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ARC SPACES AND WEDGE SPACES FOR TORIC VARIETIES

by Ana J.REGUERA (*)

Abstract. — Let X be a normal toric variety over a perfect field k and let X_∞ be its space of arcs. Let P be a toric stable point of X_∞, i.e. defined by a toric divisorial valuation ν. We describe the irreducible components of $\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P}}$ and their respective dimensions. This description is derived from the existence of a finite family of regular toric varieties such that every wedge centered at P lifts to some of them. As a first consequence, we obtain that, in general, the ring $\mathcal{O}_{X_\infty,P}$ is neither analytically irreducible nor catenary. A second consequence is that, when X is $\mathbb{Q}$-Gorenstein, we recover the log discrepancy of ν from the space of arcs X_∞.

Résumé. — Soit X une variété torique normale sur un corps parfait k et soit X_∞ son espace d'arcs. Soit P un point stable torique de X_∞, i.e. défini par une valuation divisorielle torique ν. Nous décrivons les composantes irréductibles de $\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P}}$ et leur dimension respective. Cette description est déduite de l'existence d'une famille finie de variétés toriques régulières telles que tout coin centré en P se relève à l'une d'elles. Comme première conséquence, nous obtenons que l'anneau $\mathcal{O}_{X_\infty,P}$ n'est ni analytiquement irréductible ni caténaire en général. Une deuxième conséquence est que, lorsque X est $\mathbb{Q}$-Gorenstein, nous récupérons la log-discrépance de ν à partir de l'espace d'arcs X_∞.

1. Introduction

The space of arcs X_∞ of an algebraic variety X was introduced by J. Nash in the 60's [27]. His aim was, when X is defined over a field k of characteristic zero, to recover properties of the resolutions of singularities of X from invariants of its space of arcs. His work was done just after the proof of Resolution of Singularities in characteristic zero by H. Hironaka. Nash's work was made known by H. Hironaka in the 70's and afterwards by M. Lejeune-Jalabert [23]. Later, in the 90's, M. Kontsevich [22] and J. Denef

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and F. Loeser [8] set up a theory of motivic integration on X_∞ based on the existence of resolutions of singularities of X. Their development provided strong techniques for studying the space of arcs as a scheme.

The arc families considered by Nash, as well as the irreducible subsets of X_∞ of nonzero motivic measure considered in [8] correspond to certain fat points of the scheme X_∞: they are stable points ([31, 3.1]), i.e. the stability property in [8] holds in a nonempty open subset of their set of zeroes. These stable points are points of finite codimension in X_∞, and we proved in [30] the following finiteness property: if P is a stable point, then the complete local ring $\widehat{\mathcal{O}_{X_\infty,P}}$ is a Noetherian ring, i.e. has finite embedding dimension. This algebraic result led us to prove a Curve Selection Lemma in the space of arcs ([30, Corollary 4.8]) that has frequently been applied in the last ten years, especially dealing with Nash's question of understanding the decomposition of the set X_∞^{Sing} of arcs centered in $\operatorname{Sing} X$ into its irreducible components. In the same direction, we have given a minimal system of coordinates of $(X_\infty)_{\mathrm{red}}$ at a stable point P and computed the embedding dimension of $\widehat{\mathcal{O}_{X_\infty,P}}$ when $\operatorname{char} k = 0$ ([32] and [26]). This last result was extended to positive characteristic in [13]. The technique we applied in [26] is a study of the graded algebra associated to a divisorial valuation, following the line started in [34]. From this study we have also obtained in [26] a lower bound for $\dim \widehat{\mathcal{O}_{X_\infty,P}}$.

It is our purpose to understand algebraic properties of the local rings $\mathcal{O}_{X_\infty,P}$ where P is a stable point of the space of arcs X_∞ of a variety X over a perfect field k of any characteristic. In particular, our interest in $\dim \widehat{\mathcal{O}_{X_\infty,P}}$ and in the property of irreducibility of $\widehat{\mathcal{O}_{X_\infty,P}}$ is due to the following fact: We know that, assuming the existence of a resolution of singularities of X, the ring $\widehat{\mathcal{O}_{X_\infty,P}}$ is irreducible and one dimensional if and only if for every resolution of singularities $\widetilde{X}$ of X, every wedge on X centered at P lifts to $\widetilde{X}$ and, if this holds, then P is the generic point of an irreducible component of X_∞^{Sing} ([31, Corollary 5.12]). The 1-dimensionality and irreducibility of $\widehat{\mathcal{O}_{X_\infty,P}}$ has been proved to hold when P is the stable point defined by any essential valuation of a toric variety ([19, Theorem 3.16]), by nonuniruled ([24, Theorem 3.3]) and by terminal valuations if $\operatorname{char} k = 0$ ([12, Theorem 1.1]), giving a partial answer to the Nash proposal. Here the following idea is underlying: based on our Curve Selection Lemma we have translated the question of knowing whether an essential valuation is defined by an irreducible component of X_∞^{Sing} into a problem of lifting wedges to a resolution of singularities of X. But there are examples of essential valuations for which this does not hold ([19, 10, 20]).

A natural question arises here: interpreting each irreducible component of $\widehat{\mathcal{O}_{X_\infty,P}}$ as a family of arcs lifting to some morphism $Y_i \to X$, not necessarily a resolution of singularities, where i runs through a finite set.

The purpose of this article is to study higher dimensional local rings $\mathcal{O}_{X_\infty,P}$, where P is a stable point of X_∞. We will give an example to show that, in general, the rings $\mathcal{O}_{X_\infty,P}$ and $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P}$ are not catenary. Moreover, even though $\mathcal{O}_{X_\infty,P}$ is irreducible, we will show that the complete ring $\widehat{\mathcal{O}_{X_\infty,P}}$ may have irreducible components of different dimensions. We had already given examples of Noetherian 1-dimensional local rings $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P}$ which are analytically ramified ([31, Corollary 5.6]), but the fact that $\widehat{\mathcal{O}_{X_\infty,P}}$ is not equidimensional was not known until now. It opens a new question: understanding the analytic irreducible components of the rings $\widehat{\mathcal{O}_{X_\infty,P}}$ and their geometric sense.

In this article we go back to the setting in S. Ishii and J. Kollár's work [19] and [16]. We also import valuative techniques, developed by J. Novacoski and M. Spivakovsky [28] for the Local Uniformization Problem. Precisely, we consider a normal toric variety X and a toric stable point P, i.e. defined by a multiple of a toric divisorial valuation, or equivalently, by a lattice element v in the cone σ defining the corresponding affine chart. Then, each chain of toric prime ideals contained in P gives rise to a (finite) partition $\mathbf{w} : v = \sum_j n_j w_j$ of v, where the w_j's are minimal lattice elements of σ and the n_j's are positive integers. Partitions of v have already appeared in [4] in order to determine the components of the minimal model of a suitably general k-rational arc lying in the subset of X_∞ defined by v (Remark 5.7). Our main result (Theorem 5.6) states that $\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P}}$ has as many irreducible components as possible partitions $\mathbf{w}$ of v, and the dimension of the corresponding irreducible component $\mathcal{I}_{\mathbf{w}}$ is the length $\sum_j n_j$ of $\mathbf{w}$. We conclude that the dimension of $\widehat{\mathcal{O}_{X_\infty,P}}$ is equal to the *toric heigth* $\operatorname{tcht} P$ of P, i.e. the maximal length of chains of toric prime ideals contained in P.

Moreover, to each $\mathbf{w}$ we associate a toric morphism $\rho_{\mathbf{w}} : Y^{\mathbf{w}} \to X$, where $Y^{\mathbf{w}}$ is a smooth variety, and a stable point $Q^{\mathbf{w}}$ of $Y^{\mathbf{w}}_\infty$ whose image in X_∞ is P; the set $\mathcal{I}_{\mathbf{w}}$ is the image of the morphism $\operatorname{Spec}\widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}} \to \operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_v}}$ induced by $\rho^{\mathbf{w}}_\infty$. Then, every wedge centered at P lifts to some of the $Y^{\mathbf{w}}$'s. That is, we have obtained a property of lifting wedges to a finite family of regular varieties in this toric case. Applying this, a going up theorem is proved (Proposition 5.3), of which a consequence is Theorem 5.6. We present an alternative proof of this going up theorem as an interesting application of local uniformization of valuations which are composition of discrete valuations, which is a theorem due to Novacoski and Spivakovsky [28], and the finiteness property of the stable points of the space of arcs of any variety [30].

As consequence of Theorem 5.6, we obtain that the invariant tcht P_{ev}, v minimal lattice element of σ and $e \geqslant 1$, is greater or equal to e times the Mather–Jacobian log discrepancy of the prime divisor D_v defined by v. Moreover, when X is $\mathbb{Q}$-Gorenstein, we prove (Theorem 6.2) that the maximum of $\frac{\dim \widehat{\mathcal{O}_{X_\infty,P_{ev}}}}{e}$, $e \geqslant 1$, is greater or equal to the log discrepancy $a(D_v;X)$ of X with respect to D_v and, in fact, $a(D_v;X)$ is equal to $\frac{\dim \mathcal{I}}{e}$ for some $e \geqslant 1$ and some irreducible component $\mathcal{I}$ of $\operatorname{Spec} \widehat{\mathcal{O}_{X_\infty,P_{ev}}}$. This result for toric varieties opens new questions and new ideas to study the space of arcs for more general classes of varieties.

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2. Preliminaires

2.1. On the space of arcs

Let k be a perfect field and let X be a *variety over* k, i.e. X is a reduced separated k-scheme of finite type. Given a field extension $k \subseteq K$, a K-*arc* on X is a k-morphism $\operatorname{Spec} K[[t]] \to X$. Let X_∞ denote the *space of arcs of* X. More precisely, if, for $n \in \mathbb{N}$, X_n denotes the k-scheme of n-jets, whose K-rational points are the k-morphisms $\operatorname{Spec} K[t]/(t)^{n+1} \to X$, then $X_\infty = \varprojlim X_n$. We denote by $j_n : X_\infty \to X_n$, $n \geqslant 0$, the natural projections. The space of arcs X_∞ is a k-scheme whose K-rational points are the K-arcs on X, for any field extension $k \subseteq K$. Moreover, for every k-algebra A, we have a natural isomorphism

$$\operatorname{Hom}_k(\operatorname{Spec} A, X_\infty) \cong \operatorname{Hom}_k(\operatorname{Spec} A[[t]], X) \tag{2.1}$$

([2, Corollary 1.2]). In fact, through this article we will apply (2.1) for A a local ring: in this case (2.1) follows straightforwardly. For an introduction to jet and arc spaces see [11], [6, Chapter 3].

Given $P \in X_\infty$, with residue field $\kappa(P)$, we denote the induced $\kappa(P)$-arc on X by $h_P : \operatorname{Spec} \kappa(P)[[t]] \to X$. The image P_0 in X of the closed point of $\operatorname{Spec} \kappa(P)[[t]]$ is called the *center* of P. The image P_η of the generic point of Spec $\kappa(P)[[t]]$ is the generic point of $\operatorname{Im} h_P$. Then, h_P induces a morphism of k-algebras $h_P^\sharp : \mathcal{O}_{X,P_0} \to \kappa(P)[[t]]$, or an injective morphism $h_P^\sharp : \mathcal{O}_{\overline{\{P_\eta\}},P_0} \to \kappa(P)[[t]]$. We denote by ν_P the order function $\operatorname{ord}_t h_P^\sharp : \mathcal{O}_{X,P_0} \to \mathbb{N} \cup \{\infty\}$.

The space of arcs of the affine space $\mathbb{A}_k^m = \operatorname{Spec} k[x_1, \ldots, x_m]$ is $(\mathbb{A}_k^m)_\infty = \operatorname{Spec} k[\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_n, \ldots]$ where for $n \geqslant 0$, $\underline{X}_n = (X_{1,n}, \ldots, X_{m,n})$ is an m-uple of variables. For any $f \in k[x_1, \ldots, x_m]$, let $\sum_{n=0}^\infty F_n\, t^n$ be the Taylor expansion of $f(\sum_n \underline{X}_n\, t^n)$, hence $F_n \in k[\underline{X}_0, \ldots, \underline{X}_n]$. If $X \subseteq \mathbb{A}_k^m$ is affine, and $I_X \subset k[x_1, \ldots, x_m]$ is the ideal defining X in $\mathbb{A}_k^m$, then we have

$$X_\infty = \operatorname{Spec} k[\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_n, \ldots] \;/\; (\{F_n\}_{n \geqslant 0, f \in I_X}).$$

2.2. On the stable points of the space of arcs

If X is affine and irreducible, a point P of X_∞ is a *stable point of* X_∞ if there exist $n_1 \in \mathbb{N}$, and $G \in \mathcal{O}_{X_\infty} \setminus P$, $G \in \mathcal{O}_{X_{n_1}}$ such that, for $n \geqslant n_1$, the map $X_{n+1} \longrightarrow X_n$ induces a trivial fibration

$$\overline{j_{n+1}(Z(P))} \cap (X_{n+1})_G \longrightarrow \overline{j_n(Z(P))} \cap (X_n)_G$$

with fiber $\mathbb{A}_k^d$, where $d = \dim X$, $(X_n)_G$ is the open subset $X_n \setminus Z(G)$ of X_n and $\overline{j_n(Z(P))}$ is the closure of $j_n(Z(P))$ in X_n with the reduced structure (3.1 in [31], see also the stability property [8, Lemma 4.1]). This definition extends to a variety X, not necessarily affine and irreducible, so that the set of stable points of X_∞ is the disjoint union of the sets of stable points of the spaces of arcs of the irreducible components of X ([31] and [33]). Applying [8, Lemma 4.1] and [30, Lemma 4.2] it can be shown that stable points of a variety X, whose irreducible components are $\{X^i\}_{i=1}^c$, are characterized as follows: $P \in X_\infty$ is stable if and only if $Z(P)$ is not contained in $\cup_{i=1}^c (\operatorname{Sing} X^i)_\infty$ and there exists an open affine subscheme W_0 of X_∞, such that $N \cap W_0$ is a nonempty closed subset of W_0 whose defining ideal is the radical of a finitely generated ideal.

Stable points are fat points in the following sense: if $P \in X_\infty$ is stable then the image of the arc $h_P : \operatorname{Spec} \kappa(P)[[t]] \to X$ is dense on an irreducible component of X ([31, Proposition 3.7(i)]). The local ring $\mathcal{O}_{X_\infty,P}$ of X_∞ at a stable point P is irreducible of finite dimension ([31, Theorem 2.9 and Proposition 3.7(iv)]), but in general it is not reduced and non Noetherian ([31, Example 3.16]). However we have:

Finiteness property of the stable points ([30, Theorem 4.1]*).* — Let P be a stable point of X_∞, then the formal completion $\widehat{\mathcal{O}_{(X_\infty)_{\mathrm{red}},P}}$ of the local ring of $(X_\infty)_{\mathrm{red}}$ at P is a Noetherian ring. Moreover, if X is affine, then there exists $G \in \mathcal{O}_{X_\infty} \setminus P$ such that the ideal $P\left(\mathcal{O}_{(X_\infty)_{\mathrm{red}}}\right)_G$ is a finitely generated ideal of $\left(\mathcal{O}_{(X_\infty)_{\mathrm{red}}}\right)_G$.

