

# ANNALES DE L'INSTITUT FOURIER 

Hayato Chiba

Weights, Kovalevskaya exponents and the Painlevé property
Article à paraître, mis en ligne le 3 juillet 2023, 38 p.

Article mis à disposition par son auteur selon les termes de la licence Creative Commons attribution - pas de modification 3.o France

MERSENNE

# WEIGHTS, KOVALEVSKAYA EXPONENTS AND THE PAINLEVÉ PROPERTY 

by Hayato CHIBA


#### Abstract

Weighted degrees of quasihomogeneous Hamiltonian functions of the Painlevé equations are investigated. A t-uple of positive integers, called a regular weight, satisfying certain conditions related to singularity theory is classified. For each polynomial Painlevé equation a regular weight is associated. Conversely, for 2 and 4 -dim cases, it is shown that there exists a differential equation satisfying the Painlevé property associated with each regular weight. Kovalevskaya exponents of quasihomogeneous Hamiltonian systems are also investigated by means of regular weights, singularity theory and dynamical systems theory. It is shown that there is a one-to-one correspondence between Laurent series solutions and stable manifolds of the associated vector field obtained by the blow-up of the system. For 4-dim autonomous Painlevé equations, the level surface of Hamiltonian functions can be decomposed into a disjoint union of stable manifolds.

RÉSumé. - On étudie les degrés pondérés des fonctions hamiltoniennes quasi homogènes des équations de Painlevé. On classifie un t-uplet d'entiers positifs, appelé poids régulier, satisfaisant certaines conditions liées à la théorie de la singularité. A chaque équation polynomiale de Painlevé est associé un poids régulier. Inversement, pour les cas des dimensions 2 et 4 , on montre qu'il existe une équation différentielle satisfaisant la propriété de Painlevé associée à chaque poids régulier. Les exposants de Kovalevskaya des systèmes hamiltoniens quasi homogènes sont également étudiés au moyen de poids réguliers, la théorie des singularités et la théorie des systèmes dynamiques. On montre qu'il existe une correspondance biunivoque entre les solutions de la série de Laurent et des variétés stables du champ vectoriel associé obtenues par l'éclatement du système. Pour des équations autonomes de Painlevé en dimension 4, la surface de niveau des fonctions hamiltoniennes peut être décomposée en une union disjointe de variétés stables.


## 1. Introduction

A differential equation defined on a complex region is said to have the Painlevé property if any movable singularity (a singularity of a solution

Keywords: Painlevé equation, quasihomogeneous vector field, regular weight, Kovalevskaya exponent.
2020 Mathematics Subject Classification: 34M55.
which depends on an initial condition) of any solution is a pole. Painlevé and his group classified second order ODEs having the Painlevé property and found new six differential equations called the Painlevé equations $\mathrm{P}_{\mathrm{I}}, \ldots, \mathrm{P}_{\mathrm{VI}}$. Nowadays, it is known that they are written in Hamiltonian forms

$$
\left(\mathrm{P}_{\mathrm{J}}\right): \frac{\mathrm{d} q}{\mathrm{~d} z}=\frac{\partial H_{J}}{\partial p}, \quad \frac{\mathrm{~d} p}{\mathrm{~d} z}=-\frac{\partial H_{J}}{\partial q}, \quad J=\mathrm{I}, \ldots, \mathrm{VI} .
$$

Among six Painlevé equations, the Hamiltonian functions of the first, second and fourth Painlevé equations are polynomials in both of the independent variable $z$ and the dependent variables $(q, p)$. They are given by

$$
\begin{align*}
H_{\mathrm{I}} & =\frac{1}{2} p^{2}-2 q^{3}-z q \\
H_{\mathrm{II}} & =\frac{1}{2} p^{2}-\frac{1}{2} q^{4}-\frac{1}{2} z q^{2}-\alpha q  \tag{1.1}\\
H_{\mathrm{IV}} & =-p q^{2}+p^{2} q-2 p q z-\alpha p+\beta q
\end{align*}
$$

respectively, where $\alpha, \beta \in \mathbb{C}$ are arbitrary parameters.
In general, a polynomial $H\left(x_{1}, \ldots, x_{n}\right)$ is called a quasihomogeneous polynomial if there are positive integers $a_{1}, \ldots, a_{n}$ and $h$ such that

$$
\begin{equation*}
H\left(\lambda^{a_{1}} x_{1}, \ldots, \lambda^{a_{n}} x_{n}\right)=\lambda^{h} H\left(x_{1}, \ldots, x_{n}\right) \tag{1.2}
\end{equation*}
$$

for any $\lambda \in \mathbb{C}$. A polynomial $H$ is called a semi-quasihomogeneous if $H$ is decomposed into two polynomials as $H=H^{P}+H^{N}$, where $H^{P}$ satisfies (1.2) and $H^{N}$ satisfies

$$
H^{N}\left(\lambda^{a_{1}} x_{1}, \ldots, \lambda^{a_{n}} x_{n}\right) \sim o\left(\lambda^{h}\right), \quad|\lambda| \rightarrow \infty
$$

The integer $\operatorname{deg}(H):=h$ is called the weighted degree of $H$ with respect to the weight $\operatorname{deg}\left(x_{1}, \ldots, x_{n}\right):=\left(a_{1}, \ldots, a_{n}\right) . H^{P}$ and $H^{N}$ are called the principal part and the non-principal part of $H$, respectively. The weight of $H, H^{P}$ and $H^{N}$ are also calculated by the Newton diagram as follows. Plot all exponents $\left(r_{1}, \ldots, r_{n}\right)$ of monomials $x_{1}^{r_{1}} x_{2}^{r_{2}} \cdots x_{n}^{r_{n}}$ included in $H^{P}$ on the integer lattice in $\mathbb{R}^{n}$. If they lie on a unique hyperplane $a_{1} x_{1}+\cdots+a_{n} x_{n}=$ $h$, then $\operatorname{deg}\left(H^{P}\right)=h$ with respect to the weight $\left(a_{1}, \ldots, a_{n}\right)$. Exponents of monomials included in $H^{N}$ should be on the lower side of the hyperplane. See [3] for the detail.

The Hamiltonian functions for $\mathrm{P}_{\mathrm{I}}, \mathrm{P}_{\text {II }}$ and $\mathrm{P}_{\text {IV }}$ are semi-quasihomogeneous. If we define degrees of variables by $\operatorname{deg}(q, p, z)=(2,3,4)$ for $H_{\mathrm{I}}$, $\operatorname{deg}(q, p, z)=(1,2,2)$ for $H_{\mathrm{II}}$ and $\operatorname{deg}(q, p, z)=(1,1,1)$ for $H_{\mathrm{IV}}$, then Hamiltonian functions have the weighted degrees 6,4 and 3 , respectively, (Table 1.1) with $H_{\mathrm{I}}^{N}=0, H_{\mathrm{II}}^{N}=-\alpha q$ and $H_{\mathrm{IV}}^{N}=-\alpha p+\beta q$.

The Hamiltonian functions of the third, fifth and sixth Painlevé equations are not polynomials in $z$, and their weights include nonpositive integers (Table 1.1). They are not treated in this paper, while the analysis of them using weighted projective spaces is given in [7].

Higher dimensional Painlevé equations have not been classified yet, however, a lot of such equations have been reported in the literature. A list of four dimensional Painlevé equations derived from the monodromy preserving deformation is given in [20,21]. Lie-algebraic approach is often employed to find new Painlevé equations [12, 14, 18, 27]. Several Painlevé hierarchies, which are hierarchies of $2 n$-dimensional Painlevé equations, are obtained by the similarity reductions of soliton equations such as the KdV equation. Among them, it is known that Hamiltonian functions of the the first Painlevé hierarchy $\left(\mathrm{P}_{\mathrm{I}}\right)_{n}[22,23,31]$, the second-first Painlevé hierarchy $\left(\mathrm{P}_{\mathrm{II}-1}\right)_{n}[9,10,22,23]$, the second-second Painlevé hierarchy $\left(\mathrm{P}_{\mathrm{II}-2}\right)_{n}$ and the fourth Painlevé hierarchy $\left(\mathrm{P}_{\mathrm{IV}}\right)_{n}[15,22]$ can be expressed as polynomials with respect to both of the dependent variables and the independent variables. They are Hamiltonian PDEs of the form

$$
\left\{\begin{array}{l}
\frac{\partial q_{j}}{\partial z_{i}}=\frac{\partial H_{i}}{\partial p_{j}}, \frac{\partial p_{j}}{\partial z_{i}}=-\frac{\partial H_{i}}{\partial q_{j}}, \quad j=1, \ldots, n ; i=1, \ldots, n  \tag{1.3}\\
H_{i}=H_{i}\left(q_{1}, \ldots, q_{n}, p_{1}, \ldots, p_{n}, z_{1}, \ldots, z_{n}\right)
\end{array}\right.
$$

consisting of $n$ Hamiltonians $H_{1}, \ldots, H_{n}$ with $n$ independent variables $z_{1}, \ldots, z_{n}$. When $n=1,\left(\mathrm{P}_{\mathrm{I}}\right)_{1}$ and $\left(\mathrm{P}_{\mathrm{IV}}\right)_{1}$ are reduced to the first and fourth Painlevé equations, respectively. Both of $\left(\mathrm{P}_{\mathrm{II}-1}\right)_{1}$ and $\left(\mathrm{P}_{\mathrm{II}-2}\right)_{1}$ coincide with the second Painlevé equation, while they are different systems for $n \geqslant 2$. When $n=2$, Hamiltonians of $\left(\mathrm{P}_{\mathrm{I}}\right)_{2},\left(\mathrm{P}_{\mathrm{II}-1}\right)_{2},\left(\mathrm{P}_{\mathrm{II}-2}\right)_{2}$ and $\left(\mathrm{P}_{\mathrm{IV}}\right)_{2}$ are given by
$\left(\mathrm{P}_{\mathrm{I}}\right)_{2}\left\{\begin{array}{c}H_{1}^{9 / 2}=2 p_{2} p_{1}+3 p_{2}^{2} q_{1}+q_{1}^{4}-q_{1}^{2} q_{2}-q_{2}^{2}-z_{1} q_{1}+z_{2}\left(q_{1}^{2}-q_{2}\right) \\ H_{2}^{9 / 2}=p_{1}^{2}+2 p_{2} p_{1} q_{1}-q_{1}^{5}+p_{2}^{2} q_{2}+3 q_{1}^{3} q_{2}-2 q_{1} q_{2}^{2} \\ \\ +z_{1}\left(q_{1}^{2}-q_{2}\right)+z_{2}\left(z_{2} q_{1}+q_{1} q_{2}-p_{2}^{2}\right)\end{array}\right.$
$(1.5)\left(\mathrm{P}_{\mathrm{II}-1}\right)_{2}\left\{\begin{array}{c}H_{1}^{7 / 2+1}=2 p_{1} p_{2}-p_{2}^{3}-p_{1} q_{1}^{2}+q_{2}^{2}-z_{1} p_{2}+z_{2} p_{1}+2 \alpha q_{1}, \\ H_{2}^{7 / 2+1}=-p_{1}^{2}+p_{1} p_{2}^{2}+p_{1} p_{2} q_{1}^{2}+2 p_{1} q_{1} q_{2}+z_{1} p_{1} \\ +z_{2}\left(z_{2} p_{1}-p_{1} q_{1}^{2}+p_{1} p_{2}\right)-\alpha\left(2 p_{2} q_{1}+2 q_{2}+2 z_{2} q_{1}\right),\end{array}\right.$
$\left(\mathrm{P}_{\mathrm{II}-2}\right)_{2}\left\{\begin{array}{c}H_{1}^{5}=p_{1} p_{2}-p_{1} q_{1}^{2}-2 p_{1} q_{2}+p_{2} q_{1} q_{2}+q_{1} q_{2}^{2} \\ \quad+q_{2} z_{1}+z_{2}\left(q_{1} q_{2}-p_{1}\right)+\alpha q_{1}, \\ H_{2}^{5}=p_{1}^{2}-p_{1} p_{2} q_{1}+p_{2}^{2} q_{2}-2 p_{1} q_{1} q_{2}-p_{2} q_{2}^{2}+q_{1}^{2} q_{2}^{2} \\ +z_{1}\left(q_{1} q_{2}-p_{1}\right)-z_{2}\left(p_{1} q_{1}+q_{2}^{2}+q_{2} z_{2}\right)+\alpha p_{2},\end{array}\right.$
$(1.7) \quad\left(\mathrm{P}_{\mathrm{IV}}\right)_{2}\left\{\begin{array}{c}H_{1}^{4+1}=p_{1}^{2}+p_{1} p_{2}-p_{1} q_{1}^{2}+p_{2} q_{1} q_{2}-p_{2} q_{2}^{2} \\ \quad-z_{1} p_{1}+z_{2} p_{2} q_{2}+\alpha q_{2}+\beta q_{1}, \\ H_{2}^{4+1}=p_{1} p_{2} q_{1}-2 p_{1} p_{2} q_{2}-p_{2}^{2} q_{2}+p_{2} q_{1} q_{2}^{2}+p_{2} q_{2} z_{1} \\ +z_{2}\left(p_{1} p_{2}-p_{2} q_{2}^{2}+p_{2} q_{2} z_{2}\right)+\left(p_{1}-q_{1} q_{2}+q_{2} z_{2}\right) \alpha-\beta p_{2},\end{array}\right.$
respectively, with arbitrary parameters $\alpha, \beta \in \mathbb{C}$ (these notations for Hamiltonian functions are related to the spectral type of a monodromy preserving deformation [21]). The weighted degrees of these hierarchies determined by the Newton diagrams are shown in Table 1.2 (see also Table 1.3). From Tables 1.1 and 1.2 and the equations, we deduce the following properties.

- $\operatorname{deg}\left(q_{i}\right)+\operatorname{deg}\left(p_{i}\right)=\operatorname{deg}\left(H_{1}\right)-1$.
- $\operatorname{deg}\left(z_{1}\right)=\operatorname{deg}\left(H_{1}\right)-2$.
- $\operatorname{deg}\left(z_{i}\right)+\operatorname{deg}\left(H_{i}\right)$ is independent of $i=1, \ldots, n$.
- $\min _{1 \leqslant i \leqslant n}\left\{\operatorname{deg}\left(q_{i}\right), \operatorname{deg}\left(p_{i}\right)\right\}=1$ or 2 .
- The equation (1.3) is invariant under the $\mathbb{Z}_{s}$-action

$$
\left(q_{i}, p_{i}, z_{i}\right) \mapsto\left(\omega^{\operatorname{deg}\left(q_{i}\right)} q_{i}, \omega^{\operatorname{deg}\left(p_{i}\right)} p_{i}, \omega^{\operatorname{deg}\left(z_{i}\right)} z_{i}\right)
$$

where

$$
s:=\operatorname{deg}\left(H_{1}\right)-1 \text { and } \omega:=\mathrm{e}^{2 \pi \mathrm{i} / s} .
$$

- The symplectic form

$$
\sum_{i=1}^{n} \mathrm{~d} q_{i} \wedge \mathrm{~d} p_{i}+\sum_{i=1}^{n} \mathrm{~d} z_{i} \wedge \mathrm{~d} H_{i}
$$

is also invariant under the same $\mathbb{Z}_{s}$-action, for which $H_{i} \mapsto \omega^{\operatorname{deg}\left(H_{i}\right)} H_{i}$.
We decompose the Hamiltonian function $H_{i}$ into the principal part $H_{i}^{P}$ and the non-principal part $H_{i}^{N}$. Then, we further deduce

- The non-principal part $H_{i}^{N}$ consists of monomials including arbitrary parameters.
- $\operatorname{deg}\left(H_{i}^{N}\right)=\operatorname{deg}\left(H_{i}\right)-\operatorname{deg}\left(H_{1}\right)+1$. In particular $\operatorname{deg}\left(H_{1}^{N}\right)=1$.
- The variety defined by

$$
H_{1}^{P}\left(q_{1}, \ldots, q_{n}, p_{1}, \ldots, p_{n}, 0, \ldots, 0\right)=0
$$

in $\mathbb{C}^{2 n}$ has a unique singularity at the origin.

In Section 2.4, several above properties will be proved from the others. For $\left(\mathrm{P}_{\mathrm{I}}\right),\left(\mathrm{P}_{\mathrm{II}}\right)$ and $\left(\mathrm{P}_{\mathrm{IV}}\right)$, we have

$$
\begin{aligned}
& H_{\mathrm{I}}^{P}(q, p, 0)=\frac{1}{2} p^{2}-2 q^{3} \\
& H_{\mathrm{II}}^{P}(q, p, 0)=\frac{1}{2} p^{2}-\frac{1}{2} q^{4} \\
& H_{\mathrm{IV}}^{P}(q, p, 0)=-p q^{2}+p^{2} q
\end{aligned}
$$

They define $A_{2}, A_{3}$ and $D_{4}$ singularities at the origin, respectively. In singularity theory, it is known that if a singularity defined by a quasihomogeneous polynomial $H\left(x_{1}, \ldots, x_{n}\right)=0$ is isolated, then the rational function

$$
\begin{equation*}
\chi(T):=\frac{\left(T^{h-a_{1}}-1\right) \cdots\left(T^{h-a_{n}}-1\right)}{\left(T^{a_{1}}-1\right) \cdots\left(T^{a_{n}}-1\right)} \tag{1.8}
\end{equation*}
$$

becomes a polynomial (Poincaré polynomial), where $\operatorname{deg}\left(x_{i}\right)=a_{i}$ and $\operatorname{deg}(H)=h$.

Motivated by these observation, we classify regular weights, a tuple of integers $\left(a_{1}, \ldots, a_{n}, b_{1}, \ldots, b_{n} ; h\right)$ satisfying certain conditions in Section 2. In particular, for $n=1$ and 2 , we will show that there is a corresponding Painlevé equation for each weight such that $\operatorname{deg}\left(q_{i}\right)=a_{i}, \operatorname{deg}\left(p_{i}\right)=b_{i}$ and $\operatorname{deg}(H)=h$. In Section 2.4, a Hamiltonian system, whose Hamiltonian function satisfies certain assumptions on the quasihomogeneity, will be considered. Then, some of the above properties of weights will be proved.

