|\kappa|}\frac{\partial}{\partial z}$ depend continuously on the twisted differential) that allows to lift inclusions of contractible open sets $U \to \mathfrak{W}^k_{\mathrm{na}}(\widehat{\Gamma})$ uniquely to maps $U \to \mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$. Requiring that these lifts are holomorphic local homeomorphism provides $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$ with a complex structure so that $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma}) \to \mathfrak{W}^k_{\mathrm{na}}(\widehat{\Gamma})$ is a covering map.

3.3. The level rotation torus

Our compactification combines the geometry of moduli spaces of k-differentials of lower complexity and aspects of a toroidal compactification. The torus action for the latter is given by the level rotation torus that we now define. To describe various group actions on prong-matchings, we view $\widehat{\Gamma}$ as a graph with L level *passages*, the first from level 0 to level -1, the second from level -1 to level -2 etc. This is summarized by:

Convention 3.1. — *Levels are indexed by negative integers* $0, -1, \dots, -L$*, while level passages are indexed by positive integers* $1, \dots, L$*.*

The unit vector e_i in the *level rotation group* $R_{\widehat{\Gamma}} \cong \mathbb{Z}^L$ acts on the set of prong-matchings by shifting the prong-matching for each edge crossing the i-th level passage by one counterclockwise turn. Of particular importance is the subgroup $\mathrm{Tw}_{\widehat{\Gamma}}$ of $R_{\widehat{\Gamma}}$ that fixes all prongs, the *twist group*. The *(reduced) level rotation torus* $\mathrm{T}_{\widehat{\Gamma}}$ is the quotient

$$\mathrm{T}_{\widehat{\Gamma}} = \mathbb{C}^L/\mathrm{Tw}_{\widehat{\Gamma}}.$$

(Here reduced refers to the fact that $\mathrm{T}_{\widehat{\Gamma}}$ does not rotate the top level. We will introduce this action separately for projectivization and usually drop the adjective "reduced".) The level rotation torus can also be characterized ([4, Proposition 5.4]) by its joint action on edge and level parameters, namely as the connected component of the identity of the subtorus

$$\Big\{((r_i, \rho_e))_{i,e} \in (\mathbb{C}^*)^L \times (\mathbb{C}^*)^{E(\widehat{\Gamma})} \Big| r_{|e^-|} \dots r_{|e^+|+1} = \rho_e^{\kappa_e} \text{ for all } e \in E(\widehat{\Gamma})\Big\}. \tag{3.1}$$

This characterization makes the reason for introducing $\mathrm{T}_{\widehat{\Gamma}}$ apparent, as there is a natural action of the level rotation torus on $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$ given by

$$\begin{aligned} \mathrm{T}_{\widehat{\Gamma}} \times \mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma}) \quad &\to \quad \mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma}) \\ (r_{|i|}, \rho_e) * \Big(\widehat{X}, (\omega_{(i)}), (\sigma_e)\Big) &= \Big(\widehat{X}, \big(r_{|i|} \dots r_1 \omega_{(i)}\big), (\rho_e * \sigma_e)\Big) \end{aligned} \tag{3.2}$$

where $\rho_e * \sigma_e$ is the prong-matching σ_e post-composed with the rotation by $\arg(\rho_e)$ (if the full Dehn twist around e corresponds to angle 2π, equivalently by the rotation by $\kappa \arg(\rho_e)$ for the angle in the flat metric). We alert the reader that this action uses the "triangular" basis, where the i-th component of $\mathrm{T}_{\widehat{\Gamma}}$ rotates the i-th level and all level below it by the amount $r_{|i|}$.

To obtain orbifold charts we need to define the *simple twist group* $\mathrm{Tw}^s_{\widehat{\Gamma}} \subseteq \mathrm{Tw}_{\widehat{\Gamma}}$ as the twist group elements generated by rotations of one level at a time, i.e.

$$\mathrm{Tw}^s_{\widehat{\Gamma}} = \bigoplus_{i=1}^{L} \mathrm{Tw}_{\delta_i(\widehat{\Gamma})} \,.$$

We can now define the *(reduced) simple level rotation torus* as

$$\mathrm{T}^s_{\widehat{\Gamma}} = \mathbb{C}^L / \mathrm{Tw}^s_{\widehat{\Gamma}} \,. \tag{3.3}$$

In order to describe the action of these tori we will need the integers

$$\ell_i = \mathrm{lcm}_{e \in E(\delta_i(\widehat{\Gamma}))} k_e \,, \quad \text{and} \quad m_{e,i} = \ell_i / \kappa_e \tag{3.4}$$

for $i = 1, \dots, L$ and $e \in E(\widehat{\Gamma})$. Now Proposition 5.4 in loc. cit. moreover shows that there is an identification $\mathrm{T}^s_{\widehat{\Gamma}} \cong (\mathbb{C}^*)^L$ such that the projection $\mathrm{T}^s_{\widehat{\Gamma}} \to \mathrm{T}_{\widehat{\Gamma}}$ is given in coordinates by

$$(t_i) \mapsto (r_i, \rho_e) = \left(t_i^{\ell_i} \,, \prod_{i=|e^-|}^{|e^+|+1} t_i^{\ell_i/\kappa_e} \right) \tag{3.5}$$

The composition of this parametrization (3.5) of $\mathrm{T}_{\widehat{\Gamma}}$ by $\mathrm{T}^s_{\widehat{\Gamma}}$ with the action (3.2) gives an action of $\mathbf{t} = (t_i) \in \mathrm{T}^s_{\widehat{\Gamma}}$ on welded surfaces and we denote the image of $\widehat{X}_\sigma$ under the action of $\mathbf{s}$ by $\widehat{X}_{\mathbf{s}\cdot\sigma}$.

3.4. The compactification as topological space

We start with the definition of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ as a set. For each k-cyclic covering $\pi : \widehat{\Gamma} \to \Gamma$ we define the *boundary stratum* $\Omega^k\mathcal{B}_{\hat{\Gamma}} = \mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})/\mathrm{T}_{\widehat{\Gamma}}$ and we define the set

$$\Xi^k\overline{\mathcal{M}}_{g,n}(\mu) = \coprod_{\pi:\widehat{\Gamma}\to\Gamma} \Omega^k\mathcal{B}_{\hat{\Gamma}} \,. \tag{3.6}$$

This union includes $\Omega^k\mathcal{M}_{g,n}$ for π being the trivial covering of a point to a point. Points of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ are called *multi-scale k-differentials*, i.e. the preceding definition completes the specification of the equivalence relation

stated in the introduction. Points of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ are thus given by a tuple $(\widehat{X}, \widehat{\mathbf{z}}, \widehat{\Gamma}, \boldsymbol{\omega}, \widehat{\sigma}, \tau)$ where $\boldsymbol{\omega} = (\omega_{(-i)})_{i=0}^L$ is a tuple of one forms ω_i on the subcurve corresponding to the vertices at level i. We often write just $(\widehat{X}, \boldsymbol{\omega}, \tau)$ or $(\widehat{X}, \boldsymbol{\omega}, \widehat{\Gamma})$. The equivalence classes are given by the orbits of the action (3.2) on $(\boldsymbol{\omega}, \sigma)$.

We now provide this space with a topology by exhibiting all converging sequences. The basic idea is the conformal topology on $\overline{\mathcal{M}}_g$ where sequences converge if there is an exhaustion of the complement of nodes and punctures and conformal maps of the exhaustion to neighboring surfaces, see (ii) below. For multi-scale differentials we require moreover the convergence of the differentials as in (iii) after a rescaling, where the magnitude of rescaling is compatible with the level structure, see (i) and (iii). Since the conformal topology only requires the comparison maps to be diffeomorphisms near the nodes, which can twist arbitrarily, we need to add (iv) to avoid constructing a non-Hausdorff space. In the sequel we write $\widehat{X}_{\sigma_n}$ for $(\widehat{X}_n)_{\sigma_n}$ in a sequence of welded surfaces.

We say that a sequence $(\widehat{X}_n, \boldsymbol{\omega}_n, \widehat{\Gamma}_n)$ converges to $(\widehat{X}, \boldsymbol{\omega}, \widehat{\Gamma})$, if there exist representatives of all the equivalence classes (that we denote by the same letters), a sequence $\varepsilon_n \to 0$ and a sequence $\mathbf{t}_n = (t_{n,i})_{i=1}^L \in (\mathbb{C}^*)^L$ of tuples such that:

(i) For sufficiently large n there is an undegeneration $\delta_n = (\delta_n^{\mathrm{ver}}, \delta_n^{\mathrm{hor}})$ with $\delta_n(\widehat{\Gamma}) = \widehat{\Gamma}_n$.

(ii) For sufficiently large n there is an almost-diffeomorphism $g_n\colon \widehat{X}_{\mathbf{t}_n\cdot\sigma} \to \widehat{X}_{\sigma_n}$ that is conformal on the ϵ_n-thick part of $(\widehat{X}, \widehat{\mathbf{z}})$ and that respects the marked points, up to relabeling in π-fibers.

(iii) The restriction of $\prod_{j=1}^i t_{n,j}^{\ell_j} \cdot g_n^*(\omega_n)$ to the ϵ_n-thick part of the level $-i$ subsurface of $(\widehat{X}, \widehat{\mathbf{z}})$ converges uniformly to $\omega_{(-i)}$.

(iv) For any $i, j \in L(\widehat{\Gamma})$ with $i > j$, and any subsequence along which $\delta_n^{\mathrm{ver}}(i) = \delta_n^{\mathrm{ver}}(j)$, we have

$$\lim_{n\to\infty} \prod_{k=|i|+1}^{|j|} |t_{n,k}|^{-\ell_k} = 0\,.$$

(v) The almost-diffeomorphism g_n are nearly turning-number preserving, i.e. for every good arc γ in $\widehat{X}_\sigma$, the difference $\rho(g_n \circ F_{\mathbf{t}_n} \circ \gamma) - \rho(F_{\mathbf{t}_n} \circ \gamma)$ of turning numbers converges to zero, where $F_{\mathbf{t}_n}$ is the fractional Dehn twist around the edge e by the angle $\prod_{j=1}^i t_{n,j}^{\ell_j/\kappa_e}$.

This topology is exactly the topology defined in [4] of the compactification of the moduli spaces $\Omega\mathcal{M}_g(\widehat{\mu})$ quotiented by $S_{\hat{\mu}}$ and restricted to

the subspace of k-cyclic covers. Note that the inclusion of the covering enhanced level graph $\widehat{\Gamma}$ into the datum of a multi-scale k-differentialimplies that even boundary points have canonically determined k-covers. We thus obtain a map

$$\Xi^k\overline{\mathcal{M}}_{g,n}(\mu) \longrightarrow \Xi\overline{\mathcal{M}}_{\hat{g},\{\hat{n}\}}(\widehat{\mu}) := \Xi\overline{\mathcal{M}}_{\hat{g},\hat{n}}(\widehat{\mu})/S_{\widehat{\mu}}. \tag{3.7}$$

Proposition 3.2. — *The moduli space $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ is a Hausdorff topological space and its projectivization $\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ is a compact Hausdorff space.*

The map to the partially marked stratum of abelian differentials in (3.7) *restricted ot $\Omega^k\mathcal{M}_{g,n}$ is a closed immersion.*

Proof. — The proof that $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ is Hausdorff and has compact quotient can be taken verbatim from the abelian case [3, Theorem 9.4 and Proposition 14.2], since the topology is defined in the same way. The second statement is local, so that we can use a marking of relative homology and period coordinates, see e.g. [3, Theorem 2.1]. □

We do not enter the discussion about the local structure of (3.7) near the boundary, since for local computations in Section 4 we use the model domain we now define and the resulting perturbed period coordinates.

3.5. Model differentials and modifying differentials

In order to provide $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ with a complex structure we use a local model space that automatically has a complex structure (as a finite cover of a product of spaces of non-zero k-differentials). The degeneration of differentials on lower components is emulated in the model space by vanishing of auxiliary parameters t_i.

A first attempt would be to consider that action (3.2) of the level rotation torus $\mathrm{T}_{\widehat{\Gamma}}$, that makes $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$ into a principal $(\mathbb{C}^*)^L$-bundle over the quotient space. However $\mathrm{T}_{\widehat{\Gamma}}$ is in general not naturally isomorphic to $(\mathbb{C}^*)^L$ and so there is no natural associated $\mathbb{C}^L$-bundle that could serve as local compactification. But the simple level rotation torus $\mathrm{T}^s_{\widehat{\Gamma}}$ has such an isomorphism, as remarked along with (3.5).

The successful strategy is to construct a $K_{\widehat{\Gamma}} := \mathrm{Tw}_{\widehat{\Gamma}}/\mathrm{Tw}^s_{\widehat{\Gamma}}$-cover of $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$ on which now $\mathrm{T}^s_{\widehat{\Gamma}}$ acts as a lift of the action (3.2). In [4, Section 5] such a space is constructed using an additional Teichmüller marking, and here we use the subspace where the flat surfaces admit a τ-action. On this space even the universal cover of $\mathrm{T}_{\widehat{\Gamma}}$ acts, and one can then quotient by $\mathrm{Tw}^s_{\widehat{\Gamma}}$

and forget the marking of the rest of the surface to obtain the requested *(uncompactified) simple model domain* $\mathfrak{W}^{k,s}_{\mathrm{pm}}(\widehat{\Gamma})$.