Furthermore, we have $\widehat{\mathcal{O}_{X_\infty,P}} \cong \widehat{\mathcal{O}_{(X_\infty)_{\mathrm{red}},P}}$ ([31, Theorem 3.13] if $\operatorname{char} k = 0$; if $\operatorname{char} k > 0$ the proof in [31] holds if we take Hasse–Schmidt derivations).

The following is still an open question:

Question 2.1 ([31, Question 3.17]). — *Let P be a stable point of X. Is the ring $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P}$ a Noetherian ring?*

Even this weaker question is still unsolved:

Question 2.2. — *Given a variety X and a stable point P of X_∞, is it true that* $\dim \mathcal{O}_{X_\infty,P} = \dim \widehat{\mathcal{O}_{X_\infty,P}}$ *?*

2.3. Stable points defined by divisorial valuations

We will deal with the following stable points: Let ν be a divisorial valuation on X. There exists a proper and birational morphism $\pi : Y \to X$ with Y normal, and a divisor E on Y such that $\nu = \nu_E$ is the valuation defined by E. For every $e \geqslant 1$, let $Y_\infty^{eE_{\mathrm{reg}}} := \{Q \in Y_\infty \ / \ \nu_Q(I_E) = e\}$, where I_E is the ideal defining E in an open affine subset of $\operatorname{Reg}(Y)$. Then $Y_\infty^{eE_{\mathrm{reg}}}$ is an irreducible subset of Y_∞, let N_{eE} be the closure of $\pi_\infty(Y_\infty^{eE_{\mathrm{reg}}})$, which is irreducible, and let P_{eE} be the generic point of N_{eE}. Note that P_{eE} only depends on e and on the divisorial valuation $\nu = \nu_E$. We have that P_{eE} is a stable point of X_∞ ([31, Proposition 4.1], see also [30, Proposition 3.8]).

2.4. Wedges

We will also deal with wedges: Given $r \in \mathbb{N}$, a *r-dimensional K-wedge*, or a *K-r-wedge* is a k-morphism $\Phi : \operatorname{Spec} K[[\underline{\xi}, t]] \to X$, or equivalently (see (2.1)), a k-morphism $\varphi : \operatorname{Spec} K[[\underline{\xi}]] \to X_\infty$, where $\underline{\xi} := (\xi_1, \dots, \xi_r)$ are variables, that is, a K-$(r-1)$-wedge (resp. an arc) on X_∞ if $r \geqslant 2$ (resp. $r = 1$). Given a K-r-wedge Φ, the image in X_∞ of the closed point (resp. generic point) of $\operatorname{Spec} K[[\underline{\xi}]]$ by φ will be called the *special arc* (resp. *generic arc*) of Φ. Note that a wedge Φ whose special arc is $P \in X_\infty$ is equivalent to a local k-morphism $\mathcal{O}_{X_\infty,P} \to K[[\underline{\xi}]]$. If P is a stable point of X_∞, it is also equivalent to a local k-morphism $\widehat{\mathcal{O}_{X_\infty,P}} \to K[[\underline{\xi}]]$ (this follows from [30, Theorem 4.1] and [3, Chapter III, Section 2.12, Corollary 2]).

3. On stable points of the space of arcs of a toric variety

In this section we will study stable points of the space of arcs of normal toric varieties. For more details on toric varieties see [21, 29, 15, 7]. See also [19, 16] for the study of the space of arcs of a normal toric variety.

Let N be the free $\mathbb{Z}$-module $\mathbb{Z}^d$ and let M be its dual $\operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Let $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $\langle\ ,\ \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ the canonic bilinear map. Given a fan Σ in $N_{\mathbb{R}}$, let $X := X_\Sigma$ be the corresponding normal toric variety. Let $T \cong (k^*)^d$ be the d-dimensional torus seating inside X_Σ. Then T acts on X_Σ.

An *equivariant resolution* of X is a resolution of singularities $\pi : Y \to X$ of X (i.e. π is a proper, birational k-morphism, with Y smooth, such that the induced morphism $Y \setminus \pi^{-1}(\operatorname{Sing} X) \to X \setminus \operatorname{Sing} X$ is an isomorphism) which respects the action of the torus. Each equivariant resolution of X_Σ is a morphism $X_{\Sigma'} \to X_\Sigma$ where Σ' is a regular subdivision of Σ. Here recall that a subdivision Σ' of Σ is regular if it consists of regular cones, and a cone is regular if its generators can be extended to a basis of N. It follows that any normal toric variety has an equivariant resolution.

A *toric divisorial valuation* is a divisorial valuation ν on X_Σ for which there exists an equivariant resolution $\pi : X_{\Sigma'} \to X_\Sigma$ such that the center of ν on $X_{\Sigma'}$ is a divisor. Hence the center of ν on $X_{\Sigma'}$ is the closure $D_{\alpha'}$ of the orbit $O_{\alpha'}$ of a 1-dimensional cone $\alpha' \in \Sigma'$. The divisorial valuation $\nu = \nu_{D_{\alpha'}}$ is centered on $\operatorname{Sing} X$ if and only if $\alpha' \in \Sigma' \setminus \Sigma$.

An *essential divisorial valuation* on X is a divisorial valuation ν on X such that, for every resolution of singularities $\pi : Y \to X$, the center of ν on Y is an irreducible component of the exceptional locus of π. We know that the essential divisorial valuations are precisely the toric divisorial valuations which are essential for the equivariant resolutions ([19, Corollary 3.17], see also [5]).

Suppose that X is an affine normal toric variety. Hence $X = X_\Sigma$ (also denoted $X = X_\sigma$) where Σ is the fan defined by a (convex rational) cone σ in $N_\mathbb{R}$ and its faces. More precisely, $X := \operatorname{Spec} k[\sigma^\vee \cap M]$ where $\sigma^\vee \subset M_\mathbb{R}$ is the dual cone of σ. The semigroup $\sigma^\vee \cap M$ is finitely generated and $k[\sigma^\vee \cap M]$ is a k-algebra of finite type which is generated as a k-vector space by $\{\aleph^u\}_{u\in\sigma^\vee\cap M}$, where $\aleph^u \cdot \aleph^{u'} := \aleph^{u+u'}$. In fact, we may suppose with no loss of generality that σ is a strongly convex cone (i.e. $\sigma \cap (-\sigma) = \{0\}$, or equivalently, $\dim X = d$), then the torus $T = X_{\{0\}} = \operatorname{Spec} k[M]$ is inside $X = X_\sigma$. More precisely, there exists $\{u_1, \dots, u_d\} \subset \sigma^\vee \cap M$ which is a basis of the free $\mathbb{Z}$-module M. Let us extend it to a system of generators $\{u_1, \dots, u_m\}$ of the semigroup $\sigma^\vee \cap M$, so that the morphism of k-algebras given by

$$k[x_1, \dots, x_m] \to k[\sigma^\vee \cap M], \quad x_i \mapsto \aleph^{u_i} \quad \text{for } 1 \leqslant i \leqslant m,$$

is surjective. Then $T = \operatorname{Spec} k[\{\aleph^{u_i}, \aleph^{-u_i}\}_{i=1}^d] = \operatorname{Spec} k[x_1, \dots, x_d]_{x_1\cdots x_d}$ $\cong \operatorname{Spec}(k^*)^d$ is a torus and $\mathcal{O}_X$ is a quotient of $k[x_1, \dots, x_m]$, that is,

$$X = \operatorname{Spec} k[x_1, \dots, x_m] \,/\, I_X.$$

Here I_X is defined as follows: let

$$\Lambda := \left\{ \underline{a} = (a_1, \dots, a_m) \in \mathbb{Z}^m \,/\, \sum_{i=1}^m a_i u_i = 0 \right\}. \tag{3.1}$$

For every $\underline{a} \in \Lambda$ set $\mathcal{J}^+(\underline{a}) = \{i \,/\, 1 \leqslant i \leqslant m, a_i > 0\}$, $\mathcal{J}^-(\underline{a}) = \{i \,/\, 1 \leqslant i \leqslant m, a_i < 0\}$ and

$$f_{\underline{a}} := \prod_{i\in\mathcal{J}^+(\underline{a})} x_i^{a_i} - \prod_{j\in\mathcal{J}^-(\underline{a})} x_j^{-a_j}. \tag{3.2}$$

Then I_X is the ideal generated by $\{f_{\underline{a}}\}_{\underline{a}\in\Lambda}$. Moreover, I_X is finitely generated, hence it is generated by $\{f_{\underline{a}}\}_{\underline{a}\in G_\Lambda}$ where G_Λ is some finite subset of Λ. The torus $T = k[x_1, \dots, x_d]_{x_1\cdots x_d} \cong (k^*)^d$ is a dense open subset of X and the action of T lifts to X. In fact, for any $u \in (\sigma^\vee)^\circ \cap M$, where $^\circ$ means the relative interior, we have

$$(\mathcal{O}_X)_l \cong (k[x_1, \dots, x_m] \,/\, I_X)_l \cong k[x_1, \dots, x_d]_{x_1\cdots x_d} \tag{3.3}$$

where $l = \aleph^u$. In particular, we may take $l = x_1 \cdots x_d$ in (3.3).

In this affine case, the essential divisorial valuations are precisely the divisorial valuations defined by $D_v := \overline{O_{\langle v\rangle}}$ where v runs between the minimal elements of $S := N \cap (\bigcup_{\tau\in\Sigma,\tau\text{singular}} \overset{\circ}{\tau})$ (see [5] and [19]). Here recall that a cone τ is singular if it is not regular. The order in $\sigma \cap N$ is defined as follows: given $v, v' \in \sigma \cap N$, we define:

$$v \leqslant v' \quad \text{iff} \quad v' \in v + \sigma.$$

Then, for any subset A of $\sigma \cap N$, $v \in A$ is a minimal element in A if there does not exist $v' \in A$ with $v' \leqslant v$, $v' \neq v$.

If P is a stable point of X_∞ then $\nu_P : k[\sigma^\vee \cap M] \to \mathbb{N} \cup \{\infty\}$ is a valuation ([31, Proposition 3.7(i), see Sections 2.1 and 2.2]), hence it determines an element of $\sigma \cap N$. In fact, the map $M \to \mathbb{Z}$ given by $u \mapsto \nu_P(\aleph^u)$ defines an element v_P of $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \cong N$. Then we have

$$\langle u, v_P \rangle = \nu_P(\aleph^u) \quad \text{for all } u \in M. \tag{3.4}$$

Therefore, $\langle u, v_P \rangle \geqslant 0$ for $u \in \sigma^\vee \cap M$, and thus $v_P \in \sigma \cap N$. Here note that ν_P may not be the divisorial valuation defined by v_P.

Conversely we will next define, for each $v \in \sigma \cap N$, a stable point P_v of X_∞. It will satisfy the property that it is infimum, with respect to inclusion, between stable points P of X_∞ such that $v_P = v$. That is, $v_{P_v} = v$ and, for any stable point P of X_∞ such that $v_P = v$ we have $P_v \subseteq P$.

Lemma 3.1. — *Let $X = X_\sigma$ be an affine normal toric variety. Given $v \in \sigma \cap N$, set $c_i \coloneqq \langle u_i, v \rangle$, which is a nonnegative integer, for $1 \leqslant i \leqslant m$. Then, the ideal*

$$\left(\{X_{i,0}, \ldots, X_{i,c_i - 1}\}_{i=1}^m\right) \mathcal{O}_{(X_\infty)_{\mathrm{red}}}$$

is a prime ideal of $\mathcal{O}_{(X_\infty)_{\mathrm{red}}}$. Moreover, its preimage P_v in $\mathcal{O}_{X_\infty}$ is a stable point of X_∞ and the valuation ν_{P_v} is the divisorial valuation defined by v.

Proof. — First, we have

$$\mathcal{O}_{(X_\infty)_{\mathrm{red}}} = k[\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_n, \ldots] \Big/ \sqrt{(\{F_{\underline{a},n}\}_{n \geqslant 0, \underline{a} \in G_\Lambda})}$$

and hence

$$\begin{aligned} &\mathcal{O}_{(X_\infty)_{\mathrm{red}}} \,/\, \left(\{X_{i,0}, \ldots, X_{i,c_i - 1}\}_{i=1}^m\right) \\ &\qquad = k\left[\{X_{i,n_i}\}_{1 \leqslant i \leqslant m, n_i \geqslant c_i}\right] \Big/ \left(\left\{\overline{H} \,/\, H \in \sqrt{(\{F_{\underline{a},n}\}_{n \geqslant 0, \underline{a} \in G_\Lambda})}\right\}\right) \end{aligned}$$

where, given

$$H \in k[\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_n, \ldots],$$

by $\overline{H}$ we mean the element in $k[\{X_{i,n_i}\}_{1 \leqslant i \leqslant m, n_i \geqslant c_i}]$ representing the class of H modulo $\left(\{X_{i,0}, \ldots, X_{i,c_i - 1}\}_{i=1}^m\right)$.

Let us consider the isomorphism of k-algebras

$$\delta : k[\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_n, \ldots] \longrightarrow k[\{X_{i,n_i}\}_{1 \leqslant i \leqslant m, n_i \geqslant c_i}]$$

given by $\delta(X_{i,n}) = X_{i,c_i+n}$, $1 \leqslant i \leqslant m$, $n \geqslant 0$. Note that, given $\underline{a} \in \Lambda$, if we set $c_{\underline{a}} := \sum_{i \in \mathcal{J}^+(\underline{a})} a_i c_i$, then we have

$$\delta(F_{\underline{a},n}) = \overline{F_{\underline{a},c_{\underline{a}}+n}} \quad \text{for all } n \geqslant 0$$

(apply the definition of Λ in (3.1), of $f_{\underline{a}}$ in (3.2) and Taylor's expansion). From this it follows that

$$\delta\left(\sqrt{(\{F_{\underline{a},n}\}_{n\geqslant 0,\underline{a}\in G_\Lambda})}\right) = \left(\left\{\overline{H} \ / \ H \in \sqrt{(\{F_{\underline{a},n}\}_{n\geqslant 0,\underline{a}\in G_\Lambda})}\right\}\right).$$

Therefore δ induces an isomorphism

$$\mathcal{O}_{(X_\infty)_{\mathrm{red}}} \ / \ \left(\{X_{i,0},\ldots,X_{i,c_i-1}\}_{i=1}^m\right) \cong \mathcal{O}_{(X_\infty)_{\mathrm{red}}} \tag{3.5}$$

and, since $\mathcal{O}_{(X_\infty)_{\mathrm{red}}}$ is a domain, we conclude the first assertion of the lemma.

