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WILD MONODROMY OF THE FIFTH PAINLEVÉ EQUATION AND ITS ACTION ON WILD CHARACTER VARIETY: AN APPROACH OF CONFLUENCE

by Martin KLIMEŠ

ABSTRACT. — The article studies the nonlinear Stokes phenomenon at the irregular singularity of the Fifth Painlevé equation from the point of view of confluence from the Sixth Painlevé equation. This approach is developed separately on both sides of the Riemann–Hilbert correspondence. On the side of the Painlevé–Okamoto foliation, the relation between the nonlinear monodromy group of Painlevé VI and the "nonlinear wild monodromy pseudogroup" of Painlevé V (the pseudogroup generated by nonlinear Stokes operators and nonlinear exponential torus) is explained. On the side of the associated linear isomonodromic problems, the "wild" character variety (the space of the linear monodromy and Stokes data) of Painlevé V is constructed through a birational transformation from the one of Painlevé VI. Explicit formulas for the action of the "nonlinear wild monodromy" of Painlevé V on its character variety are then obtained by transporting the description of the action of the nonlinear monodromy of Painlevé VI on its character variety to that of Painlevé V.

RÉSUMÉ. — L'article étudie le phénomène de Stokes non linéaire à la singularité irrégulière de la Cinquième équation de Painlevé du point de vue de la confluence à partir de la Sixième équation de Painlevé. Cette approche est développée séparément des deux côtés de la correspondance de Riemann–Hilbert. Du côté du feuilletage de Painlevé–Okamoto, la relation entre le groupe de monodromie non-linéaire de Painlevé VI et le « pseudogroupe de monodromie sauvage non-linéaire » de Painlevé V (le pseudogroupe engendré par les opérateurs de Stokes non-linéaires et le tore exponentiel non-linéaire) est expliquée. Du côté des problèmes isomonodromiques linéaires associés, la variété de caractères « sauvages » (l'espace de la monodromie linéaire et des données de Stokes) de Painlevé V est construite par une transformation birationnelle à partir de celle de Painlevé VI. On obtient alors des formules explicites de l'action de la « monodromie sauvage non-linéaire » de Painlevé V sur sa variété de caractères en transportant la description de l'action de la monodromie non-linéaire de Painlevé VI sur sa variété de caractères à celle de Painlevé V.

Keywords: Painlevé equations, wild character variety, confluence, nonlinear Stokes phenomenon.
2020 *Mathematics Subject Classification:* 34M40, 34M55, 34M56.

1. Introduction

The six families of Painlevé equations $P_I, \dots, P_{VI}$ can be written in the form of non-autonomous Hamiltonian systems

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H_\bullet(q,p,t)}{\partial p}, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H_\bullet(q,p,t)}{\partial q}, \qquad \bullet = I, \dots, VI,$$

the solutions of which define a foliation in the (q, p, t)-space (or better, in the Okamoto's semi-compactification of it). By virtue of the Painlevé property, there are well-defined *nonlinear monodromy* operators acting on the foliation by analytic continuation of solutions along loops in the t-variable. In the case of the sixth Painlevé equation P_{VI} the associated monodromy group carries a great deal of information about the foliation and hence about the equation. The other Painlevé equations $P_I, \dots, P_V$ are obtained from P_{VI} by a limiting process through confluences of singularities and other degenerations. When different singularities merge, as is the case in the degeneration $P_{VI} \to P_V$, one loses some of the monodromy. It is known that the lost information should reappear in some way as a *nonlinear Stokes phenomenon* at the confluent irregular singularity. Roughly speaking, the source of this nonlinear phenomenon is the existence of local normalizing transformations above certain sectors at the singularity and therefore of canonical 2-parameter families of solutions with well-behaved exponential asymptotics over the sectors. This was proved originally in the works of Takano [63, 64] and Yoshida [65, 66], and recently also by Bittmann [3]. Passing from one sector to a neighboring one the family changes – this is encoded by *nonlinear Stokes operators*. The "good" analogue of the nonlinear monodromy group for the equations $P_I, \dots, P_V$ is the, so called, "*wild monodromy pseudogroup*", which is generated not only by the nonlinear monodromy operators but also by the nonlinear Stokes operators and nonlinear exponential tori (Lie groups of bounded sectorial symmetries) associated to each irregular singularity. This is a non-linear equivalent to the wild monodromy group of meromorphic linear differential systems (or meromorphic connections over Riemann surfaces) of Martinet and Ramis [41, 55], whose closure is the differential Galois group [54]. The main goal of this paper is to describe in explicit terms the dynamics of this wild monodromy pseudogroup in the case of the non-degenerated fifth Painlevé equation P_V.

In the paper [33], the author has shown that in the case of confluence of two regular singularities to an irregular one in non-autonomous Hamiltonian systems, such as is the case of the degeneration $P_{VI} \to P_V$, both the nonlinear Stokes phenomenon and the wild monodromy pseudogroup can be reconstructed from a parametric family of limit "*wild monodromy operators*" to which the usual nonlinear monodromy operators accumulate along certain discrete sequences of the parameter of confluence. We will use this confluence approach to describe the nonlinear Stokes phenomenon in the fifth Painlevé equation P_V on the other side of the Riemann–Hilbert correspondence: as operators acting on a "*wild character variety*". This article provides one of the very first results in the general program of study of such wild monodromy actions on the character varieties of isomonodromic deformations of linear differential systems, that was sketched in [52].

The approach to Painlevé equations through a Riemann–Hilbert correspondence is a well established and fruitful method based on the fact that these equations govern isomonodromic (and iso-Stokes) deformations of certain linear differential systems with meromorphic coefficients on $\mathbb{CP}^1$. This means that the usual Riemann–Hilbert correspondence between linear systems and their generalized linear monodromy representations (consisting of monodromy and Stokes data), can be interpreted as a map between the space of local solutions of the given Painlevé equation with fixed values of parameters, and the space of generalized monodromy representations with fixed local multipliers. This latter space is called the *wild character variety* [6]. In this setting, the Riemann–Hilbert correspondence conjugates the transcendental flow of the Painlevé equations to a locally constant flow on the corresponding character variety [25]. The nonlinear monodromy of the sixth Painlevé equation P_{VI} is represented by an action of the a pure-braid group $\mathcal{B}_3$ on the character variety of P_{VI} [13, 15, 26, 27].

Our goal is to use the confluence from P_{VI} to P_V in order to describe the action of the nonlinear wild monodromy pseudogroup of P_V on the associated wild character variety of P_V. In order to do that, we will need to describe the confluence procedure on both sides of the Riemann–Hilbert correspondence: on the Painlevé–Okamoto foliation and on the linear isomonodromic problem on one side and on the associated character variety on the other side. This obviously brings certain level technicality to this paper. The part dealing with the Painlevé–Okamoto foliation has been already treated in [33]. We will recall these results in Section 2, where we introduce the nonlinear monodromy group of P_{VI} and the nonlinear wild monodromy pseudogroup of P_V, and in Section 3, where we explain the confluence and the relation between these nonlinear monodromy (pseudo)groups.

The main part of this paper is devoted to the study of the confluence in the associated linear isomonodromy problem, and of the dynamics on the wild character variety. In Section 4 we recall the usual approach to P_{VI} as an equation governing the isomonodromic deformations of 2×2 linear systems with four Fuchsian singularities at $0, 1, t, \infty \in \mathbb{CP}^1$, and some classical results concerning the geometry of the character variety $\mathcal{S}_{VI}$ of P_{VI} and the braid group action on it. In Section 5 we will study in detail the confluence $P_{VI} \to P_V$ through the viewpoint of isomonodromic deformations. We show that the wild character variety $\mathcal{S}_V$ of P_V is obtained from the character varieties of P_{VI} by birational changes of coordinates (in fact, a blow-down) depending on the parameter of confluence. This part is fundamentally based on the theory of confluence of singularities in linear systems of Hurtubise, Lambert & Rousseau [23, 36].

An alternative approach is provided in the Appendix using isomonodromic deformations of 3×3 linear systems in Birkhoff normal form with an irregular singularity of Poincaré rank 1 at the origin (of eigenvalues $0, 1, t$) and a Fuchsian singularity at infinity, where the degeneration from P_{VI} to P_V happens through a confluence of eigenvalues $t \to 1$, the description of which is based on the author's work [34]. Ultimately this leads to the same picture as the approach of Section 5 (this is not surprising since the 2×2 and 3×3 systems are related one to the other by a middle convolution and a Laplace transform, or by a Harnad duality [4, 21, 22, 43]).

Using the birational transformation between the character varieties we transfer the known actions of pure-braid group from $\mathcal{S}_{VI}$ to $\mathcal{S}_V$, and then push them to the limit along discrete sequences of the parameter of confluence. This leads to our main result Theorem 6.5, which gives explicit formulas for the action of the nonlinear monodromy, of the nonlinear Stokes operators and of the nonlinear exponential torus of P_V on the wild character variety.

It is expected that the nonlinear wild monodromy pseudogroup described in this paper should have a natural interpretation in terms of a differential Galois theory (e.g. the differential Galois groupoid of Malgrange [10, 39]), perhaps in a similar way to [9], and that our results could be applied to construct and classify certain special type solutions of P_V in an analogy with the construction and classification of algebraic solutions of P_{VI} [4, 5, 15, 37]. However this is well beyond the scope of this paper.

While the main motivation of this paper is to describe in precise terms the confluence $P_{VI} \to P_V$, and to recover the nonlinear action of the wild monodromy pseudogroup, there are several side results that are obtained along the way which are worth of independent attention since they hint to more general phenomena regarding the whole hierarchy of Painlevé equations. For example, it is known that the irregular singularity of P_V has a special pole free solution, “tronquée” solution, on each of the two sectors of normalizations, which correspond to the sectorial center manifold of the saddle-node singularity of the foliation. We show is that this pair of sectorial center manifold solutions unfolds to a single solution in the confluent family of P_{VI}, characterized by its asymptotics at both of the two confluent singularities, and which is pole free on certain “unfolded sectorial” domain attached to the two singularities. This solution, both in P_{VI} and at the limit in P_V, corresponds through the Riemann–Hilbert correspondence to a point on the intersection of two lines on the character variety (which is a cubic surface containing up to 27 lines in case of P_{VI}, resp. up to 21 lines in case of P_V). These kind of solutions, which seem to be new, might be expected to play a special role in physics, similar to the one that the “tronquée” solutions play. Another side result worth of mentioning are Propositions 4.5 and 5.6 which give explicit formulas for all the lines on the character varieties of P_{VI} and P_V and their interpretations in terms of the isomonodromic problem.

Finally, let us remark that our confluence approach can be well extended to the other Painlevé equations (with increasing complexity of the description, the further the equation is from P_{VI} in the degeneration process), and some of this shall be done in future works. As of now, a general theory of confluence in linear systems has been developed only for non-resonant irregular singularities [23], therefore allowing to deal with only about half of the isomonodromic systems associated to Painlevé equations. However, in the case of traceless linear 2×2 systems with resonant irregular singularities, this theory has a natural generalization along the lines of [34]: this is a subject of a paper in preparation by the author.

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2. Nonlinear monodromy and Stokes phenomenon in P_{VI} and P_V

2.1. The Painlevé equations

The Painlevé equations originated from the effort of Painlevé [50] and Gambier [17] to classify all second order ordinary differential equations of type $q'' = R(q', q, t)$, with R rational, possessing the so called *Painlevé property* which controls the ramification points of solutions:

Painlevé property. — *Each germ of a solution can be meromorphically continued along any path avoiding the singular points of the equation (fixed singularities). In other words, solutions cannot have any other movable singularities other than poles.*

Painlevé and Gambier [17] produced a list of 50 canonical forms of equations to which any such equation can be reduced. Aside of equations solvable in terms of classical special functions, the list contained six new families of equations, $P_I, \dots, P_{VI}$, whose general solutions provided a new kind of special functions. In many aspects they may be regarded as nonlinear analogues of the hypergeometric equations [28].

In a modern approach to the Painlevé equations, following on Okamoto's works, the traditional families P_{II}, P_{III}, P_V are further divided to subfamilies by specification of some redundant parameters, and are classified according to the affine Weyl group of their Bäcklund symmetries [44, 49, 57], as well as according to the type of the isomonodromic problem they control [12, 46, 53]. The equation P_{VI} is a mother equation for the other Painlevé equations, which can be obtained through degeneration and confluence following the diagram [46]

$$\begin{array}{ccccccccc}
 & & & & P_{III}^{D_6} & \to & P_{III}^{D_7} & \to & P_{III}^{D_8} \\
 & & & \nearrow & & \times & & \searrow & \\
P_{VI} & \to & P_V & \to & P_V^{\deg} & & P_{II}^{JM} & \to & P_I \\
 & & & \searrow & & \times & & \nearrow & \\
 & & & & P_{IV} & \to & P_{II}^{FN} & &
\end{array}$$

according to which also the associated isomonodromy problems degenerate. A good understanding of the degeneration procedures should allow to transfer information along the diagram. The main obstacle is that as the nature of the singularities changes at the limit, it causes the naive limit of most local objects to diverge – this is a common rule in confluence problems. One therefore needs to find for each of the arrows in the above diagram a description that allows to deal with this divergence. This article studies some aspects of the confluence $P_{VI} \to P_V$ through the Riemann–Hilbert correspondence.

Each of the Painlevé equations is equivalent to a time dependent *Hamiltonian system*

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial}{\partial p} H_\bullet(q,p,t), \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial}{\partial q} H_\bullet(q,p,t), \qquad \bullet = I,\dots,VI, \tag{2.1}$$

from which it is obtained by reduction to the q-variable [48].

The general form of the *sixth Painlevé equation* is [30]

$$\begin{aligned} P_{VI}(\vartheta) : q'' = \frac{1}{2}\Big(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\Big)(q')^2 - \Big(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\Big) q' \\ + \frac{q(q-1)(q-t)}{2\,t^2(t-1)^2}\left[(\vartheta_\infty - 1)^2 - \vartheta_0^2 \frac{t}{q^2} + \vartheta_1^2 \frac{(t-1)}{(q-1)^2} + (1-\vartheta_t^2)\frac{t(t-1)}{(q-t)^2}\right], \end{aligned}$$

where $(\,\cdot\,)' = \frac{\mathrm{d}}{\mathrm{d}t}$, and where $\vartheta = (\vartheta_0, \vartheta_t, \vartheta_1, \vartheta_\infty) \in \mathbb{C}^4$ are complex constants related to the eigenvalues of the associated isomonodromic problem (4.1). The Hamiltonian function $H_{VI}(q,p,t)$ of its associated Hamiltonian system (2.1) is given by

$$\begin{aligned} H_{VI} = \frac{q(q-1)(q-t)}{t(t-1)}\bigg[p^2 - \Big(\frac{\vartheta_0}{q} + \frac{\vartheta_1}{q-1} + \frac{\vartheta_t - 1}{q-t}\Big) p \\ + \frac{(\vartheta_0 + \vartheta_1 + \vartheta_t - 1)^2 - (\vartheta_\infty - 1)^2}{4q(q-1)}\bigg]. \end{aligned} \tag{2.2}$$

The Hamiltonian system of P_{VI} has three simple (regular) singular points on the Riemann sphere $\mathbb{CP}^1$ at $t = 0, 1, \infty$.

The non-degenerate *fifth Painlevé equation* P_V (more precisely the fifth Painlevé equation with a parameter $\eta_1 = -1$, the general form is obtained by scaling $t \mapsto -\frac{t}{\eta_1}$) is

$$\begin{aligned} P_V(\widetilde{\vartheta}) : \quad q'' = \Big(\frac{1}{2q} + \frac{1}{q-1}\Big)(q')^2 - \frac{1}{\widetilde{t}} q' \\ + \frac{(q-1)^2}{2\widetilde{t}^2}\left((\vartheta_\infty - 1)^2 q - \frac{\vartheta_0^2}{q}\right) + \widetilde{\vartheta}_1 \frac{q}{\widetilde{t}} - \frac{q(q+1)}{2(q-1)}, \end{aligned}$$

where $()' = \frac{\mathrm{d}}{\mathrm{d}\widetilde{t}}$ and $\widetilde{\vartheta} = (\vartheta_0, \widetilde{\vartheta_1}, \vartheta_\infty)$. It is obtained from P_{VI} as a limit $\epsilon \to 0$ after the change of the independent variable and a substitution of the parameters

$$t = 1 + \epsilon \widetilde{t}, \qquad \vartheta_t = \frac{1}{\epsilon}, \quad \vartheta_1 = -\frac{1}{\epsilon} + \widetilde{\vartheta}_1, \tag{2.3}$$

which sends the three singularities to $\widetilde{t} = -\frac{1}{\epsilon}, 0, \infty$. At the limit, the two simple singular points $-\frac{1}{\epsilon}$ and ∞ merge into a double (irregular) singularity at the infinity. The change of variables (2.3), changes the function $\epsilon\cdot H_{VI}$ to

$$H_{VI}^{\mathrm{conf}}=\frac{q(q-1)(q-1-\epsilon\widetilde{t})}{\widetilde{t}(1+\epsilon\widetilde{t})}\Bigg[p^2-\left(\frac{\vartheta_0}{q}+\frac{\widetilde{\vartheta}_1-1}{q-1}+\frac{(1-\epsilon)\widetilde{t}}{(q-1)(q-1-\epsilon\widetilde{t})}\right)p \\ +\frac{(\vartheta_0+\widetilde{\vartheta}_1-1)^2-(\vartheta_\infty-1)^2}{4q(q-1)}\Bigg],$$

and the Hamiltonian system to

$$\frac{\mathrm{d}q}{\mathrm{d}\widetilde{t}}=\frac{\partial}{\partial p}H_{VI}^{\mathrm{conf}}(q,p,\widetilde{t}),\qquad \frac{\mathrm{d}p}{\mathrm{d}\widetilde{t}}=-\frac{\partial}{\partial q}H_{VI}^{\mathrm{conf}}(q,p,\widetilde{t}),\tag{2.4}$$

whose limit $\epsilon\to 0$ is the Hamiltonian system of P_V, $H_V=\lim_{\epsilon\to 0}H_{VI}^{\mathrm{conf}}$.

2.2. Nonlinear monodromy of P_{VI}

Consider the foliation in the (q,p,t)-space given by the solutions of the Hamiltonian system of $P_j(\vartheta)$, $j=I,\dots,VI$. As general solutions may have many poles the flow of P_j on $\mathbb{C}^2\times(\mathbb{CP}^1\smallsetminus\mathrm{Sing}(P_j))$, where $\mathrm{Sing}(P_j)\subseteq\{0,1,\infty\}$ is the set of fixed singularities of $P_j(\vartheta)$, is not complete. Okamoto [47] has constructed a semi-compactification $\mathcal{M}_j(\vartheta)$ of this space in form of a fibration over $\mathbb{CP}^1\smallsetminus\mathrm{Sing}(P_j)$ (corresponding to the projection $(q,p,t)\mapsto t$) on which the foliation is analytic and transverse to the fibers. We will skip the details of this construction as we won't need them. We will denote

$$\mathcal{M}_{j,t}(\vartheta):=\textit{the Okamoto space of initial conditions of } P_j(\vartheta),\tag{2.5}$$

the fiber of $\mathcal{M}_j(\vartheta)$ above a point $t\in\mathbb{CP}^1\smallsetminus\mathrm{Sing}(P_j)$. It is a complex surface sitting inside a compact rational surface as a complement of some anti-canonical divisor [47, 57], and endowed with a symplectic structure given by the standard symplectic form

$$\omega=\mathrm{d}q\wedge\mathrm{d}p,\tag{2.6}$$

in the local coordinate (q,p).

The Painlevé property of P_j means that for each path $t_0\xrightarrow{\gamma}t_1$ in the t-space $\mathbb{CP}^1\smallsetminus\mathrm{Sing}(P_j)$ the flow induces a symplectomorphisms $\mathcal{M}_{j,t_0}(\vartheta)\to\mathcal{M}_{j,t_1}(\vartheta)$ between the fibers corresponding to analytic continuation of the solutions of (2.1) along the path (see Figure 2.1). In particular, for any given base-point $t_0\in\mathbb{CP}^1\smallsetminus\mathrm{Sing}(P_j)$ the loops $\gamma\in\pi_1\big(\mathbb{CP}^1\smallsetminus\mathrm{Sing}(P_j),t_0\big)$ induce

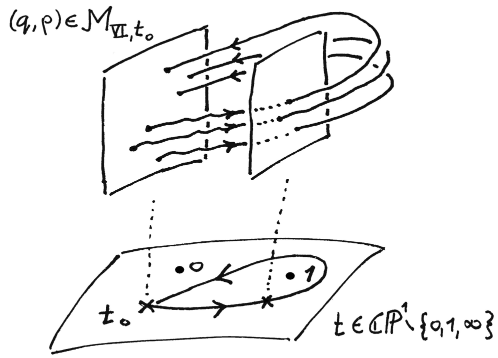


Figure 2.1. The Painlevé flow and its monodromy in case of P_{VI}.

a nonlinear *monodromy* action which is a representation of the fundamental group of the base space into the group of symplectomorphisms of $\mathcal{M}_{j,t_0}(\vartheta)$,

$$\begin{aligned}\pi_1(\mathbb{CP}^1 \smallsetminus \mathrm{Sing}(P_j), t_0) &\longrightarrow \mathrm{Aut}_\omega(\mathcal{M}_{j,t_0}(\vartheta)),\\ \gamma &\longmapsto \mathfrak{M}_\gamma(\,\cdot\,, t_0).\end{aligned}$$

In case of P_{VI} this nonlinear monodromy $\pi_1(\mathbb{CP}^1 \smallsetminus \{0,1,\infty\}, t_0) \to \mathrm{Aut}_\omega(\mathcal{M}_{VI,t_0}(\vartheta))$ carries (together with the analytic invariants of the singular fibers $t = 0, 1, \infty$ which are determined by the parameter ϑ) a complete information about the foliation and thus about the equation. For example, algebraic solutions of P_{VI} correspond to finite orbits of the monodromy action (see e.g. [5, 15, 37])). The nonlinear monodromy also plays an important role in the differential Galois theory: the Malgrange–Galois groupoid of the foliation is "generated" by the monodromy. Cantat and Loray [9] have showed the irreducibility of P_{VI} in the sense of Malgrange (i.e. maximality of the Malgrange–Galois groupoid), which then implies transcendentness of general solutions [10], by studying the action of this monodromy. The important fact on which these studies are based is that the nonlinear monodromy of P_{VI} has a well known explicit representation as a braid group action on an $\mathrm{SL}_2(\mathbb{C})$-character variety (more about this in Section 4).

On the other hand, for the other Painlevé equations the nonlinear monodromy operator doesn't carry enough information: it is cyclic for P_V, P_V^{deg}, $P_{III}^{D_i}$, and trivial for P_{IV}, P_{II}^{FN}, P_{II}^{JM}, P_I. In case of the confluence $P_{VI} \to P_V$, the merging of the two singularities $-\frac{1}{\epsilon}, \infty$ in the confluent family (2.4) of P_{VI} into a single singularity ∞ of P_V means that an important part of the monodromy group is lost and therefore also the information carried by it. This lost information reappears in the nonlinear Stokes phenomenon at the irregular singularity.

2.3. Nonlinear Stokes phenomenon of P_V

In the local coordinate $x = \widetilde{t}^{-1}$ near $\widetilde{t} = \infty$, the Hamiltonian system (2.1) of P_V has the form

$$\begin{aligned}
x^2\frac{\mathrm{d}}{\mathrm{d}x}q &= x\vartheta_0 + \big(1 - x(2\vartheta_0+\widetilde{\vartheta}_1-1)\big)q \\
&\qquad + x\big((\vartheta_0+\widetilde{\vartheta}_1-1)q^2 - 2pq + 4pq^2 - 2pq^3\big), \\
x^2\frac{\mathrm{d}}{\mathrm{d}x}p &= x\frac{(\vartheta_0+\widetilde{\vartheta}_1-1)^2-(\vartheta_\infty-1)^2}{4} - \big(1 - x(2\vartheta_0+\widetilde{\vartheta}_1-1)\big)p \\
&\qquad + x\big(-2(\vartheta_0+\widetilde{\vartheta}_1-1)pq + p^2 - 4p^2q + 3p^2q^2\big),
\end{aligned} \tag{2.7}$$

with an irregular singularity at $x = 0$.

Theorem 2.1 (Takano [63, 64], Shimomura [61, 62], Yoshida [66]).

(i) Formal normalization: *The above system* (2.7) *can be brought to a formal normal form*

$$x^2\frac{\mathrm{d}}{\mathrm{d}x}u = \big(1 - (2\vartheta_0+\widetilde{\vartheta}_1-1)x + 4u_1u_2x\big)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}u, \qquad u = \begin{pmatrix}u_1\\ u_2\end{pmatrix}, \tag{2.8}$$

by means of a formal transversely symplectic (w.r.t. the canonical forms (2.6) *and* $\mathrm{d}u_1\wedge\mathrm{d}u_2$*) change of coordinates*

$$\begin{pmatrix}q\\ p\end{pmatrix} = \widehat{\mathbf{\Psi}}(u,x,0) = \sum_{k\geqslant 0}\psi^{(k0)}(u)x^k,$$

where $\psi^{(k0)}(u)$ *are analytic on some polydisc* $\mathsf{U} = \{|u_1|,|u_2| < \delta_u\}$, $\delta_u > 0$.

(ii) Sectorial normalization: *The formal series* $\widehat{\mathbf{\Psi}}$ *of* x *is divergent but Borel summable, with a pair of Borel sums* $\mathbf{\Psi}^{\blacktriangle}(u,x,0)$ *and* $\mathbf{\Psi}^{\blacktriangledown}(u,x,0)$ *defined respectively above the sectors*

$$x\in\begin{cases}\mathsf{X}^{\blacktriangle}(0) = \{|\arg x - \frac{\pi}{2}| < \pi-\eta,\ |x| < \delta_x\},\\ \mathsf{X}^{\blacktriangledown}(0) = \{|\arg x + \frac{\pi}{2}| < \pi-\eta,\ |x| < \delta_x\},\end{cases}$$

for some $0 < \eta < \frac{\pi}{2}$ *arbitrarily small and some* $\delta_x > 0$ *(depending on* η*), and* $u\in\mathsf{U}$. *The sectorial transformations* $\begin{pmatrix}q\\ p\end{pmatrix} = \mathbf{\Psi}^{\bullet}(u,x,0)$, $\bullet = \blacktriangle, \blacktriangledown$, *bring the system* (2.7) *to its formal normal form* (2.8).

In particular, the formal transformation $\widehat{\mathbf{\Psi}}(u,x,0)$ and the sectorial transformations $\mathbf{\Psi}^{\bullet}(u,x,0)$ satisfy the same $(\frac{\partial}{\partial u},\frac{\partial}{\partial x})$-differential relations over the field of germs of meromorphic functions.

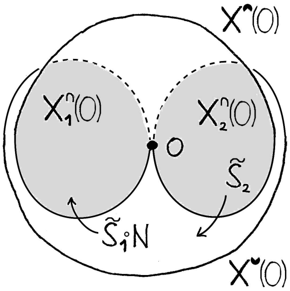


Figure 2.2. The sectorial domains $\mathsf{X}^{\frown}(0)$ and $\mathsf{X}^{\smile}(0)$ in the x-coordinate and the connecting transformations between the canonical solutions $\big(q^{\frown},p^{\frown}\big)(x,0;c)$ and $\big(q^{\smile},p^{\smile}\big)(x,0;c)$ on the left $\mathsf{X}_1^{\cap}(0)$ and right $\mathsf{X}_2^{\cap}(0)$ intersection sectors.

Canonical 2-parameter family of solutions

The system (2.8), which is Hamiltonian for the time-x-dependent Hamiltonian function

$$\frac{\big(1-(2\vartheta_0+\widetilde{\vartheta}_1-1)x\big)u_1u_2+2\big(u_1u_2\big)^2}{x^2}$$

with respect to the standard symplectic form $\mathrm{d}u_1\wedge\mathrm{d}u_2$, has a *canonical 2-parameter family of solutions*

$$\begin{aligned} u_1(x,0;c)&=c_1\,\mathrm{e}^{-\frac{1}{x}}x^{-(2\vartheta_0+\tilde{\vartheta}_1-1)+4c_1c_2},\\ u_2(x,0;c)&=c_2\,\mathrm{e}^{\frac{1}{x}}x^{(2\vartheta_0+\tilde{\vartheta}_1-1)-4c_1c_2},\end{aligned}\qquad c={}^{\mathrm{t}}(c_1,c_2)\in\mathbb{C}^2, \tag{2.9}$$

and an *analytic first integral*

$$h=u_1u_2=c_1c_2. \tag{2.10}$$

Let $u^{\frown}(x,0;c)$, resp. $u^{\smile}(x,0;c)$, be fixed branches of (2.9) restricted to $\mathsf{X}^{\frown}(0)$, resp. $\mathsf{X}^{\smile}(0)$, then we define[(1)]

$$\begin{pmatrix} q^{\bullet}\\ p^{\bullet}\end{pmatrix}(x,0;c)=\mathbf{\Psi}^{\bullet}(\,\cdot\,,x,0)\circ u^{\bullet}(x,0;c),\qquad \bullet=\frown,\smile, \tag{2.11}$$

[(1)] The notation $\mathbf{\Psi}^{\bullet}(\,\cdot\,,x,0)$ stands for the function $u\mapsto\mathbf{\Psi}^{\bullet}(u,x,0)$, i.e. $\mathbf{\Psi}^{\bullet}(\,\cdot\,,x,0)\circ u^{\bullet}(x,0;c)$ denotes the substitution $\mathbf{\Psi}^{\bullet}(u,x,0)\big|_{u=u^{\bullet}(x,0;c)}$.

as the corresponding *canonical 2-parameter family of solutions* to the Hamiltonian Painlevé system (2.7). We call the parameter $c = {}^{\mathrm{t}}(c_1, c_2)$ an *initial condition.*

Sectorial center manifold solutions

Corollary 2.2. — *The system* (2.7) *has a unique bounded analytic solution on each of the two sectors* $\mathsf{X}^\bullet(0)$, $\bullet = \blacktriangle, \blacktriangledown$, *given by* $\begin{pmatrix} q^\bullet \\ p^\bullet \end{pmatrix}(x,0;0) = \mathbf{\Psi}^\bullet(0,x,0)$, *corresponding to the initial condition* $c = 0$.

We call these solutions the "*sectorial center manifold*" solutions, since they correspond to a sectorial center manifold of the saddle-node singularity of the foliation. Each of them can be characterized as the pole-free solution on its respective sector. These solutions are also known as the "*bi-tronquée*" (double-truncated) solutions of P_V [1].