The action of $\mathrm{T}^s_{\widehat{\Gamma}}$ on $\mathfrak{W}^{k,s}_{\mathrm{pm}}(\widehat{\Gamma})$ make this space a principal $(\mathbb{C}^*)^L$-bundle over the quotient space, which we call with hindsight the "simple" version of the boundary stratum $\Omega^k\mathcal{B}^s_{\widehat{\Gamma}} = \mathfrak{W}^{k,s}_{\mathrm{pm}}(\widehat{\Gamma})/\mathrm{T}^s_{\widehat{\Gamma}}$. We define the *simple model domain* to be the associated $\mathbb{C}^L$-bundle,

$$\overline{\mathfrak{W}^{\mathfrak{k},\mathfrak{s}}_{\mathrm{pm}}(\widehat{\mathfrak{d}})} = (\mathbb{C}^L \times \mathfrak{W}^{k,s}_{\mathrm{pm}}(\widehat{\Gamma}))/\sim, \quad (\mathbf{t},\boldsymbol{\eta}) \sim (\rho\cdot\mathbf{t}, \rho^{-1}\cdot\boldsymbol{\eta}) \ \text{ for } \rho \in \mathrm{T}^s_{\widehat{\Gamma}} \tag{3.8}$$

over $\Omega^k\mathcal{B}^s_{\widehat{\Gamma}}$. The smoothness of strata of k-differentials and the smoothness of the associated $\mathbb{C}^L$-bundle directly implies:

Proposition 3.3. — *The compactified simple model domain* $\overline{\mathfrak{W}^{\mathfrak{k},\mathfrak{s}}_{\mathrm{pm}}(\widehat{\mathfrak{d}})}$ *is smooth with normal crossing boundary divisor given by the divisors* $D_i = \{t_i = 0\}$ *with* t_i *the local coordinates on* $\mathrm{T}^s_{\widehat{\Gamma}}$ *as in* (3.5).

Note that the boundary of the compactified simple model domain comes with a natural *stratification* given by the subset of $\{1,\dots,L\}$ of the t_i that are zero.

Our next goal is to exhibit the universal curve and a universal family of differentials over $\overline{\mathfrak{W}^{\mathfrak{k},\mathfrak{s}}_{\mathrm{pm}}(\widehat{\mathfrak{d}})}$. After adding modifying differentials and performing a plumbing construction this gives a family of multi-scale differentials in the proof of the main theorem below.

A change of notation seems adequate to illustrate the process: So far we denoted a twisted k-differentials by $(\widehat{X},\boldsymbol{\omega},\tau)$, since they arose from limits of canonical covers of k-differentials. Subsequently we will however denote curves with differentials over the simple model domain $\overline{\mathfrak{W}^{\mathfrak{k},\mathfrak{s}}_{\mathrm{pm}}(\widehat{\mathfrak{d}})}$ by $(\widehat{Y},\boldsymbol{\eta})$ (as we did already in (3.8)) and call them *model differentials*, even though the space $\overline{\mathfrak{W}^{\mathfrak{k},\mathfrak{s}}_{\mathrm{pm}}(\widehat{\mathfrak{d}})}$ is made from a space of twisted differentials by adding prong data and compactification. We will then call $(X,\boldsymbol{\omega})$ the curves with differentials obtained from plumbing. These will yield multi-scale k-differentials. To memorize: the model differentials $\boldsymbol{\eta}$ are always non-zero, the curves are equisingular and the boundary is specified by vanishing of the auxiliary parameters t_i. The differentials $\boldsymbol{\omega}$ will be constructed below on degenerating curves, they will vanish on lower level components and the boundary is given by the appearance of nodes.

The space $\mathfrak{W}^k_{\mathrm{na}}(\widehat{\Gamma})$, being the subspace defined by global residue conditions in a product of moduli spaces, obviously comes with a universal family of curves and k-differentials that we can pull back to $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$. Since the level rotation torus only acts on differentials and prong-matchings, not

on the curve, the universal curve descends to a family of curves $f\colon \widehat{\mathcal{Y}} \to \overline{\mathfrak{W}^{\mathfrak{k},\mathfrak{s}}_{\mathrm{pm}}(\widehat{\mathfrak{d}})}$.

Next we turn to the differentials. It will be convenient to fix the scale of the $\mathrm{T}^{\mathfrak{s}}_{\widehat{\Gamma}}$-orbits of the differentials in (3.8) rather than working with equivalence classes. That is, over small enough open sets $W \subset \overline{\mathfrak{W}^{\mathfrak{k},\mathfrak{s}}_{\mathrm{pm}}(\widehat{\mathfrak{d}})}$ we may assume that we work with a fixed collection $\boldsymbol{\eta} = (\eta_{(-i)})_{i=0}^{L}$ of families of differentials representing in each fiber the equivalence class of the corresponding point in $\Omega^k\mathcal{B}^{\mathfrak{s}}_{\widehat{\Gamma}}$. Consequently the collection of functions $\mathbf{t} = (t_i)$ is part of a coordinate system on W.

The modifying differentials we now define will be used for plumbing and also for perturbed period coordinates on charts of $\overline{\mathfrak{W}^{\mathfrak{k},\mathfrak{s}}_{\mathrm{pm}}(\widehat{\mathfrak{d}})}$. In the sequel we check that the construction of [4, Section 11] works in the k-equivariant setup. We define

$$\mathbf{t} * \boldsymbol{\eta} = \left(t_{\lceil i \rceil} \cdot \eta_{(-i)}\right)_{i=0}^{L} = \left(t_1^{\ell_1} \dots t_i^{\ell_i} \cdot \eta_{(-i)}\right)_{i=0}^{L}, \tag{3.9}$$

for $\mathbf{t} = (t_1, \dots, t_L) \in (\mathbb{C}^*)^L$ and use by definition the trivial rescaling $t_0 = 1$ on top level unless specified otherwise.

Definition 3.4. — *An* equivariant family of modifying differentials *on the universal family $f\colon \widehat{\mathcal{Y}} \to W$ restricted to W equipped with the universal differential $\mathbf{t} * \boldsymbol{\eta}$ is a family of meromorphic differentials $\boldsymbol{\xi} = (\xi_{(-i)})_{i=0}^{L}$ on $f\colon \widehat{\mathcal{Y}} \to W$, such that*

(i) *the equivariance $\tau^*\boldsymbol{\xi} = \zeta \cdot \boldsymbol{\xi}$ holds,*
(ii) *the differentials $\xi_{(-i)}$ are holomorphic, except for possible simple poles along both horizontal and vertical nodal sections, and except for marked poles,*
(iii) *the component $\xi_{(-L)}$ vanishes identically and moreover $\xi_{(-i)}$ is divisible by $t_{\lceil i+1 \rceil}$ for each $i = 1, \dots, L-1$, and*
(iv) *the sum $\mathbf{t} * \boldsymbol{\eta} + \boldsymbol{\xi}$ has opposite residues at every node.*

Proposition 3.5. — *The universal family $f\colon \widehat{\mathcal{Y}} \to W$ equipped with the universal differential $\mathbf{t} * \boldsymbol{\eta}$ admits an equivariant family of modifying differentials.*

Proof. — The proof of [4, Proposition 11.3] works in the situation where the edges of $\widehat{\Gamma}$ are images of the pinched multicurve Λ via a family of markings $\Sigma \to \widehat{Y}$ by a reference surface Σ. Choosing W contractible, we may assume that we have such a marking here as well.

The proof in loc. cit. starts by taking the subspace $V = \langle \lambda \in \Lambda \rangle_{\mathbb{Q}}$ and the subspace V_P spanned by the loops around the marked poles inside

$H_1(\widehat{Y} \smallsetminus \widehat{P}, \mathbb{Q})$. We define $V_N = V + V_P$. The proof proceeds by searching for a complementary subspace V_C (i.e. with $V_N \cap V_C = 0$) such that the projection $p(V')$ to $H_1(\widehat{Y}, \mathbb{Q})$ is a Lagrangian subspace, where $V' = V_N + V_C$. The proof then constructs $\boldsymbol{\xi} = (\xi_{(i)})$ for each $w \in W$ from assignments $\rho_i : V_i \to \mathbb{C}$ determined by the periods of the fiber $\boldsymbol{\eta}_w$ on subspaces V_i of $V + V_P$ generated by multicurves associated to edges whose lower level is below i. Relevant here is that those $\boldsymbol{\xi}$ satisfy all properties of Definition 3.4 except possibly the equivariance in (i). Moreover, $\boldsymbol{\xi}$ depends uniquely on an extension ρ_i' of ρ_i, that we may chose to be zero on V_C.

If we can find a subspace V_C which is τ-invariant, then the extended residue assignment ρ_i' is τ-equivariant (with τ acting by multiplication by ζ on the range) and thus $\boldsymbol{\xi}$ satisfies (i). To find such a V_C, we enlarge V_C and thus $V' = V_C + V_N$ step by step, staying τ-invariant at each step, until $p(V')$ is a Lagrangian subspace. If at some step V' is τ-invariant, but $p(V')$ is strictly contained in a Lagrangian subspace, we may find an element γ that pairs trivially with $p(V')$. But then $\tau^i(\gamma)$ also pairs trivially with $p(V')$ for all i and we add to V_C the span of all $\tau^i(\gamma)$. □

3.6. The perturbed period map

Periods give local coordinates on $\mathfrak{W}^k(\widehat{\Gamma})$ and thus on $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$. Together with the tuple of degeneration parameters $\mathbf{t}$ and deprived of one coordinate per level to fix the scale of projectivization they give local coordinates of $\overline{\mathfrak{W}^{\mathbf{t},\mathfrak{s}}_{\mathrm{pm}}(\widehat{\mathfrak{d}})}$. We introduce some perturbation of these coordinates here and show that this still gives local coordinates. The reason for this procedure is that the perturbed period coordinates can still be used after plumbing, see Section 3.7. Together with horizontal plumbing parameters it will provide coordinates on an orbifold chart of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$. Except for the use of appropriate eigenspaces this is exactly [4, Section 11].

We use for the rest of the discussion a small enough W as above and a fixed modifying differential provided by Proposition 3.5 after choosing a curve system, as in its proof. Near the marked point e^+ corresponding to the upper end (say on level $i = i(e^+)$) of each of the vertical nodes, choose an auxiliary section $s_e^+ : W \to \widehat{\mathcal{Y}}$ such that

$$\int_{e^+}^{s_e^+(w)} \boldsymbol{\eta}_{(i)} = \mathrm{const}\,, \tag{3.10}$$

where the constant is sufficiently small, depending on W, and constrained by the plumbing construction later. Near each zero marked z_j of $\boldsymbol{\eta}$ (say

on level $i = i(z_j)$) choose an auxiliary section $s_j : W \to \widehat{\mathcal{Y}}$ that coincides with the barycenter of the zeros of $\eta_{(i)} + t_{\lceil -i \rceil}^{-1} \cdot \xi_{(i)}$ that result from the deformation of z_j. We let γ_{ij} for $i = 0, \dots, -L$ and $j = 1, \dots, \dim E_{(i)}^{\mathrm{grc}}$ be a basis of the subspaces $E_{(i)}^{\mathrm{grc}}$ of homology. Since the contribution of each level to the twisted differentials compatible with a level graph is positive-dimensional (by the rescaling of the differential), the definition of periods coordinates along the boundary in Proposition 2.1 implies that for each i there exists some j such that $\int_{\gamma_{i,j}} \eta_{(i)} \neq 0$. We use this to fix the scale of the projectivization and assume that the periods for $j = 1$ are normalized on each level, i.e. $\int_{\gamma_{i,1}} \eta_{(i)} = 1$.

The i-th level component of the perturbed period map is now given by

$$\mathrm{PPer}_i \colon \begin{cases} W \to \mathbb{C}^{\dim E_{(i)}^{\mathrm{grc}} - 1 + \delta_{i,0}}, \\ [(\widehat{Y}, \boldsymbol{\eta}, \mathbf{t})] \mapsto \left(\int_{\gamma_{i,j}} \eta_{(i)} + t_{\lceil -i \rceil}^{-1} \cdot \xi_{(i)} \right)_{j=2-\delta_{i,0}}^{\dim E_{(i)}^{\mathrm{grc}}}, \end{cases} \tag{3.11}$$

where $\delta_{i,0}$ is Kronecker's delta and where the integrals are to be interpreted starting and ending at the nearby points determined by the sections s_e^+ and s_j rather than the true zeros of $\boldsymbol{\eta}$. The reason for this technical step is that those nearby points are still present after the surfaces has been plumbed ("Step 2" below). Recall that we defined $D_i = \{t_i = 0\}$.

