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Narcisse RANDRIANANTOANINA

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# TRIPLE OPERATOR VERSION OF THE GOLDEN-THOMPSON INEQUALITY FOR TRACES ON VON NEUMANN ALGEBRAS

by Narcisse RANDRIANANTOANINA

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**ABSTRACT.** — We provide a generalization of Lieb’s triple matrix extension of the Golden–Thompson inequality from matrix algebras to the setting of traces on finite von Neumann algebras. More precisely, assume that  $\mathcal{M}$  is a finite von Neumann algebra equipped with a tracial state  $\tau$ . If  $1 \leq p, q \leq \infty$  with  $1/p + 1/q = 1$ , it is shown that whenever  $a$ ,  $b$ , and  $c$  are self-adjoint  $\tau$ -measurable operators satisfying  $a \in \mathcal{M}$ ,  $e^b \in L_p(\mathcal{M}, \tau)$ , and  $e^c \in L_q(\mathcal{M}, \tau)$ , then the following inequality holds:

$$\tau(e^{a+b+c}) \leq \int_0^\infty \tau(e^{c/2}(e^{-a} + t\mathbf{1})^{-1}e^b(e^{-a} + t\mathbf{1})^{-1}e^{c/2}) dt$$

where  $\mathbf{1}$  denotes the identity in  $\mathcal{M}$ . We also present other results related to the Wigner–Yanase–Dyson–Lieb concavity in the context of general tracial state.

We use the above version of the Golden–Thompson inequality for three operators to prove an extension of the Prokhorov arcsinh inequality to noncommutative martingales in general noncommutative probability spaces.

**RÉSUMÉ.** — Nous prouvons une généralisation de l’extension de Lieb à trois matrices de l’inégalité de Golden–Thompson de l’algèbre des matrices à des traces associés à des algèbres de von Neumann finies. Plus précisément, supposons que  $\mathcal{M}$  est une algèbre de von Neumann finie munie d’un état tracé  $\tau$ . Soient  $1 \leq p, q \leq \infty$  tels que  $1/p + 1/q = 1$ . Alors pour tout  $a$ ,  $b$ , et  $c$  opérateurs auto-adjoints et  $\tau$ -mesurables satisfaisant  $a \in \mathcal{M}$ ,  $e^b \in L_p(\mathcal{M}, \tau)$ , et  $e^c \in L_q(\mathcal{M}, \tau)$ , on a l’inégalité suivante:

$$\tau(e^{a+b+c}) \leq \int_0^\infty \tau(e^{c/2}(e^{-a} + t\mathbf{1})^{-1}e^b(e^{-a} + t\mathbf{1})^{-1}e^{c/2}) dt$$

où  $\mathbf{1}$  denote l’identité de  $\mathcal{M}$ . Nous présentons également d’autres résultats liés à Wigner–Yanase–Dyson–Lieb concavité dans le contexte général d’état tracé.

Nous utilisons la version ci-dessus de l’inégalité de Golden–Thompson pour trois opérateurs pour démontrer une extension de inégalité arcsinh de Prokhorov aux martingales non commutatives dans des espaces de probabilité non commutatif.

## 1. Introduction

In 1965, Golden [9] and Thompson [45] discovered independently one of the most powerful trace inequalities that is nowadays referred to as *the Golden–Thompson inequality* (for short, GT-inequality). It states that for any two Hermitian matrices  $A$  and  $B$ , the following tracial inequality holds:

$$\mathrm{Tr}(e^{A+B}) \leq \mathrm{Tr}(e^A e^B).$$

Here  $\mathrm{Tr}(\cdot)$  denotes the usual trace on matrix algebras. The GT-inequality was originally motivated by considerations in statistical mechanics but since its inception, it has found applications in many fields of mathematics such as mathematical physics, random matrices ([8, 47]), quantum information theory ([26]), among others.

Two facts that motivate our consideration in this article are: first, the GT-inequality generalizes to tracial states on finite von Neumann algebras and second, it does not formally extend to three noncommuting Hermitian matrices in the sense that there are three Hermitian matrices  $A$ ,  $B$ , and  $C$  with

$$\mathrm{Tr}(e^{A+B+C}) \not\leq \mathrm{Tr}(e^A e^B e^C).$$

The version of GT-inequality to general traces on von Neumann algebras was due to Ruskai in [38] and counterexamples for the second fact already appeared in [45] (see also [3, p. 279]). More general form of the GT-inequality were also considered in [1, 24] in the context of general  $\sigma$ -finite von Neumann algebras. We refer to the survey [30] for more perspective on the GT-inequality.

A nontrivial extension of the GT-inequality involving three noncommuting Hermitian matrices was given by Lieb in the seminal article [25]. This version of GT-inequality for three matrices does not appeared to be as well-known as the original GT-inequality. In the spirit of Ruskai's result, it is a natural question to consider if Lieb's triple matrices version of the GT-inequality is valid for tracial states on arbitrary finite von Neumann algebras. The primary result of this note is a generalization of Lieb's result to traces of exponential of sum of three noncommuting self-adjoint measurable operators with respect to traces on finite von Neumann algebras. We refer to the preliminary section below for notation used in the statement of the following result:

**THEOREM A.** — *Let  $\mathcal{M}$  be a finite von Neumann algebra equipped with a tracial state  $\tau$  and  $1 \leq p, q \leq \infty$  with  $1/p + 1/q = 1$ . Assume that  $a$ ,  $b$ , and*

$c$  are self-adjoint  $\tau$ -measurable operators satisfying:  $a \in \mathcal{M}$ ,  $e^b \in L_p(\mathcal{M}, \tau)$ , and  $e^c \in L_q(\mathcal{M}, \tau)$ . The following inequality holds:

$$\tau(e^{a+b+c}) \leq \int_0^\infty \tau(e^{c/2}(e^{-a} + t\mathbf{1})^{-1}e^b(e^{-a} + t\mathbf{1})^{-1}e^{c/2}) dt.$$

One can verify that under the assumptions of the theorem, the integral on the right hand side is finite. If we consider the special case of matrix algebra with the usual trace, then Theorem A is precisely Lieb's result from [25, Theorem 7]. We also observe that if one considers only two self-adjoint  $\tau$ -measurable operators, that is, if one of the three operators is equal to zero or two of them commute, then Theorem A reduces to Ruskai's result [38, Theorem 4]. Thus, we may treat Theorem A as a common generalization of Lieb's result and Ruskai's result. We should mention here that extension of the GT-inequality for arbitrary number of Hermitian matrices was recently discovered in [44] but this is outside of the scope of our consideration.

Our approach follows the same pattern as in the argument used in [25] which is primarily based on the Wigner–Yanase–Dyson–Lieb concavity. Although the Wigner–Yanase–Dyson–Lieb concavity has been established for general  $\sigma$ -finite von Neumann algebras ([2]), considerable technical details are still required in implementing those ideas to the general setting of  $\tau$ -measurable operators. Most notably our use of ultrapowers of von Neumann algebras. In the process, we also establish two other convexity results from [25] in the more general context of finite von Neumann algebras which we believe are of interests on their own right. We refer to Theorem 3.11 and Theorem 3.13 for detailed formulations.

It is already known that the GT-inequality is quite useful in dealing with noncommutative martingale inequalities. We refer to the work of Junge and Zeng [20, 21] among others for applications of GT-inequalities for proving concentration results for martingales. One extra motivation in considering Theorem A was in our attempt to obtain the so-called Prokhorov inequality for noncommutative martingales. It turned out that the Golden–Thompson inequality with two operators is not sufficient for our argument. For more on history and development on noncommutative martingales, we refer to the articles [4, 14, 16, 18, 31, 34, 37]. Recall that the Prokhorov inequality first appeared in [33] as a tail bound of sums of independent random variables. Extension to martingales was obtained by Johnson, Schechtman, and Zinn [15] which was later refined by Hitczenko [13]. The first noncommutative Prokhorov inequality was proved in [20, Corollary 0.2] for the so-called successively independent mean-zero self-adjoint sequences. This is a variant of the various types of independences introduced in [19] for noncommuting

sequences. We use Theorem A to prove a version of the Prokhorov inequality for noncommutative self-adjoint martingales. We refer to the body of the paper for notation used for noncommutative martingales used below. It reads as follows:

**THEOREM B.** — *Let  $x = (x_n)_{n \geq 1}$  be a mean-zero self-adjoint noncommutative martingale. For every  $t > 0$  and  $n \geq 1$ , the following holds:*

$$\tau(\chi_{(t, \infty)}(x_n)) \leq \exp\left(-\frac{t}{2M} \operatorname{arcsinh}\left(\frac{tM}{2\|\sum_{k \geq 1} \mathcal{E}_{k-1}|dx_k|^2\|_\infty}\right)\right)$$

where  $M = \sup_{k \geq 1} \|dx_k\|_\infty$ .

Theorem B is a noncommutative analogue of the version proved by Hitzchenko in [13]. Using basic scalar inequality, Theorem B implies a noncommutative version of the Bernstein inequality. In fact, using this route, we obtain better constants compared to the version of the Bernstein inequality due to Junge and Zeng in [21, Theorem 1.1 (ii)].

The paper is organized as follows: in the next preliminary section, we review the basics of noncommutative spaces and present some necessary background on various ways of producing ultrapowers of von Neumann algebras that are relevant and essential for the presentation of the paper. Section 3 contains the detailed proof of Theorem A. We should point out that Theorem A is deduced from two convexity results that are fully detailed in the section. We also discuss in this section possible extensions to semifinite and  $\sigma$ -finite von Neumann algebras. Section 4 is dedicated to application of the GT-inequality for three operators to noncommutative martingale inequalities. Here, we provide a proof of Theorem B and discuss how it fits with previously known concentration type results for noncommutative martingales.

## 2. Preliminaries and notation

### 2.1. Noncommutative spaces

Throughout,  $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$  will denote a finite von Neumann algebra on some Hilbert space  $\mathcal{H}$  (here  $\mathcal{B}(\mathcal{H})$  is the algebra of all bounded operators on  $\mathcal{H}$  equipped with the usual operator norm). It is assumed that  $\mathcal{M}$  is equipped with a tracial state  $\tau$ . The identity on  $\mathcal{M}$  will be denoted by  $\mathbf{1}$ . A linear operator  $x : \mathcal{D}(x) \rightarrow \mathcal{H}$  with domain  $\mathcal{D}(x) \subseteq \mathcal{H}$ , is said to be *affiliated with  $\mathcal{M}$*  if  $xu \subset ux$  for all unitary  $u$  in the commutant  $\mathcal{M}'$

of  $\mathcal{M}$ . We recall that since  $\mathcal{M}$  is finite, every operator affiliated with  $\mathcal{M}$  is  $\tau$ -measurable in the sense of [28]. The collection of all  $\tau$ -measurable operators is denoted by  $\widetilde{\mathcal{M}}$ . It is well-known that  $\widetilde{\mathcal{M}}$  is a  $*$ -algebra. For  $\varepsilon, \delta > 0$ , we denote by  $V(\varepsilon, \delta)$  the set of all  $x \in \widetilde{\mathcal{M}}$  for which there exists an orthogonal projection  $p \in \mathcal{M}$  such that  $p(\mathcal{H}) \subseteq \mathcal{D}(x)$ ,  $\|xp\|_\infty \leq \varepsilon$ , and  $\tau(\mathbf{1} - p) \leq \delta$ . The collection of sets  $\{V(\varepsilon, \delta) : \varepsilon, \delta\}$  forms a base at 0 for a metrizable Hausdorff topology on  $\widetilde{\mathcal{M}}$  called the *measure topology*. Equipped with the measure topology,  $\widetilde{\mathcal{M}}$  becomes a complete topological  $*$ -algebra. We refer to [28] for these facts. If  $a$  is a self-adjoint operator in  $\widetilde{\mathcal{M}}$  and  $a = \int_{-\infty}^{\infty} s dE_s^{(a)}$  is its spectral decomposition, then for any Borel subset  $B \subseteq \mathbb{R}$ , we denote by  $\chi_B(a)$  the corresponding spectral projection  $\int_{-\infty}^{\infty} \chi_B(s) dE_s^{(a)}$ . For  $x \in \widetilde{\mathcal{M}}$ , we recall that the *generalized singular value function*  $\mu(x) = \mu(|x|)$  is defined by

$$\mu_t(x) := \inf\{s \geq 0 : \tau(\chi_{(s, \infty)}(|x|)) \leq t\}, \quad t \in (0, 1].$$

Note that  $t \mapsto \mu_t(x)$  is decreasing and since  $\tau(\mathbf{1}) = 1$ ,  $\mu_1(x) = 0$ . We refer to [7] for more details on relevant properties of singular values that we use. We recall here that a sequence  $\{x_n\}$  converges to zero for the measure topology if and only if  $\mu_t(x_n) \rightarrow 0$  for all  $t > 0$ . The following lemma is a consequence of a result of Tikhonov and will be repeatedly used in the sequel.

LEMMA 2.1 ([46]). — Assume that  $\{a_n\}_{n \geq 1} \subseteq \widetilde{\mathcal{M}}$  is a sequence of self-adjoint operators and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous. If  $a$  is a self-adjoint operator in  $\widetilde{\mathcal{M}}$  and  $a_n \rightarrow a$  for the measure topology then  $f(a_n) \rightarrow f(a)$  for the measure topology.

We denote by  $\widetilde{\mathcal{M}}_h$  the set  $\{x \in \widetilde{\mathcal{M}} : x = x^*\}$ . The real vector space  $\widetilde{\mathcal{M}}_h$  is a partially ordered vector space with the partial order defined by setting  $x \geq 0$  if and only if  $\langle x\xi, \xi \rangle \geq 0$  for all  $\xi \in \mathcal{D}(x)$ . The positive cone in  $\widetilde{\mathcal{M}}_h$  with respect to this order will be denoted by  $\widetilde{\mathcal{M}}_+$ .

It is important to note that the trace  $\tau$  extends to  $\widetilde{\mathcal{M}}_+$  as a non-negative extended real-valued functional which is positively homogeneous, additive, unitary invariant, and normal. This extension is given by:

$$\tau(x) = \int_0^1 \mu_t(x) dt, \quad x \in \widetilde{\mathcal{M}}_+.$$

This functional satisfies  $\tau(y^*y) = \tau(yy^*)$  for  $y \in \widetilde{\mathcal{M}}$  which we will refer below as *tracial property*.

For  $1 \leq p < \infty$ , we will denote by  $L_p(\mathcal{M}, \tau)$  (or simply  $L_p(\mathcal{M})$ ), the noncommutative  $L_p$ -space associated to the pair  $(\mathcal{M}, \tau)$ . We recall that

$L_p(\mathcal{M}, \tau)$  is defined as the set of all  $x \in \widetilde{\mathcal{M}}$  such that  $\|x\|_p = \tau(|x|^p)^{1/p} < \infty$ . The space  $L_p(\mathcal{M})$  equipped with the norm  $\|\cdot\|_p$  is a Banach space. As is customary,  $L_\infty(\mathcal{M}, \tau)$  is  $\mathcal{M}$  with the operator norm. We refer to the survey article [32] for basic properties and further references concerning noncommutative spaces.

Below, we use the notation  $\mathcal{M}_h$  (resp.  $\mathcal{M}_+$ ) for  $\mathcal{M} \cap \widetilde{\mathcal{M}}_h$  (resp.  $\mathcal{M} \cap \widetilde{\mathcal{M}}_+$ ). We will also use  $\mathcal{M}_{++}$  to denote the collection of all  $x \in \mathcal{M}_+$  for which its spectrum is a subset of  $(0, \infty)$ . That is,  $a \in \mathcal{M}_{++}$  if and only if  $a \in \mathcal{M}_+$  and it admits a bounded inverse.

We conclude this subsection by recording an inequality on distribution functions that we will use in the subsequent sections.

LEMMA 2.2. — *For every  $x, y \in \widetilde{\mathcal{M}}_h$ , we have*

$$\tau(\chi_{(t+s, \infty)}(x+y)) \leq \tau(\chi_{(t, \infty)}(x)) + \tau(\chi_{(s, \infty)}(y)), \quad t, s \in \mathbb{R}.$$

*Proof.* — The lemma was proved in [43, Lemma 16] for  $t, s > 0$ . The argument given in [43] actually applies to arbitrary real numbers. We reproduce it here for convenience.

Let  $t, s \in \mathbb{R}$ . Set  $a = (x - t\mathbf{1})_+$  and  $b = (y - s\mathbf{1})_+$ . Clearly,  $x - t\mathbf{1} \leq a$  and  $y - s\mathbf{1} \leq b$ . Thus,  $x + y \leq a + b + (t + s)\mathbf{1}$ . We have

$$\tau(\chi_{(t+s, \infty)}(x+y)) \leq \tau(\chi_{(t+s, \infty)}(a+b+(t+s)\mathbf{1})) = \tau(\text{supp}(a+b))$$

where  $\text{supp}(a+b)$  denotes the support projection of  $a+b$ . It follows from [43, Lemma 15] that

$$\tau(\chi_{(t+s, \infty)}(x+y)) \leq \tau(\text{supp}(a)) + \tau(\text{supp}(b)).$$

On the other hand,  $\tau(\text{supp}(a)) = \tau(\chi_{(0, \infty)}(a)) = \tau(\chi_{(0, \infty)}(x - t\mathbf{1})) = \tau(\chi_{(t, \infty)}(x))$ . Similarly,  $\tau(\text{supp}(b)) = \tau(\chi_{(s, \infty)}(y))$ . The lemma is verified.  $\square$

## 2.2. Ultrapowers of von Neumann algebras

In this subsection, we will review some basics of ultrapowers of Banach spaces with particular attention given to some variants that apply to some classes of von Neumann algebras.