In Section 3, a brief review of Kovalevskaya exponents of quasihomogeneous vector fields, which seems to be closely related to regular weights, is given. A list of Kovalevskaya exponents of 4-dim Painlevé equations are shown in Table 3.1. From the table, it is expected that Painlevé equations defined by semi-quasihomogeneous Hamiltonian functions can be classified by their weights and Kovalevskaya exponents.

In Section 4, Kovalevskaya exponents of quasihomogeneous systems are further studied by means of singularity theory and dynamical systems theory. In general, the level surface of quasihomogeneous Hamiltonian functions has a singularity at the origin. The weighted blow-up of the singularity at the origin induces a vector field on the exceptional divisor. Then, Laurent series solutions, Kovalevskaya exponents and the level surface are investigated via the vector field. In particular, it is shown that there is a one-to-one correspondence between Laurent series solutions and fixed points of the vector field, and the eigenvalues of the Jacobi matrix of the vector field at the fixed point precisely coincide with Kovalevskaya exponents. With the aid of these results, it is shown for several 4-dim Painlevé equations
that the level surface of Hamiltonian functions can be decomposed into a disjoint sum of stable manifolds of the fixed points.

Table 1.1. $\operatorname{deg}(H)$ denotes the weighted degree of the Hamiltonian function with respect to the weight $\operatorname{deg}(q, p, z) . \kappa$ denotes the Kovalevskaya exponent defined in Section 3.

|  | $\operatorname{deg}(q, p, z)$ | $\operatorname{deg}(H)$ | $\kappa$ |
| :---: | :---: | :---: | :---: |
| $\mathrm{P}_{\mathrm{I}}$ | $(2,3,4)$ | 6 | 6 |
| $\mathrm{P}_{\mathrm{II}}$ | $(1,2,2)$ | 4 | 4 |
| $\mathrm{P}_{\mathrm{IV}}$ | $(1,1,1)$ | 3 | 3 |
| $\mathrm{P}_{\mathrm{III}\left(D_{8}\right)}$ | $(-1,2,4)$ | 2 | 2 |
| $\mathrm{P}_{\mathrm{III}\left(D_{7}\right)}$ | $(-1,2,3)$ | 2 | 2 |
| $\mathrm{P}_{\mathrm{III}\left(D_{6}\right)}$ | $(0,1,2)$ | 2 | 2 |
| $\mathrm{P}_{\mathrm{V}}$ | $(1,0,1)$ | 2 | 2 |
| $\mathrm{P}_{\mathrm{VI}}$ | $(1,0,0)$ | 2 | 2 |

Table 1.2. Weights for four classes of the Painlevé hierarchies.

|  | $\operatorname{deg}\left(q_{j}, p_{j}\right)$ | $\operatorname{deg}\left(z_{i}\right)$ | $\operatorname{deg}\left(H_{i}\right)$ |
| :---: | :---: | :---: | :---: |
| $\left(\mathrm{P}_{\mathrm{I}}\right)_{n}$ | $(2 j, 2 n+3-2 j)$ | $2 n-2 i+4$ | $2 n+2 i+2$ |
| $\left(\mathrm{P}_{\mathrm{II}-1}\right)_{n}$ | $(2 j-1,2 n+2-2 j)$ | $2 n-2 i+2$ | $2 n+2 i$ |
| $\left(\mathrm{P}_{\mathrm{II}-2}\right)_{n}$ | $(j, n+2-j)$ | $n-i+2$ | $n+i+2$ |
| $\left(\mathrm{P}_{\mathrm{IV}}\right)_{n}$ | $(j, n+1-j)$ | $n-i+1$ | $n+i+1$ |

Table 1.3. Weightsfor four classes of the Painlevé hierarchies when $n=2,3$, where $\operatorname{deg}\left(q_{j}\right), \operatorname{deg}\left(p_{j}\right)$ 's are shown in ascending order.

|  | $\left\{\operatorname{deg}\left(q_{j}\right), \operatorname{deg}\left(p_{j}\right)\right\}$ | $\operatorname{deg}\left(z_{i}\right)$ | $\operatorname{deg}\left(H_{i}\right)$ |
| :---: | :---: | :---: | :---: |
| $\left(\mathrm{P}_{\mathrm{I}}\right)_{2}$ | $(2,3,4,5)$ | 6,4 | 8,10 |
| $\left(\mathrm{P}_{\mathrm{I}}\right)_{3}$ | $(2,3,4,5,6,7)$ | $8,6,4$ | $10,12,14$ |
| $\left(\mathrm{P}_{\mathrm{II}-1}\right)_{2}$ | $(1,2,3,4)$ | 4,2 | 6,8 |
| $\left(\mathrm{P}_{\mathrm{II}-}\right)_{3}$ | $(1,2,3,4,5,6)$ | $6,4,2$ | $8,10,12$ |
| $\left(\mathrm{P}_{\mathrm{II}-2}\right)_{2}$ | $(1,2,2,3)$ | 3,2 | 5,6 |
| $\left(\mathrm{P}_{\mathrm{II}-2}\right)_{3}$ | $(1,2,2,3,3,4)$ | $4,3,2$ | $6,7,8$ |
| $\left(\mathrm{P}_{\mathrm{IV}}\right)_{2}$ | $(1,1,2,2)$ | 2,1 | 4,5 |
| $\left(\mathrm{P}_{\mathrm{IV}}\right)_{3}$ | $(1,1,2,2,3,3)$ | $3,2,1$ | $5,6,7$ |

## 2. Classification of regular weights

Let $a_{1}, \ldots, a_{n}, b_{1}, \ldots, b_{n}$ and $h$ be positive integers such that $1 \leqslant a_{i}, b_{i}<$ $h$. Motivated by the observation in Section 1, we suppose the following.
(W1) $\min _{1 \leqslant i \leqslant n}\left\{a_{i}, b_{i}\right\}=1$ or 2 .
(W2) $a_{i}+b_{i}=h-1$ for $i=1, \ldots, n$.
(W3) A function

$$
\begin{equation*}
\chi(T)=\frac{\left(T^{h-a_{1}}-1\right)\left(T^{h-b_{1}}-1\right) \cdots\left(T^{h-a_{n}}-1\right)\left(T^{h-b_{n}}-1\right)}{\left(T^{a_{1}}-1\right)\left(T^{b_{1}}-1\right) \cdots\left(T^{a_{n}}-1\right)\left(T^{b_{n}}-1\right)} \tag{2.1}
\end{equation*}
$$

is polynomial.
In Saito [29], a tuple of integers $\left(a_{1}, \ldots, a_{n}, b_{1}, \cdots, b_{n} ; h\right)$ satisfying (W3) is called a regular weight. In this paper, a tuple is called a regular weight if it satisfies (W1) to (W3). In this section, we will classify all regular weights for $n=1,2,3$. In particular, for $n=1$ and $n=2$, we will show that there are Hamiltonians of Painlevé equations associated with regular weights such that $\operatorname{deg}\left(q_{i}\right)=a_{i}, \operatorname{deg}\left(p_{i}\right)=b_{i}$ and $\operatorname{deg}(H)=h$.

## 2.1. $n=1$

Proposition 2.1. - When $n=1$, regular weights satisfying (W1) to (W3) are only

$$
(a, b ; h)=(2,3 ; 6), \quad(1,2 ; 4), \quad(1,1 ; 3)
$$

They coincide with the weights $(\operatorname{deg}(q), \operatorname{deg}(p) ; \operatorname{deg}(H))$ of $H_{\mathrm{I}}, H_{\mathrm{II}}$ and $H_{\mathrm{IV}}$ for 2-dim Painlevé equations, respectively, given in Section 1.

Hence, there is a one to one correspondence between regular weights and the 2-dim Painlevé equations written in polynomial Hamiltonians. Note that $\operatorname{deg}(z)$ is recovered by the rule $\operatorname{deg}(z)=\operatorname{deg}(H)-2$ (see also Proposition 2.5). Now we show that $H_{\mathrm{I}}, H_{\text {II }}$ and $H_{\text {IV }}$ can be reconstructed from the regular weights with the aid of singularity theory.

Step 1. - Consider generic polynomials $H(q, p)$ whose weighted degrees are $\operatorname{deg}(q, p ; H)=(2,3 ; 6),(1,2 ; 4)$ and $(1,1 ; 3)$. They are given by

$$
\begin{aligned}
& H=c_{1} p^{2}+c_{2} q^{3} \\
& H=c_{1} p^{2}+c_{2} q^{2} p+c_{3} q^{4} \\
& H=c_{1} q^{3}+c_{2} p q^{2}+c_{3} p^{2} q+c_{4} p^{3}
\end{aligned}
$$

respectively, with arbitrary constants $c_{1}, \ldots, c_{4}$.

Step 2. - Simplify by symplectic transformations. One of the results are

$$
\begin{aligned}
& H=\frac{1}{2} p^{2}-2 q^{3} \\
& H=\frac{1}{2} p^{2}-\frac{1}{2} q^{4} \\
& H=-p q^{2}+p^{2} q
\end{aligned}
$$

respectively.

Step 3. - Consider the versal deformations of them [3]. We obtain

$$
\begin{aligned}
& H=\frac{1}{2} p^{2}-2 q^{3}+\alpha_{4} q+\alpha_{6} \\
& H=\frac{1}{2} p^{2}-\frac{1}{2} q^{4}+\alpha_{2} q^{2}+\alpha_{3} q+\alpha_{4} \\
& H=-p q^{2}+p^{2} q+\alpha_{1} p q+\alpha_{2} p+\beta_{2} q+\alpha_{3}
\end{aligned}
$$

respectively, where $\alpha_{i}, \beta_{i} \in \mathbb{C}$ are deformation parameters. The subscripts $i$ of $\alpha_{i}, \beta_{i}$ denote the weighted degrees of $\alpha_{i}, \beta_{i}$ so that $H$ becomes a quasihomogeneous.

Step 4. - Now we use the ansatz $\operatorname{deg}(z)=\operatorname{deg}(H)-2$ observed in Section 1. If there is a parameter $\alpha_{i}$ such that $i=\operatorname{deg}(H)-2$, then replace it by $z$. The results are

$$
\begin{aligned}
& H=\frac{1}{2} p^{2}-2 q^{3}+z q+\alpha_{6} \\
& H=\frac{1}{2} p^{2}-\frac{1}{2} q^{4}+z q^{2}+\alpha_{3} q+\alpha_{4} \\
& H=-p q^{2}+p^{2} q+z p q+\alpha_{2} p+\beta_{2} q+\alpha_{3},
\end{aligned}
$$

respectively. They are equivalent to $H_{\mathrm{I}}, H_{\mathrm{II}}$ and $H_{\mathrm{IV}}$ up to the scaling of $z$ (constant terms in Hamiltonians such as $\alpha_{6}$ do not play a role).

Hence, when $n=1$, there is a one to one correspondence between the regular weights and 2-dim polynomial Painlevé equations, and we can recover one of them from the other.

## 2.2. $n=2$

Proposition 2.2. - When $n=2$, regular weights satisfying (W1) to (W3) are only

$$
\begin{aligned}
\left(a_{1}, a_{2}, b_{2}, b_{1} ; h\right)= & (2,3,4,5 ; 8), \\
& (1,2,3,4 ; 6), \\
& (2,2,3,3 ; 6), \\
& (1,2,2,3 ; 5), \\
& (1,1,2,2 ; 4), \\
& (1,1,1,1 ; 3),
\end{aligned}
$$

where we assume without loss of generality that $a_{1} \leqslant a_{2} \leqslant b_{2} \leqslant b_{1}$. For each weight, there exists a polynomial Hamiltonian of a 4-dim Painlevé equation (not unique).

Explicit forms of Hamiltonian functions are given as follows.

The first Hamiltonian $H_{1}^{9 / 2}$ of $\left(\mathrm{P}_{\mathrm{I}}\right)_{2}$ shown in (1.4) has this weight with $\operatorname{deg}\left(q_{1}, q_{2}, p_{1}, p_{2}\right)=(2,4,5,3)$. Another example is
$H_{\text {Cosgrove }}=-4 p_{1} p_{2}-2 p_{2}^{2} q_{1}-\frac{73}{128} q_{1}^{4}+\frac{11}{8} q_{1}^{2} q_{2}-\frac{1}{2} q_{2}^{2}-q_{1} z-\frac{1}{48}\left(q_{1}+\frac{\alpha}{6}\right) q_{1}^{2} \alpha$.
This Hamiltonian system is derived by a Lie-algebraic method of type $B_{2}$ and can be written in Lax form [8], thus it enjoys the Painlevé property. It seems that it does not appear in the list of 4 -dim Painlevé equations in $[20,21,26]$. If we rewrite the system as the fourth order single equation of $q_{1}=y$, we obtain

$$
\begin{equation*}
y^{\prime \prime \prime \prime}=18 y y^{\prime \prime}+9\left(y^{\prime}\right)^{2}-24 y^{3}+16 z+\alpha y\left(y+\frac{1}{9} \alpha\right) . \tag{2.3}
\end{equation*}
$$

This equation was first given in Cosgrove [11], denoted by F-VI. He conjectured that this equation defines a new Painlevé transcendents (i.e. it is not reduced to known equations).

The first Hamiltonian $H_{1}^{7 / 2+1}$ of $\left(\mathrm{P}_{\mathrm{II}-1}\right)_{2}$ shown in (1.5) has this weight $\operatorname{deg}\left(q_{1}, q_{2}, p_{1}, p_{2}\right)=(1,3,4,2)$. Another example is the matrix Painlevé equation of the first type $H_{\mathrm{I}}^{\text {Mat }}[20,21]$ defined by

$$
\begin{equation*}
H_{\mathrm{I}}^{\mathrm{Mat}}=\frac{1}{2} p_{1}^{2}-2 q_{1}^{3}-2 p_{2}^{2} q_{2}+6 q_{1} q_{2}-2 q_{1} z+2 \alpha p_{2} \tag{2.4}
\end{equation*}
$$

with $\operatorname{deg}\left(q_{1}, q_{2}, p_{1}, p_{2}\right)=(2,4,3,1)$.

For $H_{1}^{7 / 2+1}$ and $H_{2}^{7 / 2+1}$ of $\left(\mathrm{P}_{\mathrm{II}-1}\right)_{2}$ shown in (1.5), perform the symplectic transformation

$$
\begin{equation*}
q_{1}=-\frac{y_{1}}{2 x_{1}}, p_{1}=-x_{1}^{2}, q_{2}=\frac{y_{2}}{2}, p_{2}=2 x_{2} \tag{2.5}
\end{equation*}
$$

Then we obtain the Hamiltonians

$$
\left\{\begin{align*}
H_{1}^{(2,3,2,3)} & =-4 x_{1}^{2} x_{2}-8 x_{2}^{3}+\frac{y_{1}^{2}}{4}+\frac{y_{2}^{2}}{4}-2 z_{1} x_{2}-z_{2} x_{1}^{2}-\frac{\alpha y_{1}}{x_{1}}  \tag{2.6}\\
H_{2}^{(2,3,2,3)} & =-x_{1}^{4}-4 x_{1}^{2} x_{2}^{2}-\frac{x_{2} y_{1}^{2}}{2}+\frac{x_{1} y_{1} y_{2}}{2}-z_{1} x_{1}^{2} \\
& -z_{2}^{2} x_{1}^{2}-2 z_{2} x_{1}^{2} x_{2}+\frac{z_{2} y_{1}^{2}}{4}-\frac{\alpha z_{2} y_{1}}{x_{1}}+\frac{2 \alpha x_{2} y_{1}}{x_{1}}-\alpha y_{2}
\end{align*}\right.
$$

Thus, putting $\alpha=0$ yields semi-quasihomogeneous Hamiltonians of $\operatorname{deg}\left(H_{1}^{(2,3,2,3)}, H_{2}^{(2,3,2,3)}\right)=(6,8)$ with respect to $\operatorname{deg}\left(x_{1}, y_{1}, x_{2}, y_{2}\right)=$ $(2,3,2,3)$ and $\operatorname{deg}\left(z_{1}, z_{2}\right)=(4,2)$. Although this is equivalent to $\left(\mathrm{P}_{\mathrm{II}-1}\right)_{2}$ for $\alpha=0$, they should be distinguished from each other from a viewpoint of a geometric classification of Painlevé equations (i.e. a classification based on the spaces of initial conditions) because the above symplectic transformation is not a one-to-one mapping. The direct product of two ( $\mathrm{P}_{\mathrm{I}}$ ) also has this weight, see Example 4.14.

The first Hamiltonian $H_{1}^{5}$ of $\left(\mathrm{P}_{\mathrm{II}-2}\right)_{2}$ shown in (1.6) has this weight with $\operatorname{deg}\left(q_{1}, q_{2}, p_{1}, p_{2}\right)=(1,2,3,2)$.

The first Hamiltonian $H_{1}^{4+1}$ of $\left(\mathrm{P}_{\mathrm{IV}}\right)_{2}$ shown in (1.7) has this weight with $\operatorname{deg}\left(q_{1}, q_{2}, p_{1}, p_{2}\right)=(1,1,2,2)$. Another example is the matrix Painlevé equation of the second type $H_{\mathrm{II}}^{\text {Mat }}[20,21]$ defined by

$$
\begin{equation*}
H_{\mathrm{II}}^{\mathrm{Mat}}=\frac{1}{2} p_{1}^{2}-p_{1} q_{1}^{2}+p_{1} q_{2}-2 p_{2}^{2} q_{2}-4 p_{2} q_{1} q_{2}-p_{1} z+2 \alpha p_{2}+2 \beta\left(p_{2}+q_{1}\right) \tag{2.7}
\end{equation*}
$$ with $\operatorname{deg}\left(q_{1}, q_{2}, p_{1}, p_{2}\right)=(1,2,2,1)$. The direct product of two ( $\mathrm{P}_{\text {II }}$ ) also has this weight.

