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POSITIVE SUPERSOLUTIONS OF NON-AUTONOMOUS QUASILINEAR ELLIPTIC EQUATIONS WITH MIXED REACTION

by Asadollah AGHAJANI & Vicenţiu D. RĂDULESCU (*)

Abstract. — We provide a simple method for obtaining new Liouville-type theorems for positive supersolutions of the elliptic problem $-\Delta_p u + b(x)|\nabla u|^{\frac{pq}{q+1}} = c(x)u^q$ in Ω, where Ω is an exterior domain in $\mathbb{R}^N$ with $N > p > 1$ and $q > p-1$. In the case $q = p-1$, we mainly deal with potentials of the type $b(x) = |x|^a$, $c(x) = \lambda|x|^\sigma$, where $\lambda > 0$ and $a, \sigma \in \mathbb{R}$. We show that positive supersolutions do not exist in some ranges of the parameters p, q, a, σ, which turn out to be optimal. When $q = p-1$, we consider the above problem with general weights $b(x) > 0$, $c(x) > 0$ and we assume that $c(x) - \frac{b^p(x)}{p^p} > 0$ for large $|x|$, but we also allow the case $\lim_{|x|\to\infty}[c(x) - \frac{b^p(x)}{p^p}] = 0$. The weights b and c are allowed to be unbounded. We prove that if this equation has a positive supersolution, then the potentials must satisfy a related differential inequality not depending on the supersolution. We also establish sufficient conditions for the nonexistence of positive supersolutions in relationship with the values of $\tau := \limsup_{|x|\to\infty} |x|b(x) \leq \infty$. A key ingredient in the proofs is a generalized Hardy-type inequality associated to the p-Laplace operator.

Résumé. — Nous proposons une méthode simple pour obtenir de nouveaux théorèmes du type Liouville pour les supersolutions positives du problème elliptique $-\Delta_p u + b(x)|\nabla u|^{\frac{pq}{q+1}} = c(x)u^q$ dans Ω, où Ω est un domaine extérieur dans $\mathbb{R}^N$ avec $N > p > 1$ et $q > p-1$. Dans le cas $q = p-1$, on traite principalement des potentiels du type $b(x) = |x|^a$, $c(x) = \lambda|x|^\sigma$, où $\lambda > 0$ et $a, \sigma \in \mathbb{R}$. Nous montrons que les supersolutions positives n'existent pas dans certaines gammes de paramètres p, q, a, σ, qui s'avèrent optimales. Si $q = p-1$, on considère le problème ci-dessus avec des poids généraux $b(x) > 0$, $c(x) > 0$ et on suppose que $c(x) - \frac{b^p(x)}{p^p} > 0$ si $|x|$ est assez large, mais on admet aussi le cas $\lim_{|x|\to\infty}[c(x) - \frac{b^p(x)}{p^p}] = 0$. Les potentiels b et c sont autorisés à être non bornés. Nous prouvons que si cette équation a une supersolution positive, alors les potentiels doivent satisfaire une certaine inégalité

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différentielle ne dépendant pas de la supersolution. Nous établissons également des conditions suffisantes pour la non-existence de supersolutions positives en relation avec les valeurs de $\tau := \limsup_{|x|\to\infty} |x|b(x) \leq \infty$. Un ingrédient clé des preuves est une inégalité généralisée de type Hardy associée à l'opérateur p-Laplace.

1. Introduction

Consider the quasilinear elliptic equation

$$-\Delta_p u + b(x)|\nabla u|^{\frac{pq}{q+1}} = c(x)u^q \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an exterior domain, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p \leq N$, and $q > p-1$. We are concerned with the existence or non-existence of solutions $u \in C^2(\Omega, \mathbb{R}^+)$, but without assuming neither any hypothesis on the asymptotic behavior of the supersolutions near infinity, nor that they are bounded or radial.

In the case $q > p-1$, we shall be mainly concerned with potentials of the type $b(x) = |x|^a$, $c(x) = \lambda|x|^\sigma$, hence we study the problem

$$-\Delta_p u + |x|^a|\nabla u|^{\frac{pq}{q+1}} = \lambda|x|^\sigma u^q \quad \text{in } \Omega, \tag{P_λ}$$

where a, σ are real numbers and λ is a positive parameter.

In the case $q = p-1$, we consider Eq. (1.1) with general weights $b(x) > 0$, $c(x) > 0$, hence we study the problem

$$-\Delta_p u + b(x)|\nabla u|^{p-1} = c(x)u^{p-1} \text{ in } \Omega. \tag{Q}$$

Before we summarize our results, we give a brief history of the problem. Problem (1.1) in the semilinear autonomous case without convection corresponding to $p = 2$, $b(x) \equiv 0$, $c(x) \equiv 1$ and $\Omega = \mathbb{R}^N$ does not admit any positive solution provided that $q < (N+2)/(N-2)$ $(N > 3)$, see Gidas and Spruck [30]. It was later proved by Gidas and Spruck [31] that even positive supersolutions of this equation cannot exist with the more restrictive assumption $q \leq N/(N-2)$ (see Quittner and Souplet [36, Theorem 8.4] for a simple proof of this assertion; this restriction on the exponent is optimal). In the autonomous quasilinear case corresponding to $b(x) \equiv c(x) \equiv 1$, problem (1.1) reduces to the following elliptic equation with convection

$$-\Delta_p u + |\nabla u|^s = \lambda u^q \quad \text{in } \Omega. \tag{1.2}$$

The semilinear case

$$-\Delta u + |\nabla u|^s = \lambda u^q \text{ in } \Omega, \tag{1.3}$$

where $s > 1$, has been the subject of many works, where a basic observation is that the two terms λu^q and $|\nabla u|^s$ are in competition; see Serrin and

Zou [38]. However, Serrin and Zou [38] were interested in the existence and nonexistence of radial solutions in $\mathbb{R}^N$. The general case corresponding to problem (1.2) has been much less considered in the literature, see for example [26, 27]). In particular, Galakhov [27] studied the quasilinear elliptic equation

$$-\Delta_p u = u^q - |\nabla u|^s \text{ in } \mathbb{R}^N,$$

and considered the existence or nonexistence of positive classical solutions with radial symmetry such that $u(x) \to 0$ as $|x| \to \infty$. If $s > 1$, then under the scaling transformation $u_\mu := \mu^{\frac{p-s}{s-1}} u(\mu x)$ for $\mu > 0$ (see [15, 16] for the case $p = 2$), the equation (1.2) becomes

$$-\Delta_p u_\mu + |\nabla u_\mu|^s = \mu^{\frac{s(p+q-1)-pq}{s-1}} \lambda u_\mu^q.$$

Therefore, if $s = \frac{pq}{p+q-1}$, then problem (1.2) reduces to

$$-\Delta_p u + |\nabla u|^s = u^q.$$

If $s > \frac{pq}{p+q+1}$ then the limit equation obtained as $\mu \to 0$ is the Riccati equation

$$-\Delta_p u_\mu + |\nabla u_\mu|^s = 0,$$

where the exponent s is dominant. The other scaling transformation $v_\mu := \mu^{\frac{p}{q-1}} u(\mu x)$ for $\mu > 0$, transforms equation (1.2) into

$$-\Delta_p v_\mu + \mu^{\frac{pq-s(p+q-1)}{q-1}} |\nabla v_\mu|^s = \lambda v_\mu^q.$$

Moreover, when $s < \frac{pq}{p+q-1}$, the limit equation when $\mu \to 0$ is the generalized Lane–Emden equation

$$-\Delta_p v = \lambda v^q,$$

where the exponent q is dominant.