Proposition 3.6. — *The perturbed period map*

$$\mathrm{PPer}_{\hat{\Gamma}}^{\mathrm{MD}} \colon W \to \mathbb{C}^L \times \prod_{i=0}^{-L} \mathbb{C}^{\dim E_{(i)}^{\mathrm{grc}} - 1 + \delta_{i,0}},$$
$$[(\widehat{Y}, \boldsymbol{\eta}, \mathbf{t})] \mapsto \left(\mathbf{t} \,;\, \coprod_{i=0}^{-L} \mathrm{PPer}_i(\widehat{Y}, \boldsymbol{\eta}, \mathbf{t}) \right)$$

is open and locally injective on a neighborhood of the most degenerate boundary stratum $W_{\hat{\Gamma}} = \bigcap_{i=1}^{L} D_i$ in the compactified model domain $\overline{\mathfrak{W}_{\mathrm{pm}}^{\mathfrak{k},\mathfrak{s}}(\widehat{\mathfrak{d}})}$.

We will write $(\mathbf{t}, \mathbf{w}) = \mathrm{PPer}_{\hat{\Gamma}}^{\mathrm{MD}}(\widehat{Y}, \boldsymbol{\eta}, \mathbf{t})$.

Proof. — As in [4, Proposition 11.6] it suffices to show that the derivative is surjective, by dimension comparison. For the tangent directions to the boundary this follows from Proposition 2.1 (and the fact that we have projectivized the lower levels). For the transverse direction this follows since the t_i are the local coordinates of the $\mathbb{C}^L$-bundles used to construct the compactifications. □

The reader should keep in mind, that in the model domain with its equisingular family of curves horizontal nodes are untouched. They enter in Proposition 1.2 only after plumbing horizontal nodes, see below.

3.7. The complex structure and the proof of Theorem 1.1

The outline of the proof of Theorem 1.1 consists of the following steps.

(1) Construct locally covers $U^s \to U$ for small open sets $U \subset \Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ that will be used as orbifold charts.
(2) Perform a plumbing construction on the pullback of the universal family $f\colon \widehat{\mathcal{Y}} \to \overline{\mathfrak{W}^{\mathfrak{k},\mathfrak{s}}_{\mathrm{pm}}(\widehat{\mathfrak{d}})}$ to small open sets W and via the second projection to $W \times \Delta^h_\varepsilon$ in order to obtain a family $\mathcal{X} \to W \times \Delta^h_\varepsilon$ together with a family of differentials.
(3) Use the moduli properties of the strata of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ to obtain moduli maps $\Omega\mathrm{Pl} : W \times \Delta^h_\varepsilon \to U^s$ for an appropriately chosen target set U^s, defined stratum by stratum.
(4) Show that $\Omega\mathrm{Pl}$ is a homeomorphism near a central point $P \times (0,\dots,0) \in W \times \Delta^h_\varepsilon$ and thus provide charts there.

The charts constructed in this way depend on many choices, in the construction of the modifying differential and the parameters for plumbing. However, the induced complex structures fit together and that's all we need since $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ already exists as a topological space. We provide the details for Step 2), since there the τ-equivariance needs to be respected and since we need this in the next section. The technical Step 1, Step 3 and Step 4 proceed exactly as in [4].

Step 1. — In order to provide $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ with a complex structure we consider the neighborhood U of a point $(\widehat{X}, \widehat{\mathbf{z}}, \boldsymbol{\omega}, \tau, \widehat{\Gamma})$ that we may assume to be at the boundary, say for level graph $\widehat{\Gamma}$. (The following description assumes that $(\widehat{X}, \boldsymbol{\omega}, \tau)$ has no automorphisms. In general we should start from an orbifold chart, and add the extra orbifold structure described below.) The compactified simple model domain is a $K_{\widehat{\Gamma}} = \mathrm{Tw}_{\widehat{\Gamma}}/\mathrm{Tw}^s_{\widehat{\Gamma}}$-cover of the (in general) singular space that we would get by compactifying the $\mathrm{T}_{\widehat{\Gamma}}$-quotient of $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$. Consequently, we have to pass locally near $(X, \boldsymbol{\omega}, \widehat{\Gamma})$ to a K-cover of U. We define this cover U^s as follows. Define the auxiliary *simple boundary stratum* to be $\Omega^k\mathcal{B}^s_{\widehat{\Gamma}} = \mathfrak{W}^{k,s}_{\mathrm{pm}}(\widehat{\Gamma})/\mathrm{T}^s_{\widehat{\Gamma}}$. As a set

$$U^s = \Big\{ (X', \boldsymbol{\omega}', \widehat{\Gamma}') \in \bigcup_{\hat{\Gamma}' \rightsquigarrow \hat{\Gamma}} \Omega^k\mathcal{B}^s_{\hat{\Gamma}'} \ : \ \varphi((X', \boldsymbol{\omega}', \widehat{\Gamma}')) \in U \Big\},$$

where φ is induced by the quotient maps $\Omega^k\mathcal{B}^s_{\widehat{\Gamma}'} \to \Omega^k\mathcal{B}_{\widehat{\Gamma}'}$. We provide U^s with a topology where convergence is formally given exactly by the same conditions as for $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ in Section 3.4, but where now the "existence of representatives of the equivalence classes" is up to the torus $\mathrm{T}^s_{\widehat{\Gamma}}$ rather than the quotient torus $\mathrm{T}_{\widehat{\Gamma}}$.

Step 2. — To start the plumbing construction we first define the *plumbing fixture* for each vertical edge $e \in E(\widehat{\Gamma})$ to be the degenerating family of annuli

$$\mathbb{V}_e = \Big\{(\mathbf{w}, \mathbf{t}, u, v) \in W \times \Delta^2_\delta : uv = \prod_{i=-e^+ +1}^{-e^-} t_i^{m_{e,i}}\Big\}, \tag{3.12}$$

that only depends on the $\mathbf{t}$-part of the perturbed period coordinates $(\mathbf{t}, \mathbf{w})$ of W. Recall that by definition (3.4) we have set $m_{e,i} = \ell_i/\kappa_e$. We equip $\mathbb{V}_e$ with the family of differentials

$$\Omega_e = \left(t_{\lceil -e^+ \rceil} \cdot u^{\kappa_e - 1} - \frac{r'_e}{u}\right) \mathrm{d}u = \left(-t_{\lceil -e^- \rceil} \cdot v^{-\kappa_e - 1} + \frac{r'_e}{v}\right) \mathrm{d}v\,, \tag{3.13}$$

where we recall that $t_{\lceil i \rceil} = t_i^{\ell_i} \dots t_1^{\ell_1}$ and where $r'_e = r'_e(\mathbf{w}, \mathbf{t})$ are the residues of the universal family over model domain. Inside the plumbing fixture we define the gluing annuli $\mathcal{A}^\pm_e$ by $\delta/R < |u| < \delta$ and $\delta/R < |v| < \delta$ respectively. The sizes δ, R and the size of the neighborhood W will be determined in terms of the geometry of the universal family, to ensure for example that plumbing annuli are not overlapping.

Suppose we only have vertical nodes. The plumbing construction proceeds bottom up. Near each of the nodes of bottom level we put the family of differentials $\eta_{(-L)}$ in standard form $(v^{-\kappa_e - 1} + \frac{r_e}{v})\mathrm{d}v$ so that after rescaling with $t_{\lceil -e^- \rceil}$ it can be glued to Ω_e for $r'_e = t_{\lceil -e^- \rceil} r_e$. That such a normal form exists in families is the content of [4, Theorem 3.3]. The functions r'_e determine the modifying differential $\xi_{(-L+2)}$ as the proof of Proposition 3.5 shows, see [4, Corollary 11.4]. We will thus put $t_{\lceil -e^+ \rceil}\eta_{(-L)} + \xi_{(-L+1)}$ in standard form near e^+ using the normal form on the deformation of an annulus ([2, Theorem 4.5] or [4, Theorem 12.2]) and this glues with the form (3.13) on the upper end of the annulus. Iterating the procedure allows to plumb the collection of families of one-forms

$$\mathbf{t} * \boldsymbol{\eta} + \boldsymbol{\xi} = \big(t_{\lceil i \rceil} \cdot \eta_{(-i)} + \xi_{(-i)}\big)_{i=0}^{L} \tag{3.14}$$

on the equisingular family of curves $\mathcal{Y} \to W$ to a family of one-forms $\boldsymbol{\omega}$ on a degenerating family of curves $\mathcal{X} \to W$. The zeros of higher order of $\boldsymbol{\eta}$ may have split up in $\boldsymbol{\omega}$ when adding $\boldsymbol{\xi}$. A local surgery merges them back to the barycenter [2, Lemma 4.7].

In the preceding construction we have neglected so far that the choice of the normal form is unique only up multiplication by a κ_e-th root of unity. The prong-matching that is part of the datum of the universal family over the model domain determines this choice. Many more details, using reference sections to make the construction rigorous, are given in [4, Section 12].

The whole construction can obviously performed τ-equivariantly, since the modifying differential is τ-equivariant and since the sizes of the neighborhoods and plumbing annuli are determined by the rates of degeneration of $\mathbf{t} * \boldsymbol{\eta} + \boldsymbol{\xi}$, i.e. by τ-equivariant data.

Finally, we investigate horizontal nodes of $\widehat{\Gamma}$, that come in τ-orbits of length k and that we thus label as $e_1^{(a)}, \dots, e_{|E^h|}^{(a)}$ for $0 \leqslant a < k$. We parameterize the plumbing by additional plumbing parameters $\mathbf{x} = (x_1, \dots, x_{|E^h|}) \in \Delta_\epsilon^{|E^h|}$ and define the *(horizontal) plumbing fixture* to be

$$\mathbb{W}_j = \left\{ (\mathbf{w}, \mathbf{t}, \mathbf{x}, u, v) \in W \times \Delta_\epsilon^{|E^h|} \times \Delta_\delta^2 : uv = x_j \right\}, \tag{3.15}$$

independently of the upper label a of $e_j^{(a)}$, equipped with the family of holomorphic one-forms

$$\Omega_j = -r'_{e_j}(\mathbf{w}, \mathbf{t}) \mathrm{d}u/u = r'_{e_j}(\mathbf{w}, \mathbf{t}) \mathrm{d}v/v\,, \tag{3.16}$$

where $\pm r'_{e_j}(\mathbf{w}, \mathbf{t})$ is the residue of $\mathbf{t} * \boldsymbol{\eta} + \boldsymbol{\xi}$ at the j-th horizontal node. Here the gluing happens along annuli $\mathcal{B}_j^\pm$ by $\delta/R < |u| < \delta$ and $\delta/R < |v| < \delta$.

Step 3. — The existence of moduli maps on each stratum of the simple model domain to $\Xi^k \overline{\mathcal{M}}_{g,n}(\mu)$ is immediate from the construction of $\Xi^k \overline{\mathcal{M}}_{g,n}(\mu)$ as union of strata $\Omega^k \mathcal{B}_{\hat{\Gamma}} = \mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})/\mathrm{T}_{\widehat{\Gamma}}$ and the property of $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$ as moduli space of k-differentials. We let U be the range of the union of these maps. The map factors through U^s since both this space and the simple model domain are defined as $\mathrm{T}^s_{\widehat{\Gamma}}$-equivalence classes. [4, Section 12.5] provides more details.

Step 4. — To show that the resulting map $\Omega\mathrm{Pl}\colon W \times \Delta^{|E^h|} \to U^s$ is continuous we have to invoke the definition of the topology on U^s to show that the images of a converging sequence converges. This entails exhibiting the almost-diffeomorphisms g_n with the properties (i)–(v). These g_n are construct level by level, bottom up, using conformal identifications of flat surfaces with the same periods ([4, Theorem 2.7]), a C^1-quasi-conformal extension of these maps across the plumbing cylinder and the equivalence of the conformal and C^1-quasi-conformal topology on strata of abelian differential ([4, Section 2]).

To show that $\Omega\mathrm{Pl}$ is a homeomorphism we need to show that this map is open and locally injective. Openness amounts to showing that for any converging sequence in U^s, say converging to $(\widehat{X}, \widehat{\mathbf{z}}, \omega, \tau, \widehat{\Gamma})$, we can eventually undo the plumbing construction and find $\Omega\mathrm{Pl}$-preimages in the model domain $\overline{\mathfrak{W}^{\mathfrak{k},\mathfrak{s}}_{\mathrm{pm}}(\widehat{\mathfrak{d}})}$. These preimages are again found level by level, the scales t_i of the model differentials being determined by the scales s_i in the definition of convergence in U^s. Local injectivity amounts to checking uniqueness of the previous unplumbing steps using perturbed period coordinates and can be checked applying [4]. See Section 12.5–12.7 in loc. cit. for details on these steps.