The Noumi-Yamada system of type $A_{4}[21,27]$ defined by

$$
\begin{align*}
H_{\mathrm{NY}}^{A_{4}}=2 p_{1} p_{2} q_{1}+p_{1} q_{1}\left(p_{1}-q_{1}-z\right)+ & p_{2} q_{2}\left(p_{2}-q_{2}-z\right)  \tag{2.8}\\
& +\alpha p_{1}+\beta q_{1}+\gamma p_{2}+\delta q_{2}
\end{align*}
$$

has the weight $\operatorname{deg}\left(q_{1}, q_{2}, p_{1}, p_{2}\right)=(1,1,1,1)$, where $\alpha, \beta, \gamma, \delta$ are arbitrary parameters. The direct product of two $\left(\mathrm{P}_{\mathrm{IV}}\right)$ also has this weight.

$$
\text { 2.3. } n=3
$$

To determine all regular weights satisfying (W1) to (W3), the following lemma is useful. Without loss of generality, we assume $a_{1} \leqslant a_{2} \leqslant \cdots \leqslant$ $a_{n} \leqslant b_{n} \leqslant \cdots \leqslant b_{2} \leqslant b_{1}$. There exist integers $N$ and $j(1), \ldots, j(N)$ such that

$$
\begin{aligned}
& a_{1}=\cdots=a_{j(1)}<a_{j(1)+1}=\cdots=a_{j(2)}<\cdots<a_{j(N)+1}=\cdots=a_{n} \\
& \leqslant b_{n}=\cdots=b_{j(N)+1}<\cdots<b_{j(2)}=\cdots=b_{j(1)+1}<b_{j(1)}=\cdots=b_{1} .
\end{aligned}
$$

We put $J_{l}=j(l)-j(l-1)(l=1, \ldots, N+1)$, where $j(0)=0$ and $j(N+1)=n$.

## Lemma 2.3 .

(i) When $N=0$ (i.e. $a_{1}=a_{n}$ ), then

$$
\begin{aligned}
\left(a_{1}, \ldots, a_{n}, b_{n}, \ldots, b_{1} ; h\right) & =(1, \ldots, 1,1, \ldots, 1 ; 3) \\
& =(1, \ldots, 1,2, \ldots, 2 ; 4) \\
& =(2, \ldots, 2,3, \ldots, 3 ; 6)
\end{aligned}
$$

(ii) When $N \geqslant 1$, the equality $b_{j(i)}=b_{j(i+1)}+1$ holds for $i=1, \ldots, N$ and $J_{i+1} \geqslant J_{i}$ holds for $i=1, \ldots, N-1$. If $a_{n} \neq b_{n}$, further $b_{n}=a_{n}+1$ and $J_{N+1} \geqslant J_{N}$ hold.
(iii) If $a_{i}<a_{i+1}$ for any $i=1, \ldots, n-1$, then

$$
\begin{aligned}
\left(a_{1}, \ldots, a_{n}, b_{n}, \ldots, b_{1} ; h\right) & =(1, \ldots, n, n, \ldots, 2 n-1 ; 2 n+1) \\
& =(1, \ldots, n, n+1, \ldots, 2 n ; 2 n+2) \\
& =(2, \ldots, n+1, n+2, \ldots, 2 n+1 ; 2 n+4)
\end{aligned}
$$

Proof. - Because of (W2), (2.1) is rewritten as

$$
\begin{equation*}
\chi(T)=\frac{\left(T^{a_{1}+1}-1\right) \cdots\left(T^{a_{n}+1}-1\right)\left(T^{b_{n}+1}-1\right) \cdots\left(T^{b_{1}+1}-1\right)}{\left(T^{a_{1}}-1\right) \cdots\left(T^{a_{n}}-1\right)\left(T^{b_{n}}-1\right) \cdots\left(T^{b_{1}}-1\right)} \tag{2.9}
\end{equation*}
$$

(i). - In this case, $a_{1}=a_{n} \leqslant b_{n}=b_{1}$ due to (W2), which implies

$$
\chi(T)=\frac{\left(T^{a_{1}+1}-1\right)^{n}\left(T^{b_{1}+1}-1\right)^{n}}{\left(T^{a_{1}}-1\right)^{n}\left(T^{b_{1}}-1\right)^{n}}
$$

Since it is polynomial, either $b_{1}+1$ or $a_{1}+1$ is a multiple of $b_{1}$. If $b_{1} m=b_{1}+1$, then $\left(m, b_{1}\right)=(2,1)$ and we obtain $\left(a_{1}, \ldots, a_{n}, b_{n}, \ldots, b_{1}\right)=$ $(1, \cdots 1,1, \ldots, 1)$. If $a_{1}=b_{1}$, the same result is obtained. Now suppose that $b_{1} m=a_{1}+1<b_{1}+1$. It is easy to verify that $m=1$ and $b_{1}=a_{1}+1$. Then,

$$
\chi(T)=\frac{\left(T^{a_{1}+2}-1\right)^{n}}{\left(T^{a_{1}}-1\right)^{n}}
$$

Since $a_{1}+2$ is a multiple of $a_{1}$, we have $a_{1} m=a_{1}+2$. This provides $a_{1}=1$ or 2 (we need not use (W1)).
(ii). - In what follows, we suppose that $b_{1}>1$. In this case, $b_{j}>1$ for any $j=1, \ldots, n$ due to the assumption $1 \leqslant a_{1} \leqslant \cdots \leqslant a_{n} \leqslant b_{n} \leqslant \cdots \leqslant b_{1}$ and (W2).

Step 1. - Since $\chi(T)$ is polynomial, there is a multiple of $b_{j(1)}$ among exponents $b_{j(l)}+1$ in the numerator. If $b_{j(1)} m=b_{j(1)}+1$, then $\left(m, b_{j(1)}\right)=$ $(2,1)$ and it contradicts the assumption $b_{j(1)}=b_{1}>1$.

If $b_{j(1)} m=b_{j(l)}+1<b_{j(1)}+1$ for some $l>1$, it is easy to verify $m=1, l=2$ and $b_{j(1)}=b_{j(2)}+1$. There are $J_{1}$ factors $T^{b_{j(1)}}-1$ in the denominator. This implies that $2 J_{2} \geqslant J_{1}$ when $N=1$ and $a_{n}=b_{n}$, and $J_{2} \geqslant J_{1}$ otherwise.

Step 2. - Now we assume that for some $r \leqslant N, b_{j(i)}=b_{j(i+1)}+1$ holds for $i=1, \ldots, r-1$. There exists a multiple of $b_{j(r)}$ among $b_{j(l)}+1$. If $l \leqslant r$, we have

$$
b_{j(r)} m=b_{j(l)}+1=b_{j(l+1)}+2=\cdots=b_{j(r)}+r-l+1,
$$

which yields

$$
1<b_{j(r)} \leqslant r-l+1 \leqslant r
$$

This proves $b_{j(r)}=b_{n}=a_{n}=r$ (otherwise, $a_{1}$ becomes nonpositive). Hence, $r=N+1$, which contradicts the assumption $r \leqslant N$.

If $b_{j(r)} m=b_{j(l)}+1$ for some $l>r$, then $m=1, l=r+1$ and $b_{j(r)}=$ $b_{j(r+1)}+1$. There are $J_{r}$ factors $T^{b_{j(r)}}-1$ in the denominator. This implies that $2 J_{r+1} \geqslant J_{r}$ when $r=N$ and $a_{n}=b_{n}$, and $J_{r+1} \geqslant J_{r}$ otherwise.

Step 3. - By induction, we obtain $b_{j(i)}=b_{j(i+1)}+1$ for $i=1, \ldots, N$, and $J_{i+1} \geqslant J_{i}$ for $i=1, \ldots, N-1$. In particular, if $a_{n} \neq b_{n}, J_{N+1} \geqslant J_{N}$ also holds.

Step 4. - There exists a multiple of $b_{j(N+1)}=b_{n}$ among exponents of the numerator. Suppose $b_{j(N+1)} m=b_{j(l)}+1$ for some $l=1, \ldots, N+1$. The same argument as Step 2 shows that $a_{n}=b_{n}$. Suppose $b_{j(N+1)} m=a_{j(l)}+$ $1<b_{j(N+1)}+1$ for some $l=1, \ldots, N+1$. Then, we obtain $m=1, l=N+1$ and $b_{j(N+1)}=b_{n}=a_{n}+1$. This completes the proof of (ii).
(iii). - This is verified by a direct calculation with the aid of (ii).

Proposition 2.4. - When $n=3$, regular weights satisfying (W1) to (W3) are only

$$
\begin{aligned}
\left(a_{1}, a_{2}, a_{3}, b_{3}, b_{2}, b_{1} ; h\right)= & (2,3,4,5,6,7 ; 10) \\
& (2,3,3,4,4,5 ; 8) \\
& (1,2,3,4,5,6 ; 8) \\
& (1,2,3,3,4,5 ; 7) \\
& (2,2,2,3,3,3 ; 6) \\
& (1,2,2,3,3,4 ; 6) \\
& (1,1,2,2,3,3 ; 5) \\
& (1,1,1,2,2,2 ; 4) \\
& (1,1,1,1,1,1 ; 3)
\end{aligned}
$$

where we assume without loss of generality that $a_{1} \leqslant a_{2} \leqslant a_{3} \leqslant b_{3} \leqslant$ $b_{2} \leqslant b_{1}$.

This proposition is easily obtained with the aid of Lemma 2.3 To find corresponding Painlevé equations is a future work. The weights of 6 -dim Painlevé equations $\left(\mathrm{P}_{\mathrm{I}}\right)_{3},\left(\mathrm{P}_{\mathrm{II}-1}\right)_{3},\left(\mathrm{P}_{\mathrm{II}-2}\right)_{3}$ and $\left(\mathrm{P}_{\mathrm{IV}}\right)_{3}$ shown in Table 1.3 are included in Proposition 2.4 The author does not know a Painlevé equation whose Hamiltonian function is semi-quasihomogeneous but its degree does not satisfy (W1) to (W3).

### 2.4. Properties of weights for semi-quasihomogeneous Hamiltonian systems

We gave the definition of a regular weight which is independent of differential equations so far. Now let us consider the $2 n$-dimensional Hamiltonian system

$$
\begin{equation*}
\frac{\mathrm{d} q_{i}}{\mathrm{~d} z}=\frac{\partial H}{\partial p_{i}}, \quad \frac{\mathrm{~d} p_{i}}{\mathrm{~d} z}=-\frac{\partial H}{\partial q_{i}}, \quad i=1, \ldots, n, \tag{2.10}
\end{equation*}
$$

with the Hamiltonian function $H\left(q_{1}, \ldots, q_{n}, p_{1}, \ldots, p_{n}, z\right)$ with a single time variable $z$ for simplicity. We suppose the following.
(A1) $H$ is semi-quasihomogeneous; it is decomposed into two polynomials as $H=H^{P}+H^{N}$. For the principal part $H^{P}$, there exist integers $1 \leqslant a_{i}, b_{i}, r<h$ such that

$$
\begin{equation*}
H^{P}\left(\lambda^{a} q, \lambda^{b} p, \lambda^{r} z\right)=\lambda^{h} H^{P}(q, p, z) \tag{2.11}
\end{equation*}
$$

for any $\lambda \in \mathbb{C}$, where

$$
\lambda^{a} q=\left(\lambda^{a_{1}} q_{1}, \ldots, \lambda^{a_{n}} q_{n}\right)
$$

and

$$
\lambda^{b} p=\left(\lambda^{b_{1}} p_{1}, \ldots, \lambda^{b_{n}} p_{n}\right)
$$

(A2) The Hamiltonian vector field of $H^{P}$ satisfies

$$
\begin{aligned}
\frac{\partial H^{P}}{\partial p_{i}}\left(\lambda^{a} q, \lambda^{b} p, \lambda^{r} z\right) & =\lambda^{1+a_{i}} \frac{\partial H^{P}}{\partial p_{i}}(q, p, z), \\
\frac{\partial H^{P}}{\partial q_{i}}\left(\lambda^{a} q, \lambda^{b} p, \lambda^{r} z\right) & =\lambda^{1+b_{i}} \frac{\partial H^{P}}{\partial q_{i}}(q, p, z) .
\end{aligned}
$$

(A3) The non-principal part satisfies $H^{N}\left(\lambda^{a} q, \lambda^{b} p, \lambda^{r} z\right) \sim o\left(\lambda^{h}\right)$ as $|\lambda| \rightarrow \infty$.
(A4) The Hamiltonian vector field of $H=H^{P}+H^{N}$ is invariant under the $\mathbb{Z}_{s}$ action

$$
\begin{equation*}
\left(q_{j}, p_{j}, z\right) \mapsto\left(\omega^{a_{j}} q_{j}, \omega^{b_{j}} p_{j}, \omega^{r} z\right) \tag{2.12}
\end{equation*}
$$

where $s=h-1$ and $\omega:=\mathrm{e}^{2 \pi \mathrm{i} / s}$.
(A5) The symplectic form $\sum_{j=1}^{n} \mathrm{~d} q_{j} \wedge \mathrm{~d} p_{j}+\mathrm{d} z \wedge \mathrm{~d} H$ is also invariant under the same $\mathbb{Z}_{s}$-action, for which $H \mapsto \omega^{h} H$.

From these assumptions, we will explain some of the properties of weights shown in Section 1.

Remark. - The assumption (A2) is used to define the Kovalevskaya exponents in the next section. In this case, we can construct Laurent series solutions of (2.10) systematically. Due to the assumptions (A1), (A2) and (A5), it is easy to show that the Hamiltonian vector field of $H^{P}$ is invariant under the action (2.12). The assumption (A4) requires that the vector field of $H^{N}$ is also invariant under the action. Then, (2.10) induces a rational differential equation on the weighted projective space $\mathbb{C} P^{2 n+1}(a, b, r, s)[5,6]$.

In what follows, we assume $h \geqslant 3$ (if $h \leqslant 2,(2.10)$ is linear).
Proposition 2.5. - Suppose that (2.10) satisfies (A1) to (A5) and $h \geqslant 3$. Then,
(i) $a_{i}+b_{i}=h-1$ for $i=1, \ldots, n$,
(ii) $r=h-2$,
(iii) $\operatorname{deg}\left(H^{N}\right)=1$,
(iv) if (2.10) is non-autonomous, $\min _{1 \leqslant i \leqslant n}\left\{a_{i}, b_{i}\right\}=1$ or 2 .

Proof.
(i). - The first statement immediately follows from (A1) and (A2).
(ii). - Because of (A5), there exists an integer $N$ such that $r+h=$ $N(h-1)$. Since $r<h$, we obtain $0<r=N(h-1)-h<h$. This yields $h<N /(N-2)$ if $N \neq 2$. This contradicts the assumption $h \geqslant 3$. Therefore, $N=2$, which proves $r=h-2$.
(iii). - Let $q_{1}^{\mu_{1}} \cdots q_{n}^{\mu_{n}} p_{1}^{\nu_{1}} \cdots p_{n}^{\nu_{n}} z^{\eta}$ be a monomial included in $H^{N}$. Due to (A3), the exponents satisfy

$$
0 \leqslant \sum_{i=1}^{n}\left(a_{i} \mu_{i}+b_{i} \nu_{i}\right)+r \eta \leqslant h-1
$$

Further, (A4) implies that there exists an integer $N$ such that

$$
\sum_{i=1}^{n}\left(a_{i} \mu_{i}+b_{i} \nu_{i}\right)+r \eta-a_{j}-b_{j}+r=N(h-1)
$$

This and (i), (ii) give

$$
\sum_{i=1}^{n}\left(a_{i} \mu_{i}+b_{i} \nu_{i}\right)+r \eta=N(h-1)+1
$$

Hence, we obtain $0 \leqslant N(h-1)+1 \leqslant h-1$. This proves $N=0$ and $\sum_{i=1}^{n}\left(a_{i} \mu_{i}+b_{i} \nu_{i}\right)+r \eta=1$.
(iv). - Suppose that $H$ includes $z$. Since $\operatorname{deg}(H)=h$ and $\operatorname{deg}(z)=$ $h-2, z$ is multiplied by a function whose weighted degree is 2 . It exists only when $\min _{1 \leqslant i \leqslant n}\left\{a_{i}, b_{i}\right\}=1$ or 2 .

## 3. Kovalevskaya exponents

Kovalevskaya exponents are the most important invariants of a quasihomogeneous vector field related to the Painlevé test. Here, we give a brief review of properties of them according to [5]. Let us consider the system of differential equations on $\mathbb{C}^{m}$

$$
\begin{equation*}
\frac{\mathrm{d} x_{i}}{\mathrm{~d} z}=f_{i}\left(x_{1}, \ldots, x_{m}, z\right)+g_{i}\left(x_{1}, \ldots, x_{m}, z\right), \quad i=1, \ldots, m \tag{3.1}
\end{equation*}
$$

where $f_{i}$ and $g_{i}$ are polynomials in $\left(x_{1}, \ldots, x_{m}, z\right) \in \mathbb{C}^{m+1}$. We suppose that
(K1) $\left(f_{1}, \ldots, f_{m}\right)$ is a quasihomogeneous vector field satisfying

$$
\begin{equation*}
f_{i}\left(\lambda^{a_{1}} x_{1}, \ldots, \lambda^{a_{m}} x_{m}, \lambda^{r} z\right)=\lambda^{1+a_{i}} f_{i}\left(x_{1}, \ldots, x_{m}, z\right) \tag{3.2}
\end{equation*}
$$

for any $\lambda \in \mathbb{C}$ and $i=1, \ldots, m$, where $\left(a_{1}, \ldots, a_{m}, r\right) \in \mathbb{Z}_{>0}^{m+1}$.
(K2) $\left(g_{1}, \ldots, g_{m}\right)$ satisfies

$$
g_{i}\left(\lambda^{a_{1}} x_{1}, \ldots, \lambda^{a_{m}} x_{m}, \lambda^{r} z\right)=o\left(\lambda^{a_{i}+1}\right), \quad|\lambda| \rightarrow \infty .
$$

Put $f_{i}^{A}\left(x_{1}, \ldots, x_{m}\right):=f_{i}\left(x_{1}, \ldots, x_{m}, 0\right)$ and $f_{i}^{N A}:=f_{i}-f_{i}^{A}$ (i.e. $f_{i}^{A}$ and $f_{i}^{N A}$ are autonomous and nonautonomous parts, respectively). We also consider the truncated system

$$
\begin{equation*}
\frac{\mathrm{d} x_{i}}{\mathrm{~d} z}=f_{i}^{A}\left(x_{1}, \ldots, x_{m}\right), \quad i=1, \ldots, m \tag{3.3}
\end{equation*}
$$

By substituting $x_{i}(z)=c_{i}\left(z-z_{0}\right)^{-a_{i}}$ into the truncated system, we find the following definition.