In most of the above mentioned works (studied in the semilinear case $p = 2$) the analysis deals with the case $s = \frac{2q}{q+1}$. In the critical case $s = \frac{2q}{q+1}$, then the value of λ plays a fundamental role, with a delicate interaction with the exponents p, q. Bidaut-Véron, Garcia-Huidobro and Véron [15] studied local and global properties of positive solutions of problem (1.3) in $\Omega = \Omega \setminus \{0\}$ where Ω is an open subset of R^N, and existence or non-existence of entire positive solutions in $\mathbb{R}^N$. A related analysis was developed in [16] in the framework of radial solutions to problem (1.3) in $\Omega = \mathbb{R}^N$ or $\mathbb{R}^N \setminus \{0\}$ for $s, q > 1$ and λ is a real parameter. Alarcon, García-Melián and Quaas in [2] considered problem (1.3) with $s > 1$ and obtained Liouville-type theorems for positive classical supersolutions in exterior domains without making any assumption about boundedness or asymptotic behavior of the supersolutions near infinity.

Consider problem (P_λ) and observe that a supersolution for some value of $\overline{\lambda}$ is also a supersolution for every value $\lambda < \overline{\lambda}$. Thus, we can define

$$\lambda^* = \lambda^*(p,q,\sigma,a) := \inf\left\{\lambda > 0;\ \text{there are no positive supersolutions for problem } (P_\lambda)\right\}.$$

If $p = 2$, $a = \sigma = 0$ and $\Omega = R^N \setminus B_{R_0}$, it is proved in [2] that

$$\lambda^* = \left(\frac{2}{N - q(N-2)}\right)^q \frac{q^q}{(q+1)^{q+1}} \tag{1.4}$$

and that for any $\lambda \in (0, \lambda^*)$ there are positive supersolutions which do not blow up at infinity. The method relies in analyzing the function $m(R) = \min_{|x|=R} u(x)$. By means of a device introduced in [11] but slightly different and more involved approach (due to the nonhomogeneity of the left-hand side in (P_λ) that makes hard to obtain a Hadamard-type property), Alarcon, García-Melián and Quaas in [2] obtained some suitable upper and lower bounds for $m(R)$. These estimates are useful to get nonexistence results.

In this paper we first consider problem (P_λ) in an exterior domain Ω for all $p > 1$ and determine the exact value of λ^* for which the problem does not admit any positive supersolution for $\lambda > \lambda^*$ while for $\lambda \leq \lambda^*$ there always exist a positive supersolution for (P_λ).

In the case $q = p - 1$, the problem of existence or nonexistence of supersolution for problem (Q) has been studied in the literature, mostly in the case $p = 2$, namely

$$-\Delta u + b(x)|\nabla u| = c(x)u \text{ in } \Omega, \tag{1.5}$$

where Ω is an exterior domain $\mathbb{R}^N$, $N > 3$.

As a consequence of the study of eigenvalue problems in $\mathbb{R}^N$, Berestycki, Hamel and Nadirashvili [11] proved that if the vector field b and the function c are continuous, then the problem

$$-\Delta u + b(x)\cdot\nabla u > c(x)u \text{ in } \mathbb{R}^N \tag{1.6}$$

does not admit any positive solution provided that b and c are bounded and satisfy

$$\liminf_{|x|\to\infty} \left(c(x) - \frac{|b(x)|^2}{4}\right) > 0. \tag{1.7}$$

Berestycki, Hamel and Rossi [12] extended the results of [11] to non-autonomous elliptic equations of the type

$$-\operatorname{tr}(A(x)D^2u) - b(x)\cdot Du > c(x)u \text{ in } \mathbb{R}^N. \tag{1.8}$$

They showed that if $A(x)$ is a smooth uniformly elliptic matrix field, $b : \mathbb{R}^N \to \mathbb{R}^N$ and $c : \mathbb{R}^N \to \mathbb{R}$ are bounded and smooth and

$$\liminf_{|x|\to\infty}(4\lambda(x)c(x) - |b(x)|^2) > 0,$$

where $\lambda(x) := \min_{|\xi|=1} A(x)\xi\cdot\xi$, then the only nonnegative function u satisfying (1.8) in the classical sense is $u \equiv 0$.

Rossi [37] generalized the above nonexistence results to the framework of fully nonlinear elliptic operators, showing that the above assumption can be relaxed (by applying the result to a subdomain); in particular, the case

$$\liminf_{|x|\to\infty}\left(c(x) - \frac{|b(x)|^2}{4}\right) < 0$$

is allowed. However, all the above results require that

$$\limsup_{|x|\to\infty}\left(c(x) - \frac{|b(x)|^2}{4}\right) > 0. \tag{1.9}$$

Aghajani and Cowan [1] refined some of these results by using a nontrivial application of a generalized Hardy inequality obtained by Cowan [24]. In particular, the case

$$\lim_{|x|\to\infty}\left(c(x) - \frac{|b(x)|^2}{4}\right) = 0$$

is allowed. We also refer to [20] and [21], where the authors proved some Hadamard and Liouville type properties for nonnegative viscosity supersolutions of fully non linear uniformly elliptic partial differential inequalities in the whole space or in an exterior domain; see [2, 3, 4, 5, 6, 19, 20, 21, 22, 23, 25]. Alarcon, García-Melián and Quaas [4] considered positive classical supersolutions of problem (1.5) for more general unbounded weights b and c. They proved that if $b, c \in C(\mathbb{R}^N \setminus B_{R_0})$ verify (1.7) and satisfy a further restriction related to the fundamental solutions of the homogeneous problem (see Theorems 1.1 and 1.2 in [4]) then there are no classical positive supersolutions to (1.5) which do (or do not) blow up at infinity. Their proof of nonexistence results depends on properties of the function $m(R) = \inf_{|x|=R} u(x)$ and the fundamental solution of the equation $-\Delta v + \bar{b}(|x|)|\nabla v| = 0$ in $\mathbb{R}^N \setminus B_{R_0}$, where $\bar{b}(r) := \sup_{|x|=r} b(x)$.

In this paper, we consider the more general problem (Q) in exterior domains with continuous functions b, c satisfying $c(x) - \frac{b^p(x)}{p^p} > 0$ for $|x|$ large. Using a different approach, by employing a generalized version of the Hardy inequality, we obtain new Liouville-type results, that seems to be sharp in some sense. In particular, we may allow the degenerate case

$$\lim_{|x|\to\infty} c(x) \frac{b^p(x)}{p^p} = 0;$$

and without the boundedness assumption on the weights b and c.