Given a Banach space  $X$  and an index set  $I$ , we let

$$\ell_\infty(I; X) = \left\{ (x_i)_{i \in I} : x_i \in X \text{ for all } i \in I, \text{ and } \sup_{i \in I} \|x_i\|_X < \infty \right\}$$

equipped with its usual norm

$$\|(x_i)_{i \in I}\| = \sup_{i \in I} \|x_i\|_X.$$

For a given free ultrafilter  $\mathcal{U}$  over the index set  $I$ , the *ultrapower*  $X^I/\mathcal{U}$  (also denoted by  $X_{\mathcal{U}}$ ) is defined to be the quotient space  $\ell_{\infty}(I; X)/\mathcal{N}_{\mathcal{U}}$ , where

$$\mathcal{N}_{\mathcal{U}} = \left\{ (x_i)_{i \in I} \in \ell_{\infty}(I; X) : \lim_{i, \mathcal{U}} \|x_i\|_X = 0 \right\}.$$

We note that the norm on the Banach space  $X_{\mathcal{U}}$  may be evaluated as limit over the ultrafilter  $\mathcal{U}$ . That is, if  $(x_i)_{i \in I} \in \ell_{\infty}(I, X)$  then

$$\|(x_i)_{i \in I}^{\bullet}\|_{X_{\mathcal{U}}} = \lim_{i, \mathcal{U}} \|x_i\|_X$$

where  $(x_i)_{i \in I}^{\bullet}$  denotes the equivalent class in  $X_{\mathcal{U}}$  containing  $(x_i)_{i \in I}$ . For extensive discussions on ultrapowers of Banach spaces and their use, we refer to [12, 41]. Below, we consider only the case where the Banach space  $X$  is a von Neumann algebra.

It is well-known that if  $\mathcal{A}$  is a  $C^*$ -algebra then its ultrapower  $\mathcal{A}_{\mathcal{U}}$  is also a  $C^*$ -algebra. On the other hand, the class of von Neumann algebras is not closed under ultrapowers. However, the class of preduals of von Neumann algebras and more generally the class of noncommutative  $L_p$ -spaces is closed under ultrapowers. For the von Neumann algebra  $\mathcal{M}$ , the ultrapower  $(\mathcal{M}_*)_{\mathcal{U}}$  of  $\mathcal{M}_*$  is the predual of a von Neumann algebra which we denote by

$$\overline{\mathcal{M}}_{\mathcal{U}} = ((\mathcal{M}_*)_{\mathcal{U}})^*.$$

It is known that the  $C^*$ -algebra  $\mathcal{M}_{\mathcal{U}}$  identifies as a weak\*-dense subalgebra of  $\overline{\mathcal{M}}_{\mathcal{U}}$ . We refer to the work of Groh ([10]) and Raynaud ([36]) for more in depth discussion.

There is a substantial difficulty associated with the von Neumann algebra  $\overline{\mathcal{M}}_{\mathcal{U}}$  due to the fact that it need not be semifinite. This is a well-known fact that we will refer to [36]. We will use instead another model that we will refer to as the Ocneanu construction [29, Chapter 5]. It is built on earlier considerations of McDuff ([27]) and Vesterstrøm ([48]). We now recall this construction.

For the free ultrafilter  $\mathcal{U}$  over an index set  $I$ , consider the two-sided closed ideal

$$\mathcal{J}_{\mathcal{U}} = \left\{ (x_i)_{i \in I} \in \ell_{\infty}(I; \mathcal{M}) : \lim_{i, \mathcal{U}} \tau(x_i^* x_i) = 0 \right\}.$$

The quotient space  $\mathcal{M}^{\omega} = \mathcal{M}_{\mathcal{U}}^{\omega} = \ell_{\infty}(I; \mathcal{M})/\mathcal{J}_{\mathcal{U}}$  is a von Neumann algebra. The most important fact for our purpose is that since  $\mathcal{M}$  is a finite von Neumann algebra, so is  $\mathcal{M}^{\omega}$ . In this case, the faithful normal tracial state on  $\mathcal{M}^{\omega}$  is given by

$$\tau_{\mathcal{U}}(\tilde{x}) = \lim_{i, \mathcal{U}} \tau(x_i)$$



where  $\tilde{x} = (x_i)_{i \in I}^\bullet$  is the equivalent class in  $\mathcal{M}^\omega$  containing  $(x_i)_{i \in I} \in \ell_\infty(I; \mathcal{M})$ . For more information on the construction and properties of the ultrapower  $\mathcal{M}^\omega$ , we refer to [17, 29].

### 3. Golden–Thompson inequality

In this section, we provide the detailed account of our approach to the main theorem along with discussions on closely related matter.

#### 3.1. Proof of Theorem A

Before we proceed, let us first verify that under the given assumptions, the integral on the right hand side of the inequality from the statement is finite. Indeed, we observe first that since  $a \in \mathcal{M}$ , we may write,

$$\|(e^{-a} + t\mathbf{1})^{-1}\|_\infty \leq \begin{cases} e^{\|a\|_\infty}, & 0 < t < 1; \\ t^{-1}, & 1 \leq t. \end{cases}$$

Using Hölder's inequality, we have

$$\begin{aligned} \int_0^\infty \tau(e^{c/2}(e^{-a} + t\mathbf{1})^{-1}e^b(e^{-a} + t\mathbf{1})^{-1}e^{c/2}) \, dt \\ \leq \|e^b\|_p \|e^c\|_q \int_0^\infty \|(e^{-a} + t\mathbf{1})^{-1}\|_\infty^2 \, dt. \end{aligned}$$

It follows that

$$\int_0^\infty \tau(e^{c/2}(e^{-a} + t\mathbf{1})^{-1}e^b(e^{-a} + t\mathbf{1})^{-1}e^{c/2}) \, dt \leq \|e^b\|_p \|e^c\|_q (e^{2\|a\|_\infty} + 1).$$

As in [25], our approach is based on the celebrated Wigner–Yanase–Dyson–Lieb concavity theorem which we will refer below as the (WYDL)-concavity. This was originally proved in [25] for matrix algebra with the usual trace. In [2], the (WYDL)-concavity was extended to the setting of general  $\sigma$ -finite von Neumann algebra. We outline the statement below:

Assume that  $\mathfrak{M}$  is a  $\sigma$ -finite von Neumann with a standard form given by  $(\mathfrak{M}, H, J, P)$ . That is,  $\mathfrak{M}$  is standardly acting on a Hilbert space  $H$ , and  $P$  is a self dual cone in  $H$  such that  $P \ni \xi \mapsto w\xi \in \mathfrak{M}_*^+$  is a bijection, a unitary involution  $J$  on  $H$  satisfying  $J\mathfrak{M}J = \mathfrak{M}'$  and  $J\xi = \xi$  for  $\xi \in P$ . Each  $\phi \in \mathfrak{M}_*^+$  can be implemented by a vector in  $P$  which we denote by  $\xi_\phi$ . Given  $\phi, \psi$  in  $\mathfrak{M}_*^+$  (with  $\phi$  being faithful), there is a unique positive operator  $\Delta_{\psi\phi}$  on  $H$  such that  $J\Delta_{\psi\phi}^{1/2}x\xi_\phi = x^*\xi_\psi$  for  $x \in \mathfrak{M}$  known as the

relative modular operator of  $\psi$  relative to  $\phi$ . The (WYDL)-concavity for general  $\sigma$ -finite von Neumann algebra states that for  $0 < \theta \leq 1$  and  $x \in \mathfrak{M}$ , the map  $\mathfrak{M}_+^* \times \mathfrak{M}_+^* \rightarrow \mathbb{R}$  given by:

$$(3.1) \quad (\phi, \psi) \mapsto \|\Delta_{\psi\phi}^{\theta/2} x \xi_\phi\|_H$$

is jointly concave.

When specializing to the finite von Neumann algebra  $(\mathcal{M}, \tau)$ , (3.1) reduces to the following statement:

**PROPOSITION 3.1.** — *Let  $0 < \theta < 1$ . For every  $x \in \mathcal{M}$ , the map  $\mathcal{M}_+ \times \mathcal{M}_+ \rightarrow \mathbb{R}$  defined by*

$$(a, b) \mapsto \tau(a^{1-\theta} x^* b^\theta x)$$

*is jointly concave.*

For the passage from (3.1) to the assertion in Proposition 3.1, we can apply mutatis mutandis the explanation given in [23, Example 5.1]. Proposition 3.1 plays significant role in the series of arguments below.

Another key-point of Lieb's argument is the following general statement about convex functions defined on convex cone in a topological vector space. We record it here for convenience.

**LEMMA 3.2** ([25, Lemma 5]). — *Let  $\mathcal{C}$  be a convex cone in a topological vector space and let  $F : \mathcal{C} \rightarrow \mathbb{R}$  be a convex function which is also right differentiable in the sense that*

$$\lim_{s \rightarrow 0^+} \frac{1}{s} [F(A + sB) - F(A)] := G(A, B)$$

*exists for all  $A, B \in \mathcal{C}$ . Assume also that  $F$  is homogeneous of order 1. That is,  $F(tA) = tF(A)$  for all  $A \in \mathcal{C}$  and  $t > 0$ . Then*

$$G(A, B) \leq F(B).$$

*Conversely, if  $F$  is two sided differentiable,  $G(A + xB, B)$  is measurable on  $\{x : x \geq 0\}$ ,  $G(A, B) \leq F(B)$ , and  $F$  is homogeneous of order 1, then  $F$  is convex.*

For the next statement, we recall that for given Banach spaces  $X$  and  $Y$ , a continuous function  $F : X \rightarrow Y$  is said to be *Fréchet differentiable* if for each  $x \in X$ , there exists a bounded linear operator  $D(F)(x) : X \rightarrow Y$  such that

$$\lim_{h \rightarrow 0} \|h\|^{-1} (F(x + h) - F(x) - D(F)(x)(h)) = 0.$$

In this case, the bounded linear operator  $D(F)(x)$  is called the *Fréchet derivative* of  $F$  at  $x$ . We record the next result for further use.

LEMMA 3.3 ([11, Proposition 1.3]). — *If  $\mathfrak{A}$  is a Banach algebra, then the exponential function  $\Psi : \mathfrak{A} \rightarrow \mathfrak{A}$  defined by  $\Psi(x) = e^x$  is Fréchet differentiable and for every  $x \in \mathfrak{A}$ , the Fréchet derivative  $D(\Psi)(x) : \mathfrak{A} \rightarrow \mathfrak{A}$  is given by*

$$D(\Psi)(x)(y) = \int_0^1 e^{tx} y e^{(1-t)x} dt, \quad y \in \mathfrak{A}.$$

We now introduce the following linear transformation on  $\mathcal{M}$  that is central to the formulation of the GT-inequality. This is a generalization of the same transformation introduced in [25] for matrix algebra. For  $a \in \mathcal{M}_{++}$ , set

$$(3.2) \quad \mathcal{T}_a(x) = \int_0^\infty (a + s\mathbf{1})^{-1} x (a + s\mathbf{1})^{-1} ds, \quad x \in \mathcal{M},$$

where the integral can be taken in the sense of Bochner ([5]). The integral is well-defined and the resulting operator belongs to  $\mathcal{M}$ . Indeed, for a given  $x \in \mathcal{M}$ , the map  $s \mapsto (a + s\mathbf{1})^{-1} x (a + s\mathbf{1})^{-1}$  is continuous. Moreover, one can readily verify that the following estimate holds:

$$\|(a + s\mathbf{1})^{-1} x (a + s\mathbf{1})^{-1}\|_\infty \leq \begin{cases} \|a^{-1}\|_\infty^2 \|x\|_\infty, & 0 < s < 1; \\ s^{-2} \|x\|_\infty, & 1 \leq s. \end{cases}$$

It follows that

$$\int_0^\infty \|(a + s\mathbf{1})^{-1} x (a + s\mathbf{1})^{-1}\|_\infty ds < \infty.$$

This implies by [5, Theorem 2 (p. 45)] that the function  $s \mapsto (a + s\mathbf{1})^{-1} \times x (a + s\mathbf{1})^{-1}$  is Bochner-integrable. The estimates above also show that  $\mathcal{T}_a$  is a bounded linear operator with

$$\|\mathcal{T}_a\| \leq \|a\|_\infty + 1.$$

We gather below further properties of  $\mathcal{T}_a$  that we will need in the sequel.

PROPOSITION 3.4. — *Let  $a \in \mathcal{M}_{++}$  and  $b \in \mathcal{M}_h$ . Fix a neighborhood  $U$  of  $0 \in \mathbb{R}$  so that the operator  $a + sb$  is invertible for all  $s \in U$ . Define  $f : U \rightarrow \mathcal{M}$  by  $f(s) = \ln(a + sb)$ , for  $s \in U$ . Then, the following hold:*

- (i) *for every  $s \in U$ ,  $f'(s) = \mathcal{T}_{a+sb}(b)$ ;*
- (ii) *For every  $x \in \mathcal{M}$ ,*

$$\begin{aligned} \frac{d}{ds} (\mathcal{T}_{a+sb}(x)) \Big|_{s=0} &= - \int_0^\infty (a + s\mathbf{1})^{-1} b (a + s\mathbf{1})^{-1} x (a + s\mathbf{1})^{-1} ds \\ &\quad - \int_0^\infty (a + s\mathbf{1})^{-1} x (a + s\mathbf{1})^{-1} b (a + s\mathbf{1})^{-1} ds. \end{aligned}$$

In particular,

$$f''(0) = -2 \int_0^\infty (a + s\mathbf{1})^{-1} b (a + s\mathbf{1})^{-1} b (a + s\mathbf{1})^{-1} ds.$$

(iii)  $\mathcal{T}_a$  admits an inverse linear transformation  $\mathcal{T}_a^{-1}$  given by:

$$\mathcal{T}_a^{-1} : x \mapsto \int_0^1 a^t x a^{1-t} dt.$$

We need some preparation for the proof. We assume that the following representation of the log of operators from  $\mathcal{M}_{++}$  is well-known but we include a short argument for completeness. Let  $b \in \mathcal{M}_{++}$ . We have the following integral representation:

$$(3.3) \quad \ln b = \int_0^\infty (b - \mathbf{1})(b + s\mathbf{1})^{-1} \frac{ds}{s+1}.$$

We may verify the above identity as follows: for any given positive real number  $y$ , one can readily compute that

$$\ln y = \int_0^\infty \frac{y-1}{(y+s)(s+1)} ds.$$

For a given  $b \in \mathcal{M}_{++}$ , we may consider the Abelian  $C^*$ -algebra generated by  $b$ . By the Gelfand–Naimark identification, it suffices to verify the stated formula for continuous functions  $f : \Omega \rightarrow (0, \infty)$  where  $\Omega$  is a compact subset. From the scalar integral above, we may write:

$$\ln(f(w)) = \int_0^\infty (f(w) - 1)(f(w) + s)^{-1} \frac{ds}{s+1}, \quad w \in \Omega.$$

This is the desired representation for any positive function  $f \in C(\Omega)$ . By identification, (3.3) holds.

LEMMA 3.5. — *Let  $a \in \mathcal{M}_{++}$  and  $b \in \mathcal{M}$ . Fix a neighborhood  $U$  of  $0 \in \mathbb{R}$  so that the operator  $a+tb$  is invertible for all  $t \in U$ . Define  $g : U \rightarrow \mathcal{M}$  by  $g(t) = (a+tb)^{-1}$  for  $t \in U$ . Then,*

$$g'(t) = -(a+tb)^{-1} b (a+tb)^{-1}, \quad t \in U.$$

In particular,  $g'(0) = -a^{-1}ba^{-1}$ .

*Proof.* — Let  $t \in U$  and fix  $h$  such that  $t+h \in U$ . Then,

$$\begin{aligned} h^{-1}(g(t+h) - g(t)) &= h^{-1}[(a+tb+hb)^{-1} - (a+tb)^{-1}] \\ &= h^{-1}[(a+tb+hb)^{-1}((a+tb) - (a+tb+hb))(a+tb)^{-1}] \\ &= -[(a+tb+hb)^{-1}b(a+tb)^{-1}]. \end{aligned}$$

The assertion follows from taking  $h \rightarrow 0$ . □

*Proof of Proposition 3.4.* — For item (i), we use the integral representation (3.3) and write that for  $h > 0$ ,

$$\begin{aligned} f(h) - f(0) &= \int_0^\infty (a + hb - \mathbf{1})(a + hb + s\mathbf{1})^{-1} - (a - \mathbf{1})(a + s\mathbf{1})^{-1} \frac{ds}{s+1} \\ &= \int_0^\infty (a + hb - \mathbf{1})[(a + hb + s\mathbf{1})^{-1} - (a + s\mathbf{1})^{-1}] \\ &\quad + [(a + hb - \mathbf{1}) - (a - \mathbf{1})](a + s\mathbf{1})^{-1} \frac{ds}{s+1} \\ &= \int_0^\infty (a + hb - \mathbf{1})[(a + hb + s\mathbf{1})^{-1} - (a + s\mathbf{1})^{-1}] + (hb)(a + s\mathbf{1})^{-1} \frac{ds}{s+1}. \end{aligned}$$

Dividing by  $h$  and taking limit as  $h \rightarrow 0$ , it follows from Lemma 3.5 that

$$\begin{aligned} f'(0) &= \int_0^\infty -(a - \mathbf{1})[(a + s\mathbf{1})^{-1}b(a + s\mathbf{1})^{-1}] + b(a + s\mathbf{1})^{-1} \frac{ds}{s+1} \\ &= \int_0^\infty (a + s\mathbf{1})^{-1}[-(a - \mathbf{1}) + (a + s\mathbf{1})]b(a + s\mathbf{1})^{-1} \frac{ds}{s+1} \\ &= \int_0^\infty (a + s\mathbf{1})^{-1}[s\mathbf{1} + \mathbf{1}]b(a + s\mathbf{1})^{-1} \frac{ds}{s+1} \\ &= \int_0^\infty (a + s\mathbf{1})^{-1}b(a + s\mathbf{1})^{-1} ds. \end{aligned}$$

This shows that  $f'(0) = \mathcal{T}_a(b)$ . The statement for  $f'(s)$  follows by translation.