Nonlinear exponential torus

Definition 2.3. — *A (fibered)* symmetry *of the system* (2.8) *is a symplectic transformation* $u \mapsto \phi(u,x)$ *that preserves the system.*

Proposition 2.4 ([33, Proposition 31]). — *Any symmetry of* (2.8) *that is bounded and analytic above one of the sectors* $x \in \mathsf{X}^\bullet(0)$, $\bullet = \blacktriangle, \blacktriangledown$, $u \in \mathsf{U}$, *is in fact independent of* x. *It is given by the time-1 flow*

$$\mathbf{T}_\alpha : u \longmapsto \begin{pmatrix} \mathrm{e}^{\alpha(u_1u_2)} & 0 \\ 0 & \mathrm{e}^{-\alpha(u_1u_2)} \end{pmatrix} u, \tag{2.12}$$

of a Hamiltonian vector field

$$\xi = \alpha(u_1u_2)\left(u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}\right), \quad \text{with } \alpha(h) \text{ an analytic germ.} \tag{2.13}$$

The Lie group of these symmetries is commutative and connected.

Following the terminology of the differential Galois theory of linear systems, the Lie group of symmetries (2.12) is called the *nonlinear exponential torus*. Its Lie algebra of infinitesimal symmetries (2.13) is called the *nonlinear infinitesimal torus*.

Corollary 2.5. — *The normalizing sectorial transformations* $\widehat{\mathbf{\Psi}}$ *and* $\mathbf{\Psi}^\bullet$ *of Theorem 2.1 are unique up to a right composition with the same analytic symmetry* $\mathbf{T}_\alpha(u)$.

The nonlinear *exponential torus* (2.12) acts on the Painlevé system (2.7) by the sectorial isotropies

$$\mathfrak{T}_{\alpha}^{\bullet}(q,p,x) := \boldsymbol{\Psi}^{\bullet}(\,\cdot\,,x,0)\circ \mathbf{T}_{\alpha}(u), \qquad \bullet = \frown, \smile,$$

given by the time-1-flow of the vector field which is the sectorial pullback of (2.13) by $(\boldsymbol{\Psi}^{\bullet})^{\circ(-1)}$. Its right action on the canonical solutions $\mathfrak{T}_{\alpha}^{\bullet}(\,\cdot\,,x)\circ\binom{q^{\bullet}}{p^{\bullet}}(x,0;c) = \binom{q^{\bullet}}{p^{\bullet}}(x,0;\cdot\,)\circ\mathbf{T}_{\alpha}(c)$ is given by the corresponding action of $\mathbf{T}_{\alpha}$ on the initial condition parameter c,

$$\mathbf{T}_{\alpha} : c \longmapsto \begin{pmatrix} \mathrm{e}^{\alpha(c_1c_2)} & 0 \\ 0 & \mathrm{e}^{-\alpha(c_1c_2)} \end{pmatrix} c, \qquad \alpha\in\mathcal{O}(\mathbb{C},0). \tag{2.14}$$

In particular, the nonlinear *formal monodromy* (monodromy of the formal solution $\widehat{\boldsymbol{\Psi}}(\,\cdot\,,x,0)\circ u(x,0;c)$), acting on the initial condition c as

$$\mathbf{N}(c) = \mathbf{T}_{2\pi\mathrm{i}(-2\vartheta_0-\tilde{\vartheta}_1+1+4h)}(c) = \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}(-2\vartheta_0-\tilde{\vartheta}_1+4c_1c_2)} & 0 \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}(-2\vartheta_0-\tilde{\vartheta}_1+4c_1c_2)} \end{pmatrix} c, \tag{2.15}$$

is an element of the exponential torus.

Nonlinear Stokes operators

Let

$$\begin{aligned} \mathsf{X}_1^{\cap}(0) &= \left\{|\arg x| < \frac{\pi}{2}-\eta,\, |x|<\delta_x\right\} \subset \mathsf{X}^{\frown}(0),\\ \mathsf{X}_2^{\cap}(0) &= \left\{|\arg x+\pi| < \frac{\pi}{2}-\eta,\, |x|<\delta_x\right\} \subset \mathsf{X}^{\smile}(0), \end{aligned}$$

be the left and right intersection sectors of the overlapping sectors $\mathsf{X}^{\frown}(0)$, $\mathsf{X}^{\smile}(0)$ (see Figure 2.2). The two transition maps

$$\begin{aligned} \mathfrak{S}_1(q,p,x,0) &= \boldsymbol{\Psi}^{\smile}(\,\cdot\,,x,0)\circ(\boldsymbol{\Psi}^{\frown})^{\circ(-1)}(q,p,x,0), \qquad x\in\mathsf{X}_1^{\cap}(0),\\ \mathfrak{S}_2(q,p,x,0) &= \boldsymbol{\Psi}^{\frown}(\,\cdot\,,x,0)\circ(\boldsymbol{\Psi}^{\smile})^{\circ(-1)}(q,p,x,0), \qquad x\in\mathsf{X}_2^{\cap}(0), \end{aligned} \tag{2.16}$$

are called the *nonlinear Stokes operators* of the Painlevé system acting on the foliation (2.7) while preserving the fibers $x = \mathit{const}$. They are exponentially close to identity on the sectors of their definition $\mathsf{X}_1^{\cap}(0)$, resp. $\mathsf{X}_2^{\cap}(0)$. Let us remark, that unlike the monodromy, these Stokes operators are defined only locally on the foliation. Their action on the canonical solutions (2.11) is represented by a pair of transformations $\widetilde{\mathbf{S}}_1(c)$, $\widetilde{\mathbf{S}}_2(c)$ of the initial condition c defined by

$$\begin{aligned} \mathfrak{S}_1(\,\cdot\,,x,0)\circ\binom{q^{\frown}}{p^{\frown}}(x,0;c) &= \binom{q^{\frown}}{p^{\frown}}(x,0;\cdot\,)\circ\widetilde{\mathbf{S}}_1(c),\\ \mathfrak{S}_2(\,\cdot\,,x,0)\circ\binom{q^{\smile}}{p^{\smile}}(x,0;c) &= \binom{q^{\smile}}{p^{\smile}}(x,0;\cdot\,)\circ\widetilde{\mathbf{S}}_2(c), \end{aligned} \tag{2.17}$$

which are symplectic w.r.t. the canonical form $\mathrm{d}c_1 \wedge \mathrm{d}c_2$ and independent of x. This pair $\big(\widetilde{\mathbf{S}}_1(c), \widetilde{\mathbf{S}}_2(c)\big)$ is well defined up to a simultaneous conjugation by some symmetry $\mathbf{T}_\alpha$. It provides a local analytic invariant of the system (2.7) with respect to fiber-preserving transversely symplectic transformations (e.g. [33, Theorem 36]).

Remark 2.6. — The operators $\widetilde{\mathbf{S}}_1$, resp. $\widetilde{\mathbf{S}}_2$, are the nonlinear Stokes operators corresponding to the singular directions $-\pi$, resp. 0, whereas $\mathbf{N}^{\circ(-1)} \circ \widetilde{\mathbf{S}}_2 \circ \mathbf{N}$ is the nonlinear Stokes operator corresponding to the singular direction π.

Nonlinear monodromy operator:

Analytic continuation of solutions along a simple positive loop around the origin in the x-coordinate acts locally on the space of solutions as the nonlinear monodromy operator $\mathfrak{M}(q,p,x,0)$. Its representation by its action $\widetilde{\mathbf{M}}^\bullet(c)$ on the initial condition of either of the solutions $\begin{pmatrix} q^\bullet \\ p^\bullet \end{pmatrix}$, $\bullet = \frown, \smile$,

$$\mathfrak{M}(\,\cdot\,,x,0)\circ\begin{pmatrix} q^\bullet \\ p^\bullet \end{pmatrix}(x,0;c) = \begin{pmatrix} q^\bullet \\ p^\bullet \end{pmatrix}(\mathrm{e}^{2\pi\mathrm{i}}x,0;c) = \begin{pmatrix} q^\bullet \\ p^\bullet \end{pmatrix}(x,0;\cdot\,)\circ\widetilde{\mathbf{M}}^\bullet(c), \tag{2.18}$$

is expressed (see [33]) as the composition

$$\widetilde{\mathbf{M}}^\frown = \widetilde{\mathbf{S}}_2 \circ \widetilde{\mathbf{S}}_1 \circ \mathbf{N}, \qquad \widetilde{\mathbf{M}}^\smile = \widetilde{\mathbf{S}}_1 \circ \mathbf{N} \circ \widetilde{\mathbf{S}}_2. \tag{2.19}$$

Note that $\widetilde{\mathbf{M}}^\frown(c)$ and $\widetilde{\mathbf{M}}^\smile(c)$ are conjugated, being two representations of the same monodromy map $\mathfrak{M}(q,p,x,0)$.

Nonlinear wild monodromy pseudogroup

Definition 2.7. — *The pseudogroup action on the foliation of P_V that is generated by both the Stokes operators, the monodromy operator, and the exponential torus*

$$\big\langle \mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{M}, \{\mathfrak{T}_\alpha^\frown, \mathfrak{T}_\alpha^\smile \mid \alpha \in \mathcal{O}(\mathbb{C},0)\}\big\rangle$$

is called the nonlinear wild monodromy pseudo-group. *Its representation on the c-space of initial conditions is generated by*

$$\big\langle \widetilde{\mathbf{S}}_1, \widetilde{\mathbf{S}}_2, \{\mathbf{T}_\alpha \mid \alpha \in \mathcal{O}(\mathbb{C},0)\}\big\rangle.$$

The central problem of this paper is to relate this wild monodromy of P_V to the monodromy of P_{VI} and to describe its dynamics.

3. Unfolding of the nonlinear Stokes phenomenon

It is not very surprising that the only monodromy operators of the confluent family of P_{VI} that converge when $\epsilon \to 0$ are those corresponding to the loops in $\pi_1\big(\mathbb{CP}^1 \smallsetminus \{-\frac{1}{\epsilon}, 0, \infty\}, t_0\big)$ that persist to the limit as loops in $\pi_1\big(\mathbb{CP}^1 \smallsetminus \{0, \infty\}, t_0\big)$, while those corresponding to the vanishing loops diverge. In fact, for each $\epsilon \neq 0$ the nonlinear monodromy group of P_{VI} is discretely generated, while for $\epsilon = 0$ the nonlinear wild monodromy group of P_V is generated by a continuous family. However, the author's paper [33] shows that generators of the wild monodromy pseudogroup can be obtained through the accumulation of monodromy when $\epsilon \to 0$ along the sequences $\{\epsilon_n\}_{n\in\pm\mathbb{N}}$,

$$\frac{1}{\epsilon_n} = \frac{1}{\epsilon_0} + n.$$

In this section we will summarize the results of [33] on the confluence $P_{VI} \to P_V$.

3.1. Sectorial normalization and the unfolded nonlinear Stokes operators

In the coordinate

$$x = \widetilde{t}^{\,-1} + \epsilon,$$

the confluent Painlevé system (2.4) is written as

$$x(x-\epsilon)\frac{\mathrm{d}q}{\mathrm{d}x} = \frac{\partial}{\partial p}H(q,p,x,\epsilon), \qquad x(x-\epsilon)\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{\partial}{\partial q}H(q,p,x,\epsilon), \tag{3.1}$$

with

$$\begin{aligned} H(q,p,x,\epsilon) &= -(1+\epsilon\widetilde{t})H_{VI}^{\mathrm{conf}}(q,p,\widetilde{t},\epsilon) \\ &= \big(1-\epsilon-(x-\epsilon)\vartheta_0 - x(\vartheta_0+\widetilde{\vartheta}_1-1)\big)qp + (2x-\epsilon)(qp)^2 \\ &\quad + \frac{(\vartheta_0+\widetilde{\vartheta}_1-1)^2-(\vartheta_\infty-1)^2}{4}\big((x-\epsilon)q - x\big) + x\vartheta_0 p \\ &\quad + (x-\epsilon)(\vartheta_0+\widetilde{\vartheta}_1-1)q^2p - xqp^2 - (x-\epsilon)q^3p^2. \end{aligned} \tag{3.2}$$

An essential tool in understanding the relation between the monodromy of this system for $\epsilon \neq 0$ and the wild monodromy of the limit system with $\epsilon = 0$ is the following theorem which is an unfolded generalization of Theorem 2.1.

THEOREM 3.1 ([33, Theorems 17 & 43]).

(i) Formal normalization: *The confluent Painlevé system* (3.1) *can be brought to a formal normal form*

$$x(x-\epsilon)\frac{\mathrm{d}u}{\mathrm{d}x} = \big(1-\epsilon-(x-\epsilon)\vartheta_0-x(\vartheta_0+\widetilde{\vartheta}_1-1)+2(2x-\epsilon)u_1u_2\big)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}u. \tag{3.3}$$

by means of a formal transversely symplectic change of coordinates

$$\begin{pmatrix} q \\ p \end{pmatrix} = \widehat{\mathbf{\Psi}}(u,x,\epsilon) = \sum_{k,l\geqslant 0} \psi^{(kl)}(u)\, x^k \epsilon^l, \tag{3.4}$$

where $\psi^{(kl)}(u)$ *are analytic on some fixed polydisc* $\mathsf{U} = \{|u_1|, |u_2| < \delta_u\}$, $\delta_u > 0$.

The restriction of $\widehat{\mathbf{\Psi}}(u,x,\epsilon)$ *to the strong invariant manifolds* $x = 0$: $\widehat{\mathbf{\Psi}}(u,0,\epsilon) = \sum_{l\geqslant 0}\psi^{(0l)}(u)\,\epsilon^l$, *and* $x=\epsilon$: $\widehat{\mathbf{\Psi}}(u,\epsilon,\epsilon) = \sum_{k,l\geqslant 0}\psi^{(kl)}(u)\,\epsilon^{k+l}$, *are convergent for* $(u,\epsilon) \in \mathsf{U}\times\{|\epsilon| < \delta_\epsilon\}$ *for some* $\delta_\epsilon > 0$.

(ii) Unfolded sectorial normalization: *Let* $\eta > 0$ *be some arbitrarily small constant, and let* $\delta_x \gg \delta_\epsilon > 0$ *denote the radii of small discs at the origin in the* x*-and* ϵ*-space (depending on* η*). Let*

$$\mathsf{E}_\pm := \{|\epsilon| < \delta_\epsilon,\ |\arg(\pm\epsilon)| < \pi - 2\eta\} \tag{3.5}$$

be two sectors in the ϵ*-space. For* $\epsilon \in \mathsf{E}_\pm$*, define a "spiraling domain"* $\mathsf{X}_\pm(\epsilon)$ *(see Figure 3.1) as a simply connected ramified domain spanned by the complete real-time trajectories of the vector fields*

$$\mathrm{e}^{\mathrm{i}\omega_\pm} x(x-\epsilon)\frac{\partial}{\partial x} \tag{3.6}$$

that never leave the disc of radius δ_x*, where* $\omega_\pm$ *is varying in the interval*

$$\begin{cases} \max\{0,\arg(\pm\epsilon)\} - \frac{\pi}{2} + \eta < \omega_\pm < \min\{0,\arg(\pm\epsilon)\} + \frac{\pi}{2} - \eta, & \\ & \text{for } \epsilon \neq 0, \\ |\omega_\pm| < \frac{\pi}{2} - \eta, & \text{for } \epsilon = 0. \end{cases} \tag{3.7}$$

On this domain, there exists a bounded transversely symplectic change of coordinates

$$\begin{pmatrix} q \\ p \end{pmatrix} = \mathbf{\Psi}_\pm(u,x,\epsilon), \qquad u \in \mathsf{U},\ x \in \mathsf{X}_\pm(\epsilon),\ \epsilon \in \mathsf{E}_\pm,$$

analytic on the interior of the domain, which brings the confluent Painlevé system (3.1) *to its formal normal form* (3.3).

When ϵ *tends radially to* 0 *with* $\arg\epsilon = \beta$*, then* $\mathbf{\Psi}_\pm(u,x,\epsilon)$ *converges to* $\mathbf{\Psi}_\pm(u,x,0)$ *uniformly on compact sets of the sub-domains*

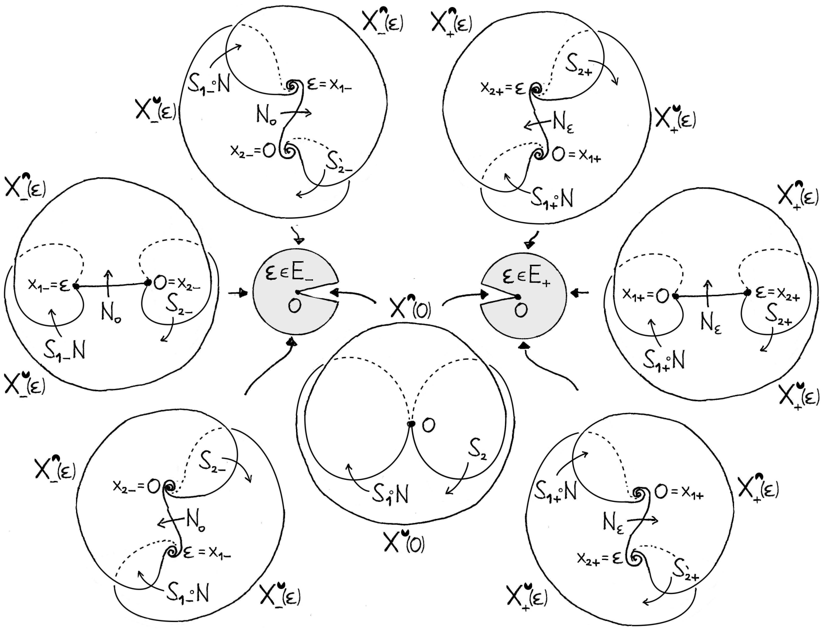


Figure 3.1. Connecting transformation between the general solutions $(q_{\pm}^{\bullet}, p_{\pm}^{\bullet})(x,\epsilon;c) = \mathbf{\Psi}_{\pm}(u^{\bullet}(x,\epsilon;c),x,\epsilon)$, on the two parts $\mathsf{X}_{\pm}^{\frown}(\epsilon), \mathsf{X}_{\pm}^{\smile}(\epsilon)$ of the domain $\mathsf{X}_{\pm}(\epsilon)$.

$\lim_{\substack{\epsilon\to 0\\ \arg\epsilon=\beta}} \mathsf{X}_{\pm}(\epsilon) \subseteq \mathsf{X}_{\pm}(0)$. *The limit domain* $\mathsf{X}_{+}(0) = \mathsf{X}_{-}(0)$ *consists of a pair of sectors* $\mathsf{X}^{\frown}(0)$, $\mathsf{X}^{\smile}(0)$ *with a common point at* 0, *and the transformation* $\mathbf{\Psi}_{+}(u,x,0) = \mathbf{\Psi}_{-}(u,x,0)$ *consists in fact of a pair of sectorial transformations* $\mathbf{\Psi}^{\frown}(u,x,0)$, $\mathbf{\Psi}^{\smile}(u,x,0)$ *of Theorem 2.1 (it is a functional cochain in the terminology of* [40]*).*

The transformations $\mathbf{\Psi}_{\pm}(u,x,\epsilon)$ *are asymptotic to* $\widehat{\mathbf{\Psi}}(u,x,\epsilon)$ *when* $(x,\epsilon)\to 0$ *inside the domain* $\coprod_{\epsilon\in\mathsf{E}_{\pm}} \mathsf{X}_{\pm}(\epsilon) \ni$, *and they satisfy the same* $(\frac{\partial}{\partial u}, \frac{\partial}{\partial x}, \frac{\partial}{\partial \epsilon})$*-differential relations with meromorphic coefficients.*

Remark 3.2.

(1) Strictly speaking, the domains $\mathsf{X}_{\pm}(\epsilon)$ are defined on the universal covering of $\{|x| < \delta_x\} \smallsetminus \{0,\epsilon\}$.

(2) The definition of the domains $\mathsf{X}_{\pm}(\epsilon)$ in Theorem 3.1 is such that when x approaches a singular point $x_{i\pm}(\epsilon)$, where

$$x_{1+}(\epsilon) = x_{2-}(\epsilon) = 0, \qquad x_{1-}(\epsilon) = x_{2+}(\epsilon) = \epsilon, \tag{3.8}$$

from within the domain, then the corresponding components of the general solution (3.9) tend to

$$u_i(x,\epsilon) \longrightarrow \infty, \quad u_{3-i}(x,\epsilon) = \frac{h}{u_i(x,\epsilon)} \longrightarrow 0,$$

when $x \to x_{i\pm}(\epsilon)$ along a real trajectory of (3.6).

(3) Alternatively, the form of the domains $\mathsf{X}_{\pm}(\epsilon)$ could be also understood from the point of view of exact WKB analysis of the system (3.1) after a change of variable $x = \epsilon z$.

Canonical 2-parameter family of solutions

The Hamiltonian (3.2) satisfies $H \circ \widehat{\boldsymbol{\Psi}}(u,x,\epsilon) = G(u,x,\epsilon) + O(x(x-\epsilon))$, where

$$\begin{aligned} G(u,x,\epsilon) = {} & \frac{(\vartheta_0 + \widetilde{\vartheta}_1 - 1)^2 - (\vartheta_\infty - 1)^2}{4} x \\ & + \big(1-\epsilon-(x-\epsilon)\vartheta_0 - x(\vartheta_0+\widetilde{\vartheta}_1-1)\big)u_1u_2 + (2x-\epsilon)(u_1u_2)^2, \end{aligned}$$

and $\frac{G(u,x,\epsilon)}{x(x-\epsilon)}$ is a time-x-dependent Hamiltonian for the normal form (3.3). The general solutions of (3.3) are of the form

$$\begin{aligned} u_1(x,\epsilon;c) &= c_1 E(c_1c_2,x,\epsilon), \\ u_2(x,\epsilon;c) &= c_2 E(c_1c_2,x,\epsilon)^{-1}, \end{aligned} \qquad c = {}^{\mathrm{t}}(c_1,c_2) \in \mathbb{C}^2, \tag{3.9}$$

where

$$E(h,x,\epsilon) = \begin{cases} x^{-\frac{1}{\epsilon}+1-\vartheta_0+2h}(x-\epsilon)^{\frac{1}{\epsilon}-\vartheta_0-\tilde{\vartheta}_1+2h}, & \text{for } \epsilon \neq 0, \\ \mathrm{e}^{-\frac{1}{x}} x^{-2\vartheta_0-\tilde{\vartheta}_1+1+4h}, & \text{for } \epsilon = 0. \end{cases} \tag{3.10}$$

In order for a branch of the solution $u(x,\epsilon;c)$ (3.9) to have a limit when $\epsilon \to 0$, one needs to further cut the domains $\mathsf{X}_{\pm}(\epsilon)$, $\epsilon \neq 0$, to two parts

$$\text{an upper part } \mathsf{X}^{\frown}_{\pm}(\epsilon), \qquad \text{and a lower part } \mathsf{X}^{\smile}_{\pm}(\epsilon), \tag{3.11}$$

corresponding to the two parts $\mathsf{X}^{\frown}(0)$ and $\mathsf{X}^{\smile}(0)$ of $\mathsf{X}_+(0) = \mathsf{X}_-(0)$, by a cut in between the singular points 0 and ϵ along a suitable trajectory of (3.6) (see Figure 3.1). The two parts of $\mathsf{X}_{\pm}(\epsilon)$ then intersect in two outer spiraling sectors

$$\mathsf{X}^{\cap}_{i\pm}(\epsilon) = \{x \in \mathsf{X}_{\pm}(\epsilon) : x_{i\pm} + \mathrm{e}^{2\pi\mathrm{i}}(x - x_{i\pm}) \in \mathsf{X}_{\pm}(\epsilon)\}, \qquad i = 1,2,$$

attached to the singularities $x_{i\pm}$ (3.8), and along the central cut between the singularities $\{x_{1\pm}, x_{2\pm}\} = \{0, \epsilon\}$.

We now take $E_{\pm}^{\boldsymbol{\cap}}(h, x, \epsilon)$ and $E_{\pm}^{\boldsymbol{\cup}}(h, x, \epsilon)$ as two branches of $E(h, x, \epsilon)$ (3.10) on the two parts (3.11) of the domain, that agree on the right intersection sector $\mathsf{X}_{2\pm}^{\cap}$, and have a limit when $\epsilon \to 0$. They determine a pair of general solutions (3.9) of the model system

$$u_{\pm}^{\bullet}(x, \epsilon; c), \qquad \bullet = \boldsymbol{\cap}, \boldsymbol{\cup},$$

and a pair of *canonical 2-parameter families of solutions* of the original system (3.1)

$$\begin{pmatrix} q_{\pm}^{\bullet} \\ p_{\pm}^{\bullet} \end{pmatrix}(x, \epsilon; c) := \boldsymbol{\Psi}_{\pm}(u_{\pm}^{\bullet}(x, \epsilon; c), x, \epsilon), \qquad \bullet = \boldsymbol{\cap}, \boldsymbol{\cup}. \tag{3.12}$$

Unfolded center manifold solution

Corollary 3.3. — *The system* (3.1) *has a unique bounded analytic solution on the domain* $\mathsf{X}_{\pm}^{\bullet}(\epsilon)$, $\epsilon \in \mathsf{E}_{\pm}$, $\bullet = \boldsymbol{\cap}, \boldsymbol{\cup}$, *given by* $\begin{pmatrix} q_{\pm}^{\bullet} \\ p_{\pm}^{\bullet} \end{pmatrix}(x, \epsilon; 0) = \boldsymbol{\Psi}_{\pm}(0, x, \epsilon)$ *(which agrees between* $\mathsf{X}_{\pm}^{\boldsymbol{\cap}}(\epsilon)$ *and* $\mathsf{X}_{\pm}^{\boldsymbol{\cup}}(\epsilon)$ *along the central intersection). We call it the* "unfolded sectorial center manifold" *of the foliation.*

Remark 3.4. — This kind of solution seems to be new in the literature. In our opinion this is an analogue of the bi-tronquée solution for P_{VI}. While we have obtained it by the confluence, this solution is simply characterized by the fact that it is bounded at both of the singular points when approached along certain logarithmic spiral. Due to the overall symmetry of P_{VI} there should be in general one such solution for every selection of a pair of singular points together with suitable paths of approach.

Unfolded exponential torus

Proposition 3.5 ([33, Proposition 31]). — *For* $\epsilon \in \mathsf{E}_{\pm}$, *any symmetry of* (3.3) *that is bounded and analytic on* $(u, x) \in \mathsf{U} \times \mathsf{X}_{\pm}(\epsilon)$ *is of the form* $\mathbf{T}_{a_\epsilon}$ (2.12) *with* $a_\epsilon(h)$ *analytic.*

Corollary 3.6 ([33, Corollary 32]). — *The normalizing transformations* $\widehat{\boldsymbol{\Psi}}$ *and* $\boldsymbol{\Psi}_{\pm}$ *of Theorem 3.1 are unique up to a right composition with the same analytic symmetry* $\mathbf{T}_{\alpha}(u, \epsilon)$ *where* $\alpha(h, \epsilon)$ *is an analytic germ in* (h, ϵ) *at the origin which is uniquely determined by the convergent series* $\widehat{\boldsymbol{\Psi}}(u, 0, \epsilon)$.

Unfolded nonlinear Stokes operators

The connecting transformations between the 2-parameter families of solutions (3.12) $\begin{pmatrix} q_\pm^\bullet \\ p_\pm^\bullet \end{pmatrix}$ on the 3 intersections between $\mathsf{X}_\pm^{\cap}(\epsilon)$ and $\mathsf{X}_\pm^{\cup}(\epsilon)$, when represented by an action on the initial condition c, are as in Figure 3.1.

There, the operators $\mathbf{S}_{1\pm}(c,\epsilon)$, $\mathbf{S}_{2\pm}(c,\epsilon)$ are the representations of *unfolded Stokes operators* $\mathfrak{S}_1(\,\cdot\,,x,\epsilon)$, $\mathfrak{S}_2(\,\cdot\,,x,\epsilon)$, defined in the same way as in (2.16), (2.17) on the intersection sectors $\mathsf{X}_{i\pm}^{\cap}(\epsilon)$, $i=1,2$, and converge to the Stokes operators (2.17) when $\epsilon\to 0$ in the sector $\mathsf{E}_\pm$

$$\mathbf{S}_{i\pm}(c,\epsilon) \longrightarrow \mathbf{S}_{i\pm}(c,0) = \widetilde{\mathbf{S}}_i(c) \quad \text{as} \quad \mathsf{E}_\pm \ni \epsilon \longrightarrow 0.$$

On the other hand the *formal monodromies* (corresponding to the monodromies of the solutions (3.9) of the formal normal form (3.3))

$$\begin{aligned} \mathbf{N}_0(c,\epsilon) &= \mathbf{T}_{2\pi\mathrm{i}(-\frac{1}{\epsilon}-\vartheta_0+2h)}(c) = \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}(-\frac{1}{\epsilon}-\vartheta_0+2c_1c_2)} & 0 \\ 0 & \mathrm{e}^{2\pi\mathrm{i}(\frac{1}{\epsilon}+\vartheta_0-2c_1c_2)} \end{pmatrix} c, \\ \mathbf{N}_\epsilon(c,\epsilon) &= \mathbf{T}_{2\pi\mathrm{i}(\frac{1}{\epsilon}-\vartheta_0-\tilde\vartheta_1+1+2h)}(c) = \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}(\frac{1}{\epsilon}-\vartheta_0-\tilde\vartheta_1+2c_1c_2)} & 0 \\ 0 & \mathrm{e}^{2\pi\mathrm{i}(-\frac{1}{\epsilon}+\vartheta_0+\tilde\vartheta_1-2c_1c_2)} \end{pmatrix} c, \end{aligned} \tag{3.13}$$

diverge when $\epsilon\to 0$, except for the formal monodromy operator

$$\mathbf{N} = \mathbf{N}_0 \circ \mathbf{N}_\epsilon = \mathbf{T}_{2\pi\mathrm{i}(-2\vartheta_0-\tilde\vartheta_1+4h)}. \tag{3.14}$$

Decomposition of nonlinear monodromy operators

For $\epsilon \neq 0$, one can represent the action of the *nonlinear monodromy operators* $\mathfrak{M}_0(q,p,x,\epsilon)$ and $\mathfrak{M}_\epsilon(q,p,x,\epsilon)$ around the singular points 0 and ϵ in positive direction by their action on the initial condition c of the general solutions (3.12),

$$\mathfrak{M}_{x_{i\pm}}(\,\cdot\,,x,\epsilon)\circ \begin{pmatrix} q_\pm^\bullet \\ p_\pm^\bullet \end{pmatrix}(x,\epsilon;c) = \begin{pmatrix} q_\pm^\bullet \\ p_\pm^\bullet \end{pmatrix}(x,\epsilon;\cdot\,)\circ \mathbf{M}_{x_{i\pm}}^\bullet(c,\epsilon), \qquad \bullet = \cap, \cup,$$

and express it as

$$\begin{aligned} \mathbf{M}_{x_{1\pm}}^{\cap} &= \mathbf{N}_{x_{2\pm}}^{\circ(-1)} \circ \mathbf{S}_{1\pm} \circ \mathbf{N}, & \qquad \mathbf{M}_{x_{1\pm}}^{\cup} &= \mathbf{S}_{1\pm}\circ \mathbf{N}_{x_{1\pm}}, \\ \mathbf{M}_{x_{2\pm}}^{\cap} &= \mathbf{S}_{2\pm} \circ \mathbf{N}_{x_{2\pm}}, & \qquad \mathbf{M}_{x_{2\pm}}^{\cup} &= \mathbf{N}_{x_{2\pm}}\circ \mathbf{S}_{2\pm}, \end{aligned} \tag{3.15}$$

see Figure 3.1.