2. Statement of main results

We now describe the main results of this paper and consider both problems (P) and (Q).

2.1. Results for problem (P)

Here we formulate our first main result for problem (P) with $q > p - 1$ in exterior domains.

Define

$$Q := \frac{(N + \ )(p - 1)}{N - p}. \tag{2.1}$$

First note that every solution of the inequality

$$\ _p u \geq |x| \ u^q; \tag{2.2}$$

is also a supersolution of (P). When $q > Q$, by the results of Bidaut-Véron and Pohozaev [17] or Bidaut-Véron [13], there exists a nontrivial solution of problem (2.2) in exterior domains (see Theorem 4.1 in Appendix), hence in this case we simply get $\ = 0$. The following theorem gives a rather complete description of existence and nonexistence of positive supersolutions of problem (P) for $p - 1 < q < Q$.

Theorem 2.1. Consider problem (P) in an exterior domain $\ = \mathbb{R}^N \setminus B_{R_0}$. Assume that $1 < p \leqslant N$ and $p - 1 < q < Q$. Then the following properties hold.

(i) If $\ = a(q+1)$ then

$$= \left(\frac{Q - p + 1}{Q - q}\right)^q \frac{q^q}{(p - 1)^q (q+1)^{q+1}}.$$

Moreover, for all 2 (0;] there exist positive supersolutions of problem (P) and every positive supersolution satis es

$$\limsup_{jxj!1} \frac{jr\ uj^p}{u^{q+1}} > T_1 \quad \frac{Q \quad p+1}{q \quad p+1}^{q+1}$$

(2.3) and

$$\liminf_{jxj!1} \frac{jr\ uj^p}{u^{q+1}} \; 6 \; T_2 \quad \frac{Q \quad p+1}{q \quad p+1}^{q+1} ;$$

where $0 < T_1 < T_2$ are two roots of the following equation in $[0; 1\ )$

$$\frac{q \quad p+1}{Q \quad p+1}^{q} + \frac{(p \quad 1)(Q \quad q)}{q \quad p+1} T \quad T^{\frac{q}{q+1}} = 0:$$

(ii) If $> a\ (q+1)$, then $= 0$.

(iii) If $< a\ (q+1)$, then for any $< \ 1$ a positive supersolution of (P) can always be constructed for suitable large R_0.

Remark. Note that by the above theorem, in the special case when a = = 0 and if is an exterior domain, we obtain

$$= \frac{p(p \quad 1)}{N(p \quad 1) \quad q(N \quad p)}^{q} \frac{q^q}{(q+1)^{q+1}}; \text{ when } p \quad 1 < q < \frac{N(p \quad 1)}{N \quad p}:$$

In particular, if $p = 2$ we have

$$= \frac{2}{N \quad q(N \quad 2)}^{q} \frac{q^q}{(q+1)^{q+1}}; \text{ when } 1 < q < \frac{N}{N \quad 2};$$

which coincides with (1.4) obtained in [2].

2.2. Results for problem (Q)

As we have already pointed out, when $q = p \quad 1$ we consider problem (Q) with general weights $b(x) > 0;\ c(x) > 0$.

Theorem 2.2. Let $1 < p < N$, $b; c\,2\ C(R^N \, nB_{R_0})$ with $c(x) \quad \frac{b(x)^p}{p^p} >$ 0 for large jxj. Then problem (Q) does not have any positive supersolutions if

(2.4) $$\liminf_{jxj!1} jxj^p \quad \frac{c(x)}{(k+1)^{p \quad 1}} \quad \frac{b(x)^p}{p^p k^{p \quad 1}} > \left(\frac{N \quad p}{p}\right)^p; \text{ for some } k > 0;$$

or

$$(2.5)\qquad \inf_{\frac{R}{2}<|x|<2R}\left(1-\frac{b(x)}{p\sqrt[p]{c(x)}}\right)^p \inf_{R<|x|<\theta R}|x|^p c(x) > \frac{2(\theta+4)^p\ln 2}{\ln\theta}+\gamma^p;\ \text{where}\ \gamma:=\frac{N-p}{p};$$

for some $\theta>1$ and $R>2R_0$. In particular, if $\lambda:=\limsup_{|x|\to\infty}|x|b(x)<\infty$ then problem (Q) does not have any positive supersolution if

$$(2.6)\qquad \liminf_{|x|\to\infty}|x|^p c(x) > \left(\frac{N-p+\lambda}{p}\right)^p.$$

Moreover, if

$$(2.7)\qquad \limsup_{|x|\to\infty}|x|^p c(x) < \left(\frac{N-p+\lambda}{p}\right)^p;$$

then problem (Q) has a positive supersolution in $\mathbb{R}^N\setminus B_{R_1}$ for R_1 sufficiently large. Furthermore, when $N=p$ we have the same nonexistence result if

$$(2.8)\qquad \liminf_{|x|\to\infty}|x|^\alpha\left(\frac{c(x)}{(k+1)^{p-1}}-\frac{b(x)^p}{p^pk^{p-1}}\right)>0;$$

for some $\alpha<p$ and $k>0$.

Corollary 2.3. Assume that $N>p$, $p>1$. Consider the equation

$$(2.9)\qquad \Delta_p u+b|x|^a|\nabla u|^{p-1}=c|x|^{-\sigma}u^{p-1}\ \text{in}\ \Omega;$$

where Ω is an exterior domain, $b,c\in\mathbb{R}$, $c>0$. Then the above equation does not admit any positive supersolution in the following four sets of parameters

$$[\sigma>p;\ \sigma>ap;\ c>0];\quad [\sigma>p;\ \sigma=ap;\ c>b^p/p^p]$$

$$[\sigma=p;\ \sigma>ap;\ c>(N-p)^p/p^p]$$

or

$$[\sigma=p;\ \sigma=ap;\ c>(N-p+b)^p/p^p]:$$

Corollary 2.4. Assume $N>p>1$. Consider the problem

$$(2.10)\qquad \Delta_p u+be^{a|x|}|\nabla u|^{p-1}=ce^{\sigma|x|}u^{p-1}\ \text{in}\ \Omega;$$

where Ω is an exterior domain, $b,c\in\mathbb{R}$, $\sigma,a>0$. Then the above equation does not admit any positive supersolution when $\sigma>ap$ and $c>0$ or $\sigma=ap$ and $c>b^p/p^p$.

Here we give an example of a Problem (Q) where $c(x)-\frac{b(x)^p}{p^p}>0$ but $\liminf_{|x|\to\infty}\left[c(x)-\frac{b(x)^p}{p^p}\right]=0$.

Example 2.5. Consider the problem

$$-\Delta_p u + |x|^a|\nabla u|^{p-1} = \left(\frac{|x|^{pa}}{p^p} + \frac{1}{|x|^\sigma}\right) u^{p-1}, \quad x \in \mathbb{R}^N \setminus B_1 \tag{2.11}$$

where $\sigma, a > 0$ and $(p-1)a + \sigma < 1$. Notice that we have

$$\lim_{|x|\to\infty}\left(c(x) - \frac{b(x)^p}{p^p}\right) = \lim_{|x|\to\infty}\frac{1}{|x|^\sigma} = 0.$$

However, for a fixed $\lambda > 1$, it is not hard to see that

$$\inf_{\frac{R}{2}<|x|<2\lambda R}\left(1 - \frac{b(x)}{p\sqrt[p]{c(x)}}\right)^p \inf_{R<|x|<\lambda R}|x|^p c(x) = O(R^{p(1-\sigma-(p-1)a)}).$$

Hence, condition (2.5) of Theorem 2.2 holds and the problem does not admit any positive supersolution.