The action of $\mathbb{C}^*$ on the k-th root ω defines an action on the space $\Omega^k\mathcal{M}_{g,n}$ that is equivariant via $\lambda \mapsto \lambda^k$ with a $\mathbb{C}^*$-action on $\Omega^k\mathcal{M}_{g,n}$. The quotients of both actions is the same space $\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$. We encourage the reader to revisit all the steps to check that the first action extends equivariantly to all auxiliary spaces, multiplying simultaneously all forms at all levels. The resulting quotient of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ by $\mathbb{C}^*$ is the compactification $\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ claimed in (iii) of Theorem 1.1.

The *proof of Proposition 1.2* is contained in these statements, since Proposition 3.6 together with the disc coordinates x_j used in (3.15) gives local coordinates on $\overline{\mathfrak{W}^{\mathfrak{k},\mathfrak{s}}_{\mathrm{pm}}(\widehat{\mathfrak{d}})} \times \Delta^{|E^h|}$. Consequently, the perturbed period coordinates are given by

$$\begin{aligned} \mathrm{PPer}\colon \quad U^s &\xrightarrow{\Omega\mathrm{Pl}^{-1}} W \times \Delta^{|E^h|} \qquad \longrightarrow \mathbb{C}^h \times \mathbb{C}^{L+1} \times \prod_{i=0}^{-L} \mathbb{C}^{\dim E^{\mathrm{grc}}_{(i)}-1} \\ \left[\widehat{X}, \boldsymbol{\omega}\right] &\xmapsto{\Omega\mathrm{Pl}^{-1}} \left[\left(\widehat{Y}, \boldsymbol{\eta}, \mathbf{t}, \mathbf{x}\right)\right] \longmapsto \left(\mathbf{x}\,;\mathbf{t}\,;\, \prod_{i=0}^{-L} \mathrm{PPer}_i(\widehat{Y}, \boldsymbol{\eta}, \mathbf{t})\right) \end{aligned} \tag{3.17}$$

on open orbifold charts U^s of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$, using the inverse of the homeomorphism $\Omega\mathrm{Pl}$ constructed in Step 3 and 4.

4. The area form is good enough

Here we prove our main Theorem 1.4. We place ourselves in the setting of the theorem and recall that now $m_i > -k$ and thus the sets P and $\widehat{P}$ as defined in Section 2 are empty. The first step is to determine where the metric tends to infinity and then to give a convenient expression of the metric. Arguing inductively on k, we may also suppose that we are dealing with primitive k-differentials, i.e. that the canonical k-cover is connected.

We start with the definition of the corresponding hermitian form. For a symplectic basis $\alpha_1, \dots, \alpha_{\hat{g}}, \beta_1, \dots, \beta_{\hat{g}}$ of the absolute homology $H_1(\widehat{\Sigma}, \mathbb{Z})$ and for $\omega, \eta \in H^1(\widehat{X}, \mathbb{C})$ we define hermitian form

$$\langle \omega, \eta \rangle = \frac{i}{2} \sum_{i=1}^{\hat{g}} \left(\omega(\alpha_i)\, \overline{\eta(\beta_i)} - \omega(\beta_i)\, \overline{\eta(\alpha_i)} \right) \tag{4.1}$$

with the abbreviations $\omega(\alpha_i) = \int_{\alpha_i} \omega$ etc. By Riemann's bilinear relations we can rewrite the metric defined in (1.1) using the above hermitian form as

$$h(X,q)^{1/k} = \langle \omega, \omega \rangle = \frac{i}{2} \sum_i (a_i \overline{b_i} - b_i \overline{a_i}) \,,$$

where we introduce another abbreviation $a_i = \omega(\alpha_i)$ and $b_i = \omega(\beta_i)$, to be used if ω is the only one-form that appears. We recall from (3.9) the notation

$$t_{\lceil i \rceil} = t_1^{\ell_1} \cdots t_i^{\ell_i}$$

for any $i = 0, 1, \dots, L$ (in particular $t_{\lceil 0 \rceil} = 1$).

Lemma 4.1. — *On a neighborhood U of boundary point whose corresponding level graph $\widehat{\Gamma}$ has only vertical edges and $L+1$ levels, the metric h extends to a function of the form*

$$h(X,q)^{1/k} = \sum_{i=0}^{L} \left| t_{\lceil i \rceil} \right|^2 \left(h^{\mathrm{tck}}_{(-i)} - \sum_{p=1}^{i} R^{\mathrm{ver}}_{(-i),p} \log |t_p| \right) , \tag{4.2}$$

where $h^{\mathrm{tck}}_{(-i)}$ is a smooth positive function bounded away from zero and $R^{\mathrm{ver}}_{(-i),p}$ is a smooth non-negative function.

Proof. — A neighborhood of the point $(\widehat{X}, \boldsymbol{\omega})$ is also a neighborhood U of that point in the model domain. There, $\boldsymbol{\omega}$ is interpreted as a collection ω_{-i} of non-zero differential forms on the subsurface $\widehat{X}_{(-i)}$ on the $-i$-th level. The neighborhood of $(\widehat{X}, \boldsymbol{\omega})$ consists of the stable differentials obtained via the plumbing construction applied to the differential forms $(\prod_{j=1}^{i} t_j^{\ell_j}) \eta_{(-i)} + \xi_{(-i)}$ on the universal family over model domain restricted to the small neighborhood. Here $\mathbf{t} = (t_i)_{i=1}^{L}$ is the collection of "opening-up" parameters in the polydisc and the positive integers ℓ_j are determined by the enhanced level graph Γ via (3.4). The central fiber of this family agrees with $\boldsymbol{\omega}$ by construction.

We compute the local expression of the metric in the neighborhood U. We decompose a plumbed surface $(\widehat{X}_u, \omega_u)$ over $u \in U$ in the following way. Let $\widehat{\mathcal{X}}_{(-i),u}$ be the fiber over $u \in U$ of the $-i$-th level subsurface over the model domain. For any edge e connected to level $-i$, consider the complement

of the interior of the gluing annuli $\mathcal{A}_e^{+/-}$ (defined in Step 2 above), so that we remove neighborhoods of the points where $\eta_{(i)}$ has zeroes or poles corresponding to edges of $\widehat{\Gamma}$. We thus decompose the plumbed surface as

$$\widehat{X}_u = \bigsqcup_{i=0}^{L} \widehat{\mathcal{X}}^{\circ}_{(-i),u} \quad \sqcup \quad \bigsqcup_{e\in E(\widehat{\Gamma})} \mathbb{V}_e \,,$$

where $\mathbb{V}_e$ are the plumbing fixtures at the edges defined in (3.12). We have

$$\begin{aligned}(4.3)\quad & \operatorname{area}_{\widehat{X}_u}(\omega_u) \\ &= \sum_{i=0}^{L} \operatorname{area}_{\hat{\mathcal{X}}^{\circ}_{(-i),u}}(t_{\lceil i\rceil}\eta_{(-i),u} + \xi_{(-i),u}) \;+\; \sum_{e\in E(\widehat{\Gamma})} \operatorname{area}_{\mathbb{V}_e}\big(\Omega_e\big)\,,\end{aligned}$$

where the differential form Ω_e was defined in (3.13). The first summands above give a smooth function in U. Moreover, since the components $\xi_{(-i),u}$ of the modification differentials are divisible by $t_{\lceil i+1\rceil}$ (see Definition 3.4(iii)), we can write

$$\operatorname{area}_{\hat{\mathcal{X}}^{\circ}_{(-i),u}}(t_{\lceil i\rceil}\eta_{(-i),u} + \xi_{(-i),u}) = \big|t_{\lceil i\rceil}\big|^2 h^{\mathrm{tck}}_{1,(-i)}$$

where $h^{\mathrm{tck}}_{1,(-i)}$ is a smooth function bounded away from zero on U.

It remains to compute the second summands in the expression (4.3). For each edge e of the graph $\widehat{\Gamma}$, let $t_{\Delta(e)} \coloneqq \prod_{p=-e^+ +1}^{-e^-} t_p^{m_{e,p}}$, where recall from (3.4) that $m_{e,p} = \ell_p/\kappa_e$. Hence we obtain using $r_e = r'_e/t_{\lceil e^-\rceil}$ as defined after (3.13) that

$$\begin{aligned}(4.4)\quad & \operatorname{area}_{\mathbb{V}_e}\big(\Omega_e\big) \\ &= \frac{i}{2}\int\limits_{\frac{1}{\delta}|t_{\Delta(e)}|\leqslant|z|\leqslant\delta} \left| t_{\lceil e^+\rceil}\cdot z^{\kappa_e-1} - \frac{t_{\lceil e^-\rceil} r_e(u)}{z}\right|^2 \mathrm{d}z\,\mathrm{d}\bar{z} \\ &= 2\pi\int_{\frac{1}{\delta}|t_{\Delta(e)}|}^{\delta} \left(\big|t_{\lceil e^+\rceil}\big|^2 r^{2(\kappa_e-1)} + \big|t_{\lceil e^-\rceil}\big|^2 |r_e(u)|^2 \frac{1}{r^2}\right) r\,\mathrm{d}r \\ &\qquad - \big|t_{\lceil e^+\rceil}t_{\lceil e^-\rceil}\big|\,|r_e(u)|^2 \int_0^{2\pi}\int_{\frac{1}{\delta}|t_{\Delta(e)}|}^{\delta} r^{\kappa_e-2}\Big(\mathrm{e}^{i\kappa_e\theta} + \mathrm{e}^{-i\kappa_e\theta}\Big) r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \big|t_{\lceil e^+\rceil}\big|^2 h^v_e(u) - 2\pi|r_e(u)|^2\,\big|t_{\lceil e^-\rceil}\big|^2 \sum_{p=-e^+ +1}^{-e^-} m_{e,p}\log|t_p|\end{aligned}$$

where $h^v_e(u)$ is a smooth function on U. Here the third line integrates to zero, the $|t_{\Delta(e)}|$-part of the lower bound of the $\mathrm{d}r/r$-integral in the second line gives the last term and all the rest has been subsumed in $h^v_e(u)$.

We now rearrange the sum over all the vertical plumbing fixtures contributions according to the level of the top end of the edge. Moreover we group together the sum of the areas of the thick part and the non-residue part of the plumbing fixtures to get an expression

$$h^{\mathrm{tck}}_{(-i)} \coloneqq h^{\mathrm{tck}}_{1,(-i)} + \sum_{e\colon e^+=-i} h^v_e(u)\,.$$

This is a smooth function bounded away from zero. We also group the residue terms, the last terms in (4.4), but now according to the bottom end of the edge. We set for all $0 < p \leqslant i$:

$$R^{\mathrm{ver}}_{(-i),p} = -\sum_{e\colon -e^-=i,\, p<-e^-} 2\pi |r_e(u)|^2 m_{e,p}\,.$$

This gives (4.2). □

Now suppose we work in a neighborhood U of a boundary point with only horizontal nodes. Assume there are $E^h = E^h_{(0)}$ horizontal nodes and let x_j for $j = 1, \dots, E^h$ be their opening-up parameters.

Lemma 4.2. — *On the neighborhood U the metric h has the form*

$$h(X,q)^{1/k} = h^{\mathrm{tck}}_{(0)} - \sum_{j=1}^{E^h_{(0)}} R^{\mathrm{hor}}_{(0),j} \log(|x_j|^2)\,, \tag{4.5}$$

where $h^{\mathrm{tck}}_{(0)}$ and $R^{\mathrm{hor}}_{(0),j}$ are smooth functions independent of the x_j parameters, both bounded above and away from zero.

Proof. — The total space of the line bundle $\mathcal{O}(-1)$ defined in the introduction as the φ-pullback of $\mathcal{O}(-1)$ from the incidence variety compactification, is nothing but the total space of the projection $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu) \to \mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$. Our goal is thus to find an expression for the area of a point in $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ near a boundary point $(\widehat{X}, \boldsymbol{\omega}, \widehat{\Gamma}) \in \partial\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$.

For notational simplicity we consider first the case *that X has only one horizontal node that we moreover suppose to be non-separating*. Consequently, $\widehat{X}$ has k nodes. We pick a convenient basis of $H_1(\widehat{\Sigma}, \mathbb{Z})$ on a smooth model $\widehat{\Sigma}$ (connected by our standing primitivity assumption) that is pinched to $\widehat{X}$. The k pinched curves $\alpha_i \in \widehat{\Sigma}$ are linearly independent and form a τ-orbit in homology. Next, we take the symplectic dual curves β_i with the intersection pairing $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$. Note that β_i is well-defined in a neighborhood of $(\widehat{X}, \boldsymbol{\omega}, \widehat{\Gamma})$ (only) up to adding an integer multiple of α_i. We arbitrarily complement these elements by $\alpha_i, \beta_i \in H_1(\widehat{\Sigma}, \mathbb{Z})$ for $i = k+1, \dots, \widehat{g}$ to a symplectic basis.