3.2. Accumulation of monodromy

We can now formulate the essential result of [33] which allows to obtain a representation of the wild monodromy pseudogroup of the limit system as an accumulation of the monodromy pseudogroup of the system when $\epsilon \to 0$.

Let $\{\epsilon_n\}_{n\in\pm\mathbb{N}}$ be sequence in $\mathsf{E}_\pm \smallsetminus \{0\}$ defined by

$$\frac{1}{\epsilon_n} = \frac{1}{\epsilon_0} + n, \qquad \epsilon_0 \in \mathsf{E}_\pm \smallsetminus \{0\},\ n \in \pm\mathbb{N}, \tag{3.16}$$

along which the divergent exponential factor $\mathrm{e}^{\frac{2\pi\mathrm{i}}{\epsilon}}$ in the formal monodromy (3.13) stays constant. Let

$$\widetilde{\mathfrak{M}}_{x_{i\pm}}(q,p,x;\kappa) := \lim_{n\to\pm\infty} \mathfrak{M}_{x_{i\pm}}(q,p,x,\epsilon_n), \quad \kappa := \mathrm{e}^{\frac{2\pi\mathrm{i}}{\epsilon_0}},$$

and let $\widetilde{\mathbf{M}}^{\bullet}_{x_{i\pm}}(c;\kappa)$ be the corresponding limit action (2.18) on the initial conditions of the general solution $\begin{pmatrix} q^{\bullet}_\pm \\ p^{\bullet}_\pm \end{pmatrix}(x,0;c)$. Then

$$\begin{aligned}
&\widetilde{\mathbf{M}}^{\curlywedge}_{0+}(c;\kappa) = \widetilde{\mathbf{N}}_\epsilon(\,\cdot\,;\kappa)^{\circ(-1)}\circ\widetilde{\mathbf{S}}_1\circ\mathbf{N}(c), && \widetilde{\mathbf{M}}^{\curlyvee}_{0+}(c;\kappa) = \widetilde{\mathbf{S}}_1\circ\widetilde{\mathbf{N}}_0(c;\kappa),\\
&\widetilde{\mathbf{M}}^{\curlywedge}_{\epsilon+}(c;\kappa) = \widetilde{\mathbf{S}}_2\circ\widetilde{\mathbf{N}}_\epsilon(c;\kappa), && \widetilde{\mathbf{M}}^{\curlyvee}_{\epsilon+}(c;\kappa) = \widetilde{\mathbf{N}}_\epsilon(\,\cdot\,;\kappa)\circ\widetilde{\mathbf{S}}_2(c),\\
&\widetilde{\mathbf{M}}^{\curlywedge}_{\epsilon-}(c;\kappa) = \widetilde{\mathbf{N}}_0(\,\cdot\,;\kappa)^{\circ(-1)}\circ\widetilde{\mathbf{S}}_1\circ\mathbf{N}(c), && \widetilde{\mathbf{M}}^{\curlyvee}_{\epsilon-}(c;\kappa) = \widetilde{\mathbf{S}}_1\circ\widetilde{\mathbf{N}}_\epsilon(c;\kappa),\\
&\widetilde{\mathbf{M}}^{\curlywedge}_{0-}(c;\kappa) = \widetilde{\mathbf{S}}_2\circ\widetilde{\mathbf{N}}_0(c;\kappa), && \widetilde{\mathbf{M}}^{\curlyvee}_{0-}(c;\kappa) = \widetilde{\mathbf{N}}_0(\,\cdot\,;\kappa)\circ\widetilde{\mathbf{S}}_2(c),
\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}
\widetilde{\mathbf{N}}_0(c;\kappa) &= \mathbf{T}_{2\pi\mathrm{i}(-\vartheta_0+2h)-\log\kappa}(c) = \begin{pmatrix} \frac{1}{\kappa}\mathrm{e}^{2\pi\mathrm{i}(-\vartheta_0+2c_1c_2)} & 0 \\ 0 & \kappa\mathrm{e}^{2\pi\mathrm{i}(\vartheta_0-2c_1c_2)} \end{pmatrix} c,\\
\widetilde{\mathbf{N}}_\epsilon(c;\kappa) &= \mathbf{T}_{2\pi\mathrm{i}(-\vartheta_0-\tilde\vartheta_1+2h)+\log\kappa}(c) = \begin{pmatrix} \kappa\mathrm{e}^{2\pi\mathrm{i}(-\vartheta_0-\tilde\vartheta_1+2c_1c_2)} & 0 \\ 0 & \frac{1}{\kappa}\mathrm{e}^{2\pi\mathrm{i}(\vartheta_0+\tilde\vartheta_1-2c_1c_2)} \end{pmatrix} c,
\end{aligned} \tag{3.18}$$

and $\mathbf{N}(c) = \widetilde{\mathbf{N}}_0(\,\cdot\,;\kappa)\circ\widetilde{\mathbf{N}}_\epsilon(c;\kappa)$, are elements of the nonlinear exponential torus. (The subscripts $0\pm$, $\epsilon\pm$ in (3.17), (3.18) are purely symbolical and no longer related to the parameter ϵ.)

In order to express the *nonlinear Stokes operators* from the monodromy one can substitute in (3.17) $\mathrm{e}^{2\pi\mathrm{i}(-\vartheta_0+2c_1c_2)}$ for κ in $\widetilde{\mathbf{M}}^{\bullet}_{0\pm}$ to kill the factor

$\widetilde{\mathbf{N}}_0(\kappa)$, resp. $\mathrm{e}^{2\pi\mathrm{i}(\vartheta_0+\tilde{\vartheta}_1-2c_1c_2)}$ in $\widetilde{\mathbf{M}}^{\bullet}_{\epsilon\pm}$ to kill the factor $\widetilde{\mathbf{N}}_\epsilon$, hence e.g.

$$\begin{aligned}
\widetilde{\mathbf{S}}_1(c) &= \widetilde{\mathbf{M}}^{\smile}_{0+}\big(c;\mathrm{e}^{2\pi\mathrm{i}(-\vartheta_0+2h(c))}\big),\\
\widetilde{\mathbf{S}}_2(c) &= \widetilde{\mathbf{M}}^{\smile}_{\epsilon+}\big(c;\mathrm{e}^{2\pi\mathrm{i}(\vartheta_0+\tilde{\vartheta}_1-2h(c))}\big)\\
&= \widetilde{\mathbf{M}}^{\frown}_{\epsilon+}\big(c;\mathrm{e}^{2\pi\mathrm{i}(\vartheta_0+\tilde{\vartheta}_1-2h(c))}\big),\\
\mathbf{N}^{\circ(-1)}\circ\widetilde{\mathbf{S}}_1\circ\mathbf{N}(c) &= \widetilde{\mathbf{M}}^{\frown}_{0+}\big(c;\mathrm{e}^{2\pi\mathrm{i}(-\vartheta_0+2h(c))}\big),
\end{aligned}\tag{3.19}$$

where $h(c)=c_1c_2$.

Proposition 3.7. — *The wild monodromy pseudogroup of P_V (Definition 2.7) is generated by the limit operators*

$$\big\langle\widetilde{\mathfrak{M}}_{0\pm}(q,p,x;\mathrm{e}^{\alpha(\mathfrak{h}^\bullet)}),\ \widetilde{\mathfrak{M}}_{\epsilon\pm}(q,p,x;\mathrm{e}^{\alpha(\mathfrak{h}^\bullet)})\ \big|\ \alpha\in\mathcal{O}(\mathbb{C},0)\big\rangle,$$

where $\mathfrak{h}^\bullet$ is defined by $\mathfrak{h}^\bullet\circ\Psi^\bullet=u_1u_2$. Its representation on the c-space of initial conditions over $\mathsf{X}^\bullet(0)$, $\bullet=\frown,\smile$, is generated by,

$$\big\langle\widetilde{\mathbf{M}}^{\bullet}_{0\pm}(c;\mathrm{e}^{\alpha(h(c))}),\ \widetilde{\mathbf{M}}^{\bullet}_{\epsilon\pm}(c;\mathrm{e}^{\alpha(h(c))})\ \big|\ \alpha\in\mathcal{O}(\mathbb{C},0)\big\rangle,\quad \textit{where}\ \ h(c)=c_1c_2.$$

Proof. — Let for example $\mathsf{X}^\bullet_\pm=\mathsf{X}^{\frown}_+$. Then by (3.17), (3.18) both maps $\widetilde{\mathbf{M}}^{\frown}_{0+}(c;\mathrm{e}^{\alpha(h)})$ and $\widetilde{\mathbf{M}}^{\frown}_{\epsilon+}(c;\mathrm{e}^{\alpha(h)})$ belong to the representation of the wild monodromy pseudogroup. Conversely, one can express

$$\mathbf{T}_\alpha=\widetilde{\mathbf{M}}^{\frown}_{0+}(\,\cdot\,;\mathrm{e}^{-\alpha(\,\cdot\,)})\circ(\widetilde{\mathbf{M}}^{\frown}_{0+})^{\circ(-1)}(c;\mathrm{e}^{\alpha(h)}),$$

and $\widetilde{\mathbf{S}}_1(c)$, $\widetilde{\mathbf{S}}_2(c)$ using (3.19). □

The vector field $(c_1\frac{\partial}{\partial c_1}-c_2\frac{\partial}{\partial c_2})$, which is in a sense an *"infinitesimal generator" of the exponential torus* (2.14), is Hamiltonian vector field of $h=c_1c_2$ with respect to $\mathrm{d}c_1\wedge\mathrm{d}c_2$. It can be expressed as e.g.

$$\begin{aligned}
\dot c &= -\left(\kappa\frac{\mathrm{d}}{\mathrm{d}\kappa}\widetilde{\mathbf{N}}_0(\,\cdot\,;\kappa)\right)\circ\widetilde{\mathbf{N}}_0(c;\kappa)^{\circ(-1)}\\
&= -\left(\kappa\frac{\mathrm{d}}{\mathrm{d}\kappa}\widetilde{\mathbf{M}}^{\frown}_{0+}(\,\cdot\,;\kappa)\right)\circ\widetilde{\mathbf{M}}^{\frown}_{0+}(c;\kappa)^{\circ(-1)}.
\end{aligned}\tag{3.20}$$

Among its Hamiltonians, which are determined only up to an additive constant, the function $h=c_1c_2$ is characterized by vanishing on the center manifold solution of Corollary 3.3.

4. The character variety of P_{VI} and the nonlinear monodromy action on it

In this section we recall the usual approach to P_{VI} through isomonodromic deformations of 2×2 traceless linear systems with four Fuchsian singularities on $\mathbb{CP}^1$, the Riemann–Hilbert correspondence between Fuchsian systems and their monodromy representations, the character variety of P_{VI} and the modular group action on it, and gather some facts that will serve us later. Our main reference for this part are the articles of Iwasaki [27] and of Inaba, Iwasaki and Saito [25].

Notation 4.1. — A triple of indices (i,j,k) will always denote a permutation of $(0,t,1)$, and a quadruple (i,j,k,l) will denote a permutation of $(0,t,1,\infty)$.

4.1. Isomonodromic deformations of 2×2 systems and the Riemann–Hilbert correspondence

The sixth Painlevé equation $P_{VI}(\vartheta)$ with a parameter

$$\vartheta = (\vartheta_0,\vartheta_t,\vartheta_1,\vartheta_\infty) \in \mathbb{C}^4$$

governs isomonodromic deformations of traceless 2×2 linear differential systems with four Fuchsian singularities on $\mathbb{CP}^1$

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = \left[\frac{A_0(t)}{z} + \frac{A_t(t)}{z-t} + \frac{A_1(t)}{z-1}\right]\phi \tag{4.1}$$

with the residue matrices $A_l \in \mathfrak{sl}_2(\mathbb{C})$ having $\pm\frac{\vartheta_l}{2}$ as eigenvalues. In general (if each A_l is semi-simple and the system is irreducible), one can write

$$A_i = \begin{pmatrix} v_i+\frac{\vartheta_i}{2} & -u_iv_i \\ \frac{v_i+\vartheta_i}{u_i} & -v_i-\frac{\vartheta_i}{2}\end{pmatrix},\quad i=0,t,1,\quad -A_0-A_t-A_1 = A_\infty = \begin{pmatrix} \frac{\vartheta_\infty}{2} & 0 \\ 0 & -\frac{\vartheta_\infty}{2}\end{pmatrix},$$

for some functions $u_i(t), v_i(t)$. The isomonodromicity of such system is expressed by the Schlesinger equations

$$\frac{\mathrm{d}A_0}{\mathrm{d}t} = \frac{[A_t,A_0]}{t},\quad \frac{\mathrm{d}A_t}{\mathrm{d}t} = \frac{[A_0,A_t]}{t} + \frac{[A_1,A_t]}{t-1},\quad \frac{\mathrm{d}A_1}{\mathrm{d}t} = \frac{[A_t,A_1]}{t-1}, \tag{4.2}$$

corresponding to the integrability conditions on the logarithmic connection in variables (z,t)

$$\nabla(z,t) = \mathrm{d} - \Big[A_0(t)\,\mathrm{d}\,\log(z) + A_t(t)\,\mathrm{d}\,\log(z-t) + A_1(t)\,\mathrm{d}\,\log(z-1)\Big] \tag{4.3}$$

on the trivial rank 2 vector bundle. Denoting $[A(z,t)]_{ij}$ the (i,j)-component of the matrix of the system (4.1), then if the system is irreducible the 1-form $[A(z,t)]_{12}\,\mathrm{d}z$ is non-null, and so it must have a unique zero at some point $z=q(t)$

$$q(t) = \frac{-t[A_0]_{12}}{t[A_t]_{12} + [A_1]_{12}}.$$

This point is an apparent singularity of the second order linear ODE solved by the first component of any solution $\phi(z,t)$ of (4.1). Denoting

$$p(t) = [A(q,t)]_{11} + \frac{\vartheta_0}{2q} + \frac{\vartheta_t}{2(q-t)} + \frac{\vartheta_1}{2(q-1)},$$

then the Schlesinger equations (4.2) are equivalent to the Hamiltonian system (2.1) of P_{VI} [29, 30], whose the Hamiltonian function (2.2) is given by

$$H_{VI}\big(q(t), p(t), t\big) = \operatorname{tr}\left[\left(\frac{A_0(t)}{t} + \frac{A_1(t)}{t-1}\right) A_t(t)\right] - \frac{\vartheta_0\vartheta_t}{2t} - \frac{\vartheta_t\vartheta_1}{2(t-1)}.$$

The *tau function* τ_{VI} of P_{VI} is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\tau_{VI}(t) = \operatorname{tr}\left[\left(\frac{A_0(t)}{t} + \frac{A_1(t)}{t-1}\right) A_t(t)\right],$$

which is the coefficient of $\frac{1}{z-t}$ in $\frac{1}{2}\operatorname{tr} A(z,t)^2$.

Choosing a germ of a fundamental matrix solution $\Phi(z,t)$ of the system (4.1) near some nonsingular point z_0, one has a linear monodromy representation (anti-homomorphism)

$$\rho : \pi_1\big(\mathbb{CP}^1 \smallsetminus \{0,t,1,\infty\}, z_0\big) \longrightarrow \mathrm{SL}_2(\mathbb{C}),$$

such that the analytic continuation $\Phi(z,t)$ along a path γ defines another fundamental matrix solution $\Phi(z,t)\rho(\gamma)$. The conjugation class of such monodromy representation in $\mathrm{SL}_2(\mathbb{C})$ is independent of the choice of Φ. The *isomonodromic condition* (4.2) on the system (4.1) is equivalent to the conjugation class of the monodromy being locally constant with respect to t, or equivalently to the existence of a fundamental matrix solution $\Phi(z,t)$ whose actual monodromy is locally independent of t [7].

The *Riemann–Hilbert correspondence* in this setting is given by the monodromy map between the space of linear systems (4.1) with prescribed poles and local eigenvalue data $\pm\frac{\vartheta_l}{2}$ modulo global gauge transformations (conjugation by $\mathrm{SL}_2(\mathbb{C})$) on one side, and the space of monodromy representations with prescribed local exponents e_l, e_l^{-1}

$$e_l := \mathrm{e}^{\pi\vartheta_l}, \quad l \in \{0,t,1,\infty\}, \tag{4.4}$$

modulo the adjoint action (conjugation) of $\mathrm{SL}_2(\mathbb{C})$ on the other side (see [25] for much more precise setting of the correspondence). Therefore it can be also translated as a correspondence between solutions of P_{VI} and equivalence classes of monodromy representations.

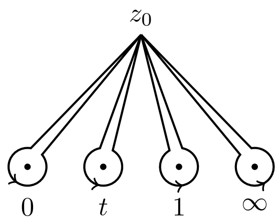


Figure 4.1. The loops $\gamma_0, \gamma_t, \gamma_1, \gamma_\infty \in \pi_1(\mathbb{CP}^1 \smallsetminus \{0, t, 1, \infty\}, z_0)$.

4.2. The character variety of P_{VI}

Given a representation (anti-homomorphism)

$$\rho : \pi_1(\mathbb{CP}^1 \smallsetminus \{0, t, 1, \infty\}, z_0) \longrightarrow \mathrm{SL}_2(\mathbb{C}), \qquad \rho(\gamma_i\gamma_j) = \rho(\gamma_j)\rho(\gamma_i), \tag{4.5}$$

(where $\gamma_i\gamma_j$ denotes the concatenation of paths, i.e. the path following first γ_i and then γ_j), let $\gamma_0, \gamma_t, \gamma_1, \gamma_\infty$ be simple loops in the z-space around $0, t, 1, \infty$ respectively such that $\gamma_0\gamma_t\gamma_1\gamma_\infty = \mathrm{id}$ (see Figure 4.1), and denote $M_l = \rho(\gamma_l)$ the corresponding monodromy matrices

$$M_\infty M_1 M_t M_0 = I.$$

The conjugacy class of an irreducible monodromy representation is completely determined by its trace coordinates by a theorem of Fricke, Klein and Vogt (cf. [38]). These coordinates are given by the four parameters

$$a_l = \mathrm{tr}(M_l) = e_l + \frac{1}{e_l} = 2\cos(\pi\vartheta_l), \quad l = 0, t, 1, \infty, \tag{4.6}$$

and the three variables

$$X_0 = \mathrm{tr}(M_1 M_t), \quad X_t = \mathrm{tr}(M_0 M_1), \quad X_1 = \mathrm{tr}(M_t M_0), \tag{4.7}$$

satisfying the *Fricke relation*

$$F(X, \theta(a)) = 0, \tag{4.8}$$

where

$$F(X, \theta) := X_0 X_t X_1 + X_0^2 + X_t^2 + X_1^2 - \theta_0 X_0 - \theta_t X_t - \theta_1 X_1 + \theta_\infty, \tag{4.9}$$

with

$$\theta_i(a) = a_i a_\infty + a_j a_k, \;\; i = 0, t, 1, \quad \theta_\infty(a) = a_0 a_t a_1 a_\infty + a_0^2 + a_t^2 + a_1^2 + a_\infty^2 - 4.$$

Definition 4.2. — *We call* the character variety of P_{VI} *the complex cubic surface*

$$\mathcal{S}_{VI}(\theta) = \{X \in \mathbb{C}^3 : F(X, \theta) = 0\}. \tag{4.10}$$

In this setting, the Riemann–Hilbert correspondence can be seen as a map between the Hamiltonian flow of the Painlevé system (defined on the Okamoto fibration $\mathcal{M}_{VI}(\vartheta) \to \mathbb{CP}^1 \smallsetminus \{0,1,\infty\}$) on one side and a locally constant flow on the character variety on the other side

$$RH_{VI,t} : \mathcal{M}_{VI,t}(\vartheta) \longrightarrow \mathcal{S}_{VI}(\theta).$$

Under this correspondence the Okamoto space of initial conditions $\mathcal{M}_{VI,t}(\vartheta)$ is a minimal resolution of singularities of $\mathcal{S}_{VI}(\theta)$ [25].

The character variety $\mathcal{S}_{VI}(\theta)$ is equipped with a natural algebraic symplectic form

$$\omega_{\mathcal{S}_{VI}} = \frac{\mathrm{d}X_t \wedge \mathrm{d}X_0}{2\pi \mathrm{i} F_{X_1}} = \frac{\mathrm{d}X_1 \wedge \mathrm{d}X_t}{2\pi \mathrm{i} F_{X_0}} = \frac{\mathrm{d}X_0 \wedge \mathrm{d}X_1}{2\pi \mathrm{i} F_{X_t}}, \tag{4.11}$$

where

$$F_{X_i} = \frac{\mathrm{d}F}{\mathrm{d}X_i} = X_j X_k + 2X_i - \theta_i.$$

The Poisson bracket associated to $-2\pi\mathrm{i}\,\omega_{\mathcal{S}_{VI}}$ is the *Goldman bracket*

$$\{X_i, X_j\} = F_{X_k}, \quad (i,j,k) \text{ a cyclic permutation of } (0,t,1).$$

PROPOSITION 4.3. — *The standard symplectic form ω* (2.6) *on the (q,p)-space corresponds through the Riemann–Hilbert correspondence to the symplectic form* (4.11).

PROPOSITION 4.4 (*Jimbo's asymptotic formula* [29]). — *Given a monodromy representation ρ as above and its associated coordinate X* (4.7) *on the character variety, then the solution of $P_{VI}(\vartheta)$ corresponding to a point $X \in \mathcal{S}_{VI}(\theta)$ has the following asymptotics when $t \to 0$ in the sector $|\arg t| < \pi$:*

$$\begin{aligned} t^{\sigma_1 - 1} q &= \alpha(X,\vartheta) + O(t^{\sigma_1}) + O(t^{1-\sigma_1}) \\ t^{1-\sigma_1} p &= \frac{\vartheta_0 + \vartheta_t - \sigma_1}{2\alpha(X,\vartheta)} + O(t^{\sigma_1}) + O(t^{1-\sigma_1}), \end{aligned} \tag{4.12}$$

where $\sigma_1 \neq 0$ is such that $2\cos\pi\sigma_1 = X_1$ and $0 \leqslant \Re\sigma_1 < 1$,

$$\alpha = \frac{(\vartheta_0 + \vartheta_t + \sigma_1)(-\vartheta_0 + \vartheta_t + \sigma_1)(\vartheta_\infty + \vartheta_1 + \sigma_1) \cdot d(\sigma_1,\vartheta)}{4\sigma_1^2\,(\vartheta_\infty + \vartheta_1 - \sigma_1) \cdot c(\sigma_1,\vartheta)\,\big(a(\sigma_1, X_0, \vartheta) + b(\sigma_1, X_1, \vartheta)\big)},$$

with

$$\begin{aligned}
a &= \frac{1}{4}\mathrm{e}^{\pi\mathrm{i}\sigma_1}\big(2\mathrm{i}\sin\pi\sigma_1\cdot X_0-\theta_t\big)=\frac{1}{4}\big((\mathrm{e}^{\pi\mathrm{i}\sigma_1}X_1-2)X_0-\theta_t\big),\\
b &= \frac{1}{4}\big(2\mathrm{i}\sin\pi\sigma_1\cdot X_t+\theta_0\big)=\frac{1}{4}\big((2\mathrm{e}^{\pi\mathrm{i}\sigma_1}-X_1)X_t+\theta_0\big),\\
c &= \frac{\Gamma\big(\frac{2+\vartheta_0+\vartheta_t+\sigma_1}{2}\big)\Gamma\big(\frac{2-\vartheta_0+\vartheta_t+\sigma_1}{2}\big)\Gamma\big(\frac{2+\vartheta_\infty+\vartheta_1+\sigma_1}{2}\big)\Gamma\big(\frac{2-\vartheta_\infty+\vartheta_1+\sigma_1}{2}\big)\Gamma(1-\sigma_1)^2}{\Gamma\big(\frac{2+\vartheta_0+\vartheta_t-\sigma_1}{2}\big)\Gamma\big(\frac{2-\vartheta_0+\vartheta_t-\sigma_1}{2}\big)\Gamma\big(\frac{2+\vartheta_\infty+\vartheta_1-\sigma_1}{2}\big)\Gamma\big(\frac{2-\vartheta_\infty+\vartheta_1-\sigma_1}{2}\big)\Gamma(1+\sigma_1)^2},\\
d &= 4\sin\frac{\pi(\vartheta_0+\vartheta_t-\sigma_1)}{2}\sin\frac{\pi(-\vartheta_0+\vartheta_t-\sigma_1)}{2}\sin\frac{\pi(\vartheta_\infty+\vartheta_1-\sigma_1)}{2}\\
&\qquad\times\sin\frac{\pi(-\vartheta_\infty+\vartheta_1-\sigma_1)}{2},
\end{aligned}$$

under the assumption that

$$\vartheta_0,\ \vartheta_t,\vartheta_1,\vartheta_\infty,\frac{\vartheta_0+\vartheta_t\pm\sigma_1}{2},\ \frac{-\vartheta_0+\vartheta_t\pm\sigma_1}{2},\ \frac{\vartheta_\infty+\vartheta_1\pm\sigma_1}{2},\ \frac{-\vartheta_\infty+\vartheta_1\pm\sigma_1}{2}\ \notin\mathbb{Z},$$

see [4, p. 191].

Proof of Proposition 4.3. — We will use the Jimbo's asymptotic formula (4.12). Since the point $X\in\mathcal{S}_{VI}(\theta)$ corresponding to a given solution $(q(t),p(t))$ of P_{VI} is locally independent of t, we can take limit $t\to 0$ and ignore higher order terms. Restricting to the subvariety where $\Re\sigma_1\neq 0$, we have

$$\begin{aligned}
\omega=\mathrm{d}q\wedge\mathrm{d}p=\frac{\mathrm{d}\sigma_1\wedge\mathrm{d}\alpha}{2\alpha}&=\frac{\mathrm{d}X_1\wedge(\mathrm{d}a+\mathrm{d}b)}{4(a+b)\pi\sin\pi\sigma_1}\\
&=-\frac{\mathrm{d}X_1\wedge(\mathrm{e}^{\pi\mathrm{i}\sigma_1}\mathrm{d}X_0+\mathrm{d}X_t)}{2\pi\mathrm{i}(\mathrm{e}^{\pi\mathrm{i}\sigma_1}F_{X_t}-F_{X_0})}=\frac{\mathrm{d}X_1\wedge\mathrm{d}X_t}{2\pi\mathrm{i}F_{X_0}},
\end{aligned}$$

using that

$$4(a+b)=\mathrm{e}^{\pi\mathrm{i}\sigma_1}F_{X_t}-F_{X_0},\tag{4.13}$$

and the identity $\frac{\mathrm{d}X_1\wedge\mathrm{d}X_0}{F_{X_t}}=\frac{\mathrm{d}X_t\wedge\mathrm{d}X_1}{F_{X_0}}$. □

4.3. Lines and singularities of $\mathcal{S}_{VI}(\theta)$

The projective completion of the character variety $\mathcal{S}_{VI}(\theta)$ in $\mathbb{CP}^3$ is a compact cubic surface. In the classical theory of classification of cubic surfaces a major role is played by the configurations of complex lines inside the surface.[2] We have:

Proposition 4.5 (Lines of $\mathcal{S}_{VI}(\theta)$). — *The Fricke polynomial* $F(X,\theta)$ (4.9) *can be decomposed as*

$$\begin{aligned}
F &= \left(X_k - \frac{e_i}{e_j} - \frac{e_j}{e_i}\right)\left(F_{X_k} - X_k + \frac{e_i}{e_j} + \frac{e_j}{e_i}\right) \\
&\quad + \frac{1}{e_ie_j}(e_iX_i + e_jX_j - a_\infty - e_ie_ja_k)(e_iX_j + e_jX_i - a_k - e_ie_ja_\infty), \\
&= \left(X_k - e_ie_j - \frac{1}{e_ie_j}\right)\left(F_{X_k} - X_k + e_ie_j + \frac{1}{e_ie_j}\right) \\
&\quad + \frac{1}{e_ie_j}(e_ie_jX_i + X_j - e_ja_\infty - e_ia_k)(e_ie_jX_j + X_i - e_ja_k - e_ia_\infty), \\
&= \left(X_k - \frac{e_k}{e_\infty} - \frac{e_\infty}{e_k}\right)\left(F_{X_k} - X_k + \frac{e_k}{e_\infty} + \frac{e_\infty}{e_k}\right) \\
&\quad + \frac{1}{e_ke_\infty}(e_\infty X_i + e_kX_j - a_i - e_ke_\infty a_j)(e_kX_i + e_\infty X_j - a_j - e_ke_\infty a_i), \\
&= \left(X_k - e_ke_\infty - \frac{1}{e_ke_\infty}\right)\left(F_{X_k} - X_k + e_ke_\infty + \frac{1}{e_ke_\infty}\right) \\
&\quad + \frac{1}{e_ke_\infty}(X_j + e_ke_\infty X_i - e_ka_i - e_\infty a_j)(X_i + e_ke_\infty X_j - e_ka_j - e_\infty a_i).
\end{aligned}$$

In particular, the following 24 lines (counted with multiplicity) are contained in $\mathcal{S}_{VI}(\theta)$*:*

$$\left\{X_k = \frac{e_i}{e_j} + \frac{e_j}{e_i},\quad e_iX_i + e_jX_j = a_\infty + e_ie_ja_k\right\},$$

$$\left\{X_k = \frac{e_i}{e_j} + \frac{e_j}{e_i},\quad e_iX_j + e_jX_i = a_k + e_ie_ja_\infty\right\},$$

$$\left\{X_k = e_ie_j + \frac{1}{e_ie_j},\quad X_i + e_ie_jX_j = e_ja_k + e_ia_\infty\right\},$$

$$\left\{X_k = e_ie_j + \frac{1}{e_ie_j},\quad X_j + e_ie_jX_i = e_ja_\infty + e_ia_k\right\},$$

[2] The author is grateful to E. Paul and J.P. Ramis for pointing out the major significance of these lines in the theory of Painlevé equations and for illuminating discussions on this subject.

$$\left\{X_k = \frac{e_k}{e_\infty} + \frac{e_\infty}{e_k}, \quad e_\infty X_i + e_k X_j = a_i + e_k e_\infty a_j\right\},$$
$$\left\{X_k = \frac{e_k}{e_\infty} + \frac{e_\infty}{e_k}, \quad e_k X_i + e_\infty X_j = a_j + e_k e_\infty a_i\right\},$$
$$\left\{X_k = e_k e_\infty + \frac{1}{e_k e_\infty}, \quad X_i + e_k e_\infty X_j = e_k a_j + e_\infty a_i\right\},$$
$$\left\{X_k = e_k e_\infty + \frac{1}{e_k e_\infty}, \quad X_j + e_k e_\infty X_i = e_k a_i + e_\infty a_j\right\}.$$

For each pair $l, m \in \{0, t, 1, \infty\}$ each of the two planes

$$\begin{cases} X_n = e_l e_m + \frac{1}{e_l e_m}, \\ X_n = \frac{e_l}{e_m} + \frac{e_m}{e_l}, \end{cases} \qquad \text{with} \quad (l, m, n) = \begin{cases} (i, j, k) \\ (k, \infty, k), \end{cases} \tag{4.14}$$

intersects $\mathcal{S}_{VI}(\theta)$ at 2 lines. The resulting 4 lines correspond to the reducibility of the pair of matrices $\{M_l, M_m\}$, i.e. to the existence of common invariant space for the pair.