The proof of Theorem 2.2 relies on Lemma 3.1 and the following results, which have their own interest. The first one is a generalized form of the Hardy inequality.

Proposition 2.6. Let $E > 0$ be a smooth function on a domain Ω of $\mathbb{R}^N$, then we have

$$\int_\Omega \frac{-\Delta_p E}{E^{p-1}}|\phi|^p \leqslant \int_\Omega |\nabla\phi|^p, \tag{2.12}$$

for every $\phi \in C_c^1(\Omega)$.

Remark. Note that when $1 < p < N$ if we take $E(x) = (|x| + \varepsilon)^{\frac{p-N}{p}}$ in (2.12) and then we set $\varepsilon \to 0$ we get the Hardy inequality

$$\left(\frac{N-p}{p}\right)^p \int_\Omega \frac{|\phi|^p}{|x|^p} \leqslant \int_\Omega |\nabla\phi|^p, \quad \phi \in C_c^1(\Omega).$$

Also note that the above general Hardy-type inequality (2.12) for the case $p = 2$ and with the restriction that $-\Delta E > 0$ is well-known; see Cowan [24] or the book of Ghoussoub Moradifam [28] and Ghoussoub Robert [29].

Proposition 2.7. Assume problem(Q) has a positive supersolution in an arbitrary domain Ω in $\mathbb{R}^N$. Then the functions $b(x)$ and $c(x)$ must satisfy the inequality

$$\sqrt[p]{\int_\Omega c(x)|\phi|^p} \leqslant \sqrt[p]{\int_\Omega |\nabla\phi|^p} + \sqrt[p]{\int_\Omega \frac{b(x)^p}{p^p}|\phi|^p}, \tag{2.13}$$

for every $\phi \in C_c^1(\Omega)$, which also implies that

$$\inf_{\operatorname{supp}\phi}\left(1 - \frac{b(x)}{p\sqrt[p]{c(x)}}\right)^p \int_\Omega c(x)|\phi|^p \leqslant \int_\Omega |\nabla\phi|^p, \tag{2.14}$$

and

$$(2.15)\qquad \int_\Omega c(x)^{1-p}\frac{b(x)^p}{p^p}|\varphi|^p \leqslant \alpha^{1-p}\int_\Omega |\nabla\varphi|^p;$$

$$\text{with } \alpha+\beta=1;\ \alpha;\beta>0.$$

In order to apply the results of Proposition 2.7 to the problem (Q) we need the following property.

Theorem 2.8. — Let $1<p<N$. If a function $c(x)>0$ satisfies

$$(2.16)\qquad \int_\Omega c(x)|\varphi|^p \leqslant \int_\Omega |\nabla\varphi|^p;$$

for every $0\leqslant\varphi\in C_c^1(\Omega)$, where Ω is an exterior domain in $\mathbb{R}^N$, then we have

$$(2.17)\qquad \inf_{R<|x|<\lambda R}(|x|^p c(x)) \leqslant \frac{2(\lambda+4)^p\ln 2}{\ln\lambda}+\gamma^p;\ \text{with}\ \gamma:=\frac{N-p}{p};$$

for all $\lambda>1$ and R sufficiently large. In particular, we have

$$(2.18)\qquad \liminf_{x\to\infty}|x|^p c(x) \leqslant \left(\frac{N-p}{p}\right)^p.$$

As a consequence, the inequality problem

$$(2.19)\qquad -\Delta_p w \geqslant c(x)w^{p-1};\ \text{in }\Omega;$$

does not admit any positive solution if

$$(2.20)\qquad \liminf_{x\to\infty}|x|^p c(x) > \left(\frac{N-p}{p}\right)^p.$$

Corollary 2.9. — Let E be a positive smooth function in an exterior domain Ω of $\mathbb{R}^N$ $(N\geqslant 3)$ with $-\Delta_p E\geqslant 0$. Then

$$(2.21)\qquad \liminf_{|x|\to\infty}|x|^p\frac{-\Delta_p E}{E^{p-1}} \leqslant \left(\frac{N-p}{p}\right)^p.$$

Example 2.10. — As an application of the above property, consider the equation

$$(2.22)\qquad -\Delta_p u = |x|^a u^q\ \text{in }\Omega;$$

where $a\in\mathbb{R}$, $q>p-1$ and Ω is an exterior domain in $\mathbb{R}^N$ $(N\geqslant 3)$. If u is a positive classical supersolution of this equation then we get, by Corollary 2.9,

$$\liminf_{|x|\to\infty}|x|^p\frac{-\Delta_p u}{u^{p-1}} = \liminf_{|x|\to\infty}|x|^{a+p}u^{q-p+1} \leqslant \left(\frac{N-p}{p}\right)^p.$$

However, we know that a super p-harmonic function u satisfies $u(x) \geq C|x|^{\frac{p-N}{p-1}}$ in Ω (see [7] or [38]), hence we must have

$$a + p + (q - p + 1)\frac{p-N}{p-1} \leq 0$$

Thus, the above equation does not admit any positive supersolution if $q < \frac{(N+a)(p-1)}{N-p}$, which is a known result. Also, by a similar argument from Corollary 2.9 we see that the equation

$$-\Delta_p u = \frac{\lambda u}{|x|^p} \quad (\lambda > 0)$$

does not admit any positive supersolution in an exterior domain if $\lambda > (N-p)^p/p^p$.

3. Proof of main results

The following simple observation is crucial for the proof of the main results.

Lemma 3.1. *Let u be a classical positive supersolution of problem (1.1) in an arbitrary domain Ω. Then for any $k > 0$ the function w_k defined by $u(x) = \frac{w_k^{k+1}}{k+1}$ is a solution of*

$$-\Delta_p w_k \geq \lambda_k(x) w_k^{k(q-p+1)+q} \quad \text{in } \Omega; \tag{3.1}$$

where

$$\lambda_k(x) := \frac{c(x)}{(k+1)^q} - \frac{\left(\frac{b(x)}{q+1}\right)^{q+1}\left(\frac{q}{p-1}\right)^q}{k^q}. \tag{3.2}$$