In the current case the multi-scale differential case $\boldsymbol{\omega} = (\omega_0) = (\eta_0)$ consists of a single one-form. Recall from Step 2 in Section 3.7 that points in a neighborhood of $(\widehat{X}, \boldsymbol{\omega}, \widehat{\Gamma})$ are obtained from surfaces

$$(\widehat{X}', \eta') \in \Omega\mathcal{M}_{\hat{g}-k,\hat{n}+2k}(\widehat{\mu}, (-1)^{2k})$$

that admit an action by $\langle\tau\rangle \cong \mathbb{Z}/k$, by gluing in k times each of the plumbing fixtures $\mathbb{W}$ in a τ-equivariant way, parameterized by a coordinate $\mathbf{x} = (x) \in \Delta$ as in Step 2 above. By Proposition 1.2 and explicitly (3.17) the coordinates near the boundary point are $\mathbf{x}$ and the periods in the ζ_k-eigenspace of η'. We denote by ω' the differential obtained from η' after the plumbing construction. Notice that ω' is a holomorphic differential on the plumbed surfaces having all plumbing parameters x_i different from zero. Our aim is to rewrite the area form, which is defined using ω' periods in the interior, in terms of the perturbed period coordinates, i.e. $\mathbf{x}$ and η' periods, which give charts near the boundary. We abbreviate $a_i = \omega'(\alpha_i)$ and $b_i = \omega'(\beta_i)$.

Next we decompose $\beta_j = \beta_j^X + \beta_j^\circ$ into the "eXterior" part β_j^X outside the plumbing fixture and the part β_j° between the two seams of the plumbing fixture, as in Figure 4.1.

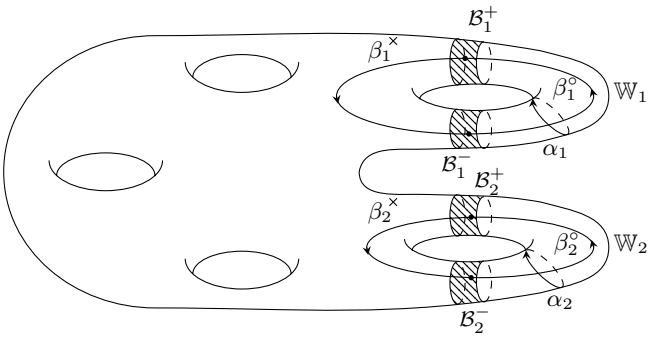

Figure 4.1. Decomposing the β_i into exterior and interior of the plumbing fixture

The separation happens at fixed sections (of the universal family over the stratum $\Omega\mathcal{M}_{g-kh,\hat{n}+2kh}(\widehat{\mu}, (-1)^{2kh})$) in the neighborhood of $(\widehat{X}', \eta')$ in the plumbing annuli $\mathcal{B}_j$, say at the points $u = \delta_0$ and $v = \delta_0$. Equation (3.16) simplifies in the one-level case to $\Omega_j = r_j \mathrm{d}v/v$ where $r_j = \zeta^j a_1/2\pi i =$

$a_j/2\pi i$. We compute

$$b_j = \int_{\beta_j} \omega' = \int_{\beta_j^X} \eta' + \int_{\delta_0}^{x/\delta_0} \Omega_j = \int_{\beta_j^X} \eta' + r_j(\log x - 2\log\delta_0)\,,$$

which is well-defined in $\mathbb{C}+r_j\mathbb{Z}$ because of the ambiguity of β_j. By definition of the area form and since ω' and η' agree outside the plumbing fixture,

$$\begin{aligned} h(X,q)^{1/k} &= \frac{i}{2}\sum_{j=1}^{k}(a_j\overline{b_j} - b_j\overline{a_j}) \;+\; \frac{i}{2}\sum_{j=k+1}^{\hat g}(a_j\overline{b_j} - b_j\overline{a_j}) \\ &= C \;+\; \frac{i}{2}\sum_{j=k+1}^{\hat g}(a_j\overline{b_j} - b_j\overline{a_j}) - \frac{k}{4\pi}\cdot|a_1|^2\log(|x|^2) \end{aligned} \tag{4.6}$$

is independent of the ambiguity of b_j. Here C is some function that stems from the integration in the thick part and that is independent of x. We may now let $h^{\mathrm{tck}}_{(0)} = C + \frac{i}{2}\sum_{j=k+1}^{\hat g}(a_j\overline{b_j} - b_j\overline{a_j})$ and $R^{\mathrm{hor}}_{(0),1} = \frac{k}{4\pi}\cdot\pi|a_1|^2$. Both functions are smooth and bounded away from zero near $(\widehat{X},\boldsymbol{\omega},\widehat{\Gamma})$, in fact they correspond to the volume of the region outside the handles and the residue at the handle respectively.

For a *general X that has only horizontal nodes* we arrive at a similar formula. We decompose the plumbed surface of the canonical covering into the thick part and the plumbing fixtures $\mathbb{W}_j$, for $j = 1,\dots,n(0)$. Since the flat area is additive, we can write it as a sum of the contribution of the flat area of the thick part and the flat area of the $\mathbb{W}_j$, as we did in the previous case. The area of the thick part is clearly a smooth function of the period coordinates and bounded away from zero. For each node j of X, we get k-plumbing fixtures $\mathbb{W}_{j,l}$. Since the residues $r_{j,l}$ of the associated simple pole differential are τ-conjugates for each fixed j, they all have the same modulus that we denote by $|r_j|$. From the computation in the plumbing fixtures as in the previous case we see that the flat area is given by

$$h(X,q)^{1/k} = h^{\mathrm{tck}}_{(0)} - \frac{k}{4\pi}\sum_{j=1}^{E^h_{(0)}}|r_j|^2\log(|x_j|^2)$$

which is of the shape we claimed. □

Now suppose we work in a neighborhood U of a general boundary point with notations $\left(\mathbf{x}\,;\mathbf{t}\,;\,\coprod_{i=0}^{L}\mathrm{PPer}_{(-i)}\right)$ for the perturbed period coordinates as in Proposition 1.2 and in detail in (3.17). More precisely, we group the vector $\mathbf{x}$ of coordinates for opening the horizontal nodes as $\mathbf{x} = (x_{(-i),j})$, where $-i$ denotes the level that contains the nodes and where $j = 1,\dots,E^h_{(i)}$ labels these nodes.

Proposition 4.3. — *On the neighborhood U the metric h has the form*

$$h(X,q)^{1/k} = \sum_{i=0}^{L} |t_{\lceil i \rceil}|^2 \left(h^{\mathrm{tck}}_{(-i)} + h^{\mathrm{ver}}_{(-i)} + h^{\mathrm{hor}}_{(-i)} \right) \tag{4.7}$$

where $h^{\mathrm{tck}}_{(-i)}$ are smooth positive functions bounded away from zero and

$$h^{\mathrm{ver}}_{(-i)} := -\sum_{p=1}^{i} R^{\mathrm{ver}}_{(-i),p} \log |t_p| \,, \quad h^{\mathrm{hor}}_{(-i)} := -\sum_{j=1}^{E^h_{(-i)}} R^{\mathrm{hor}}_{(-i),j} \log |x_{(-i),j}| \,, \tag{4.8}$$

with $R^{\mathrm{ver}}_{(-i),p}$ is a smooth non-negative function and $R^{\mathrm{hor}}_{(-i),j}$ is a smooth positive function bounded away from zero, both involving only coordinates in $\mathrm{PPer}_{(-i)}$.

Proof. — Recall that the plumbing construction decomposes the surface along the vertical edges into various levels (and the plumbing cylinders between the levels). For each level, we decompose the plumbed surface as the union of the horizontal plumbing fixtures, the vertical plumbing fixtures and the thick part.

In Lemma 4.1, we have investigated the contribution of the vertical fixtures, while in Lemma 4.2 we have investigated the contribution of the horizontal fixtures. By defining $h^{\mathrm{tck}}_{(-i)}$ the area of the thick part and since the contribution of the horizontal plumbing fixtures at level $-i$ have to be rescaled by $|t_{\lceil i \rceil}|^2$, by summing together the contribution of (4.2) and (4.5), we have shown the claim. □

Before proceeding to the proof of Theorem 1.4 we recall as an aside and for comparison the definition of a good metric in the sense of [8] on a smooth r-dimensional variety (or orbifold) $\overline{X}$.

Suppose that $\overline{X}$ is the compactification of X with a normal crossing boundary divisor $\partial X = \overline{X} \setminus X$. Let $\mathcal{L}$ be a line bundle on $\overline{X}$. A metric h on $\mathcal{L}|_X$ is *good*, if for each point $p \in \partial X$ there is a neighborhood Δ^r with coordinates such that $\partial X = \{\prod_{i=1}^k x_i = 0\}$ and such that the function $h_s = h(s,s)$ for a local generating section s of $\mathcal{L}$ has the following properties:

(i) There exist $C > 0$ and $n \in \mathbb{N}$ such that $|h_s| < C\Big(\sum_{i=1}^k \log |x_i|\Big)^{2n}$ and $|h_s^{-1}| < C\Big(\sum_{i=1}^k \log |x_i|\Big)^{2n}$.

(ii) the connection one-form $\partial \log h$ and the curvature two-form $\overline{\partial}\partial \log h$ have Poincaré growth.

Here a p-form η is said to have *Poincaré growth* on Δ^r if for any choice of sections v_i of $T_{\overline{X}}(\Delta^r)$ there is C such that

$$|\eta(v_1,\dots,v_p)|^2 \leqslant C\prod_{i=1}^{p}\omega_P(v_i,v_i)$$

holds for ω_P the product of the Poincaré metrics $|\mathrm{d}x_i|^2/x_i^2\log|x_i|^2$ in the coordinates x_i for $i\leqslant k$ and the euclidean metric in the other coordinates.

Mumford shows ([8, Theorem 1.4]) that for a good metric h the curvature form $\frac{i}{2\pi}[F_h]$ defines a closed $(1,1)$-current that represents the first Chern class of $\mathcal{L}$. This estimate boils down to the observation that the "Poincaré" integral

$$\frac{1}{2\pi i}\int_{\Delta_\varepsilon}\frac{\mathrm{d}x\mathrm{d}\overline{x}}{|x|^2(\log|x|^2)^2} = -\int_0^\varepsilon\frac{\mathrm{d}s}{s(\log(s))^2} = \frac{1}{\log\varepsilon} < \infty. \tag{4.9}$$

of the Poincaré metric on the (punctured) disc Δ_ε is finite and goes to zero as $\varepsilon\to 0$.

The metric h is indeed good if there is only one level, i.e. if the graph has no vertical edges, as one can deduce from the estimates in the propositions below. However *the metric h fails to be good* if there are several levels and horizontal nodes on lower level. Consider the simplest such case of a graph with two levels, one vertex at each level and two edges, one edge joining the levels and a horizontal edge on lower level. Simplifying the situation by assuming $k=1$ and that $R^{\mathrm{ver}}_{(-1),1}=0$, that $R^{\mathrm{hor}}_{(-1),1}=2$ and that the other bounded functions are 1, the metric is then given by

$$h(X,q) = 1+|t_1|^2(1-2\log|x|)\,.$$

We observe that this metric is not good in the sense of Mumford near the point $(t_1,x)=(0,0)$, when considering the natural boundary $\{x=0\}\cup\{t_1=0\}$ consisting of the complement of the locus where the metric smoothly extends.

Suppose the metric were good. Then we would have a constant C such that

$$\Big|\partial\log h\Big(\frac{\partial}{\partial t_1}\Big)\Big|^2 \leqslant \frac{C}{|t_1|^2(\log|t_1|)^2}$$

on the neighborhood U of $(t_1,x)=(0,0)$, which is equivalent to the inequality of the square roots

$$\frac{|t_1|(1-2\log|x|)}{1+|t_1|^2(1-2\log|x|)} \leqslant \frac{C^{1/2}}{|t_1|\log|t_1|} \tag{4.10}$$

Choosing a sequence tending to $(0,0)$ with $1-\log|x| = |t_1|^{-2}$ we get a contradiction.

Instead of aiming for a bound as in the definition of good, integrability statements are sufficient. In fact the coefficients of

$$\partial \log(h) = \frac{|t_1|^2(1-2\log|x|)}{h}\frac{\mathrm{d}t_1}{t_1} - \frac{|t_1|^2}{h}\frac{\mathrm{d}x}{x}$$

and

$$\bar{\partial}\partial \log h = \frac{|t_1|^2(1-2\log|x|)}{h^2}\frac{\mathrm{d}\bar{t}_1}{\bar{t}_1}\frac{\mathrm{d}t_1}{t_1} - \frac{|t_1|^4}{h^2}\frac{\mathrm{d}\bar{x}}{\bar{x}}\frac{\mathrm{d}x}{x} - \frac{|t_1|^2}{h^2}\left(\frac{\mathrm{d}\bar{t}_1}{\bar{t}_1}\frac{\mathrm{d}x}{x} + \frac{\mathrm{d}\bar{x}}{\bar{x}}\frac{\mathrm{d}t_1}{t_1}\right)$$

are locally integrable, and thus define currents. To see that the current $F_h = [\bar{\partial}\partial \log h]$ given by the curvature form is closed, we have to show that we can apply the derivative (in the sense of currents) inside the brackets, on the differential form, where it gives zero. This requires an application of Stokes' theorem, and thus an integral over the boundary T_δ of a shrinking tubular neighborhood around the locus where the metric is not smooth, i.e. around $\{x = 0\}$ and $\{t_1 = 0\}$. To see that this current represents the first chern class $c_1(\mathcal{O}(-1))$, we compare with the curvature form of a smooth metric. To see that the difference is zero in cohomology, the term $[\mathrm{d}\partial \log(h)]$ appears and we'd like to invoke say that this is $\mathrm{d}[\partial \log(h)]$, i.e. a coboundary of a current. This is gives a second application of Stokes' theorem, justified by another integration over T_δ. To justify that we can pass to wedge powers is a third application of Stokes' theorem. This integrals are estimated in the general case in the following proofs.