Remark 4.6. — To be more precise, if for each $l = 0, t, 1, \infty$ the monodromy matrix M_l is diagonalizable, then there exists a pair of invariant subspaces of the space of solutions of (4.1), giving rise to a basis of solutions (sometimes called Levelt basis) with respect to which M_l is diagonal. When each of these basic solutions is analytically continued towards the base-point z_0 of $\pi_1(\mathbb{CP}^1 \smallsetminus \{0, t, 1, \infty\}, z_0)$ along the path encircled by the loop γ_i defining the monodromy (Figure 4.1), then for every pair $\{M_l, M_m\}$ there are 4 different possibilities how one can form a *mixed basis* out of the two pairs of solutions. Each of the 4 lines at the intersection of $\mathcal{S}_{VI}(\theta)$ with the planes (4.14) correspond to the degeneracy of one of these 4 mixed bases.

The projective completion of $\mathcal{S}_{VI}(\theta)$ in $\mathbb{CP}^3$ contains 3 additional lines at infinity, giving the total of 27 lines provided by the classical Cayley–Salmon theorem [8, 11]. They are all distinct if and only if $\mathcal{S}_{VI}(\theta)$ is non-singular.

Remark 4.7. — By the classic theory of cubic surfaces, there are 45 tritangent planes, i.e. containing 3 lines forming a triangle, and each line belongs to exactly 5 of these planes. The above 12 decompositions of the cubic corresponds to the 3×4 planes that contain one of the 3 lines at infinity, the fifth plane being the plane at infinity.

Singular points of $\mathcal{S}_{VI}(\theta)$

The surface $\mathcal{S}_{VI}(\theta)$ is simply connected (cf. [9]), and it may or may not have singular points depending on a, but it never has more than 4 singular points [45, Corollary 4.6]. The singularities that appear correspond to unstable monodromy representations, which are of two kinds:

- Either $M_l = \pm I$ for some $l \in \{0, t, 1, \infty\}$, hence $e_l = \pm 1$.
  If $l = i \in \{0, t, 1\}$, then $a_i = \pm 2$, $X_i = \pm a_\infty$, $X_j = \pm a_k$, $X_k = \pm a_j$.
  If $l = \infty$, then $a_\infty = \pm 2$, $X_i = \pm a_i$, $i = 0, t, 1$.
- Or the representation is reducible, in which case $M_l = \begin{pmatrix} e_l^{\delta_l} & * \\ 0 & e_l^{-\delta_l} \end{pmatrix}$, $l = 0, t, 1, \infty$, for some quadruple of signs $(\delta_0, \delta_t, \delta_1) \in \{\pm 1\}^3$, $\delta_\infty = 1$, and
$$e_0^{\delta_0} e_t^{\delta_t} e_1^{\delta_1} e_\infty = 1, \qquad X_i = e_j^{\delta_j} e_k^{\delta_k} + e_j^{-\delta_j} e_k^{-\delta_k}.$$

The surface is therefore singular if and only if
$$\prod_{l \in \{0,t,1,\infty\}} (a_l^2 - 4) \cdot w(a) = 0,$$
where
$$\begin{aligned} w(a) &:= \prod_{\{j,k\} \subset \{0,t,1\}} (a_i - a_j - a_k + a_\infty) - \prod_{i \in \{0,t,1\}} (a_i a_\infty - a_j a_k) \\ &= \frac{1}{e_\infty^4} \prod_{(\delta_0, \delta_t, \delta_1) \in \{\pm 1\}^3} (e_0^{\delta_0} e_t^{\delta_t} e_1^{\delta_1} e_\infty - 1), \end{aligned}$$
see [26]. All the singularities of the projective completion of $\mathcal{S}_{VI}(\theta)$ are contained in its finite part, where they are situated on the intersection of several lines.

The *singular locus* of $\mathcal{S}_{VI}(\theta)$ corresponds through the Riemann–Hilbert correspondence to so called *Riccati solutions* of P_{VI} [25].

4.4. The braid group action on $\mathcal{S}_{VI}(\theta)$

The nonlinear *monodromy action* on the space of $\mathrm{SL}_2(\mathbb{C})$-monodromy representations, is given by the action of moving t along loops in $\mathbb{CP}^1 \smallsetminus \{0, 1, \infty\}$ while keeping the representation constant. When t returns to the initial position t_0, the loops generating $\pi_1(\mathbb{CP}^1 \smallsetminus \{0, t, 1, \infty\}, z_0)$ will not be the same as before. It induces an automorphism of the fundamental group through which it acts on the space of monodromy representations.

The movement of t can be also seen as an action of the pure-braid group $\mathcal{P}_3$ on three strands $(0, t, 1)$, generated by the pure braids $\beta_{0t}^2, \beta_{t1}^2 \in \mathcal{P}_3$ (Figure 4.2). These actions were considered by Dubrovin [13, 14] and de-

Figure 4.2. Elementary pure braids.

scribed in detail by Dubrovin and Mazzocco [15, 42] and Iwasaki [27], and their dynamics was further studied by Cantat & Loray [9] in connection to the problem of transcendentness of P_{VI}.

It is advantageous to consider the elementary monodromy actions as square iterates of "half-monodromy" actions, and to investigate the action of the whole braid group $\mathcal{B}_3$ on three strands $(0, t, 1)$, generated by two braids β_{0t}, β_{t1} (Figure 4.3). However such "half-monodromy" actions don't act on the given Painlevé foliation, but rather on a whole set of such foliations since they change the parameter ϑ by permuting its components. As such there are different possible ways how to define such actions and how to interpret them.

Instead of the four singularities $(0, t, 1, \infty)$, let us consider an ordered quadruple of distinct points (t_i, t_j, t_l, t_m) in $\mathbb{CP}^1$, where (i, j, l, m) is a cyclic permutation of $(0, t, 1, \infty)$. The action of the braid β_{ij} consists in turning the two points t_i, t_j, around each other by a half turn while fixing t_l, t_m.

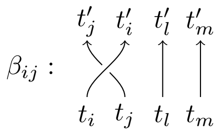
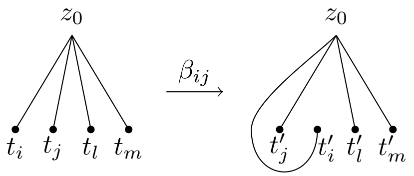


Figure 4.3. Elementary braid β_{ij}, (i, j, l, m) is a cyclic permutation of $(0, t, 1, \infty)$.

The braids act on the fundamental group $\pi_1(\mathbb{CP}^1 \smallsetminus \{t_i, t_j, t_l, t_m\}, z_0)$ by transforming the loops, $\beta : \gamma \mapsto \gamma^\beta$,



$$\begin{aligned} \beta_{ij} :\ \gamma_i &\longmapsto \gamma'_i = \gamma_i\gamma_j\gamma_i^{-1}, \\ \gamma_j &\longmapsto \gamma'_j = \gamma_i, \\ \gamma_l &\longmapsto \gamma'_l = \gamma_l, \\ \gamma_m &\longmapsto \gamma'_m = \gamma_m, \end{aligned}$$

(in the above picture we draw just the connecting paths from z_0 to the singularities i, j that are encircled by the loops γ_i, γ_j), preserving the relation $\gamma_i\gamma_j\gamma_l\gamma_m = \mathrm{id}$.

This in turn induces an action $\beta_* : \rho \mapsto \rho^\beta$ on the monodromy representation (4.5) defined by $\rho^\beta(\gamma^\beta) = \rho(\gamma)$, and satisfying $(\beta\beta')_* = \beta_* \circ \beta'_* : \rho \mapsto \rho^{\beta\beta'} = (\rho^\beta)^{\beta'}$,

$$\begin{aligned} (\beta_{ij})_* :\ M_i &\longmapsto \rho^{\beta_{ij}}(\gamma_i) = M_j, \\ M_j &\longmapsto \rho^{\beta_{ij}}(\gamma_j) = M_j M_i M_j^{-1}, \\ M_l &\longmapsto \rho^{\beta_{ij}}(\gamma_l) = M_l, \\ M_m &\longmapsto \rho^{\beta_{ij}}(\gamma_m) = M_m. \end{aligned} \tag{4.15}$$

The pure braid actions of the two square iterates $\beta_{ij}^{\circ 2}$ and $\beta_{lm}^{\circ 2}$ are conjugated, and therefore induce the same action on the character variety. However the braid actions of β_{ij} and β_{lm} are different.

Now for each of $\{i, j\} = \left\{\begin{smallmatrix}\{0,t\}\\ \{t,1\}\end{smallmatrix}\right.$ let k be the third finite index, $\{i, j, k\} = \{0, t, 1\}$. Then the map between the character varieties induced by β_{ij} is given by $g_{ij} :\ \mathcal{S}_{VI}(\theta) \to \mathcal{S}_{VI}\big(g_{ij}(\theta)\big)$

$$\begin{aligned} g_{ij} :\ \theta_i &\longmapsto \theta_j, & X_i &\longmapsto X_j - F_{X_j}, & F_{X_i} &\longmapsto -F_{X_j}, \\ \theta_j &\longmapsto \theta_i, & X_j &\longmapsto X_i, & F_{X_j} &\longmapsto F_{X_i} - F_{X_j} X_k, \\ \theta_k &\longmapsto \theta_k, & X_k &\longmapsto X_k, & F_{X_k} &\longmapsto F_{X_k} - F_{X_j} X_i, \\ \theta_\infty &\longmapsto \theta_\infty, \end{aligned} \tag{4.16}$$

where the action on the trace parameters is

$$g_{ij} :\ (a_i, a_j, a_k, a_\infty) \longmapsto (a_j, a_i, a_k, a_\infty).$$

Moreover in this notation:

$$g_{ij}^{\circ -1} = g_{ji}.$$

On the other hand for each of $\{l, m\} = \left\{\begin{smallmatrix}\{1,\infty\}\\ \{\infty,0\}\end{smallmatrix}\right.$ let $k = \left\{\begin{smallmatrix}1\\ 0\end{smallmatrix}\right.$ be the corresponding finite index and $\{i, j\} = \left\{\begin{smallmatrix}\{0,t\}\\ \{t,1\}\end{smallmatrix}\right.$ the remaining two indices.

Then the induced map by β_{lm} between the character varieties is given by $g_{lm}:\ \mathcal{S}_{VI}(\theta)\to\mathcal{S}_{VI}\big(g_{lm}(\theta)\big)$

$$\begin{aligned} g_{lm}:\ \theta_i &\longmapsto \theta_j, & X_i &\longmapsto X_j - F_{X_j}, & F_{X_i} &\longmapsto\ -F_{X_j},\\ \theta_j &\longmapsto \theta_i, & X_j &\longmapsto X_i, & F_{X_j} &\longmapsto\ F_{X_i} - F_{X_j}X_k,\\ \theta_k &\longmapsto \theta_k, & X_k &\longmapsto X_k, & F_{X_k} &\longmapsto\ F_{X_k} - F_{X_j}X_i,\\ \theta_\infty &\longmapsto \theta_\infty, \end{aligned} \tag{4.17}$$

which is the same as g_{ij} (4.16) except for the way it acts on a,

$$g_{lm}:\ (a_i,a_j,a_l,a_m)\longmapsto(a_i,a_j,a_m,a_l).$$

It differs from g_{ij} by composition with the involutive permutation

$$(a_i,a_j,a_l,a_m)\longmapsto(a_j,a_i,a_m,a_l)$$

which preserves $\theta(a)$ and commutes with both g_{ij} and g_{lm}. This subtle difference between g_{ij} (4.16) and g_{lm} (4.17) will be amplified during the confluence. Nevertheless, the square iterates of both actions are equal,

$$g_{0t}^{\circ 2}=g_{1\infty}^{\circ 2},\qquad g_{t1}^{\circ 2}=g_{\infty 0}^{\circ 2}.$$

Proposition 4.8 (Dubrovin & Mazzocco [15], Iwasaki [26, 27]).

(1) *For any permutation (i,j,k) of $(0,t,1)$ the above* half-monodromy *actions $g_{ij}:\mathcal{S}_{\mathrm{VI}}(\theta)\to\mathcal{S}_{\mathrm{VI}}\big(g_{ij}(\theta)\big)$ preserve the Fricke relation* (4.8)*, $F\circ g_{ij}=F$, and the 2-form $\omega_{\mathcal{S}_{\mathrm{VI}}}$. The group $\Gamma=\langle g_{0t},g_{t1}\rangle$ generated by the actions of the braids β_{0t} and β_{t1} (generators of $\mathcal{B}_3$) is isomorphic to the* modular group $\mathrm{PSL}_2(\mathbb{Z})$*, with the standard generators*

$$S=g_{0t}^{\circ 2}\circ g_{t1},\quad T=g_{0t}^{\circ(-1)},\qquad \textit{satisfying}\qquad S^{\circ 2}=\mathrm{id}=(T\circ S)^{\circ 3}.$$

(2) *The action of the* monodromy group of P_{VI} *on the character variety $\mathcal{S}_{VI}(\theta)$ is induced by the action of the pure braids $\beta_{0t}^2,\beta_{t1}^2\in\mathcal{P}_3$ on the monodromy representations. It is isomorphic to the* principal congruence subgroup *of the modular group*

$$\Gamma(2)=\langle g_{0t}^{\circ 2},g_{t1}^{\circ 2}\rangle\subseteq\mathrm{Aut}_{\omega_{\mathcal{S}_{VI}}}(\mathcal{S}_{VI}(\theta)),$$

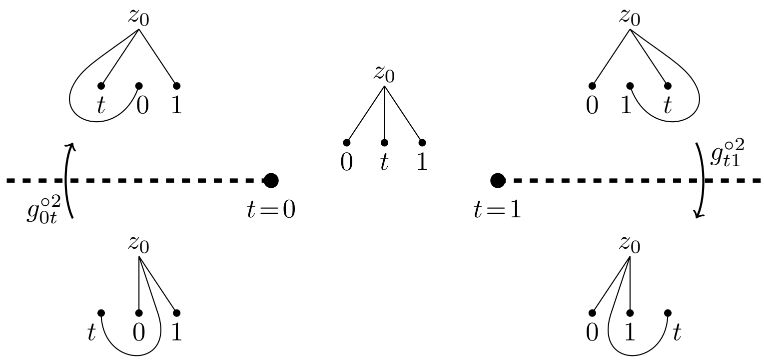


Figure 4.4. The paths around which are taken the loops $\gamma_0, \gamma_t, \gamma_1$ defining the monodromy representation ρ and therefore the coordinate X on $\mathcal{S}_{VI}(\theta)$ in dependence on $t \in \mathbb{CP}^1 \smallsetminus \{0,1,\infty\}$, and the corresponding transition maps $g_{0t}^{\circ 2}$, $g_{t1}^{\circ 2} : \mathcal{S}_{VI}(\theta) \to \mathcal{S}_{VI}(\theta)$.

where

$$
\begin{aligned}
g_{ij}^{\circ 2} : X_i &\longmapsto X_i - F_{X_i} + X_k F_{X_j}, \\
X_j &\longmapsto X_j - F_{X_j}, \\
X_k &\longmapsto X_k, \\
F_{X_i} &\longmapsto -F_{X_i} + X_k F_{X_j}, \\
F_{X_j} &\longmapsto -F_{X_j} - X_k F_{X_i} + X_k^2 F_{X_j}, \\
F_{X_k} &\longmapsto F_{X_k} - X_i F_{X_j} - X_j F_{X_i} + F_{X_j} F_{X_i} + X_k X_j F_{X_j} - X_k F_{X_j}^2 .
\end{aligned}
\tag{4.18}
$$

while preserving the parameter $a = (a_0, a_t, a_1, a_\infty)$.

The fixed points of this $\Gamma(2)$-action are exactly the singularities of $\mathcal{S}_{VI}$, and its restriction on the smooth locus of $\mathcal{S}_{VI}(\theta)$ represents faithfully the nonlinear monodromy action on the non-Riccati locus of space of initial conditions $\mathcal{M}_{VI,t_0}(\vartheta)$ (i.e. on the initial conditions corresponding to non-Riccati solutions of P_{VI}).

As t varies in $\mathbb{CP}^1 \smallsetminus \{0,1,\infty\}$ the character varieties define a fibration above $\mathbb{CP}^1 \smallsetminus \{0,1,\infty\}$ with fibers $\mathcal{S}_{VI}(\theta)$, for which the coordinates $X = (X_0, X_t, X_1)$ (4.7) are local trivializations. The transformation $g_{0t}^{\circ 2}$, $g_{t1}^{\circ 2}$ are the gluing maps when t makes a round around 0 or 1, see Figure 4.4.

5. The confluence $P_{VI} \to P_V$ and the character varieties

We will study the degeneration of the $\mathrm{SL}_2(\mathbb{C})$-isomonodromic problem (4.1) of P_{VI} to the one of P_V [30] using the description of confluence in linear systems by Hurtubise, Lambert & Rousseau [23, 36] in order to understand the degeneration of the character variety of P_{VI} to the wild character variety of P_V. The goal is to be able to consider the sequential limits (3.16) in the parameter ϵ of the braid group actions on the character varieties of P_{VI}, in order to apply the results of Section 3.2 on their accumulation to the generators of the wild monodromy pseudogroup of P_V, which will be done in Section 6.

The wild character variety of P_V in the form of a generalized Fricke formula was constructed by van der Put & Saito [53], as well as by Chekhov, Mazzocco & Rubtsov [12] who also describe the confluence but from a very different point of view. The idea of sequential limits (discretization) in the parameter ϵ from P_{VI} to P_V was previously exploited by Kitaev [32] in relation to the asymptotics of the corresponding Riemann–Hilbert problem and the tau function. But our approach is different.

5.1. Confluence of isomonodromic systems

The substitution

$$t = 1 + \epsilon\widetilde{t}, \quad \vartheta_t = \frac{1}{\epsilon}, \quad \vartheta_1 = -\frac{1}{\epsilon} + \widetilde{\vartheta}_1, \tag{5.1}$$

in the system (4.1) with

$$\begin{aligned} & v_t = \frac{\widetilde{v}_1}{\epsilon\widetilde{t}}, \quad v_1 = -\frac{\widetilde{v}_1}{\epsilon\widetilde{t}} - v_0 + \kappa_2, \\ & u_t = \widetilde{u}_1, \quad u_1 = \widetilde{u}_1 + \epsilon\widetilde{t}\frac{u_0 v_0 - \widetilde{u}_1(v_0-\kappa_2)}{\widetilde{v}_1 + \epsilon\widetilde{t}(v_0-\kappa_2)}, \end{aligned} \tag{5.2}$$

$$\text{where} \quad \kappa_2 = -\frac{\vartheta_0 + \widetilde{\vartheta}_1 + \vartheta_\infty}{2},$$

gives a parametric family (depending on the parameter ϵ) of isomonodromic deformations:

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = \left[\frac{\widetilde{A}_0(\widetilde{t})}{z} + \frac{\widetilde{A}_1^{(0)}(\widetilde{t}) + (z-1-\epsilon\widetilde{t})\widetilde{A}_1^{(1)}(\widetilde{t})}{(z-1)(z-1-\epsilon\widetilde{t})}\right]\phi, \tag{5.3}$$

where

$$\begin{aligned}
\widetilde{A}_0 &= A_0,\\
\widetilde{A}_1^{(0)} &= \epsilon\widetilde{t}A_t = \begin{pmatrix} \tilde{v}_1+\frac{\tilde{t}}{2} & -\tilde{u}_1\tilde{v}_1 \\ \frac{\tilde{v}_1+t}{\tilde{u}_1} & -\tilde{v}_1-\frac{\tilde{t}}{2} \end{pmatrix},\\
\widetilde{A}_1^{(1)} &= A_t + A_1 = -A_0 - A_\infty = \begin{pmatrix} -v_0-\frac{\vartheta_0+\vartheta_\infty}{2} & u_0 v_0 \\ -\frac{v_0+\vartheta_0}{u_0} & v_0+\frac{\vartheta_0+\vartheta_\infty}{2} \end{pmatrix},
\end{aligned}$$

which then have well defined limits when $\epsilon \to 0$. The matrix $\widetilde{A}_1^{(0)}$ has eigenvalues $\pm\frac{\tilde{t}}{2}$, therefore the matrix function $\frac{\tilde{A}_1^{(0)}(\tilde{t})+(z-1-\epsilon\tilde{t})\tilde{A}_1^{(1)}(\tilde{t})}{(z-1)(z-1-\epsilon\tilde{t})}$ can be diagonalized on a uniform neighborhood of $z = 1$ for ϵ small (for fixed $\widetilde{t} \neq 0$), with eigenvalues

$$\pm\frac{1}{2}\left(\frac{\vartheta_1}{z-1} + \frac{\vartheta_t}{z-t}\right) + O(1) = \pm\frac{\widetilde{t} + \widetilde{\vartheta}_1(z-1-\epsilon\widetilde{t})}{2(z-1)(z-1-\epsilon\widetilde{t})} + O(1), \tag{5.4}$$

which form the set of local formal invariants for the confluent pair of singularities at $z = 1$ and $z = 1 + \epsilon\widetilde{t}$.

The meromorphic connection (4.3) becomes

$$\mathrm{d} - \widetilde{A}(z,t)\mathrm{d}z + \frac{\widetilde{A}_1^{(0)}(t)}{\widetilde{t}(z-1-\epsilon\widetilde{t})}\mathrm{d}\widetilde{t},$$

the flatness of which is the isomonodromicity condition on (5.3). The variables (q,p) of the confluent Painlevé system (2.4) are defined as before by

$$\begin{aligned}
q &= -(1+\epsilon\widetilde{t})\frac{[\widetilde{A}_0]_{12}}{[\widetilde{A}_1^0]_{12} + [\widetilde{A}_1^1]_{12}},\\
p &= [\widetilde{A}(q,t)]_{11} + \frac{\vartheta_0}{2q} + \frac{\widetilde{t}}{2(q-1)(q-1-\epsilon\widetilde{t})} + \frac{\widetilde{\vartheta}_1}{2(q-1)}.
\end{aligned}$$

5.2. Confluence on character varieties

The confluence of singularities in linear systems has been studied by many authors, including Garnier [18], Ramis [56], Schäfke [59], Duval [16], Glutsyuk [19, 20], Zhang [67], etc.. Here we will use a description following from a theorem of sectorial normalization for unfolding of non-resonant irregular singularities due to Hurtubise, Lambert and Rousseau [23, 36] and Parise [51].

The local analytic invariants of the limit system (5.3) with $\epsilon = 0$ are usually expressed in terms of a pair of Stokes matrices. The work [23, 36] shows that one can define "unfolded Stokes matrices" also for small $\epsilon \neq 0$.

More precisely, for ϵ in each of the sectors $\mathsf{E}_+, \mathsf{E}_-$ (3.5) (the same sectors as those for normalization of the confluent family of nonlinear Painlevé equations!), the system (5.3) has certain privileged fundamental solution matrix with respect to which the two monodromies M_t, M_1 are one upper triangular and the other lower triangular. This allows to decompose each of the matrices M_t, M_1 as a product of a diagonal "formal monodromy matrix" (with divergent limit $\epsilon \to 0$) and of a unipotent "unfolded Stokes matrix" (with convergent limit $\epsilon \to 0$).

In fact the two columns of this privileged fundamental solution matrix form a *mixed basis of solutions* (cf. Remark 4.6): the first column, which is an "eigensolution" for the eigenvalue e_t of M_t, is characterized as a *subdominant solution* which vanishes when z approaches $t = 1+\epsilon\widetilde{t}$ along some suitable curve, while the second column, which is an "eigensolution" for the eigenvalue e_1 of M_1, is a *subdominant solution* which vanishes when z approaches 1 along some suitable curve. Here the suitable curves of approach of the singular points $1+\epsilon\widetilde{t}$ and 1 are the real time trajectories of the vector field

$$\frac{\mathrm{d}z}{\mathrm{d}\widetilde{t}} = \mathrm{e}^{\mathrm{i}\omega_\pm} \frac{(z-1)(z-1-\epsilon\widetilde{t})}{\widetilde{t}+\widetilde{\vartheta}_1(z-1-\epsilon\widetilde{t})}, \quad \widetilde{t} \in \mathbb{R},$$

or equivalently, leaves of the *horizontal foliation* of the meromorphic differential form $\mathrm{e}^{-\mathrm{i}\omega_\pm} \frac{\widetilde{t}+\widetilde{\vartheta}_1(z-1-\epsilon\widetilde{t})}{(z-1)(z-1-\epsilon\widetilde{t})}\mathrm{d}z$ which expresses the formal invariants (5.4), where $\omega_\pm$ is as in (3.7) (in particular, for $\epsilon \in \{|\arg(\pm\epsilon)| < \frac{\pi}{2}-\eta\} \subset \mathsf{E}_\pm$ one can take $\omega_\pm = 0$). When appropriately normalized, such mixed solution basis has a well-defined limit when $\epsilon \to 0$, $\epsilon \in \mathsf{E}_\pm$, which is the canonical sectorial basis of solutions at the limit irregular singularity.