Proof. We set $w = w_k$, then substituting $u = \frac{w^{k+1}}{k+1}$ in (1.1), using the formulas $\nabla u = w^k \nabla w$ and

$$-\Delta_p u = -k(p-1)w^{k(p-1)-1}|\nabla w|^p + w^{k(p-1)}(-\Delta_p w);$$

we deduce that w satisfies

$$-\Delta_p w \geq \frac{c(x)}{(k+1)^q} w^{k(q-p+1)+q} + k(p-1)\frac{|\nabla w|^p}{w} - b(x) w^{\frac{k(q-p+1)}{q+1}} |\nabla w|^{\frac{pq}{q+1}}$$

$$= w^{k(q-p+1)+q}\left[\frac{c(x)}{(k+1)^q} + k(p-1)\frac{|\nabla w|^p}{w^{k(q-p+1)+q+1}} - b(x)\frac{|\nabla w|^{\frac{pq}{q+1}}}{w^{\frac{qk(q-p+1)}{q+1}+q}}\right]:$$

Now taking

$$T(x) := \frac{|\nabla w|^p}{w^{k(q-p+1)+q+1}};$$

we see that the equation above turns into

$$(3.3)\qquad \begin{aligned} -\Delta_p w &\geq w^{k(q-p+1)+q}\left[\frac{c(x)}{(k+1)^q} + k(p-1)T(x) - b(x)T(x)^{\frac{q}{q+1}}\right] \\ &:= w^{k(q-p+1)+q} H_k(x). \end{aligned}$$

Set $f(T) := A + BT - CT^{\frac{q}{q+1}}$ with $A, B \geq 0$, $C > 0$. Then

$$\min_{T \geq 0} f(T) = A - \left(\frac{C}{q+1}\right)^{q+1}\left(\frac{q}{B}\right)^{q};$$

hence

$$H_k(x) \geq \left[\frac{c(x)}{(k+1)^q} - \left(\frac{b(x)}{q+1}\right)^{q+1}\left(\frac{q}{k(p-1)}\right)^{q}\right] = \Lambda_k(x);$$

hence w satisfies (3.1).

3.1. Proof of Theorem 2.1

Let u be a supersolution of (P) in an exterior domain Ω, and define the function w_k on Ω via $u = \frac{w_k^{k+1}}{k+1}$, $k > 0$. Then by Proposition 2.6 (with $b(x) = |x|^a$ and $c(x) = \mu|x|^\sigma$) we see that w_k satisfies

$$(3.4)\qquad -\Delta_p w_k \geq \Lambda_k(x) w_k^{k(q-p+1)+q}, \text{ in } \Omega;$$

where

$$\Lambda_k(x) := \frac{\mu|x|^\sigma}{(k+1)^q} - \left(\frac{|x|^a}{q+1}\right)^{q+1}\left(\frac{q}{k(p-1)}\right)^{q}$$

$$(3.5)\qquad = |x|^\sigma\left[\frac{\mu}{(k+1)^q} - \left(\frac{1}{q+1}\right)^{q+1}\left(\frac{q}{k(p-1)}\right)^{q}|x|^{a(q+1)-\sigma}\right].$$

Now assume that we are in the case (i), that is, $\sigma = a(q+1)$. Then we have

$$(3.6)\qquad \Lambda_k(x) = |x|^\sigma\left(\frac{\mu}{(k+1)^q} - \left(\frac{1}{q+1}\right)^{q+1}\left(\frac{q}{k(p-1)}\right)^{q}\right).$$

Thus, by Theorem 4.1 in Appendix, if $k(q-p+1)+q \leq Q$, or equivalently $k \leq \frac{Q-q}{q-p+1}$, this is impossible if the term inside the parentheses in (3.6) is positive. This happens if

$$(3.7)\qquad \mu > \left(1+\frac{1}{k}\right)^{q}\frac{q^q}{(q+1)^{q+1}(p-1)^q}.$$

Taking $k := \frac{Q \quad q}{q \quad p+1}$ we see that the right-hand side of (3.7) becomes . Thus, we proved that for all $>$, problem (P) does not admit a positive supersolution. Note that in the above case we used the fact that if $q > p \quad 1$, then $k(q \quad p+1) + q > q > p \quad 1$ for any $k > 0$, hence we need $q < Q$ in order to have $k > 0$.

Now we construct a positive supersolution for any $2\ (0;\quad]$, provided that $p \quad 1 < q < Q$. Set

$$u(x) = A|x| \quad ; \quad > 0:$$

Then

$$_p u + |x|^a |r u|^s > \quad |x| \, u^q \text{ in}$$

is equivalent to

$$A^{p \ 1 \ q} |x|^{t_1} + A^{s \ q \ s} |x|^{t_2} > \quad \text{in} \quad ; \tag{3.8}$$

where

$$:= \ ^p \ \frac{N \quad p}{} \quad p+1 \quad ;$$

$$t_1 := \quad (q \quad p+1) \quad p \quad ; \text{ and } t_2 := a \quad + \quad (q \quad s) \quad s:$$

We now observe that if $s = \frac{pq}{q+1}$ then taking $:= \frac{p+}{q \quad p+1}$ (which is positive by the assumption $p \quad 1 < q < Q$) we get $t_1 = 0$, $t_2 = a \quad \frac{}{q+1}$. If we set $T := A^{p \ 1 \ q \ p}$ then relation (3.8) becomes

$$\frac{N \quad p}{} \quad p+1 \quad T + T^{\frac{q}{q+1}} |x|^{a \quad \frac{}{q+1}} > \quad \text{in} \quad : \tag{3.9}$$

Now assume that $a = \frac{}{q+1}$. Then relation (3.9) is equivalent to

$$f(T) := \frac{N \quad p}{} \quad p+1 \quad T + T^{\frac{q}{q+1}} > \quad :$$

By the assumption $p \quad 1 < q < Q$ we have

$$\frac{N \quad p}{} \quad p+1 = \frac{(Q \quad q)}{Q \quad p+1} < 0:$$

Then we see that the function $f(T)$ attains its maximum at the point

$$T_0 = \left(\frac{q}{(q+1)(p \quad 1 \quad \frac{N \quad p}{})} \right)^{q+1} = (q \quad)^{\frac{q+1}{q}}$$

with

$$f(T_0) = \frac{q^q}{(q+1)^{q+1}} \left(\frac{1}{p \quad 1 \quad \frac{N \quad p}{}} \right)^q = \quad :$$

Hence, choosing A so that $A^{p \ 1 \ q \ p} = T_0$, we see then that the function $u(x) = A|x|$ is a supersolution of (P) for 6 .

To prove the estimates (2.3), let $\lambda \in (0, \lambda^*]$ and u_λ be a positive supersolution of (P_λ). Then, by (3.3), the function $w = ((k+1)u)^{\frac{1}{k+1}}$ with $k := \frac{Q-q}{q-p+1}$ satisfies

$$\Delta_p w \geq |x|^{\sigma} w^Q H(T(x)); \tag{3.10}$$

where

$$T(x) := \frac{|\nabla w|^p}{w^{Q+1}}$$

and

$$H(T) = \left(\frac{q-p+1}{Q-p+1}\right)^q + \frac{(p-1)(Q-q)}{q-p+1}T - T^{\frac{q}{q+1}}.$$

Note that when $\lambda \leq \lambda^*$, then H has exactly two roots in $[0, \infty)$, say $T_1 < T_2$, and $H(T) \leq 0$ in $[T_1, T_2]$ and positive for other T's. By Theorem 4.1 in Appendix, if $H(T(x))$ is positive at infinity, bounded away from zero, then (3.10) has not any positive solution, a contradiction. It follows that $H(T(x_n)) \leq 0$ for a sequence $x_n \to \infty$ as $n \to \infty$, hence $T(x_n) \in [T_1, T_2]$ for every n. We deduce that

$$\limsup_{x\to\infty} T(x) \geq T_1 \text{ and } \liminf_{x\to\infty} T(x) \leq T_2,$$

and writing this out in terms of u we get the estimate (2.3).