For the second one, since P_v is the ideal $\sqrt{\left(\{X_{i,0},\ldots,X_{i,c_i-1}\}_{i=1}^m\right)}\mathcal{O}_{X_\infty}$, hence it is the radical of a finitely generated ideal, to prove that it is a stable point of X_∞ it suffices to show that

$$P_v \notin (\operatorname{Sing} X)_\infty. \tag{3.6}$$

Let $\nu := \nu_v$ be the toric divisorial valuation defined by the orbit of $\langle v\rangle$ in $X_{\Sigma'}$ where Σ' is a subdivision of σ which contains $\langle v\rangle$. Given $l \in k[\sigma^\vee \cap M]$, we have

$$L_0 \notin \sqrt{(\{F_{\underline{a},n}\}_{n\geqslant 0,\underline{a}\in G_\Lambda})}.$$

In addition, if $l = \sum_{u\in\sigma^\vee\cap M} \lambda_u \aleph^u$, then $\nu_v(l) = \inf\{\langle u, v\rangle \ / \ \lambda_u \neq 0\}$ and hence $\overline{L_{\nu(l)}} = \delta(L'_0)$ where l' is the initial form of l with respect to ν (see [15, 3.3] or [5, Section 1.1]). Since $L'_0 \neq 0$ in $\mathcal{O}_{(X_\infty)_{\mathrm{red}}}$, this implies that $\overline{L_{\nu(l)}} \notin \left(\{X_{i,0},\ldots,X_{i,c_i-1}\}_{i=1}^m\right)\mathcal{O}_{(X_\infty)_{\mathrm{red}}}$ and hence $L_{\nu(l)} \notin P_v$. From this it follows that ν_v is the valuation ν_{P_v} defined by P_v and also that P_v is not in the space of arcs of the hypersurface $l = 0$. In particular, taking l an element of the Jacobian ideal of X, this implies that (3.6) holds. □

Remark 3.2. — Let l be an element of the Jacobian ideal of X and keep the notation as before. Then δ induces an isomorphism

$$\left(\mathcal{O}_{X_\infty} \ / \ \left(\{X_{i,0},\ldots,X_{i,c_i-1}\}_{i=1}^m\right)\right)_{L_{\nu(l)}} \cong \left(\mathcal{O}_{X_\infty}\right)_{L_0}.$$

Since $\left(\mathcal{O}_{X_\infty}\right)_{L_0}$ is a domain, it follows that

$$P_v\left(\mathcal{O}_{X_\infty}\right)_{L_{\nu(l)}} = \left(\{X_{i,0},\ldots,X_{i,c_i-1}\}_{i=1}^m\right)\left(\mathcal{O}_{X_\infty}\right)_{L_{\nu(l)}}$$

is finitely generated in $\left(\mathcal{O}_{X_\infty}\right)_{L_{\nu(l)}}$.

Definition 3.3. — *With the notation in Lemma 3.1, the ideal P_v will be called the toric stable point of X_∞ associated to $v \in \sigma \cap N$.*

In general, if $X = X_\Sigma$ is a normal toric variety, a stable point P of X_∞ is called a toric stable point if there exists $\sigma \in \Sigma$ such that P lies in $(X_\sigma)_\infty$ and P is the toric stable point of $(X_\sigma)_\infty$ associated to some $v \in \sigma \cap N$.

Remark 3.4. — Note that, in this general case $X = X_\Sigma$, if P is a stable point of $(X_\Sigma)_\infty$, there exists a unique $\sigma \in \Sigma$ such that P is a point in $(X_\sigma)_\infty$. In addition σ is a cone of dimension $d = \dim X$. Moreover, if P is a toric stable point, then the element $v \in \sigma \cap N$ such that $P = P_v$ is precisely v_P.

Note also that, if ν_E is the toric divisorial valuation defined by the 1-dimensional cone $\langle v \rangle$ determined by v in a subdivision of σ, v_E is a primitive element in $\langle v \rangle$ and $e \in \mathbb{N}$ is such that $v = e\, v_E$ then, with the notation in Section 2.3, we have $P_v = P_{eE}$. With the terminology in [14], P_v is the generic point of the maximal divisorial set N_{eE}, denoted by $W(E, e)$ in [14].

Definition 3.5. — *Let $X = X_\Sigma$ be a normal toric variety and let P be a toric stable point of X_∞. We define the* toric height $\operatorname{tcht} P$ *of P as the superior of the lengths r of chains of toric stable points of X_∞ contained in P. Note that, if P is a point in $(X_\sigma)_\infty$ where σ is a cone in Σ, hence $P = P_v$ where $v \in \sigma \cap N$, then the chain is in fact a chain of toric stable points of $(X_\sigma)_\infty$*

$$P_{v_0} \subset P_{v_1} \subset \cdots \subset P_{v_{r-1}} \subset P_{v_r} = P_v, \tag{3.7}$$

where $v_0, \ldots, v_{r-1}, v_r = v \in \sigma \cap N$.

Given $v \in \sigma \cap N$, by a partition of v *we mean $\mathbf{w} = \{(w_j; n_j)\}_{1 \leqslant j \leqslant s}$ where $s \in \mathbb{N}$, $w_1, \ldots, w_s$ are minimal elements of $\sigma \cap N$ and $n_1, \ldots, n_s \in \mathbb{N} \setminus \{0\}$ are such that*

$$v = n_1 w_1 + \cdots + n_s w_s.$$

We denote by $\mathcal{W}_v$ the set of all partitions of v.

Given a partition $\mathbf{w} = \{(w_j; n_j)\}_{1 \leqslant j \leqslant s}$ of v, we define the length of $\mathbf{w}$ *by*

$$\mathrm{l}(\mathbf{w}) := \sum_{j=1}^{s} n_j.$$

Corollary 3.6. — *Let $X = X_\sigma$ be an affine normal toric variety and let $P = P_v$ be a toric stable point of X_∞. Then we have*

$$\dim \mathcal{O}_{X_\infty, P} = \dim \mathcal{O}_{(X_\infty)_{\mathrm{red}}, P} \geqslant \operatorname{tcht} P.$$

Moreover, if $v = 0 \in N$ *then* $\operatorname{tcht} P = 0$ *and, if* $v \neq 0$ *then*

$$\begin{aligned}\operatorname{tcht} P &= \sup\{r / \exists\, w_1, \ldots, w_r \in \sigma \cap N \setminus \{0\} \text{ such that } v = w_1 + \cdots + w_r\} \\ &= \sup\{\mathrm{l}(\mathbf{w}) \;/\; \mathbf{w} \in \mathcal{W}_v\}\,.\end{aligned}$$

Proof. — The first assertion follows from the definition of Krull dimension. For the second one note that, given $v_1, v_2 \in \sigma \cap N$, we have $P_{v_1} \subset P_{v_2}$ if and only if $\langle u, v_1\rangle \leqslant \langle u, v_2\rangle$ for every $u \in \sigma^\vee \cap M$, or equivalently, $v_2 - v_1 \in (\sigma^\vee)^\vee \cap N = \sigma \cap N$ ([19, Section 3]). Hence, given a chain of toric prime ideals (3.7) of maximal length, we must have $v_0 = 0 \in N$ and $v = w_1 + \cdots + w_r$ where $w_i = v_i - v_{i-1} \in \sigma \cap N \setminus \{0\}$. Conversely, if $v = w_1 + \cdots + w_r$ where $w_i \in \sigma \cap N \setminus \{0\}$ then take $v_0 = 0$ and $v_i = w_1 + \cdots + w_i$, $1 \leqslant i \leqslant r$, in (3.7). From this the result follows. □

In the next section we will show that, in general, $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P}$ is not a catenary ring (Corollary 4.4). We will also show that, if $X = X_\Sigma$ is a normal toric variety and P is a toric stable point of X_∞, then

$$\dim \widehat{\mathcal{O}_{X_\infty,P}} = \dim \widehat{\mathcal{O}_{(X_\infty)_{\mathrm{red}},P}} = \operatorname{tcht} P.$$

Therefore, following Question 2.2, we ask:

Question 3.7. — *Let* $X = X_\sigma$ *be an affine normal toric variety and let* $P = P_v$, $v \in \sigma \cap N$, *be a toric stable point of* X_∞. *Is it true that* $\dim \mathcal{O}_{X_\infty,P_v} = \operatorname{tcht} P_v$ *?*

Now, let us describe the completion $\widehat{\mathcal{O}_{X_\infty,P}}$ of the local ring $\mathcal{O}_{X_\infty,P}$ of the space of arcs X_∞ of a toric variety X at a toric stable point P. First, we may suppose with no loss of generality that that X is affine, that is $X = X_\sigma$ where σ is a cone in $N_\mathbb{R}$. Moreover, we may suppose that σ is a strongly convex cone. Let $v \in \sigma \cap M$ be defining P, i.e. $P = P_v$. We will first embed X in a complete intersection variety $X' \subset \mathbb{A}^m_k$ of the same dimension d, so that we have

$$(3.8) \qquad \mathcal{O}_{(X_\infty)_{\mathrm{red}},P} \cong \mathcal{O}_{(X'_\infty)_{\mathrm{red}},P} \quad \text{and} \quad \widehat{\mathcal{O}_{X_\infty,P}} \cong \widehat{\mathcal{O}_{X'_\infty,P}}$$

where we also denote by P the point induced by P in $(X_\infty)_{\mathrm{red}}$, X'_∞ and $(X'_\infty)_{\mathrm{red}}$ (see [31, Proposition 3.7(ii) and Theorem 3.13]).

Keep the notation at the begining of the section, i.e. $\{u_1, \ldots, u_d\} \subset \sigma^\vee \cap M$ is a basis of the free $\mathbb{Z}$-module M, $\{u_1, \ldots, u_d, \ldots, u_m\}$ a system of generators of the semigroup $\sigma^\vee \cap M$, and $\mathcal{O}_X = k[\sigma^\vee \cap M] \cong k[x_1, \ldots, x_m] / I_X$ where we identify the class of x_i with $\aleph^{u_i}$ and I_X is the ideal generated by $\{f_{\underline{a}}\}_{\underline{a} \in \Lambda}$ (see (3.1) and (3.2)). Now, since $\{u_1, \ldots, u_d\}$ is a $\mathbb{Z}$-basis of M, for every j, $d + 1 \leqslant j \leqslant m$, there exists $\underline{a}_j = (a_{j,1}, \ldots, a_{j,m}) \in \Lambda$ with

$a_{j,j} = 1$ and $a_{j,l} = 0$ for $d+1 \leqslant l \leqslant m$, $l \neq j$. That is,

$$u_j + \sum_{i \in \mathcal{J}_j^+ \setminus \{j\}} a_{j,i}\, u_i = \sum_{i \in \mathcal{J}_j^-} b_{j,i}\, u_i. \tag{3.9}$$

where $\mathcal{J}_j^+ \coloneqq \mathcal{J}^+(\underline{a}_j) \subseteq \{1,\dots,d\} \cup \{j\}$, $\mathcal{J}_j^- \coloneqq \mathcal{J}^-(\underline{a}_j) \subseteq \{1,\dots,d\}$ and $b_{j,i} = -a_{j,i}$ for $i \in \mathcal{J}_j^-$. For $d+1 \leqslant j \leqslant m$, let $f_j \coloneqq f_{\underline{a}_j} \in I_X$ (see (3.2)), that is,

$$f_j = \prod_{i \in \mathcal{J}_j^+} x_i^{a_{j,i}} - \prod_{i \in \mathcal{J}_j^-} x_i^{b_{j,i}} = x_j \prod_{i \in \mathcal{J}_j^+ \setminus \{j\}} x_i^{a_{j,i}} - \prod_{i \in \mathcal{J}_j^-} x_i^{b_{j,i}}. \tag{3.10}$$

We define X' to be the complete intersection variety in $\mathbb{A}_k^m$ given by

$$X' = \operatorname{Spec} k[x_1,\dots,x_d,x_{d+1},\dots,x_m] \,/\, (\{f_j\}_{j=d+1}^m).$$

Then X' contains X and, if we set $l \coloneqq x_1 \cdots x_d$ then $X' \setminus Z(l) = X \setminus Z(l)$, precisely

$$(\mathcal{O}_{X'})_l = (\mathcal{O}_X)_l \cong k[x_1,\dots,x_d]_{x_1 \cdots x_d}$$

(see (3.3)). From this it follows that, if, with the notation in Lemma 3.1,

$$c_i \coloneqq \langle u_i, v\rangle \quad \text{for} \quad 1 \leqslant i \leqslant m,$$

then

$$\left(\mathcal{O}_{(X_\infty)_{\mathrm{red}}}\right)_{L_c} \cong \left(\mathcal{O}_{(X'_\infty)_{\mathrm{red}}}\right)_{L_c}$$

where $L_c \coloneqq X_{1,c_1} \cdots X_{d,c_d}$, hence $L_c \notin P_v$ and we conclude (3.8) (see [31, Proof of Proposition 3.7(ii) and Theorem 3.13]).