For simplicity we will consider only the confluence in the sector $\epsilon \in \mathsf{E}_+$. Then for $0 \neq \epsilon$ we have a monodromy representation

$$\rho_+ : \pi_1\left(\mathbb{C} \smallsetminus \{0, 1+\epsilon\widetilde{t}, 1\}, z_0\right) \longrightarrow \mathrm{SL}_2(\mathbb{C})$$

with respect to this privileged mixed basis of solutions, which takes the form:

$$\begin{aligned}
M_{0+} &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \\
M_{t+} &= N_t S_{2+} = \begin{pmatrix} e_t & e_t s_{2+} \\ 0 & \frac{1}{e_t} \end{pmatrix}, \\
M_{1+} &= S_{1+} N_1 = \begin{pmatrix} e_1 & 0 \\ e_1 s_{1+} & \frac{1}{e_1} \end{pmatrix}, \\
M_{\infty+} &= \begin{pmatrix} e_t e_1(\beta s_{1+}+\delta s_{1+}s_{2+})+\frac{\delta_+}{e_t e_1}, & -e_t e_1(\beta+\delta s_{2+}) \\ -e_t e_1(\alpha s_{1+}+\gamma s_{1+}s_{2+})-\frac{\gamma}{e_t e_1}, & e_t e_1(\alpha+\gamma s_{2+}) \end{pmatrix},
\end{aligned} \tag{5.5}$$

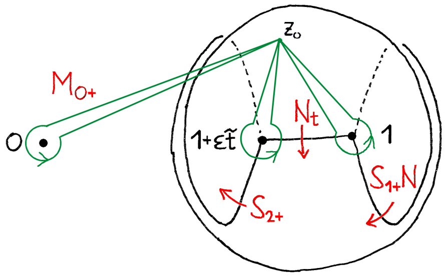


Figure 5.1. Monodromy of the system (5.3) when $\epsilon \to 0+$.

see Figure 5.1, where

$$S_{1+}(\epsilon) = \begin{pmatrix} 1 & 0 \\ s_{1+}(\epsilon) & 1 \end{pmatrix}, \qquad S_{2+}(\epsilon) = \begin{pmatrix} 1 & s_{2+}(\epsilon) \\ 0 & 1 \end{pmatrix},$$

are *unfolded Stokes matrices* of the family (5.3), which tend to the *Stokes matrices* of the limit system $\lim_{\mathsf{E}_+\ni\epsilon\to 0} S_{i+}(\epsilon) = S_i(0)$, $i=1,2$, and where

$$N_i(\epsilon) = \begin{pmatrix} e_i & 0 \\ 0 & \frac{1}{e_i} \end{pmatrix}, \quad i = t, 1, \qquad N = N_t N_1 = \begin{pmatrix} e_t e_1 & 0 \\ 0 & \frac{1}{e_t e_1} \end{pmatrix},$$

with

$$e_t = (M_{t+})_{11} = \mathrm{e}^{\frac{\pi \mathrm{i}}{\epsilon}}, \qquad e_1 = (M_{1+})_{11} = \mathrm{e}^{\pi\mathrm{i}\tilde{\vartheta}_1 - \frac{\pi\mathrm{i}}{\epsilon}},$$

are the *formal monodromy matrices* around the points $t = 1 + \epsilon t$ and 1 for $\epsilon \neq 0$. *Such monodromy representation is determined uniquely up to conjugation by diagonal matrices.*

The monodromy matrices are subject to the conditions

$$\begin{aligned}
1 &= \det\, M_{0+} = \alpha\delta - \gamma\beta, \\
a_0 &= \mathrm{tr}\, M_{0+} = \alpha + \delta, \\
a_\infty &= \mathrm{tr}\, M_{\infty+} = \frac{\delta}{e_t e_1} + e_t e_1(\alpha + \beta s_{1+} + \gamma s_{2+} + \delta s_{1+} s_{2+}), \\
\widetilde{e}_1 &:= \mathrm{e}^{\pi\mathrm{i}\tilde{\vartheta}_1} = [M_{1+}M_{t+}]_{11} = e_t e_1.
\end{aligned}$$

The trace coordinates X_i (4.7) for $0 \neq \epsilon \in \mathsf{E}_+$ are given by

$$\begin{aligned}
X_0 &= \mathrm{tr}(M_{1+}M_{t+}) = e_t e_1 + \frac{1}{e_t e_1} + e_t e_1 s_{1+} s_{2+}, \\
X_t &= \mathrm{tr}(M_{1+}M_{0+}) = e_1(\alpha + \beta s_{1+}) + \frac{\delta}{e_1}, \qquad (5.6)\\
X_1 &= \mathrm{tr}(M_{0+}M_{t+}) = e_t(\alpha + \gamma s_{2+}) + \frac{\delta}{e_t},
\end{aligned}$$

Only the parameters

$$a_0 = 2\cos(\pi\vartheta_0), \qquad \widetilde{e}_1 = e_t e_1 = \mathrm{e}^{\pi\mathrm{i}\tilde{\vartheta}_1}, \qquad a_\infty = 2\cos(\pi\vartheta_\infty),$$

have well defined limits when $\epsilon \to 0$, while $e_t = \mathrm{e}^{\frac{\pi\mathrm{i}}{\epsilon}}$ and $e_1 = \mathrm{e}^{\pi\mathrm{i}\tilde{\vartheta}_1 - \frac{\pi\mathrm{i}}{\epsilon}}$ diverge. Therefore the coordinate X_0 passes well to the limit, but not X_t, X_1 which need be replaced by new coordinates. The new coordinates should be functions of $M_{0+}, S_{1+}, S_{2+}, N, M_{\infty+}$ invariant by diagonal conjugation (since the decomposition (5.5) for $\epsilon \neq 0$, as well as the pair of Stokes matrices S_{1+}, S_{2+} for $\epsilon = 0$, are determined only up to conjugation by diagonal matrices). Following [53], we choose them as the lower diagonal elements of $M_{\infty+}$ and M_{0+}:

$$\begin{aligned} \widetilde{X}_1 &= \mathrm{tr}(M_{1+}M_{t+0}) = X_0, \\ \widetilde{X}_0 &= [M_{0+}]_{22} = \delta, \\ \widetilde{X}_\infty &= [M_{\infty+}]_{22} = e_t e_1(\alpha + \gamma s_{2+}). \end{aligned} \tag{5.7}$$

A substitution in the identity

$$(e_t e_1)^2 s_{1+} s_{2+} (\alpha\delta - \beta\gamma - 1) = 0,$$

gives the Fricke relation in the new coordinates $\widetilde{X} = (\widetilde{X}_0, \widetilde{X}_1, \widetilde{X}_\infty)$

$$\widetilde{F}(\widetilde{X}, \widetilde{\theta}(\widetilde{a})) = 0,$$

where

$$\widetilde{F}(\widetilde{X}, \widetilde{\theta}) := \widetilde{X}_0\widetilde{X}_1\widetilde{X}_\infty + \widetilde{X}_0^2 + \widetilde{X}_\infty^2 - \widetilde{\theta}_0\widetilde{X}_0 - \widetilde{\theta}_1\widetilde{X}_1 - \widetilde{\theta}_\infty\widetilde{X}_\infty + \widetilde{\theta}_t = 0 \tag{5.8}$$

where $\widetilde{\theta} = (\widetilde{\theta}_0, \widetilde{\theta}_1, \widetilde{\theta}_\infty, \widetilde{\theta}_t)$ is a function of the parameter $\widetilde{a} = (a_0, \widetilde{e}_1, a_\infty)$, independent of ϵ,

$$\widetilde{\theta}_0 = a_0 + \widetilde{e}_1 a_\infty, \quad \widetilde{\theta}_1 = \widetilde{e}_1, \quad \widetilde{\theta}_\infty = a_\infty + \widetilde{e}_1 a_0, \quad \widetilde{\theta}_t = 1 + \widetilde{e}_1 a_0 a_\infty + \widetilde{e}_1^2. \tag{5.9}$$

For $\epsilon = 0$, the relation (5.8) for the character variety of P_V was derived in [53, Section 3.2].

Definition 5.1. — *The* wild character variety of P_V *is the complex cubic surface*

$$\mathcal{S}_V(\widetilde{\theta}) = \{\widetilde{X} \in \mathbb{C}^3 : \widetilde{F}(\widetilde{X}, \widetilde{\theta}) = 0\}, \tag{5.10}$$

where $\widetilde{F}$ *is* (5.8), *endowed with the algebraic symplectic form*

$$\widetilde{\omega}_{\mathcal{S}_V} = \frac{\mathrm{d}\widetilde{X}_0 \wedge \mathrm{d}\widetilde{X}_1}{2\pi\mathrm{i}\widetilde{F}_{\tilde{X}_\infty}} = \frac{\mathrm{d}\widetilde{X}_\infty \wedge \mathrm{d}\widetilde{X}_0}{2\pi\mathrm{i}\widetilde{F}_{\tilde{X}_1}} = \frac{\mathrm{d}\widetilde{X}_1 \wedge \mathrm{d}\widetilde{X}_\infty}{2\pi\mathrm{i}\widetilde{F}_{\tilde{X}_0}}, \tag{5.11}$$

where

$$\begin{aligned}
\widetilde{F}_{\tilde{X}_1} &:= \frac{\partial \widetilde{F}}{\partial \widetilde{X}_1} = \widetilde{X}_\infty \widetilde{X}_0 - \widetilde{\theta}_1,\\
\widetilde{F}_{\tilde{X}_0} &:= \frac{\partial \widetilde{F}}{\partial \widetilde{X}_0} = \widetilde{X}_1 \widetilde{X}_\infty + 2\widetilde{X}_0 - \widetilde{\theta}_0,\\
\widetilde{F}_{\tilde{X}_\infty} &:= \frac{\partial \widetilde{F}}{\partial \widetilde{X}_\infty} = \widetilde{X}_1 \widetilde{X}_0 + 2\widetilde{X}_\infty - \widetilde{\theta}_\infty.
\end{aligned}$$

The trace coordinates X (5.6) on the character variety $\mathcal{S}_{VI}(\theta)$ for $\epsilon \neq 0$ and the new coordinates $\widetilde{X}$ (5.7) on the wild character variety $\mathcal{S}_V(\widetilde{\theta})$ are related by the following birational transformations:

Theorem 5.2 ($\epsilon \neq 0$). — *The cubic surfaces $\mathcal{S}_{VI}(\theta)$ and $\mathcal{S}_V(\widetilde{\theta})$ are birationally equivalent through the change of variables*

$$X = \Phi_+(\widetilde{X}),$$

where

$$\begin{aligned}
\Phi_+(\,\cdot\,, a_0, e_t, e_1, a_\infty) : \mathcal{S}_V(\widetilde{\theta}) &\longrightarrow \mathcal{S}_{VI}(\theta)\\
\widetilde{X} &\longmapsto X
\end{aligned}$$

is given by

(5.12)

$$\begin{aligned}
&X_0 = \widetilde{X}_1, && F_{X_0}\circ\Phi_+ = \frac{\widetilde{X}_0}{e_t}\left(\frac{\widetilde{F}_{\tilde{X}_0}}{e_1} - \frac{\widetilde{F}_{\tilde{X}_\infty}}{e_t}\right) - \frac{\widetilde{F}_{\tilde{X}_1}}{e_t e_1}\left(\widetilde{X}_1 - \frac{e_t}{e_1} - \frac{e_1}{e_t}\right)\\
&X_t = \frac{\widetilde{X}_0}{e_1} + \frac{\widetilde{X}_\infty - \widetilde{F}_{\tilde{X}_\infty}}{e_t}, && F_{X_t}\circ\Phi_+ = \frac{\widetilde{F}_{\tilde{X}_0}}{e_1} - \frac{\widetilde{F}_{\tilde{X}_\infty}}{e_t},\\
&X_1 = \frac{\widetilde{X}_\infty}{e_1} + \frac{\widetilde{X}_0}{e_t}, && F_{X_1}\circ\Phi_+ = \frac{\widetilde{F}_{\tilde{X}_0}}{e_t} + \frac{\widetilde{F}_{\tilde{X}_\infty}}{e_1} - \frac{\widetilde{X}_1\widetilde{F}_{\tilde{X}_\infty}}{e_t}.
\end{aligned}$$

The inverse map $\Phi_+^{\circ(-1)} : \mathcal{S}_{VI}(\theta) \dashrightarrow \mathcal{S}_V(\widetilde{\theta})$, $X \mapsto \widetilde{X}_+$, is given by

$$\begin{aligned}
\widetilde{X}_0 &= -\frac{e_t X_t + e_1 X_1 - a_\infty - e_t e_1 a_0}{X_0 - \frac{e_t}{e_1} - \frac{e_1}{e_t}}\\
&= -\frac{e_t e_1\left(X_0 - \frac{e_t}{e_1} - \frac{e_1}{e_t} - F_{X_0}\right)}{e_1 X_t + e_t X_1 - a_0 - e_t e_1 a_\infty},\\
\widetilde{X}_1 &= X_0,\\
\widetilde{X}_\infty &= -\frac{e_1 X_t + e_t X_1 - a_0 - e_t e_1 a_\infty - e_1 F_{X_t}}{X_0 - \frac{e_t}{e_1} - \frac{e_1}{e_t}},
\end{aligned} \tag{5.13}$$

and is singular on the line:

$$L_0 := \left\{ X_0 = \frac{e_t}{e_1} + \frac{e_1}{e_t}, \quad e_1 X_t + e_t X_1 = \widetilde{\theta}_0 \right\}. \tag{5.14}$$

The two Fricke relations are related by $F \circ \Phi_+ = \frac{-1}{e_t e_1}(X_0 - \frac{e_t}{e_1} - \frac{e_1}{e_t}) \cdot \widetilde{F}$. *The restriction*

$$\Phi_+ : \mathcal{S}_V(\widetilde{\theta}) \longrightarrow \mathcal{S}_{VI}(\theta) \smallsetminus L_0$$

is an isomorphism. The pull-back of the symplectic form $\omega_{\mathcal{S}_{VI}}$ (4.11) *by* Φ_+ *is the symplectic form* (5.11).

Proof. — From (5.6) and (5.7) by a direct calculation. □

Remark 5.3 ($\epsilon \neq 0$). — In the trace coordinates (4.6) on the space of monodromy representations, the eigenvalues e_i and $\frac{1}{e_i}$ of M_i are interchangeable. On the other hand in the above confluent description of the monodromy data this is no longer true for $i = t, 1$. Indeed, during the confluence we are fixing an appropriately normalized mixed basis of solutions consisting of one "eigensolution" for the eigenvalue e_t of M_t and another for e_1 of M_1. The general theory of confluence [23] tells us that this mixed basis tends, when $\epsilon \to 0$ in the sector E_+, to the appropriate sectorial basis at the irregular singularity of the limit system. This means that, for small enough ϵ, the chosen mixed basis can never be degenerate. On the other hand, the singular line L_0 (5.14) for $\epsilon \neq 0$ corresponds by Remark 4.6 precisely to the monodromy representations for which our mixed basis degenerates, so there is no place for it in the confluent picture.

Remark 5.4. — A very simple way to obtain the coordinates $\widetilde{X}$ on the wild character variety $\mathcal{S}_V(\widetilde{\theta})$ is by taking the following limit:

(i) When $\epsilon \to 0$ in a sector $\eta < \arg \epsilon < \pi - \eta$, $\eta > 0$, then $e_t \to \infty$, $e_1 \to 0$, hence $\frac{a_t}{a_1} \to \widetilde{e}_1$,

$$\left(\frac{X_t}{a_1}, X_0, \frac{X_1}{a_1}\right) \longrightarrow (\widetilde{X}_0, \widetilde{X}_1, \widetilde{X}_\infty),$$
$$\left(\frac{\theta_t}{a_1}, \frac{\theta_0}{a_1^2}, \frac{\theta_1}{a_1}, \frac{\theta_\infty}{a_1^2}\right) \longrightarrow (\widetilde{\theta}_0, \widetilde{\theta}_1, \widetilde{\theta}_\infty, \widetilde{\theta}_t),$$

and $\frac{1}{a_1^2} F(X, \theta) \to \widetilde{F}(\widetilde{X}, \widetilde{\theta})$.

(ii) When $\epsilon \to 0$ in a sector $-\pi+\eta < \arg\epsilon < -\eta$, $\eta>0$, then $e_t \to 0$, $e_1 \to \infty$, hence $\frac{a_1}{a_t} \to \widetilde{e}_1$,

$$\left(\frac{X_1}{a_t}, X_0, \frac{X_t - F_{X_t}}{a_t}\right) \longrightarrow (\widetilde{X}_0, \widetilde{X}_1, \widetilde{X}_\infty),$$
$$\left(\frac{\theta_1}{a_t}, \frac{\theta_0}{a_t^2}, \frac{\theta_t}{a_t}, \frac{\theta_\infty}{a_t^2}\right) \longrightarrow (\widetilde{\theta}_0, \widetilde{\theta}_1, \widetilde{\theta}_\infty, \widetilde{\theta}_t),$$

$\frac{X_t}{a_t} \to \widetilde{X}_\infty - \widetilde{F}_{\tilde{X}_\infty}$, and $\frac{1}{a_t^2}F(X,\theta) \to \widetilde{F}(\widetilde{X},\widetilde{\theta})$.

These two limits are related by the action of the half-monodromy operator $g_{1t} = g_{t1}^{\circ(-1)}$ (4.16) on the left side (and identity on the right side). This limiting procedure should be related to Glutsyuk's description of confluence [19, 20] and has been well known (for example [12]). Whenever it leads to a convergent limit it can be used as a simple way to obtain limit formulas. However it doesn't allow to treat the confluence when $\epsilon \to 0$ along the sequences $\{\epsilon_n\}_{n\in\mathbb{N}}$ with $\frac{1}{\epsilon_n} = \frac{1}{\epsilon_0} + n$, that accumulate to 0 asymptotically tangent to $\mathbb{R}_{>0}$, which is essential for us in order to obtain the generators of the wild monodromy.

5.3. The monodromy action on $\mathcal{S}_V(\widetilde{\theta})$

Both of the two different half-monodromy operators g_{t1} and $g_{\infty 0}$ (4.16), (4.17), satisfying $g_{t1}^{\circ 2} = g_{\infty 0}^{\circ 2}$, and acting on the space of monodromy representations associated to P_{VI}, pass well to the confluent limit if properly interpreted. However, the action $g_{\infty 0}$ (which differs from g_{t1} in how it acts on $(a_0, a_t, a_1, a_\infty)$) seems to be the one that is better suited: this has to do with the fact that its action is on the points $z = 0, \infty$, thus away from where the confluence happens.

Proposition 5.5.

(i) *The pullback of the half-monodromy operators* $g_{ij} : \mathcal{S}_{VI}(\theta) \to \mathcal{S}_{VI}(g_{ij}(\theta))$, $(i,j) = (t,1), (\infty,0)$, *and their inverses* $g_{ji} = g_{ij}^{\circ(-1)}$, *by the transformation* Φ_+ (5.13) *is given by*

$$\widetilde{g}_{ij} = \Phi_+^{\circ(-1)} \circ g_{ij} \circ \Phi_+ : \mathcal{S}_V(\widetilde{\theta}) \longrightarrow \mathcal{S}_V(\widetilde{g}_{ij}(\widetilde{\theta}))$$

$$\begin{aligned}
\widetilde{g}_{t1}:\ &\widetilde{\theta}_0 \longmapsto \frac{\widetilde{\theta}_\infty}{\widetilde{\theta}_1}, \quad && \widetilde{X}_0 \longmapsto \frac{1}{\widetilde{\theta}_1}\big(\widetilde{X}_\infty - \widetilde{F}_{\tilde{X}_\infty}\big), \quad && \widetilde{F}_{\tilde{X}_0} \longmapsto -\frac{1}{\widetilde{\theta}_1}\widetilde{F}_{\tilde{X}_\infty},\\
&\widetilde{\theta}_1 \longmapsto \frac{1}{\widetilde{\theta}_1}, && \widetilde{X}_1 \longmapsto \widetilde{X}_1, && \widetilde{F}_{\tilde{X}_1} \longmapsto \widetilde{F}_{\tilde{X}_1} - \widetilde{F}_{\tilde{X}_\infty}\widetilde{X}_0,\\
&\widetilde{\theta}_\infty \longmapsto \frac{\widetilde{\theta}_0}{\widetilde{\theta}_1}, && \widetilde{X}_\infty \longmapsto \frac{1}{\widetilde{\theta}_1}\widetilde{X}_0, && \widetilde{F}_{\tilde{X}_\infty} \longmapsto \frac{1}{\widetilde{\theta}_1}\big(\widetilde{F}_{\tilde{X}_0} - \widetilde{F}_{\tilde{X}_\infty}\widetilde{X}_1\big),\\
&\widetilde{\theta}_t \longmapsto \frac{\widetilde{\theta}_t}{\widetilde{\theta}_1^2}.
\end{aligned}$$

$$\begin{aligned}
\widetilde{g}_{1t}:\ &\widetilde{\theta}_0 \longmapsto \frac{\widetilde{\theta}_\infty}{\widetilde{\theta}_1}, \quad && \widetilde{X}_0 \longmapsto \frac{1}{\widetilde{\theta}_1}\widetilde{X}_\infty, \quad && \widetilde{F}_{\tilde{X}_0} \longmapsto \frac{1}{\widetilde{\theta}_1}\big(\widetilde{F}_{\tilde{X}_\infty} - \widetilde{F}_{\tilde{X}_0}\widetilde{X}_1\big),\\
&\widetilde{\theta}_1 \longmapsto \frac{1}{\widetilde{\theta}_1}, && \widetilde{X}_1 \longmapsto \widetilde{X}_1, && \widetilde{F}_{\tilde{X}_1} \longmapsto \widetilde{F}_{\tilde{X}_1} - \widetilde{F}_{\tilde{X}_0}\widetilde{X}_\infty,\\
&\widetilde{\theta}_\infty \longmapsto \frac{\widetilde{\theta}_0}{\widetilde{\theta}_1}, && \widetilde{X}_\infty \longmapsto \frac{1}{\widetilde{\theta}_1}\big(\widetilde{X}_0 - \widetilde{F}_{\tilde{X}_0}\big), && \widetilde{F}_{\tilde{X}_\infty} \longmapsto -\frac{1}{\widetilde{\theta}_1}\widetilde{F}_{\tilde{X}_0}\\
&\widetilde{\theta}_t \longmapsto \frac{\widetilde{\theta}_t}{\widetilde{\theta}_1^2}.
\end{aligned}$$

$$\begin{aligned}
\widetilde{g}_{\infty 0}:\ &\widetilde{\theta}_0 \longmapsto \widetilde{\theta}_\infty, \quad && \widetilde{X}_0 \longmapsto \widetilde{X}_\infty - \widetilde{F}_{\tilde{X}_\infty}, \quad && \widetilde{F}_{\tilde{X}_0} \longmapsto -\widetilde{F}_{\tilde{X}_\infty},\\
&\widetilde{\theta}_1 \longmapsto \widetilde{\theta}_1, && \widetilde{X}_1 \longmapsto \widetilde{X}_1, && \widetilde{F}_{\tilde{X}_1} \longmapsto \widetilde{F}_{\tilde{X}_1} - \widetilde{F}_{\tilde{X}_\infty}\widetilde{X}_0,\\
&\widetilde{\theta}_\infty \longmapsto \widetilde{\theta}_0, && \widetilde{X}_\infty \longmapsto \widetilde{X}_0, && \widetilde{F}_{\tilde{X}_\infty} \longmapsto \widetilde{F}_{\tilde{X}_0} - \widetilde{F}_{\tilde{X}_\infty}\widetilde{X}_1\\
&\widetilde{\theta}_t \longmapsto \widetilde{\theta}_t.
\end{aligned}$$

$$\begin{aligned}
\widetilde{g}_{0\infty}:\ &\widetilde{\theta}_0 \longmapsto \widetilde{\theta}_\infty, \quad && \widetilde{X}_0 \longmapsto \widetilde{X}_\infty, \quad && \widetilde{F}_{\tilde{X}_0} \longmapsto \widetilde{F}_{\tilde{X}_\infty} - \widetilde{F}_{\tilde{X}_0}\widetilde{X}_1,\\
&\widetilde{\theta}_1 \longmapsto \widetilde{\theta}_1, && \widetilde{X}_1 \longmapsto \widetilde{X}_1, && \widetilde{F}_{\tilde{X}_1} \longmapsto \widetilde{F}_{\tilde{X}_1} - \widetilde{F}_{\tilde{X}_0}\widetilde{X}_\infty,\\
&\widetilde{\theta}_\infty \longmapsto \widetilde{\theta}_0, && \widetilde{X}_\infty \longmapsto \widetilde{X}_0 - \widetilde{F}_{\tilde{X}_0}, && \widetilde{F}_{\tilde{X}_\infty} \longmapsto -\widetilde{F}_{\tilde{X}_0}\\
&\widetilde{\theta}_t \longmapsto \widetilde{\theta}_t.
\end{aligned}$$

(ii) *The pullback* $\widetilde{g}_{t1}^{\circ 2} = \widetilde{g}_{\infty 0}^{\circ 2} \in \mathrm{Aut}_{\omega_{\mathcal{S}_V}}(\mathcal{S}_V(\widetilde{\theta}))$ *of the monodromy operator* $g_{t1}^{\circ 2} = g_{\infty 0}^{\circ 2} \in \mathrm{Aut}_{\omega_{\mathcal{S}_{VI}}}(\mathcal{S}_{VI}(\theta))$ *by the transformation* Φ_+ (5.13) *is given by the square iterate of the above operators* $\widetilde{g}_{t1}$, $\widetilde{g}_{\infty 0}$

$$\begin{aligned}
\widetilde{g}_{\infty 0}^{\circ 2}:\ &\widetilde{X}_0 \longmapsto \widetilde{X}_0 - \widetilde{F}_{\tilde{X}_0} + \widetilde{X}_1\widetilde{F}_{\tilde{X}_\infty},\\
&\widetilde{X}_1 \longmapsto \widetilde{X}_1,\\
&\widetilde{X}_\infty \longmapsto \widetilde{X}_\infty - \widetilde{F}_{\tilde{X}_\infty},
\end{aligned}$$

$$\begin{aligned}
\widetilde{F}_{\tilde{X}_0} &\longmapsto -\widetilde{F}_{\tilde{X}_0} + \widetilde{X}_1\widetilde{F}_{\tilde{X}_\infty},\\
\widetilde{F}_{\tilde{X}_1} &\longmapsto \widetilde{F}_{\tilde{X}_1} - \widetilde{X}_0\widetilde{F}_{\tilde{X}_\infty} - \widetilde{X}_\infty\widetilde{F}_{\tilde{X}_0} + \widetilde{F}_{\tilde{X}_0}\widetilde{F}_{\tilde{X}_\infty} + \widetilde{X}_1\widetilde{X}_\infty\widetilde{F}_{\tilde{X}_\infty} - \widetilde{X}_1\widetilde{F}^2_{\tilde{X}_\infty},\\
\widetilde{F}_{\tilde{X}_\infty} &\longmapsto -\widetilde{F}_{\tilde{X}_\infty} - \widetilde{X}_1\widetilde{F}_{\tilde{X}_0} + \widetilde{X}_1^2\widetilde{F}_{\tilde{X}_\infty},
\end{aligned}$$

and its inverse

$$\begin{aligned}
\widetilde{g}^{\circ 2}_{0\infty}:\ \widetilde{X}_0 &\longmapsto \widetilde{X}_0 - \widetilde{F}_{\tilde{X}_0},\\
\widetilde{X}_1 &\longmapsto \widetilde{X}_1,\\
\widetilde{X}_\infty &\longmapsto \widetilde{X}_\infty - \widetilde{F}_{\tilde{X}_\infty} + \widetilde{X}_1\widetilde{F}_{\tilde{X}_0},\\
\widetilde{F}_{\tilde{X}_0} &\longmapsto -\widetilde{F}_{\tilde{X}_0} - \widetilde{X}_1\widetilde{F}_{\tilde{X}_\infty} + \widetilde{X}_1^2\widetilde{F}_{\tilde{X}_0},\\
\widetilde{F}_{\tilde{X}_1} &\longmapsto \widetilde{F}_{\tilde{X}_1} - \widetilde{X}_0\widetilde{F}_{\tilde{X}_\infty} - \widetilde{X}_\infty\widetilde{F}_{\tilde{X}_0} + \widetilde{F}_{\tilde{X}_0}\widetilde{F}_{\tilde{X}_\infty} + \widetilde{X}_1\widetilde{X}_0\widetilde{F}_{\tilde{X}_0} - \widetilde{X}_1\widetilde{F}^2_{\tilde{X}_0},\\
\widetilde{F}_{\tilde{X}_\infty} &\longmapsto -\widetilde{F}_{\tilde{X}_\infty} + \widetilde{X}_1\widetilde{F}_{\tilde{X}_0}.
\end{aligned}$$

They preserve the Fricke relation: $\widetilde{F}\circ\widetilde{g}^{\circ 2}_{\infty 0} = \widetilde{F}$.

The "half-monodromy" action $\widetilde{g}_{t1}$ was previously described in [52], and the monodromy action $g_{t1}^{\circ 2} = g_{\infty 0}^{\circ 2}$ was considered in [52, 53].

Proof. — One way to obtain the action $\widetilde{g}_{t1}$ is by the limiting procedure of Remark 5.4. For example in (i), for $\Im\epsilon > 0$:

$$\begin{tikzcd}
\left(\frac{X_t}{a_1}, X_0, \frac{X_1}{a_1}\right) \arrow[r, mapsto, "g_{t1}"] \arrow[d, "\epsilon\to 0"] & \left(\frac{X_1 - F_{X_1}}{a_t}, X_0, \frac{X_t}{a_t}\right) \arrow[d, "\epsilon\to 0"] \\
\left(\widetilde{X}_0, \widetilde{X}_1, \widetilde{X}_\infty\right) \arrow[r, mapsto, "\tilde{g}_{t1}"] & \left(\frac{\tilde{X}_\infty - \tilde{F}_{\tilde{X}_\infty}}{\tilde{e}_1}, \widetilde{X}_1, \frac{\tilde{X}_0}{\tilde{e}_1}\right),
\end{tikzcd}$$

and similarly in (ii), for $\Im\epsilon < 0$.

A different wayto obtain the action $\widetilde{g}_{t1}$ is to look on how the braid β_{t1} acts on the monodromy representation ρ_+ (5.5). After a conjugation by a matrix PM_{1+}^{-1} the action (4.15) is written as

$$\begin{aligned}
(\beta_{t1})_*:\ M_{0+} &\longmapsto \rho^{\beta_{t1}}(\gamma_0) = PM_{1+}^{-1}M_{0+}M_{1+}P,\\
M_{t+} &\longmapsto \rho^{\beta_{t1}}(\gamma_t) = PM_{1+}P,\\
M_{1+} &\longmapsto \rho^{\beta_{t1}}(\gamma_1) = PM_{0+}P,\\
M_{\infty+} &\longmapsto \rho^{\beta_{t1}}(\gamma_0) = PM_{1+}^{-1}M_{\infty+}M_{1+}P,
\end{aligned}$$

where $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so that $\rho^{\beta_{t1}}(\gamma_t)$ is upper-triangular, and $\rho^{\beta_{t1}}(\gamma_1)$ is lower-triangular. Therefore g_{t1} maps $(e_t, e_1) \mapsto (\frac{1}{e_1}, \frac{1}{e_t})$, and we have

$$\begin{aligned}
\widetilde{e}_1 &\longmapsto \frac{1}{\widetilde{e}_1}, \\
\widetilde{X}_1 &\longmapsto \widetilde{X}_1, \\
\widetilde{X}_0 &\longmapsto \left(M_{1+}^{-1} M_{0+} M_{1+}\right)_{11} = \alpha + \beta s_{1+} = \frac{\widetilde{X}_\infty - \widetilde{F}_{\tilde{X}_\infty}}{\widetilde{e}_1}, \\
\widetilde{X}_\infty &\longmapsto \left(M_{1+}^{-1} M_{\infty+} M_{1+}\right)_{11} = \frac{\delta}{e_t e_1} = \frac{\widetilde{X}_0}{\widetilde{e}_1}.
\end{aligned}$$

Likewise, the formula for the action of $\widetilde{g}_{\infty 0}$ can be obtained from $g_{\infty 0}$ by the limiting procedure of Remark 5.4. $\square$

5.4. Lines and singularities of $\mathcal{S}_V(\widetilde{\theta})$

Proposition 5.6 (Lines of $\mathcal{S}_V(\widetilde{\theta})$). — *The polynomial $\widetilde{F}(\widetilde{X}, \widetilde{\theta})$ (5.8) can be decomposed as*

$$\begin{aligned}
\widetilde{F} &= (\widetilde{X}_1 - \widetilde{e}_1 - \tfrac{1}{\tilde{e}_1})\widetilde{F}_{\tilde{X}_1} + \widetilde{e}_1(\widetilde{X}_\infty + \tfrac{\tilde{X}_0}{\tilde{e}_1} - a_\infty)(\widetilde{X}_0 + \tfrac{\tilde{X}_\infty}{\tilde{e}_1} - a_0), \\
&= (\widetilde{X}_1 - e_0 e_\infty - \tfrac{1}{e_0 e_\infty})\widetilde{F}_{\tilde{X}_1} \\
&\qquad + (e_0 \widetilde{X}_\infty + \tfrac{\tilde{X}_0}{e_\infty} - \widetilde{e}_1 - \tfrac{e_0}{e_\infty})(\tfrac{\tilde{X}_\infty}{e_0} + e_\infty \widetilde{X}_0 - \widetilde{e}_1 - \tfrac{e_\infty}{e_0}), \\
&= (\widetilde{X}_1 - \tfrac{e_0}{e_\infty} - \tfrac{e_\infty}{e_0})\widetilde{F}_{\tilde{X}_1} \\
&\qquad + (e_0 \widetilde{X}_\infty + e_\infty \widetilde{X}_0 - \widetilde{e}_1 - e_0 e_\infty)(\tfrac{\tilde{X}_\infty}{e_0} + \tfrac{\tilde{X}_0}{e_\infty} - \widetilde{e}_1 - \tfrac{1}{e_0 e_\infty}), \\
&= (\widetilde{X}_0 - e_0)(\widetilde{F}_{\tilde{X}_0} - \widetilde{X}_0 + e_0) + (\widetilde{X}_\infty - \tfrac{\tilde{e}_1}{e_0})(e_0 \widetilde{X}_1 + \widetilde{X}_\infty - \widetilde{e}_1 e_0 - a_\infty), \\
&= (\widetilde{X}_0 - e_0)(\tfrac{\tilde{e}_1}{e_0}\widetilde{X}_1 + \widetilde{X}_0 - \tfrac{1}{e_0} - \widetilde{e}_1 a_\infty) + (\widetilde{X}_\infty - \tfrac{\tilde{e}_1}{e_0})(\widetilde{F}_{\tilde{X}_\infty} - \widetilde{X}_\infty + \tfrac{\tilde{e}_1}{e_0}), \\
&= (\widetilde{X}_0 - \tfrac{1}{e_0})(\widetilde{F}_{\tilde{X}_0} - \widetilde{X}_0 + \tfrac{1}{e_0}) + (\widetilde{X}_\infty - \widetilde{e}_1 e_0)(\tfrac{\tilde{X}_1}{e_0} + \widetilde{X}_\infty - \tfrac{\tilde{e}_1}{e_0} - a_\infty), \\
&= (\widetilde{X}_0 - \tfrac{1}{e_0})(\widetilde{e}_1 e_0 \widetilde{X}_1 + \widetilde{X}_0 - e_0 - \widetilde{e}_1 a_\infty) + (\widetilde{X}_\infty - \widetilde{e}_1 e_0)(\widetilde{F}_{\tilde{X}_\infty} - \widetilde{X}_\infty + \widetilde{e}_1 e_0), \\
&= (\widetilde{X}_\infty - e_\infty)(\widetilde{F}_{\tilde{X}_\infty} - \widetilde{X}_\infty + e_\infty) + (\widetilde{X}_0 - \tfrac{\tilde{e}_1}{e_\infty})(e_\infty \widetilde{X}_1 + \widetilde{X}_0 - \widetilde{e}_1 e_\infty - a_0), \\
&= (\widetilde{X}_\infty - e_\infty)(\tfrac{\tilde{e}_1}{e_\infty}\widetilde{X}_1 + \widetilde{X}_\infty - \tfrac{1}{e_\infty} - \widetilde{e}_1 a_0) + (\widetilde{X}_0 - \tfrac{\tilde{e}_1}{e_\infty})(\widetilde{F}_{\tilde{X}_0} - \widetilde{X}_0 + \tfrac{\tilde{e}_1}{e_\infty}), \\
&= (\widetilde{X}_\infty - \tfrac{1}{e_\infty})(\widetilde{F}_{\tilde{X}_\infty} - \widetilde{X}_\infty + \tfrac{1}{e_\infty}) + (\widetilde{X}_0 - \widetilde{e}_1 e_\infty)(\tfrac{\tilde{X}_1}{e_\infty} + \widetilde{X}_0 - \tfrac{\tilde{e}_1}{e_\infty} - a_0), \\
&= (\widetilde{X}_\infty - \tfrac{1}{e_\infty})(\widetilde{e}_1 e_\infty \widetilde{X}_1 + \widetilde{X}_\infty - e_\infty - \widetilde{e}_1 a_0) \\
&\qquad + (\widetilde{X}_0 - \widetilde{e}_1 e_\infty)(\widetilde{F}_{\tilde{X}_0} - \widetilde{X}_0 + \widetilde{e}_1 e_\infty).
\end{aligned}$$

defining thus 18 lines on $\mathcal{S}_V(\widetilde{\theta})$ (note that some lines appear twice in the above decomposition).