For item (ii), we see from the first item that for a given  $h$ ,

$$\begin{aligned} \mathcal{T}_{a+hb}(x) - \mathcal{T}_a(x) &= \int_0^\infty (a + hb + s\mathbf{1})^{-1}x(a + hb + s\mathbf{1})^{-1} - (a + s\mathbf{1})^{-1}x(a + s\mathbf{1})^{-1} ds \\ &= \int_0^\infty (a + hb + s\mathbf{1})^{-1}x[(a + hb + s\mathbf{1})^{-1} - (a + s\mathbf{1})^{-1}] ds \\ &\quad + \int_0^\infty [(a + hb + s\mathbf{1})^{-1} - (a + s\mathbf{1})^{-1}]x(a + s\mathbf{1})^{-1} ds. \end{aligned}$$

Dividing by  $h$  and taking limit as  $h \rightarrow 0$  leads to

$$\frac{d}{ds}(\mathcal{T}_{a+sb}(x))\big|_{s=0} = \int_0^\infty (a + s\mathbf{1})^{-1}xg'_s(0) + g'_s(0)x(a + s\mathbf{1})^{-1} ds$$

where for each  $s > 0$ ,  $g_s(t) = (a + tb + s\mathbf{1})^{-1}$ . Item (ii) follows from Lemma 3.5 by using  $a + s\mathbf{1}$  in place of  $a$ .

We now verify item (iii). For a positive operator  $x \in \mathcal{M}$ , fix a neighborhood  $V$  of  $0 \in \mathbb{R}$  so that  $\ln(a + sx)$  is well-defined for all  $s \in V$ . Let  $f_x : V \rightarrow \mathcal{M}_h$  be the function defined by  $f_x(s) = \ln(a + sx)$ . Consider

$\Psi : \mathcal{M} \rightarrow \mathcal{M}$  defined by  $\Psi(z) = e^z$ . Clearly,  $\Psi(f_x(s)) = a + sx$  and therefore we have  $\frac{d}{ds}(\Psi(f_x(s))) = x$ .

On the other hand, if we denote by  $D(\Psi)$  the Fréchet derivative of  $\Psi$  then by the chain rule, we may state:

$$(3.4) \quad D(\Psi)(f_x(0))(f'_x(0)) = \frac{d}{ds}(\Psi(f_x(s)))|_{s=0} = x.$$

Next, we note that  $f'_x(0) = \mathcal{T}_a(x)$ . Moreover, we have from Lemma 3.3 that

$$D(\Psi)(\alpha)(y) = \int_0^1 e^{t\alpha} y e^{(1-t)\alpha} dt, \quad y \in \mathcal{M}.$$

As  $f_x(0) = \ln a$ , we have

$$D(\Psi)(\ln a)(y) = \int_0^1 a^t y a^{1-t} dt, \quad y \in \mathcal{M}.$$

These facts combined with (3.4) shows that  $D(\Psi)(\ln a)(\mathcal{T}_a(x)) = x$ . Since this is valid for any arbitrary positive operator  $x$ , we obtain by linearity that  $D(\Psi)(\ln a) \circ \mathcal{T}_a$  is the identity map on  $\mathcal{M}$ .

On the other hand,  $D(\Psi)(\ln a)$  is one to one. Indeed, if  $y \in \mathcal{M}$  is such that  $D(\Psi)(\ln a)(y) = 0$ , then  $\tau(D(\Psi)(\ln a)(y)y^*) = 0$ . This is equivalent to  $\int_0^1 \tau(a^t y a^{1-t} y^*) dt = 0$ . By tracial property, we have

$$\int_0^1 \tau((a^{t/2} y a^{(1-t)/2})(a^{(1-t)/2} y^* a^{t/2})) dt = \int_0^1 \|a^{t/2} y a^{(1-t)/2}\|_2^2 dt = 0.$$

This implies that for a.e  $t \in [0, 1]$ ,  $a^{t/2} y a^{(1-t)/2} = 0$ . But since  $a$  is invertible, so are  $a^{t/2}$  and  $a^{(1-t)/2}$ . This shows that  $y = 0$ .

Now fix  $x \in \mathcal{M}$  and set  $\mathcal{T}_a(D(\Psi)(\ln a)(x)) = y$ . Then, by applying  $D(\Psi)(\ln a)$  on both sides and using the fact that  $D(\Psi)(\ln a) \circ \mathcal{T}_a$  is the identity map, we have  $D(\Psi)(\ln a)(x) = D(\Psi)(\ln a)(y)$ . Since  $D(\Psi)(\ln a)$  is one to one, we must have  $x = y$ . That is,  $\mathcal{T}_a(D(\Psi)(\ln a)(x)) = x$ , showing that  $\mathcal{T}_a \circ D(\Psi)(\ln a)$  is the identity map on  $\mathcal{M}$ .

We can now conclude that  $\mathcal{T}_a$  admits an inverse transformation with  $\mathcal{T}_a^{-1} = D(\Psi)(\ln a)$  which is the desired statement.  $\square$

We will now consider ultrapowers of the class of operators discussed above. First, we state the following elementary lemma.

LEMMA 3.6. — For  $a \in \mathcal{M}_{++}$  and  $1 \leq p \leq \infty$ ,  $\mathcal{T}_a$  extends into a bounded operator from  $L_p(\mathcal{M})$  onto  $L_p(\mathcal{M})$ .

Similarly,  $\mathcal{T}_a^{-1}$  extends to a bounded linear operator from  $L_p(\mathcal{M})$  into  $L_p(\mathcal{M})$ .

*Proof.* — For  $x$  and  $y$  in  $\mathcal{M}$ , we have  $\tau(\mathcal{T}_a(x)y) = \tau(x\mathcal{T}_a(y))$ . Then,

$$\begin{aligned} \|\mathcal{T}_a(x)\|_1 &= \sup\{\tau(\mathcal{T}_a(x)y) : \|y\|_\infty \leq 1\} \\ &= \sup\{\tau(x\mathcal{T}_a(y)) : \|y\|_\infty \leq 1\} \\ &\leq \|\mathcal{T}_a : \mathcal{M} \rightarrow \mathcal{M}\| \cdot \|x\|_1. \end{aligned}$$

Since  $\mathcal{M}$  is dense in  $L_1(\mathcal{M})$ , this proves the case  $p = 1$ . The other values of  $p$  follow by complex interpolations.

For the inverse map  $\mathcal{T}_a^{-1}$ , we have from its expression in Proposition 3.4 that if  $x \in \mathcal{M}$ , then for  $1 \leq p < \infty$ ,

$$\|\mathcal{T}_a^{-1}(x)\|_p \leq \int_0^1 \|a^t x a^{1-t}\|_p dt \leq \|a\|_\infty \|x\|_p.$$

If we still denote the extensions by  $\mathcal{T}_a$  and  $\mathcal{T}_a^{-1}$ , then we clearly have  $\mathcal{T}_a \mathcal{T}_a^{-1}(x) = \mathcal{T}_a^{-1} \mathcal{T}_a(x) = x$  for  $x \in L_p(\mathcal{M})$ .  $\square$

Let  $\mathcal{U}$  be a free ultrafilter over  $\mathbb{N}$ . If  $(\alpha_n)_{n \in \mathbb{N}}$  belongs to the ideal  $\mathcal{J}_{\mathcal{U}}$ , then it follows from Lemma 3.6 (applied to  $p = 2$ ) that  $(\mathcal{T}_a(\alpha_n))_{n \in \mathbb{N}}$  and  $(\mathcal{T}_a^{-1}(\alpha_n))_{n \in \mathbb{N}}$  both belong to  $\mathcal{J}_{\mathcal{U}}$ . As a consequence, the following map

$$(3.5) \quad (x_n)_{n \in \mathbb{N}}^\bullet \mapsto (\mathcal{T}_a(x_n))_{n \in \mathbb{N}}^\bullet$$

is well-defined and is bounded from  $\mathcal{M}^\omega$  into  $\mathcal{M}^\omega$ . We will denote such map by  $\mathcal{T}_a^\bullet$ . We note that  $\mathcal{T}_a^\bullet$  is also bounded on  $L_p(\mathcal{M}^\omega)$  for  $1 \leq p < \infty$  but this fact will not be needed below.

Similarly, we may also define a bounded linear map  $(\mathcal{T}_a^{-1})^\bullet : \mathcal{M}^\omega \rightarrow \mathcal{M}^\omega$  by setting:

$$(\mathcal{T}_a^{-1})^\bullet((x_n)_{n \in \mathbb{N}}^\bullet) = (\mathcal{T}_a^{-1}(x_n))_{n \in \mathbb{N}}^\bullet.$$

One can readily verify that  $(\mathcal{T}_a^{-1})^\bullet \mathcal{T}_a^\bullet = \mathcal{T}_a^\bullet (\mathcal{T}_a^{-1})^\bullet$  and both are equal to the identity map on  $\mathcal{M}^\omega$ . In other word,  $\mathcal{T}_a^\bullet$  is invertible with  $(\mathcal{T}_a^\bullet)^{-1} = (\mathcal{T}_a^{-1})^\bullet$ . These facts will be used below.

We now introduce another map that is essential for the proof of Theorem A. Following [25], we consider the map  $\mathcal{Q} : \mathcal{M}_{++} \times \mathcal{M} \rightarrow [0, \infty)$  by setting:

$$(3.6) \quad \mathcal{Q}(a, x) = \tau \left( \int_0^\infty (a + s\mathbf{1})^{-1} x^* (a + s\mathbf{1})^{-1} x ds \right) = \tau(\mathcal{T}_a(x)x^*).$$

As observed earlier, the integral is convergent for any  $(a, x) \in \mathcal{M}_{++} \times \mathcal{M}$  and therefore  $\mathcal{Q}$  is well-defined. From tracial properties, we have

$$\mathcal{Q}(a, x) = \int_0^\infty \|(a + s\mathbf{1})^{-1/2} x (a + s\mathbf{1})^{-1/2}\|_2^2 ds.$$

This shows in particular that  $\mathcal{Q}(a, x) \geq 0$  for all  $(a, x) \in \mathcal{M}_{++} \times \mathcal{M}$ . It also implies that  $\mathcal{Q}(\cdot, \cdot)$  can be extended as continuous function on  $\mathcal{M}_{++} \times$

$L_2(\mathcal{M})$  into  $[0, \infty)$ . We record here an elementary but crucial property of  $\mathcal{Q}(\cdot, \cdot)$  that we will need.

LEMMA 3.7. — *The map  $\mathcal{Q}(\cdot, \cdot)$  is homogeneous of order 1.*

*Proof.* — Let  $t > 0$  and  $(a, x) \in \mathcal{M}_{++} \times \mathcal{M}$ . Then, by linearity, we have  $\mathcal{Q}(ta, tx) = \tau(\mathcal{T}_{ta}(tx)(tx)^*) = t^2\tau(\mathcal{T}_{ta}(x)x^*)$ . Moreover, a simple use of change of variable on the integral in the definition of  $\mathcal{T}_a$  shows that  $\mathcal{T}_{ta}(\cdot) = t^{-1}\mathcal{T}_a(\cdot)$  which when combined with the previous assertion gives  $\mathcal{Q}(ta, tx) = t\mathcal{Q}(a, x)$ .  $\square$

The next result constitutes an important step toward the proof of Theorem A. This is the generalization of [25, Theorem 3].

PROPOSITION 3.8. — *The map  $\mathcal{Q} : \mathcal{M}_{++} \times \mathcal{M} \rightarrow [0, \infty)$  is jointly convex.*

*Proof.* — Let  $0 < \lambda < 1$  and fix  $a, b \in \mathcal{M}_{++}$ . Set  $c = \lambda a + (1 - \lambda)b$ . We need to verify that for every  $x, y \in \mathcal{M}$ , the following inequality holds:

$$(3.7) \quad \mathcal{Q}(c, z) \leq \lambda \mathcal{Q}(a, x) + (1 - \lambda) \mathcal{Q}(b, y)$$

where  $z = \lambda x + (1 - \lambda)y$ .

We will assume first that  $(x, y) \in B_{\mathcal{M}_h} \times B_{\mathcal{M}_h}$  where  $B_{\mathcal{M}_h}$  denotes the closed unit ball of  $\mathcal{M}_h$ . Let

$$\mathfrak{D}(x, y) := \lambda \mathcal{Q}(a, x) + (1 - \lambda) \mathcal{Q}(b, y) \quad \text{and} \quad \mathfrak{N}(x, y) := \mathcal{Q}(c, z).$$

Fix  $\varepsilon > 0$ . For  $x, y \in B_{\mathcal{M}_h}$ , we define the following map:

$$\Gamma_\varepsilon(x, y) = \frac{\mathfrak{N}(x, y)}{\varepsilon + \mathfrak{D}(x, y)}.$$

It is clear that  $\Gamma_\varepsilon(x, y) \geq 0$  and  $\Gamma_\varepsilon(x, y) = 0$  if and only if  $z = 0$ .

The idea of the proof is to estimate the supremum of the function  $\Gamma_\varepsilon$  over all  $(x, y) \in B_{\mathcal{M}_h} \times B_{\mathcal{M}_h}$ . For convenience, let

$$(3.8) \quad \gamma := \sup\{\Gamma_\varepsilon(x, y); x, y \in B_{\mathcal{M}_h}\}.$$

Taking  $x = y = \mathbf{1}$ , we see that  $\gamma > 0$ . We claim that  $0 < \gamma \leq 1$ .

The main difficulty in verifying the above claim lies in the fact that contrary to the case of matrix algebra where the supremum is actually a maximum by compactness of the unit ball, the supremum may not be necessarily attained. To circumvent this inconvenience, we appeal to ultrapower technique.



Let  $\mathcal{U}$  be a free ultrafilter over  $\mathbb{N}$ . We introduce an ultrapower version of  $\Gamma_\varepsilon$ . Consider the map  $\Gamma_{\varepsilon, \mathcal{U}} : B_{\mathcal{M}_h^\omega} \times B_{\mathcal{M}_h^\omega} \longrightarrow [0, \infty)$  by setting:

$$\Gamma_{\varepsilon, \mathcal{U}}(\cdot, \cdot) := \frac{\mathfrak{N}_{\mathcal{U}}(\cdot, \cdot)}{\varepsilon + \mathfrak{D}_{\mathcal{U}}(\cdot, \cdot)}$$

where for  $\tilde{\zeta} = (\zeta_n)_{n \in \mathbb{N}}^\bullet$  and  $\tilde{\eta} = (\eta_n)_{n \in \mathbb{N}}^\bullet$  in  $B_{\mathcal{M}_h^\omega}$ , we define

$$\mathfrak{D}_{\mathcal{U}}(\tilde{\zeta}, \tilde{\eta}) = \lim_{n, \mathcal{U}} \mathfrak{D}(\zeta_n, \eta_n) \quad \text{and} \quad \mathfrak{N}_{\mathcal{U}}(\tilde{\zeta}, \tilde{\eta}) = \lim_{n, \mathcal{U}} \mathfrak{N}(\zeta_n, \eta_n).$$

These are equivalent to setting:

$$\mathfrak{D}_{\mathcal{U}}(\tilde{\zeta}, \tilde{\eta}) = \lambda \tau_{\mathcal{U}}(\mathcal{T}_a^\bullet(\tilde{\zeta})\tilde{\zeta}) + (1 - \lambda) \tau_{\mathcal{U}}(\mathcal{T}_b^\bullet(\tilde{\eta})\tilde{\eta})$$

and if we let  $\tilde{\xi} = \lambda \tilde{\zeta} + (1 - \lambda) \tilde{\eta}$ , then

$$\mathfrak{N}_{\mathcal{U}}(\tilde{\zeta}, \tilde{\eta}) = \tau_{\mathcal{U}}(\mathcal{T}_c^\bullet(\tilde{\xi})\tilde{\xi}).$$

We make the crucial observation that

$$(3.9) \quad \gamma = \sup \left\{ \Gamma_{\varepsilon, \mathcal{U}}(\tilde{\zeta}, \tilde{\eta}) : (\tilde{\zeta}, \tilde{\eta}) \in B_{\mathcal{M}_h^\omega} \times B_{\mathcal{M}_h^\omega} \right\}.$$

The advantage in working with the map  $\Gamma_{\varepsilon, \mathcal{U}}$  is that the supremum is attained. We state this in the next lemma.

LEMMA 3.9. — *There exists  $(\tilde{x}, \tilde{y}) \in B_{\mathcal{M}_h^\omega} \times B_{\mathcal{M}_h^\omega}$  such that*

$$\gamma = \Gamma_{\varepsilon, \mathcal{U}}(\tilde{x}, \tilde{y}).$$

*Proof.* — Choose two sequences  $(x_n)$  and  $(y_n)$  in  $B_{\mathcal{M}_h}$  so that for every  $n \geq 1$ ,

$$\gamma - \frac{1}{2^n} < \Gamma_\varepsilon(x_n, y_n) \leq \gamma.$$

Clearly,  $\gamma = \lim_{n \rightarrow \infty} \Gamma_\varepsilon(x_n, y_n)$ . Consider  $\tilde{x} = (x_n)_{n \in \mathbb{N}}^\bullet$  and  $\tilde{y} = (y_n)_{n \in \mathbb{N}}^\bullet$  in  $B_{\mathcal{M}_h^\omega}$ . We see that  $\gamma = \Gamma_{\varepsilon, \mathcal{U}}(\tilde{x}, \tilde{y})$  which verifies the lemma.  $\square$

Before we proceed, we would like to clarify some of the considerations we have taken in our argument. First, the restriction to  $B_{\mathcal{M}} \times B_{\mathcal{M}}$  is used in order to facilitate the passage to elements in the ultrapower. Second, the role of  $\varepsilon > 0$  in the denominator is to prevent the possibility of reaching  $\gamma$  with  $0/0$  type limit.