Let us now follow the procedure in [31, Corollary 5.6] to describe the ring $\widehat{\mathcal{O}_{X'_\infty,P_v}}$. First, for $d+1 \leqslant j \leqslant m$, set $f'_j \coloneqq \frac{\partial f_j}{\partial x_j} = \prod_{i \in \mathcal{J}_j^+ \setminus \{j\}} x_i^{a_{j,i}}$ and

$$\epsilon_j \coloneqq \nu_v(f'_j) = \sum_{i \in \mathcal{J}_j^+ \setminus \{j\}} a_{j,i} c_i = \sum_{i \in \mathcal{J}_j^-} b_{j,i} c_i - c_j$$

(recall (3.9)). Note that F'_{j,ϵ_j} does not belong to P_v (Lemma 3.1). For $n \geqslant 0$, we have

$$\begin{aligned} &\frac{\partial F_{j,\epsilon_j+n}}{\partial X_{j,n}} = F'_{j,\epsilon_j} \mod \left(\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m\right) \\ &\frac{\partial F_{j,\epsilon_j+n}}{\partial X_{j,n'}} = 0 \mod \left(\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m\right) \quad \text{for } n' > n. \end{aligned} \tag{3.11}$$

This implies that from F_{j,ϵ_j+n}, $n \geqslant c_j$, we can eliminate $X_{j,n}$ modulo $\left(\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m\right)$ in the ring $\left(\mathcal{O}_{X'_\infty}\right)_{\prod_{j=d+1}^m F'_{j,\epsilon_j}}$. In addition, we have

$$F_{j,n} \in \left(\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m\right) \quad \text{for} \quad 0 \leqslant n \leqslant \epsilon_j + c_j - 1. \tag{3.12}$$

Therefore

$$\left(\mathcal{O}_{X'_\infty}\right)_{\prod_{j=d+1}^m F'_{j,\epsilon_j}} \Big/ \left(\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m\right) \\ \cong k\left[\{X_{i,n_i}\}_{1\leqslant i\leqslant d,\, n_i\geqslant c_i}\right]_{\prod_{j=d+1}^m F'_{j,\epsilon_j}}$$

is a domain, and hence $\left(\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m\right)\left(\mathcal{O}_{X'_\infty}\right)_{\prod_{j=d+1}^m F'_{j,\epsilon_j}}$ is a prime ideal. We conclude that

$$P_v\left(\mathcal{O}_{X'_\infty}\right)_{\prod_{j=d+1}^m F'_{j,\epsilon_j}} = \left(\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m\right)\left(\mathcal{O}_{X'_\infty}\right)_{\prod_{j=d+1}^m F'_{j,\epsilon_j}}$$

and the residue field of P_v in $\mathcal{O}_{X'_\infty}$ is

$$\kappa(P_v) \cong k\left(\{X_{i,n_i}\}_{1\leqslant i\leqslant d,\, n_i\geqslant c_i}\right).$$

We consider the embedding $\kappa(P_v) \hookrightarrow \widehat{\mathcal{O}_{X'_\infty,P_v}}$ which sends $X_{i,n}$, $1 \leqslant i \leqslant d$, $n \geqslant c_i$, to $X_{i,n} \in \widehat{\mathcal{O}_{X'_\infty,P_v}}$ and we identify $\kappa(P_v)$ with $k\left(\{X_{i,n_i}\}_{1\leqslant i\leqslant d,\, n_i\geqslant c_i}\right)$. In particular, for $d+1 \leqslant j \leqslant m$, $n \geqslant c_j$, we have defined $X_{j,n}^{(0)} \in \kappa(P_v)$ such that $X_{j,n} - X_{j,n}^{(0)} \in \left(\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m\right)$, even more,

$$F_{j,\epsilon_j+n} \equiv F'_{j,\epsilon_j}\cdot\left(X_{j,n} - X_{j,n}^{(0)}\right) \mod \left(\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m\right).$$

Arguing recursively on $r \geqslant 1$ and $n \geqslant c_j$, with the lexicographical order on (r,n), and reasoning as in Corollary 5.6 in [31] it follows that, for $d+1 \leqslant j \leqslant m$, $r \geqslant 0, n \geqslant c_j$, there exists

$$X_{j,n}^{(r)} \in \kappa(P_v)\left[\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m\right]$$

such that,

$$F_{j,\epsilon_j+n} \equiv F'_{j,\epsilon_j}\cdot\left(X_{j,n} - X_{j,n}^{(r)}\right) \mod \left(\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m\right)^{r+1}. \tag{3.13}$$

Precisely, set (r,n), $r \geqslant 1$, $n \geqslant c_j$, and suppose that the statement holds for $(r',n') < (r,n)$. Thus, for $d+1 \leqslant j \leqslant m$, $X_{j,n'}^{(r)}$, $n' < n$, and $X_{j,n''}^{(r-1)}$, $n'' \geqslant n$, are defined. By (3.11), F_{j,ϵ_j+n} is equal, modulo $\left(\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m\right)^{r+1}$, to the element of $\kappa(P_v)[\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m][Y_{j,n}]$ obtained by replacing $X_{j,n'}$ by $X_{j,n'}^{(r)}$ for $n' < n$ and $X_{j,n''}$ by $X_{j,n''}^{(r-1)}$ for $n'' > n$, and moreover, it is equal to

$$F'_{j,\epsilon_j}\cdot\left(X_{j,n} - X_{j,n}^{(r)}\right) \mod \left(\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m\right)^{r+1}$$

for some $X_{j,n}^{(r)} \in \kappa(P_v)[\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m]$. In particular, $X_{j,n}^{(r)} \equiv X_{j,n}^{(r-1)}$ mod $\left(\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m\right)^r$. Therefore, the above equalities define series $\widetilde{X}_{j,n} \in \kappa(P_v)\left[\left[\{X_{i,0},\dots,X_{i,c_i-1}\}_{i=1}^m\right]\right]$, $n \geqslant c_j$, and we conclude:

Proposition 3.8. — *The following holds:*

$$\widehat{\mathcal{O}_{X_\infty,P_v}} \cong \widehat{\mathcal{O}_{X'_\infty,P_v}} \cong \kappa(P_v)[[\{X_{i,0},\ldots,X_{i,c_i-1}\}_{i=1}^m]] \Big/ \left(\{\widetilde{F}_{j,n'_j}\}_{\substack{d+1\leqslant j\leqslant m\\ 0\leqslant n'_j\leqslant \epsilon_j+c_j-1}}\right) \tag{3.14}$$

where

$$\kappa(P_v) \cong k\left(\{X_{i,n_i}\}_{1\leqslant i\leqslant d,\, n_i\geqslant c_i}\right). \tag{3.15}$$

and, for $d+1\leqslant j\leqslant m$, $0\leqslant n\leqslant \epsilon_j+c_j-1$, $\widetilde{F}_{j,n}$ *is obtained from* $F_{j,n}$ *by substituting* $X_{j,n'}$ *by the series* $\widetilde{X}_{j,n'}\in\kappa(P_v)\left[\left[\{X_{i,0},\ldots,X_{i,c_i-1}\}_{i=1}^m\right]\right]$ *for* $c_j\leqslant n'\leqslant n$.

Let us give another description of the ring $\widehat{\mathcal{O}_{X_\infty,P_v}}$ which is motivated by the Weierstrass factorization of the series defined by a deformation of a general element in the subset $Z(P)$ of X_∞. Recall that $\{u_i\}_{i=1}^d\subset\sigma^\vee\cap M$ is a $\mathbb{Z}$-basis of M. Let $\{u_i^*\}_{i=1}^d\subset N$ be its dual basis. For $1\leqslant j\leqslant m$ set

$$\bar{x}_j(t) \coloneqq t^{c_j}+\sum_{n=0}^{c_j-1}\overline{X}_{j,n}t^n,\quad x_j(t)\coloneqq\prod_{i=1}^d\left(\sum_{n\geqslant 0}U_{i,n}t^n\right)^{\langle u_j,u_i^*\rangle}\cdot\bar{x}_j(t) \tag{3.16}$$

where $U_{i,n}$ and $\overline{X}_{j,n}$ are variables. Note that, for the *binomial* equations f_j, $d+1\leqslant j\leqslant m$, we have

$$f_j(x_1(t),\ldots,x_m(t))=\left(\sum_{n\geqslant 0}\Lambda_{j,n}t^n\right)f_j(\bar{x}_1(t),\ldots,\bar{x}_m(t)) \tag{3.17}$$

where $\Lambda_{j,n}\in k\left[\{U_{i,n'}\}_{\substack{1\leqslant i\leqslant d\\ 0\leqslant n'\leqslant n}}\right]$ are such that

$$\sum_{n\geqslant 0}\Lambda_{j,n}t^n=\prod_{i=1}^d\left(\sum_{n\geqslant 0}U_{i,n}t^n\right)^{\left\langle u_j+\sum_{r\in\mathcal{J}_j^+\setminus\{j\}}a_{j,r}u_r,u_i^*\right\rangle}$$

(recall (3.9)). Let us consider the Taylor development of $f_j(\bar{x}_1(t),\ldots,\bar{x}_m(t))$:

$$f_j(\bar{x}_1(t),\ldots,\bar{x}_m(t))=\overline{F}_{j,0}+\overline{F}_{j,1}t+\cdots+\overline{F}_{j,c_j+\epsilon_j-1}t^{c_j+\epsilon_j-1}. \tag{3.18}$$

where

$$\overline{F}_{j,n}\in k\left[\{\overline{X}_{i,n'_i}\}_{1\leqslant i\leqslant m,\,0\leqslant n'_i\leqslant\min\{c_j-1,n\}}\right].$$

In fact, if we set $\overline{X}_{j,c_j}\coloneqq 1$, $\overline{X}_{j,n}\coloneqq 0$ for $n>c_j$ and $\underline{X}_n=(\overline{X}_{1,n},\ldots,\overline{X}_{m,n})$ then $\overline{F}_{j,n}=F_{j,n}(\underline{X}_0,\ldots,\underline{X}_n)$. Note that $\overline{F}_{j,n}=0$ for $n\geqslant c_j+\epsilon_j$.

Proposition 3.9. — *The following holds:*

$$\begin{aligned}\widehat{\mathcal{O}_{X_\infty,P_v}} &\cong \widehat{\mathcal{O}_{X'_\infty,P_v}} \\ &\cong \kappa(P_v)\left[\left[\left\{\overline{X}_{i,0},\dots,\overline{X}_{i,c_i-1}\right\}_{i=1}^m\right]\right] \Big/ \left(\{\overline{F}_{j,n'_j}\}_{\substack{d+1\leqslant j\leqslant m \\ 0\leqslant n'_j\leqslant \epsilon_j+c_j-1}}\right)\end{aligned} \tag{3.19}$$

where the $\overline{F}_{j,n}$*'s are the polynomials in* $k\left[\left\{\overline{X}_{i,0},\dots,\overline{X}_{i,c_i-1}\right\}_{i=1}^m\right]$ *defined in* (3.18) *and*

$$\kappa(P_v) \cong k\left(\{U_{i,n}\}_{1\leqslant i\leqslant d,\, n\geqslant 0}\right). \tag{3.20}$$

Proof. — First note that there is an isomorphism of fields

$$k\left(\{X_{i,n_i}\}_{1\leqslant i\leqslant d,\, n_i\geqslant c_i}\right) \cong k\left(\{U_{i,n}\}_{1\leqslant i\leqslant d,\, n\geqslant 0}\right) \quad X_{i,n}\mapsto U_{i,n-c_i}.$$

Therefore, applying (3.15), we obtain the isomorphism (3.20).

Now, set

$$\widehat{A} := k\left(\{U_{i,n}\}_{\substack{1\leqslant i\leqslant d\\ n\geqslant 0}}\right)\left[\left[\left\{\overline{X}_{i,0},\dots,\overline{X}_{i,c_i-1}\right\}_{i=1}^m\right]\right] \Big/ \left(\{\overline{F}_{j,n'_j}\}_{\substack{d+1\leqslant j\leqslant m \\ 0\leqslant n'_j\leqslant \epsilon_j+c_j-1}}\right)$$

Since $\widehat{A}$ is a complete local ring, in order to define a local morphism $\Theta : \widehat{\mathcal{O}_{X'_\infty,P_v}} \to \widehat{A}$ it suffices to define a local morphism $\mathcal{O}_{X'_\infty,P_v} \to \widehat{A}$. By the representability property of X_∞ (see (2.1)), it is equivalent to define $\theta : \mathcal{O}_{X'} \to \widehat{A}[[t]]$ such that the arc defined by the morphism $\bar\theta : \mathcal{O}_{X'} \to \kappa(P_v)[[t]]$ obtained by composition of θ with

$$\widehat{A}[[t]] \to \widehat{A}/\left(\left\{\overline{X}_{i,0},\dots,\overline{X}_{i,c_i-1}\right\}_{i=1}^m\right)[[t]],$$

is the point P_v of X'_∞.

Recall that

$$\mathcal{O}_{X'} = k[x_1,\dots,x_d,x_{d+1},\dots,x_m] \,/\, \left(\{f_j\}_{j=d+1}^m\right).$$

Let us define $\theta : \mathcal{O}_{X'} \to \widehat{A}[[t]]$ by

$$\theta(x_j) := \prod_{i=1}^d \left(\sum_{n\geqslant 0} U_{i,n}t^n\right)^{\langle u_j,u_i^*\rangle} \cdot \left(t^{c_j} + \sum_{n=0}^{c_j-1} \overline{X}_{j,n}t^n\right) \in \widehat{A}[[t]] \tag{3.21}$$

for $1\leqslant j\leqslant m$ (recall the equalities (3.16)). By (3.17) and (3.18), we have that, for $d+1\leqslant j\leqslant m$,

$$\theta(f_j) = \left(\sum_{n\geqslant 0}\Lambda_{j,n}t^n\right)\left(\sum_{n=0}^{c_j+\epsilon_j-1} \overline{F}_{j,n}t^n\right) = 0 \in \widehat{A}[[t]].$$

Hence θ is well defined. Moreover, the arc $\bar{\theta} : \mathcal{O}_{X'} \to \kappa(P_v)[[t]]$ is given by

$$\bar{\theta}(x_j) \coloneqq \prod_{i=1}^{d} \left(\sum_{n \geqslant 0} U_{i,n} t^n \right)^{\langle u_j, u_i^* \rangle} \cdot t^{c_j} \quad \text{for } 1 \leqslant j \leqslant m$$

which defines the point P_v of X'_∞ (recall the isomorphism (3.20)). Therefore θ defines a morphism of local rings $\Theta : \widehat{\mathcal{O}_{X'_\infty, P_v}} \to \widehat{A}$.

Now let $\widehat{B}$ be the ring

$$k\left(\{X_{i,n_i}\}_{\substack{1 \leqslant i \leqslant d \\ n_i \geqslant c_i}} \right) \left[\left[\left\{ X_{j,0}, \dots, X_{j,c_j-1} \right\}_{j=1}^{m} \right]\right] \Big/ \left(\{\widetilde{F}_{j,n'_j}\}_{\substack{d+1 \leqslant j \leqslant m \\ 0 \leqslant n'_j \leqslant \epsilon_j + c_j - 1}} \right).$$

Applying the isomorphism (3.14) in Proposition 3.8, we have defined a morphism $\Theta : \widehat{B} \longrightarrow \widehat{A}$. In fact, Θ is induced by the morphism of k-algebras

$$\begin{aligned} \mathbf{\Theta} : k\left(\{X_{i,n_i}\}_{\substack{1 \leqslant i \leqslant d \\ n_i \geqslant c_i}} \right) &\left[\left[\left\{ X_{j,0}, \dots, X_{j,c_j-1} \right\}_{j=1}^{m} \right]\right] \\ &\longrightarrow k\left(\{U_{i,n}\}_{\substack{1 \leqslant i \leqslant d \\ n \geqslant 0}} \right) \left[\left[\left\{ \overline{X}_{j,0}, \dots, \overline{X}_{j,c_j-1} \right\}_{j=1}^{m} \right]\right] \end{aligned}$$

obtained by identifying the coefficients in t^n, $n \geqslant 0$, in the series

$$\sum_{n \geqslant 0} X_{i,n} t^n = \left(\sum_{n \geqslant 0} U_{i,n} t^n \right) \cdot \left(t^{c_i} + \sum_{n=0}^{c_i - 1} \overline{X}_{i,n} t^n \right) \quad \text{for } 1 \leqslant i \leqslant d$$

and, for $d+1 \leqslant j \leqslant m$, identifying

$$\sum_{n=0}^{c_j-1} X_{j,n} t^n = \prod_{i=1}^{d} \left(\sum_{n \geqslant 0} U_{i,n} t^n \right)^{\langle u_j, u_i^* \rangle} \cdot \left(t^{c_j} + \sum_{n=0}^{c_j-1} \overline{X}_{j,n} t^n \right) \mod (t^{c_j})$$

that is, Θ sends $\{X_{j,n}\}_{n=0}^{c_j-1}$ to the first c_j terms of the right hand side term in the previous equality.