Now consider the case (ii), that is, $\sigma > a(q+1)$ and fixed $\lambda > 0$. If u is a positive supersolition of (P_λ), we define again the function w_k on Ω via $u = \frac{w_k^{k+1}}{k+1}$ $(k > 0)$. With similar arguments we deduce that w_k satisfies

$$\Delta_p w_k \geq \Phi_k(x) w_k^{k(q-p+1)+q} \text{ in } \Omega;$$

where $\Phi_k(x)$ is given in (3.5). However, since $a(q+1) - \sigma < 0$ then for any $k > 0$ we have $\Phi_k(x) \geq \frac{\lambda}{2(k+1)^q}|x|^{\sigma}$ in $\mathbb{R}^N \setminus \{B_{R_1}\}$ for $R_1 > R_0$ sufficiently large depending on k. But by Theorem 4.1 in Appendix this is impossible if we choose k small so that $k(q-p+1)+q \leq Q$.

Finally, assume we are in the case (iii), that is, $\sigma < a(q+1)$. To construct a positive supersolution we set again $u = A|x|^{-\alpha}$ with $\alpha = \frac{p+\sigma}{q-p-1}$, as in the first part above. Recall that u is a supersolution if it satisfies (3.9). Since in this case we have $a > \frac{\sigma}{q+1}$, then for a given $\lambda < \infty$ the inequality (3.9) holds for a suitable large R_0 that depends on λ.

3.2. Proof of Theorem 2.2

First we prove the result under the condition (2.4). Taking $u = \frac{w^{k+1}}{k+1}$ with $k > 0$ and applying Proposition 2.6, we see that w satisfies

$$\Delta_p w \geq \Phi_k(x) w^{p-1} \text{ in } \Omega; \tag{3.11}$$

where

$$(3.12)\qquad {}_k(x) := \frac{c(x)}{(k+1)^{p-1}} \quad \frac{b(x)}{p}^{p} \frac{1}{k^{p-1}} :$$

If $N > p$ then by Theorem 2.8 we see that problem (1.1) does not admit any positive supersolution if for some $k > 0$ we have

$$\liminf_{|x|\to\infty} |x|^p \quad \frac{c(x)}{(k+1)^{p-1}} \quad \frac{b(x)}{p}^{p} \frac{1}{k^{p-1}} > \frac{N-p}{p}^{p} :$$

Note that we can also get the above result by using Proposition 2.7. Indeed, by (2.15), the potentials b and c must satisfy

$$(3.13)\qquad \int c(x) \quad {}^{1-p} \frac{b(x)^p}{p^p} \quad | \quad |^p$$
$$\le \quad {}^{1-p} \int |r \quad |^p \text{ with } \quad + \quad = 1; \quad ; \quad > 0:$$

Dividing both sides of (3.13) by ${}^{1-p}$ and then setting $:= \frac{1}{k+1}$ for $k > 0$ we get

$$\int {}_k(x)| \quad |^p \le \int |r \quad |^p;$$

where ${}_k$ is given in (3.12), hence the result follows by Theorem 2.8.

Now we prove the result under condition (2.5). Note that by Proposition 2.7, if problem (Q) has a solution $u > 0$ then we have for all $\in C^1_c( )$

$$(3.14)\qquad \inf_{\text{supp}} \quad 1 - \frac{b(x)}{p^{p}\sqrt[p]{c(x)}}^{p} \int c(x)| \quad |^p \le \int |r \quad |^p:$$

Now consider the same test function as in the proof of Theorem 2.8. Since we have supp $= [\frac{R}{2} \le |x| \le 2R]$ then

(3.15)
$$\inf_{\frac{R}{2}<|x|<2R} \quad 1 - \frac{b(x)}{p^{p}\sqrt[p]{c(x)}}^{p} \inf_{R<|x|<R} |x|^p c(x) \le \frac{2(+4)^p \ln 2}{\ln} + {}^p;$$

hence the problem does not admit any positive solution if the above inequality does not hold.

To prove the second part, let $:= \limsup_{|x|\to\infty} |x| b(x) < \infty$ and set

$$:= \liminf_{|x|\to\infty} |x|^p c(x):$$

If $= +\infty$, then (2.6) obviously holds. Thus, we assume $< \infty$. Then we see that (2.4) holds if

$$\frac{}{(k+1)^{p-1}} \quad \frac{{}^p}{p^p k^{p-1}} > \frac{N-p}{p}^{p}$$

for some $k > 0$. Taking $k = \left(p^{\frac{1}{p}} \right)^{1}$ we see that the above inequality holds if

$$> \left(\frac{N \quad p +}{p} \right)^{p} :$$

Now set $:= \limsup_{|x|\to\infty} |x|^{p} c(x)$ and assuming that $< (N \quad p +)^{p} = p^{p}$, we construct a positive supersolution. Choose $_1 <$ and $_1 <$ so that

$$_1 \leq \left(\frac{N \quad p + \ _1}{p} \right)^{p} : \tag{3.16}$$

Now we look for some $m > 0$ so that the function $u(x) = |x|^{m}$ is a supersolution of Problem (Q) in $\mathbb{R}^N \setminus B_{R_1}$ for R_1 sufficiently large. For this purpose we need

$$\begin{aligned} &_p u + b(x)|\nabla u|^{p \ 1} \quad c(x)u^{p \ 1} \\ &= |x|^{p \ m(p \ 1)} \ m^{p \ 1}(\ m(p \ 1) + N \quad p) + m^{p \ 1}|x|b(x) \ | \ x|^{p}c(x) \\ &> 0 \end{aligned}$$

for $|x|$ sufficiently large. By the definition of $;$ we have

$$\begin{aligned} m^{p \ 1}(\ m(p \ 1) + N \quad p) &+ m^{p \ 1}|x|b(x) \ | \ x|^{p}c(x) \\ &> m^{p \ 1}(\ m(p \ 1) + N \quad p) + m^{p \ 1} \ _1 \quad _1 \\ &= \ (p \ 1)m^{p} + (N \quad p + \ _1)m^{p \ 1} \quad _1 \end{aligned}$$

for $|x|$ sufficiently large. Note the last term is nonnegative for some $m > 0$ if and only if (3.16) holds. The proof is now complete.