We now go back to the full space. For  $(\tilde{\zeta}, \tilde{\eta}) \in \mathcal{M}_h^\omega \times \mathcal{M}_h^\omega$ , let

$$\hat{\Gamma}_{\varepsilon, \mathcal{U}}(\tilde{\zeta}, \tilde{\eta}) := \frac{\tau_{\mathcal{U}}(\mathcal{T}_c^\bullet(\tilde{\xi})\tilde{\xi})}{\varepsilon + \lambda \tau_{\mathcal{U}}(\mathcal{T}_a^\bullet(\tilde{\zeta})\tilde{\zeta}) + (1 - \lambda) \tau_{\mathcal{U}}(\mathcal{T}_b^\bullet(\tilde{\eta})\tilde{\eta})}$$

where  $\tilde{\xi} = \lambda \tilde{\zeta} + (1 - \lambda) \tilde{\eta}$ . Then,  $\hat{\Gamma}_{\varepsilon, \mathcal{U}}$  is a real-valued map defined on the real vector space  $\mathcal{M}_h^\omega \times \mathcal{M}_h^\omega$ . Clearly, the restriction of  $\hat{\Gamma}_{\varepsilon, \mathcal{U}}$  to  $B_{\mathcal{M}_w^\omega} \times B_{\mathcal{M}_w^\omega}$  is  $\Gamma_{\varepsilon, \mathcal{U}}$ .

Let  $(\tilde{x}, \tilde{y})$  be as in Lemma 3.9. Then a local maximum of the function  $\widehat{\Gamma}_{\varepsilon, \mathcal{U}}$  defined on the real vector space  $\mathcal{M}_h^\omega \times \mathcal{M}_h^\omega$  occurs at  $(\tilde{x}, \tilde{y})$ .

Fix  $(\tilde{u}, \tilde{v}) \in \mathcal{M}_h^\omega \times \mathcal{M}_h^\omega$ . One can easily compute that the directional derivative of the numerator of  $\widehat{\Gamma}_{\varepsilon, \mathcal{U}}$  at  $(\tilde{x}, \tilde{y})$  along  $(\tilde{u}, \tilde{v})$  is given by:

$$2\tau_{\mathcal{U}}((\mathcal{T}_c^\bullet(\lambda\tilde{x} + (1-\lambda)\tilde{y})(\lambda\tilde{u} + (1-\lambda)\tilde{v})).$$

Similarly, the directional derivative of the denominator of  $\widehat{\Gamma}_{\varepsilon, \mathcal{U}}$  at  $(\tilde{x}, \tilde{y})$  along  $(\tilde{u}, \tilde{v})$  is given by

$$2\lambda\tau_{\mathcal{U}}(\mathcal{T}_a^\bullet(\tilde{x})\tilde{u}) + 2(1-\lambda)\tau_{\mathcal{U}}(\mathcal{T}_b^\bullet(\tilde{y})\tilde{v}).$$

Since  $(\tilde{x}, \tilde{y})$  is a critical point for  $\widehat{\Gamma}_{\varepsilon, \mathcal{U}}$ , we have from the quotient rule of differentiation that for every  $(\tilde{u}, \tilde{v})$ :

$$\begin{aligned} \mathfrak{N}_{\mathcal{U}}(\tilde{x}, \tilde{y}) [\lambda\tau_{\mathcal{U}}(\mathcal{T}_a^\bullet(\tilde{x})\tilde{u}) + (1-\lambda)\tau_{\mathcal{U}}(\mathcal{T}_b^\bullet(\tilde{y})\tilde{v})] \\ = (\varepsilon + \mathfrak{D}_{\mathcal{U}}(\tilde{x}, \tilde{y})) [\tau_{\mathcal{U}}(\mathcal{T}_c^\bullet(\lambda\tilde{x} + (1-\lambda)\tilde{y})(\lambda\tilde{u} + (1-\lambda)\tilde{v}))]. \end{aligned}$$

This is equivalent to:

$$\gamma\lambda\tau_{\mathcal{U}}(\mathcal{T}_a^\bullet(\tilde{x})\tilde{u}) + \gamma(1-\lambda)\tau_{\mathcal{U}}(\mathcal{T}_b^\bullet(\tilde{y})\tilde{v}) = \tau_{\mathcal{U}}(\mathcal{T}_c^\bullet(\lambda\tilde{x} + (1-\lambda)\tilde{y})(\lambda\tilde{u} + (1-\lambda)\tilde{v})).$$

Since  $\tilde{u}$  and  $\tilde{v}$  are arbitrary, we can deduce that the following two equations hold:

$$(3.10) \quad \begin{cases} \gamma\mathcal{T}_a^\bullet(\tilde{x}) = \mathcal{T}_c^\bullet(\lambda\tilde{x} + (1-\lambda)\tilde{y}), \\ \gamma\mathcal{T}_b^\bullet(\tilde{y}) = \mathcal{T}_c^\bullet(\lambda\tilde{x} + (1-\lambda)\tilde{y}). \end{cases}$$

Let  $\tilde{z} = \lambda\tilde{x} + (1-\lambda)\tilde{y}$ . It is important to note that since  $\gamma > 0$ , we have

$$(3.11) \quad \tau_{\mathcal{U}}(\mathcal{T}_c^\bullet(\tilde{z})\tilde{z}) > 0.$$

Otherwise,  $\mathfrak{N}_{\mathcal{U}}(\tilde{x}, \tilde{y}) = \tau_{\mathcal{U}}(\mathcal{T}_c^\bullet(\tilde{z})\tilde{z}) = 0$  and this implies that  $\Gamma_{\varepsilon, \mathcal{U}}(\tilde{x}, \tilde{y}) = 0$  which is a contradiction.

Let  $\tilde{\zeta} = (\zeta_n)_{n \in \mathbb{N}}^\bullet := \mathcal{T}_c^\bullet(\tilde{z})$ . Then,  $(\mathcal{T}_c^{-1})^\bullet(\tilde{\zeta}) = \tilde{z}$ ,  $(\mathcal{T}_a^{-1})^\bullet(\tilde{\zeta}) = \gamma\tilde{x}$ , and  $(\mathcal{T}_b^{-1})^\bullet(\tilde{\zeta}) = \gamma\tilde{y}$ . It follows that

$$\gamma(\mathcal{T}_c^{-1})^\bullet(\tilde{\zeta}) = \lambda(\mathcal{T}_a^{-1})^\bullet(\tilde{\zeta}) + (1-\lambda)(\mathcal{T}_b^{-1})^\bullet(\tilde{\zeta}).$$

Multiplying by  $\tilde{\zeta}$  from the right and taking traces, we have

$$\gamma\tau_{\mathcal{U}}[(\mathcal{T}_c^{-1})^\bullet(\tilde{\zeta})\tilde{\zeta}] = \lambda\tau_{\mathcal{U}}[(\mathcal{T}_a^{-1})^\bullet(\tilde{\zeta})\tilde{\zeta}] + (1-\lambda)\tau_{\mathcal{U}}[(\mathcal{T}_b^{-1})^\bullet(\tilde{\zeta})\tilde{\zeta}].$$

Using the description of the inverse transformations from Proposition 3.4, this is equivalent to

$$\lim_{n, \mathcal{U}} \int_0^1 \lambda\tau[a^s\zeta_n a^{1-s}\zeta_n] + (1-\lambda)\tau[b^s\zeta_n b^{1-s}\zeta_n] - \gamma\tau[c^s\zeta_n c^{1-s}\zeta_n] ds = 0.$$

By the concavity of the map  $w \mapsto \tau(w^s \zeta w^{1-s} \zeta)$  for  $0 < s < 1$  from Proposition 3.1, we obtain that

$$(1 - \gamma) \lim_{n, \mathcal{U}} \int_0^1 \tau[c^s \zeta_n c^{1-s} \zeta_n] ds \geq 0.$$

This is equivalent to

$$(1 - \gamma) \tau_{\mathcal{U}}[(\mathcal{T}_c^{-1})^\bullet(\tilde{\zeta})\tilde{\zeta}] = (1 - \gamma) \tau_{\mathcal{U}}[\mathcal{T}_c^\bullet(\tilde{z})\tilde{z}] \geq 0.$$

We can now conclude from (3.11) that  $\gamma \leq 1$  and thus proving the claim.

The fact that  $\gamma \leq 1$  shows in particular that for every  $(x, y) \in B_{\mathcal{M}_h} \times B_{\mathcal{M}_h}$ , we have

$$\mathfrak{N}(x, y) \leq \varepsilon + \mathfrak{D}(x, y).$$

Since  $\varepsilon$  is arbitrary, we conclude that  $\mathfrak{N}(x, y) \leq \mathfrak{D}(x, y)$  which is precisely (3.7) when  $x, y \in B_{\mathcal{M}_h}$ .

Let  $x$  and  $y$  be two arbitrary self-adjoint operators in  $\mathcal{M}$ . For  $t \in (0, \infty)$ , it follows from Lemma 3.7 that  $\mathfrak{N}(tx, ty) = t\mathfrak{N}(x, y)$  and  $\mathfrak{D}(tx, ty) = t\mathfrak{D}(x, y)$ . This clearly implies that  $\mathfrak{N}(x, y)/\mathfrak{D}(x, y) = \mathfrak{N}(tx, ty)/\mathfrak{D}(tx, ty)$  so it reduces to the previous case.

The extension of the self-adjoint case to the general case follows from the simple calculation that for arbitrary  $\alpha \in \mathcal{M}_{++}$  and  $w \in \mathcal{M}$ , we have  $\mathcal{Q}(\alpha, w) = \mathcal{Q}(\alpha, \operatorname{Re}(w)) + \mathcal{Q}(\alpha, \operatorname{Im}(w))$ . The proof of Proposition 3.8 is complete.  $\square$

We use the map  $\mathcal{Q}$  to prove another inequality involving the transform  $\mathcal{T}$  that we need in the proof of the next theorem.

LEMMA 3.10. — *For  $a, b \in \mathcal{M}_{++}$  and  $x, y \in \mathcal{M}_h$ , the following inequality holds:*

$$-2 \int_0^\infty \tau(b(a + s\mathbf{1})^{-1} x (a + s\mathbf{1})^{-1} x (a + s\mathbf{1})^{-1}) ds + 2\tau(y\mathcal{T}_a(x)) \leq \tau(y\mathcal{T}_b(y)).$$

*Proof.* — Consider the convex function  $\mathcal{Q}(\cdot, \cdot)$  defined on the cone  $\mathcal{C} = \mathcal{M}_{++} \times \mathcal{M}_h$ . We recall that  $\mathcal{Q}(\cdot, \cdot)$  is 1-homogeneous. By Lemma 3.2, we have

$$\lim_{s \rightarrow 0^+} \frac{1}{s} [\mathcal{Q}(a + sb, x + sy) - \mathcal{Q}(a, x)] \leq \mathcal{Q}(b, y).$$

A simple calculation gives

$$\begin{aligned} \mathcal{Q}(a + sb, x + sy) - \mathcal{Q}(a, x) &= [\mathcal{Q}(a + sb, x) - \mathcal{Q}(a, x)] \\ &\quad + 2s\tau(y\mathcal{T}_{a+sb}(x)) + s^2\tau(y\mathcal{T}_{a+sb}(y)). \end{aligned}$$

This implies,

$$\begin{aligned} \lim_{s \rightarrow 0^+} \frac{1}{s} [\mathcal{Q}(a + sb, x + sy) - \mathcal{Q}(a, x)] \\ = \lim_{s \rightarrow 0^+} \frac{1}{s} [\mathcal{Q}(a + sb, x) - \mathcal{Q}(a, x)] + 2\tau(y\mathcal{T}_a(x)) \\ = \frac{d}{ds} (\tau(x\mathcal{T}_{a+sb}(x))) \Big|_{s=0} + 2\tau(y\mathcal{T}_a(x)). \end{aligned}$$

The lemma follows from Proposition 3.4(ii).  $\square$

The next result is the extension of [25, Theorem 6] to the case of finite von Neumann algebras and could be of independent interest.

**THEOREM 3.11.** — *For any given  $v \in \mathcal{M}_h$ , the map from  $\mathcal{M}_{++}$  into  $(0, \infty)$ , defined by*

$$a \mapsto \tau(\exp(v + \ln a))$$

*is concave.*

*Proof.* — Fix  $a \in \mathcal{M}_{++}$  and  $b \in \mathcal{M}_h$ . Consider the real-valued function

$$\varphi(s) = \tau(\exp(v + \ln(a + sb)))$$

which is defined and differentiable on the real variable  $s$  in some neighborhood of  $0 \in \mathbb{R}$ . The theorem is equivalent to  $d^2\varphi/ds^2 \leq 0$  when  $s = 0$ .

First, by using chain rule and the Fréchet derivative of the exponential function from Lemma 3.3 and Proposition 3.4, we have

$$\varphi'(s) = \tau(\exp(v + \ln(a + sb))\mathcal{T}_{a+sb}(b)).$$

After rearrangement, we obtain that

$$\begin{aligned} \varphi'(s) - \varphi'(0) = \tau([\exp(v + \ln(a + sb)) - \exp(v + \ln a)]\mathcal{T}_{a+sb}(b)) \\ + \tau(\exp(v + \ln a)[\mathcal{T}_{a+sb}(b) - \mathcal{T}_a(b)]). \end{aligned}$$

Applying one more time the Fréchet derivative of the exponential function on the first term and the derivative of  $s \mapsto \mathcal{T}_{a+sb}(b)$  from Proposition 3.4(ii) (with  $x = b$ ) for the second term, we have

$$\begin{aligned} \varphi''(0) = \tau([\int_0^1 c^t(\mathcal{T}_a(b))c^{1-t} dt]\mathcal{T}_a(b)) \\ - 2 \int_0^\infty \tau(c(a + s\mathbf{1})^{-1}b(a + s\mathbf{1})^{-1}b(a + s\mathbf{1})^{-1}) ds \\ = \tau([\mathcal{T}_c^{-1}(\mathcal{T}_a(b))]\mathcal{T}_a(b)) \\ - 2 \int_0^\infty \tau(c(a + s\mathbf{1})^{-1}b(a + s\mathbf{1})^{-1}b(a + s\mathbf{1})^{-1}) ds \end{aligned}$$

where  $c = \exp(v + \ln a)$ . Applying Lemma 3.10 with  $w = \mathcal{T}_c^{-1}(\mathcal{T}_a(b))$ , we arrive at

$$\varphi''(0) \leq -\tau(w\mathcal{T}_a(b)) + \tau(w\mathcal{T}_c(w)) = 0.$$

This completes the proof.  $\square$

*Proof of Theorem A.* — Assume first that  $(a, b, c)$  is a triple of self-adjoint bounded operators in  $\mathcal{M}$ . Set  $\alpha = e^{-a}$ ,  $\beta = e^b$ , and  $v = a + c$ . By the previous theorem,  $x \mapsto -\tau(\exp(v + \ln x))$  is convex on the cone  $\mathcal{M}_{++}$ . We also note that it is homogeneous of order 1. Applying Lemma 3.2 to this convex function, we conclude that

$$\begin{aligned} \tau(\exp(a + b + c)) &= \tau(\exp(v + \ln \beta)) \leq (d/ds)(\tau(\exp(v + \ln(\alpha + s\beta)))|_{s=0} \\ &= \tau(e^c \mathcal{T}_\alpha(\beta)). \end{aligned}$$

This proves Theorem A for the case where all three operators are bounded.

Assume now that  $1 \leq p, q \leq \infty$  with  $1/p + 1/q = 1$ . Let  $(a, b, c)$  be a triplet of self-adjoint operators satisfying  $a \in \mathcal{M}$ ,  $e^b \in L_p(\mathcal{M})$ , and  $e^c \in L_q(\mathcal{M})$ .

Recall that if  $b = \int_{-\infty}^{\infty} \lambda dE_\lambda^{(b)}$  is the spectral decomposition of  $b$  then  $e^b = \int_{-\infty}^{\infty} e^\lambda dE_\lambda^{(b)}$ . Similarly, if  $c = \int_{-\infty}^{\infty} \lambda dE_\lambda^{(c)}$ , then  $e^c = \int_{-\infty}^{\infty} e^\lambda dE_\lambda^{(c)}$ . For  $j \geq 1$ , we set

$$b_j = \int_{-j}^j \lambda dE_\lambda^{(b)} \quad \text{and} \quad c_j = \int_{-j}^j \lambda dE_\lambda^{(c)}.$$

Then, the operators  $b_j$ 's and  $c_j$ 's are bounded and self-adjoint that belong to  $\mathcal{M}$ . Moreover, if  $\pi_j = \chi_{[-j, j]}(b)$  and  $\vartheta_j = \chi_{[-j, j]}(c)$ , then  $b_j = \pi_j b = \pi_j b \pi_j$  and  $c_j = \vartheta_j c = \vartheta_j c \vartheta_j$ . Furthermore,  $\pi_j \uparrow^j \mathbf{1}$  and  $\vartheta_k \uparrow^k \mathbf{1}$  where the convergences are taken with respect to the strong operator topology.