Note that $\mathbf{\Theta}$ is an isomorphism. Let $\mathbf{\Theta}^{-1}$ be its inverse. Then, from the second equality in (3.11), (3.12), (3.13) and (3.17) it follows that, for $d+1 \leqslant j \leqslant m$, $0 \leqslant n \leqslant \epsilon_j + c_j - 1$,

$$\begin{aligned} \mathbf{\Theta}(\widetilde{F}_{j,n}) = \widetilde{\Lambda}_{j,n;n} \overline{F}_{j,0} + \dots + \widetilde{\Lambda}_{j,0;n} \overline{F}_{j,n} \\ + \Gamma_{j,n+1;n} \overline{F}_{j,n+1} + \dots + \Gamma_{j,\epsilon_j+c_j-1;n} \overline{F}_{j,\epsilon_j+c_j-1} \end{aligned}$$

where $\widetilde{\Lambda}_{j,n';n}, \Gamma_{j,r;n} \in k\left(\{U_{i,n}\}_{1 \leqslant i \leqslant d, n \geqslant 0}\right) \left[\left[\left\{ \overline{X}_{j,0}, \dots, \overline{X}_{j,c_j-1} \right\}_{j=1}^{m} \right]\right]$ and, for $0 \leqslant n' \leqslant n, n+1 \leqslant r \leqslant \epsilon_j + c_j - 1$,

$$\widetilde{\Lambda}_{j,n';n} = \Lambda_{j,n'}, \quad \Gamma_{j,r;n} = 0 \mod \left(\left\{ \overline{X}_{j,0}, \dots, \overline{X}_{j,c_j-1} \right\}_{j=1}^{m} \right).$$

Since $\Lambda_{j,0}=\prod_{i=1}^{d}U_{i,0}^{\langle u_j+\sum_{r\in\mathcal{J}_j^+\setminus\{j\}}a_{j,r}u_r,u_i^*\rangle}$ is invertible, $\widetilde{\Lambda}_{j,0;n}$ is also invertible, and from this it follows that

$$\mathbf{\Theta}^{-1}\left(\{\overline{F}_{j,n'_j}\}_{\substack{d+1\leqslant j\leqslant m\\ 0\leqslant n'_j\leqslant\epsilon_j+c_j-1}}\right)\subseteq\left(\{\widetilde{F}_{j,n'_j}\}_{\substack{d+1\leqslant j\leqslant m\\ 0\leqslant n'_j\leqslant\epsilon_j+c_j-1}}\right).$$

Therefore $\mathbf{\Theta}^{-1}$ induces a morphism $\widehat{A}\to\widehat{B}$ which is the inverse of $\Theta:\widehat{B}\to\widehat{A}$. That is, Θ is an isomorphism and this concludes the proof. □

Remark 3.10. — For every $\underline{a}\in\Lambda$ (see (3.1)) we have

$$f_{\underline{a}}(x_1(t),\dots,x_m(t))$$
$$=\left(\prod_{i=1}^{d}\left(\sum_{n\geqslant 0}U_{i,n}t^n\right)^{\left\langle\sum_{r\in\mathcal{J}^+(\underline{a})}a_{j,r}u_r,u_i^*\right\rangle}\right)f_{\underline{a}}(\overline{x}_1(t),\dots,\overline{x}_m(t))$$

where

$$f_{\underline{a}}(\overline{x}_1(t),\dots,\overline{x}_m(t))=\overline{F}_{\underline{a},0}+\overline{F}_{\underline{a},1}t+\cdots+\overline{F}_{\underline{a},n_{\underline{a}}-1}t^{c_{\underline{a}}-1},$$

$c_{\underline{a}}=\sum_{i\in\mathcal{J}^+(\underline{a})}a_ic_i\in\mathbb{N}$ and $\overline{F}_{\underline{a},n}\in k\left[\{\overline{X}_{i,n'}\}_{1\leqslant i\leqslant m,\,0\leqslant n'\leqslant\min\{c_j-1,n\}}\right]$. Since $f_{\underline{a}}\in I_X$, the image of $f_{\underline{a}}\in\mathcal{O}_{\mathbb{A}_k^m}$ in $\widehat{\mathcal{O}_{X_\infty,P_v}}$, hence in $\widehat{A}$, is zero. Therefore

$$\overline{F}_{\underline{a},n}\in\left(\{\overline{F}_{j,n'_j}\}_{\substack{d+1\leqslant j\leqslant m\\ 0\leqslant n'_j\leqslant\epsilon_j+c_j-1}}\right)\quad\text{for } 0\leqslant n<c_{\underline{a}}$$

and (3.19) in Proposition 3.9 can be stated as

$$\widehat{\mathcal{O}_{X_\infty,P_v}}\cong\kappa(P_v)\left[\left[\left\{\overline{X}_{i,0},\dots,\overline{X}_{i,c_i-1}\right\}_{i=1}^{m}\right]\right]\Big/\left(\{\overline{F}_{\underline{a},n}\}_{\underline{a}\in\Lambda,\ 0\leqslant n<n_{\underline{a}}}\right).$$

In relation with the Drinfeld–Grinbeg–Kazhdan theorem, and with the previous notation, Theorem 5.2 in [4] asserts that, for every k-point $\gamma\in X_\infty$ in the open subset of $Z(P_v)$ defined by the conditions $\gamma\notin(X\setminus T)_\infty$ and $\operatorname{ord}_t h_\gamma^\sharp(x_i)=c_i$ for $1\leqslant i\leqslant m$, we have

$$\widehat{\mathcal{O}_{X_\infty,\gamma}}$$
$$\cong k\left[\left[\left\{\overline{X}_{i,0},\dots,\overline{X}_{i,c_i-1}\right\}_{i=1}^{m}\right]\right]\Big/\left(\{\overline{F}_{\underline{a},n}\}_{\underline{a}\in\Lambda,\ 0\leqslant n<n_{\underline{a}}}\right)\widehat{\otimes}\,k[[\{T_i\}_{i\in\mathbb{N}}]].$$

The first ring in the right hand side of the previous isomorphism is then called a finite dimensional formal model of $\widehat{\mathcal{O}_{X_\infty,\gamma}}$.

4. Applying wedges to compute the dimension of $\widehat{\mathcal{O}_{X_\infty,P_v}}$

In this section we will prove that, given a toric stable point P of the space of arcs of a normal toric variety X, the dimension of the ring $\widehat{\mathcal{O}_{X_\infty,P}}$ is equal to the toric height of P. The main idea in this section is to apply wedges in order to understand $\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P}}$.

Assume that $X = X_\sigma$ is affine and let $P = P_v$ where $v \in \sigma \cap N$. Recall that a K-r-wedge on X is a k-morphism $\Phi : \operatorname{Spec} K[[\underline{\xi}, t]] \to X$ where $\underline{\xi} = (\xi_1, \dots, \xi_r)$, or equivalently $\varphi : \operatorname{Spec} K[[\underline{\xi}]] \to X_\infty$. The special arc of Φ is P_v if the image by φ of the closed point of $K[[\underline{\xi}]]$ is P_v, or equivalently, it induces a morphism $\widehat{\varphi} : \operatorname{Spec} K[[\underline{\xi}]] \to \operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_v}}$ (see Section 2.4).

Lemma 4.1. — *Given a K-r-wedge $\Phi : \operatorname{Spec} K[[\underline{\xi}, t]] \to X$, there exist $\{w_j\}_{j=1}^s \subset \sigma \cap N$ and irreducible elements $\{p_j\}_{j=1}^s$ of $K[[\underline{\xi}, t]]$ such that $(p_j, p_{j'}) = 1$ for $j \neq j'$ and the morphism of rings $\Phi^\sharp : k[\sigma^\vee \cap M] \to K[[\underline{\xi}, t]]$ induced by Φ is given by*

$$\aleph^u \mapsto o_u \prod_{j=1}^{s} p_j^{\langle u, w_j\rangle} \quad \text{for } u \in \sigma^\vee \cap M, \tag{4.1}$$

where o_u is a unit in $K[[\underline{\xi}, t]]$, and moreover, the morphism $\aleph^u \mapsto o_u$, $u \in \sigma^\vee \cap M$, defines a wedge on the torus

$$\psi : \operatorname{Spec} K[[\underline{\xi}, t]] \longrightarrow T = \operatorname{Spec} k[M].$$

Furthermore, $\{w_j\}_{j=1}^s \subset \sigma \cap N$ are uniquely determined and the irreducible elements p_j, $1 \leqslant j \leqslant s$, are uniquely determined modulo product by a unit.

In addition, if the wedge Φ is centered at a stable point P then we have

$$v_P = \sum_{j=1}^{s} \operatorname{ord}_t p_j(\underline{0}, t)\, w_j. \tag{4.2}$$

Proof. — Recall that $\{u_i\}_{i=1}^d \subset \sigma^\vee \cap M$ is a $\mathbb{Z}$-basis of M and $\{u_i^*\}_{i=1}^d \subset N$ its dual basis. Since $K[[\underline{\xi}, t]]$ is factorial, by looking at the factorization of the images of $x_i = \aleph^{u_i}$, $1 \leqslant i \leqslant d$, by $\Phi^\sharp : k[\sigma^\vee \cap M] \to K[[\underline{\xi}, t]]$, we obtain a finite number of irreducible elements $\{p_j\}_{j=1}^s$ in $K[[\underline{\xi}, t]]$, with $(p_j, p_{j'}) = 1$ for $j \neq j'$, uniquely determined modulo product by a unit, such that, for every $u \in \sigma^\vee \cap M$, $\Phi^\sharp(\aleph^u)$ factors in $K[[\underline{\xi}, t]]$ as a product of powers of $\{p_j\}_{j=1}^s$ modulo a unit. Moreover, for $1 \leqslant j \leqslant s$,

$$w_j \coloneqq \sum_{i=1}^{d} \operatorname{ord}_{p_j} \Phi^\sharp(\aleph^{u_i})\, u_i^*$$

is the unique element in N which satisfies

$$\operatorname{ord}_{p_j} \Phi^\sharp(\aleph^u) = \langle u, w_j\rangle \quad \text{for all } u \in \sigma^\vee \cap M.$$

Since $\langle u, w_j\rangle \geqslant 0$ for all $u \in \sigma^\vee$, we have that $w_j \in \sigma \cap N$ for $1 \leqslant j \leqslant s$. Thus, $\Phi^\sharp$ is defined by (4.1) where o_u is a unit in $K[[\underline{\xi}, t]]$. In addition, the morphism $\aleph^u \mapsto o_u$, $u \in \sigma^\vee \cap M$, defines a wedge on the torus $\psi : \operatorname{Spec} K[[\underline{\xi}, t]] \longrightarrow T$.

Now, the condition that Φ is centered at P implies that

$$\begin{aligned}\langle u, v_P\rangle = \nu_P(\aleph^u) &= \operatorname{ord}_t \left(\prod_{j=1}^{s} p_j(\underline{0}, t)^{\langle u, w_j\rangle}\right)\\ &= \sum_{j=1}^{s} \operatorname{ord}_t p_j(\underline{0}, t)\langle u, w_j\rangle\end{aligned}$$

for all $u \in \sigma^\vee \cap M$ (recall (3.4) and (4.1)). Therefore (4.2) holds. □

Corollary 4.2. — *If v is a minimal element in $\sigma \cap N \setminus \{0\}$ then the following holds:*

(i) $\operatorname{Spec} \widehat{\mathcal{O}_{X_\infty, P_v}}$ *is irreducible and* $\dim \widehat{\mathcal{O}_{X_\infty, P_v}} = 1$.
(ii) $\dim \mathcal{O}_{X_\infty, P_v} = 1$.

Proof. — First, Corollary 5.8 in [31], applied to the toric variety X and the stable point P_v of X_∞, asserts that (i) and (ii) are equivalent conditions. In addition, let Σ be the fan defined by σ and let us denote by $\Sigma(1)$ the set of 1-dimensional cones of Σ. On the one hand, if $\langle v\rangle \in \Sigma(1)$ then it defines a divisor D_v on X. Since X is regular at the generic point of D_v, we have $\widehat{\mathcal{O}_{X_\infty, P_v}} \cong \kappa(P_v)[[U_0]]$, therefore (i) and (ii) hold in this case. On the other hand, if σ is a regular cone, for any minimal element v in $\sigma \cap N \setminus \{0\}$ we have $\langle v\rangle \in \Sigma(1)$, and hence (i) and (ii) hold.

Now suppose that $\langle v\rangle \notin \Sigma(1)$. Then there exists $\tau \in \Sigma \setminus \Sigma(1)$ such that $v \in \overset{\circ}{\tau}$. Since v is a minimal element in $\sigma \cap N \setminus \{0\}$, τ is singular. Thus v defines an essential divisorial valuation. We will apply [31, Corollary 5.12]. Precisely, we will prove that:

(iii) If Σ' is a regular subdivision of Σ and $Y = X_{\Sigma'} \to X$ the corresponding resolution of singularities, then for every wedge $\Phi : \operatorname{Spec} K[[\xi, t]] \to X$ with special arc P_v, Φ lifts to Y.

Note that Corollary 5.12 in [31], applied to the toric variety X and the stable point P_v of X_∞, asserts that (i), (ii) and (iii) are equivalent conditions.