Definition 3.1.- $A$ root $c=\left(c_{1}, \ldots, c_{m}\right) \in \mathbb{C}^{m}$ of the equation

$$
\begin{equation*}
-a_{i} c_{i}=f_{i}^{A}\left(c_{1}, \ldots, c_{m}\right), \quad i=1, \ldots, m \tag{3.4}
\end{equation*}
$$

is called the indicial locus.
For each indicial locus, $x_{i}(z)=c_{i}\left(z-z_{0}\right)^{-a_{i}}$ is an exact solution of the truncated system for any $z_{0} \in \mathbb{C}$. Due to the assumption (K1), $c=0$ is always an indicial locus, which corresponds to the fixed point at the origin. Usually, we assume $c \neq 0$ for an indicial locus. Considering the variational equation along the exact solution $x_{i}(z)=c_{i}\left(z-z_{0}\right)^{-a_{i}}$ suggests the following definition.

Definition 3.2. - For an indicial locus $c=\left(c_{1}, \ldots, c_{m}\right) \neq 0$, the matrix

$$
\begin{equation*}
K=K(c):=\left\{\frac{\partial f_{i}^{A}}{\partial x_{j}}\left(c_{1}, \ldots, c_{m}\right)+a_{i} \delta_{i j}\right\}_{i, j=1}^{m} \tag{3.5}
\end{equation*}
$$

and its eigenvalues are called the Kovalevskaya matrix and the Kovalevskaya exponents, respectively, of the system (3.1) associated with $c$.

Proposition 3.3 (see $[2,5,16]$ for the detail). - Suppose (K1) and (K2).
(i) -1 is always a Kovalevskaya exponent with the eigenvector $\left(a_{1} c_{1}, \ldots, a_{m} c_{m}\right)^{T}$.
(ii) $\lambda=0$ is a Kovalevskaya exponent associated with $c$ if and only if $c$ is not an isolated root of the equation $-a_{i} c_{i}=f_{i}^{A}\left(c_{1}, \ldots, c_{m}\right)$.
(iii) The Kovalevskaya exponents are invariant under weight preserving diffeomorphisms.

Consider a formal power series solution of (3.1) of the form

$$
\begin{equation*}
x_{i}=c_{i}\left(z-z_{0}\right)^{-a_{i}}+b_{i, 1}\left(z-z_{0}\right)^{-a_{i}+1}+b_{i, 2}\left(z-z_{0}\right)^{-a_{i}+2}+\cdots \tag{3.6}
\end{equation*}
$$

Coefficients $b_{i, j}$ are determined by substituting it into (3.1). The column vector $b_{j}=\left(b_{1, j}, \ldots, b_{m, j}\right)^{T}$ satisfies

$$
\begin{equation*}
(K-j I) b_{j}=\left(\text { a function of } c_{i} \text { and } b_{i, k} \text { with } k<j\right) . \tag{3.7}
\end{equation*}
$$

If a positive integer $j$ is not an eigenvalue of $K, b_{j}$ is uniquely determined. If a positive integer $j$ is an eigenvalue of $K$ and (3.7) has no solutions, we have to introduce a logarithmic term $\log \left(z-z_{0}\right)$ into the coefficient $b_{j}$. In this case, the system (3.1) has no Laurent series solution of the form (3.6) with a given indicial locus $c$. If a positive integer $j$ is an eigenvalue of $K$ and (3.7) has a solution $b_{j}$, then $b_{j}+v$ is also a solution for any eigenvectors $v$. This implies that the power series solution (3.6) includes a free parameter in $\left(b_{1, j}, \ldots, b_{m, j}\right)$. Therefore, if (3.6) represents a $k$-parameter family of formal Laurent series solutions which includes $k-1$ free parameters other than $z_{0}$, at least $k-1$ Kovalevskaya exponents have to be nonnegative integers. Hence, the classical Painlevé test [1, 16, 33] for the necessary condition for the Painlevé property is stated as follows;

## Classical Painlevé test

If the system (3.1) satisfying (K1) and (K2) has the Painlevé property in a sense that any solutions are meromorphic, then there exists an indicial
locus $c=\left(c_{1}, \ldots, c_{m}\right)$ such that all Kovalevskaya exponents except for one -1 are nonnegative integers (such an indicial locus is called principal), and the Kovalevskaya matrix is semisimple. In this case, (3.6) represents an $m$-parameter family of formal Laurent series solutions.

Due to (K1), the system $\mathrm{d} x_{i} / \mathrm{d} t=f_{i}\left(x_{1}, \ldots, x_{m}, z\right)$ is invariant under the $\mathbb{Z}_{s}$ action

$$
\begin{equation*}
\left(x_{1}, \ldots, x_{m}, z\right) \mapsto\left(\omega^{a_{1}} x_{1}, \ldots, \omega^{a_{m}} x_{m}, \omega^{r} z\right), \quad \omega:=\mathrm{e}^{2 \pi \mathrm{i} / s} \tag{3.8}
\end{equation*}
$$

where $s=r+1$. We assume that the full system (3.1) is also invariant under the same action (i.e. the perturbation term $g_{i}$ admit the same $\mathbb{Z}_{s}$ action as $f_{i}$;
(K3) The system (3.1) is invariant under the above $\mathbb{Z}_{s}$ action.
Proposition 3.4. - Suppose (K1) to (K3). If the system (3.1) has a formal power series solution (3.6), then it is a convergent power series on $0<\left|z-z_{0}\right|<\varepsilon$ for some $\varepsilon>0$. In particular, when $g_{i}=0(i=1, \ldots, m)$ this is true without the assumption (K3).

This proposition is shown in [17] for autonomous systems and extended to nonautonomous systems (3.1) in [5] by using the weighted projective space $\mathbb{C} P^{m+1}\left(a_{1}, \ldots, a_{m}, r, s\right)$. The assumption (K1) and (K3) are used to confirm that the system (3.1) induces a rational vector field on the space $\mathbb{C} P^{m+1}\left(a_{1}, \ldots, a_{m}, r, s\right)$. The classical Painlevé test gives the necessary condition that (3.1) has an $m$-parameter family of formal Laurent series solutions. Proposition 3.4 means that if a formal power series solution of the form (3.6) exists, it is convergent. In Chiba [5, Proposition 3.5], the necessary and sufficient condition that (3.1) has a $k$-parameter family of convergent Laurent series solution (3.6) is given under the assumption (K1) to (K3) with the aid of the weighted projective space, Kovalevskaya exponents and the normal form theory of dynamical systems.

For the next theorem, we further assume that
(S) The origin is the only fixed point of the truncated system (3.3), i.e,

$$
\begin{equation*}
f_{i}^{A}\left(x_{1}, \ldots, x_{m}\right)=0 \quad(i=1, \ldots, m) \Rightarrow\left(x_{1}, \ldots, x_{m}\right)=(0, \ldots, 0) \tag{3.9}
\end{equation*}
$$

Theorem 3.5 (see [5]). - If the system (3.1) satisfies (K1), (K2) and (S), any formal Laurent series solutions with a pole at $z=z_{0}$ are of the form (3.6) such that $\left(c_{1}, \ldots, c_{m}\right) \neq(0, \ldots, 0)$. If we further assume (K3), they are convergent (due to Proposition 3.4).

This theorem implies that there does not exist Laurent series solutions $\left(x_{1}(z), \ldots, x_{m}(z)\right)$ of (3.1) such that the order of a pole of $x_{i}$ is larger
than $a_{i}$ for some $i$ (For the proof, ( S ) is essentially used). Furthermore, if $\left(c_{1}, \ldots, c_{m}\right)=0$ (i.e. the orders of a pole of $x_{1}, \ldots, x_{m}$ are smaller than $\left.a_{1}, \ldots, a_{m}\right)$, it should be a local analytic solution. Therefore, the leading term of a Laurent series solution is strictly given by $c_{i}\left(z-z_{0}\right)^{-a_{i}}$ with a given weight $\left(a_{1}, \ldots, a_{m}\right)$ and an indicial locus $\left(c_{1}, \ldots, c_{m}\right) \neq 0$.

In the rest of this section, we consider the semi-quasihomogeneous Hamiltonian system (2.10). If it satisfies (A1) to (A5), then it also satisfies (K1) to (K3) and the above results are applicable. Further, the assumption (S) implies that a singularity of the algebraic variety defined by $\{H=0\}$ is isolated. This fact is used to study a relationship between the Painlevé equations and singularity theory (see (1.8)). The next lemma is well known [4, $16,19]$.

Lemma 3.6. - For a semi-quasihomogeneous Hamiltonian system (2.10) of $\operatorname{deg}(H)=h$ satisfying (A1) and (A2), if $\kappa$ is a Kovalevskaya exponent, so is $\mu$ given by $\kappa+\mu=h-1$. In particular, $h$ is always a Kovalevskaya exponent for any indicial loci.

Example 3.7. - The first Painlevé equation in Hamiltonian form is given by

$$
\left(\mathrm{P}_{\mathrm{I}}\right)\left\{\begin{array}{l}
\frac{\mathrm{d} x}{\mathrm{~d} z}=6 y^{2}+z \\
\frac{\mathrm{~d} y}{\mathrm{~d} z}=x
\end{array}\right.
$$

It satisfies the assumptions (A1) to (A5) as is mentioned with $\left(a_{1}, a_{2} ; h\right)=$ $(3,2 ; 6)$ (Table 1.1). The indicial locus is uniquely given by $\left(c_{1}, c_{2}\right)=$ $(-2,1)$. The associated Laurent series solution is given by
$\binom{x}{y}=\binom{-2}{0} T^{-3}+\binom{0}{1} T^{-2}-\binom{z_{0} / 5}{0} T-\binom{1 / 2}{z_{0} / 10} T^{2}+\binom{A_{6}}{-1 / 6} T^{3}+\cdots$,
where $T=z-z_{0}$ and $A_{6}$ is an arbitrary constant. Lemma 3.6 shows that $h=6$ is a Kovalevskaya exponent. As a result, an arbitrary constant appears in the sixth place from the beginning (i.e. in the coefficient of $T^{-3+h}=T^{3}$.

We give a list of Kovalevskaya exponents of 4-dim Painlevé equations shown in Section 2.2. In Table 3.1, $H_{1}^{9 / 2}, H_{1}^{7 / 2+1}, H_{1}^{5}$ and $H_{1}^{4+1}$ denote the first Hamiltonians of $\left(\mathrm{P}_{\mathrm{I}}\right)_{2},\left(\mathrm{P}_{\mathrm{II}-1}\right)_{2},\left(\mathrm{P}_{\mathrm{II}-2}\right)_{2}$ and $\left(\mathrm{P}_{\mathrm{IV}}\right)_{2}$, respectively, given in Section 1 (this notation is related to the spectral type of a monodromy preserving deformation [21]). For example, $(-1,2,3,6) \times 2$ in Table $3.1 \mathrm{im}-$ plies that there are two indicial loci $c$, for which the associated Kovalevskaya exponents are $\kappa=-1,2,3$ and 6 . Since Kovalevskaya exponents are invariant under weight preserving diffeomorphisms, we can conclude that two

Hamiltonian systems having the same weights are actually different systems if their Kovalevskaya exponents are different from each other.

For a differential equation $\mathrm{d} x / \mathrm{d} z=f(x, z)$ on $(x, z) \in \mathbb{C}^{m+1}$, an $m$-dim manifold $\mathcal{M}(z)$ parameterized by $z$ is called the space of initial conditions if any solutions of the system give global holomorphic sections of the fiber bundle $\mathcal{P}=\{(x, z) \mid x \in \mathcal{M}(z), z \in \mathbb{C}\}$ over $\mathbb{C}$ [28]. In particular, the space of initial conditions exists for a system having the Painlevé property in the sense that any solutions are meromorphic. Many experts believe that the Painlevé equations can be classified by the geometry of the space of initial conditions, which was confirmed for two dimensional Painlevé equations by Sakai [30] and Takano et al. [24, 25, 32]. In Chiba [5], an algorithm to construct the space of initial conditions for semi-quasihomogeneous systems is obtained by the weighted blow-up of the weighted projective space. The weight for the weighted projective space is just the weight of the variables, and the weight for the blow-up is given by Kovalevskaya exponents. This suggests the conjecture that polynomial systems having the Painlevé property can be classified by their weights and Kovalevskaya exponents.

For 2-dim Painlevé equations, we have constructed the Painlevé equations $P_{\mathrm{I}}, P_{\mathrm{II}}$ and $P_{\mathrm{IV}}$ from the weights (Proposition 2.1). In this case, the Kovalevskaya exponent is given by $h$ (Lemma 3.6), which is included in the information of the weight $(a, b ; h)$. For 4-dim Painlevé equations listed in Table 3.1, they are classified by the weights with Kovalevskaya exponents. Thus, the above conjecture looks true at least up to four dimensional quasihomogeneous systems.

As a convenience for readers, we provide a few 4-dim Painlevé equations whose Hamiltonian functions are polynomial, but the weights are not positive integers. Thus, they do not satisfy the assumption (W3).

$$
\begin{gather*}
H_{\mathrm{IV}}^{\mathrm{Mat}}=\frac{p_{1}^{2} q_{1}}{2}-p_{1} q_{1}^{2}+p_{1} q_{2}+2 p_{1} p_{2} q_{2}-4 p_{2} q_{1} q_{2}-2 p_{2}^{2} q_{1} q_{2}  \tag{3.10}\\
-p_{1} q_{1} z-2 p_{2} q_{2} z+2 p_{2} q_{1} \theta_{0}-p_{1} \theta_{1}+2 p_{2} q_{1} \theta_{1} \\
\\
-p_{1} \theta_{2}+2 q_{1} \theta_{2}+2 p_{2} q_{1} \theta_{2}  \tag{3.11}\\
H^{(1,2,1,0)}=-p_{1}^{2} q_{1}-2 p_{1} q_{1}^{2}+2 p_{1} q_{2}-2 p_{1} p_{2} q_{2} \\
-2 p_{2} q_{1} q_{2}+\left(2 p_{1} q_{1}+2 p_{2} q_{2}\right) z+\left(2 \alpha_{2}+2 \beta_{2}\right) q_{1} \\
+  \tag{3.12}\\
+2 \beta_{2} p_{1}+2 \beta_{3} p_{2}, \\
H^{(-1,1,4,2)}=p_{1}-p_{2}^{2}-2 p_{1} q_{1} q_{2}-p_{2} q_{2}^{2}+2 \beta_{3} q_{2}+2 \beta_{5} q_{1}+p_{2} z .
\end{gather*}
$$

The first one $H_{\mathrm{IV}}^{\mathrm{Mat}}$, whose degree is $\operatorname{deg}\left(q_{1}, q_{2}, p_{1}, p_{2} ; h\right)=(1,2,1,0 ; 3)$, is the matrix Painlevé equation of the fourth type $H_{\mathrm{IV}}^{\mathrm{Mat}}[20,21] . H^{(1,2,1,0)}(h=$ 3) and $H^{(-1,1,4,2)}(h=4)$ are obtained in [8] by a Lie algebraic method as well as $H_{\text {Cosgrove. }}$. Although the weights are nonpositive, they still satisfy (W2) and (A1) to (A5). See also Table 3.1.

## 4. Blow-up of quasihomogeneous systems

Let us investigate the role of Kovalevskaya exponents for quasihomogeneous systems from a view point of dynamical systems theory. Since the Kovalevskaya exponents are defined by the autonomous part of a quasihomogeneous system, we consider the following autonomous system

$$
\begin{equation*}
\frac{\mathrm{d} x_{i}}{\mathrm{~d} z}=f_{i}\left(x_{1}, \ldots, x_{m}\right), \quad i=1, \ldots, m \tag{4.1}
\end{equation*}
$$

satisfying the assumptions (K1) and (S) for the weight $\left(a_{1}, \ldots, a_{m}\right) \in \mathbb{Z}_{>0}^{m}$. For an indicial locus $c=\left(c_{1}, \ldots, c_{m}\right) \neq 0 \in \mathbb{C}^{m}$ given as a root of $-a_{i} c_{i}=$ $f_{i}\left(c_{1}, \ldots, c_{m}\right), x_{i}(z)=c_{i} z^{-a_{i}}$ is an exact solution.