By commutativity, we have for every  $j \geq 1$ ,

$$e^{b_j} = \pi_j e^b \pi_j + (\mathbf{1} - \pi_j) \quad \text{and} \quad e^{c_j} = \vartheta_j e^c \vartheta_j + (\mathbf{1} - \vartheta_j).$$

We begin by stating that since  $\{b_j\}_{j \geq 1}$  and  $\{c_k\}_{k \geq 1}$  are sequences in  $\mathcal{M}$ , the previous case implies that for every  $j \geq 1$  and  $k \geq 1$ , the following holds:

$$\tau(e^{a+b_j+c_k}) \leq \int_0^\infty \tau(e^{c_k/2}(e^{-a} + s\mathbf{1})^{-1}e^{b_j}(e^{-a} + s\mathbf{1})^{-1}e^{c_k/2}) ds.$$

The conclusion will be deduced using limit arguments. We start by estimating the right-hand side of the inequality. For  $j, k \in \mathbb{N}$  and  $s > 0$ , let

$$R_{j,k}(s) = \tau(e^{c_k/2}(e^{-a} + s\mathbf{1})^{-1}e^{b_j}(e^{-a} + s\mathbf{1})^{-1}e^{c_k/2}).$$

From tracial property and commutativity, we have,

$$\begin{aligned}
 R_{j,k}(s) &= \tau(e^{c_k/2}(e^{-a} + s\mathbf{1})^{-1}\pi_j e^b \pi_j (e^{-a} + s\mathbf{1})^{-1}e^{c_k/2}) \\
 &\quad + \tau(e^{c_k/2}(e^{-a} + s\mathbf{1})^{-1}(\mathbf{1} - \pi_j)(e^{-a} + s\mathbf{1})^{-1}e^{c_k/2}) \\
 &\leq \tau(e^{c_k}(e^{-a} + s\mathbf{1})^{-1}e^b(e^{-a} + s\mathbf{1})^{-1}) \\
 &\quad + \tau(e^{c_k}(e^{-a} + s\mathbf{1})^{-1}(\mathbf{1} - \pi_j)(e^{-a} + s\mathbf{1})^{-1}).
 \end{aligned}$$

Splitting further using the index  $k$ ,

$$\begin{aligned}
 R_{j,k}(s) &\leq \tau(\vartheta_k e^c \vartheta_k (e^{-a} + s\mathbf{1})^{-1}e^b(e^{-a} + s\mathbf{1})^{-1}) \\
 &\quad + \tau((\mathbf{1} - \vartheta_k)(e^{-a} + s\mathbf{1})^{-1}e^b(e^{-a} + s\mathbf{1})^{-1}) \\
 &\quad + \tau(\vartheta_k e^c \vartheta_k (e^{-a} + s\mathbf{1})^{-1}(\mathbf{1} - \pi_j)(e^{-a} + s\mathbf{1})^{-1}) \\
 &\quad + \tau((\mathbf{1} - \vartheta_k)(e^{-a} + s\mathbf{1})^{-1}(\mathbf{1} - \pi_j)(e^{-a} + s\mathbf{1})^{-1}) \\
 &\leq \tau(e^c(e^{-a} + s\mathbf{1})^{-1}e^b(e^{-a} + s\mathbf{1})^{-1}) \\
 &\quad + \tau((\mathbf{1} - \vartheta_k)(e^{-a} + s\mathbf{1})^{-1}e^b(e^{-a} + s\mathbf{1})^{-1}) \\
 &\quad + \tau(e^c(e^{-a} + s\mathbf{1})^{-1}(\mathbf{1} - \pi_j)(e^{-a} + s\mathbf{1})^{-1}) \\
 &\quad + \tau((\mathbf{1} - \vartheta_k)(e^{-a} + s\mathbf{1})^{-1}(\mathbf{1} - \pi_j)(e^{-a} + s\mathbf{1})^{-1}).
 \end{aligned}$$

Thus, taking integrals, we have for every  $j, k \geq 1$ ,

$$\begin{aligned}
 (3.12) \quad \tau(e^{a+b_j+c_k}) &\leq \int_0^\infty \tau(e^{c/2}(e^{-a} + s\mathbf{1})^{-1}e^b(e^{-a} + s\mathbf{1})^{-1}e^{c/2}) \, ds \\
 &\quad + \tau((\mathbf{1} - \vartheta_k)\mathcal{T}_{e^{-a}}(e^b)) + \tau((\mathbf{1} - \pi_j)\mathcal{T}_{e^{-a}}(e^c)) \\
 &\quad + \tau((\mathbf{1} - \vartheta_k)\mathcal{T}_{e^{-a}}(\mathbf{1} - \pi_j)).
 \end{aligned}$$

Fix  $k \geq 1$ . From the boundedness of  $\mathcal{T}_{e^{-a}}$  on  $L_r(\mathcal{M})$  for  $1 \leq r \leq \infty$ , we see that  $\lim_{j \rightarrow \infty} \tau((\mathbf{1} - \pi_j)\mathcal{T}_{e^{-a}}(e^c)) = 0$ . Similarly, we have  $\lim_{j \rightarrow \infty} \tau((\mathbf{1} - \vartheta_k)\mathcal{T}_{e^{-a}}(\mathbf{1} - \pi_j)) = 0$ . Thus, the right hand side of (3.12) converges to  $\int_0^\infty \tau(e^{c/2}(e^{-a} + s\mathbf{1})^{-1}e^b(e^{-a} + s\mathbf{1})^{-1}e^{c/2}) \, ds + \tau((\mathbf{1} - \vartheta_k)\mathcal{T}_{e^{-a}}(e^b))$  when  $j \rightarrow \infty$ .

On the other hand, for the left hand side of (3.12), consider the sequence of self-adjoint operators  $(a + b_j + c_k)_{j \geq 1}$ . We have  $\lim_{j \rightarrow \infty} a + b_j + c_k = a + b + c_k$  for the measure topology. As  $t \mapsto e^t$  is a continuous function, it follows from Lemma 2.1 that  $\lim_{j \rightarrow \infty} e^{a+b_j+c_k} = e^{a+b+c_k}$  for the measure topology. Using [7, Theorem 3.5], we have,

$$\tau(e^{a+b+c_k}) \leq \varliminf_{j \rightarrow \infty} \tau(e^{a+b_j+c_k}).$$

Applying these facts to (3.12), we obtain that for  $k \geq 1$  fixed,

$$\begin{aligned} \tau(e^{a+b+c_k}) &\leq \int_0^\infty \tau(e^{c/2}(e^{-a} + s\mathbf{1})^{-1}e^b(e^{-a} + s\mathbf{1})^{-1}e^{c/2}) \, ds \\ &\quad + \tau((1 - \vartheta_k)\mathcal{T}_{e^{-a}}(e^b)). \end{aligned}$$

Repeating the same argument with the index  $k$ , we arrive at the desired conclusion.  $\square$

The assumptions in the statement of Theorem A were used to insure that the right hand side of the inequality is finite. This can be extended by considering more general noncommutative spaces.

Denote by  $L_0$ , the space of Lebesgue measurable functions on the interval  $[0, 1]$ . A Banach function space  $(E, \|\cdot\|_E)$  of measurable functions on the interval  $[0, 1]$  is called *symmetric* if for any  $g \in E$  and any  $f \in L_0$  with  $\mu(f) \leq \mu(g)$ , we have  $f \in E$  and  $\|f\|_E \leq \|g\|_E$ . For such symmetric Banach function space, we define the corresponding noncommutative space by setting:

$$E(\mathcal{M}, \tau) = \left\{ x \in \widetilde{\mathcal{M}} : \mu(x) \in E \right\}.$$

Equipped with the norm  $\|x\|_{E(\mathcal{M}, \tau)} := \|\mu(x)\|_E$ , the linear space  $E(\mathcal{M}, \tau)$  becomes a complex Banach space ([22, 42, 49]) and is usually referred to as the *noncommutative symmetric space* associated with  $(\mathcal{M}, \tau)$  corresponding to  $E$ . We recall that a symmetric Banach function space  $E$  is said to have the *Fatou property* if, whenever  $0 \leq f_\alpha \uparrow_\alpha \subseteq E$  is an upwards directed net with  $\sup_\alpha \|f_\alpha\|_E < \infty$ , it follows that  $f = \sup_\alpha f_\alpha$  exists in  $E$  and  $\|f\|_E = \sup_\alpha \|f_\alpha\|_E$ .

The *Köthe dual* of a symmetric space  $E$  is the function space defined by setting:

$$E^\times = \left\{ f \in L_0 : \int_0^1 |f(t)g(t)| \, dt < \infty, \forall g \in E \right\}.$$

When equipped with the norm

$$\|f\|_{E^\times} := \sup \left\{ \int_0^1 |f(t)g(t)| \, dt : \|g\|_E \leq 1 \right\},$$

$E^\times$  is a symmetric Banach function space. If the symmetric Banach space  $E$  is separable then  $E^* = E^\times$ .

We note that if  $1 \leq p \leq \infty$  and  $E = L_p$ , then  $E(\mathcal{M}, \tau)$  is exactly the space  $L_p(\mathcal{M}, \tau)$  associated with the pair  $(\mathcal{M}, \tau)$ . Extensive discussions on the various properties of the noncommutative spaces  $E(\mathcal{M}, \tau)$  can be

found in [6, 32, 49]. The important fact we need is that if  $x \in E(\mathcal{M}, \tau)$  and  $y \in E^\times(\mathcal{M}, \tau)$ , then

$$\tau(xy) \leq \|x\|_{E(\mathcal{M}, \tau)} \cdot \|y\|_{E^\times(\mathcal{M}, \tau)}.$$

Using this fact, the argument in the preceding  $L_p$ -case can be readily adjusted to provide a slight improvement of the assumptions used in Theorem A for the context of symmetric spaces which we formulate in the next statement.

*Remark 3.12.* — Let  $E$  be a symmetric Banach function space on  $[0, 1]$  with the Fatou property. Assume that  $a$ ,  $b$ , and  $c$  are self-adjoint operators satisfying:  $a \in \mathcal{M}$ ,  $e^b \in E(\mathcal{M}, \tau)$ , and  $e^c \in E^\times(\mathcal{M}, \tau)$ . The following inequality holds:

$$(3.13) \quad \tau(e^{a+b+c}) \leq \int_0^\infty \tau(e^{c/2}(e^{-a} + t\mathbf{1})^{-1}e^b(e^{-a} + t\mathbf{1})^{-1}e^{c/2}) dt.$$

As an example, let  $\Phi$  be an Orlicz function on  $[0, \infty)$  in the sense that  $\Phi$  is a continuous convex function satisfying  $\Phi(0) = 0$  and  $\lim_{t \rightarrow \infty} \Phi(t) = \infty$ . We also assume that  $\Phi$  satisfies the  $\Delta_2$ -condition. That is, for some  $C > 0$ ,

$$\Phi(2t) \leq C\Phi(t), \quad t \geq 0.$$

Denote by  $\Phi^*$  the Orlicz complementary to  $\Phi$ . We recall that the Orlicz function space  $L_\Phi[0, 1]$  (or simply  $L_\Phi$ ) is the set of all Lebesgue measurable functions  $f$  defined on  $[0, 1]$  such that for some constant  $c > 0$ ,

$$\int_0^1 \Phi(|f(t)|/c) dt < \infty.$$

If we equip  $L_\Phi$  with the Luxemburg norm

$$\|f\|_{L_\Phi} = \inf \left\{ c > 0 : \int_0^1 \Phi(|f(t)|/c) dt \leq 1 \right\},$$

then  $L_\Phi$  is a symmetric Banach function space on  $[0, 1]$ . We also have  $(L_\Phi)^\times = L_{\Phi^*}$ . Under the above condition, a  $\tau$ -measurable operator  $x$  belongs to  $L_\Phi(\mathcal{M}, \tau)$  if and only if  $\tau(\Phi(|x|)) < \infty$ . We refer to [35] for all these facts. We see that (3.13) applies to any triplet of self-adjoint operators  $(a, b, c)$  satisfying  $a \in \mathcal{M}$ ,  $\Phi(e^b) \in L_1(\mathcal{M})$ , and  $\Phi^*(e^c) \in L_1(\mathcal{M})$ .

We take the opportunity to show that the idea used in the proof of Proposition 3.8 can be adapted to prove another convexity result related to the WYDL-concavity. The following result extends [25, Theorem 2] to the case of finite von Neumann algebra.



THEOREM 3.13. — Assume that  $q \geq 0$  and  $r \geq 0$  with  $q + r = 1$ . The function  $F : \mathcal{M}_{++} \times \mathcal{M} \rightarrow \mathbb{R}_+$  defined by

$$F(a, x) = \tau(a^{-q} x^* a^{-r} x)$$

is jointly convex.

*Sketch of the proof.* — Let  $a, b \in \mathcal{M}_{++}$ ,  $x, y \in \mathcal{M}$ , and  $0 < \lambda < 1$  be fixed. Set  $c = \lambda a + (1 - \lambda)b$  and  $z = \lambda x + (1 - \lambda)y$ . We need to verify that

$$F(c, z) \leq \lambda F(a, x) + (1 - \lambda)F(b, y).$$

As in Proposition 3.8, it suffices to verify the above inequality for  $x, y \in B_{\mathcal{M}_h}$ . We only outline the argument as it is similar to the proof of Proposition 3.8.

For  $(x, y) \in B_{\mathcal{M}_h} \times B_{\mathcal{M}_h}$ , let

$$D(x, y) := \lambda F(a, x) + (1 - \lambda)F(b, y) \quad \text{and} \quad N(x, y) := F(c, z).$$

For  $\varepsilon > 0$ , we may define the function

$$\psi_\varepsilon(x, y) = \frac{N(x, y)}{\varepsilon + D(x, y)}, \quad x, y \in B_{\mathcal{M}_h}.$$

We estimate the supremum of the function  $\psi_\varepsilon(x, y)$  over all  $x, y \in B_{\mathcal{M}_h}$ . If we consider the ultrapower version  $\psi_{\varepsilon, \mathcal{U}}$  of  $\psi_\varepsilon$  as in the proof of Proposition 3.8, then one can show that the supremum of  $\psi_{\varepsilon, \mathcal{U}}$  is attained at some  $(\tilde{x}, \tilde{y}) \in B_{\mathcal{M}_h^\omega} \times B_{\mathcal{M}_h^\omega}$ . Let

$$\gamma := \psi_{\varepsilon, \mathcal{U}}(\tilde{x}, \tilde{y}) = \sup \{ \psi_\varepsilon(x, y) : x, y \in B_{\mathcal{M}_h} \}.$$

Repeating the argument used in the proof of Proposition 3.8 based on computing directional derivatives at  $(\tilde{x}, \tilde{y})$ , we deduce that if  $(x_n)_{n \in \mathbb{N}}$  and  $(y_n)_{n \in \mathbb{N}}$  are in the equivalent classes of  $\tilde{x}$  and  $\tilde{y}$  respectively and  $(z_n)_{n \in \mathbb{N}} = \lambda(x_n)_{n \in \mathbb{N}} + (1 - \lambda)(y_n)_{n \in \mathbb{N}}$ , then we have:

$$\begin{cases} \gamma(a^{-r} x_n a^{-q})_{n \in \mathbb{N}}^\bullet = (c^{-r} z_n c^{-q})_{n \in \mathbb{N}}^\bullet, \\ \gamma(b^{-r} y_n b^{-q})_{n \in \mathbb{N}}^\bullet = (c^{-r} z_n c^{-q})_{n \in \mathbb{N}}^\bullet. \end{cases}$$

For  $n \in \mathbb{N}$ , let  $w_n := c^{-r} z_n c^{-q}$ . Then,

$$\tilde{x} = \gamma^{-1}(a^r w_n a^q)_{n \in \mathbb{N}}^\bullet \quad \text{and} \quad \tilde{y} = \gamma^{-1}(b^r w_n b^q)_{n \in \mathbb{N}}^\bullet.$$

We see that

$$\gamma(c^r w_n c^q)_{n \in \mathbb{N}}^\bullet = \lambda(a^r w_n a^q)_{n \in \mathbb{N}}^\bullet + (1 - \lambda)(b^r w_n b^q)_{n \in \mathbb{N}}^\bullet.$$

Multiplying the preceding equation on the right by  $(w_n)_{n \in \mathbb{N}}^\bullet$  and taking traces on both sides, we obtain

$$\gamma \lim_{n, \mathcal{U}} \tau(c^r w_n c^q w_n) = \lim_{n, \mathcal{U}} [\lambda \tau(a^r w_n a^q w_n) + (1 - \lambda) \tau(b^r w_n b^q w_n)].$$

By the concavity of the function in (3.1), we deduce that

$$(1 - \gamma) \lim_{n, \mathcal{U}} \tau(c^r w_n c^q w_n) \geq 0.$$

Note that for every  $n \geq 1$ ,

$$\tau(c^r w_n c^q w_n) = \tau(z_n c^{-r} z_n c^{-q}) = \|c^{-r/2} z_n c^{-q/2}\|_2^2.$$

Since  $\tilde{z} \neq 0$  and the linear operator  $x \mapsto c^{-r/2} x c^{-q/2}$  is invertible in  $L_2(\mathcal{M})$ , we have that  $\lim_{n, \mathcal{U}} \tau(c^r w_n c^q w_n) > 0$ . Consequently,  $\gamma \leq 1$ . As  $\gamma$  is the supremum of  $\psi_\varepsilon(\cdot, \cdot)$  and  $\varepsilon$  is arbitrary, we arrive at

$$F(c, z) \leq \lambda F(a, x) + (1 - \lambda) F(b, y), \quad x, y \in B_{\mathcal{M}_h}.$$

The extension to the general case follows the same argument as in Proposition 3.8 for which we omit the details.  $\square$

For a complete picture of the type of convexity/concavity results in the spirit of Theorem 3.13 (still for the case of matrix algebras), we refer to the recent article [50].

### 3.2. The infinite case

In this subsection, we explore possible generalizations of Theorem A for infinite semifinite von Neumann algebras. The infinite situation is rather delicate since having exponential of measurable operators living inside non-commutative  $L_p$ -spaces is quite restrictive. Nevertheless, despite these obstacles, a version of Golden–Thompson inequality in the spirit of Theorem A subsists in this context. We refer to [40, Chapter 8] for some discussions on Golden–Thompson inequality for trace class operators.