Finally, let us prove (iii). Let Σ' be a regular subdivision of Σ. Let $\Phi : \operatorname{Spec} K[[\xi, t]] \to X$ be a wedge with special arc P_v. Then, from

Lemma 4.1 it follows that $\Phi^\sharp$ is defined by

$$\aleph^u \mapsto o_u p^{\langle u,v\rangle}$$

where $p \in K[[\xi,t]]$ is irreducible and $\operatorname{ord}_t p(0,t) = 1$. In fact, since $v_{P_v} = v$ is a minimal element of $\sigma \cap N$, in (4.2) only one term appears. Then, if σ' is a d-dimensional cone in Σ' such that $v \in \sigma'$, the wedge Φ lifts to $X_{\sigma'}$. Therefore Φ lifts to $X_{\Sigma'}$. This concludes the proof. □

Corollary 4.3. — *Given $v \in \sigma \cap N \setminus \{0\}$, let us consider a chain of prime ideals in $\mathcal{O}_{X_\infty}$*

$$P_{v_0} \subset P_{v_1} \subset \cdots \subset P_{v_{r-1}} \subset P_{v_r} = P_v, \tag{4.3}$$

where $v_0 = 0 \in N$, $v_1, \ldots, v_{r-1}, v_r = v \in \sigma \cap N$. If $v_{l+1} - v_l$ is a minimal element of $\sigma \cap N$ for $0 \leqslant l < r$, then (4.3) *defines a saturated chain of prime ideals in $\mathcal{O}_{X_\infty, P_v}$.*

Proof. — First note that $P_0 = \sqrt{(0)}$ (see Lemma 3.1). Now, fix l, $0 \leqslant l \leqslant r-1$, and set $c_{l,i} := \langle u_i, v_l\rangle$ for $1 \leqslant i \leqslant m$. By the definition of P_{v_l}, the natural morphism $\mathcal{O}_{X_\infty} \to \mathcal{O}_{(X_\infty)_{\mathrm{red}}}$ induces an isomorphism

$$\mathcal{O}_{X_\infty} \,/\, P_{v_l} \cong \mathcal{O}_{(X_\infty)_{\mathrm{red}}} \,/\, \left(\left\{X_{i,0}, \ldots, X_{i,c_{l,i}-1}\right\}_{i=1}^{m}\right).$$

Hence, applying (3.5), we obtain an isomorphism $\mathcal{O}_{X_\infty}/P_{v_l} \cong \mathcal{O}_{(X_\infty)_{\mathrm{red}}}$. The image of $P_{v_{l+1}}$ is $P_{w_{l+1}}\mathcal{O}_{(X_\infty)_{\mathrm{red}}}$ where $w_{l+1} := v_{l+1} - v_l$. Therefore

$$(\mathcal{O}_{X_\infty} \,/\, P_{v_l})_{P_{v_{l+1}}} \cong \mathcal{O}_{(X_\infty)_{\mathrm{red}}, P_{w_{l+1}}}$$

and this is a 1-dimensional ring, since w_{l+1} is a minimal element of $\sigma \cap N$ (Corollary 4.2). Therefore there is no prime ideal strictly contained between P_l and P_{l+1}, hence (4.3) is a saturated chain of prime ideals in $\mathcal{O}_{X_\infty}$. Since all these prime ideals are contained in P_v, it defines a saturated chain of prime ideals in $\mathcal{O}_{X_\infty, P_v}$. □

Corollary 4.4. — *In this corollary, let X be the toric 3-dimensional variety defined by the cone $\sigma = \langle (1,1,0), (1,0,1), (0,1,1)\rangle$ in $\mathbb{R}^3$. Let $P = P_{(2,2,2)}$ be the stable point of X_∞ defined by the element $(2,2,2)$ of $\sigma \cap \mathbb{Z}^3$. Then the following two chains define saturated chains of prime ideals in the ring $\mathcal{O}_{X_\infty, P}$*

(i) $\sqrt{(0)} = P_{(0,0,0)} \subset P_{(1,1,1)} \subset P_{(2,2,2)}$.

(ii) $\sqrt{(0)} = P_{(0,0,0)} \subset P_{(1,1,0)} \subset P_{(2,1,1)} \subset P_{(2,2,2)}$.

Therefore the rings $\mathcal{O}_{X_\infty, P}$ and $\mathcal{O}_{(X_\infty)_{\mathrm{red}}, P}$ are not catenary.

Proof. — It follows from Corollary 4.3 because, for (i), $(1,1,1)$ is a minimal element of $\sigma \cap N$ and, for (ii), $(1,1,0), (1,0,1)$ and $(0,1,1)$ are minimal elements of $\sigma \cap N$. □

Remark 4.5. — The toric variety in Corollary 4.4 appears in [12, Example 6.3], to give a an example of an essential valuation ν_E which is not terminal but belongs to the image of the Nash map, i.e. the set N_E (see Section 2.3) is an irreducible component of the set X_∞^{Sing} of arcs centered in $\operatorname{Sing} X$.

The fact that the ring $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P}$ is not in general catenary was found out in a joint discussion with M. Mustata.

Given $v \in \sigma \cap N$, recall the definition of $\mathcal{W}_v$ (Definition 3.5). For each $\mathbf{w} = \{(w_j; n_j)\}_{1\leqslant j\leqslant s} \in \mathcal{W}_v$, we define a morphism

$$\rho_{\mathbf{w}} : Y^{\mathbf{w}} := \operatorname{Spec}\left(k[y_1,\ldots,y_s,z_1,\ldots,z_d]\right)_{z_1\cdots z_d} \longrightarrow X = \operatorname{Spec} k[\sigma^\vee \cap M].$$

given by

$$\aleph^u \mapsto y_1^{\langle u,w_1\rangle}\cdots y_s^{\langle u,w_s\rangle}\cdot z_1^{\langle u,u_1^*\rangle}\cdots z_d^{\langle u,u_d^*\rangle} \quad \text{for } u\in\sigma^\vee\cap M. \tag{4.4}$$

where recall that $\{u_i^*\}_{i=1}^d \subset N$ is the dual basis of $\{u_i\}_{i=1}^d$. Note that $\rho_{\mathbf{w}}$ is a dominant morphism. In fact, $\rho_{\mathbf{w}}$ is the toric morphism induced by the map of fans $\widetilde{\rho}_{\mathbf{w}} : (\widetilde{N}_{\mathbf{w}}, \widetilde{\sigma}_{\mathbf{w}}) \to (N,\sigma)$ where $\widetilde{N}_{\mathbf{w}} := \mathbb{Z}^{s+d}$, $\widetilde{\sigma}_{\mathbf{w}}$ is the cone $(\mathbb{R}_{\geqslant 0})^s \times \{\underline{0}\}$ and $\widetilde{\rho}_{\mathbf{w}} : \widetilde{N}_{\mathbf{w}} \longrightarrow N$ is defined by

$$\begin{cases} \widetilde{u}_j \mapsto w_j & 1\leqslant j\leqslant s,\\ \widetilde{u}_{s+i} \mapsto u_i^* & 1\leqslant i\leqslant d \end{cases}$$

where $\widetilde{u}_l := (0,\ldots,0,1,0,\ldots,0) \in \mathbb{Z}^{s+d}$, the 1 in the l-th position. The induced map $M \to \widetilde{M_{\mathbf{w}}} = \operatorname{Hom}(\widetilde{N}_{\mathbf{w}},\mathbb{Z})$ is injective.

The morphism $\rho_{\mathbf{w}}$ induces a morphism $(\rho_{\mathbf{w}})_\infty : Y_\infty^{\mathbf{w}} \longrightarrow X_\infty$. We have

$$Y_\infty^{\mathbf{w}} = \operatorname{Spec}\left(k[\underline{Y}_0,\underline{Z}_0]\right)_{Z_{1,0}\cdots Z_{d,0}}[\underline{Y}_1,\underline{Z}_1\ldots,\underline{Y}_n,\underline{Z}_n,\ldots]$$

where $\underline{Y}_n = (Y_{1,n},\ldots,Y_{s,n}), \underline{Z}_n = (Z_{1,n},\ldots,Z_{d,n})$ are uples of variables. Hence,

$$Q^{\mathbf{w}} := (Y_{1,0},\ldots,Y_{1,n_1-1},\ldots,Y_{s,0},\ldots,Y_{s,n_s-1})$$

is a prime ideal of $Y_\infty^{\mathbf{w}}$. In fact, $Q^{\mathbf{w}}$ is the toric stable point of $Y_\infty^{\mathbf{w}}$ associated to $\widetilde{v} := \sum_{1\leqslant j\leqslant s} n_j\widetilde{u}_j$. In an analogous way as in Propositions 3.8 and 3.9, we have:

Proposition 4.6. — *The following holds:*

$$\widehat{\mathcal{O}_{Y_\infty^{\mathbf{w}},Q^{\mathbf{w}}}} \cong \kappa(Q^{\mathbf{w}})[[Y_{1,0},\ldots,Y_{1,n_1-1},\ldots,Y_{s,0},\ldots,Y_{s,n_s-1}]] \tag{4.5}$$

where

$$\kappa(Q^{\mathbf{w}}) \cong k\left(\{Y_{j,n_j'}\}_{\substack{1\leqslant j\leqslant s\\ n_j'\geqslant n_j}} \cup \{Z_{i,n}\}_{\substack{1\leqslant i\leqslant d\\ n\geqslant 0}}\right) \tag{4.6}$$

is the residue field of $Q^{\mathbf{w}}$*. Moreover, we also have*

$$\widehat{\mathcal{O}_{Y_\infty^{\mathbf{w}},Q^{\mathbf{w}}}} \cong k\left(\{V_{j,n}, Z_{i,n}\}_{\substack{1\leqslant j\leqslant s, 1\leqslant i\leqslant d\\ n\geqslant 0}}\right)\left[\left[\left\{\overline{Y}_{j,0},\dots,\overline{Y}_{j,n_j-1}\right\}_{j=1}^{s}\right]\right] \tag{4.7}$$

and the isomorphism

$$k\left(\{Y_{j,n'_j}, Z_{i,n}\}_{\substack{1\leqslant j\leqslant s, 1\leqslant i\leqslant d\\ n\geqslant 0, n'_j\geqslant n_j}}\right)\left[\left[\left\{Y_{j,\overline{n}_j}\right\}_{\substack{1\leqslant j\leqslant s\\ 0\leqslant \overline{n}_j\leqslant n_j-1}}\right]\right] \longrightarrow k\left(\{V_{j,n}, Z_{i,n}\}_{\substack{1\leqslant j\leqslant s, 1\leqslant i\leqslant d\\ n\geqslant 0}}\right)\left[\left[\left\{\overline{Y}_{j,\overline{n}_j}\right\}_{\substack{1\leqslant j\leqslant s\\ 0\leqslant \overline{n}_j\leqslant n_j-1}}\right]\right]$$

is defined by $Z_{i,n}\mapsto Z_{i,n}$ *for* $1\leqslant i\leqslant d, n\geqslant 0$*, and, for* $1\leqslant j\leqslant s$*,* $n\geqslant 0$*, the image of* $Y_{j,n}$ *is determined by identifying the coefficients in* t^n *in the following equality*

$$\sum_{n\geqslant 0} Y_{j,n}\, t^n = \left(\sum_{n\geqslant 0} V_{j,n}\, t^n\right)\left(t^{n_j} + \sum_{n=0}^{n_j-1} \overline{Y}_{j,n}\, t^n\right). \tag{4.8}$$

The image of the prime ideal $Q^{\mathbf{w}}$ of $Y_\infty^{\mathbf{w}}$ by $(\rho_{\mathbf{w}})_\infty$ is the stable point P_v of X_∞. In fact, the image of $\widetilde{v} = \sum_{1\leqslant j\leqslant s} n_j \widetilde{u}_j$ by $\widetilde{\rho}_{\mathbf{w}} : \widetilde{N}_{\mathbf{w}} \longrightarrow N$ is v, because $\mathbf{w}\in\mathcal{W}_v$. Hence the valuation given by $\widetilde{v}$ on $Y^{\mathbf{w}}$ induces the valuation given by v on X. Precisely, from the injectivity of the map $M\to\widetilde{M}$ induced by $\widetilde{\rho}_{\mathbf{w}}$ it follows that $\kappa(P_v)\subset\kappa(Q^{\mathbf{w}})$ (see (3.15) and (4.6)). Hence the point in X_∞ defined by the arc $\rho_{\mathbf{w}}\circ h_{Q^{\mathbf{w}}} : \operatorname{Spec}\kappa(Q^{\mathbf{w}})[[t]]\to X$ is P_v.

Therefore $(\rho_{\mathbf{w}})_\infty$ induces a morphism $(Y_\infty^{\mathbf{w}}, Q^{\mathbf{w}}) \longrightarrow (X_\infty, P_v)$ hence, a morphism of local rings:

$$\widehat{\rho}_{\mathbf{w}}^{\sharp} : \widehat{\mathcal{O}_{X_\infty,P_v}} \longrightarrow \widehat{\mathcal{O}_{Y_\infty^{\mathbf{w}},Q^{\mathbf{w}}}}.$$

Let $\widehat{P}_{\mathbf{w}}$ be the kernel of this morphism and let

$$\mathcal{I}_{\mathbf{w}} := \operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_v}}\ /\ \widehat{P}_{\mathbf{w}}.$$

Since $\widehat{\mathcal{O}_{Y_\infty^{\mathbf{w}},Q^{\mathbf{w}}}}$ is a domain, $\mathcal{I}_{\mathbf{w}}$ is reduced. Thus $\mathcal{I}_{\mathbf{w}}$ is the closure of the image $\widehat{\rho}_{\mathbf{w}}(\operatorname{Spec}\widehat{\mathcal{O}_{Y_\infty^{\mathbf{w}},Q^{\mathbf{w}}}})$ of the morphism $\widehat{\rho}_{\mathbf{w}} : \operatorname{Spec}\widehat{\mathcal{O}_{Y_\infty^{\mathbf{w}},Q^{\mathbf{w}}}} \to \operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_v}}$ induced by $\widehat{\rho}_{\mathbf{w}}^{\sharp}$. Finally set

$$\widehat{R}_{\mathbf{w}} := \kappa(P_v)\left[\left[\left\{\overline{Y}_{j,0},\dots,\overline{Y}_{j,n_j-1}\right\}_{1\leqslant j\leqslant s}\right]\right] \tag{4.9}$$

and let $\iota : \widehat{R}_{\mathbf{w}} \hookrightarrow \widehat{\mathcal{O}_{Y_\infty^{\mathbf{w}},Q^{\mathbf{w}}}}$ be the inclusion induced by the field extension $\kappa(P_v)\subset\kappa(Q^{\mathbf{w}})$ defined by $(\rho_{\mathbf{w}})_\infty$ and the isomorphism

$$\widehat{\mathcal{O}_{Y_\infty^{\mathbf{w}},Q^{\mathbf{w}}}} \cong \kappa(Q^{\mathbf{w}})\left[\left[\left\{\overline{Y}_{j,0},\dots,\overline{Y}_{j,n_j-1}\right\}_{1\leqslant j\leqslant s}\right]\right]$$

(see (4.7)).

Lemma 4.7. — *The morphism* $\widehat{\rho}^{\sharp}_{\mathbf{w}} : \widehat{\mathcal{O}_{X_\infty,P_v}} \to \widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}}$ *factors through* $\iota : \widehat{R}_{\mathbf{w}} \hookrightarrow \widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}}$. *That is, there exists a morphism of local rings*

$$\widehat{\varrho}^{\sharp}_{\mathbf{w}} : \widehat{\mathcal{O}_{X_\infty,P_v}} \to \widehat{R}_{\mathbf{w}}$$

such that $\iota \circ \widehat{\varrho}^{\sharp}_{\mathbf{w}} = \widehat{\rho}^{\sharp}_{\mathbf{w}}$. *We conclude that* $\mathcal{I}_{\mathbf{w}}$ *is the closure of the image of the induced morphism* $\widehat{\varrho}_{\mathbf{w}} : \operatorname{Spec} \widehat{R}_{\mathbf{w}} \to \operatorname{Spec} \widehat{\mathcal{O}_{X_\infty,P_v}}$.

Moreover, the extension of rings $\widehat{\mathcal{O}_{X_\infty,P_v}}/\widehat{P}_{\mathbf{w}} \hookrightarrow \widehat{R}_{\mathbf{w}}$ *induced by* $\widehat{\varrho}^{\sharp}_{\mathbf{w}}$ *is integral.*

Proof. — Given a complete local ring $(\widehat{R},\mathcal{M})$, in order to define a local morphism $\widehat{\mathcal{O}_{X_\infty,P_v}} \to \widehat{R}$ it suffices to define a local morphism $\mathcal{O}_{X_\infty,P_v} \to \widehat{R}$. By the representability property of X_∞, it is equivalent to define $\theta : \mathcal{O}_X \to \widehat{R}[[t]]$ such that the arc defined by the morphism $\bar{\theta} : \mathcal{O}_X \to \widehat{R}/\mathcal{M}[[t]]$ obtained by composition of θ with $\widehat{R}[[t]] \to \widehat{R}/\mathcal{M}\,[[t]]$, is the point P_v of X_∞.