Remark 5.7. — Each of the monodromies M_0, M_∞ (assuming diagonalizable) define a pair of invariant eigenspaces of solutions, while each of the Stokes matrices S_{1+}, S_{2+} (assuming nontrivial) define one invariant eigenspace of solutions. As in Remark 4.6, the lines in $\mathcal{S}_V(\widetilde{\theta})$ correspond to degeneration of some associated mixed bases of solutions. More about this in a future work.

The projective completion of $\mathcal{S}_V(\widetilde{\theta})$ in $\mathbb{CP}^3$ contains 3 additional lines at infinity, giving a total 21 lines, and there is a singularity of type A_1 at the infinity (see [8]). The surface $\mathcal{S}_V(\widetilde{\theta})$ can have additional singularities, which happens if and only if some the 18 lines coincide.

Proposition 5.8 (Singular points of $\mathcal{S}_V(\widetilde{\theta})$). — *The affine cubic variety $\mathcal{S}_V(\widetilde{\theta})$ has singular points if and only if*

$$(a_0^2 - 4)(a_\infty^2 - 4)\widetilde{w}(\widetilde{\theta}) = 0,$$

where

$$\begin{aligned}\widetilde{w}(\widetilde{a}) &= (a_0^2 + \widetilde{a}_1^2 + a_\infty^2 - a_0\widetilde{a}_1 a_\infty - 4), \quad \textit{with} \quad \widetilde{a}_1 = \widetilde{e}_1 + \frac{1}{\widetilde{e}_1}, \\ &= \frac{1}{\widetilde{e}_1^2} \prod_{(\delta_0,\delta_\infty)\in\{\pm1\}^2} (e_0^{\delta_0}\widetilde{e}_1 e_\infty^{\delta_\infty} - 1),\end{aligned}$$

(cf. [53, Section 3.2.2]*). The corresponding possible singularities are the following:*

- *if* $a_\infty = \pm 2$*:* $\widetilde{X}_0 = \pm\widetilde{e}_1, \quad \widetilde{X}_1 = \pm a_0, \quad \widetilde{X}_\infty = \pm 1,$
- *if* $a_0 = \pm 2$*:* $\widetilde{X}_0 = \pm 1, \quad \widetilde{X}_1 = \pm a_\infty, \quad \widetilde{X}_\infty = \pm\widetilde{e}_1,$
- *if* $\widetilde{e}_1 = e_0^{\delta_0} e_\infty^{\delta_\infty}$, $(\delta_0,\delta_\infty) \in \{\pm1\}^2$*:* $\widetilde{X}_0 = e_0^{\delta_0}$, $\widetilde{X}_1 = \widetilde{e}_1 + \frac{1}{\widetilde{e}_1}$, $\widetilde{X}_\infty = e_\infty^{\delta_\infty}$.

Setting $\widetilde{X} = (\frac{x_0}{v}, \frac{x_1}{v}, \frac{x_\infty}{v})$, the projective completion of $\mathcal{S}_V(\widetilde{\theta})$ in $\mathbb{CP}^3$ has also a singularity at the point $(x_0 : x_1 : x_\infty : v) = (0 : 1 : 0 : 0)$ for any value of the parameters.

Proof. — By Theorem 5.2, the surface $\mathcal{S}_V(\widetilde{\theta})$ is isomorphic to $\mathcal{S}_{VI}(\theta) \smallsetminus L_0$, one can therefore use the description of the singular points of $\mathcal{S}_{VI}(\theta)$ given in Section 4.3. □

5.5. The center manifold solution

Proposition 5.9. — *For $\epsilon \in \mathsf{E}_+$ let $\widetilde{X}$ be the coordinate* (5.7) *on $\mathcal{S}_V(\widetilde{\theta})$ which for $\epsilon \neq 0$ is given by the monodromy representation ρ_+ depending* *on* $x = \frac{1}{\tilde t} + \epsilon = \frac{\epsilon t}{t-1}$ *as in Figure 5.2. Then the upper "sectorial center manifold" solution* $\Psi_+^{\spadesuit}(0, x, \epsilon)$ *of Corollaries 3.3 and 2.2 over the domain* $\mathsf{X}_+^{\spadesuit}$ *corresponds to the point*

$$(\widetilde{X}_0, \widetilde{X}_1, \widetilde{X}_\infty) = \left(\frac{1}{e_0},\ \widetilde{e}_1 + e_0 a_\infty - e_0^2\widetilde{e}_1,\ e_0\widetilde{e}_1\right) \tag{5.15}$$

for $x \in \mathsf{X}_+^{\spadesuit}(\epsilon)$, $\epsilon \in \mathsf{E}_+$*.*

The proof of Proposition 5.9 is based on the following reformulation of the Jimbo's asymptotic formula.

Proposition 5.10 (Jimbo's formula for the confluent system). — *For* $\epsilon \in \mathsf{E}_+ \smallsetminus \{0\}$ *let* $\widetilde{X}$ *be the coordinate* (5.7) *on* $\mathcal{S}_V(\widetilde{\theta})$ *given by the monodromy representation* ρ_+ *depending on* $x = \frac{1}{\tilde t} + \epsilon = \frac{\epsilon t}{t-1}$ *as in Figure 5.2. Let* ${}^{\mathrm{t}}(q(x), p(x)) = {}^{\mathrm{t}}(q_+^{\spadesuit}(x), p_+^{\spadesuit}(x))$ *be the solution of the confluent Painlevé system* (3.1) *corresponding to a point* $\widetilde{X} \in \mathcal{S}_V(\widetilde{\theta})$ *over the domain* $x \in \mathsf{X}_+^{\spadesuit}$*, which corresponds (up to analytic extension) to the upper half-plane in Figure 5.2.*

(1) *When* $x \to 0$, $x \in \mathsf{X}_+^{\spadesuit}(\epsilon)$, $\epsilon \in \mathsf{E}_+$*:*

$$q \sim \alpha(\vartheta, X)\left(\frac{x}{x-\epsilon}\right)^{1-\sigma_1} + \ O\left(\left(\frac{x}{x-\epsilon}\right)^{2-2\sigma_1}\right),$$
$$p \sim \frac{\vartheta_0 + \vartheta_t - \sigma_1}{2\alpha(\vartheta, X)}\left(\frac{x}{x-\epsilon}\right)^{\sigma_1 - 1} + \ O(1),$$

where σ_1*, defined by* $X_1 = \mathrm{e}^{\pi \mathrm{i}\sigma_1} + \mathrm{e}^{-\pi\mathrm{i}\sigma_1}$*, and* $\alpha(X, \vartheta)$ *are as in Proposition 4.4, and* (X, ϑ) *is related to* $(\widetilde{X}, \widetilde{\vartheta}, \epsilon)$ *by the birational transformation* (5.12) *and* (5.1)*.*

(2) *When* $x \to \epsilon$, $x \in \mathsf{X}_+^{\spadesuit}(\epsilon)$, $\epsilon \in \mathsf{E}_+$*:*

$$q \sim \frac{1}{\alpha(\vartheta', X')}\left(\frac{x-\epsilon}{x}\right)^{\sigma_1' - 1} + \ O(1),$$
$$p \sim \alpha(\vartheta', X')\frac{\vartheta_0 + \vartheta_1 + \sigma_1' - 2}{2}\left(\frac{x-\epsilon}{x}\right)^{1-\sigma_1'} + \ O\left(\left(\frac{x-\epsilon}{x}\right)^{2-2\sigma_1'}\right),$$

where $(X', \vartheta') = g_{\infty 0}(X, \theta)$*, i.e.*

$$\vartheta' = (\vartheta_\infty, \vartheta_t, \vartheta_1, \vartheta_0), \qquad X' = (X_0,\ X_1 - F_{X_1},\ X_t),$$

and (X, ϑ) *are related to* $(\widetilde{X}, \widetilde{\vartheta}, \epsilon)$ *by* (5.12) *and* (5.1)*.*

Proof of Proposition 5.9. — By Proposition 5.10, in order for the solution ${}^{\mathrm{t}}(q_+^{\spadesuit}(x), p_+^{\spadesuit}(x))$ to be bounded at $x = 0$ we need to have $\sigma_1 = \vartheta_0 + \vartheta_t$, while $\alpha(\vartheta, X) \neq 0, \infty$. Using (4.13) let us write $\alpha(\vartheta, X) = \frac{A(\vartheta,\sigma_1)}{\mathrm{e}^{\pi\mathrm{i}\sigma_1} F_{X_t} - F_{X_0}}$ for some function A. This gives us the following condition:

$$X_1 = e_0 e_t + \frac{1}{e_0 e_t}, \qquad e_0 e_t F_{X_t} - F_{X_0} \neq 0. \tag{5.16}$$

Similarly, in order for the solution ${}^{\mathrm{t}}(q_+^{\spadesuit}(x), p_+^{\spadesuit}(x))$ to be bounded at $x = \epsilon$ we need to have $\sigma_1' = 2 - \vartheta_0 - \vartheta_1$, while $\alpha(\vartheta', X') \neq 0, \infty$, meaning that $\mathrm{e}^{\pi\mathrm{i}\sigma_1'} F'_{X'_t} - F'_{X'_0} \neq 0$. This gives us the following condition:

$$X_t = e_0 e_1 + \frac{1}{e_0 e_1}, \qquad e_0 e_1 F_{X_1} - F_{X_0} \neq 0. \tag{5.17}$$

By Proposition 4.5, the identity in (5.17) means that either

$$\begin{cases} \text{(a)} \quad e_0 e_1 X_0 + X_1 - e_1 a_\infty - e_0 a_t = 0, \\ \text{(b)} \quad e_0 e_1 X_1 + X_0 - e_0 a_\infty - e_1 a_t = 0, \end{cases}$$

from which we have respectively

$$\begin{cases} \text{(a)} \quad X_0 = \frac{1}{e_t e_1} + \frac{a_\infty}{e_0} - \frac{1}{e_0^2 e_t e_1}, \\ \text{(b)} \quad X_0 = e_t e_1 + e_0 a_\infty - e_0^2 e_t e_1. \end{cases}$$

The case (a) doesn't satisfy neither the condition (5.16) nor (5.17). In the case (b) using the formulas (5.13) we obtain that

$$\widetilde{X}_0 = \frac{1}{e_0}, \quad \widetilde{X}_1 = e_t e_1 + e_0 a_\infty - e_0^2 e_t e_1, \quad \widetilde{X}_\infty = e_0 e_t e_1. \qquad \square$$

Proof of Proposition 5.10. — The asymptotics at $x = 0$ is simply the Jimbo's formula of Proposition 4.4, which for $\epsilon \in \mathbb{R}_{>0}$ is defined on the sector $|\arg(-x)| < \pi$.

The asymptotics at $x = \epsilon$ is obtained from the Jimbo's formula using the Okamoto's transformation

$$t' = t^{-1}, \quad q' = q^{-1}, \quad p' = q \cdot \left(\frac{\vartheta_0 + \vartheta_t + \vartheta_1 + \vartheta_\infty}{2} - 1 - qp \right),$$
$$\vartheta' = (\vartheta_\infty, \vartheta_t, \vartheta_1, \vartheta_0),$$

which preserves the Hamiltonian system of P_{VI}, i.e.

$$x' = \epsilon - x, \quad q' = q^{-1}, \quad p' = q \cdot \left(\frac{\vartheta_0 + \widetilde{\vartheta}_1 + \vartheta_\infty}{2} - 1 - qp \right), \tag{5.18}$$

which preserves the system (3.1). This corresponds to the transformation

$$z' = z^{-1}, \quad t' = t^{-1}, \quad (A_0', A_{t'}', A_1', A_\infty') = (A_\infty, A_t, A_1, A_0),$$

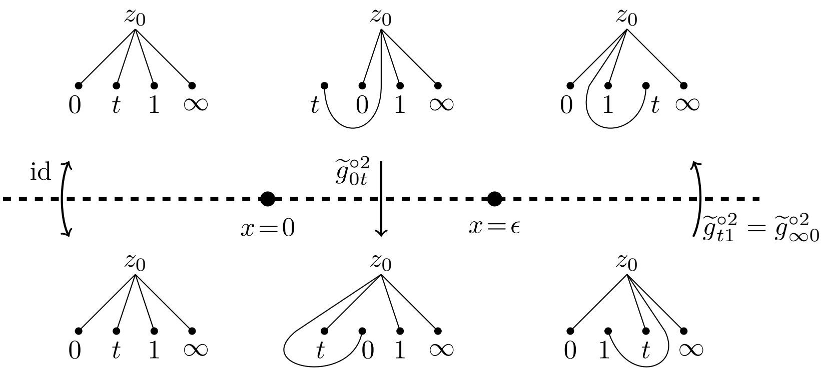


Figure 5.2. The paths around which are taken the loops $\gamma_0, \gamma_t, \gamma_1$ defining the confluent monodromy representation ρ_+ and therefore the coordinate $\widetilde{X}$ on $\mathcal{S}_V(\widetilde{\theta})$ for $\epsilon \in \mathsf{E}_+ \smallsetminus \{0\}$ in dependence on $x = \frac{\epsilon t}{t-1} \in \mathbb{C} \smallsetminus \{0, \epsilon\}$ (cf. Figure 4.4), and the corresponding transition maps $\widetilde{g}_{0t}^{\circ 2}$, $\widetilde{g}_{\infty 0}^{\circ 2}(\mathrm{e}^{\frac{2\pi \mathrm{i}}{\epsilon}}) : \mathcal{S}_V(\widetilde{\theta}) \to \mathcal{S}_V(\widetilde{\theta})$.

of the isomonodromic problem (4.1). It transforms the monodromy representation ρ_+ (Figure 5.2) to ρ', $\rho'(\gamma'_{j'}) = M_j$, $j = 0, t, 1, \infty$, where for $x \in \mathsf{X}_+^{\spadesuit}(\epsilon)$, $\epsilon \in \mathsf{E}_+$, one has up to a conjugation (see Figure 5.3)

$$\gamma'_{0'} = \gamma_\infty, \quad \gamma'_{t'} = \gamma_t, \quad \gamma'_{1'} = \gamma_1, \quad \gamma'_{\infty'} = \gamma_\infty \gamma_0 \gamma_\infty^{-1}.$$

This means that up to a conjugation

$$\begin{aligned} M'_0 &= \rho'(\gamma_0) = M_0 M_\infty M_0^{-1}, \\ M'_t &= \rho'(\gamma_t) = M_t, \\ M'_1 &= \rho'(\gamma_1) = M_1, \\ M'_\infty &= \rho'(\gamma_\infty) = M_0, \end{aligned}$$

i.e. $\rho' = (\beta_{\infty 0})_* \rho_+$ (4.15), and hence (4.17)

$$X' = g_{\infty 0}(X) = (X_0,\ X_1 - F_{X_1},\ X_t). \qquad \square$$

6. The wild monodromy action on $\mathcal{S}_V(\widetilde{\theta})$

The only nonlinear monodromy actions (pure braid actions) on $\mathcal{S}_V(\widetilde{\theta})$ that converge during the confluence are those generated by $\widetilde{g}_{\infty 0}^{\circ 2}$ (in Proposition 5.5). As was explained in Section 3.2, for the other actions we need

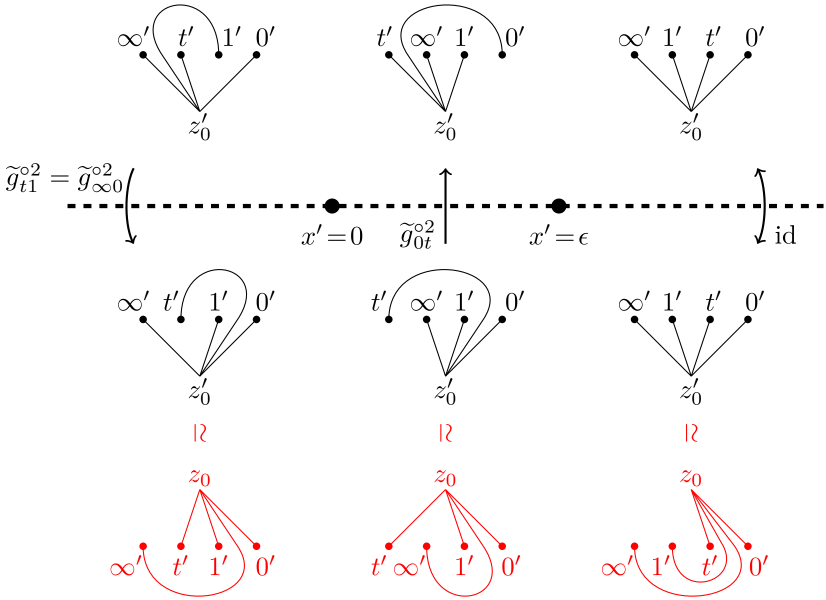


Figure 5.3. The transform of the Figure 5.2 by $x' = \epsilon - x$, $z' = z^{-1}$, $(0', t', 1', \infty') = (\infty, t^{-1}, 1, 0)$, showing the paths around which are taken the loops $\gamma'_{0'}, \gamma'_{t'}, \gamma'_{1'}, \gamma'_{\infty'}$ defining the confluent monodromy representation ρ' and therefore the coordinate X' on x. We are only interested in the lower half-plane part of this picture. The diagrams in the bottom row (red) are conjugated to those above (black) by moving the basepoint from z'_0 to z_0 in order to compare with those in Figure 5.2.

instead to consider limits of their pullbacks by Φ_+ (5.12) along the sequences

$$(\epsilon_n)_{n\in\pm\mathbb{N}}, \qquad \frac{1}{\epsilon_n} = \frac{1}{\epsilon_0} + n$$

along which the divergent parameter $e_t^2 = \mathrm{e}^{\frac{2\pi\mathrm{i}}{\epsilon}}$ stays constant. This amounts to replacing e_t^2 by a new independent parameter $\kappa \in \mathbb{C}^*$, i.e. writing

$$e_t e_1 = \widetilde{e}_1, \qquad \frac{e_t}{e_1} = \frac{\kappa}{\widetilde{e}_1}. \tag{6.1}$$

The idea of taking the limit of the Riemann–Hilbert correspondence for the isomonodromic problem (5.3) along such discrete sequences (ϵ_n) was already considered by Kitaev [32] but with a different aim. The limit of a nonlinear monodromy operator accumulating towards a one-parameter family of wild monodromy operators acting on the wild character variety have not been studied before.

Proposition 6.1. — *For $\epsilon \neq 0$, let Φ_+ be the transformation* (5.12)*, and let*

$$\widetilde{g}_{ij}^{\circ 2}(\,\cdot\,;\kappa) := \Phi_+^{-1} \circ g_{ij}^{\circ 2} \circ \Phi_+ : \mathcal{S}_V(\widetilde{\theta}) \longrightarrow \mathcal{S}_V(\widetilde{\theta}), \quad \{i,j\} = \{0,t\},$$

be the pullback of the monodromy action $g_{ij}^{\circ 2}$,composed with the substitution (6.1)*. Then*

$$\begin{aligned}
\widetilde{g}_{0t}^{\circ 2}(\,\cdot\,;\kappa):\quad & \widetilde{X}_0 \longmapsto \frac{\widetilde{e}_1}{\widetilde{X}_\infty},\\
& \widetilde{X}_1 \longmapsto \widetilde{X}_1 + \frac{1}{\widetilde{e}_1}(\widetilde{X}_1\widetilde{F}_{\tilde{X}_1} - \widetilde{X}_\infty\widetilde{F}_{\tilde{X}_\infty}) + \frac{\kappa}{\widetilde{e}_1^2}\widetilde{X}_\infty\widetilde{F}_{\tilde{X}_0} - \widetilde{F}_{\tilde{X}_1}\left(\frac{1}{\kappa} + \frac{\kappa}{\widetilde{e}_1^2}\right),\\
& \widetilde{X}_\infty \longmapsto \widetilde{X}_\infty + \frac{\widetilde{e}_1}{\kappa\widetilde{X}_\infty}\widetilde{F}_{\tilde{X}_1},
\end{aligned}$$

$$\begin{aligned}
\widetilde{g}_{t0}^{\circ 2}(\,\cdot\,;\kappa):\quad & \widetilde{X}_0 \longmapsto \widetilde{X}_0 + \frac{\kappa}{\widetilde{e}_1\widetilde{X}_0}\widetilde{F}_{\tilde{X}_1},\\
& \widetilde{X}_1 \longmapsto \widetilde{X}_1 + \frac{1}{\widetilde{e}_1}(\widetilde{X}_1\widetilde{F}_{\tilde{X}_1} - \widetilde{X}_0\widetilde{F}_{\tilde{X}_0}) + \frac{1}{\kappa}\widetilde{X}_0\widetilde{F}_{\tilde{X}_\infty} - \widetilde{F}_{\tilde{X}_1}\left(\frac{1}{\kappa} + \frac{\kappa}{\widetilde{e}_1^2}\right),\\
& \widetilde{X}_\infty \longmapsto \frac{\widetilde{e}_1}{\widetilde{X}_0}.
\end{aligned}$$

They preserve the symplectic form $\widetilde{\omega}_{\mathcal{S}_V}$, and change the Fricke relation by a factor:

$$\widetilde{F} \circ \widetilde{g}_{t0}^{\circ 2}(\widetilde{X};\kappa) = \frac{(\widetilde{X}_1 - \frac{\tilde{e}_1}{\kappa} - \frac{\kappa}{\tilde{e}_1})}{\left(\widetilde{X}_1 - \frac{\tilde{e}_1}{\kappa} - \frac{\kappa}{\tilde{e}_1} - (F_{X_0} \circ \Phi_+)\right)}\widetilde{F}(\widetilde{X},\widetilde{\theta}).$$

Proof. — The formulas are obtained by plugging (5.12), (5.13) to $g_{0t}^{\circ 2}$ and $g_{t0}^{\circ 2}$ (4.18), where the action of $g_{0t}^{\circ 2}$ fixes (e_t, e_1), and hence also $(\widetilde{e}_1, \kappa)$.

Alternatively, the action (4.15) of $(\beta_{0t}^2)_*$ on the monodromy representation ρ_+ is

$$\begin{aligned}
(\beta_{0t}^2)_* :\ M_{0+} &\longmapsto (M_{t+}M_{0+})M_{0+}(M_{t+}M_{0+})^{-1},\\
M_{t+} &\longmapsto (M_{t+}M_{0+})M_{t+}(M_{t+}M_{0+})^{-1},\\
M_{1+} &\longmapsto M_{1+},\\
M_{\infty+} &\longmapsto M_{\infty+},
\end{aligned}$$

where by (5.5) $M_{t+}M_{0+} = N_t S_{2+} M_{0+}$. Decompose $S_{2+}M_{0+} = LU$ with

$$L = \begin{pmatrix} \alpha+\gamma s_{2+} & 0 \\ \gamma & \delta-\gamma\frac{\beta+\delta s_{2+}}{\alpha+\gamma s_{2+}} \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & \frac{\beta+\delta s_{2+}}{\alpha+\gamma s_{2+}} \\ 0 & 1 \end{pmatrix},$$

and conjugate the above monodromy representation by N_tL, so that the image of $M_{t+} = N_tS_{2+}$, resp. $M_{1+} = S_{1+}N_1$, is again upper, resp. lower, triangular:

$$\begin{aligned} M_{0+} &\longmapsto UM_{0+}U^{-1}, \\ M_{t+} &\longmapsto UM_{t+}U^{-1}, & S_{2+} &\longmapsto N_t^{-1}UN_tS_{2+}U^{-1} \\ M_{1+} &\longmapsto L^{-1}N_t^{-1}M_{1+}N_tL, & S_{1+} &\longmapsto L^{-1}N_t^{-1}S_{1+}NLN^{-1}N_t, \\ M_{\infty+} &\longmapsto L^{-1}N_t^{-1}M_{\infty+}N_tL, \end{aligned}$$

where

$$N_t(\kappa) = \begin{pmatrix} \kappa^{\frac{1}{2}} & 0 \\ 0 & \kappa^{-\frac{1}{2}} \end{pmatrix}, \qquad N = \begin{pmatrix} \tilde{e}_1 & 0 \\ 0 & \tilde{e}_1^{-1} \end{pmatrix},$$

and in particular the images of M_{t+} and M_{1+} have the same diagonal parts as M_{t+} and M_{1+} have. From this one can express the action of $\widetilde{g}^{\circ 2}_{0t}(\kappa)$ on $\widetilde{X}$ (5.7) in terms of $\alpha, \beta, \gamma, \delta, s_{1+}, s_{2+}$ and then re-express it in terms of $\widetilde{X}, \widetilde{\theta}$. Similarly for the inverse $\widetilde{g}^{\circ 2}_{t0}(\kappa)$. □

The initial condition $c = (c_1, c_2)$ of the canonical sectorial solutions of P_V of Section 2.3 define through the map $(x, c) \mapsto {}^{\mathrm{t}}(q^{\spadesuit}(x; c), p^{\spadesuit}(x; c))$ (2.11) local coordinates on the space of leaves of the Painlevé foliation over the sector $x \in \mathsf{X}^{\spadesuit}(0)$, $|c| < \delta_c$. Therefore, c defines local coordinates also on the Okamoto space of initial conditions $\mathcal{M}_{V,\tilde{t}}$ (2.5) of P_V for each $x = \widetilde{t}^{-1} \in \mathsf{X}^{\spadesuit}(0)$ on a neighborhood of the point which corresponds to the sectorial center manifold $c = 0$. The Okamoto space $\mathcal{M}_{V,\tilde{t}}$ and the wild character variety $\mathcal{S}_V(\widetilde{\theta})$ are isomorphic on some Zariski open set

$$\mathcal{M}_{V,\tilde{t}}(\widetilde{\vartheta}) \dashrightarrow \mathcal{S}_V(\widetilde{\theta}).$$

It has been conjectured in [53] that in fact $\mathcal{M}_{V,\tilde{t}}$ is a isomorphic to a minimal strict resolution of $\mathcal{M}_{V,\tilde{t}}$ if singular (for P_{VI} this is known to be true, see [25] and the references there). Anyway, it means that the wild monodromy pseudogroup of Definition 2.7 acts locally on $\mathcal{S}_V(\widetilde{\theta})$ near the point (5.15). It turns out that this action is in fact global (Proposition 6.2 below). By Proposition 3.7, the wild monodromy pseudogroup is obtained as a limit of the nonlinear monodromy group through the accumulation along the discrete sequences (3.16) of the parameter ϵ.

Proposition 6.2. — *For $\epsilon = 0$, the action of the nonlinear wild monodromy operators $\widetilde{\mathbf{M}}^{\spadesuit}_{0+}(c;\kappa)$ and $\widetilde{\mathbf{M}}^{\spadesuit}_{\epsilon+}(c;\kappa)$ (3.17), which are defined locally near the point corresponding to $c = 0$ on the Okamoto fiber $\mathcal{M}_{V,\tilde{t}}$ over $x = \widetilde{t}^{-1} \in \mathsf{X}^{\spadesuit}(0)$, extend as bimeromorphic maps to the whole fiber.*

In the coordinate $\widetilde{X}$ (5.7) *on* $\mathcal{S}_V(\widetilde{\theta})$ *over* $\widetilde{t}^{-1} \in \mathsf{X}^{\spadesuit}(0)$ *the nonlinear wild monodromy operator*

- $\widetilde{\mathbf{M}}^{\spadesuit}_{0+}(c;\kappa)$ *corresponds to the action of* $\widetilde{g}^{\circ 2}_{0t}(\widetilde{X};\kappa)$ *on* $\mathcal{S}_V(\widetilde{\theta})$,
- $\widetilde{\mathbf{M}}^{\spadesuit}_{\epsilon+}(c;\kappa)$ *corresponds to the action of* $\widetilde{g}^{\circ 2}_{0\infty} \circ \widetilde{g}^{\circ 2}_{t0}(\widetilde{X};\kappa)$ *on* $\mathcal{S}_V(\widetilde{\theta})$.