We introduce the weighted blow-up $\pi: B \rightarrow \mathbb{C}^{m}$ of the system (4.1) at the origin by the coordinates transformations

$$
\left(\begin{array}{c}
x_{1}  \tag{4.2}\\
x_{2} \\
\vdots \\
x_{m}
\end{array}\right)=\left(\begin{array}{c}
r_{1}^{a_{1}} \\
r_{1}^{a_{2}} X_{2}^{(1)} \\
\vdots \\
r_{1}^{a_{m}} X_{m}^{(1)}
\end{array}\right)=\left(\begin{array}{c}
r_{2}^{a_{1}} X_{1}^{(2)} \\
r_{2}^{a_{2}} \\
\vdots \\
r_{2}^{a_{m}} X_{m}^{(2)}
\end{array}\right)=\cdots=\left(\begin{array}{c}
r_{m}^{a_{1}} X_{1}^{(m)} \\
r_{m}^{a_{2}} X_{2}^{(m)} \\
\vdots \\
r_{m}^{a_{m}}
\end{array}\right)
$$

and the blow-up space $B$ by

$$
B=B_{1} \cup B_{2} \cup \cdots \cup B_{m}, \quad B_{j} \simeq \mathbb{C}^{m} / \mathbb{Z}_{a_{j}}
$$

Here, the space $\mathbb{C}^{m} / \mathbb{Z}_{a_{j}}$ is defined as follows: Let $\left(r_{1}, X_{2}^{(1)}, \ldots, X_{m}^{(1)}\right)$ be the coordinates of $\mathbb{C}^{m}$. Then, $B_{1}$ is defined as a quotient space by the $\mathbb{Z}_{a_{1}}$ action

$$
\begin{align*}
\left(r_{1}, X_{2}^{(1)}, \ldots,\right. & \left.X_{m}^{(1)}\right)  \tag{4.3}\\
& \mapsto\left(\mathrm{e}^{2 \pi \mathrm{i} / a_{1}} r_{1}, \mathrm{e}^{-2 \pi \mathrm{i} a_{2} / a_{1}} X_{2}^{(1)}, \ldots, \mathrm{e}^{-2 \pi \mathrm{i} a_{m} / a_{1}} X_{m}^{(1)}\right)
\end{align*}
$$

and similar for $B_{2}, \ldots, B_{m}$. Let $\pi: B \rightarrow \mathbb{C}^{m}$ be the surjection defined through (4.2). The exceptional divisor

$$
\begin{equation*}
D:=\pi^{-1}(\{0\})=\left\{r_{1}=0\right\} \cup\left\{r_{2}=0\right\} \cup \cdots \cup\left\{r_{m}=0\right\} \subset B \tag{4.4}
\end{equation*}
$$

Table 3.1. Weights and Kovalevskaya exponents $\kappa$ of 4-dim Painlevé equations. The weights for dependent variables $q_{1}, q_{2}, p_{1}, p_{2}$ are shown in ascending order. For example, $(-1,2,3,6) \times 2$ means that there are two indicial loci whose Kovalevskaya exponents are given by $\kappa=$ $-1,2,3,6$.

|  | $\left(a_{1}, a_{2}, b_{2}, b_{1} ; h\right)$ | $\kappa$ |  |
| :--- | :---: | :--- | :--- |
| $H_{1}^{9 / 2}(1.4)$ | $(2,3,4,5 ; 8)$ | $(-1,2,5,8)$ | $\times 1$ |
|  |  | $(-3,-1,8,10)$ | $\times 1$ |
| $H_{\text {Cosgrove }}(2.2)$ | $(2,3,4,5 ; 8)$ | $(-1,3,4,8)$ | $\times 1$ |
|  |  | $(-5,-1,8,12)$ | $\times 1$ |
| $H_{1}^{7 / 2+1}(1.5)$ | $(1,2,3,4 ; 6)$ | $(-1,2,3,6)$ | $\times 2$ |
|  |  | $(-3,-1,6,8)$ | $\times 2$ |
| $H_{\mathrm{I}}^{\text {Mat }}(2.4)$ | $(1,2,3,4 ; 6)$ | $(-1,2,3,6)$ | $\times 2$ |
|  |  | $(-2,-1,6,7)$ | $\times 1$ |
|  |  | $(-7,-1,6,12)$ | $\times 1$ |
| $H_{1}^{(2,3,2,3)}(2.6)$ | $(2,2,3,3 ; 6)$ | $(-1,1,4,6)$ | $\times 1$ |
|  |  | $(-3,-1,6,8)$ | $\times 2$ |
| $H_{1}^{5}(1.6)$ | $(1,2,2,3 ; 5)$ | $(-1,1,3,5)$ | $\times 2$ |
|  |  | $(-2,-1,5,6)$ | $\times 3$ |
| $H_{1}^{4+1}(1.7)$ | $(1,1,2,2 ; 4)$ | $(-1,1,2,4)$ | $\times 3$ |
|  |  | $(-2,-1,4,5)$ | $\times 5$ |
| $H_{\text {II }}^{\text {Mat }}(2.7)$ | $(1,1,2,2 ; 4)$ | $(-1,1,2,4)$ | $\times 3$ |
|  |  | $(-2,-1,4,5)$ | $\times 2$ |
|  |  | $(-5,-1,4,8)$ | $\times 2$ |
|  | $(-1,-1,4,4)$ | $\times 1$ |  |
| $H_{\text {NY }}^{A_{4}}(2.8)$ | $(1,1,1,1 ; 3)$ | $(-1,1,1,3)$ | $\times 5$ |
|  |  | $(-1,-1,3,3)$ | $\times 5$ |
|  |  | $(-3,-1,3,5)$ | $\times 5$ |
| $H_{\text {IV }}^{\text {Mat }}(3.10)$ | $(0,1,1,2 ; 3)$ | $(-1,1,1,3)$ | $\times 2$ |
|  | $(-1,-1,3,3)$ | $\times 3$ |  |
|  |  | $(-2,-1,3,4)$ | $\times 2$ |
|  |  | $(-4,-1,3,6)$ | $\times 3$ |
| $H^{(1,2,1,0)}(3.11)$ | $(0,1,1,2 ; 3)$ | $(-1,1,1,3)$ | $\times 2$ |
|  | $(-2,-1,3,4)$ | $\times 4$ |  |
|  | $(-1,1,2,4 ; 4)$ | $(-1,1,2,4)$ | $\times 2$ |
|  | $(-3,-1,4,6)$ | $\times 2$ |  |

is isomorphic to the $m-1$ dimensional weighted projective space $\mathbb{C} P^{m-1}\left(a_{1}, \ldots, a_{m}\right)$, and $\left.\pi\right|_{B \backslash D}: B \backslash D \rightarrow \mathbb{C}^{m} \backslash\{0\}$ is a diffeomorphism. Since $\left(c_{1}, \ldots, c_{m}\right) \neq(0, \ldots, 0)$, we assume $c_{1} \neq 0$ and denote the first local coordinates $\left(r_{1}, X_{2}^{(1)}, \ldots, X_{m}^{(1)}\right)$ on the chart $B_{1}$ as $\left(r, X_{2}, \ldots, X_{m}\right)$ for simplicity. In this coordinates, (4.1) is written as

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} r}{\mathrm{~d} z}=\frac{1}{a_{1}} r^{2} f_{1}\left(1, X_{2}, \ldots, X_{m}\right) \\
\frac{\mathrm{d} X_{i}}{\mathrm{~d} z}=r f_{i}\left(1, X_{2}, \ldots, X_{m}\right)-\frac{a_{i}}{a_{1}} r X_{i} f_{1}\left(1, X_{2}, \ldots, X_{m}\right), \quad i=2, \ldots, m
\end{array}\right.
$$

A new independent variable $t$ satisfying the relation $\mathrm{d} / \mathrm{d} z=r \cdot \mathrm{~d} / \mathrm{d} t$ is introduced, that results in

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} r}{\mathrm{~d} t}=\frac{1}{a_{1}} r f_{1}\left(1, X_{2}, \ldots, X_{m}\right)  \tag{4.5}\\
\frac{\mathrm{d} X_{i}}{\mathrm{~d} t}=f_{i}\left(1, X_{2}, \ldots, X_{m}\right)-\frac{a_{i}}{a_{1}} X_{i} f_{1}\left(1, X_{2}, \ldots, X_{m}\right), \quad i=2, \ldots, m
\end{array}\right.
$$

We regard it as a vector field on $B_{1}$. The set $\left\{\left(0, X_{2}, \ldots, X_{m}\right)\right\} \subset D$ is an invariant manifold.

Lemma 4.1.
(i) For an indicial locus $\left(c_{1}, \ldots, c_{m}\right)$ of (4.1) with $c_{1} \neq 0$,

$$
\begin{equation*}
\left(r, X_{2}, \ldots, X_{m}\right)=\left(0, c_{1}^{-a_{2} / a_{1}} c_{2}, \ldots, c_{1}^{-a_{m} / a_{1}} c_{m}\right) \tag{4.6}
\end{equation*}
$$

is a fixed point of the vector field (4.5). Conversely, for any fixed point $\left(0, X_{2}, \ldots, X_{m}\right)$ of (4.5) on the divisor, there exists an indicial locus $\left(c_{1}, \ldots, c_{m}\right)$ satisfying (4.6).
(ii) For an indicial locus $c$, the exact solution $x_{i}(z)=c_{i} z^{-a_{i}},(i=$ $1, \ldots, m)$ on the blow-up space converges to the fixed point (4.6) as $z \rightarrow \infty$.

## Proof.

(i). - A fixed point satisfying $r=0$ is given by a root of the equation
(4.7) $a_{1} f_{i}\left(1, X_{2}, \ldots, X_{m}\right)-a_{i} X_{i} f_{1}\left(1, X_{2}, \ldots, X_{m}\right)=0,(i=2, \ldots, m)$.

If there is a root $\left(X_{2}, \ldots, X_{m}\right)$ satisfying $f_{1}\left(1, X_{2}, \ldots, X_{m}\right)=0$, then $f_{i}\left(1, X_{2}, \ldots, X_{m}\right)=0$ for $i=2, \ldots, m$. This contradicts the assumption (S). Thus, there is a number $\lambda \neq 0$ such that (4.7) is equivalent to

$$
\left\{\begin{array}{l}
f_{i}\left(1, X_{2}, \ldots, X_{m}\right)=-a_{i} X_{i} \lambda^{-1}  \tag{4.8}\\
f_{1}\left(1, X_{2}, \ldots, X_{m}\right)=-a_{1} \lambda^{-1}
\end{array}\right.
$$

By using the assumption (K1), we rewrite the above equation as

$$
\begin{aligned}
\lambda^{a_{i}+1} f_{i}\left(1, X_{2}, \ldots, X_{m}\right) & =f_{i}\left(\lambda^{a_{1}}, \lambda^{a_{2}} X_{2}, \ldots, \lambda^{a_{m}} X_{n}\right) \\
& =-\lambda^{a_{i}+1} a_{i} X_{i} \lambda^{-1}=-a_{i} \lambda^{a_{i}} X_{i}
\end{aligned}
$$

for $i=2, \ldots, m$, and

$$
\begin{aligned}
\lambda^{a_{1}+1} f_{1}\left(1, X_{2}, \ldots, X_{m}\right) & =f_{1}\left(\lambda^{a_{1}}, \lambda^{a_{2}} X_{2}, \ldots, \lambda^{a_{m}} X_{n}\right) \\
& =-\lambda^{a_{1}+1} a_{1} \lambda^{-1}=-a_{1} \lambda^{a_{1}}
\end{aligned}
$$

By putting $\lambda^{a_{1}}=c_{1}$ and $\lambda^{a_{i}} X_{i}=c_{i}$, it turns out that (4.8) is equivalent to the equation $-a_{i} c_{i}=f_{i}\left(c_{1}, \ldots, c_{m}\right)$ to determine an indicial locus. A proof of (ii) is straightforward.

Note that the choice of a branch of $c_{1}^{-a_{j} / a_{1}}$ does not matter because of the $\mathbb{Z}_{a_{1}}$ action (4.3). When $c_{1}=0$, there are no fixed points in the chart $B_{1}$ but exists in $B_{j}$ when $c_{j} \neq 0$. In this manner, there is a one-to-one correspondence between indicial loci $c$ and fixed points of the vector field induced on the divisor, denoted by $\mathrm{P}(c)$. If we do not assume ( S ), there is a fixed point of (4.5) on the divisor, which results not from an indicial locus but from a fixed point of (4.1) other than the origin. The next proposition associates the Kovalevskaya exponents with the local dynamics around a fixed point of the vector field.

Proposition 4.2. - Let $\kappa_{1}=-1, \kappa_{2}, \ldots, \kappa_{m}$ be Kovalevskaya exponents of the system (4.1) associated with an indicial locus $c=\left(c_{1}, \ldots, c_{m}\right)$. The eigenvalues of the Jacobi matrix of the vector field (4.5) at the fixed point $\mathrm{P}(c)$ are given by

$$
\begin{equation*}
\lambda_{1}=-c_{1}^{-1 / a_{1}}, \lambda_{2}=c_{1}^{-1 / a_{1}} \kappa_{2}, \ldots, \lambda_{m}=c_{1}^{-1 / a_{1}} \kappa_{m} . \tag{4.9}
\end{equation*}
$$

Hence, the ratio of eigenvalues is the same as that of the Kovalevskaya exponents.

Proof. - Let $K$ be the Kovalevskaya matrix for an indicial locus $c$. Set $v_{1}=a_{1} c_{1}$ and $v_{2}=\left(a_{2} c_{2}, \ldots, a_{m} c_{m}\right)$. Then, $\left(v_{1}, v_{2}\right)^{T}$ is an eigenvector of $K$ associated with $\kappa_{1}=-1$ (Proposition 3.3). Define

$$
P=\left(\begin{array}{cc}
v_{1} & 0 \\
v_{2}^{T} & \mathrm{id}
\end{array}\right), \quad P^{-1}=\left(\begin{array}{cc}
v_{1}^{-1} & 0 \\
-v_{1}^{-1} v_{2}^{T} & \mathrm{id}
\end{array}\right), \quad K=\left(\begin{array}{ll}
K_{1} & K_{2} \\
K_{3} & K_{4}
\end{array}\right) .
$$

We obtain

$$
\left.P^{-1} K P=\left(\begin{array}{cc}
-1 & v_{1}^{-1} K_{2} \\
0 & K_{4}-v_{1}^{-1} v_{2}^{T} K_{2}
\end{array}\right)\right)=:\left(\begin{array}{cc}
-1 & v_{1}^{-1} K_{2} \\
0 & \widetilde{K}
\end{array}\right)
$$

where an $(m-1) \times(m-1)$ matrix $\widetilde{K}=\left(\widetilde{K}_{i j}\right)_{i, j=2}^{m}$ is given by

$$
\begin{equation*}
\widetilde{K}_{i j}=\frac{\partial f_{i}}{\partial x_{j}}(c)-\frac{a_{i} c_{i}}{a_{1} c_{1}} \frac{\partial f_{1}}{\partial x_{j}}(c)+a_{i} \delta_{i j} . \tag{4.10}
\end{equation*}
$$

By the definition, eigenvalues of $\widetilde{K}$ are $\kappa_{2}, \ldots, \kappa_{m}$.

On the other hand, the Jacobi matrix of (4.5) at the fixed point $\mathrm{P}(c)$ is given by

$$
J=\left(\begin{array}{cc}
\lambda_{1} & 0 \\
0 & \widetilde{J}
\end{array}\right)
$$

where

$$
\begin{aligned}
\lambda_{1} & =\frac{1}{a_{1}} f_{1}\left(1, c_{1}^{-a_{2} / a_{1}} c_{2}, \ldots, c_{1}^{-a_{m} / a_{1}} c_{m}\right) \\
& =\frac{1}{a_{1}} c_{1}^{-\left(a_{1}+1\right) / a_{1}} f_{1}\left(c_{1}, \ldots, c_{m}\right)=-c_{1}^{-1 / a_{1}}
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{J}_{i j}= & \frac{\partial f_{i}}{\partial x_{j}}\left(1, c_{1}^{-a_{2} / a_{1}} c_{2}, \ldots, c_{1}^{-a_{m} / a_{1}} c_{m}\right) \\
& \quad-\frac{a_{i}}{a_{1}} X_{i} \frac{\partial f_{1}}{\partial x_{j}}\left(1, c_{1}^{-a_{2} / a_{1}} c_{2}, \ldots, c_{1}^{-a_{m} / a_{1}} c_{m}\right) \\
& \quad-\frac{a_{i}}{a_{1}} f_{1}\left(1, c_{1}^{-a_{2} / a_{1}} c_{2}, \ldots, c_{1}^{-a_{m} / a_{1}} c_{m}\right) \delta_{i j} \\
= & c_{1}^{-\left(a_{i}+1-a_{j}\right) / a_{1}} \frac{\partial f_{i}}{\partial x_{j}}(c)-\frac{a_{i}}{a_{1}} c_{1}^{-a_{i} / a_{1}} c_{i} c_{1}^{-\left(a_{1}+1-a_{j}\right) / a_{1}} \frac{\partial f_{1}}{\partial x_{j}}(c) \\
& \quad-\frac{a_{i}}{a_{1}} c_{1}^{-\left(a_{1}+1\right) / a_{1}} f_{1}(c) \delta_{i j} \\
= & c_{1}^{-1 / a_{1}} c_{1}^{-\left(a_{i}-a_{j}\right) / a_{1}}\left(\frac{\partial f_{i}}{\partial x_{j}}(c)-\frac{a_{i} c_{i}}{a_{1} c_{1}} \frac{\partial f_{1}}{\partial x_{j}}(c)+a_{i} \delta_{i j}\right) .
\end{aligned}
$$

This shows

$$
c_{1}^{1 / a_{1}} \widetilde{J}_{i j}=c_{1}^{-\left(a_{i}-a_{j}\right) / a_{1}} \widetilde{K}_{i j}
$$

Let $\kappa$ be an eigenvalue of $\widetilde{K}$ with the eigenvector $u=\left(u_{2}, \ldots, u_{m}\right)^{T}$ satisfying $\sum \widetilde{K}_{i j} u_{j}=\kappa u_{i}$. Putting $u_{j}=c_{1}^{a_{j} / a_{1}} \widetilde{u}_{j}$ yields

$$
\sum c_{1}^{-\left(a_{i}-a_{j}\right) / a_{1}} \widetilde{K}_{i j} \widetilde{u}_{j}=\kappa \widetilde{u}_{i}
$$