The *Proof of Theorem 1.4* is now contained in the following two propositions.

Proposition 4.4. — *The differential forms $\Omega = \partial \log h$ and $F_h = \bar{\partial}\partial \log h$, and more generally the forms F_h^d and $\Omega \wedge F_h^d$ have coefficients in L^1_{loc}.*

In particular F_h^d defines a current of type (d,d) for any $d \in \mathbb{N}$.

Proposition 4.5. — *The current $[\bar{\partial}\partial \log h]$ is closed and $\frac{1}{2\pi i}$ times the curvature $(1,1)$-form $F_h = \bar{\partial}\partial \log h$ represents the first Chern class $c_1(\mathcal{O}(-1))$ in cohomology.*

More generally, the wedge powers $\wedge^d(\frac{1}{2\pi i}F_h)$ represent the class of $c_1(\mathcal{O}(-1))^d$ in cohomology.

We calculate the relevant differential forms explicitly. More specifically, we first determine explicitly the types of differential forms that we can encounter in ∂h, $\bar{\partial} h$ and $\bar{\partial}\partial h$ up to continuous factors that don't affect integrability.

Recall that by Proposition 4.3, the function h near a boundary point is given as the sum of three contributions given by the thick part and the vertical and horizontal plumbing fixtures.

For every level $(-i)$, the thick part contribution $\sum_{i=0}^{L} |t_{\lceil i \rceil}|^2 h^{\mathrm{tck}}_{(-i)}$ is a smooth positive function bounded away from zero. Hence all of its derivatives are in particular smooth.

Then we analyze the contribution $\sum_{i=0}^{L} |t_{\lceil i \rceil}|^2 h^{\mathrm{ver}}_{(-i)}$ from the vertical plumbing fixtures. An important remark is that the functions $|t_{\lceil i \rceil}|^2 h^{\mathrm{ver}}_{(-i)}$ are continuous, since they are given by sums of functions of type $|t_p|^2 \log |t_p|$, for $p \leqslant i$. By using the explicit expression of $h^{\mathrm{ver}}_{(-i)}$ given in (4.8) we compute

$$\begin{aligned}\partial \left(|t_{\lceil i \rceil}|^2 h^{\mathrm{ver}}_{(-i)} \right) = &- |t_{\lceil i \rceil}|^2 \sum_{p,p_1=1}^{i} \ell_{p_1} R^{\mathrm{ver}}_{(-i),p} \log |t_p| \, \frac{\mathrm{d}t_{p_1}}{t_{p_1}} \\ &- |t_{\lceil i \rceil}|^2 \sum_{p=1}^{i} \left(\frac{1}{2} R^{\mathrm{ver}}_{(-i),p} \frac{\mathrm{d}t_p}{t_p} + \log |t_p| \cdot \partial R^{\mathrm{ver}}_{(-i),p} \right).\end{aligned}$$

It is clear that all the 1-forms appearing in the previous expression are continuous. The same is obviously true for the analogous form given by the $\bar{\partial}$ derivative. We compute now the second derivative

$$\begin{aligned}\bar{\partial}\partial \left(|t_{\lceil i \rceil}|^2 h^{\mathrm{ver}}_{(-i)} \right) = &-|t_{\lceil i \rceil}|^2 \sum_{p,p_1,p_2=1}^{i} \ell_{p_1} \ell_{p_2} R^{\mathrm{ver}}_{(-i),p} \log |t_p| \, \frac{\mathrm{d}\bar{t}_{p_2}}{\bar{t}_{p_2}} \frac{\mathrm{d}t_{p_1}}{t_{p_1}} \\ &- |t_{\lceil i \rceil}|^2 \sum_{p,p_1=1}^{i} \ell_{p_1} \left(\frac{1}{2} R^{\mathrm{ver}}_{(-i),p} \frac{\mathrm{d}\bar{t}_{p_1}}{\bar{t}_{p_1}} \frac{\mathrm{d}t_p}{t_p} + \log |t_p| \, \frac{\mathrm{d}\bar{t}_{p_1}}{\bar{t}_{p_1}} \wedge \partial R^{\mathrm{ver}}_{(-i),p} \right) \\ &- |t_{\lceil i \rceil}|^2 \sum_{p,p_1=1}^{i} \ell_{p_1} \left(\log |t_p| \, \bar{\partial} R^{\mathrm{ver}}_{(-i),p} \wedge \frac{\mathrm{d}t_{p_1}}{t_{p_1}} + \frac{1}{2} R^{\mathrm{ver}}_{(-i),p} \frac{\mathrm{d}\bar{t}_{p_1}}{\bar{t}_{p_1}} \frac{\mathrm{d}t_p}{t_p} \right) \\ &- |t_{\lceil i \rceil}|^2 \sum_{p=1}^{i} \left(\frac{1}{2} \bar{\partial} R^{\mathrm{ver}}_{(-i),p} \wedge \frac{\mathrm{d}t_p}{t_p} + \frac{1}{2} \frac{\mathrm{d}\bar{t}_p}{\bar{t}_p} \wedge \partial R^{\mathrm{ver}}_{(-i),p} + \log |t_p| \cdot \bar{\partial}\partial R^{\mathrm{ver}}_{(-i),p} \right).\end{aligned}$$

By inspecting the terms of the previous expression, we can see that the only non-continuous coefficients that can appear stem from the first line. Indeed if $p = p_1 = p_2$ and $\ell_p = 1$, then we can have 2-forms (up to multiplication by smooth positive functions) of type

$$\log |t_p| \, \mathrm{d}\bar{t}_p \mathrm{d}t_p. \tag{4.11}$$

We finally analyze the contribution $\sum_{i=0}^{L} |t_{\lceil i \rceil}|^2 h^{\mathrm{hor}}_{(-i)}$ from the horizontal plumbing fixtures. Again by using the explicit expression of $h^{\mathrm{hor}}_{(-i)}$ given

in (4.8) we compute

$$\partial\left(|t_{\lceil i\rceil}|^2 h^{\mathrm{hor}}_{(-i)}\right) = -|t_{\lceil i\rceil}|^2 \sum_{p=1}^{i}\sum_{j=1}^{E^h_{(-i)}} R^{\mathrm{hor}}_{(-i),j}\log|x_{(-i),j}|\frac{\mathrm{d}t_p}{t_p}$$
$$- |t_{\lceil i\rceil}|^2 \sum_{j=1}^{E^h_{(-i)}}\left(\log|x_{(-i),j}|\cdot\partial R^{\mathrm{hor}}_{(-i),j} + \frac{1}{2}R^{\mathrm{hor}}_{(-i),j}\frac{\mathrm{d}x_{(-i),j}}{x_{(-i),j}}\right).$$

The non-continuous 1-forms appearing in the previous expression (up to continuous coefficients) are

$$|t_{\lceil i\rceil}|^2\log|x_{(-i),j}|\frac{\mathrm{d}t_p}{t_p},\quad |t_{\lceil i\rceil}|^2\log|x_{(-i),j}|\partial R^{\mathrm{hor}}_{(-i),j},\quad |t_{\lceil i\rceil}|^2\frac{\mathrm{d}x_{(-i),j}}{x_{(-i),j}} \tag{4.12}$$

for $p \leqslant i$. We compute now the second derivative

$$\bar\partial\partial\left(|t_{\lceil i\rceil}|^2 h^{\mathrm{hor}}_{(-i)}\right) = -|t_{\lceil i\rceil}|^2 \sum_{p_1,p_2=1}^{i}\sum_{j=1}^{E^h_{(-i)}} R^{\mathrm{hor}}_{(-i),j}\log|x_{(-i),j}|\frac{\mathrm{d}\bar t_{p_2}}{\bar t_{p_2}}\frac{\mathrm{d}t_{p_1}}{t_{p_1}}$$
$$- |t_{\lceil i\rceil}|^2 \sum_{p=1}^{i}\sum_{j=1}^{E^h_{(-i)}}\left(\log|x_{(-i),j}|\frac{\mathrm{d}\bar t_p}{\bar t_p}\wedge\partial R^{\mathrm{hor}}_{(-i),j} + \frac{1}{2}R^{\mathrm{hor}}_{(-i),j}\frac{\mathrm{d}\bar t_p}{\bar t_p}\wedge\frac{\mathrm{d}x_{(-i),j}}{x_{(-i),j}}\right)$$
$$- |t_{\lceil i\rceil}|^2 \sum_{p=1}^{i}\sum_{j=1}^{E^h_{(-i)}}\left(\log|x_{(-i),j}|\bar\partial R^{\mathrm{hor}}_{(-i),j}\wedge\frac{\mathrm{d}t_p}{t_p} + \frac{1}{2}R^{\mathrm{hor}}_{(-i),j}\frac{\mathrm{d}\bar x_{(-i),j}}{\bar x_{(-i),j}}\wedge\frac{\mathrm{d}t_p}{t_p}\right)$$
$$- |t_{\lceil i\rceil}|^2 \sum_{j=1}^{n(-i)}\left(\frac{1}{2}\frac{\mathrm{d}\bar x_{(-i),j}}{\bar x_{(-i),j}}\wedge\partial R^{\mathrm{hor}}_{(-i),j} + \log|x_{(-i),j}|\right.$$
$$\left.\cdot\,\bar\partial\partial R^{\mathrm{hor}}_{(-i),j} + \frac{1}{2}\bar\partial R^{\mathrm{hor}}_{(-i),j}\wedge\frac{\mathrm{d}x_{(-i),j}}{x_{(-i),j}}\right).$$

The non-continuous 2-forms appearing in the previous expression (up to continuous coefficients) are

$$\begin{aligned}
&|t_{\lceil i\rceil}|^2\log|x_{(-i),j}|\frac{\mathrm{d}\bar t_{p_2}}{\bar t_{p_2}}\frac{\mathrm{d}t_{p_1}}{t_{p_1}}, && |t_{\lceil i\rceil}|^2\log|x_{(-i),j}|\partial R^{\mathrm{hor}}_{(-i),j}\frac{\mathrm{d}\bar t_p}{\bar t_p},\\
&|t_{\lceil i\rceil}|^2\frac{\mathrm{d}\bar t_p}{\bar t_p}\frac{\mathrm{d}x_{(-i),j}}{x_{(-i),j}} && |t_{\lceil i\rceil}|^2\log|x_{(-i),j}|\bar\partial\partial R^{\mathrm{hor}}_{(-i),j},\\
&|t_{\lceil i\rceil}|^2\partial R^{\mathrm{hor}}_{(-i),j}\frac{\mathrm{d}\bar x_{(-i),j}}{\bar x_{(-i),j}}
\end{aligned} \tag{4.13}$$

and their complex conjugates, where $p, p_1, p_2 \leqslant i$. Recall also that by Proposition 4.3 the functions $R^{\mathrm{hor}}_{(-i),j}$ are smooth and involve only coordinates in $\mathrm{PPer}_{(-i)}$.

In order to analyze the general expression for the differential forms associated with the metric h, note that

$$\begin{aligned} \partial \log(h) &= k\frac{\partial h^{1/k}}{h^{1/k}}, \\ F_h = \bar\partial\partial \log h &= k\frac{\bar\partial\partial h^{1/k}}{h^{1/k}} - k\frac{\bar\partial h^{1/k}}{h^{1/k}} \wedge \frac{\partial h^{1/k}}{h^{1/k}}. \end{aligned} \tag{4.14}$$

Hence the non-continuous 1-forms appearing in $\partial \log(h)$ consist of linear combinations of the building blocks given by the quotient by $h^{1/k}$ of the terms appearing in (4.12), while the non-continuous 2-forms appearing in $\bar\partial\partial \log(h)$ consist of the two-fold wedge products of type $(1,1)$ of the quotient by h of the terms appearing in (4.12) and of the terms appearing in (4.11) and (4.13), together with their complex conjugates.

We now fix some more notation. We may assume that our neighborhood U is the product of the polydiscs

$$D^{\mathbf{t}} = \{\mathbf{t} : t_i \in \Delta_\varepsilon\} \quad \text{and} \quad D^{\mathbf{x}} = \{\mathbf{x} : x_{(-i),j} \in \Delta_\varepsilon\}\,. \tag{4.15}$$

in the corresponding variables times a ball B corresponding to all the variables in $\mathrm{PPer}_{(-i)}$ for $i = 0, \dots, L$.