Proof of Corollaries 2.3 and 2.4. Consider the equation (2.9). By Theorem 2.2, we see that this equation does not have any positive supersolution if

$$\liminf_{|x|\to\infty} |x|^{p+} \left(\frac{c}{(k+1)^{p \ 1}} \quad \frac{b^{p}|x|^{ap}}{p^{p}k^{p \ 1}} \right) > \left(\frac{N \quad p}{p} \right)^{p} \quad \text{for some } k > 0.$$

We first assume that $> \ p$. If $ap <$ and $c > 0$, then (2.10) obviously holds. If $= ap$ then we need

$$\frac{c}{(k+1)^{p \ 1}} \quad \frac{b^{p}}{p^{p}k^{p \ 1}} > 0$$

or, equivalently,

$$\frac{p^{p}c}{b^{p}} > \left(1 + \frac{1}{k} \right)^{p \ 1}$$

for some $k > 0$. This is the case if $cp^{p} > b^{p}$.

Now let $\sigma = \beta p$. If $ap < \sigma$, then (2.10) holds if

$$\frac{c}{(k+1)^{p-1}} > \left(\frac{N-p}{p}\right)^p;$$

for some $k > 0$, and this is the case if $cp^p > (N-p)^p$. Note that in this situation the result can be also followed by the last part of Theorem 2.2 as in this case we must have $a < -1$ hence $\ell := \lim_{x\to\infty} |x|(b|x|^a) = 0$.

If $\sigma = ap$ (that is, $a = -1$) then we have $\ell := \lim_{x\to\infty} |x|(b|x|^a) = b$ and the result follows by the last part of Theorem 2.2.

The proof of Corollary 2.4 can be done in a similar way by the fact that, by Theorem 2.2, Equation (2.10) does not admit any positive supersolution if

$$\liminf_{|x|\to\infty} |x|^p e^{-\sigma|x|}\left(\frac{c}{(k+1)^{p-1}} - \frac{b^p e^{(ap-\sigma)|x|}}{p^p k^{p-1}}\right) > \left(\frac{N-p}{p}\right)^p \quad \text{for some } k > 0.$$

Also, when $N = p$ then for some $\sigma < p$ we can rewrite (3.11) as

$$-\Delta_p w > |x|^{-\sigma}\phi_k(x)\,|x|^{\sigma-p} w^{p-1}.$$

If $C := \liminf_{|x|\to\infty} |x|^{-\sigma}\phi_k(x) > 0$ we see that w is a solution of the problem

$$-\Delta_p w > \frac{C}{2}|x|^{\sigma-p} w^{p-1}$$

in an exterior domain. Since $\sigma - p > -p$ we can then use part (ii) of Theorem 4.1 in Appendix. The proof is now complete.

Proof of Proposition 2.6. First let $F : \Omega \to \mathbb{R}$ be a smooth function and $\phi \in C_c^1(\Omega)$. For simplicity take $\phi > 0$. Using the divergence theorem we have

$$\int \Delta_p F\,\phi^p = -p\int (|\nabla F|^{p-2}\nabla F\,\phi^{p-1})\cdot\nabla\phi \le p\int (|\nabla F|\phi)^{p-1}|\nabla\phi|.$$

By the Young inequality we have

$$(|\nabla F|\phi)^{p-1}|\nabla\phi| \le \frac{p-1}{p}(|\nabla F|\phi)^p + \frac{|\nabla\phi|^p}{p}$$

and use this in the first inequality we get

$$\int|\nabla\phi|^p \ge \int(\Delta_p F - (p-1)|\nabla F|^p)|\phi|^p.$$

Now setting $F = \ln E$ $(E > 0)$ we obtain

$$\int|\nabla\phi|^p \ge \int\frac{\Delta_p E}{E^{p-1}}|\phi|^p.$$

This completes the proof.

Proof of Proposition 2.7. — Let $u > 0$ be a positive supersolution of problem (Q), hence

$$-\Delta_p u + b(x)|\nabla u|^{p-1} \geq c(x)u^{p-1} \text{ in } \Omega. \tag{3.17}$$

For $t > 1$ set $v := (tu)^{\frac{1}{t}}$ or equivalently, $u = \frac{v^t}{t}$. Then from (3.17) and using the formula

$$\Delta_p u = (t-1)(p-1)v^{(t-1)(p-1)-1}|\nabla v|^p + v^{(t-1)(p-1)}\Delta_p v,$$

we obtain

$$-(p-1)(t-1)v^{(p-1)(t-1)-1}|\nabla v|^p - v^{(p-1)(t-1)}\Delta_p v + b(x)v^{p(t-1)}|\nabla v|^p \geq c(x)\frac{v^{t(p-1)}}{t^{p-1}}.$$

Dividing both sides of the above inequality by $v^{t(p-1)}$, we get

$$-\frac{\Delta_p v}{v^{p-1}} \geq \frac{c(x)}{t^{p-1}} + (p-1)(t-1)\frac{|\nabla v|^p}{v^p} - b(x)\frac{|\nabla v|^{p-1}}{v^{p-1}}.$$

Since for all $A, B > 0$ we have

$$\min_{t>0}(AT^p - BT^{p-1}) = -\frac{(p-1)^{p-1}}{p^p}\frac{B^p}{A^{p-1}}$$

we obtain

$$-\frac{\Delta_p v}{v^{p-1}} \geq \frac{c(x)}{t^{p-1}} - \frac{|b(x)|^p}{p^p(t-1)^{p-1}}. \tag{3.18}$$

Multiplying both sides of (3.18) by $|\phi|^p$, integrating over Ω and applying the Hardy-type inequality (2.12) in Proposition 2.6 we obtain

$$\int_\Omega c(x)|\phi|^p \leq t^{p-1}\int_\Omega |\nabla\phi|^p + \frac{t^{p-1}}{p^p(t-1)^{p-1}}\int_\Omega b(x)^p|\phi|^p.$$

Now we set

$$t = 1 + \left(\frac{\int_\Omega b(x)^p|\phi|^p}{\int_\Omega |\nabla\phi|^p}\right)^{\frac{1}{p}}$$

to get

$$\int_\Omega c(x)|\phi|^p \leq \left(\sqrt[p]{\int_\Omega |\nabla\phi|^p} + \sqrt[p]{\int_\Omega b(x)^p|\phi|^p}\right)^p$$

or

$$\sqrt[p]{\int_\Omega c(x)|\phi|^p} \leq \sqrt[p]{\int_\Omega |\nabla\phi|^p} + \sqrt[p]{\int_\Omega \frac{b(x)^p}{p^p}|\phi|^p},$$

which is the desired result.

To see (2.14) it suffices to write

$$\int_\Omega \frac{b(x)^p}{p^p}|\varphi|^p = \int_\Omega \frac{b(x)^p}{p^p c(x)} c(x)|\varphi|^p \le \left(\sup_{\operatorname{supp}\varphi} \frac{b(x)^p}{p^p c(x)}\right)\int_\Omega c(x)|\varphi|^p.$$

To prove (2.15), first note that by the concavity of the function $\sqrt[p]{t}$ $(p > 1)$ on $(0, \infty)$ we have

$$\alpha\sqrt[p]{t_1}+\beta\sqrt[p]{t_2}\le\sqrt[p]{\alpha t_1+\beta t_2},\quad \alpha, \beta, t_1, t_2 > 0, \text{ with } \alpha+\beta=1,$$

and then by changing $t_1 \to \frac{t_1}{\alpha^p}$ and $t_2 \to \frac{t_2}{\beta^p}$ in the inequality above we get

$$\sqrt[p]{t_1}+\sqrt[p]{t_2}\le\sqrt[p]{\alpha^{1-p}t_1+\beta^{1-p}t_2}. \tag{3.19}$$

Now by using (3.19) in (2.13) we obtain

$$\int_\Omega c(x)|\varphi|^p \le \alpha^{1-p}\int_\Omega|\nabla\varphi|^p+\beta^{1-p}\int_\Omega\frac{b(x)^p}{p^p}|\varphi|^p$$

or

$$\int_\Omega\left(c(x)-\beta^{1-p}\frac{b(x)^p}{p^p}\right)|\varphi|^p\le\alpha^{1-p}\int_\Omega|\nabla\varphi|^p,$$

which is the desired result.