In this way, the morphism $\widehat{\rho}^{\sharp}_{\mathbf{w}} : \widehat{\mathcal{O}_{X_\infty,P_v}} \to \widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}}$ is defined by $\theta_\rho : \mathcal{O}_X \to \widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}}[[t]]$, given by

$$\theta_\rho(\aleph^u) = \prod_{i'=1}^{d}\left(\sum_{n\geqslant 0} Z_{i',n}t^n\right)^{\langle u,u^*_{i'}\rangle} \cdot \prod_{j=1}^{s}\left(\left(\sum_{n\geqslant 0} V_{j,n}t^n\right)\cdot\left(t^{n_j} + \sum_{n=0}^{n_j-1}\overline{Y}_{j,n}t^n\right)\right)^{\langle u,w_j\rangle} \tag{4.10}$$

for $u \in \sigma^\vee \cap M$. Here we are applying the isomorphism (4.7). On the other hand, under the isomorphisms (3.19) and (3.20) in Proposition 3.9, the morphism $\theta_{id} : \mathcal{O}_X \to \widehat{\mathcal{O}_{X_\infty,P_v}}[[t]]$

$$x_i \mapsto \prod_{i'=1}^{d}\left(\sum_{n\geqslant 0} U_{i',n}t^n\right)^{\langle u_i,u^*_{i'}\rangle} \cdot \left(t^{c_i} + \sum_{n=0}^{c_i-1}\overline{X}_{i,n}t^n\right) \tag{4.11}$$

for $1 \leqslant i \leqslant m$, defines the identity in $\widehat{\mathcal{O}_{X_\infty,P_v}}$ (see (3.21) and recall that $c_i = \langle u_i, v\rangle$).

We conclude that θ_ρ is the composition of θ_{id} with the morphism

$$\widehat{\mathcal{O}_{X_\infty,P_v}}[[t]] \to \widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}}[[t]]$$

induced by $\widehat{\rho}_{\mathbf{w}}^{\sharp}$ which, under the isomorphisms (3.19) and (4.7) is determined by

$$\prod_{i'=1}^{d}\left(\sum_{n\geqslant 0} U_{i',n}t^n\right)^{\langle u_i,u_{i'}^*\rangle}\cdot\left(t^{c_i}+\sum_{n=0}^{c_i-1}\overline{X}_{i,n}t^n\right)$$
$$\mapsto \prod_{i'=1}^{d}\left(\sum_{n\geqslant 0} Z_{i',n}t^n\right)^{\langle u_i,u_{i'}^*\rangle}\cdot\prod_{j=1}^{s}\left(\left(\sum_{n\geqslant 0} V_{j,n}t^n\right)\cdot\left(t^{n_j}+\sum_{n=0}^{n_j-1}\overline{Y}_{j,n}t^n\right)\right)^{\langle u_i,w_j\rangle}$$

for $1\leqslant i\leqslant m$. From this, and by the uniqueness part in the Weierstrass preparation theorem ([3, Chapter VII, 3.8, Proposition 6]), it follows that

$$\prod_{i'=1}^{d}\left(\sum_{n\geqslant 0}\widehat{\rho}_{\mathbf{w}}^{\sharp}(U_{i',n})t^n\right)^{\langle u_i,u_{i'}^*\rangle} = \prod_{i'=1}^{d}\left(\sum_{n\geqslant 0} Z_{i',n}t^n\right)^{\langle u_i,u_{i'}^*\rangle}\cdot\prod_{j=1}^{s}\left(\sum_{n\geqslant 0} V_{j,n}t^n\right)^{\langle u_i,w_j\rangle} \tag{4.12}$$

and

$$t^{c_i}+\sum_{n=0}^{c_i-1}\widehat{\rho}_{\mathbf{w}}^{\sharp}(\overline{X}_{i,n})\,t^n=\prod_{j=1}^{s}\left(t^{n_j}+\sum_{n=0}^{n_j-1}\overline{Y}_{j,n}t^n\right)^{\langle u_i,w_j\rangle} \tag{4.13}$$

for $1\leqslant i\leqslant m$. Therefore, $\widehat{\rho}_{\mathbf{w}}^{\sharp}$ induces $\kappa(P_v)[[t]]\to\kappa(Q^{\mathbf{w}})[[t]]$, determined by (4.12) for $1\leqslant i\leqslant d$. Setting $\widetilde{\iota}(\overline{Y}_{j,n})=\overline{Y}_{j,n}$ for $1\leqslant j\leqslant s$, $0\leqslant n<n_j$, it defines a morphism

$$\widetilde{\iota}:\widehat{R}_{\mathbf{w}}[[t]]\longrightarrow\widehat{O_{Y_\infty^{\mathbf{w}},Q^{\mathbf{w}}}}[[t]]$$

which is the extension of $\iota:\widehat{R}_{\mathbf{w}}\to\widehat{O_{Y_\infty^{\mathbf{w}},Q^{\mathbf{w}}}}$ to $\widehat{R}_{\mathbf{w}}[[t]]\to\widehat{O_{Y_\infty^{\mathbf{w}},Q^{\mathbf{w}}}}[[t]]$ by $\widetilde{\iota}(t)=t$.

Now, let $\vartheta_{\varrho}:\mathcal{O}_X\to\widehat{R}_{\mathbf{w}}[[t]]$ be the morphism defined by

$$\aleph^u\mapsto\prod_{i'=1}^{d}\left(\sum_{n\geqslant 0} U_{i',n}t^n\right)^{\langle u,u_{i'}^*\rangle}\cdot\prod_{j=1}^{s}\left(t^{n_j}+\sum_{n=0}^{n_j-1}\overline{Y}_{j,n}t^n\right)^{\langle u,w_j\rangle} \tag{4.14}$$

for $u\in\sigma^\vee\cap M$. Let $\widehat{\varrho}_{\mathbf{w}}^{\sharp}:\widehat{\mathcal{O}_{X_\infty,P_v}}\to\widehat{R}_{\mathbf{w}}$ be the induced morphism. From (4.10) and (4.12) it follows that $\widetilde{\iota}\circ\vartheta_{\varrho}=\theta_{\rho}$. Therefore $\iota\circ\widehat{\varrho}_{\mathbf{w}}^{\sharp}=\widehat{\rho}_{\mathbf{w}}^{\sharp}$. From this the first assertion of the lemma follows.

For the second assertion, we consider the embedding of $\widehat{R}_{\mathbf{w}}$ in the formal power series ring

$$\widehat{S}_{\mathbf{w}} := \kappa(P_v)\left[\left[\left\{\overline{\overline{Y}}_{(j,1)},\ldots,\overline{\overline{Y}}_{(j,n_j)}\right\}_{1\leqslant j\leqslant s}\right]\right]$$

defined sending $t^{n_j}+\sum_{n=0}^{n_j-1}\overline{Y}_{j,n}t^n$ to $\prod_{r=1}^{n_j}(t-\overline{\overline{Y}}_{(j,r)})$. Then, we have

$$\widehat{\mathcal{O}_{X_\infty,P_v}}/\widehat{P}_{\mathbf{w}} \hookrightarrow \widehat{R}_{\mathbf{w}} \hookrightarrow \widehat{S}_{\mathbf{w}}.$$

It suffices to prove that the induced embedding $\widehat{\varrho}^\sharp : \widehat{\mathcal{O}_{X_\infty,P_v}}/\widehat{P}_{\mathbf{w}} \hookrightarrow \widehat{S}_{\mathbf{w}}$ is integral. The morphism $\widehat{\varrho}^\sharp$ is defined by

$$t^{c_i}+\sum_{n=0}^{c_i-1}\widehat{\varrho}^\sharp(\overline{X}_{i,n})\,t^n = \prod_{j=1}^{s}\prod_{r=1}^{n_j}\left(t-\overline{\overline{Y}}_{(j,r)}\right)^{\langle u_i,w_j\rangle} \qquad \text{for } 1\leqslant i\leqslant m.$$

Thus, if $\langle u_i,w_j\rangle>0$, then

$$(\overline{\overline{Y}}_{(j,r)})^{c_i}+\sum_{n=0}^{c_i-1}\widehat{\varrho}^\sharp(\overline{X}_{i,n})\,(\overline{\overline{Y}}_{(j,r)})^n=0 \tag{4.15}$$

is an integral relation of $\overline{\overline{Y}}_{(j,r)}$ in $\widehat{A}:=\widehat{\varrho}^\sharp(\widehat{\mathcal{O}_{X_\infty,P_v}}/\widehat{P}_{\mathbf{w}})$. Since $\{u_i\}_{i=1}^d\subset\sigma^\vee\cap M$ is a $\mathbb{Z}$-basis of M, we conclude that $\overline{\overline{Y}}_{(j,r)}$ is integral in the image $\widehat{A}$ of $\widehat{\varrho}^\sharp$, for $1\leqslant j\leqslant s, 1\leqslant r\leqslant n_j$. Therefore, the ring

$$\widehat{A}\left[\overline{\overline{\underline{Y}}}\right] := \widehat{A}\left[\left\{\overline{\overline{Y}}_{(j,r)}\right\}_{\substack{1\leqslant j\leqslant s\\ 1\leqslant r\leqslant n_j}}\right] \subset \kappa(P_v)\left[\left[\left\{\overline{\overline{Y}}_{(j,r)}\right\}_{\substack{1\leqslant j\leqslant s\\ 1\leqslant r\leqslant n_j}}\right]\right] = \widehat{S}_{\mathbf{w}}$$

is integral over $\widehat{A}$. In addition we have

$$\widehat{S}_{\mathbf{w}} = \widehat{A}\left[\overline{\overline{\underline{Y}}}\right] \tag{4.16}$$

and from this the last assertion of the lemma follows.

For convenience of the reader we give a detailed proof of (4.16). First recall that $\widehat{A}$ is a complete local ring with maximal ideal

$$M_{\widehat{A}} = \left(\left\{\widehat{\varrho}^\sharp(\overline{X}_{i,n})\right\}_{\substack{1\leqslant i\leqslant m\\ 0\leqslant n\leqslant c_i-1}}\right).$$

From (4.15) we obtain that

$$\left(\left\{\left(\overline{\overline{Y}}_{(j,r)}\right)^c\right\}_{\substack{1\leqslant j\leqslant s\\ 1\leqslant n\leqslant n_j}}\right) \subset M_{\widehat{A}}\,\widehat{A}\left[\overline{\overline{\underline{Y}}}\right]$$

where $c\geqslant c_i$, $1\leqslant i\leqslant m$. From this, and since the extension $\widehat{A}\subset\widehat{A}[\overline{\overline{\underline{Y}}}]$ is integral, it follows that $\widehat{A}[\overline{\overline{\underline{Y}}}]$ is a local ring. It also follows that the $M_{\widehat{A}}$-adic topology in $\widehat{A}[\overline{\overline{\underline{Y}}}]$ is the topology given by its maximal ideal.

Therefore $\widehat{A}[\overline{\overline{\underline{Y}}}]$ is complete for this topology ([25, Theorem 8.7]). This implies (4.16) and concludes the proof. ☐

Lemma 4.8. — *Given a K-r-wedge $\Phi : \operatorname{Spec} K[[\underline{\xi}, t]] \to X$ with special arc P_v, there exists $\mathbf{w} \in \mathcal{W}_v$ such that Φ lifts to a K-r-wedge $\Phi_{\mathbf{w}} : \operatorname{Spec} K[[\underline{\xi}, t]] \to Y^{\mathbf{w}}$ in $Y^{\mathbf{w}}$.*

Moreover, the induced morphism $\widehat{\varphi} : \operatorname{Spec} K[[\underline{\xi}]] \to \operatorname{Spec} \widehat{\mathcal{O}_{X_\infty, P_v}}$ lifts to $\operatorname{Spec} \widehat{R}_{\mathbf{w}}$. Precisely, there exists a morphism of local rings $\widehat{\psi}^\sharp_{\mathbf{w}} : \widehat{R}_{\mathbf{w}} \to K[[\underline{\xi}]]$ whose induced morphism $\widehat{\psi}_{\mathbf{w}}$ makes commutative the following diagram

$$\begin{array}{ccc} & & \operatorname{Spec} \widehat{R}_{\mathbf{w}} \\ & \overset{\widehat{\psi}_{\mathbf{w}}}{\nearrow} & \downarrow \widehat{\varrho}_{\mathbf{w}} \\ \operatorname{Spec} K[[\underline{\xi}]] & \xrightarrow{\widehat{\varphi}} & \operatorname{Spec} \widehat{\mathcal{O}_{X_\infty, P_v}} \end{array}.$$

Proof. — Let $\{w_j\}_{j=1}^s \subset \sigma \cap N$ and let $\{p_j\}_{j=1}^s$ be irreducible elements of $K[[\underline{\xi}, t]]$ with $(p_j, p_{j'}) = 1$ for $j \neq j'$, such that the morphism of rings $\Phi^\sharp : \mathcal{O}_X = k[\sigma^\vee \cap M] \to K[[\underline{\xi}, t]]$ induced by Φ is given by

$$\aleph^u \mapsto o_u \prod_{j=1}^{s} p_j^{\langle u, w_j \rangle},$$

where o_u is a unit in $K[[\underline{\xi}, t]]$ ((4.1) in Lemma 4.1). Moreover, we can suppose that the p_j's are in their Weierstrass form. Precisely, for each j, $1 \leqslant j \leqslant s$, let $\mu_j := \operatorname{ord}_t p_j(\underline{0}, t)$, which is a nonnegative integer since the special arc of Φ is the stable point P_v. By the Weierstrass preparation theorem ([3, Chapter VII, 3.8, Proposition 6]), we have

$$p_j = \vartheta_j \left(t^{\mu_j} + \lambda_{j,\mu_j-1}(\underline{\xi}) t^{\mu_j-1} + \lambda_{j,\mu_j-2}(\underline{\xi}) t^{\mu_j-2} + \cdots + \lambda_{j,0}(\underline{\xi}) \right)$$

where ϑ_j is a unit in $K[[\underline{\xi}, t]]$ and the $\lambda_i(\underline{\xi})$'s are elements of the maximal ideal $(\underline{\xi})$ of $K[[\underline{\xi}]]$. We may suppose with no loss of generality that

$$p_j = t^{\mu_j} + \lambda_{j,\mu_j-1}(\underline{\xi}) t^{\mu_j-1} + \lambda_{j,\mu_j-2}(\underline{\xi}) t^{\mu_j-2} + \cdots + \lambda_{j,0}(\underline{\xi}) \quad \text{for } 1 \leqslant j \leqslant s.$$