Proof. — In fact, for all $0 \neq \epsilon \in \mathsf{E}_+$ the corresponding action of the nonlinear monodromy of $P_{VI}(\vartheta)$ around 0, resp. ϵ, from some base-point $x_0 \in \mathsf{X}^{\spadesuit}_+(0)$ on the character variety $\mathcal{S}_V(\widetilde{\theta})$ is given by $\widetilde{g}^{\circ 2}_{0t}(\widetilde{X};e_t^2)$, resp. $\widetilde{g}^{\circ 2}_{0\infty} \circ \widetilde{g}^{\circ 2}_{t0}(\widetilde{X};e_t^2)$, see Figure 5.2 (compare also with the right side of Figure 3.1). In particular, by Propositions 5.5 and 6.1 it is rational on all $\mathcal{S}_V(\widetilde{\theta})$. □

LEMMA 6.3. — *The vector field* $c_1\frac{\partial}{\partial c_1} - c_2\frac{\partial}{\partial c_2}$ *which is Hamiltonian for* $h(c) = c_1c_2$ *with respect to* $\mathrm{d}c_1 \wedge \mathrm{d}c_2$ *corresponds to the vector field*

$$\frac{1}{\widetilde{X}_0}\left(\widetilde{F}_{\tilde{X}_1}\frac{\partial}{\partial \widetilde{X}_\infty} - \widetilde{F}_{\tilde{X}_\infty}\frac{\partial}{\partial \widetilde{X}_1}\right), \tag{6.2}$$

on the wild character variety $\mathcal{S}_V(\widetilde{\theta})$. *It is Hamiltonian with respect to* $\widetilde{\omega}_{\mathcal{S}_V}$ (5.11) *for the Hamiltonian function*

$$\widetilde{H}_0(\widetilde{X}) = \frac{1}{2\pi \mathrm{i}}\log \widetilde{X}_0 + \frac{\vartheta_0}{2}, \tag{6.3}$$

which corresponds to h.

Proof. — By (3.20) and Proposition 6.2 the vector field $c_1\frac{\partial}{\partial c_1} - c_2\frac{\partial}{\partial c_2}$ corresponds to

$$\dot{\widetilde{X}} = -\left(\kappa\frac{\partial}{\partial \kappa}\widetilde{g}^{\circ 2}_{0t}(\,\cdot\,,\kappa)\right) \circ \widetilde{g}^{\circ 2}_{t0}(\widetilde{X},\kappa)$$

which can be calculated as (6.2) using the formulas of Proposition 6.1 (note that the $\frac{\partial}{\partial \tilde{X}_0}$ component is clearly null, and it is enough to calculate the $\frac{\partial}{\partial \tilde{X}_\infty}$ component only, because then the $\frac{\partial}{\partial \tilde{X}_1}$ component is uniquely determined by the relation of tangency $\mathrm{d}\widetilde{F} = 0$).

Now calculating in the local coordinate $(\widetilde{X}_1, \widetilde{X}_\infty)$ on $\mathcal{S}_V(\widetilde{\theta})$ one has $\mathrm{d}\widetilde{X}_0 = -\frac{\tilde{F}_{\tilde{X}_1}}{\tilde{F}_{\tilde{X}_0}}\mathrm{d}\widetilde{X}_1 - \frac{\tilde{F}_{\tilde{X}_\infty}}{\tilde{F}_{\tilde{X}_0}}\mathrm{d}\widetilde{X}_\infty$, and therefore

$$\mathrm{d}\widetilde{H}_0 = \frac{1}{2\pi \mathrm{i}\widetilde{F}_{\tilde{X}_0}}\left(-\frac{\widetilde{F}_{\tilde{X}_1}}{\widetilde{X}_0}\mathrm{d}\widetilde{X}_1 - \frac{\widetilde{F}_{\tilde{X}_\infty}}{\widetilde{X}_0}\mathrm{d}\widetilde{X}_\infty\right),$$

so $\widetilde{H}_0$ (6.3) is the Hamiltonian of (6.2) with respect to $\widetilde{\omega}_{\mathcal{S}_V} = \frac{\mathrm{d}\tilde{X}_1 \wedge \mathrm{d}\tilde{X}_\infty}{2\pi \mathrm{i}\tilde{F}_{\tilde{X}_0}}$ (5.11). Such Hamiltonian is defined up to a constant, and we know that

$h(c) = c_1c_2$ vanishes on the "sectorial center manifold" over $\mathsf{X}^{\spadesuit}_{+}(0)$ (Corollaries 2.2 and 3.3), which is given by the initial condition $c = 0$. So our corresponding Hamiltonian $\widetilde{H}_0$ is uniquely determined by the condition that it vanishes on this "sectorial center manifold", which by Proposition 5.9 corresponds to the point $(\widetilde{X}_0, \widetilde{X}_1, \widetilde{X}_\infty) = (\frac{1}{e_0},\ \widetilde{e}_1 + e_0a_\infty - e_0^2\widetilde{e}_1,\ e_0\widetilde{e}_1)$. □

Remark 6.4. — The point $(c_1, c_2) = (0, 0)$ is fixed by $c_1\frac{\partial}{\partial c_1} - c_2\frac{\partial}{\partial c_2}$ and correspondingly the point (5.15) is fixed by (6.2) – in fact it is easy to verify that both $\widetilde{F}_{\tilde{X}_1}$ and $\widetilde{F}_{\tilde{X}_\infty}$ vanish at this point.

Using the formulas (3.19) we can now express also the Stokes operators.

Theorem 6.5 (Wild monodromy action on $\mathcal{S}_V(\widetilde{\theta})$).

(i) *The time-α-flow map of the vector field* (6.2), $\alpha \in \mathbb{C}$, *is given by*

$$\mathbf{t}(\widetilde{X}; \mathrm{e}^\alpha) = \widetilde{g}^{\circ 2}_{0t}(\,\cdot\,; \mathrm{e}^{-\alpha}) \circ \widetilde{g}^{\circ 2}_{t0}(\widetilde{X}; 1). \tag{6.4}$$

It factors through the exponential $\alpha \mapsto \mathrm{e}^\alpha \in \mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$ as a multiplicative action of $\mathbb{C}^*$:

$$\begin{aligned} \mathbf{t}(\,\cdot\,; \mathrm{e}^\alpha) :\quad & \widetilde{X}_0 \longmapsto \widetilde{X}_0, \\ & \widetilde{X}_1 \longmapsto \widetilde{X}_1 - (1 - \mathrm{e}^{-\alpha})\frac{\widetilde{F}_{\tilde{X}_\infty}}{\widetilde{X}_0} + (2 - \mathrm{e}^{\alpha} - \mathrm{e}^{-\alpha})\frac{\widetilde{F}_{\tilde{X}_1}}{\widetilde{X}_0^2}, \\ & \widetilde{X}_\infty \longmapsto \widetilde{X}_\infty - (1 - \mathrm{e}^{\alpha})\frac{\widetilde{F}_{\tilde{X}_1}}{\widetilde{X}_0}. \end{aligned} \tag{6.5}$$

The action of the nonlinear exponential torus (2.14) *of P_V on $\mathcal{S}_V(\widetilde{\theta})$ is given by the composition of* $\mathbf{t}(\widetilde{X}; \mathrm{e}^\alpha)$ *with any analytic germ* $\alpha(\widetilde{H}_0(\widetilde{X}))$ *of* $\widetilde{H}_0(\widetilde{X})$ (6.3).

(ii) *The action of the nonlinear* formal monodromy $\mathbf{N}$ (2.15) *on* $\mathcal{S}_V(\widetilde{\theta})$ *is given by*

$$\mathbf{n}(\widetilde{X}) := \mathbf{t}\left(\widetilde{X}; \frac{\widetilde{X}_0^4}{\widetilde{e}_1^2}\right), \tag{6.6}$$

where $\mathbf{t}(\,\cdot\,;\,\cdot\,)$ *is as above* (6.5).

(iii) *The action of the nonlinear* Stokes operator $\mathbf{N}^{\circ(-1)} \circ \widetilde{\mathbf{S}}_1 \circ \mathbf{N}$ (2.17) *on* $\mathcal{S}_V(\widetilde{\theta})$ *is given by*

$$\mathbf{n}^{\circ -1} \circ \widetilde{\mathbf{s}}_1 \circ \mathbf{n}(\widetilde{X}) := \widetilde{g}^{\circ 2}_{0t}(\widetilde{X}; \widetilde{X}_0^2), \tag{6.7}$$

and the action of the nonlinear Stokes operator $\widetilde{\mathbf{S}}_2$ (2.17) *on* $\mathcal{S}_V(\widetilde{\theta})$ *is given by*

$$\widetilde{\mathbf{s}}_2(\widetilde{X}) := \widetilde{g}^{\circ 2}_{0\infty} \circ \widetilde{g}^{\circ 2}_{t0}\left(\widetilde{X}; \frac{\widetilde{e}_1^2}{\widetilde{X}_0^2}\right), \tag{6.8}$$

where $\widetilde{g}_{0t}^{\circ 2}$, $\widetilde{g}_{t0}^{\circ 2}$ *are as in Proposition 6.1, and* $\widetilde{g}_{0\infty}^{\circ 2}$ *given by* (6.9) *is as in Proposition 5.5.*

(iv) *The action of the* monodromy operator $\widetilde{\mathbf{M}}^{\spadesuit} = \widetilde{\mathbf{S}}_2 \circ \widetilde{\mathbf{S}}_1 \circ \mathbf{N}$ (2.19) *on* $\mathcal{S}_V(\widetilde{\theta})$ *is given by*

$$\widetilde{\mathbf{s}}_2 \circ \widetilde{\mathbf{s}}_1 \circ \mathbf{n} = \widetilde{g}_{0\infty}^{\circ 2} : \quad \begin{aligned} \widetilde{X}_0 &\longmapsto \widetilde{X}_0 - \widetilde{F}_{\tilde{X}_0}, \\ \widetilde{X}_1 &\longmapsto \widetilde{X}_1, \\ \widetilde{X}_\infty &\longmapsto \widetilde{X}_\infty - \widetilde{F}_{\tilde{X}_\infty} + \widetilde{X}_1 \widetilde{F}_{\tilde{X}_0}. \end{aligned} \tag{6.9}$$

(v) *The action of the* wild monodromy pseudogroup *of* P_V *on* $\mathcal{S}_V(\widetilde{\theta})$ *is generated by*

$$\left\langle \widetilde{g}_{0\infty}^{\circ 2},\ \widetilde{g}_{0t}^{\circ 2}(\,\cdot\,; \mathrm{e}^{-\alpha(\tilde{H}_0)}) \,\middle|\, \alpha(\,\cdot\,) \text{ analytic germ} \right\rangle,$$

or equivalently by

$$\left\langle \widetilde{\mathbf{s}}_1,\ \widetilde{\mathbf{s}}_2,\ \mathbf{t}(\,\cdot\,; \mathrm{e}^{\alpha(\tilde{H}_0)}) \,\middle|\, \alpha(\,\cdot\,) \text{ analytic germ} \right\rangle.$$

It fixes the singularities of $\mathcal{S}_V(\widetilde{\theta})$*, and its restriction to the smooth locus of* $\mathcal{S}_V(\widetilde{\theta})$ *represents faithfully the nonlinear action of the wild monodromy pseudogroup on the "non-Riccati solutions" of* $P_V(\widetilde{\vartheta})$*.*

Proof.

(i). — By direct calculation one verifies easily that (6.5) is indeed the time-α-flow map of (6.2).

(ii). — From (3.14),

$$\mathbf{n}(\widetilde{X}) = \mathbf{t}(\widetilde{X}; \mathrm{e}^{2\pi \mathrm{i}(-2\vartheta_0 - \tilde{\vartheta}_1 + 4\tilde{H}_0)}),$$

where $H_0(\widetilde{X})$ is (6.3) meaning that $\mathrm{e}^{2\pi \mathrm{i}(-2\vartheta_0 - \tilde{\vartheta}_1 + 4\tilde{H}_0(\tilde{X}))} = \frac{\tilde{X}_0^4}{\tilde{e}_1^2}$.

(iii). — From (3.19) and Proposition 6.2 and Lemma 6.3,

$$\mathbf{n}^{\circ -1} \circ \widetilde{\mathbf{s}}_1 \circ \mathbf{n}(\widetilde{X}) = \widetilde{g}_{0t}^{\circ 2}(\widetilde{X}; \mathrm{e}^{2\pi \mathrm{i}(-\vartheta_0 + 2\tilde{H}_0)}),$$

where $\mathrm{e}^{2\pi \mathrm{i}(-\vartheta_0 + 2\tilde{H}_0(\tilde{X}))} = \widetilde{X}_0^2$, and

$$\widetilde{\mathbf{s}}_2(\widetilde{X}) = \widetilde{g}_{0\infty}^{\circ 2} \circ \widetilde{g}_{t0}^{\circ 2}(\widetilde{X}; \mathrm{e}^{2\pi \mathrm{i}(\vartheta_0 + \tilde{\vartheta}_1 - 2\tilde{H}_0)}),$$

where $\mathrm{e}^{2\pi \mathrm{i}(\vartheta_0 + \tilde{\vartheta}_1 - 2\tilde{H}_0(\tilde{X}))} = \frac{\tilde{e}_1^2}{\tilde{X}_0^2}$.

(iv). — We know from Proposition 5.5 that $\widetilde{\mathbf{M}}^{\spadesuit} = \widetilde{\mathbf{M}}_{\epsilon}^{\spadesuit} \circ \widetilde{\mathbf{M}}_0^{\spadesuit}$ corresponds to the monodromy operator $\widetilde{g}_{0\infty}^{\circ 2}$ around $x = \widetilde{t}^{-1} = 0$ (cf. Proposition 6.2) and is independent of κ. But let us calculate the composition $\widetilde{\mathbf{s}}_2 \circ \widetilde{\mathbf{s}}_1 \circ \mathbf{n}$ for the sake of clarity. From (6.4)

$$\widetilde{g}_{t0}^{\circ 2}(\widetilde{X}; \mathrm{e}^{\alpha}) = \widetilde{g}_{t0}^{\circ 2}(\,\cdot\,; 1) \circ \mathbf{t}(\widetilde{X}; \mathrm{e}^{\alpha}), \qquad \widetilde{g}_{0t}^{\circ 2}(\widetilde{X}; \mathrm{e}^{\alpha}) = \mathbf{t}(\,\cdot\,; \mathrm{e}^{-\alpha}) \circ \widetilde{g}_{t0}^{\circ 2}(\widetilde{X}; 1),$$

and plugging in (6.6), (6.7) and (6.8) we have

$$\widetilde{\mathbf{s}}_2 \circ \widetilde{\mathbf{s}}_1 \circ \mathbf{n}(\widetilde{X}) = \widetilde{g}^{\circ 2}_{0\infty}(\widetilde{X}) \circ \widetilde{g}^{\circ 2}_{t0}\left(\widetilde{X}; \frac{\widetilde{e}_1^2}{\widetilde{X}_0^2}\right) \circ \mathbf{t}\left(\widetilde{X}; \frac{\widetilde{X}_0^4}{\widetilde{e}_1^2}\right) \circ \widetilde{g}^{\circ 2}_{0t}(\widetilde{X}; \widetilde{X}_0^2)$$

$$= \widetilde{g}^{\circ 2}_{0\infty}(\widetilde{X}) \circ \widetilde{g}^{\circ 2}_{t0}(\widetilde{X}; 1) \circ \mathbf{t}\left(\widetilde{X}; \frac{\widetilde{e}_1^2}{\widetilde{X}_0^2}\right) \circ \mathbf{t}\left(\widetilde{X}; \frac{\widetilde{X}_0^4}{\widetilde{e}_1^2}\right) \circ \mathbf{t}\left(\widetilde{X}; \frac{1}{\widetilde{X}_0^2}\right) \circ \widetilde{g}^{\circ 2}_{0t}(\widetilde{X}; 1)$$

$$= \widetilde{g}^{\circ 2}_{0\infty}(\widetilde{X}),$$

since $\widetilde{X}_0$ is a first integral of $\mathbf{t}$. □

Appendix. Painlevé equations as isomonodromic deformations of 3×3 systems

This section exposes first how to derive the Fricke formula (4.9) for the character variety $\mathcal{S}_{VI}(\theta)$ of $P_{VI}(\vartheta)$ as the space of Stokes data corresponding to isomonodromic deformations of 3×3 systems in Okubo and Birkhoff canonical forms, and describes the braid group action on the Stokes data. Most of this can be also found in a bit different form in the article of Boalch [4, Sections 2 & 3]. This description is then used to study the confluence of eigenvalues in these systems in order to show how the Stokes data of the limit system for $\epsilon = 0$ are connected with those for $\epsilon \neq 0$ (Figure A.4), providing thus another derivation of the wild character $\mathcal{S}_V(\widetilde{\theta})$ variety of $P_V(\widetilde{\vartheta})$ (5.8) and of the formulas of the birational change of variables Φ_+ (5.13).

A.1. Systems in Okubo and Birkhoff forms

Aside of the 2×2 systems (4.1), the sixth Painlevé equation P_{VI} governs also the isomonodromic deformations 3×3 linear differential systems in an *Okubo form*

$$\left(zI - \begin{pmatrix} 0 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \frac{\mathrm{d}\psi}{\mathrm{d}z} = \Big[B(t) + \lambda I\Big]\psi, \tag{A.1}$$

where the matrix $B(t)$ can be written as

$$B(t) = \begin{pmatrix} \vartheta_0 & w_0 u_t v_t - v_t & w_0 u_1 v_1 - v_1 \\ w_t u_0 v_0 - v_0 & \vartheta_t & w_t u_1 v_1 - v_1 \\ w_1 u_0 v_0 - v_0 & w_1 u_t v_t - v_t & \vartheta_1 \end{pmatrix}, \qquad w_i = \frac{v_i + \vartheta_i}{u_i v_i},$$

where ϑ_i are the parameters of P_{VI}, and the matrix $B(t)$ has eigenvalues

$$0, \quad -\kappa_1 = \frac{1}{2}(\vartheta_0 + \vartheta_t + \vartheta_1 - \vartheta_\infty), \quad -\kappa_2 = \frac{1}{2}(\vartheta_0 + \vartheta_t + \vartheta_1 + \vartheta_\infty). \tag{A.2}$$

The isomonodromic deformation of such systems in relation to P_{VI} was first considered in the papers of Harnad [22], Dubrovin [13, 14] and Mazzocco [43]. The system (A.1) can be obtained from the 2×2 system (4.1) by the addition of $\frac{1}{2}(\frac{\vartheta_0}{z} + \frac{\vartheta_t}{z-t} + \frac{\vartheta_1}{z-1})I$ to $A(z,t)$ (this corresponds to a gauge transformation $\phi \mapsto z^{-\frac{\vartheta_0}{2}}(z-t)^{-\frac{\vartheta_t}{2}}(z-1)^{-\frac{\vartheta_1}{2}}\phi$), followed by the Katz's operation of middle convolution mc_λ with a generic parameter λ different from $0, \kappa_1, \kappa_2$ [21] (see also [4, 43]).

Equivalently, one may also consider the generalized isomonodromic problem for systems in a *Birkhoff canonical form*

$$\xi^2 \frac{\mathrm{d}y}{\mathrm{d}\xi} = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix} + \xi B(t) \right] y, \tag{A.3}$$

with a non-resonant irregular singularity at the origin. These are dual to the Okubo systems (A.1) through the Laplace transform

$$y(\xi) = \xi^{-1-\lambda} \int_0^\infty \psi(z)\, \mathrm{e}^{-\frac{z}{\xi}}\, \mathrm{d}z, \qquad |\arg(\xi) - \arg(z)| < \frac{\pi}{2}. \tag{A.4}$$

All three kinds of systems (4.1), (A.1), (A.3), and their isomonodromy problems are essentially equivalent (at least on the Zariski open set of irreducible systems (4.1)). Under an additional assumption that no ϑ_i is an integer, the condition on (generalized) isomonodromicity of the each of the above linear systems is equivalent to the Painlevé equation $P_{VI}(\vartheta)$ [21].

Notation A.1. — The entries of 3×3 matrices will be indexed by $(0,t,1)$ rather than $(1,2,3)$, in a correspondence to the eigenvalues of the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}$. As before, the triple of indices (i,j,k) will always denote a permutation of $(0,t,1)$, and (i,j,k,l) will denote a permutation of $(0,t,1,\infty)$.

A.2. Stokes matrices of the Birkhoff system

The Birkhoff system (A.3) posses a canonical formal solution

$$\widehat{Y}(\xi,t) = \widehat{T}(\xi,t) \begin{pmatrix} \xi^{\vartheta_0} & 0 & 0 \\ 0 & \mathrm{e}^{-\frac{t}{\xi}}\xi^{\vartheta_t} & 0 \\ 0 & 0 & \mathrm{e}^{-\frac{1}{\xi}}\xi^{\vartheta_1} \end{pmatrix},$$

with $\widehat{T}(\xi,t)$ an invertible formal series in ξ (with coefficients locally analytic in $t \in \mathbb{CP}^1 \smallsetminus \{0,1,\infty\}$), which is unique up to right multiplication by an invertible diagonal matrix, and unique if one demands that $\widehat{T}(0,t) = I$ [60]. It is well known that this series is Borel summable in each non-singular direction α (we remind that a *direction* $\alpha \in \mathbb{R}$ *is singular* for the system (A.3) if $(i-j) \in \mathrm{e}^{\mathrm{i}\alpha}\mathbb{R}^+$ for some $i,j \in \{0,t,1\}$, $i \neq j$). This means that for each non-singular direction α, there is an associated canonical fundamental matrix solution

$$Y_\alpha(\xi,t) = T_\alpha(\xi,t)\begin{pmatrix} \xi^{\vartheta_0} & 0 & 0 \\ 0 & \mathrm{e}^{-\frac{t}{\xi}}\xi^{\vartheta_t} & 0 \\ 0 & 0 & \mathrm{e}^{-\frac{1}{\xi}}\xi^{\vartheta_1} \end{pmatrix}, \qquad |\arg(\xi)-\alpha| < \pi, \tag{A.5}$$

where T_α is the Borel sum in ξ of $\widehat{T}$ in the direction α given by the Laplace integral

$$T_\alpha(\xi,t) = \frac{1}{\xi}\int_0^{+\infty \mathrm{e}^{\mathrm{i}\alpha}} U(z,t)\,\mathrm{e}^{-\frac{z}{\xi}}\,\mathrm{d}z,$$

where $U(z,t) = \sum_{k=0}^{+\infty}\frac{T_k(t)}{k!}z^k$ is the formal Borel transform of $\xi\,\widehat{T}(\xi,t) = \sum_{k=0}^{+\infty} T_k(t)\xi^{k+1}$. This solution does not depend on α as long as α does not cross any singular direction [2, 24, 41].

Let us restrict to $\alpha \in\,]-\pi,\pi[$, and suppose for a moment that $0,t,1$ are not collinear, i.e. that there are six distinct singular rays $(i-j)\mathbb{R}^+$ as in Figure A.1 (a). When α crosses some singular direction (in clockwise sense) the corresponding sectorial basis Y_α changes in a way that corresponds to a multiplication by a constant (with respect to ξ) invertible matrix, called *Stokes matrix*, of the form

$$S_{ij} = I + s_{ij}E_{ij}, \tag{A.6}$$

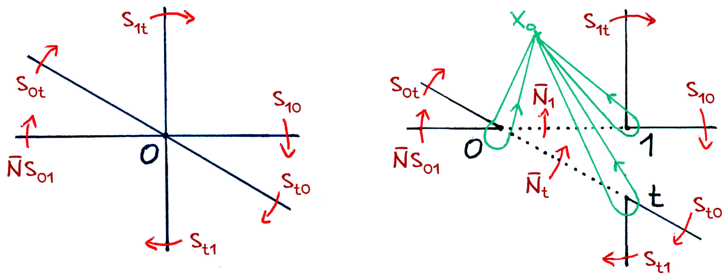
where E_{ij} denotes the matrix with 1 at the position (i,j) and zero elsewhere. For the singular ray $(0-1)\mathbb{R}^+$, one needs to take in account also the jump in the argument of ξ between $-\pi$ and π, therefore the change of basis is provided by a matrix $\overline{N}S_{01}$, where $\overline{N}$ is the *formal monodromy* of $\widehat{Y}$:

$$\overline{N} = \begin{pmatrix} e_0^2 & 0 & 0 \\ 0 & e_t^2 & 0 \\ 0 & 0 & e_1^2 \end{pmatrix}, \quad \text{where} \quad e_j := \mathrm{e}^{\pi\mathrm{i}\vartheta_j}. \tag{A.7}$$

See Figure A.1 (a).

Since in general the formal transformation $\widehat{T}(\xi,t)$, and therefore also the collection of the sectorial bases $Y_\alpha(\xi,t)$ (A.5), are unique only up to right multiplication by invertible diagonal matrices, the collection of the Stokes matrices S_{ij} is defined only up a simultaneous conjugation by diagonal matrices. The obvious invariants with respect to such conjugation are

$$s_{0t}s_{t0}, \quad s_{t1}s_{1t}, \quad s_{10}s_{01}, \quad s_{0t}s_{t1}s_{10}, \quad s_{1t}s_{t0}s_{01}, \tag{A.8}$$



(a) Birkhoff system *(b)* Okubo system

Figure A.1. Singular directions and Stokes matrices.

where s_{ij} are as in (A.6), subject to the relation

$$s_{0t}s_{t0} \cdot s_{t1}s_{1t} \cdot s_{10}s_{01} - s_{0t}s_{t1}s_{10} \cdot s_{1t}s_{t0}s_{01} = 0. \tag{A.9}$$

The *isomonodromic condition* on the family (A.3) demands that the collection of Stokes matrices is independent of t up to conjugacy by diagonal matrices, i.e. that (A.8) are constant.

A.3. Monodromy of the Okubo system

Now let us consider the Okubo system (A.1), where we chose for simplicity

$$\lambda = 0.$$

Corresponding to the canonical sectorial solutions bases $Y_\alpha = \big(Y_{\alpha,ij}\big)_{i,j}$ of (A.3), there are canonical sectorial solutions bases $\Psi_\alpha = \big(\Psi_{\alpha,ij}\big)_{i,j}$ of (A.1), given by the convolution integral

$$\Psi_{\alpha,ij}(z,t) = \frac{1}{\Gamma(\vartheta_j+\lambda)} \int_j^z U_{ij}(\zeta - j, t)(z-\zeta)^{\vartheta_j+\lambda-1}\mathrm{d}\zeta,$$

where $U(z,t) = \sum_{k=0}^{+\infty} \frac{T_k(t)}{k!} z^k$ is the formal Borel transform of $\xi\widehat{T}(\xi,t) = \sum_{k=0}^{+\infty} T_k(t)\xi^k$ [35, 58]. They are related to $Y_\alpha(\xi,t)$ by the Laplace transform (A.4)

$$Y_{\alpha,ij}(\xi,t) = \frac{1}{\xi}\int_j^{+\infty \mathrm{e}^{\mathrm{i}\alpha}} \Psi_{\alpha,ij}(z,t)\,\mathrm{e}^{-\frac{z}{\xi}}\,\mathrm{d}z, \qquad i,j \in \{0,t,1\}.$$

The sectors on which they are defined (see Figure A.1 (b)) are the different components of the complement in $\mathbb{C}$ of

$$\bigcup_{i,j\in\{0,t,1\}} \big(i+(i-j)\mathbb{R}^+\big) \,\cup\, [0,1] \,\cup\, [0,t].$$

When crossing one of the rays $i+(i-j)\mathbb{R}^+$ in clockwise sense the solution basis $\Psi_\alpha(z,t)$ changes by the same Stokes matrix S_{ij} (A.6) as before, except for the ray $-\mathbb{R}^+$, where it again changes by $\overline{N}S_{01}$ (cf. [35]). When crossing the segments $[0,t]$ the basis changes by $\overline{N}_t$, and on $[0,1]$ by $\overline{N}_1$, where

$$\overline{N}_i = I + (e_i^2-1)E_{ii}, \qquad e_i = \mathrm{e}^{\pi\mathrm{i}\vartheta_i},$$

is the monodromy matrix of $\begin{pmatrix} z^{\vartheta_0} & 0 & 0 \\ 0 & (z-t)^{\vartheta_t} & 0 \\ 0 & 0 & (z-1)^{\vartheta_1}\end{pmatrix}$ around the point $i\in\{0,t,1\}$, and $\overline{N}=\overline{N}_1\overline{N}_t\overline{N}_0$ (A.7). See Figure A.1 (b).