3.3. Proof of Theorem 2.8

Assume $c(x)$ satisfies (2.16) and for simplicity take $\Omega = R^N \setminus B_{R_0}$. Let $\gamma > 1$, $R > 2R_0$ and take a smooth function ψ in Ω with $\psi = 0$ for $R_0 < |x| < \frac{R}{2}$ and $|x| > 2\gamma R$, $\psi = 1$ in $R < |x| < \gamma R$, $0 \le \psi \le 1$ and $|\nabla\psi| \le \frac{4}{R}$. Now we consider $\varphi := |x|^{-\beta}\psi$ as a test function in (2.16), where $\beta := \frac{N-p}{p}$. We write

$$\begin{aligned}\int_\Omega|\nabla\varphi|^p &= \int_{\frac{R}{2}<|x|<2\gamma R}|\nabla\varphi|^p\\ &= \int_{\frac{R}{2}<|x|<R}|\nabla\varphi|^p+\int_{R<|x|<\gamma R}|\nabla\varphi|^p+\int_{\gamma R<|x|<2\gamma R}|\nabla\varphi|^p\\ &:= I_1(R)+I_2(R)+I_3(R).\end{aligned}$$

Since $I_3(R) = I_1(2\gamma R)$ we get

$$\int_\Omega|\nabla\varphi|^p\le 2I_1(R)+I_2(R).$$

We have

$$\nabla\varphi = -\beta|x|^{-\beta-2}\psi x+|x|^{-\beta}\nabla\psi,$$

which yields

$$|\nabla\varphi|^2 = \beta^2|x|^{-2\beta-2}\psi^2 - 2\beta|x|^{-2\beta-2}\psi x\cdot\nabla\psi + |x|^{-2\beta}|\nabla\psi|^2.$$

Then by the assumptions on ψ and $\nabla\psi$ we have the estimate

$$\begin{aligned}|\nabla\varphi|^2 &\leq \beta^2|x|^{-N} + \frac{8\beta}{R}|x|^{1-N} + \frac{16}{R^2}|x|^{2-N}\\ &= |x|^{-2\beta-2}\left(\beta + \frac{4|x|}{R}\right)^2, \quad \frac{R}{2} < |x| < R.\end{aligned}$$

Therefore

$$\begin{aligned}|\nabla\varphi|^p &\leq |x|^{(-\beta-1)p}\left(\beta + \frac{4|x|}{R}\right)^p = |x|^{-N}\left(\beta + \frac{4|x|}{R}\right)^p,\\ &\leq (\beta+4)^p|x|^{-N}, \quad \frac{R}{2} < |x| < R.\end{aligned}$$

Also, by the definition of φ we have

$$|\nabla\varphi|^p = \beta^p|x|^{-N}, \quad R < |x| < \gamma R.$$

By the estimates above and using the fact that if $\alpha + N \neq 0$ we have

$$\int_{R<|x|<T} |x|^\alpha \,\mathrm{d}x = K_N \int_R^T r^{\alpha+N-1}\,\mathrm{d}r = K_N\frac{T^{\alpha+N} - R^{\alpha+N}}{\alpha+N},$$

and, if $\alpha + N = 0$ we have

$$\int_{R<|x|<T} |x|^\alpha \,\mathrm{d}x = K_N \int_R^T r^{\alpha+N-1}\,\mathrm{d}r = K_N \ln\frac{T}{R}$$

(we set $K_N = 1$ as it appears the same in both sides of the inequality), we compute

$$I_1(R) \leq (\beta+4)^p \ln 2, \quad I_2(R) = \beta^p \ln\gamma.$$

Hence, we proved that

$$\int_\Omega |\nabla\varphi|^p \leq 2(\beta+4)^p\ln 2 + \beta^p\ln\gamma. \tag{3.20}$$

Also note that we have

$$\begin{aligned}\int_\Omega c(x)|\varphi|^p &\geq \int_{R<|x|<\gamma R} c(x)|\varphi|^p\\ &= \int_{R<|x|<\gamma R} c(x)|x|^{-\beta p} = \int_{R<|x|<\gamma R}(|x|^p c(x))|x|^{-\beta p - p}\\ &= \int_{R<|x|<\gamma R}(|x|^p c(x))|x|^{-N} \geq \inf_{R<|x|<\gamma R}(|x|^p c(x))\ln\gamma\end{aligned}$$

Thus by the inequality above and (3.20) we obtain

$$\inf_{R<|x|<\gamma R}(|x|^p c(x)) \leqslant \frac{2(\beta+4)^p \ln 2}{\ln \gamma} + \beta^p, \tag{3.21}$$

which proves (2.17). Also by letting $\gamma \to \infty$ and then $R \to \infty$ in (3.21) we get

$$\liminf_{x\to\infty} |x|^p c(x) \leqslant \beta^p,$$

which proves (2.18).

Now if the inequality (2.19) has a positive solution u then

$$\int_\Omega \frac{-\Delta_p u}{u^{p-1}}|\varphi|^p \geqslant \int_\Omega c(x)|\varphi|^p.$$

Thus, by Proposition 2.6,

$$\int_\Omega |\nabla\varphi|^p \geqslant \int_\Omega c(x)|\varphi|^p.$$

Next, by Proposition 2.6, we see that $c(x)$ must satisfy (2.16), hence from the first part, there is no positive solution if

$$\liminf_{x\to\infty} |x|^p c(x) > \left(\frac{N-p}{p}\right)^p.$$

The proof is now complete.

4. Appendix

Consider the inequality

$$-\Delta_p w \geqslant |x|^\sigma w^Q, \quad x \in \Omega_e, \tag{4.1}$$

where Ω_e is an exterior domain. The proof of the following result can be found in [17].

Theorem 4.1. — *Assume that* $N > p > 1$ *and let* Q_σ *be as defined in* (2.1)*. Then the following properties hold.*

(i) *If* $Q > p-1$ *then problem* (4.1) *has only the solution* $w \equiv 0$ *if and only if* $Q \leqslant Q_\sigma$*.*

(ii) *Assume* $Q \leqslant p-1$*. If* $\sigma > -p$*, then problem* (4.1) *has only the solution* $w \equiv 0$*. This is also true in the case* $\sigma = -p$*, provided that* $Q = p-1$*.*

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