This proves that $\kappa$ is an eigenvalue of the matrix $c_{1}^{1 / a_{1}} \widetilde{J}$.
We turn to the quasihomogeneous Hamiltonian system of degree $m$

$$
\begin{equation*}
\frac{\mathrm{d} q_{i}}{\mathrm{~d} z}=\frac{\partial H}{\partial p_{i}}, \quad \frac{\mathrm{~d} p_{i}}{\mathrm{~d} z}=-\frac{\partial H}{\partial q_{i}}, \quad i=1, \ldots, m \tag{4.11}
\end{equation*}
$$

We assume stronger conditions than (A1) and (A2) as follows.
(H0) There exist polynomials $H=H_{1}, H_{2}, \ldots, H_{k}(1 \leqslant k \leqslant m)$ that commute with respect to the canonical Poisson structure; $\left\{H_{i}, H_{j}\right\}=0$ for $i, j=1, \ldots, k$.
(H1) $H_{i}$ is quasihomogeneous; there exist positive integers $a_{j}, b_{j}$ and $h_{i}$ such that

$$
\begin{equation*}
H_{i}\left(\lambda^{a} q, \lambda^{b} p\right)=\lambda^{h_{i}} H_{i}(q, p), \quad i=1, \ldots, k \tag{4.12}
\end{equation*}
$$

for any $\lambda \in \mathbb{C}$, where

$$
\lambda^{a} q=\left(\lambda^{a_{1}} q_{1}, \ldots, \lambda^{a_{m}} q_{m}\right) \quad \text { and } \quad \lambda^{b} p=\left(\lambda^{b_{1}} p_{1}, \ldots, \lambda^{b_{m}} p_{m}\right) .
$$

(H2) $h_{1}=a_{j}+b_{j}+1$ for $j=1, \ldots, m$ and $h_{1} \leqslant h_{i}$ for $i=1, \ldots, k$.
(S) The origin is the only fixed point;

$$
\frac{\partial H_{1}}{\partial p_{i}}=\frac{\partial H_{1}}{\partial q_{i}}=0(i=1, \ldots, m) \Rightarrow\left(q_{1}, \ldots, q_{m}, p_{1}, \ldots, p_{m}\right)=0
$$

Note that (H1) and (H2) for $k=1$ is equivalent to (A1) and (A2), see Proposition 2.5(i). Let $c=\left(c_{1}, \ldots, c_{2 m}\right)$ be an indicial locus determined by $H_{1}$ as

$$
\begin{equation*}
\frac{\partial H_{1}}{\partial p_{i}}(c)=-a_{i} c_{i}, \quad \frac{\partial H_{1}}{\partial q_{i}}(c)=b_{i} c_{m+i}, \quad i=1, \ldots, m \tag{4.13}
\end{equation*}
$$

The Kovalevskaya matrix at $c$ is
$K(c)=\left(\begin{array}{cc}\frac{\partial^{2} H_{1}}{\partial p \partial q}(c) & \frac{\partial^{2} H_{1}}{\partial p \partial p}(c) \\ -\frac{\partial^{2} H_{1}}{\partial q \partial q}(c) & -\frac{\partial^{2} H_{1}}{\partial q \partial p}(c)\end{array}\right)+\left(\begin{array}{cc}\operatorname{diag}\left(a_{1}, \ldots, a_{m}\right) & 0 \\ 0 & \operatorname{diag}\left(b_{1}, \ldots, b_{m}\right)\end{array}\right)$,
where $\partial^{2} / \partial p \partial q=\left(\partial^{2} / \partial p_{i} \partial q_{j}\right)_{i, j}$.
Lemma 4.3 .

$$
\begin{equation*}
\sum_{j=1}^{m}\left(a_{j} q_{j} \frac{\partial H_{i}}{\partial q_{j}}+b_{j} p_{j} \frac{\partial H_{i}}{\partial p_{j}}\right)=h_{i} H_{i}(q, p) . \tag{4.14}
\end{equation*}
$$

Proof. - This is obtained by the derivative of (4.12) at $\lambda=1$.
Lemma 4.4. - For any $i=1, \ldots, k$ and indicial loci $c$, we have $H_{i}(c)=0$.

Proof. - Use the relations (4.13), (4.14) and $\left\{H_{1}, H_{j}\right\}=0$ at $(q, p)=c$.

In what follows, the gradient of a function $H$ is denoted by a row vector

$$
\begin{equation*}
\mathrm{d} H:=\left(\frac{\partial H}{\partial q_{1}}, \ldots, \frac{\partial H}{\partial q_{m}}, \frac{\partial H}{\partial p_{1}}, \ldots, \frac{\partial H}{\partial p_{m}}\right) . \tag{4.15}
\end{equation*}
$$

The assumption (S) implies that $\mathrm{d} H_{1}(q, p)=0$ if and only if $(q, p)=0$; i.e. the origin is a unique singularity of the variety $\left\{H_{1}=0\right\}$. The following result was first obtained by Yoshida [34]. Here, we give a simple proof.

Theorem 4.5. - For an indicial locus $c$, the following equality

$$
\begin{equation*}
\mathrm{d} H_{j}(c)\left(K(c)-h_{j} \cdot \mathrm{id}_{2 m \times 2 m}\right)=0, \quad j=1, \ldots, k \tag{4.16}
\end{equation*}
$$

holds. In particular, if $\mathrm{d} H_{j}(c) \neq 0$, then $h_{j}$ is a Kovalevskaya exponent.
Proof. - $\left\{H_{1}, H_{j}\right\}=0$ gives

$$
\begin{aligned}
0 & =\frac{\partial}{\partial q_{l}}\left\{H_{1}, H_{j}\right\} \\
& =\sum_{i=1}^{m}\left(\frac{\partial^{2} H_{1}}{\partial q_{i} \partial q_{l}} \frac{\partial H_{j}}{\partial p_{i}}+\frac{\partial H_{1}}{\partial q_{i}} \frac{\partial^{2} H_{j}}{\partial p_{i} \partial q_{l}}-\frac{\partial^{2} H_{j}}{\partial q_{i} \partial q_{l}} \frac{\partial H_{1}}{\partial p_{i}}-\frac{\partial H_{j}}{\partial q_{i}} \frac{\partial^{2} H_{1}}{\partial p_{i} \partial q_{l}}\right) .
\end{aligned}
$$

Substituting $(q, p)=c$ yields

$$
0=\sum_{i=1}^{m}\left(a_{i} c_{i} \frac{\partial^{2} H_{j}}{\partial q_{i} \partial q_{l}}+b_{i} c_{m+i} \frac{\partial^{2} H_{j}}{\partial p_{i} \partial q_{l}}+\frac{\partial^{2} H_{1}}{\partial q_{i} \partial q_{l}} \frac{\partial H_{j}}{\partial p_{i}}-\frac{\partial H_{j}}{\partial q_{i}} \frac{\partial^{2} H_{1}}{\partial p_{i} \partial q_{l}}\right)
$$

By the derivative of (4.14) with respect to $q_{l}$, we obtain

$$
\sum_{i=1}^{m}\left(a_{i} q_{i} \frac{\partial^{2} H_{j}}{\partial q_{i} \partial q_{l}}+b_{i} p_{i} \frac{\partial^{2} H_{j}}{\partial p_{i} \partial q_{l}}\right)+a_{l} \frac{\partial H_{j}}{\partial q_{l}}=h_{j} \frac{\partial H_{j}}{\partial q_{l}} .
$$

Thus, we obtain

$$
\begin{equation*}
0=\sum_{i=1}^{m}\left(\frac{\partial^{2} H_{1}}{\partial q_{i} \partial q_{l}} \frac{\partial H_{j}}{\partial p_{i}}-\frac{\partial^{2} H_{1}}{\partial p_{i} \partial q_{l}} \frac{\partial H_{j}}{\partial q_{i}}\right)+\left(h_{j}-a_{l}\right) \frac{\partial H_{j}}{\partial q_{l}} \tag{4.17}
\end{equation*}
$$

for $l=1, \ldots, m$. The derivative of (4.14) with respect to $p_{l}$ gives similar $m$ relations. The resultant $2 m$ relations are equivalent to (4.16).

Example 4.6. - We consider the Hamiltonians of degree 2

$$
\left\{\begin{array}{l}
H_{1}=2 p_{2} p_{1}+3 p_{2}^{2} q_{1}+q_{1}^{4}-q_{1}^{2} q_{2}-q_{2}^{2}  \tag{4.18}\\
H_{2}=p_{1}^{2}+2 p_{2} p_{1} q_{1}-q_{1}^{5}+p_{2}^{2} q_{2}+3 q_{1}^{3} q_{2}-2 q_{1} q_{2}^{2}
\end{array}\right.
$$

They are autonomous parts of $\left(\mathrm{P}_{\mathrm{I}}\right)_{2}$ given in (1.4). They satisfy (H0), (H1), (H2) and (S) for the weight $\left(a_{1}, a_{2}, b_{1}, b_{2}\right)=(2,4,5,3)$ and $h_{1}=$ $8, h_{2}=10$ shown in Table 1.2. There are two indicial loci $c_{1}=(1,1,1,-1)$ and $c_{2}=(3,0,27,-3)$. For the former $c_{1}$, the Kovalevskaya exponents are $\kappa=-1,2,5,8$. Thus, the corresponding Laurent series solution (3.6) represents a general solution including four free parameters (Painlevé test). Since $\kappa \neq-10$, Theorem 4.5 implies that $\mathrm{d} H_{2}\left(c_{1}\right)=0$. For the indicial locus $c_{2}$, we can verify that $\mathrm{d} H_{1}\left(c_{2}\right) \neq 0, \mathrm{~d} H_{2}\left(c_{2}\right) \neq 0$. Therefore, Theorem 4.5 and Lemma 3.6 show that the Kovalevskaya exponents are given by $\kappa=$ $-3,-1,8,10$ (see $H_{1}^{9 / 2}$ of Table 3.1).

Let $V$ be a variety defined by the level set

$$
\begin{equation*}
V=\left\{(q, p) \in \mathbb{C}^{2 m} \mid H_{j}(q, p)=0, j=1, \ldots, k\right\} \ni 0 \tag{4.19}
\end{equation*}
$$

Lemma 4.4 shows $c \in V$ for an indicial locus $c$. Because of (H1), the orbit

$$
\left\{\left(\lambda^{a_{1}} c_{1}, \ldots, \lambda^{a_{m}} c_{m}, \lambda^{b_{1}} c_{m+1}, \ldots, \lambda^{b_{m}} c_{2 m}\right) \mid \lambda \in \mathbb{C}\right\}
$$

is also included in $V$. Let us consider the weighted blow-up $\pi: B \rightarrow \mathbb{C}^{2 m}$ at the origin

$$
\begin{equation*}
B=B_{1} \cup \cdots \cup B_{2 m}, \quad B_{i}=\mathbb{C}^{2 m} / \mathbb{Z}_{a_{i}}, B_{m+i}=\mathbb{C}^{2 m} / \mathbb{Z}_{b_{i}} \tag{4.20}
\end{equation*}
$$

for $i=1, \ldots, m$. The exceptional divisor $D$ is a $2 m-1$ dimensional weighted projective space

$$
\begin{equation*}
D=\pi^{-1}(\{0\})=\mathbb{C} P^{2 m-1}\left(a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{m}\right) \tag{4.21}
\end{equation*}
$$

For an indicial locus $c$, we assume $c_{1} \neq 0$ as before. The local coordinates $\left(r, Q_{2}, \ldots, Q_{m}, P_{1}, \ldots, P_{m}\right)$ on $B_{1}$ is defined by

$$
\begin{aligned}
& q_{1}=r^{a_{1}}, q_{i}=r^{a_{i}} Q_{i} \quad(i=2, \ldots, m) \\
& p_{i}=r^{b_{i}} P_{i} \quad(i=1, \ldots, m)
\end{aligned}
$$

In particular, $D \cap B_{1}$ is given by the set $\{r=0\}$. The set $\pi^{-1}(V) \subset B$ is a disjoint union of $D$ and $\pi^{-1}(V \backslash\{0\})$. Let $\overline{\pi^{-1}(V \backslash\{0\})}$ be the closure with respect to the usual topology and

$$
V_{0}:=D \cap \overline{\pi^{-1}(V \backslash\{0\})} \subset D
$$

see Figure 4.1. On the chart $B_{1}$, we have $H_{j}(q, p)=r^{h_{j}} H_{j}\left(1, Q_{2}, \ldots, P_{m}\right)$. Hence, define
$V_{01}:=\left\{\left(0, Q_{2}, \ldots, P_{m}\right) \in D \mid H_{j}\left(1, Q_{2}, \ldots, P_{m}\right)=0, j=1, \ldots, k\right\} \subset D \cap B_{1}$.
The sets $V_{0 i}$ on the chart $B_{i}$ are also defined in the same way for $i=$ $2, \ldots, 2 m$. Then, we have

$$
\begin{aligned}
V_{0} & =V_{01} \cup V_{02} \cup \cdots \cup V_{0,2 m} \subset D \\
\pi^{-1}(V) & =D \cup \pi^{-1}(V \backslash\{0\}) \simeq D \cup\left(V_{0} \times \mathbb{C}\right) .
\end{aligned}
$$

$V$ and $V_{0}$ are $2 m-k$ and $2 m-k-1$ dimensional manifolds, respectively, with singularities. As in (4.5), the system (4.11) induces the vector field $\mathcal{X}$ on $B$ after a suitable change of the independent variable. On $B_{1}, \mathcal{X}$ is expressed as

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} r}{\mathrm{~d} t}=\frac{1}{a_{1}} r \frac{\partial H_{1}}{\partial p_{1}}\left(1, Q_{2}, \ldots, P_{m}\right)  \tag{4.22}\\
\frac{\mathrm{d} Q_{i}}{\mathrm{~d} t}=\frac{\partial H_{1}}{\partial p_{i}}\left(1, Q_{2}, \ldots, P_{m}\right)-\frac{a_{i}}{a_{1}} Q_{i} \frac{\partial H_{1}}{\partial p_{1}}\left(1, Q_{2}, \ldots, P_{m}\right) \\
\frac{\mathrm{d} P_{i}}{\mathrm{~d} t}=-\frac{\partial H_{1}}{\partial q_{i}}\left(1, Q_{2}, \ldots, P_{m}\right)-\frac{b_{i}}{a_{1}} P_{i} \frac{\partial H_{1}}{\partial p_{1}}\left(1, Q_{2}, \ldots, P_{m}\right)
\end{array}\right.
$$

## Proposition 4.7.

(i) $D$ is an invariant manifold of $\mathcal{X}$.
(ii) $V_{0} \subset D$ is an invariant manifold of $\mathcal{X}$.
(iii) All fixed points of $\mathcal{X}$ are included in $V_{0}$.
(iv) For an indicial locus $c=\left(c_{1}, \ldots, c_{2 m}\right)$, the orbit of the exact solution $q_{i}(z)=c_{i} z^{-a_{1}}, p_{i}(z)=c_{m+i} z^{-b_{i}}$ of (4.11) is included in $V$. On the blow-up space $B$, it tends to a fixed point $\mathrm{P}(c)$ on $V_{0}$ as $z \rightarrow \infty$.

Proof. - It is sufficient to prove the statements on the chart $B_{1}$. Since $D \cap B_{1}=\{r=0\}$, (i) immediately follows from (4.22). By the assumption (S), all fixed points of $\mathcal{X}$ lie on the divisor $D$. Due to Lemma 4.1(i), a fixed point $\mathrm{P}(c)$ on $D \cap B_{1}$ is of the form $\left(0, c_{1}^{-a_{2} / a_{1}} c_{2}, \ldots, c_{1}^{-b_{m} / a_{1}} c_{2 m}\right)$ for an indicial locus $c$. Then, Lemma 4.4 implies

$$
H_{j}\left(1, c_{1}^{-a_{2} / a_{1}} c_{2}, \ldots, c_{1}^{-b_{m} / a_{1}} c_{2 m}\right)=c_{1}^{-h_{j} / a_{1}} H_{j}(c)=0,
$$

which proves (iii): $\mathrm{P}(c) \in V_{01} \subset V_{0}$. Part (iv) is shown by Lemma 4.1(ii) and Lemma 4.4. Finally, let us show the statement (ii). Along an integral curve of (4.22), we have

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} t} H_{j}\left(1, Q_{2}, \ldots, P_{m}\right) \\
& \quad=\sum_{i=2}^{m} \frac{\partial H_{j}}{\partial q_{i}}\left(\frac{\partial H_{1}}{\partial p_{i}}-\frac{a_{i}}{a_{1}} Q_{i} \frac{\partial H_{1}}{\partial p_{1}}\right)+\sum_{i=1}^{m} \frac{\partial H_{j}}{\partial p_{i}}\left(-\frac{\partial H_{1}}{\partial q_{i}}-\frac{b_{i}}{a_{1}} P_{i} \frac{\partial H_{1}}{\partial p_{1}}\right) .
\end{aligned}
$$

By introducing a dummy parameter $Q_{1}=1$, it is rewritten as

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} t} H_{j}\left(1, Q_{2}, \ldots, P_{m}\right) \\
& \quad=\sum_{i=1}^{m} \frac{\partial H_{j}}{\partial q_{i}}\left(\frac{\partial H_{1}}{\partial p_{i}}-\frac{a_{i}}{a_{1}} Q_{i} \frac{\partial H_{1}}{\partial p_{1}}\right)+\sum_{i=1}^{m} \frac{\partial H_{j}}{\partial p_{i}}\left(-\frac{\partial H_{1}}{\partial q_{i}}-\frac{b_{i}}{a_{1}} P_{i} \frac{\partial H_{1}}{\partial p_{1}}\right) \\
& \quad=\sum_{i=1}^{m}\left(\frac{\partial H_{j}}{\partial q_{i}} \frac{\partial H_{1}}{\partial p_{i}}-\frac{\partial H_{1}}{\partial q_{i}} \frac{\partial H_{j}}{\partial p_{i}}\right)-\frac{1}{a_{1}} \frac{\partial H_{1}}{\partial p_{1}} \sum_{i=1}^{m}\left(a_{i} Q_{i} \frac{\partial H_{j}}{\partial q_{i}}+b_{i} P_{i} \frac{\partial H_{j}}{\partial p_{i}}\right) .
\end{aligned}
$$

Lemma 4.3 and $\left\{H_{1}, H_{j}\right\}=0$ show

$$
\frac{\mathrm{d}}{\mathrm{~d} t} H_{j}\left(1, Q_{2}, \ldots, P_{m}\right)=-\frac{1}{a_{1}} \frac{\partial H_{1}}{\partial p_{1}} h_{j} H_{j}\left(1, Q_{2}, \ldots, P_{m}\right) .
$$

This is a linear equation of $H_{j}\left(1, Q_{2}, \ldots, P_{m}\right)$ solved as
$H_{j}\left(1, Q_{2}(t), \ldots, P_{m}(t)\right)=H_{j}\left(1, Q_{2}(0), \ldots, P_{m}(0)\right) \cdot \exp \left[-\frac{h_{j}}{a_{1}} \int_{0}^{t} \frac{\partial H_{1}}{\partial p_{1}} \mathrm{~d} s\right]$.
This proves that if $\left(0, Q_{2}, \ldots, P_{m}\right) \in V_{01}$ at the initial time $t=0$, so that $H_{j}\left(1, Q_{2}(0), \ldots, P_{m}(0)\right)=0$, then $\left(0, Q_{2}, \ldots, P_{m}\right) \in V_{01}$ for any $t \in \mathbb{R}$.