Below, we assume that  $\mathcal{N}$  is a semifinite and  $\sigma$ -finite von Neumann algebra equipped with a fixed semifinite faithful normal trace  $\phi$ . All notion discussed in the preliminary section apply to the semifinite case with obvious adjustments. In particular, convergence in measure and Lemma 2.1 will be used repeatedly. Our result for the infinite case reads as follows:

**PROPOSITION 3.14.** — *Assume that  $a$  and  $c$  are two self-adjoint operators in  $\mathcal{N} \cap L_1(\mathcal{N}, \phi)$  and  $b$  is a self-adjoint  $\phi$ -measurable operator satisfying  $\phi(e^b) < \infty$ . The following inequality holds:*

$$\phi(e^{a+b+c}) \leq \int_0^\infty \phi(e^{c/2}(e^{-a} + t\mathbf{1})^{-1} e^b (e^{-a} + t\mathbf{1})^{-1} e^{c/2}) dt.$$

Before we proceed, we should observe first that the condition  $\phi(e^c) < \infty$  can only occur when the von Neumann algebra  $\mathcal{N}$  is  $\sigma$ -finite and thus this extra assumption on  $\mathcal{N}$  cannot be omitted. Second, one can repeat the simple argument presented at the beginning of the previous subsection to show that under the given assumptions the right hand side of the inequality in the statement of Proposition 3.14 is a finite number.

Below, we will use the notion of submajorization in the sense of Hardy, Littlewood, and Polya which we now review for convenience. If  $x, y \in \tilde{\mathcal{N}}$  then  $x$  is said to be *submajorized* by  $y$  if for every  $t > 0$ , the inequality

$$\int_0^t \mu_s(x) \, ds \leq \int_0^t \mu_s(y) \, ds$$

holds. In this case, we will write  $x \prec\prec y$ . We will use two submajorization inequalities which can be found for instance in [7, Theorem 4.2(ii) and Theorem 4.4(ii)]. For  $x, y \in \tilde{\mathcal{N}}$ ,

$$(3.14) \quad \mu(x + y) \prec\prec \mu(x) + \mu(y).$$

and

$$(3.15) \quad \mu(xy) \prec\prec \mu(x)\mu(y).$$

Our argument for Proposition 3.14 is based on reduction to the finite case. Since we will be working with some von Neumann subalgebras of  $\mathcal{N}$  with potentially having different unit, some care needs to be taken when dealing with exponentials of operators. For a given self-adjoint  $\phi$ -measurable operator, the notation  $e^x$  is reserved for exponential of  $x$  taken with respect to the von Neumann algebra  $\mathcal{N}$ .

Let  $f$  be a nonzero projection in  $\mathcal{N}$  with  $\phi(f) < \infty$ . Consider the von Neumann algebra  $f\mathcal{N}f$  equipped with the finite normal trace  $\phi(f \cdot f)$ . If  $x$  is a  $\phi$ -measurable operator then the operator  $fxf$  is measurable with respect to the trace  $\phi(f \cdot f)$  and one can easily see that the exponential of  $fxf$  as an operator affiliated with the von Neumann algebra  $f\mathcal{N}f$  is given by  $fe^{fxf}f$ . Our starting point is the following inequality from the finite case which when taking into account the forms of exponentials in  $f\mathcal{N}f$  reads as follows:

$$(3.16) \quad \begin{aligned} & \phi(fe^{f(a+b+c)f}f) \\ & \leq \int_0^\infty \phi(fe^{(fcf)/2}f(fe^{-faf}f + sf)^{-1} \\ & \quad \times fe^{fbf}f(fe^{-faf}f + sf)^{-1}fe^{(fcf)/2}f) \, ds \end{aligned}$$

where the inverse  $(fe^{-faf}f + sf)^{-1}$  is taken in the finite von Neumann algebra  $f\mathcal{N}f$ .

Proposition 3.14 will be deduced through some approximation process. For  $n \geq 1$ , set

$$f_n := \chi_{(-n, \infty)}(b).$$

Clearly,  $f_n \uparrow \mathbf{1}$ . We also note that  $\tau(f_n) < \infty$ . Indeed, by Chebychev inequality,  $\phi(f_n) \leq e^n \phi(e^b) < \infty$ .

Moreover, since  $b$  commutes with  $f_n$ , we have  $f_n e^{f_n b f_n} f_n \leq e^b$ . We then obtain from (3.16) and tracial property that for every  $n \geq 1$ ,

$$\begin{aligned} & \phi(f_n e^{f_n(a+b+c)f_n} f_n) \\ & \leq \int_0^\infty \phi([(f_n e^{-f_n a f_n} f_n + s f_n)^{-1} (f_n e^{(f_n c f_n)/2} f_n)^2 \\ & \quad \times (f_n e^{-f_n a f_n} f_n + s f_n)^{-1}] e^b) ds \end{aligned}$$

where as before, the inverse  $(f_n e^{-f_n a f_n} f_n + s f_n)^{-1}$  is taken in  $f_n \mathcal{N} f_n$ . Recall that  $f_n e^{(f_n c f_n)/2} f_n$  is the exponential of  $(f_n c f_n)/2$  in  $f_n \mathcal{N} f_n$ . Therefore, its square is the exponential of  $f_n c f_n$  in  $f_n \mathcal{N} f_n$  which is  $f_n e^{f_n c f_n} f_n$ . With this observation, we may state that:

$$\begin{aligned} (3.17) \quad & \phi(f_n e^{f_n(a+b+c)f_n} f_n) \\ & \leq \int_0^\infty \phi([(f_n e^{-f_n a f_n} f_n + s f_n)^{-1} f_n e^{f_n c f_n} f_n \\ & \quad \times (f_n e^{-f_n a f_n} f_n + s f_n)^{-1}] e^b) ds. \end{aligned}$$

For  $s > 0$  and  $n \in \mathbb{N}$ , let

$$A_n(s) := \phi([(f_n e^{-f_n a f_n} f_n + s f_n)^{-1} f_n e^{f_n c f_n} f_n (f_n e^{-f_n a f_n} f_n + s f_n)^{-1}] e^b)$$

and

$$A(s) := \phi([(e^{-a} + s\mathbf{1})^{-1} e^c (e^{-a} + s\mathbf{1})^{-1}] e^b).$$

We will prove the following result which will handle the right hand side of the inequality in Proposition 3.14.

LEMMA 3.15. — *The sequence of functions  $(A_n(\cdot))$  satisfies:*

$$\lim_{n \rightarrow \infty} \int_0^\infty |A_n(s) - A(s)| ds = 0.$$

Consequently,  $\lim_{n \rightarrow \infty} \int_0^\infty A_n(s) ds = \int_0^\infty A(s) ds$ .

*Proof.* — For the proof, we need to perform some basic but tedious computations. Fix  $n \geq 1$ . Below, we keep in mind that  $(f_n e^{-f_n a f_n} f_n)^{-1}$  is taken in  $f_n \mathcal{N} f_n$  while  $(e^{-a} + s\mathbf{1})^{-1}$  is taken in  $\mathcal{N}$ . We begin by writing:

$$\begin{aligned} \Phi_n(s) &= (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} f_n e^{f_n c f_n} f_n (e^{-f_n a f_n} f_n + s f_n)^{-1} \\ &= (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} e^{f_n c f_n} (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} \\ &= (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} [e^{f_n c f_n} - e^c] (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} \\ &\quad + (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} e^c (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} \\ &= I_n(s) + (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} e^c (f_n e^{-f_n a f_n} f_n + s f_n)^{-1}. \end{aligned}$$

Next, we split the second term in the last line above as follows:

$$\begin{aligned} &(f_n e^{-f_n a f_n} f_n + s f_n)^{-1} e^c (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} \\ &= [(f_n e^{-f_n a f_n} f_n + s f_n)^{-1} - (e^{-a} + s\mathbf{1})^{-1}] e^c (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} \\ &\quad + (e^{-a} + s\mathbf{1})^{-1} e^c [(f_n e^{-f_n a f_n} f_n + s f_n)^{-1} - (e^{-a} + s\mathbf{1})^{-1}] \\ &\quad + (e^{-a} + s\mathbf{1})^{-1} e^c (e^{-a} + s\mathbf{1})^{-1} \\ &= II_n(s) + III_n(s) + (e^{-a} + s\mathbf{1})^{-1} e^c (e^{-a} + s\mathbf{1})^{-1}. \end{aligned}$$

Then, we have for every  $s > 0$  and  $n \geq 1$ ,

$$(3.18) \quad A_n(s) - A(s) = \phi(I_n(s)e^b) + \phi(II_n(s)e^b) + \phi(III_n(s)e^b).$$

We will verify separately that the sequences of integrals of the absolute values of the three terms on the right hand side each converges to zero.

**SUBLEMMA 3.16.** —  $\lim_{n \rightarrow \infty} \int_0^\infty |\phi(I_n(s)e^b)| ds = 0$ .

*Proof.* — Recall that for  $n \in \mathbb{N}$  and  $s > 0$ ,

$$I_n(s) = (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} [e^{f_n c f_n} - e^c] (f_n e^{-f_n a f_n} f_n + s f_n)^{-1}.$$

We note that  $f_n e^{-f_n a f_n} f_n$  is the exponential of  $-f_n a f_n$  in  $f_n \mathcal{N} f_n$  thus, its inverse is the operator  $f_n e^{f_n a f_n} f_n$ . Since  $c \in L_1(\mathcal{N})$  by assumption, we have  $\lim_{n \rightarrow \infty} \|f_n c f_n - c\|_1 = 0$ . A fortiori,  $f_n c f_n \rightarrow c$  for the measure topology. As  $t \mapsto e^t$  is continuous, it follows from Lemma 2.1 that  $e^{f_n c f_n} \rightarrow e^c$  for the measure topology. We begin with the following estimate for  $n \geq 1$ :

$$\int_0^\infty |\phi(I_n(s)e^b)| ds \leq \int_0^\infty \|I_n(s)e^b\|_1 ds = \int_0^\infty \int_0^\infty \mu_t(I_n(s)e^b) dt ds.$$

By (3.15),  $\mu(I_n(s)e^b) \prec \prec \mu(I_n(s)).\mu(e^b)$ . We further get:

$$\int_0^\infty |\phi(I_n(s)e^b)| ds \leq \int_0^\infty \int_0^\infty \mu_t(I_n(s)) \mu_t(e^b) dt ds.$$

By properties of singular values, we have

$$\mu_t(I_n(s)) \leq \| (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} \|_\infty^2 \mu_t(e^{f_n c f_n} - e^c).$$

It follows that

$$\begin{aligned} & \int_0^\infty |\phi(I_n(s)e^b)| \, ds \\ & \leq \int_0^\infty \int_0^\infty \| (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} \|_\infty^2 \mu_t(e^{f_n c f_n} - e^c) \mu_t(e^b) \, dt \, ds \\ & = \int_0^\infty \| (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} \|_\infty^2 \, ds \cdot \int_0^\infty \mu_t(e^{f_n c f_n} - e^c) \mu_t(e^b) \, dt. \end{aligned}$$

As already used earlier, we may estimate the first integral by

$$\int_0^\infty \| (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} \|_\infty^2 \, ds \leq 1 + e^{2\|f_n a f_n\|_\infty} \leq 1 + e^{2\|a\|_\infty}.$$

On the other hand, since for every  $t > 0$ ,  $\mu_t(e^{f_n c f_n} - e^c) \rightarrow 0$  by the convergence in measure and for every  $n \geq 1$ ,  $\mu_t(e^{f_n c f_n} - e^c) \mu_t(e^b) \leq \|e^{f_n c f_n} - e^c\|_\infty \mu_t(b) \leq 2e^{\|c\|_\infty} \mu_t(e^b)$ , it follows from the Lebesgue dominated convergence theorem that  $\int_0^\infty \mu_t(e^{f_n c f_n} - e^c) \mu_t(e^b) \, dt \rightarrow 0$ . These clearly imply that  $\lim_{n \rightarrow \infty} \int_0^\infty |\phi(I_n(s)e^b)| \, ds = 0$ .  $\square$

**SUBLEMMA 3.17.** —  $\lim_{n \rightarrow \infty} \int_0^\infty |\phi(II_n(s)e^b)| \, ds = 0$ . Similarly, we have  $\lim_{n \rightarrow \infty} \int_0^\infty |\phi(III_n(s)e^b)| \, ds = 0$ .

*Proof.* — Since the proofs for the two integrals are very similar, we will only present the first one and leave the details for the second one to the reader. We begin by recalling that

$$II_n(s) = [(f_n e^{-f_n a f_n} f_n + s f_n)^{-1} - (e^{-a} + s \mathbf{1})^{-1}] e^c (f_n e^{-f_n a f_n} f_n + s f_n)^{-1}.$$

Then, by writing

$$\begin{aligned} & (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} - (e^{-a} + s \mathbf{1})^{-1} \\ & = (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} - f_n (e^{-a} + s \mathbf{1})^{-1} - (\mathbf{1} - f_n) (e^{-a} + s \mathbf{1})^{-1}, \end{aligned}$$

we see from tracial property that

$$\begin{aligned} \phi(II_n(s)e^b) & = \phi(II'_n(s)e^b) \\ & \quad - \phi((e^{-a} + s \mathbf{1})^{-1} e^c (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} e^b (\mathbf{1} - f_n)) \end{aligned}$$

where  $II'_n(s) = [(f_n e^{-f_n a f_n} f_n + s f_n)^{-1} - f_n (e^{-a} + s \mathbf{1})^{-1}] e^c (f_n e^{-f_n a f_n} f_n + s f_n)^{-1}$ . We can estimate the integral of the absolute value of the second

term as follows:

$$\begin{aligned}
& \int_0^\infty |\phi((e^{-a} + s\mathbf{1})^{-1} e^c (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} e^b (\mathbf{1} - f_n))| ds \\
& \leq \int_0^\infty \|(e^{-a} + s\mathbf{1})^{-1}\|_\infty e^{\|c\|_\infty} \|(f_n e^{-f_n a f_n} f_n + s f_n)^{-1}\|_\infty \|e^b (\mathbf{1} - f_n)\|_1 ds \\
& \leq (1 + e^{2\|a\|_\infty}) e^{\|c\|_\infty} \|e^b (\mathbf{1} - f_n)\|_1.
\end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \|e^b (\mathbf{1} - f_n)\|_1 = 0$ , it follows that

$$\lim_{n \rightarrow \infty} \int_0^\infty |\phi((e^{-a} + s\mathbf{1})^{-1} e^c (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} e^b (\mathbf{1} - f_n))| ds = 0.$$

Thus, it remains to prove that  $\lim_{n \rightarrow \infty} \int_0^\infty |\phi(H'_n(s) e^b)| ds = 0$ . For this, we note that

$$\begin{aligned}
& (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} - f_n (e^{-a} + s\mathbf{1})^{-1} \\
& = (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} [(e^{-a} + s\mathbf{1}) - (f_n e^{-f_n a f_n} f_n + s f_n)] (e^{-a} + s\mathbf{1})^{-1} \\
& = (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} [f_n (e^{-a} + s\mathbf{1}) - (f_n e^{-f_n a f_n} f_n + s f_n)] (e^{-a} + s\mathbf{1})^{-1} \\
& = (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} [f_n e^{-a} - f_n e^{-f_n a f_n} f_n] (e^{-a} + s\mathbf{1})^{-1} \\
& = (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} [f_n e^{-a} (\mathbf{1} - f_n)] (e^{-a} + s\mathbf{1})^{-1} \\
& \quad + (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} [f_n e^{-a} f_n - f_n e^{-f_n a f_n} f_n] (e^{-a} + s\mathbf{1})^{-1}.
\end{aligned}$$

Observing that  $f_n e^{-a} (\mathbf{1} - f_n) = f_n (e^{-a} - \mathbf{1}) (\mathbf{1} - f_n)$ , we get that

$$\begin{aligned}
& (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} - f_n (e^{-a} + s\mathbf{1})^{-1} \\
& = (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} [f_n (e^{-a} - \mathbf{1}) (\mathbf{1} - f_n)] (e^{-a} + s\mathbf{1})^{-1} \\
& \quad + (f_n e^{-f_n a f_n} f_n + s f_n)^{-1} [f_n e^{-a} f_n - f_n e^{-f_n a f_n} f_n] (e^{-a} + s\mathbf{1})^{-1}.
\end{aligned}$$

Using  $\|(e^{-a} + s\mathbf{1})^{-1} e^c\|_\infty \leq e^{\|a\|_\infty} e^{\|c\|_\infty}$ , we have from properties of singular values stated in (3.14) and (3.15) that for  $t > 0$ ,

$$\begin{aligned}
& \mu_t(H'_n(s) e^b) \\
& \prec\prec e^{\|a\|_\infty + \|c\|_\infty} \|(f_n e^{-f_n a f_n} f_n + s f_n)^{-1}\|_\infty^2 \mu_t(f_n (e^{-a} - \mathbf{1}) (\mathbf{1} - f_n)) \mu_t(e^b) \\
& \quad + e^{\|a\|_\infty + \|c\|_\infty} \|(f_n e^{-f_n a f_n} f_n + s f_n)^{-1}\|_\infty^2 \mu_t(f_n e^{-a} f_n - f_n e^{-f_n a f_n} f_n) \mu_t(e^b) \\
& \prec\prec e^{\|a\|_\infty + \|c\|_\infty} \|(f_n e^{-f_n a f_n} f_n + s f_n)^{-1}\|_\infty^2 \mu_t((e^{-a} - \mathbf{1}) (\mathbf{1} - f_n)) \mu_t(e^b) \\
& \quad + e^{\|a\|_\infty + \|c\|_\infty} \|(f_n e^{-f_n a f_n} f_n + s f_n)^{-1}\|_\infty^2 \mu_t(e^{-a} - e^{-f_n a f_n}) \mu_t(e^b).
\end{aligned}$$

We also recall that  $\int_0^\infty \|(f_n e^{-f_n a f_n} f_n + s f_n)^{-1}\|_\infty^2 ds \leq 1 + e^{2\|a\|_\infty}$ . Using these estimates, we have:

$$\begin{aligned} & \int_0^\infty |\phi(I_n'(s)e^b)| ds \\ & \leq \int_0^\infty \|I_n'(s)e^b\|_1 ds \\ & \leq e^{\|a\|_\infty + \|c\|_\infty} (1 + e^{2\|a\|_\infty}) \int_0^\infty \mu_t((e^{-a} - \mathbf{1})(\mathbf{1} - f_n)) \mu_t(e^b) dt \\ & \quad + e^{\|a\|_\infty + \|c\|_\infty} (1 + e^{2\|a\|_\infty}) \int_0^\infty \mu_t(e^{-a} - e^{-f_n a f_n}) \mu_t(e^b) dt. \end{aligned}$$

Since  $a \in L_1(\mathcal{N}) \cap \mathcal{N}$ , it follows that  $e^{-a} - \mathbf{1} \in L_1(\mathcal{N})$ . Consequently,  $\lim_{n \rightarrow \infty} \|(e^{-a} - \mathbf{1})(\mathbf{1} - f_n)\|_1 = 0$ . A fortiori,  $(e^{-a} - \mathbf{1})(\mathbf{1} - f_n) \rightarrow 0$  for the measure topology. It implies in particular that  $\mu_t((e^{-a} - \mathbf{1})(\mathbf{1} - f_n)) \rightarrow 0$  for all  $t > 0$ .