Proof of Proposition 4.4. — We first record the following sharp bound for the reciprocal of the metric

$$\frac{1}{h^{1/k}} \leqslant \frac{1}{h^{\mathrm{tck}}_{(0)} + |t_{\lceil i\rceil}|^2 h^{\mathrm{hor}}_{(-i)}} \leqslant \frac{1}{C - |t_{\lceil i\rceil}|^2 \log|x_{(-i),j}|}, \tag{4.16}$$

for some constant C, that stems from the fact that $h^{\mathrm{tck}}_{(0)}$ is uniformly bounded away from zero on U. This obviously implies that $\frac{1}{h^{1/k}}$ is bounded, which we refer to as the "first weak bound" and that

$$\frac{1}{h^{1/k}} \leqslant -1/|t_{\lceil i\rceil}|^2 \log|x_{(-i),j}|,$$

which we call the "second weak bound". Hence, using the expressions (4.14), all terms appearing in $\partial h^{1/k}$ and $\bar\partial\partial h^{1/k}$ with continuous coefficients induce locally integrable forms in $\partial \log h$, $\bar\partial\partial \log h$ and their powers.

We now treat the non-continuous terms appearing in $\partial \log h$ and $\bar\partial\partial \log h$, i.e. the quotient by $h^{1/k}$ of the terms displayed in (4.12) and (4.11), (4.13) respectively. We refer to these terms as "building blocks".

We first look at each term in $\partial \log h$, i.e. we focus on the quotient by $h^{1/k}$ of the coefficients (4.12). For $\mathrm{d}t_p$-differential forms we use the weak bound

for $1/h^{1/k}$ and the fact that $|t_{\lceil i\rceil}|^2/t_p$ is a polynomial expression in the t_i and $\bar{t}_i$ for $p \geqslant i$. The logarithmic contribution is unbounded, but after a change to polar coordinates we are left with

$$\begin{aligned}(4.17)\qquad \int_{B\times D^\times\times D^{\mathbf{t}}} \frac{|t_{\lceil i\rceil}|^2}{h^{1/k}\cdot t_p}\log|x_{(-i),j}|\mathrm{dvol} \\ \leqslant C_1\int_{\Delta_\varepsilon}\log|x_{(-i),j}||\mathrm{d}x_{(-i),j}|^2 \\ \leqslant C_2\int_{r=0}^{\varepsilon} r\log|r|\mathrm{d}r<\infty\,.\end{aligned}$$

The same argument applies to the $|t_{\lceil i\rceil}|^2\log|x_{(-i),j}|\partial R^{\mathrm{hor}}_{(-i),j}$ term. For the $\mathrm{d}x_{(-i),j}$-coefficient we use again the weak bound for $1/h^{1/k}$ and so we are left with a polynomial expression in the t_i and $\bar{t}_i$ and the finite integral

$$(4.18)\qquad \int_{\Delta_\varepsilon}\frac{1}{x_{(-i),j}}|\mathrm{d}x_{(-i),j}|^2<\infty.$$

The same arguments apply verbatim to the coefficients of $\bar{\partial}\log h$.

Next, we examine the coefficients in $\frac{1}{h^{1/k}}\bar{\partial}\partial h^{1/k}$. Using the weak bound and the same polar coordinates argument as in (4.17), we see that the term $\log|t_p|\,|\mathrm{d}t_p|^2/h^{1/k}$ appearing in (4.11) is locally integrable. The other terms given by the quotient by $h^{1/k}$ of the terms appearing in (4.13) are of the same shape as the one already treated, so they are locally integrable.

Finally we examine the terms that may arise from as an arbitrary wedge product of $\frac{1}{h^{1/k}}\bar{\partial}\partial h^{1/k}$, $\frac{1}{h^{1/k}}\partial h^{1/k}$ or $\frac{1}{h^{1/k}}\bar{\partial}h^{1/k}$. This relies on Fubini and a suitable organization of the order of integration, necessary since the building blocks with $\mathrm{d}t_p$ may have coefficients involving $x_{(-i),j}$ and vice versa. Ultimately we rely on the observation that in a non-zero wedge product each of the $\mathrm{d}t_p$ and $\mathrm{d}x_{(-i),j}$ and their conjugates appear only once.

To start, observe that the building block (4.11) has no $x_{(-i),j}$-dependence. Whenever such a term appear at a level below any level that has a horizontal node, we integrate $|\mathrm{d}t_p|^2$ and use the integrability of $\log|t_p|$. In general, for any level $(-i)$ and a subset $P_i\subseteq\{1,\dots,i\}$, we will use the notation

$$f_{P_i}(\mathbf{t}) \coloneqq \prod_{p\in P_i}|t_p|\log|t_p|$$

to indicate the functions appearing in the product of the building blocks of type(4.11) $\prod_{p\in P_i}\log|t_p||\mathrm{d}t_p|^2 = f_{P_i}(\mathbf{t})\prod_{p\in P_i}\mathrm{d}\theta_p\mathrm{d}|t_p|$ after passing to polar coordinates polar coordinates. We record for the sequel that the functions $f_i(\mathbf{t})$ are continuous in a neighborhood of zero.

Whenever we have a combination of the forms in (4.12) and in (4.13) not containing the $\mathrm{d}x_{(-i),j}$, we can use the first weak bound for $1/h^{1/k}$. Then we are left with $\log|x_{(-i),j}|^N$ for some $N>0$, which is integrable.

If a differential form stems from a derivative of $R^{\mathrm{hor}}_{(-i),j}$, the ratio of the coefficient over $h^{1/k}$ can be bounded by a constant using (4.16) (or rather just the second weak bound). We can thus disregard those $R^{\mathrm{hor}}_{(-i),j}$-derivatives in the rest of the discussion.

We now treat the case where we have a combination of building blocks involving $\mathrm{d}x_{(-i),j}$, i.e., coming from the first term of (4.12) and its conjugate, possibly together with other building blocks whose coefficients involve $\log(x_{(-i),j})$.

Case A: Only one of $\mathrm{d}x_{(-i),j}$ *and its conjugate appear from building blocks.* — We perform the integral over $|\mathrm{d}x_{(-i),j}|^2$ first. This leads to the integral as in (4.18), or maybe with an additional factor $\log(x_{(-i),j})^N$, that does not change the finiteness of the integral. Using the first weak bound for $h^{1/k}$ a polynomial expression in the $t_i, \bar{t}_i$ remains, which is irrelevant to the finiteness discussion for the subsequent integrals.

Case B: Both $\mathrm{d}x_{(-i),j}$ *and its conjugate appear, but with no other building block involving* $\log(x_{(-i),j})$. — Then the second weak estimate yields the same situation as the Poincaré integral (4.9), which is finite. We perform these integrals before addressing the other $\mathrm{d}x_{(-i),j}$. For simplicity of notation we label the x-variables subsequently by x_e for $e\in E^h$, keeping in mind that such a horizontal edges comes with a level $i=i(e)$. We denote the set of horizontal edges that do not belong to Case A or B by $E^h_C\subset E^h$.

Case C: Both $\mathrm{d}x_{(-i),j}$ *and its conjugate appear and there are building blocks involving* $\log(x_{(-i),j})$. — We denote by $i_{\max}:=\max_{e\in E^h_C}|i(e)|$ the largest index of a level that has a horizontal node. Define $P^1_e\subset I:=\{1,\dots,i_{\max}\}$ to be the set of indices where the product of building blocks $\frac{|t_{\lceil i\rceil}|^4\log|x_e|^2}{h^{2/k}}\frac{|\mathrm{d}t_p|^2}{|t_p|^2}$ (coming from the last term of (4.12) and its conjugate) occur. Define $P^2_e\subset I$ the set of indices parametrizing forms of type $\frac{|t_{\lceil i\rceil}|^2\log|x_e|}{h^{1/k}}\frac{|\mathrm{d}t_p|^2}{|t_p|^2}$ (coming from the first term of (4.13)). We define $P^3_e\subset I$, resp. $P^4_e\subset I$ the indices, where last term of (4.12) but not its conjugate (resp. the other way round) occur. Note that in any non-zero building block the subsets P^j_e are disjoint and that the subsets $P_e=P^1_e\cup P^2_e\cup P^3_e\cup P^4_e$ are disjoint for different e, since otherwise the wedge products of building blocks is zero. Wedging all these building blocks together we obtain the

differential form

$$\prod_{e\in E_C^h}\left(\frac{|t_{\lceil i(e)\rceil}|^{2(2+M_e)}}{h^{\frac{2+M_e}{k}}}(\log|x_e|)^{M_e}\prod_{p\in P_e^1\cup P_e^2}\frac{|\mathrm{d}t_p|^2}{|t_p|^2}\right.$$
$$\left.\times\prod_{p\in P_e^3}\frac{|\mathrm{d}t_p|^2}{t_p}\prod_{p\in P_e^4}\frac{|\mathrm{d}\bar{t}_p|^2}{\bar{t}_p}\right)\frac{|\mathrm{d}x_e|^2}{|x_e|^2}. \tag{4.19}$$

where $M_e = 2|P_e^1| + |P_e^2 \cup P_e^3 \cup P_e^4|$. We use that $\prod_{p\in P_e} t_p$ divides $t_{\lceil i(e)\rceil}$ to cancel the t_p and $\bar{t}_p$ in the denominator and apply M_e times the second weak estimate in each factor to cancel all terms raised to M_e. Note that the expression (4.19) involves all variables t_i for $i \in I$. We thus integrate also over $\prod_{p\in I\setminus P_e}$ and have to take into account the differential form that stems from the building blocks in (4.11) in the total wedge, which will contribute with the continuous function $f_{P_{i_{\max}}}(\mathbf{t})$ for some $P_{i_{\max}} \subseteq I$. We pass to polar coordinates and write $R_e = |x_e|$ and $r_p = |t_p|$. Coarsely estimating the angle integral it suffices to show the finiteness of the following expression:

$$\int_0^\varepsilon\cdots\int_0^\varepsilon f_{P_{i_{\max}}}(\mathbf{t})\cdot\prod_{e\in E_C^h}\frac{\prod_{p\leqslant i(e)} r_p^3}{\left(C-\log(R_e)\prod_{p\leqslant i(e)} r_p^2\right)^2}\frac{\mathrm{d}R_e}{R_e}\prod_{i\in I}\mathrm{d}r_i \tag{4.20}$$
$$=\int_0^\varepsilon\cdots\int_0^\varepsilon f_{P_{i_{\max}}}(\mathbf{t})$$
$$\cdot\left(\prod_{e\in E_C^h}\left(\prod_{p\leqslant i(e)} r_p\int_{-\infty}^{\log(\varepsilon)\prod_{p\geqslant i(e)} r_p^2}\frac{\mathrm{d}v_e}{(C-v_e)^2}\right)\prod_{i\in I}\mathrm{d}r_i\right)$$
$$=\int_0^\varepsilon\cdots\int_0^\varepsilon f_{P_{i_{\max}}}(\mathbf{t})\cdot\prod_{e\in E_C^h}\left(\frac{\prod_{p\leqslant i(e)} r_p}{C-\log(\varepsilon)\prod_{p\leqslant i(e)} r_p^2}\right)\prod_{i\in I}\mathrm{d}r_i<\infty$$

where we have used the change of coordinates $v_e = \log(R_e)\prod_{p\geqslant i(e)} r_p^2$ in the first equality. The last expression is finite since we integrate the product of continuous functions.

Since we have shown that all the forms F_h^d and $\Omega\wedge F_h^d$ have locally integrable coefficients, their corresponding currents of integration are well-defined. □

Proof of Proposition 4.5. — We identify the local statements needed to prove the claims and justify them simultaneously. To see that $[F_h] = [\bar{\partial}\partial\log h]$ defines a closed current we need to justify the first step in the

chain

$$\mathrm{d}\left[\bar\partial\partial\log h\right] = \left[\mathrm{d}(\bar\partial\partial\log h)\right] = 0 \tag{4.21}$$

of cohomology classes of currents. By definition we have to justify that

$$\int_{D\times B} \mathrm{d}F_h \wedge \xi = -\int_{D\times B} F_h \wedge \mathrm{d}\xi \tag{4.22}$$

for any smooth r-form ζ, where $r = \dim_{\mathbb{R}} \mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu) - 3$. By Stokes' theorem amounts to justify that

$$\lim_{\delta\to 0}\int_{T_\delta} F_h \wedge \xi = 0 \tag{4.23}$$

where T_δ is one of the tubular neighborhoods inside $B \times D^{\mathbf{t}} \times D^{\mathbf{x}}$, with tube radius δ, around the divisors defined by setting one coordinate axis to zero. We will denote such tubular neighborhoods by

$$T_\delta^{(-i),j} = B \times \left\{\begin{array}{l} |x_{(-i),j}| = \delta\,; \\ t_p \in \Delta_\varepsilon \text{ for all } p\,; \\ x_{(-i'),j'} \in \Delta_\varepsilon \text{ for all } (i',j') \neq (i,j) \end{array}\right\}$$

and

$$T_\delta^p = B \times \left\{t_p = \delta\,;\ t_{p'} \in \Delta_\varepsilon \text{ for all } p' \neq p\,;\ x_{(-i),j} \in \Delta_\varepsilon \text{ for all } (i,j)\right\}$$

and finally by

$$T_\delta^B = D \times \partial B^\delta,$$

where ∂B^δ is the union of the tubular neighborhoods around the coordinate axis in the $\mathrm{PPer}_{(-i)}$ components.