Fix an indicial locus $c=\left(c_{1}, \ldots, c_{2 m}\right) \neq 0$ with $c_{1} \neq 0$. Without loss of generality we assume that $c_{1}=1$ by a suitable scaling of the independent variable. By Lemma 4.1, the indicial locus associates the fixed point $\mathrm{P}(c)$ : $\left(r, Q_{2}, \ldots, P_{m}\right)=\left(0, c_{2}, \ldots, c_{2 m}\right)$ of the vector field (4.22) on the chart $B_{1}$. Proposition 4.2 shows that the Jacobi matrix at the fixed point written in $\left(r, Q_{2}, \ldots, P_{m}\right)$-coordinates is of the form

$$
J=\left(\begin{array}{cc}
-1 & 0  \tag{4.23}\\
0 & \widetilde{J}
\end{array}\right)
$$

and its eigenvalues coincide with the Kovalevskaya exponents $\kappa_{1}=-1$ and $\kappa_{2}, \ldots, \kappa_{2 m}$. Thus, eigenvectors of $\kappa_{2}, \ldots, \kappa_{2 m}$ are tangent to the divisor $D=\{r=0\}$. Let $\mathbf{E}_{s}, \mathbf{E}_{u}$ and $\mathbf{E}_{c}$ be the stable, unstable and center subspace at the point $\mathrm{P}=\mathrm{P}(c)$, which are eigenspaces of eigenvalues with negative real parts, positive real parts, and zero real parts, respectively. Let $W_{s}(\mathrm{P}), W_{u}(\mathrm{P})$ and $W_{c}(\mathrm{P})$ be a local stable manifold, unstable manifold and center manifold, respectively, which are tangent to $\mathbf{E}_{s}, \mathbf{E}_{u}$ and $\mathbf{E}_{c}$ at P. Because of Lemma $3.6\left(\kappa+\mu=h_{1}-1>0\right), \operatorname{dim} \mathbf{E}_{u} \geqslant m$ and $1 \leqslant \operatorname{dim} \mathbf{E}_{s} \leqslant m$.

Proposition 4.8. - Under the above situation, the unstable manifold $W_{u}(\mathrm{P})$ is included in $D$ and the stable manifold $W_{s}(\mathrm{P})$ is included in $\overline{\pi^{-1}(V \backslash\{0\})} \simeq V_{0} \times \mathbb{C}$. If there are no purely imaginary eigenvalues $(\neq 0)$, $W_{c}(\mathrm{P})$ is included in $V_{0}$.

Proof. - Since $\mathbf{E}_{u}$ is tangent to the divisor $D$ and $D$ is an invariant manifold, $W_{u}(\mathrm{P}) \subset D$. Let $x$ be a point on $W_{s}(\mathrm{P})$ and suppose $x \notin D \cup V$. Since $x \in W_{s}(\mathrm{P})$, a solution of (4.22) with an initial value at $x$ tends to the fixed point P as $t \rightarrow \infty$. Since $x \notin V, H_{j}(x) \neq 0$ for some $j$. This is a contradiction because $H_{j}=0$ at P and $H_{j}$ is a constant along a solution. Let $x^{\prime}$ be a point on $W_{s}(\mathrm{P})$ such that $x^{\prime} \in D \backslash V_{0}$. Then, there is a point $x$ on $W_{s}(\mathrm{P})$ and $x \notin D \cup V$ because the eigenvector of $\kappa_{1}=-1$ is transverse to $D$, that is again a contradiction. If the Kovalevskaya matrix at $c$ has zero eigenvalues, then an indicial locus $c$ is not isolated (Proposition 3.3). Thus, the fixed point $\mathrm{P}(c)$ is not isolated and there exists a neighborhood $U$ of $\mathrm{P}(c)$ such that $U \cap V_{0}$ consists of fixed points of the vector field. If there are no purely imaginary eigenvalues, $U \cap V_{0}$ gives a local center manifold.

Next, we consider the system (4.11) satisfying (H0), (H1), (H2) and (S) with $k=m$. In this case, $V$ and $\overline{\pi^{-1}(V \backslash\{0\})}$ are $m$ dimensional and $V_{0}$ is an $m-1$ dimensional variety with singularities. If the system satisfies the Painlevé property in a sense that any solution is meromorphic, there is an indicial locus $c$ such that all the Kovalevskaya exponents but unique -1
are positive integers (Painlevé test). Thus, a stable manifold at $\mathrm{P}(c)$ is one dimensional, which is precisely given by the orbit of the special solution $q_{i}(z)=c_{i} z^{-a_{i}}, p_{i}(z)=c_{m+i} z^{-b_{i}}$. The next theorem consider the opposite situation.

Theorem 4.9. - Suppose that the system (4.11) satisfies (H0), (H1), (H2) and (S) with $k=m$. Suppose that there exists an indicial locus $c$ such that vectors $\mathrm{d} H_{1}(c), \ldots, \mathrm{d} H_{m}(c)$ are linearly independent. Then, there exists a neighborhood $U$ of $\mathrm{P}(c)$ such that

$$
\overline{\pi^{-1}(V \backslash\{0\})} \cup U=W_{s}(\mathrm{P}(c)) \cup U
$$

Proof. - Theorem 4.5 shows that $h_{1}, \ldots, h_{m}>0$ are Kovalevskaya exponents. Due to Lemma 3.6, negative integers $\mu_{j}:=h_{1}-1-h_{j}, j=$ $1, \ldots, m$ are also Kovalevskaya exponents. Thus, a local stable manifold $W_{s}(\mathrm{P}(c))$ of the vector field (4.22) is an $m$-dimensional smooth manifold included in $\overline{\pi^{-1}(V \backslash\{0\})}$. Indeed, again Theorem 4.5 implies that the (right) eigenvectors of $K(c)$ associated with eigenvalues $\mu_{1}, \ldots, \mu_{m}$ are orthogonal to $\mathrm{d} H_{1}(c), \ldots, \mathrm{d} H_{m}(c)$. Hence, the stable subspace $\mathbf{E}_{s}$ coincides with the tangent space of $\overline{\pi^{-1}(V \backslash\{0\})}$ at $\mathrm{P}(c)$.

An indicial locus $c$ satisfying the assumption of the theorem (that implies $h_{1}, \ldots, h_{m}>0$ are Kovalevskaya exponents), for which $\operatorname{dim} \mathbf{E}_{u}=\operatorname{dim} \mathbf{E}_{s}=$ $m$, is called the lowest indicial locus. The existence of a lowest indicial locus is proved by [13] for a certain class of integrable systems called the hyperelliptically separable systems, while the existence for more general systems is not known. Let us demonstrate our results for several 4-dim systems obtained from the autonomous parts of Painlevé equations. See also Table 4.1. They have lowest indicial loci and $\overline{\pi^{-1}(V \backslash\{0\})}$ is decomposed into the disjoint union of stable manifolds at the fixed points on the divisor.

Example 4.10. - We consider the autonomous part of $\left(\mathrm{P}_{\mathrm{I}}\right)_{2}$ given in Example 4.6. Since the Kovalevskaya exponents of the indicial locus $c_{1}=$ $(1,1,1,-1)$, which corresponds to the principal Laurent series solution, are $-1,2,5,8$, the stable manifold $W_{s}\left(\mathrm{P}_{1}\right)$ at the fixed point $\mathrm{P}\left(c_{1}\right)$ is given by the orbit of the special solution $q_{1}=z^{-2}, q_{2}=z^{-4}, p_{1}=z^{-5}, p_{2}=-z^{-3}$. The Kovalevskaya exponents of the indicial locus $c_{2}=(3,0,27,-3)$, which satisfies the assumptions for Theorem 4.9, are $-3,-1,8,10$. The 2 -dim stable manifold $W_{s}\left(\mathrm{P}_{2}\right)$ locally coincides with $\overline{\pi^{-1}(V \backslash\{0\})}$. In this case, $\overline{\pi^{-1}(V \backslash\{0\})}$ is decomposed into the disjoint union of $W_{s}\left(\mathrm{P}_{1}\right)$ and $W_{s}\left(\mathrm{P}_{2}\right)$, see Figure 4.1.

On the $\left(r, Q_{2}, P_{1}, P_{2}\right)$-coordinates, $H_{1}$ and $H_{2}$ are written as

$$
\begin{aligned}
H_{1} & =r^{8}\left(1+2 P_{1} P_{2}+3 P_{2}^{2}-Q_{2}-Q_{2}^{2}\right) \\
H_{2} & =r^{10}\left(-1+P_{1}^{2}+2 P_{1} P_{2}+3 Q_{2}+P_{2}^{2} Q_{2}-2 Q_{2}^{2}\right)
\end{aligned}
$$

Thus, $V_{01}$ is defined by

$$
V_{01}=\left\{\begin{array}{l}
1+2 P_{1} P_{2}+3 P_{2}^{2}-Q_{2}-Q_{2}^{2}=0, \\
-1+P_{1}^{2}+2 P_{1} P_{2}+3 Q_{2}+P_{2}^{2} Q_{2}-2 Q_{2}^{2}=0
\end{array}\right\} .
$$

Since $\mathrm{d} H_{2}\left(c_{1}\right)=0, V_{0}$ is singular at $\mathrm{P}_{1}:\left(Q_{2}, P_{1}, P_{2}\right)=(1,1,-1)$. We can verify that it is a $A_{4}$-singularity, for example, by using blow-up of a singularity or direct suitable coordinate transformations (a singularity whose normal form of defining equation is $y^{2}+x^{5}=0$ ).


Figure 4.1. A schematic view of $\pi^{-1}(V), V_{0}$ and the dynamics on it for Example 4.10. The singularity $\mathrm{P}_{1}$ of $V_{0}$ is of type $A_{4}$.

Example 4.11. - The autonomous, quasihomogeneous part of $\left(\mathrm{P}_{\mathrm{II}-1}\right)_{2}$ given in (1.5) is defined by the Hamiltonians

$$
\left\{\begin{array}{l}
H_{1}=2 p_{1} p_{2}-p_{2}^{3}-p_{1} q_{1}^{2}+q_{2}^{2}  \tag{4.24}\\
H_{2}=-p_{1}^{2}+p_{1} p_{2}^{2}+p_{1} p_{2} q_{1}^{2}+2 p_{1} q_{1} q_{2}
\end{array}\right.
$$

The weights are $\left(a_{1}, a_{2}, b_{1}, b_{2}\right)=(1,3,4,2)$ and $h_{1}=6, h_{2}=8$. Its four indicial loci and the Kovalevskaya exponents are given by

$$
\begin{array}{ll}
c_{1}=(1,0,0,0), & \kappa=-1,2,3,6 \\
c_{2}=(-1,1,0,1), & \kappa=-1,2,3,6 \\
c_{3}=(2,1,0,1), & \kappa=-1,-3,6,8 \\
c_{4}=(-2,3,9,3), & \kappa=-1,-3,6,8 .
\end{array}
$$

Among them, $c_{3}$ and $c_{4}$ satisfy the assumptions for Theorem 4.9. Thus, the fixed points $\mathrm{P}\left(c_{3}\right)$ and $\mathrm{P}\left(c_{4}\right)$ have 2-dim stable manifolds that locally coincide with $\overline{\pi^{-1}(V \backslash\{0\})}$. The fixed points $\mathrm{P}\left(c_{1}\right)$ and $\mathrm{P}\left(c_{2}\right)$ have 1-dim stable manifolds that are given by the orbit of special solutions.

On the $\left(r, Q_{2}, P_{1}, P_{2}\right)$-coordinates, $H_{1}$ and $H_{2}$ are written as

$$
\begin{aligned}
& H_{1}=r^{6}\left(2 P_{1} P_{2}-P_{2}^{3}-P_{1}+Q_{2}^{2}\right) \\
& H_{2}=r^{8}\left(-P_{1}^{2}+P_{1} P_{2}^{2}+P_{1} P_{2}+2 P_{1} Q_{2}\right)
\end{aligned}
$$

Thus, $V_{01}$ is defined by

$$
V_{01}=\left\{2 P_{1} P_{2}-P_{2}^{3}-P_{1}+Q_{2}^{2}=0,-P_{1}^{2}+P_{1} P_{2}^{2}+P_{1} P_{2}+2 P_{1} Q_{2}=0\right\}
$$

Theorem 4.5 shows that $\mathrm{d} H_{2}\left(c_{1}\right)=\mathrm{d} H_{2}\left(c_{2}\right)=0$. Hence, $V_{0}$ is singular at $\mathrm{P}\left(c_{1}\right):\left(Q_{2}, P_{1}, P_{2}\right)=(0,0,0)$ and $\mathrm{P}\left(c_{2}\right):(-1,0,1)$. We can verify that both $\mathrm{P}\left(c_{1}\right)$ and $\mathrm{P}\left(c_{2}\right)$ are $D_{5}$-singularities (the normal form is $\left.y\left(x^{2}+y^{3}\right)=0\right)$.

Example 4.12. - The autonomous, quasihomogeneous part of $\left(\mathrm{P}_{\mathrm{II}-2}\right)_{2}$ given in (1.6) is defined by the Hamiltonians

$$
\left\{\begin{array}{l}
H_{1}=p_{1} p_{2}-p_{1} q_{1}^{2}-2 p_{1} q_{2}+p_{2} q_{1} q_{2}+q_{1} q_{2}^{2}  \tag{4.25}\\
H_{2}=p_{1}^{2}-p_{1} p_{2} q_{1}+p_{2}^{2} q_{2}-2 p_{1} q_{1} q_{2}-p_{2} q_{2}^{2}+q_{1}^{2} q_{2}^{2}
\end{array}\right.
$$

The weights are $\left(a_{1}, a_{2}, b_{1}, b_{2}\right)=(1,2,3,2)$ and $h_{1}=5, h_{2}=6$. It has five indicial loci given by

$$
\begin{array}{ll}
c_{1}=(1,0,0,0), & \kappa=-1,1,3,5 \\
c_{2}=(-1,-1,1,0), & \kappa=-1,1,3,5 \\
c_{3}=(0,-2,4,-4), & \kappa=-1,-2,5,6 \\
c_{4}=(-2,-2,0,2), & \kappa=-1,-2,5,6 \\
c_{5}=(2,0,0,2), & \kappa=-1,-2,5,6 .
\end{array}
$$

Among them, $c_{3}, c_{4}$ and $c_{5}$ satisfy the assumptions for Theorem 4.9. Thus, the fixed points $\mathrm{P}\left(c_{3}\right), \mathrm{P}\left(c_{4}\right)$ and $\mathrm{P}\left(c_{5}\right)$ have 2-dim stable manifolds that
locally coincide with $\overline{\pi^{-1}(V \backslash\{0\})}$. The fixed points $\mathrm{P}\left(c_{1}\right)$ and $\mathrm{P}\left(c_{2}\right)$ have 1-dim stable manifolds that are given by the orbit of special solutions.

On the $\left(r, Q_{2}, P_{1}, P_{2}\right)$-coordinates, $H_{1}$ and $H_{2}$ are written as

$$
\begin{aligned}
& H_{1}=r^{5}\left(P_{1} P_{2}-P_{1}-2 P_{1} Q_{2}+P_{2} Q_{2}+Q_{2}^{2}\right) \\
& H_{2}=r^{6}\left(P_{1}^{2}-P_{1} P_{2}+P_{2}^{2} Q_{2}-2 P_{1} Q_{2}-P_{2} Q_{2}^{2}+Q_{2}^{2}\right)
\end{aligned}
$$

$V_{01}$ is defined by

$$
V_{01}=\left\{\begin{array}{l}
P_{1} P_{2}-P_{1}-2 P_{1} Q_{2}+P_{2} Q_{2}+Q_{2}^{2}=0, \\
P_{1}^{2}-P_{1} P_{2}+P_{2}^{2} Q_{2}-2 P_{1} Q_{2}-P_{2} Q_{2}^{2}+Q_{2}^{2}=0
\end{array}\right\} .
$$

Since $\mathrm{d} H_{2}\left(c_{1}\right)=\mathrm{d} H_{2}\left(c_{2}\right)=0, V_{0}$ is singular at $\mathrm{P}\left(c_{1}\right):\left(Q_{2}, P_{1}, P_{2}\right)=$ $(0,0,0)$ and $\mathrm{P}\left(c_{2}\right):(-1,-1,0)$. We can verify that both $\mathrm{P}\left(c_{1}\right)$ and $\mathrm{P}\left(c_{2}\right)$ are $A_{5}$-singularities (the normal form is $y^{2}+x^{6}=0$ ).