On the other hand, as already outlined in the proof of the previous sublemma for the operator  $c$ , we also have  $e^{-f_n a f_n} \rightarrow e^{-a}$  for the measure topology and therefore  $\lim_{n \rightarrow \infty} \mu_t(e^{-a} - e^{-f_n a f_n}) = 0$  for all  $t > 0$ . We may now deduce the desired conclusion from the Lebesgue dominated convergence theorem.  $\square$

By combining (3.18), Sublemma 3.16, and Sublemma 3.17, we clearly have that  $\lim_{n \rightarrow \infty} \int_0^\infty |A_n(s) - A(s)| ds = 0$  and therefore Lemma 3.15 is verified.  $\square$

We now proceed with the conclusion of the proof of Proposition 3.14.

*End of proof of Proposition 3.14.* — We combine (3.16) and Lemma 3.15 to deduce that

$$\begin{aligned} (3.19) \quad & \overline{\lim}_{n \rightarrow \infty} \phi(f_n e^{f_n(a+b+c)f_n} f_n) \\ & \leq \int_0^\infty \phi(e^{c/2}(e^{-a} + s\mathbf{1})^{-1} e^b (e^{-a} + s\mathbf{1})^{-1} e^{c/2}) ds. \end{aligned}$$

We observe first that  $e^{a+b+c} \in L_1(\mathcal{N}, \phi)$ . This follows from the GT-inequality for two operators. Indeed,

$$\phi(e^{a+b+c}) \leq \phi(e^{a+c} e^b) \leq e^{\|a+c\|_\infty} \phi(e^b) < \infty.$$

Let  $\varepsilon > 0$ , since  $\mathbf{1} - f_n \downarrow 0$ , we may choose  $n_0 \geq 1$  so that  $\phi((\mathbf{1} - f_{n_0})e^{a+b+c}) < \varepsilon$ . This implies that

$$\phi(e^{a+b+c}) - \varepsilon \leq \phi(f_{n_0} e^{a+b+c} f_{n_0}).$$



As already observed earlier,  $f_n(a+c)f_n$  converges to  $a+c$  for the measure topology. On the other hand, from the definition of  $f_n$ ,  $f_nb f_n = b f_n = b\chi_{(-n,\infty)}(b)$  which clearly converges to  $b$  for the measure topology. Thus, we have  $f_n(a+b+c)f_n$  converges to  $a+b+c$  which implies that  $e^{f_n(a+b+c)f_n} \rightarrow e^{a+b+c}$  for the measure topology. It follows that  $f_{n_0}e^{f_n(a+b+c)f_n}f_{n_0} \rightarrow f_{n_0}e^{a+b+c}f_{n_0}$  for the measure topology. By [7, Theorem 3.5], we have

$$\phi(f_{n_0}e^{a+b+c}f_{n_0}) \leq \varliminf_{n \rightarrow \infty} \phi(f_{n_0}e^{f_n(a+b+c)f_n}f_{n_0}).$$

On the other hand, we have for  $n \geq n_0$  that  $\phi(f_{n_0}e^{f_n(a+b+c)f_n}f_{n_0}) \leq \phi(f_n e^{f_n(a+b+c)f_n} f_n)$ . Combining these series of estimates with (3.19), we arrive at

$$\phi(e^{a+b+c}) - \varepsilon \leq \int_0^\infty \phi(e^{c/2}(e^{-a} + s\mathbf{1})^{-1}e^b(e^{-a} + s\mathbf{1})^{-1}e^{c/2}) \, ds.$$

Since  $\varepsilon$  is arbitrary, this completes the proof of Proposition 3.14.  $\square$

We conclude this subsection with a remark on the assumptions imposed in Proposition 3.14. The extra assumption that  $a, c \in L_1(\mathcal{N}, \phi)$  was used in the proof to insure that  $f_n a f_n \rightarrow a$  and  $f_n c f_n \rightarrow c$  for the measure topology. This clearly can be relaxed to  $a, c \in \mathcal{N} \cap E(\mathcal{N}, \phi)$  for  $E$  being an order continuous symmetric function space on  $(0, \infty)$ . For example,  $a, c \in \mathcal{N} \cap L_p(\mathcal{N}, \phi)$  for some  $1 < p < \infty$  would be sufficient. It is reasonable to assume that Proposition 3.14 remains valid if we merely assume that  $a, c \in \mathcal{N}$  and  $e^b \in L_1(\mathcal{N}, \phi)$ . We leave this as an open problem.

#### 4. Application to noncommutative martingale inequalities

In this section, we will explore an application of Theorem A to noncommutative martingale inequalities. We begin by recalling the general setup for martingales.

Let  $(\mathcal{M}_n)_{n \geq 1}$  be an increasing sequence of von Neumann subalgebras of  $\mathcal{M}$  such that the union of the  $\mathcal{M}_n$ 's is  $w^*$ -dense in  $\mathcal{M}$ . Since  $\mathcal{M}$  is finite, for every  $n \geq 1$ , there exists a  $\tau$ -invariant conditional expectation from  $\mathcal{M}$  onto  $\mathcal{M}_n$  which we denote by  $\mathcal{E}_n$ . It is well-known that the  $\mathcal{E}_n$ 's extend to be contractive projections from  $L_p(\mathcal{M}, \tau)$  onto  $L_p(\mathcal{M}_n, \tau_n)$  for all  $1 \leq p \leq \infty$ , where  $\tau_n$  denotes the restriction of  $\tau$  on  $\mathcal{M}_n$ .

**DEFINITION 4.1.** — *A sequence  $x = (x_n)_{n \geq 1}$  in  $L_1(\mathcal{M})$  is called a noncommutative martingale with respect to the filtration  $(\mathcal{M}_n)_{n \geq 1}$  if for every  $n \geq 1$ ,*

$$\mathcal{E}_n(x_{n+1}) = x_n.$$

If in addition, all the  $x_n$ 's are in  $L_p(\mathcal{M})$  for some  $1 \leq p \leq \infty$ ,  $x$  is called an  $L_p$ -martingale. In this case, we set:

$$\|x\|_p = \sup_{n \geq 1} \|x_n\|_p.$$

If  $\|x\|_p < \infty$ ,  $x$  is called an  $L_p$ -bounded martingale. As customary, we set  $dx_1 = x_1$  and  $dx_n = x_n - x_{n-1}$  for  $n \geq 1$ . The sequence  $dx = (dx_n)_{n \geq 1}$  is called the martingale difference sequence of the martingale  $x$ . The martingale  $(x_n)_{n \geq 1}$  is said to be self-adjoint if  $x_n = x_n^*$  for all  $n \geq 1$ .

We also consider a von Neumann subalgebra  $\mathcal{M}_0$  of  $\mathcal{M}_1$  and denote by  $\mathcal{E}_0 : \mathcal{M} \rightarrow \mathcal{M}_0$  the conditional expectation onto  $\mathcal{M}_0$ . Recall that if  $\mathcal{M}_0 = \mathbb{C}\mathbf{1}$ , then the expectation  $\mathcal{E}_0$  is given by the trace  $\tau(\cdot)\mathbf{1}$ . We need the following concept:

**DEFINITION 4.2.** — *We say that a martingale  $x$  is of mean-zero (relative to  $\mathcal{M}_0$ ) if  $\mathcal{E}_0(x_n) = 0$  for all  $n \geq 1$ .*

Below, we focus on concentration type results for noncommutative self-adjoint mean-zero martingales. We refer to [20, 21, 39] for more information in this direction. Our main goal is to derive a noncommutative analogue of the so-called Prokhorov arcsinh inequality. To formally state the result, we need to recall some notation. For  $n \geq 1$ , we set:

$$\sigma_n(x)^2 = \sum_{k=1}^n \mathcal{E}_{k-1}(|dx_k|^2).$$

Also

$$\sigma(x)^2 = \sum_{k \geq 1} \mathcal{E}_{k-1}(|dx_k|^2).$$

Clearly, for  $n \geq 1$ ,  $\sigma_n(x) \in L_1(\mathcal{M}_{n-1})$ . We should note that there is a slight difference between the quantity  $\sigma(\cdot)$  and the column/row conditioned square functions introduced in [18] and are commonly used in the field. Indeed, if  $(x_n)_{n \geq 1}$  is a mean-zero self-adjoint  $L_2$ -martingale then  $s_c^2(x) = |x_1|^2 + \sigma^2(x)$  where  $s_c(\cdot)$  is the conditioned square function of  $x$ .

Below we use the notation commonly adopted in previous papers dealing with concentration results that for a self-adjoint operator  $x$  and  $t \in \mathbb{R}$ ,

$$\tau(\chi_{(t, \infty)}(x)) = \text{prob}(x > t).$$

The following is the main result of this section which is the noncommutative analogue of the Prokhorov inequality.

THEOREM 4.3. — *Let  $x$  be a self-adjoint mean-zero martingale with*

- (i)  $\|\sigma^2(x)\|_\infty = K^2$ ;
- (ii) *for every  $k \geq 1$ ,  $\|dx_k\|_\infty \leq M$ .*

*Then, for every  $\lambda > 0$ ,*

$$\text{prob}(x_n > \lambda) \leq \exp\left\{\frac{-\lambda}{2M} \operatorname{arcsinh}\left(\frac{M\lambda}{2K^2}\right)\right\}, \quad n \geq 1.$$

*Proof.* — For  $c > 0$ , let

$$y_n := \exp\left\{cx_n - \frac{c}{M} \sinh cM \cdot \sigma_n^2(x)\right\}, \quad n \geq 1.$$

Clearly,  $(y_n)_{n \geq 1}$  is such that for every  $n \geq 1$ ,  $y_n \in \mathcal{M}_n$ . We will also make use of the following sequence of positive operators:

$$z_n := \frac{c}{M} \sinh cM \cdot \sigma_n^2(x), \quad n \geq 1,$$

By assumption, the following operator inequalities hold:

$$(4.1) \quad 0 \leq z_n \leq \left(\frac{cK^2}{M} \sinh cM\right) \mathbf{1}.$$

By Lemma 2.2, we may state that for  $\lambda > 0$  and  $n \geq 1$ ,

$$\begin{aligned} \text{prob}(x_n > \lambda) &= \text{prob}(cx_n > c\lambda) \\ &\leq \text{prob}(cx_n - z_n > c\lambda - cK^2M^{-1} \sinh cM) \\ &\quad + \text{prob}(z_n > cK^2M^{-1} \sinh cM). \end{aligned}$$

We remark from (4.1) that  $\text{prob}(z_n > cK^2M^{-1} \sinh cM) = 0$ . Therefore, we obtain the following initial estimate:

$$\text{prob}(x_n > \lambda) \leq \text{prob}(cx_n - z_n > c\lambda - cK^2M^{-1} \sinh cM).$$

As already observed in the proof of Proposition 3.14, we have a Chebyshev type inequality for exponentials that would lead to

$$\text{prob}(x_n > \lambda) \leq \exp\{-c\lambda + cK^2M^{-1} \sinh cM\} \tau(\exp(cx_n - z_n)).$$

We claim that for every  $n \geq 1$ , the following holds:

$$(4.2) \quad \tau(\exp(cx_n - z_n)) \leq 1.$$

To verify this claim, fix  $n \geq 1$  and write

$$cx_n - z_n = cx_{n-1} - z_{n-1} + cdx_n - \frac{c}{M} \sinh cM \cdot \mathcal{E}_{n-1}|dx_n|^2.$$

Let

$$\alpha_n := -\frac{c}{M} \sinh cM \cdot \mathcal{E}_{n-1}|dx_n|^2, \quad \beta_n := cdx_n, \quad \text{and} \quad \eta_n := cx_{n-1} - z_{n-1}.$$

Clearly,  $cx_n - z_n = \alpha_n + \beta_n + \eta_n$ . Note that all three operators are bounded and self-adjoint. By Theorem A, we have:

$$\begin{aligned} \tau(\exp(cx_n - z_n)) &= \tau(e^{\alpha_n + \beta_n + \eta_n}) \\ &\leq \int_0^\infty \tau(e^{\eta_n}(e^{-\alpha_n} + t\mathbf{1})^{-1}e^{\beta_n}(e^{-\alpha_n} + t\mathbf{1})^{-1}) dt. \end{aligned}$$

Since  $\alpha_n$  and  $\eta_n$  belong to  $\mathcal{M}_{n-1}$ , it follows from the trace invariance of the conditional expectation  $\mathcal{E}_{n-1}$  that

$$(4.3) \quad \tau(\exp(cx_n - z_n)) \leq \int_0^\infty \tau(e^{\eta_n}(e^{-\alpha_n} + t\mathbf{1})^{-1}\mathcal{E}_{n-1}(e^{\beta_n})(e^{-\alpha_n} + t\mathbf{1})^{-1}) dt.$$

The main idea is to show that

$$(4.4) \quad \mathcal{E}_{n-1}(e^{\beta_n}) \leq e^{-\alpha_n}.$$

At this point, we adapt the arguments from the proofs of [13, Proposition 3.1] and [15, Proposition 3.1]. Since for  $s \in \mathbb{R}$ ,  $s \leq e^{s-1}$ , we have by functional calculus that

$$\begin{aligned} \mathcal{E}_{n-1}(e^{\beta_n}) &= \mathcal{E}_{n-1}(e^{cdx_n}) \leq \exp\{\mathcal{E}_{n-1}(e^{cdx_n} - \mathbf{1})\} \\ &= \exp\{\mathcal{E}_{n-1}(e^{cdx_n} - cdx_n - \mathbf{1})\}. \end{aligned}$$

Next, we use the scalar inequalities that  $e^s - s - 1 \leq e^s + e^{-s} - 2 \leq s \sinh s$  to deduce further that

$$\begin{aligned} \mathcal{E}_{n-1}(e^{cdx_n} - cdx_n - \mathbf{1}) &\leq \mathcal{E}_{n-1}(cdx_n \sinh(cdx_n)) \\ &\leq \mathcal{E}_{n-1}(c|dx_n| \sinh(c|dx_n|)) \\ &= \mathcal{E}_{n-1}(c^2|dx_n|^2(c|dx_n|)^{-1} \sinh(c|dx_n|)). \end{aligned}$$

Since the function  $s \mapsto s^{-1} \sinh s$  is increasing for  $s > 0$ , it follows from the assumption that  $(c|dx_n|)^{-1} \sinh(c|dx_n|) \leq c^{-1}M^{-1} \sinh(cM)\mathbf{1}$ . Therefore,

$$\mathcal{E}_{n-1}(e^{cdx_n} - cdx_n - \mathbf{1}) \leq \frac{c}{M} \sinh(cM) \cdot \mathcal{E}_{n-1}(|dx_n|^2) = -\alpha_n.$$

Combining all estimates above clearly gives (4.4). This allows us to conclude from (4.3) and (4.4) that the following estimate holds:

$$\tau(\exp(cx_n - z_n)) \leq \int_0^\infty \tau(e^{\eta_n}(e^{-\alpha_n} + t\mathbf{1})^{-1}e^{-\alpha_n}(e^{-\alpha_n} + t\mathbf{1})^{-1}) dt.$$

The important fact here is that for any self-adjoint operator  $\alpha \in \mathcal{M}$ , we have

$$\int_0^\infty (e^{-\alpha} + t\mathbf{1})^{-1}e^{-\alpha}(e^{-\alpha} + t\mathbf{1})^{-1} dt = \mathbf{1}$$

where as before, the integral is taken in the sense of Bochner ([5]). Indeed, by the Gelfand–Naimark representation, it suffices to verify that the assertion above holds for scalar integrals. That is, for any scalar  $s > 0$ , we need that  $\int_0^\infty \frac{s}{(s+t)^2} dt = 1$  which clearly is the case. With this fact, we get that for every  $n \geq 1$ ,

$$\tau(\exp(cx_n - z_n)) \leq \tau(\exp(cx_{n-1} - z_{n-1})).$$

By iteration, we deduce that for  $n \geq 1$ ,

$$\tau(\exp(cx_n - z_n)) \leq \tau(\exp(cx_1 - z_1)).$$

We observe that for  $n = 1$ , (4.4) is precisely the inequality  $\mathcal{E}_0(e^{cx_1}) \leq e^{z_1}$ . Therefore, from the GT-inequality, we deduce that  $\tau(\exp(cx_1 - z_1)) \leq \tau(e^{cx_1} e^{-z_1}) = \tau(\mathcal{E}_0(e^{cx_1}) e^{-z_1}) \leq 1$ . This proves (4.2).