For the second statement of the statement let h^* be a smooth (comparison) metric on $\mathcal{O}(-1)$. Then certainly $\frac{1}{2\pi i}$ times the curvature $F_h^* = \bar\partial\partial\log h^*$ represents the first Chern class of $\mathcal{O}(-1)$. To justify the equality of cohomology classes of currents

$$[\bar\partial\partial\log h^*] - [\bar\partial\partial\log h] = [d(\partial\log h^* - \partial\log h)] = d[\partial\log h^* - \partial\log h] = 0$$

we have to justify the second equality sign, i.e. that the current of i ntegration of $\bar\partial\partial\log h$ is the same as the derivative in the sense of currents of $\partial\log h$. Then the last equality follows from Proposition 4.4, showing that the expression is a coboundary in the sense of currents, since $\log h^* - \log h = \log(h^*/h)$ is independent of the scale of h and thus globally well-defined.

Writing $\Omega^* = \partial\log h^*$ and $\Omega = \partial\log h$, we have to justify that for any smooth $\dim_{\mathbb{R}} \mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu) - 2$-form the condition

$$\lim_{\delta\to 0}\int_{T_\delta} (\Omega^* - \Omega) \wedge \xi = 0$$

holds, where $T_\delta \in \{T_\delta^{(-i),j},\ T_\delta^p,\ T_\delta^B\}$ for all (i,j) and p. This follows once we have shown

$$\lim_{\delta\to 0}\int_{T_\delta}\Omega\wedge\xi = 0 \tag{4.24}$$

and from the smoothness of Ω^*.

For the generalization to wedge powers we use $F_h = \mathrm{d}\Omega$ and $F_h^* = \mathrm{d}\Omega^*$ and want to argue that there is an equality of cohomology classes of currents

$$[F_h^d] - [(F_h^*)^d] = \mathrm{d}\left[(\Omega-\Omega^*)\wedge\sum_{i+j=n-1}F_h^i(F_h^*)^j\right].$$

With this equation at hand we use that the argument of the differential operator on the right hand side defines a current by Proposition 4.4, so that $[F_h^d]$ and $[(F_h^*)^d]$ are cohomologous and $(\frac{1}{2\pi})^d[(F_h^*)^d]$ is known to represent $c_1(\mathcal{O}(-1))^d$.

To justify this equation we need again to argue about the interchange of derivative and passage to the current. Hence we need to show that for all d and for all smooth $\dim_{\mathbb{R}}\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)-2d-1$-forms ξ, the following equation

$$\lim_{\delta\to 0}\int_{T_\delta}\Omega\wedge F_h^n\wedge\xi = 0 \tag{4.25}$$

holds for $T_\delta \in \{T_\delta^{(-i),j},\ T_\delta^p,\ T_\delta^B\}$ for all (i,j) and p.

To justify these three equations (4.23), (4.24) and (4.25) we fix the tubular neighborhood T_δ around one of the boundary divisors and analyze the forms that may appear from the wedge products in (4.23), in (4.24) or in (4.25). These are wedge products of the quotient by h of the building blocks in (4.11), (4.12), (4.13) and the differentials of the coordinates themselves.

Our strategy is to apply Fubini and use Proposition 4.4 in order to ensure that the blocks not depending on the coordinate defining T_δ give a finite result, which is independent of δ. Then we need to check that the integral of arbitrary products of the building blocks appearing in (4.23), (4.24) and in (4.25) involving the special coordinate, yields an expression going to zero for δ going to zero.

First of all, we consider the tubular neighborhoods T_δ^B around the zero divisor of a coordinate function belonging to the $\mathrm{PPer}_{(-i)}$ part. Since the differential forms appearing as integrands in the three equations (4.23),

(4.24) and (4.25) are continuous in the $\mathrm{PPer}_{(-i)}$ coordinates, by Proposition 4.4 we conclude that the integrand over T_δ^B can be bounded by a constant function, so the integral goes to zero for δ going to zero.

Next, note that whenever we consider the integral over $T_\delta^{(-i),j}$, resp. T_δ^p, of a product of building blocks not depending on the variables $x_{(-i),j}$, resp. t_p, Proposition 4.4 yields finiteness independent of δ.

Hence we are left to consider integrals of product of building blocks corresponding to the variables $x_{(-i),j}$ and t_p. Note that the differential forms involving $|\mathrm{d}x_{(-i),j}|^2$, resp. $|\mathrm{d}t_p|^2$, restricts to zero on $T_\delta^{(-i),j}$, resp. T_δ^p.

We consider first the integral over T_δ^p. By the previous remark, the block (4.11) given by $\log|t_p||\mathrm{d}t_p|^2$ restricts to zero on T_δ^p. By inspecting the proof of Proposition 4.4, we see that the only possibly problematic case is the one corresponding to Case C. We have to consider an expression analogous to (4.19), but in this situation we can only have a $\mathrm{d}t_p$ or a $\mathrm{d}\bar{t}_p$. Using the same strategy as for (4.19), we consider the expression analogous to (4.20) obtained after passing to polar coordinates and estimating the angle integral. In this situation we hence have that the integral over T_δ^p of these product of building blocks is given by

$$\int_0^\varepsilon \cdots \int_0^\varepsilon f_{P_{i_{\max}}}(\mathbf{t}) \cdot \prod_{e\in E_C^h} \frac{\delta^3 \prod_{i\leqslant i(e), i\neq p} r_i^3}{\left(C-\delta^2\log(R_e)\prod_{i\leqslant i(e),i\neq p} r_i^2\right)^2} \frac{\mathrm{d}R_e}{R_e} \prod_{i\in I\setminus\{p\}} \mathrm{d}r_i$$
$$= \int_0^\varepsilon \cdots \int_0^\varepsilon f_{P_{i_{\max}}}(\mathbf{t}) \cdot \prod_{e\in E_C^h} \left(\frac{\delta \prod_{i\leqslant i(e),i\neq p} r_i}{C-\log(\varepsilon)\delta^2\prod_{i\leqslant i(e),i\neq p} r_i^2}\right) \prod_{i\in I\setminus\{p\}} \mathrm{d}r_i$$

By estimating by a constant the denominator appearing in the second line, we see that the expression is going to zero for δ going to zero.

We consider finally the integral over $T_\delta^{(-i),j}$. We already remarked that the form $|\mathrm{d}x_{(-i),j}|^2$ restricts to zero to $T_\delta^{(-i),j}$. We now describe the general case of a product of building blocks involving the $x_{(-i),j}$ variable. We consider the case where $\mathrm{d}x_{(-i),j}$ appears, the other case where $\mathrm{d}\bar{x}_{(-i),j}$ appears is clearly equivalent. Such a product of building block is given the product of the expression (4.19) (for edges e not corresponding to the special index $(-i),j$) with

$$\frac{|t_{\lceil i\rceil}|^{2(1+M_{(-i),j})}}{h^{\frac{1+M_{(-i),j}}{k}}}(\log|x_{(-i),j}|)^{M_{(-i),j}}$$
$$\times \prod_{p\in P^1\cup P^2} \frac{|\mathrm{d}t_p|^2}{|t_p|^2} \prod_{p\in P^3} \frac{|\mathrm{d}t_p|^2}{t_p} \prod_{p\in P^4} \frac{|\mathrm{d}\bar{t}_p|^2}{\bar{t}_p} \frac{\mathrm{d}x_{(-i),j}}{x_{(-i),j}}.$$

where we dropped the index $(-i), j$ of the sets P^ℓ. By proceeding as we did to reach the expression (4.20) after estimating and passing to polar coordinates, we obtain that the integral over $T_\delta^{(-i),j}$ of these product of building blocks is given by the following expression

$$\begin{aligned}
&\int_0^\varepsilon \cdots \int_0^\varepsilon f_{P_{i_{\max}}}(\mathbf{t}) \cdot \frac{\prod_{p\leqslant i} r_p}{C-\log(\delta)\prod_{p\leqslant i} r_p^2} \\
&\qquad \times \prod_{e\in E_C^h} \frac{\prod_{p\leqslant i(e)} r_p^3}{\Big(C-\log(R_e)\prod_{p\leqslant i(e)} r_p^2\Big)^2} \frac{\mathrm{d}R_e}{R_e} \prod_{p\in I} \mathrm{d}r_p \\
&\qquad = \int_0^\varepsilon \cdots \int_0^\varepsilon f_{P_{i_{\max}}}(\mathbf{t}) \cdot \frac{\prod_{p\leqslant i} r_p}{C-\log(\delta)\prod_{p\leqslant i} r_p^2} \\
&\qquad\qquad \cdot \prod_{e\in E_C^h} \left(\frac{\prod_{p\leqslant i(e)} r_p}{C-\log(\varepsilon)\prod_{p\leqslant i(e)} r_p^2} \right) \prod_{p\in I} \mathrm{d}r_p \\
&\qquad \leqslant C_1 \int_0^\varepsilon \cdots \int_0^\varepsilon \frac{\prod_{p\leqslant i} r_p}{C-\log(\delta)\prod_{p\leqslant i} r_p^2} \prod_{p=1}^{i} \mathrm{d}r_p \,.
\end{aligned}$$

The first inequality is obtained by the same substitutions as in (4.20) and the last inequality is given by estimating the product of the continuous functions by a constant C_1.

We will use notation $r_{\lceil i\rceil} := \prod_{p\leqslant i} r_p$. If $i=0$, then the last line above is simply given by $C_1/(C-\log(\delta))$ which clearly tends to 0 for δ going to zero. If $i>0$, we can integrate first the variable r_i. Using now that the antiderivative of the function $y/(C+ay^2)$, for a constant a, is given by $\log(C+ay^2)/(2a)$ we obtain

$$\begin{aligned}
(4.26)\quad & C_1 \int_0^\varepsilon \cdots \int_0^\varepsilon \frac{r_{\lceil i\rceil}}{C-\log(\delta) r_{\lceil i\rceil}^2} \prod_{p=1}^{i} \mathrm{d}r_p \\
&= \frac{C_1}{-2\log(\delta)} \int_0^\varepsilon \cdots \int_0^\varepsilon \log\Big(1-\log(\delta)\frac{\varepsilon^2}{C} r_{\lceil i-1\rceil}\Big) \prod_{p=1}^{i-1} \mathrm{d}r_p \,.
\end{aligned}$$

If $i=1$, the previous expression goes to zero for δ going to zero. If $i\geqslant 2$, we can integrate again the r_{i-1} variable. Using that the antiderivative of

$\log(1+ay)$ is given by $(1/a+y)\log(1+ay)-y$ we obtain

$$\begin{aligned}
(4.27)\quad & \frac{-C_1}{2\log(\delta)}\int_0^\varepsilon\cdots\int_0^\varepsilon \log\Big(1-\log(\delta)\frac{\varepsilon^2}{C}r_{\lceil i-1\rceil}\Big)\prod_{p=1}^{i-1}\mathrm{d}r_p \\
&= \frac{-C_1}{2\log(\delta)}\int_0^\varepsilon\cdots\int_0^\varepsilon\Bigg(\frac{1}{-\log(\delta)\frac{\varepsilon^2}{C}r_{\lceil i-2\rceil}+\varepsilon} \\
&\qquad\qquad\cdot\log\Big(1-\log(\delta)\frac{\varepsilon^3}{C}r_{\lceil i-2\rceil}\Big)-\varepsilon\Bigg)\prod_{p=1}^{i-2}\mathrm{d}r_p \\
&\leqslant \frac{C_1\varepsilon^{i-1}}{2\log(\delta)}+\frac{C_1}{-2\varepsilon\log(\delta)}\int_0^\varepsilon\cdots\int_0^\varepsilon\log\Big(1-\log(\delta)\frac{\varepsilon^3}{C}r_{\lceil i-2\rceil}\Big)\prod_{p=1}^{i-2}\mathrm{d}r_p
\end{aligned}$$

where the last inequality is given by bounding using the bound $r_{\lceil i-2\rceil}\geqslant 0$ for the denominator of the fraction appearing in front of the logarithm in the second line. The first term of (4.27) is clearly going to zero for δ going to zero. The second term of (4.27) has the same shape as the integral given by (4.26), so by induction we can show that the expression goes to zero for δ going to zero.

Hence we have shown that the initial expression given by the integral over $T_\delta^{(-i),j}$ of the product of building blocks involving the $x_{(-i),j}$ variable is going to zero for δ going to zero, as we wanted. □

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