Fixing a base-point z_0 and three simple loops $\gamma_0,\gamma_t,\gamma_1$ in positive direction around the points $0,t,1$ respectively, such that their composition $\gamma_0\gamma_t\gamma_1=\gamma_\infty^{-1}$ gives a simple loop around the infinity as in Figure A.1 (b), let $\overline{M}_l$, $l=0,t,1,\infty$, be the associated monodromy matrices along γ_l:

$$\text{(A.10)}\qquad \overline{M}_0=\overline{N}_0S_{01}S_{0t},\quad \overline{M}_t=\overline{N}_1^{-1}S_{t0}S_{t1}\overline{N}_t\overline{N}_1,\quad \overline{M}_1=S_{1t}S_{10}\overline{N}_1,$$

determined up to a simultaneous conjugation in $\mathrm{GL}_3(\mathbb{C})$. We have

$$\mathrm{tr}(\overline{M}_i)=e_i^2+2,\quad i\in\{0,t,1\}.$$

Denoting

$$X_i=\frac{\mathrm{tr}(\overline{M}_j\overline{M}_k)-1}{e_je_k},$$

and s_{ij} as in (A.6), we have

$$\text{(A.11)}\quad X_0=\frac{e_t^2+e_1^2+e_1^2s_{t1}s_{1t}}{e_te_1},\quad X_t=\frac{e_0^2+e_1^2+e_0^2s_{10}s_{01}}{e_0e_1},$$
$$X_1=\frac{e_0^2+e_t^2+e_0^2s_{0t}s_{t0}}{e_0e_t}.$$

The monodromy around all the three points equals

$$\overline{M}_\infty^{-1}=\overline{M}_1\overline{M}_t\overline{M}_0=S_{1t}S_{10}S_{t0}S_{t1}\overline{N}S_{01}S_{0t}$$
$$=\begin{pmatrix} e_0^2 & e_0^2s_{0t} & e_0^2s_{01}\\ e_0^2s_{t0} & e_t^2+e_0^2s_{t0}s_{0t} & e_1^2s_{t1}+e_0^2s_{t0}s_{01}\\ e_0^2s_{10}+e_0^2s_{1t}s_{t0} & e_t^2s_{1t}+e_0^2s_{10}s_{0t}+e_0^2s_{1t}s_{t0}s_{0t} & e_1^2+e_1^2s_{1t}s_{t1}+e_0^2s_{10}s_{01}+e_0^2s_{1t}s_{t0}s_{01}\end{pmatrix}$$

From (A.2), we know that its eigenvalues are 1, $\mathrm{e}^{-2\pi\mathrm{i}\kappa_1} = \frac{e_0e_te_1}{e_\infty}$ and $\mathrm{e}^{-2\pi\mathrm{i}\kappa_2} = e_0e_te_1e_\infty$. Expressing the coefficients of the linear term E and the quadratic term E' of the characteristic polynomial of $\overline{M}_\infty^{-1}$ leads to

$$\begin{aligned}(\mathrm{A.12})\quad e_te_1X_0 + e_0e_1X_t + e_0e_tX_1 + e_0^2s_{1t}s_{t0}s_{01} - e_0^2 - e_t^2 - e_1^2 \\ = 1 + \frac{e_0e_te_1}{e_\infty} + e_0e_te_1e_\infty := E,\end{aligned}$$

$$\begin{aligned}(\mathrm{A.13})\quad e_0^2e_te_1X_0 + e_t^2e_0e_1X_t + e_1^2e_0e_tX_1 \\ - e_0^2e_1^2s_{0t}s_{t1}s_{10} - e_0^2e_t^2 - e_t^2e_1^2 - e_1^2e_0^2 \\ = \frac{e_0e_te_1}{e_\infty} + e_0e_te_1e_\infty + e_0^2e_t^2e_1^2 := E'.\end{aligned}$$

Inserting the expression for $s_{1t}s_{t0}s_{01}$ (A.12) and for $s_{0t}s_{t1}s_{10}$ (A.13) into the relation (A.9) gives the *Fricke relation* (4.9)

$$X_0X_tX_1 + X_0^2 + X_t^2 + X_1^2 - \theta_0X_0 - \theta_tX_t - \theta_1X_1 + \theta_\infty = 0,$$

with

$$\begin{aligned}\theta_i &= \frac{e_j^2 + e_k^2 + E}{e_je_k} + \frac{E'}{e_i^2e_je_k} = a_ia_\infty + a_ja_k,\\ \theta_\infty &= 1 + \frac{E}{e_0} + \frac{E}{e_t} + \frac{E}{e_1} + \frac{E'}{e_0e_t} + \frac{E'}{e_te_1} + \frac{E'}{e_1e_0}\\ &= a_0a_ta_1a_\infty + a_0^2 + a_t^2 + a_1^2 + a_\infty^2 - 4.\end{aligned}$$

Let us remark that for (i,j,k) a cyclic permutation of $(0,t,1)$, the line

$$\left\{X_k = \frac{e_i}{e_j} + \frac{e_j}{e_i},\quad e_iX_i + e_jX_j = a_\infty + e_ie_ja_k\right\}$$

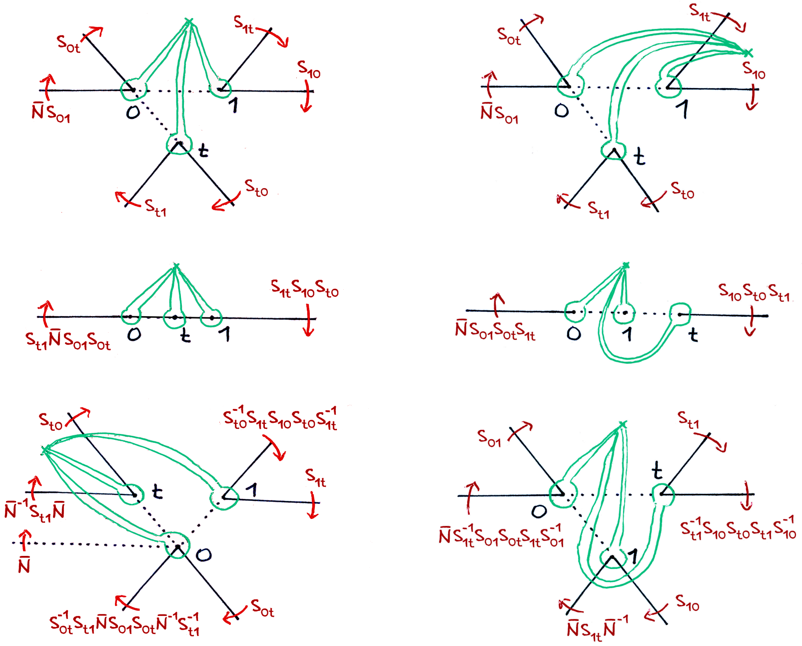
of Proposition 4.5 corresponds to $s_{ij} = 0$ in (A.13), while the line

$$\left\{X_k = \frac{e_i}{e_j} + \frac{e_j}{e_i},\quad e_iX_j + e_jX_i = a_k + e_ie_ja_\infty\right\}$$

corresponds to $s_{ji} = 0$ in (A.12).

We will now derive the induced action of the braids β_{0t} and β_{t1} (Figure A.2) on the Stokes matrices S_{ij}, providing an alternative proof of Proposition 4.8. The induced action of β_{0t}, resp. β_{t1}, on the Stokes matrices is obtained by:

(1) Tracing the connection matrices of the Okubo system (A.1) as the two corresponding points turn around each other according to the braid β_{0t}, resp. β_{t1}, and see how they change when the three points $0, t, 1$ align. See Figure A.2. We use the fact that $S_{ij}S_{kl} = S_{kl}S_{ij}$ if $j \neq k$ and $l \neq i$.



(a) Action of β_{0t}. *(b)* Action of β_{t1}.

Figure A.2. Braid actions on the Stokes matrices. The monodromy along each loop stays the same as the points $0, t, 1$ move.

(2) Swapping the names of the points $0 \leftrightarrow t$, resp. $t \leftrightarrow 1$. This permutes also the corresponding positions of all the matrices, i.e. acts on them by conjugation by $P_{0t} = \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix}$, resp. $P_{t1} = \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}$.

It follows from Figure A.2 that the action on the Stokes matrices is given (up to a simultaneous conjugation by diagonal matrices) by

$$\begin{aligned}
(\beta_{0t})_* :\ \overline{N} &\longmapsto P_{0t}\overline{N}P_{0t}, \\
S_{0t} &\longmapsto P_{0t}S_{t0}P_{0t}, \\
S_{1t} &\longmapsto P_{0t}S_{t0}^{-1}S_{1t}S_{10}S_{t0}S_{1t}^{-1}P_{0t}, \\
S_{10} &\longmapsto P_{0t}S_{1t}P_{0t}, \\
S_{t0} &\longmapsto P_{0t}S_{0t}P_{0t}, \\
S_{t1} &\longmapsto P_{0t}S_{0t}^{-1}S_{t1}\overline{N}S_{01}S_{0t}\overline{N}^{-1}S_{t1}^{-1}P_{0t}, \\
S_{01} &\longmapsto P_{0t}\overline{N}^{-1}S_{t1}\overline{N}P_{0t},
\end{aligned}
\qquad
\begin{aligned}
(\beta_{t1})_* :\ \overline{N} &\longmapsto P_{t1}\overline{N}P_{t1}, \\
S_{0t} &\longmapsto P_{t1}S_{01}P_{t1}, \\
S_{1t} &\longmapsto P_{t1}S_{t1}P_{t1}, \\
S_{10} &\longmapsto P_{t1}S_{t1}^{-1}S_{10}S_{t0}S_{t1}S_{10}^{-1}P_{t1}, \\
S_{t0} &\longmapsto P_{t1}S_{10}P_{t1}, \\
S_{t1} &\longmapsto P_{t1}\overline{N}S_{1t}\overline{N}^{-1}P_{t1}, \\
S_{01} &\longmapsto P_{t1}S_{1t}^{-1}S_{01}S_{0t}S_{1t}S_{01}^{-1}P_{t1}.
\end{aligned}$$

From this the corresponding action of g_{0t}, resp. g_{t1}, on the invariant elements (A.8) can be easily expressed, and subsequently re-expressed in terms of the coordinates X_0, X_t, X_1 (A.11). Or equivalently, one can express the induced action on the monodromy matrices (A.10):

$$\begin{aligned}(\beta_{0t})_* : \overline{M}_0 &\longmapsto P_{0t}S_{t0}^{-1}\overline{M}_t S_{t0}P_{0t}, & (\beta_{t1})_* : \overline{M}_0 &\longmapsto P_{t1}S_{1t}^{-1}\overline{M}_0 S_{1t}P_{t1},\\ \overline{M}_t &\longmapsto P_{0t}S_{t0}^{-1}\overline{M}_t\overline{M}_0\overline{M}_t^{-1}S_{t0}P_{0t}, & \overline{M}_t &\longmapsto P_{t1}S_{1t}^{-1}\overline{M}_1 S_{1t}P_{t1},\\ \overline{M}_1 &\longmapsto P_{0t}S_{t0}^{-1}\overline{M}_1 S_{t0}P_{0t}, & \overline{M}_1 &\longmapsto P_{t1}S_{1t}^{-1}\overline{M}_1\overline{M}_t\overline{M}_1^{-1}S_{1t}P_{t1}.\end{aligned}$$

A.4. Confluence of the Birkhoff systems and their character varieties

The substitution (5.1), (5.2) in the Birkhoff system (A.3) and a conjugation by $Q = \left(\begin{smallmatrix}\epsilon\tilde t & 0 & 0\\ 0 & 1 & -1\\ 0 & 0 & \epsilon\tilde t\end{smallmatrix}\right)$, corresponding to the change of variable $\widetilde{y} = Qy$, gives a parametric family of isomonodromic systems

$$\xi^2\frac{\mathrm{d}\widetilde{y}}{\mathrm{d}\xi} = \left[\begin{pmatrix}0 & 0 & 0\\ 0 & 1+\epsilon\tilde t & 1\\ 0 & 0 & 1\end{pmatrix} + \xi\widetilde{B}(\widetilde{t},\epsilon)\right]\widetilde{y}, \tag{A.14}$$

with

$$\begin{aligned}\widetilde{B} = QBQ^{-1} &= \begin{pmatrix}\vartheta_0 & \epsilon\tilde t(w_0u_tv_t - v_t) & -w_0u_0v_0 - v_1 - v_t\\ \frac{1}{\epsilon\tilde t}(w_t - w_1)u_0v_0 & \vartheta_t + v_t - w_1u_tv_t & -\frac{1}{\epsilon\tilde t}(w_t - w_1)u_0v_0\\ w_1u_0v_0 - v_0 & \epsilon\tilde t(w_1u_tv_t - v_t) & \vartheta_1 - v_t + w_1u_tv_t\end{pmatrix}\\ &= \begin{pmatrix}\vartheta_0 & \tilde u_1\tilde v_1 w_0 - \tilde v_1 & -\kappa_2 - \vartheta_0\\ -\tilde b_{t1} & \tilde\vartheta_1 + \kappa_2 + \tilde b_{10} & \tilde b_{t1}\\ \tilde b_{10} & \tilde t - \epsilon\tilde t[\tilde\vartheta_1 + \kappa_2 + \tilde b_{10}] & -\kappa_2 - \tilde b_{10}\end{pmatrix},\end{aligned}$$

where $\kappa_2 = -\frac{\vartheta_0 + \tilde\vartheta_1 + \vartheta_\infty}{2}$ and $\widetilde{u}_1, \widetilde{v}_1$ are as in (5.2), and

$$\begin{aligned}\widetilde{b}_{10} &= u_0v_0\frac{\widetilde{v}_1 + \widetilde{t} + \epsilon\widetilde{t}(v_0 - \kappa_2 - \widetilde{\vartheta}_1)}{\widetilde{u}_1\widetilde{v}_1 + \epsilon\widetilde{t}u_0v_0} - v_0,\\ \widetilde{b}_{t1} &= u_0v_0\frac{v_0 - \widetilde{\vartheta}_1 - \kappa_2 - \frac{u_0v_0}{\tilde u_1\tilde v_1}(\widetilde{t} + \widetilde{\vartheta}_1)}{\widetilde{u}_1\widetilde{v}_1 + \epsilon\widetilde{t}u_0v_0}.\end{aligned}$$

When $\epsilon \neq 0$ the irregular singular point at the origin is non-resonant and the local description of the Stokes phenomenon is the same as in the precedent section with the six Stokes matrices S_{ij} (A.6). But for $\epsilon = 0$ the singularity becomes resonant and the description changes.

For $|\epsilon t|$ small, there is a formal transformation

$$\widetilde{y} = \widehat{T}(\xi,\epsilon)\left(\begin{smallmatrix}\tilde y'\\ \tilde y''\end{smallmatrix}\right), \quad (\widetilde{y}', \widetilde{y}'') \in \mathbb{C}\times\mathbb{C}^2, \tag{A.15}$$

written as a formal power series in ξ with coefficients analytic in ϵ, that splits the system in two diagonal blocks, one corresponding to the eigenvalue 0, other corresponding to the other eigenvalues $\{1+\epsilon\widetilde{t}, 1\}$ (cf. [2]):

$$\xi^2 \frac{\mathrm{d}\widetilde{y}'}{\mathrm{d}\xi} = \xi\vartheta_0\, \widetilde{y}', \tag{A.16}$$

$$\xi^2 \frac{\mathrm{d}\widetilde{y}''}{\mathrm{d}\xi} = \left[\left(\begin{smallmatrix} 1+\epsilon\tilde{t} & 1 \\ 0 & 1 \end{smallmatrix}\right) + \xi\widetilde{B}''(\epsilon) + O(\xi^2)\right] \widetilde{y}'', \tag{A.17}$$

where $\widetilde{B}'' = \left(\begin{smallmatrix} \tilde{b}_{tt} & \tilde{b}_{t1} \\ \tilde{b}_{1t} & \tilde{b}_{11} \end{smallmatrix}\right)$ is the submatrix of $\widetilde{B} = \big(b_{ij}\big)$. This formal transformation $\widehat{T}(\xi,\epsilon)$ is Borel summable in ξ all directions except of the coalescing singular directions $\pm\mathbb{R}^+$ and $\pm(1{+}\epsilon\widetilde{t})\mathbb{R}^+$. Therefore it possesses Borel sums $T_\alpha(\xi,\epsilon)$ on four sectors: two large sectors which persist to the limit $\epsilon\to 0$, and two small ones that disappear whenever $\epsilon\widetilde{t}\in\mathbb{R}$. Only the Borel sums on the large sectors will be considered.

Confluence of eigenvalues in the subsystem (A.17)

The phenomenon of confluence of eigenvalues in 2×2 parametric systems at an irregular singular point of Poincaré rank 1 was studied previously by the author [34]. This paragraph applies some of the results to the system (A.17).

The matrix of the right side of the system has its eigenvalues equal to

$$\lambda^{(0)} + \xi\lambda^{(1)} \pm \sqrt{\alpha^{(0)} + \xi\alpha^{(1)}} \pmod{\xi^2},$$

where

$$\lambda^{(0)} = 1 + \frac{\epsilon\widetilde{t}}{2}, \qquad \lambda^{(1)} = \frac{\widetilde{b}_{tt} + \widetilde{b}_{11}}{2} = \frac{\widetilde{\vartheta}_1}{2},$$

$$\alpha^{(0)} = \left(\frac{\epsilon\widetilde{t}}{2}\right)^2, \qquad \alpha^{(1)} = \widetilde{b}_{1t} + \frac{\epsilon\widetilde{t}(\widetilde{b}_{tt} - \widetilde{b}_{11})}{2} = \epsilon\widetilde{t}\frac{\vartheta_t - \vartheta_1}{2} = \widetilde{t} - \epsilon\widetilde{t}\frac{\widetilde{\vartheta}_1}{2},$$

constitute the formal invariants of the system. In [34], it has been shown that (A.17) possess a fundamental matrix solutions of the form

$$\widetilde{Y}''_{\bullet} = R''_{\bullet}(\xi,t,\epsilon)\cdot \mathrm{e}^{-\frac{\lambda^{(0)}}{\xi}}\xi^{\lambda^{(1)}} \begin{pmatrix} \left(\alpha^{(0)}{+}\xi\alpha^{(1)}\right)^{-\frac14} & 0 \\ 0 & \left(\alpha^{(0)}{+}\xi\alpha^{(1)}\right)^{\frac14} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\Theta} & 0 \\ 0 & \mathrm{e}^{-\Theta} \end{pmatrix},$$

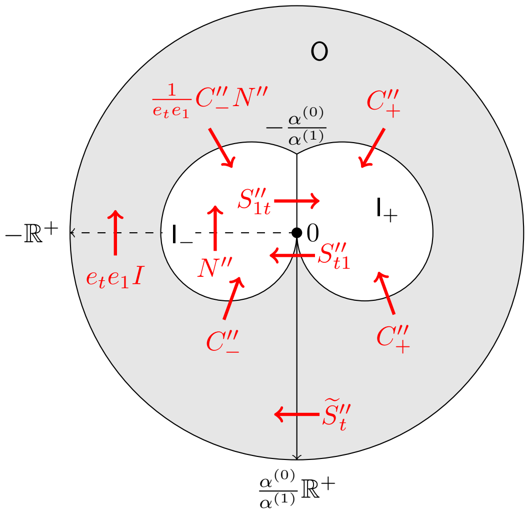


Figure A.3. The inner domains $\mathsf{I}_\pm$ (white) and the outer domain O (grey), and the connection matrices of the system (A.17). Note that for $\epsilon\widetilde{t} \neq$ the point $-\frac{\alpha^{(0)}}{\alpha^{(1)}} \neq 0$ is not a singularity of (A.17) thus solutions are regular there.

where

$$\Theta(\xi,\epsilon,\widetilde{t}) = \int_\infty^\xi \frac{\sqrt{\alpha^{(0)}+\zeta\alpha^{(1)}}}{\zeta^2}\mathrm{d}\zeta$$
$$= \begin{cases} -\frac{\sqrt{\alpha^{(0)}+\xi\alpha^{(1)}}}{\xi} - \frac{\alpha^{(1)}}{2\sqrt{\alpha^{(0)}}}\log\frac{\sqrt{\alpha^{(0)}+\xi\alpha^{(1)}}+\sqrt{\alpha^{(0)}}}{\sqrt{\alpha^{(0)}+\xi\alpha^{(1)}}-\sqrt{\alpha^{(0)}}}, & \epsilon\neq 0,\\ -\frac{2\alpha^{(0)}}{\xi}, & \epsilon = 0,\end{cases}$$

and $R''_\bullet$, $\bullet = \mathsf{I}_\pm, \mathsf{O}$, are invertible analytic transformations defined on certain domains in the ξ-space. These domains are delimited by the so called Stokes curves (in the sense of exact WKB analysis [31]): the separatrix curves of the foliation by real-time trajectories of the vector field

$$\mathrm{e}^{\mathrm{i}\omega}\frac{\xi^2}{2\sqrt{\alpha^{(0)}+\xi\alpha^{(1)}}}\frac{\partial}{\partial\xi}, \qquad \text{with some} \quad \omega\in\left]-\frac{\pi}{2}+\eta,\frac{\pi}{2}-\eta\right[,\ \eta>0,$$

emanating either from ∞ or from the "turning point" at $\xi = -\frac{\alpha^{(0)}}{\alpha^{(1)}}$, if $\epsilon\neq 0$. There are two kinds of such sectorial domains (see Figure A.3), whose shape in the coordinate $\frac{\xi}{\alpha^{(1)}}$ depends only on a parameter $\mu = \frac{\alpha^{(0)}}{(\alpha^{(1)})^2} = \left(\frac{\epsilon}{2-\epsilon\widetilde{\vartheta}_1}\right)^2$:

- A pair of *inner domains* $\mathsf{I}_\pm$ for $\epsilon \neq 0$: these are sectors at 0 of radius proportionate to $\mu \sim \frac{\epsilon^2}{4}$, separated one from another by the singular directions $\pm\epsilon t\mathbb{R}^+$. They disappear at the limit. The connection matrices between $\widetilde{Y}''_{\mathsf{I}_+}$ and $\widetilde{Y}''_{\mathsf{I}_-}$ are given by $S''_{1t} = \begin{pmatrix} 1 & 0 \\ s_{1t} & 1 \end{pmatrix}$, $S''_{t1} = \begin{pmatrix} 1 & s_{t1} \\ 0 & 1 \end{pmatrix}$, the submatrices of the Stokes matrices S_{1t}, S_{t1} (A.6), and needs to take into account also the formal monodromy $N'' = \begin{pmatrix} e_t^2 & 0 \\ 0 & e_1^2 \end{pmatrix}$, the submatrix of $\overline{N}$ (A.7). See Figure A.3.
- An *outer domain* O covering a complement of $\mathsf{I}_+ \cup \mathsf{I}_-$ in a disc of a fixed radius with a cut in the direction $\frac{\alpha(0)}{\alpha^{(1)}}\mathbb{R}^+ \sim \epsilon^2 t\mathbb{R}^+$. We are mainly interested in the limit when $\epsilon \to 0$ along the sequences $\frac{1}{\epsilon} \in \frac{1}{\epsilon_0} \pm 2\mathbb{N}$, hence the cut can be assumed to be in the direction $t\mathbb{R}^+$. The connection matrix on this cut is $\widetilde{S}''_t = \begin{pmatrix} X_0 & -i \\ -i & 0 \end{pmatrix}$ with X_0 (A.11) the trace of monodromy of the subsystem (A.17) around 0. See Figure A.3.
- The connection matrices between the outer and the inner solution bases can be expressed as

$$C''_+ = \begin{pmatrix} 1 & \frac{1}{s_{1t}} \\ 0 & -i\frac{e_t}{e_1 s_{1t}} \end{pmatrix}, \qquad C''_- = \begin{pmatrix} 0 & \frac{e_t}{e_1 s_{1t}} \\ i & -i\frac{e_t^2}{e_1^2 s_{1t}} \end{pmatrix}.$$

Returning now to the full system (A.14), one must intersect the domains $\mathsf{I}_\pm, \mathsf{O}$ with the sectors of the Borel summability of the transformation $\widehat{T}$ (A.15). The full picture is therefore that of Figure A.4. There are four inner Stokes matrices S_{1t}, S_{t1} and $S_{10}S_{t0}$, $S_{01}S_{0t}$ (A.6) between the canonical solutions on the inner domains, and three outer Stokes matrices $\widetilde{S}_0$, $\widetilde{S}_t$, $\widetilde{S}_1$ between the canonical solutions on the outer domains of the form:

(A.18)
$$\widetilde{S}_0 = \begin{pmatrix} 1 & \tilde{s}_{0t} & \tilde{s}_{01} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{S}_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & X_0 & -i \\ 0 & -i & 0 \end{pmatrix}, \quad \widetilde{S}_1 = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{s}_{t0} & 1 & 0 \\ \tilde{s}_{10} & 0 & 1 \end{pmatrix}, \quad \widetilde{N} = \begin{pmatrix} e_0^2 & 0 & 0 \\ 0 & e_t e_1 & 0 \\ 0 & 0 & e_t e_1 \end{pmatrix}.$$

The connection matrices between the canonical bases on the inner and outer domains are provided by:

$$C_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{s_{1t}} \\ 0 & 0 & \frac{-i\, e_t}{e_1 s_{1t}} \end{pmatrix}, \quad C_- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{e_t}{e_1 s_{1t}} \\ 0 & i & \frac{-i\, e_t^2}{e_1^2 s_{1t}} \end{pmatrix}. \tag{A.19}$$

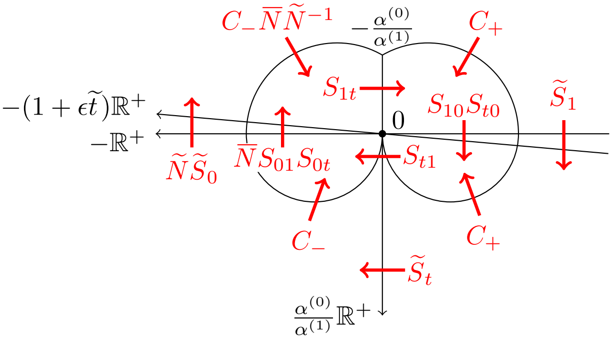


Figure A.4. The connection matrices between different canonical solution bases of the confluent system (A.14) with $\overline{N}$ (A.7), S_{ij} (A.6), $\widetilde{S}_i$, $\widetilde{N}$ (A.18) and $C_\pm$ (A.19).

Lemma A.2. — *The coefficients of the outer Stokes matrices are equal to*

$$\begin{aligned} \widetilde{s}_{0t} &= s_{0t} + s_{01}s_{1t}, & \widetilde{s}_{01} &= -i\frac{e_1}{e_t}s_{0t}, \\ \widetilde{s}_{t0} &= s_{t0} + \frac{s_{10}}{s_{1t}}, & \widetilde{s}_{10} &= -i\frac{e_t}{e_1}\frac{s_{10}}{s_{1t}}. \end{aligned} \tag{A.20}$$

Proof. — We have

$$\begin{aligned} \widetilde{S}_0 &= \widetilde{N}^{-1}C_-\overline{N}S_{01}S_{0t}\overline{N}^{-1}\widetilde{N}(C_-)^{-1}, \\ \widetilde{S}_1 &= C_+S_{10}S_{t0}(C_+)^{-1}, \end{aligned}$$

see Figure A.4. □

Remark A.3. — The inner Stokes matrices S_{ij} are determined only up to conjugation by diagonal matrices $\begin{pmatrix} d_0 & 0 & 0 \\ 0 & d_t & 0 \\ 0 & 0 & d_1 \end{pmatrix}$. This corresponds through (A.20) to conjugation of the outer Stokes matrices $\widetilde{S}_i$ by $\begin{pmatrix} d_0 & 0 & 0 \\ 0 & d_t & 0 \\ 0 & 0 & d_t \end{pmatrix}$.

The monodromy matrix of an outer solution around the origin is given by

$$\widetilde{M}_\infty^{-1} = \widetilde{S}_1\widetilde{S}_t\widetilde{N}\widetilde{S}_0 = \begin{pmatrix} e_0^2 & e_0^2\tilde{s}_{0t} & e_0^2\tilde{s}_{01} \\ e_0^2\tilde{s}_{t0} & e_te_1X_0+e_0^2\tilde{s}_{t0}\tilde{s}_{0t} & -ie_te_1+e_0^2\tilde{s}_{t0}\tilde{s}_{01} \\ e_0^2\tilde{s}_{10} & -ie_te_1+e_0^2\tilde{s}_{10}\tilde{s}_{0t} & e_0^2\tilde{s}_{10}\tilde{s}_{01} \end{pmatrix}, \tag{A.21}$$

and we know that its eigenvalues are 1, $\mathrm{e}^{-2\pi\mathrm{i}\kappa_1} = \frac{e_0e_te_1}{e_\infty}$ and $\mathrm{e}^{-2\pi\mathrm{i}\kappa_2} = e_0e_te_1e_\infty$.

The new variables $\widetilde{X}$ (5.7) are defined by

$$\widetilde{X}_1 = X_0, \qquad \widetilde{X}_0 = e_0\widetilde{s}_{10}\widetilde{s}_{01} + e_0, \qquad \widetilde{X}_\infty = ie_0\widetilde{s}_{t0}\widetilde{s}_{01} + \frac{e_te_1}{e_0}. \tag{A.22}$$

They are invariant with respect to the conjugation of Remark A.3.

Expressing the coefficients of the linear and the quadratic term of the characteristic polynomial of $\widetilde{M}_\infty^{-1}$ (A.21) gives

$$e_te_1\widetilde{X}_1 + e_0\widetilde{X}_0 + e_0^2\widetilde{s}_{0t}\widetilde{s}_{t0} = 1 + \frac{e_0e_te_1}{e_\infty} + e_0e_te_1e_\infty := E,$$

$$e_0e_te_1\widetilde{X}_1\widetilde{X}_0+e_0e_te_1\widetilde{X}_\infty+ie_0^2e_te_1\widetilde{s}_{0t}\widetilde{s}_{10} = \frac{e_0e_te_1}{e_\infty}+e_0e_te_1e_\infty+e_0^2e_t^2e_1^2 := E'.$$

Inserting these two identities into the identity

$$\begin{aligned} 0 &= e_0^2\cdot\widetilde{s}_{0t}\widetilde{s}_{10}\cdot\widetilde{s}_{t0}\widetilde{s}_{01} - e_0^2\cdot\widetilde{s}_{0t}\widetilde{s}_{t0}\cdot\widetilde{s}_{10}\widetilde{s}_{01} \\ &= -i\widetilde{s}_{0t}\widetilde{s}_{10}(e_0\widetilde{X}_\infty + e_te_1) - \widetilde{s}_{0t}\widetilde{s}_{t0}(e_0\widetilde{X}_0 - e_0^2) = 0, \end{aligned}$$

gives the Fricke relation (5.8)

$$0 = \widetilde{X}_1\widetilde{X}_0\widetilde{X}_\infty + \widetilde{X}_0^2 + \widetilde{X}_\infty^2 - \widetilde{\theta}_1\widetilde{X}_1 - \widetilde{\theta}_0\widetilde{X}_0 - \widetilde{\theta}_\infty\widetilde{X}_\infty + \widetilde{\theta}_t,$$

with

$$\widetilde{\theta}_1 = e_te_1, \qquad \widetilde{\theta}_0 = \frac{e_te_1}{e_0} + \frac{E'}{e_0e_te_1} = a_\infty + e_te_1a_0,$$

$$\widetilde{\theta}_t = E + \frac{E'}{e_0^2} = 1 + e_te_1a_0a_\infty + e_t^2e_1^2, \qquad \widetilde{\theta}_\infty = e_0 + \frac{E}{e_0} = a_0 + e_te_1a_\infty.$$

The formulas of change of variables (5.13), (5.12) of Theorem 5.2 between (X,θ) and $(\widetilde{X},\widetilde{\theta})$ are obtained from (A.22), (A.20) and (A.11). As remarked on page 181, the singular line L_0 (5.14) corresponds to $s_{1t} = 0$, i.e. to the triviality of the Stokes matrix S_{1t} (A.6).

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