Example 4.13. - The autonomous, quasihomogeneous part of $\left(\mathrm{P}_{\mathrm{IV}}\right)_{2}$ given in (1.7) is defined by the Hamiltonians

$$
\left\{\begin{array}{l}
H_{1}=p_{1}^{2}+p_{1} p_{2}-p_{1} q_{1}^{2}+p_{2} q_{1} q_{2}-p_{2} q_{2}^{2}  \tag{4.26}\\
H_{2}=p_{1} p_{2} q_{1}-2 p_{1} p_{2} q_{2}-p_{2}^{2} q_{2}+p_{2} q_{1} q_{2}^{2}
\end{array}\right.
$$

The weights are $\left(a_{1}, a_{2}, b_{1}, b_{2}\right)=(1,1,2,2)$ and $h_{1}=4, h_{2}=5$. It has eight indicial loci given by

$$
\begin{array}{ll}
c_{1}=(-1,-1,1,0), & \kappa=-1,1,2,4 \\
c_{2}=(1,0,0,0), & \kappa=-1,1,2,4 \\
c_{3}=(0,1,0,0), & \kappa=-1,1,2,4 \\
c_{4}=(0,-1,2,-4), & \kappa=-1,-2,4,5 \\
c_{5}=(2,0,0,2), & \kappa=-1,-2,4,5 \\
c_{6}=(-1,1,1,0), & \kappa=-1,-2,4,5 \\
c_{7}=(1,2,0,0), & \kappa=-1,-2,4,5 \\
c_{8}=(-2,-2,2,2), &
\end{array}
$$

Among them, $c_{4}$ to $c_{8}$ satisfy the assumptions for Theorem 4.9. To investigate the fixed points $\mathrm{P}\left(c_{1}\right)$ and $\mathrm{P}\left(c_{2}\right)$, we move to $B_{1}$ chart with the $\left(r, Q_{2}, P_{1}, P_{2}\right)$-coordinates, on which $H_{1}$ and $H_{2}$ are written as

$$
\begin{aligned}
& H_{1}=r^{4}\left(P_{1}^{2}+P_{1} P_{2}-P_{1}+P_{2} Q_{2}-P_{2} Q_{2}^{2}\right) \\
& H_{2}=r^{5}\left(P_{1} P_{2}-2 P_{1} P_{2} Q_{2}-P_{2}^{2} Q_{2}+P_{2} Q_{2}^{2}\right)
\end{aligned}
$$

$V_{01}$ is defined by

$$
V_{01}=\left\{\begin{array}{l}
P_{1}^{2}+P_{1} P_{2}-P_{1}+P_{2} Q_{2}-P_{2} Q_{2}^{2}=0 \\
P_{1} P_{2}-2 P_{1} P_{2} Q_{2}-P_{2}^{2} Q_{2}+P_{2} Q_{2}^{2}=0
\end{array}\right\}
$$

Since $\mathrm{d} H_{2}\left(c_{1}\right)=\mathrm{d} H_{2}\left(c_{2}\right)=0, V_{01}$ is singular at $\mathrm{P}\left(c_{1}\right):\left(Q_{2}, P_{1}, P_{2}\right)=$ $(1,1,0)$ and $\mathrm{P}\left(c_{2}\right):(0,0,0)$. We can verify that both $\mathrm{P}\left(c_{1}\right)$ and $\mathrm{P}\left(c_{2}\right)$ are $D_{6}$-singularities (the normal form is $y\left(x^{2}+y^{4}\right)=0$ ).

Note that $\mathrm{P}\left(c_{3}\right)$ is not included in $B_{1}$ chart because the first component of $c_{3}$ is zero. To study $\mathrm{P}\left(c_{3}\right)$ we use to $B_{2}$ chart with the coordinates ( $Q_{1}, r, P_{1}, P_{2}$ ) defined as (4.2). In this coordinates, $H_{1}$ and $H_{2}$ are written as

$$
\begin{aligned}
& H_{1}=r^{4}\left(P_{1}^{2}+P_{1} P_{2}-P_{1} Q_{1}^{2}+P_{2} Q_{1}-P_{2}\right) \\
& H_{2}=r^{5}\left(P_{1} P_{2} Q_{1}-2 P_{1} P_{2}-P_{2}^{2}+P_{2} Q_{1}\right)
\end{aligned}
$$

$V_{02}$ is defined by

$$
V_{02}=\left\{\begin{array}{l}
P_{1}^{2}+P_{1} P_{2}-P_{1} Q_{1}^{2}+P_{2} Q_{1}-P_{2}=0 \\
P_{1} P_{2} Q_{1}-2 P_{1} P_{2}-P_{2}^{2}+P_{2} Q_{1}=0
\end{array}\right\}
$$

Since $\mathrm{d} H_{2}\left(c_{3}\right)=0, V_{02}$ is singular at $\mathrm{P}\left(c_{3}\right):\left(Q_{1}, P_{1}, P_{2}\right)=(0,0,0)$, which is also a $D_{6}$-singularity.

Example 4.14. - Let us consider the following Hamiltonians

$$
\left\{\begin{array}{l}
H_{1}=\left(p_{1}^{2} / 2-2 q_{1}^{3}\right)+\left(p_{2}^{2} / 2-2 q_{2}^{3}\right)  \tag{4.27}\\
H_{2}=p_{1}^{2} / 2-2 q_{1}^{3}
\end{array}\right.
$$

The Hamiltonian equation of $H_{1}$ is a direct product of the autonomous part of the first Painleve equation. The weights are $\left(a_{1}, a_{2}, b_{1}, b_{2}\right)=(2,2,3,3)$ and $h_{1}=6, h_{2}=6$. It has three indicial loci $c_{1}, c_{2}, c_{3}$, whose Kovalevskaya exponents are $\kappa=-1,2,3,6$ for $c_{1}, c_{2}$ and $\kappa=-1,-1,6,6$ for $c_{3}$. Since $\mathrm{d} H_{2}\left(c_{1}\right)=\mathrm{d} H_{2}\left(c_{2}\right)=0, V_{0}$ is singular at $\mathrm{P}\left(c_{1}\right)$ and $\mathrm{P}\left(c_{2}\right)$. We can show that both singularities are $A_{2^{-}}$singularity.

Similarly, consider the direct product of the autonomous part of the second Painlevé equation

$$
\left\{\begin{array}{l}
H_{1}=\left(p_{1}^{2} / 2-q_{1}^{4} / 2\right)+\left(p_{2}^{2} / 2-q_{2}^{4} / 2\right)  \tag{4.28}\\
H_{2}=p_{1}^{2} / 2-q_{1}^{4} / 2
\end{array}\right.
$$

The weights are $\left(a_{1}, a_{2}, b_{1}, b_{2}\right)=(1,1,2,2)$ and $h_{1}=4, h_{2}=4$. It has eight indicial loci $c_{1}, \ldots, c_{8}$, whose Kovalevskaya exponents are $\kappa=-1,1,2,4$ for $c_{1}, c_{2}, c_{3}$ and $\kappa=-1,-1,4,4$ for the others. Since $\mathrm{d} H_{2}\left(c_{1}\right)=\mathrm{d} H_{2}\left(c_{2}\right)=$ $\mathrm{d} H_{2}\left(c_{3}\right)=0, V_{0}$ is singular at $\mathrm{P}\left(c_{1}\right)$ to $\mathrm{P}\left(c_{3}\right)$. We can show that singularities of them are $A_{3}$ - singularity.

Finally, consider the direct product of the autonomous part of the fourth Painlevé equation

$$
\left\{\begin{array}{l}
H_{1}=\left(-p_{1} q_{1}^{2}+p_{1}^{2} q_{1}\right)+\left(-p_{2} q_{2}^{2}+p_{2}^{2} q_{2}\right)  \tag{4.29}\\
H_{2}=-p_{1} q_{1}^{2}+p_{1}^{2} q_{1}
\end{array}\right.
$$

The weights are $\left(a_{1}, a_{2}, b_{1}, b_{2}\right)=(1,1,1,1)$ and $h_{1}=3, h_{2}=3$. It has fifteen indicial loci $c_{1}, \ldots, c_{15}$, whose Kovalevskaya exponents are $\kappa=-1,1,1,3$ for $c_{1}$ to $c_{5}$ and $\kappa=-1,-1,3,3$ for the others. Since $\mathrm{d} H_{2}\left(c_{1}\right) \cdots=$ $\mathrm{d} H_{2}\left(c_{5}\right)=0, V_{0}$ is singular at $\mathrm{P}\left(c_{1}\right)$ to $\mathrm{P}\left(c_{5}\right)$. They are $D_{4^{-}}$singularities.

Table 4.1. Weights, Kovalevskaya exponents and singularity-types of 4-dim autonomous Painlevé equations.

|  | $\left(a_{1}, a_{2}, b_{1}, b_{2} ; h_{1}, h_{2}\right)$ | $\kappa$ |  | sing. |
| :---: | :---: | :---: | :---: | :---: |
| $H_{\mathrm{I}} \times H_{\mathrm{I}}(4.27)$ | (2, 2, 3, 3; 6, 6) | $\begin{aligned} & (-1,2,3,6) \\ & (-1,-1,6,6) \end{aligned}$ | $\begin{aligned} & \times 2 \\ & \times 1 \end{aligned}$ | $A_{2}$ |
| $H_{\text {II }} \times H_{\text {II }}(4.28)$ | $(1,1,2,2 ; 4,4)$ | $\begin{aligned} & (-1,1,2,4) \\ & (-1,-1,4,4) \end{aligned}$ | $\begin{aligned} & \times 3 \\ & \times 5 \end{aligned}$ | $A_{3}$ |
| $H_{\text {IV }} \times H_{\text {IV }}(4.29)$ | $(1,1,1,1 ; 3,3)$ | $\begin{aligned} & (-1,1,1,3) \\ & (-1,-1,3,3) \end{aligned}$ | $\begin{aligned} & \times 5 \\ & \times 10 \end{aligned}$ | $D_{4}$ |
| $H_{1}^{9 / 2}(1.4)$ | (2, 4, 5, 3; 8, 10) | $\begin{aligned} & (-1,2,5,8) \\ & (-3,-1,8,10) \end{aligned}$ | $\begin{aligned} & \times 1 \\ & \times 1 \end{aligned}$ | $A_{4}$ |
| $H_{1}^{7 / 2+1}(1.5)$ | $(1,3,4,2 ; 6,8)$ | $\begin{aligned} & (-1,2,3,6) \\ & (-3,-1,6,8) \end{aligned}$ | $\begin{aligned} & \times 2 \\ & \times 2 \end{aligned}$ | $D_{5}$ |
| $H_{1}^{5}$ (1.6) | $(1,2,3,2 ; 5,6)$ | $\begin{aligned} & (-1,1,3,5) \\ & (-2,-1,5,6) \end{aligned}$ | $\begin{aligned} & \times 2 \\ & \times 3 \end{aligned}$ | $A_{5}$ |
| $H_{1}^{4+1}(1.7)$ | (1, 1, 2, 2; 4, 5) | $\begin{aligned} & (-1,1,2,4) \\ & (-2,-1,4,5) \end{aligned}$ | $\begin{array}{r} \times 3 \\ \times 5 \end{array}$ | $D_{6}$ |

## BIBLIOGRAPHY

[1] M. J. Ablowitz, A. Ramani \& H. Segur, "A connection between nonlinear evolution equations and ordinary differential equations of $P$-type. I", J. Math. Phys. 21 (1980), no. 4, p. 715-721.
[2] M. Adler, P. van Moerbeke \& P. Vanhaecke, Algebraic integrability, Painlevé geometry and Lie algebras, Springer-Verlag, Berlin, 2004, xii +483 pages.
[3] V. I. Arnold, V. V. Goryunov, O. V. Lyashko \& V. A. Vasil'ev, Singularity theory. I, Springer-Verlag, Berlin, 1998, iv+245 pages.
[4] A. V. Borisov \& S. L. Dudoladov, "Kovalevskaya exponents and Poisson structures", Regul. Chaotic Dyn. 4 (1999), no. 3, p. 13-20.
[5] H. Chiba, "Kovalevskaya exponents and the space of initial conditions of a quasihomogeneous vector field", J. Differential Equations 259 (2015), no. 12, p. 76817716.
[6] -, "The first, second and fourth Painlevé equations on weighted projective spaces", J. Differential Equations 260 (2016), no. 2, p. 1263-1313.
[7] -, "The third, fifth and sixth Painlevé equations on weighted projective spaces", SIGMA Symmetry Integrability Geom. Methods Appl. 12 (2016), p. 019.
[8] -, "Multi-Poisson approach to the Painlevé equations: from the isospectral deformation to the isomonodromic deformation", SIGMA Symmetry Integrability Geom. Methods Appl. 13 (2017), p. 025.
[9] P. A. Clarkson, N. Joshi \& M. Mazzocco, "The Lax pair for the mKdV hierarchy", in Théories asymptotiques et équations de Painlevé, Sémin. Congr., vol. 14, Soc. Math. France, Paris, 2006, p. 53-64.
[10] P. A. Clarkson, N. Joshi \& A. Pickering, "Bäcklund transformations for the second Painlevé hierarchy: a modified truncation approach", Inverse Problems 15 (1999), no. 1, p. 175-187.
[11] C. M. Cosgrove, "Chazy classes IX-XI of third-order differential equations", Stud. Appl. Math. 104 (2000), no. 3, p. 171-228.
[12] V. G. Drinfel'd \& V. V. Sokolov, "Lie algebras and equations of Kortewegde Vries type", in Current problems in mathematics, Vol. 24, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, p. 81-180.
[13] N. Ercolani \& E. D. Sigqia, "Painlevé property and geometry", Phys. D 34 (1989), no. 3, p. 303-346.
[14] K. Fuji \& T. Suzuki, "Drinfeld-Sokolov hierarchies of type $A$ and fourth order Painlevé systems", Funkcial. Ekvac. 53 (2010), no. 1, p. 143-167.
[15] P. R. Gordoa, N. Joshi \& A. Pickering, "On a generalized $2+1$ dispersive water wave hierarchy", Publ. Res. Inst. Math. Sci. 37 (2001), no. 3, p. 327-347.
[16] A. Goriely, Integrability and nonintegrability of dynamical systems, Advanced Series in Nonlinear Dynamics, vol. 19, World Scientific Publishing Co., Inc., River Edge, NJ, 2001, xviii+415 pages.
[17] , "Painlevé analysis and normal forms theory", Phys. D 152/153 (2001), p. 124-144.
[18] M. F. de Groot, T. J. Hollowood \& J. L. Miramontes, "Generalized Drinfel'dSokolov hierarchies", Comm. Math. Phys. 145 (1992), no. 1, p. 57-84.
[19] J. Hu \& M. Yan, "Painlevé Test and the Resolution of Singularities for Integrable Equations", https://arxiv.org/abs/1304.7982.
[20] H. Kawakami, "Four-dimensional Painlevé-type equations associated with ramified linear equations I: Matrix Painlevé systems", Funkcial. Ekvac. 63 (2020), no. 1, p. 97-132.
[21] H. Kawakami, A. Nakamura \& H. Sakai, "Degeneration scheme of 4-dimensional Painlevé-type equations", in 4-dimensional Painlevé-type equations, MSJ Mem., vol. 37, Math. Soc. Japan, Tokyo, 2018, p. 25-111.
[22] T. Koike, "On new expressions of the Painlevé hierarchies", in Algebraic analysis and the exact WKB analysis for systems of differential equations, RIMS Kôkyûroku Bessatsu, B5, Res. Inst. Math. Sci. (RIMS), Kyoto, 2008, p. 153-198.
[23] N. A. Kudryashov, "The first and second Painlevé equations of higher order and some relations between them", Phys. Lett. A 224 (1997), no. 6, p. 353-360.
[24] T. Matano, A. Matumiya \& K. Takano, "On some Hamiltonian structures of Painlevé systems. II", J. Math. Soc. Japan 51 (1999), no. 4, p. 843-866.
[25] A. Matumiya, "On some Hamiltonian structures of Painlevé systems. III", Kumamoto J. Math. 10 (1997), p. 45-73.
[26] A. Nakamura, "Autonomous limit of the 4-dimensional Painlevé-type equations and degeneration of curves of genus two", Ann. Inst. Fourier 69 (2019), no. 2, p. 845-893.
[27] M. Noumi \& Y. Yamada, "Higher order Painlevé equations of type $A_{l}^{(1)}$ ", Funkcial. Ekvac. 41 (1998), no. 3, p. 483-503.
[28] K. Окамото, "Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé", Japan. J. Math. (N.S.) 5 (1979), no. 1, p. 1-79.
[29] K. Saito, "Regular system of weights and associated singularities", in Complex analytic singularities, Adv. Stud. Pure Math., vol. 8, North-Holland, Amsterdam, 1987, p. 479-526.
[30] H. Sakai, "Rational surfaces associated with affine root systems and geometry of the Painlevé equations", Comm. Math. Phys. 220 (2001), no. 1, p. 165-229.
[31] S. Shimomura, "A certain expression of the first Painlevé hierarchy", Proc. Japan Acad. Ser. A Math. Sci. 80 (2004), no. 6, p. 105-109.
[32] T. Shioda \& K. Takano, "On some Hamiltonian structures of Painlevé systems. I", Funkcial. Ekvac. 40 (1997), no. 2, p. 271-291.
[33] H. Yoshida, B. Grammaticos \& A. Ramani, "Painlevé resonances versus Kowalevski exponents: some exact results on singularity structure and integrability of dynamical systems", Acta Appl. Math. 8 (1987), no. 1, p. 75-103.
[34] H. Yoshida, "Necessary condition for the existence of algebraic first integrals. I. Kowalevski's exponents", Celestial Mech. 31 (1983), no. 4, p. 363-379.

Manuscrit reçu le 15 octobre 2020,
révisé le 11 décembre 2021, accepté le 10 mars 2022.

## Hayato CHIBA

Advanced Institute for Materials Research
Tohoku University
Sendai, 980-8577 (Japan)
hchiba@tohoku.ac.jp