In turn, (4.2) leads to the estimate:

$$\text{prob}(x_n > \lambda) \leq \exp\{-c\lambda + cK^2 M^{-1} \sinh cM\}, \quad n \geq 1.$$

Next, if we take

$$c_0 = \frac{1}{M} \operatorname{arcsinh} \frac{M\lambda}{2K^2}$$

then

$$\frac{\lambda}{2} = \frac{\sinh c_0 \cdot M}{M} K^2.$$

This implies that

$$\text{prob}(x_n > \lambda) \leq \exp\left(-\frac{c_0 \lambda}{2}\right) = \exp\left(\frac{-\lambda}{2M} \operatorname{arcsinh}\left(\frac{\lambda M}{2K^2}\right)\right), \quad n \geq 1.$$

The proof is complete.  $\square$

*Remark 4.4.* — Repeating the same argument with the martingale  $-x$ , we obtain that under the assumption of the theorem,

$$\text{prob}(|x_n| > \lambda) \leq 2 \exp\left(\frac{-\lambda}{2M} \operatorname{arcsinh}\left(\frac{\lambda M}{2K^2}\right)\right), \quad n \geq 1.$$

Using the scalar inequality  $\operatorname{arcsinh}(u) \geq 2u/(2+u)$  for  $u \geq 0$ , one can deduce a noncommutative Bernstein inequality. We should note that the constants obtained here are slightly improved compared to [21, Theorem 1.1].

**COROLLARY 4.5.** — *Let  $x$  be a self-adjoint mean zero martingale with:*

- (i)  $\|\sigma^2(x)\|_\infty = K^2$ ;
- (ii) *for every  $k \geq 1$ ,  $\|dx_k\|_\infty \leq M$ .*

*Then, for every  $\lambda > 0$ , we have:*

$$\text{prob}(x_n \geq \lambda) \leq \exp\left\{\frac{-\lambda^2}{4K^2 + \lambda M}\right\}, \quad n \geq 1.$$

We conclude by observing that the noncommutative Bennett inequality for successively independent sequences from [20, Theorem 0.1] is valid for martingale difference sequences. The proof below follows the argument from [20] but we present it here for completeness.

**THEOREM 4.6.** — *Let  $x = (x_n)_{n \geq 1}$  be a self adjoint mean-zero martingale such that:*

- (i) *for every  $k \geq 1$ ,  $\|\mathcal{E}_{k-1}(|dx_k|^2)\|_\infty \leq b_k^2$ ;*
- (ii) *for every  $k \geq 1$ ,  $\|dx_k\|_\infty \leq M_k$ .*

*Then, for every  $\lambda > 0$ ,*

$$\text{prob}(x_n > \lambda) \leq \exp \left\{ - \frac{\sum_{j=1}^n b_j^2}{\sup_{j=1, \dots, n} M_j^2} \Phi \left( \frac{\lambda \sup_{j=1, \dots, n} M_j}{\sum_{j=1}^n b_j^2} \right) \right\}, \quad n \geq 1.$$

where  $\Phi(s) = (1+s) \log(1+s) - s$  for  $s > 0$ .

*Proof.* — Fix  $c > 0$ . For  $n \geq 1$ , we make the initial estimate,

$$\begin{aligned} \text{prob}(x_n > \lambda) &\leq e^{-c\lambda} \tau(e^{cx_n}) \\ &\leq e^{-c\lambda} \tau(e^{cx_{n-1}} e^{cdx_n}). \end{aligned}$$

Since  $e^{cx_{n-1}} \in \mathcal{M}_{n-1}$ , by trace invariance of  $\mathcal{E}_{n-1}$ , we have

$$\text{prob}(x_n > \lambda) \leq e^{-c\lambda} \tau(e^{cx_{n-1}} \mathcal{E}_{n-1}[e^{cdx_n}]).$$

Expanding the exponential together with the fact that  $\mathcal{E}_{n-1}(dx_n) = 0$ , we have

$$\begin{aligned} \mathcal{E}_{n-1}[e^{cdx_n}] &= \mathcal{E}_{n-1} \left[ \sum_{k=0}^{\infty} \frac{c^k}{k!} (dx_n)^k \right] \\ &= \mathbf{1} + \sum_{k=2}^{\infty} \frac{c^k}{k!} \mathcal{E}_{n-1}[(dx_n)^k]. \end{aligned}$$

Since  $dx_n$  is self-adjoint, we have for  $k \geq 2$ ,  $(dx_n)^k \leq |dx_n|^k$  and by the positivity of conditional expectations, it follows that

$$\begin{aligned} \mathcal{E}_{n-1}[e^{cdx_n}] &\leq \mathbf{1} + \sum_{k=2}^{\infty} \frac{c^k}{k!} \mathcal{E}_{n-1}[|dx_n|^k] \\ &= \mathbf{1} + \sum_{k=2}^{\infty} \frac{c^k}{k!} \mathcal{E}_{n-1}[|dx_n|^2 |dx_n|^{k-2}] \\ &\leq \mathbf{1} + \sum_{k=2}^{\infty} \frac{c^k}{k!} \mathcal{E}_{n-1}[(\|dx_n\|_\infty^{k-2} |dx_n|^2)]. \end{aligned}$$

Using the estimates from the assumptions, we further get

$$\begin{aligned}
 \mathcal{E}_{n-1}[e^{cdx_n}] &\leq 1 + \sum_{k=2}^{\infty} \frac{c^k}{k!} M_n^{k-2} \mathcal{E}_{n-1}(|dx_n|^2) \\
 &\leq 1 + \left( \sum_{k=2}^{\infty} \frac{c^k}{k!} M_n^{k-2} b_n^2 \right) 1 \\
 &= 1 + \left[ \frac{b_n^2}{M_n^2} (e^{cM_n} - 1 - cM_n) \right] 1 \\
 &\leq \exp \left\{ \frac{b_n^2}{M_n^2} (e^{cM_n} - 1 - \lambda M_n) \right\} 1.
 \end{aligned}$$

Since the function  $s \mapsto \exp\{s^{-2}(e^{\lambda s} - 1 - \lambda s)\}$  is increasing for  $s > 0$ , we deduce that

$$\mathcal{E}_{n-1}[e^{cdx_n}] \leq \exp \left\{ \frac{b_n^2}{M^2} (e^{cM} - 1 - cM) \right\} 1$$

where  $M = \sup_{j=1, \dots, n} M_j$ . With this estimate, we obtain that

$$\tau(e^{cx_n}) \leq \exp \left\{ \frac{b_n^2}{M^2} (e^{cM} - 1 - cM) \right\} \tau(e^{cx_{n-1}}).$$

By iteration (noting that  $x_0 = 0$ ), we arrive at

$$\tau(e^{cx_n}) \leq \exp \left\{ \frac{\sum_{j=1}^n b_j^2}{M^2} (e^{cM} - 1 - cM) \right\}$$

which then yields

$$(4.5) \quad \text{prob}(x_n > \lambda) \leq \exp \left\{ -c\lambda + \frac{\sum_{j=1}^n b_j^2}{M^2} (e^{cM} - 1 - cM) \right\}.$$

We observe as in [20] that the minimum of the right hand side is achieved by taking

$$c = M^{-1} \ln \left( 1 + \frac{\lambda M}{\sum_{j=1}^n b_j^2} \right).$$

Using this value of  $\lambda$ , (4.5) gives the desired inequality.  $\square$

*Remark 4.7.* — Since  $\Phi(u) \geq (u/2) \operatorname{arcsinh}(u/2)$  for  $u \geq 0$ , we can deduce from Theorem 4.6 that under the same assumptions on the martingale  $x = (x_n)_{n \geq 1}$ , we have for  $n \geq 1$ ,

$$\text{prob}(x_n > \lambda) \leq \exp \left\{ \frac{-\lambda}{2 \sup_{j=1, \dots, n} M_j} \operatorname{arcsinh} \left( \frac{\sup_{j=1, \dots, n} M_j \lambda}{2 \sum_{j=1}^n b_j^2} \right) \right\}.$$

However, this form is clearly weaker than Theorem 4.3 since it requires individual estimate on the size of the term  $\mathcal{E}_{j-1}(|dx_j|^2)$  for  $j \geq 1$ .

## BIBLIOGRAPHY

- [1] H. ARAKI, “Golden-Thompson and Peierls-Bogolubov inequalities for a general von Neumann algebra”, *Commun. Math. Phys.* **34** (1973), p. 167-178.
- [2] ———, “Relative entropy of states of von Neumann algebras”, *Publ. Res. Inst. Math. Sci.* **11** (1976), no. 3, p. 809-833.
- [3] R. BHATIA, *Matrix analysis*, Graduate Texts in Mathematics, vol. 169, Springer, 1996, xii+347 pages.
- [4] Z. CHEN, N. RANDRIANANTOANINA & Q. XU, “Atomic decompositions for noncommutative martingales”, <https://arxiv.org/abs/2001.08775v1>, 2020.
- [5] J. DIESTEL & J. J. UHL, JR, *Vector measures*, Math. Surveys, vol. 15, American Mathematical Society, 1977, xiii+322 pages.
- [6] P. G. DODDS, T. K.-Y. DODDS & B. DE PAGTER, “Noncommutative Köthe duality”, *Trans. Am. Math. Soc.* **339** (1993), no. 2, p. 717-750.
- [7] T. FACK & H. KOSAKI, “Generalized  $s$ -numbers of  $\tau$ -measurable operators”, *Pac. J. Math.* **123** (1986), p. 269-300.
- [8] P. J. FORRESTER & C. J. THOMPSON, “The Golden-Thompson inequality: historical aspects and random matrix applications”, *J. Math. Phys.* **55** (2014), no. 2, article no. 023503 (12 pages).
- [9] S. GOLDEN, “Lower bounds for the Helmholtz function”, *Phys. Rev., II. Ser.* **137** (1965), p. B1127-B1128.
- [10] U. GROH, “Uniform ergodic theorems for identity preserving Schwarz maps on  $W^*$ -algebras”, *J. Oper. Theory* **11** (1984), no. 2, p. 395-404.
- [11] F. HANSEN & G. K. PEDERSEN, “Perturbation formulas for traces on  $C^*$ -algebras”, *Publ. Res. Inst. Math. Sci.* **31** (1995), no. 1, p. 169-178.
- [12] S. HEINRICH, “Ultraproducts in Banach space theory”, *J. Reine Angew. Math.* **313** (1980), p. 72-104.
- [13] P. HITCZENKO, “Best constants in martingale version of Rosenthal’s inequality”, *Ann. Probab.* **18** (1990), no. 4, p. 1656-1668.
- [14] Y. JIAO, A. OSĘKOWSKI & L. WU, “Inequalities for noncommutative differentially subordinate martingales”, *Adv. Math.* **337** (2018), no. 1, p. 216-259.
- [15] W. B. JOHNSON, G. SCHECHTMAN & J. ZINN, “Best constants in moment inequalities for linear combinations of independent and exchangeable random variables”, *Ann. Probab.* **13** (1985), no. 1, p. 234-253.
- [16] M. JUNGE, “Doob’s inequality for non-commutative martingales”, *J. Reine Angew. Math.* **549** (2002), p. 149-190.
- [17] M. JUNGE & M. PERRIN, *Theory of  $\mathcal{H}_p$ -spaces for continuous filtrations in von Neumann algebras*, Astérisque, vol. 362, Société Mathématique de France, 2014, vi+134 pages.
- [18] M. JUNGE & Q. XU, “Noncommutative Burkholder/Rosenthal inequalities”, *Ann. Probab.* **31** (2003), no. 2, p. 948-995.
- [19] ———, “Noncommutative Burkholder/Rosenthal inequalities. II. Applications”, *Isr. J. Math.* **167** (2008), p. 227-282.
- [20] M. JUNGE & Q. ZENG, “Noncommutative Bennett and Rosenthal inequalities”, *Ann. Probab.* **41** (2013), no. 6, p. 4287-4316.
- [21] ———, “Noncommutative martingale deviation and Poincaré type inequalities with applications”, *Probab. Theory Relat. Fields* **161** (2015), no. 3-4, p. 449-507.



- [22] N. J. KALTON & F. SUKOCHEV, “Symmetric norms and spaces of operators”, *J. Reine Angew. Math.* **621** (2008), p. 81-121.
- [23] H. KOSAKI, “Interpolation theory and the Wigner–Yanase–Dyson–Lieb concavity”, *Commun. Math. Phys.* **87** (1982), no. 3, p. 315-329.
- [24] ———, “An inequality of Araki–Lieb–Thirring (von Neumann algebra case)”, *Proc. Am. Math. Soc.* **114** (1992), no. 2, p. 477-481.
- [25] E. LIEB, “Convex trace functions and the Wigner–Yanase–Dyson conjecture”, *Adv. Math.* **11** (1973), p. 267-288.
- [26] E. LIEB & M. B. RUSKAI, “A fundamental property of quantum-mechanical entropy”, *Phys. Rev. Lett.* **30** (1973), p. 434-436.
- [27] D. MCDUFF, “Uncountably many  $\text{II}_1$  factors”, *Ann. Math.* **90** (1969), p. 372-377.
- [28] E. NELSON, “Notes on non-commutative integration”, *J. Funct. Anal.* **15** (1974), p. 103-116.
- [29] A. OCNEANU, *Actions of discrete amenable groups on von Neumann algebras*, Lecture Notes in Mathematics, vol. 1138, Springer, 1985, iv+115 pages.
- [30] D. PETZ, “A survey of certain trace inequalities”, in *Functional analysis and operator theory (Warsaw, 1992)*, Banach Center Publications, vol. 30, Polish Academy of Sciences, 1994, p. 287-298.
- [31] G. PISIER & Q. XU, “Non-commutative martingale inequalities”, *Commun. Math. Phys.* **189** (1997), no. 3, p. 667-698.
- [32] ———, “Non-commutative  $L^p$ -spaces”, in *Handbook of the geometry of Banach spaces, Vol. 2*, North-Holland, 2003, p. 1459-1517.
- [33] Y. V. PROKHOROV, “An extremal problem in probability theory”, *Theor. Probability Appl.* **4** (1960), p. 201-203.
- [34] N. RANDRIANANTOANINA, “Non-commutative martingale transforms”, *J. Funct. Anal.* **194** (2002), p. 181-212.
- [35] N. RANDRIANANTOANINA & L. WU, “Noncommutative Burkholder/Rosenthal inequalities associated with convex functions”, *Ann. Inst. Henri Poincaré, Probab. Stat.* **53** (2017), no. 4, p. 1575-1605.
- [36] Y. RAYNAUD, “On ultrapowers of non commutative  $L_p$  spaces”, *J. Oper. Theory* **48** (2002), no. 1, p. 41-68.
- [37] É. RICARD & Q. XU, “A noncommutative martingale convexity inequality”, *Ann. Probab.* **44** (2016), no. 2, p. 867-882.
- [38] M. B. RUSKAI, “Inequalities for traces on von Neumann algebras”, *Commun. Math. Phys.* **26** (1972), p. 280-289.
- [39] G. SADEGHI & M. S. MOSLEHIAN, “Noncommutative martingale concentration inequalities”, *Ill. J. Math.* **58** (2014), no. 2, p. 561-575.
- [40] B. SIMON, *Trace ideals and their applications*, second ed., Mathematical Surveys and Monographs, vol. 120, American Mathematical Society, 2005, viii+150 pages.
- [41] B. SIMS, “Ultra”-techniques in Banach space theory, Queen’s Papers in Pure and Applied Mathematics, vol. 60, Queen’s University, 1982, iv+117 pages.
- [42] F. SUKOCHEV, “Completeness of quasi-normed symmetric operator spaces”, *Indag. Math., New Ser.* **25** (2014), no. 2, p. 376-388.
- [43] F. SUKOCHEV & D. ZANIN, “Johnson–Schlechtman inequalities in the free probability theory”, *J. Funct. Anal.* **263** (2012), no. 10, p. 2921-2948.
- [44] D. SUTTER, M. BERTA & M. TOMAMICHEL, “Multivariate trace inequalities”, *Commun. Math. Phys.* **352** (2017), no. 1, p. 37-58.
- [45] C. J. THOMPSON, “Inequality with applications in statistical mechanics”, *J. Math. Phys.* **6** (1965), p. 1812-1813.

- [46] O. E. TIKHONOV, “Continuity of operator functions in topologies connected with a trace on a von Neumann algebra”, *Izv. Vyssh. Uchebn. Zaved., Mat.* (1987), no. 1, p. 77-79.
- [47] J. A. TROPP, “User-friendly tail bounds for sums of random matrices”, *Found. Comput. Math.* **12** (2012), no. 4, p. 389-434.
- [48] J. VESTERSTRØM, “Quotients of finite  $W^*$ -algebras”, *J. Funct. Anal.* **9** (1972), p. 322-335.
- [49] Q. XU, “Analytic functions with values in lattices and symmetric spaces of measurable operators”, *Math. Proc. Camb. Philos. Soc.* **109** (1991), p. 541-563.
- [50] H. ZHANG, “From Wigner–Yanase–Dyson conjecture to Carlen–Frank–Lieb conjecture”, *Adv. Math.* **365** (2020), article no. 107053 (18 pages).

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Narcisse RANDRIANANTOANINA  
Department of Mathematics  
Miami University  
Oxford, Ohio 45056 (USA)  
randrin@miamioh.edu