Let $\phi_T : \operatorname{Spec} K[[\underline{\xi}, t]] \longrightarrow T = \operatorname{Spec} k[M]$ be the morphism on the torus induced by $\aleph^u \mapsto o_u$, $u \in \sigma^\vee \cap M$, and let $\varsigma_i := \phi_T^\sharp(\aleph^{u_i})$, $1 \leqslant i \leqslant d$. Then, for $u \in M$, $\phi_T^\sharp(\aleph^u) = \varsigma_1^{\langle u, u_1^* \rangle} \cdots \varsigma_d^{\langle u, u_d^* \rangle}$, hence,

$$o_u = \varsigma_1^{\langle u, u_1^* \rangle} \cdots \varsigma_d^{\langle u, u_d^* \rangle} \quad \text{for } u \in \sigma \cap M. \tag{4.17}$$

Since Φ is centered at P_v, we have $v = \sum_{j=1}^s \mu_j \, w_j$ ((4.2) in Lemma 4.1). Suppose first that all the w_j's are minimal elements in $\sigma \cap N$. Then $\mathbf{w} =$

$\{(w_j;\,\mu_j)\}_{1\leqslant j\leqslant s}$ is an element of $\mathcal{W}_v$ and Φ lifts to $Y^{\mathbf{w}}$. In fact, the assignment

$$y_j \mapsto p_j \quad \text{for } 1\leqslant j\leqslant s, \qquad z_i \mapsto \varsigma_i \quad \text{for } 1\leqslant i\leqslant d$$

defines a morphism $\Phi_{\mathbf{w}} : \operatorname{Spec} K[[\underline{\xi},t]] \to Y^{\mathbf{w}}$ such that $\rho_{\mathbf{w}}\circ\Phi_{\mathbf{w}} = \Phi$ (recall (4.4) and (4.17)). Hence the first assertion of the lemma is proved in this case. For the second assertion, note first that $\Phi_{\mathbf{w}}$ may not be centered in $Q^{\mathbf{w}}$. Nevertheless $\widehat{\varphi}^{\sharp} : \widehat{\mathcal{O}_{X_\infty,P_v}} \to K[[\underline{\xi}]]$ defines an inclusion on the residue fields $\widehat{\varphi}^{\sharp} : \kappa(P_v) \hookrightarrow K$. Let us define

$$\widehat{\psi}^{\sharp} : \widehat{R}_{\mathbf{w}} = \kappa(P_v)\left[\left[\{\overline{Y}_{j,0},\ldots,\overline{Y}_{j,\mu_j-1}\}_{1\leqslant j\leqslant s}\right]\right] \longrightarrow K[[\underline{\xi}]]$$

whose restriction to $\kappa(P_v)$ is $\widehat{\varphi}^{\sharp}$ and such that

$$\widehat{\psi}^{\sharp}(\overline{Y}_{j,n}) = \lambda_{j,n}(\underline{\xi}) \quad \text{for } 1\leqslant j\leqslant s,\ 0\leqslant n\leqslant \mu_j-1.$$

Then, for the induced morphism $\widehat{\psi} : \operatorname{Spec} K[[\underline{\xi}]] \to \operatorname{Spec}\widehat{R}_{\mathbf{w}}$, we have $\widehat{\varphi} = \widehat{\varrho}_{\mathbf{w}}\circ\widehat{\psi}$.

In general, i.e. if some of the w_j is not a minimal element in $\sigma\cap N$, there exist minimal elements $w'_1,\ldots,w'_{s'}$ in $\sigma\cap N$ and, for $1\leqslant j\leqslant s$, a partition

$$w_j = n_{j,1}w'_1 + \cdots + n_{j,s'}w'_{s'}$$

where the $n_{j,k}$'s are integers $\geqslant 0$. Let $n'_l = \sum_{1\leqslant j\leqslant s} n_{j,l}\mu_j$, $1\leqslant l\leqslant s'$, and set $\mathbf{w} = \{(w'_l;\,n'_l)\}_{1\leqslant l\leqslant s'}$. Then $\mathbf{w}\in\mathcal{W}_v$ and the assignment

$$y_l \mapsto \prod_{j=1}^{s} p_j^{n_{j,l}} \quad \text{for } 1\leqslant l\leqslant s', \qquad z_i \mapsto \varsigma_i \quad \text{for } 1\leqslant i\leqslant d$$

defines a lifting of Φ to $Y^{\mathbf{w}}$. For the second assertion, we define $\widehat{\psi}^{\sharp} : \widehat{R}_{\mathbf{w}} \to K[[\underline{\xi}]]$ whose restriction to $\kappa(P_v)$ is $\widehat{\varphi}^{\sharp} : \kappa(P_v)\hookrightarrow K$ and such that $\widehat{\psi}^{\sharp}(\overline{Y}_{j,n})$ is given identifying the coefficients in t^n, $0\leqslant n< n'_l$, in

$$t^{n'_l} + \sum_{n=0}^{n'_l-1}\widehat{\psi}^{\sharp}(\overline{Y}_{l,n})\,t^n = \prod_{j=1}^{s}\left(t^{\mu_j} + \sum_{n=0}^{\mu_j-1}\lambda_{j,n}(\underline{\xi})\ t^n\right)^{n_{j,l}} \qquad \text{for } 1\leqslant l\leqslant s'.$$

Then, the induced morphism $\widehat{\psi} : \operatorname{Spec} K[[\underline{\xi}]] \to \operatorname{Spec}\widehat{R}_{\mathbf{w}}$ satisfies $\widehat{\varphi} = \widehat{\varrho}_{\mathbf{w}}\circ\widehat{\psi}$. This concludes the proof. □

Remark 4.9. — Note that the special arc of the r-wedge $\Phi_{\mathbf{w}}$ may not be $Q^{\mathbf{w}}$. In fact, recall that $\kappa(Q^{\mathbf{w}}) \cong \kappa(P_v)\,(\{V_{j,n}\}_{1\leqslant j\leqslant s,n\geqslant 0})$ (see (4.12)). Thus, if K is an algebraic field extension of $\kappa(P_v)$ then K does not contain $\kappa(Q^{\mathbf{w}})$ and thus the special arc of $\Phi_{\mathbf{w}}$ is not $Q^{\mathbf{w}}$.

Remark 4.10. — Suppose that $\operatorname{char} k = 0$. Let σ be a strongly convex simplicial cone, i.e. σ is generated by d vectors $v_1, \dots, v_d$ which are linearly independent. Equivalently, $X = X_\sigma$ has only finite quotient singularities ([7, Theorem 3.1.19]). Let us consider the morphism

$$\rho : Y \coloneqq \operatorname{Spec} k[y_1, \dots, y_d] \longrightarrow X, \quad \aleph^u \mapsto y_1^{\langle u, v_1 \rangle} \cdots y_d^{\langle u, v_d \rangle} \text{ for } u \in \sigma^\vee \cap M.$$

Fix a partition of the form $\mathbf{w} = \{(v_j; n_j)\}_{1 \leqslant j \leqslant d}$, whose minimal elements are the extremal elements $\{v_i\}_{i=1}^d$ of σ, and let $v \coloneqq \sum_{i=1}^d n_i v_i$. Then, a K-r-wedge $\Phi : \operatorname{Spec} K[[\underline{\xi}, t]] \to X$ lifts to $Y^{\mathbf{w}}$ if and only if there exists a finite algebraic field extension K' of K such that the induced K'-r-wedge $\Phi' : \operatorname{Spec} K'[[\underline{\xi}, t]] \to X$ lifts to Y. In fact, for the if part, recall Lemma 4.1 and note that ρ factors by $\rho_{\mathbf{w}}$: the morphism $\zeta : Y \to Y^{\mathbf{w}}$ defined by $y_i \mapsto y_i, z_i \mapsto 1$ for $1 \leqslant i \leqslant d$ satisfies $\rho = \rho_{\mathbf{w}} \circ \zeta$.

Now, if Φ lifts to $Y^{\mathbf{w}}$, there exist irreducible elements $\{p_i\}_{i=1}^d$ of $K[[\underline{\xi}, t]]$ such that $\Phi^\sharp(\aleph^u) = o_u \prod_{j=1}^d p_j^{\langle u, v_j \rangle}$, $u \in \sigma^\vee \cap M$, where o_u is a unit in $K[[\underline{\xi}, t]]$ and $\aleph^u \mapsto o_u$ defines a wedge on T (Lemma 4.1). There exist minimal elements $v_1^+, \dots, v_d^+$ of $\sigma^\vee \cap M$ such that $\langle v_i^+, v_j \rangle = 0$ if $i \neq j$. Set $d_i \coloneqq \langle v_i^+, v_i \rangle$, and $o_i \coloneqq o_{v_i^+}$ for $1 \leqslant i \leqslant d$. Since $\operatorname{char} k = 0$, there exists a finite algebraic field extension K' of K and $o_i' \in K'[[\underline{\xi}, t]]$, $1 \leqslant i \leqslant d$, such that $(o_i')^{d_i} = o_i$. Then $\aleph^u \mapsto \prod_{j=1}^d (o_j' p_j)^{\langle u, v_j \rangle}$ defines a lifting of Φ' to Y.

The following lemma generalizes this remark. It will be applied in Section 6.

Lemma 4.11. — *Suppose that* $\operatorname{char} k = 0$. *Let σ be a strongly convex cone. Let $v \in \sigma$ and let $\mathbf{w} = \{(w_j; n_j)\}_{1 \leqslant j \leqslant s}$ be a partition of v. Suppose that $s \geqslant d$ and that $\{w_i\}_{i=1}^d$ are $\mathbb{Q}$-linearly independent. Let us consider the morphism*

$$\rho : Y \coloneqq \operatorname{Spec} k[y_1, \dots, y_s] \longrightarrow X, \quad \aleph^u \mapsto y_1^{\langle u, w_1 \rangle} \cdots y_s^{\langle u, w_s \rangle} \text{ for } u \in \sigma^\vee \cap M.$$

Let Q be the stable point in Y_∞ defined by

$$(Y_{1,0}, \dots, Y_{1,n_1-1}, \dots, Y_{s,0}, \dots, Y_{s,n_s-1}),$$

whose image in X_∞ is P_v, and let $\widehat{\rho} : \operatorname{Spec} \widehat{\mathcal{O}_{Y_\infty, Q}} \to \operatorname{Spec} \widehat{\mathcal{O}_{X_\infty, P_v}}$ be the induced morphism. Then we have $\operatorname{Im} \widehat{\rho} = \operatorname{Im} \widehat{\rho}_{\mathbf{w}} = \mathcal{I}_{\mathbf{w}}$.

Proof. — Recall that $\{u_i\}_{i=1}^d \subset \sigma^\vee \cap M$ is a basis of the free $\mathbb{Z}$-module M. Since $\{w_i\}_{i=1}^d \subset \sigma \cap N$ are $\mathbb{Q}$-linearly independent, we may suppose that

$$\det\left(\langle u_i, w_j \rangle\right)_{1 \leqslant i,j \leqslant d} > 0. \tag{4.18}$$

Let us consider the following commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} k[y_1,\ldots,y_s,z'_1,\ldots,z'_d]_{z'_1\cdots z'_d} =: Y' & \xrightarrow{\eta_{\mathbf{w}}} & Y^{\mathbf{w}} \\ \downarrow \eta & & \downarrow \rho_{\mathbf{w}} \\ \operatorname{Spec} k[y_1,\ldots,y_s] = Y & \xrightarrow{\rho} & X\ . \end{array}$$

where $\eta_{\mathbf{w}}$ is defined by $y_j \mapsto y_j$, $1 \leqslant j \leqslant s$, and $z_i \mapsto \prod_{l=1}^{d}(z'_l)^{\langle u_i, w_l\rangle}$, $1 \leqslant i \leqslant d$, and η is defined by $y_i \mapsto z'_i y_i$ for $1 \leqslant i \leqslant d$ and $y_i \mapsto y_i$ for $d+1 \leqslant i \leqslant s$. In fact, for the commutativeness of the diagram, recall the definition of $\rho_{\mathbf{w}}$ in (4.4) and the fact that, for $u \in \sigma^\vee \cap M$, we have $u = \sum_{i=1}^{d}\langle u, u_i^*\rangle u_i$, hence

$$\langle u, w_l\rangle = \sum_{i=1}^{d}\langle u, u_i^*\rangle \cdot \langle u_i, w_l\rangle, \quad 1 \leqslant l \leqslant d.$$

Note that $\eta_{\mathbf{w}}$ is a dominant and finite morphism by (4.18). Hence, since $\operatorname{char} k = 0$, it induces an inclusion

$$\mathcal{O}_{(\operatorname{Spec} k[z_1,\cdots,z_d]_{z_1\cdots z_d})_\infty} \to \mathcal{O}_{(\operatorname{Spec} k[z'_1,\cdots,z'_d]_{z'_1\cdots z'_d})_\infty}.$$

Let Q' be the stable point in Y'_∞ defined by

$$(Y_{1,0},\ldots,Y_{1,n_1-1},\ldots,Y_{s,0},\ldots,Y_{s,n_s-1}).$$

Then, its image by $\eta_{\mathbf{w}}$ is $Q^{\mathbf{w}}$. Hence we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}\widehat{\mathcal{O}_{Y'_\infty,Q'}} & \xrightarrow{\widehat{\eta}_{\mathbf{w}}} & \operatorname{Spec}\widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}} \\ \downarrow \widehat{\eta} & & \downarrow \widehat{\rho}_{\mathbf{w}} \\ \operatorname{Spec}\widehat{\mathcal{O}_{Y_\infty,Q}} & \xrightarrow{\widehat{\rho}} & \operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_v}}\ . \end{array}$$

The inclusion $\kappa(Q^{\mathbf{w}}) \subseteq \kappa(Q')$ induces an inclusion

$$\widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}} = \kappa(Q^{\mathbf{w}})[[\{Y_{j,0},\ldots,Y_{j,n_j-1}\}_{j=1}^{s}]] \hookrightarrow \widehat{\mathcal{O}_{Y'_\infty,Q'}} = \kappa(Q')[[\{Y_{j,0},\ldots,Y_{j,n_j-1}\}_{j=1}^{s}]]$$

thus $\widehat{\eta}_{\mathbf{w}}$ is dominant. Therefore

$$\operatorname{Im}\widehat{\rho}_{\mathbf{w}} = \operatorname{Im}(\widehat{\rho}_{\mathbf{w}}\circ\widehat{\eta}_{\mathbf{w}}) = \operatorname{Im}(\widehat{\rho}\circ\widehat{\eta}) \subseteq \operatorname{Im}\widehat{\rho}.$$

On the other hand, if $\zeta : Y \to Y^{\mathbf{w}}$ is defined by $y_j \mapsto y_j$, $1 \leqslant j \leqslant s$ and $z_i \mapsto 1$, $1 \leqslant i \leqslant d$, then $\rho = \rho_{\mathbf{w}} \circ \zeta$. Thus $\widehat{\rho} = \widehat{\rho}_{\mathbf{w}} \circ \widehat{\zeta}$ where $\widehat{\zeta} : \operatorname{Spec}\widehat{\mathcal{O}_{Y_\infty,Q}} \to \operatorname{Spec}\widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}}$ is induced by ζ. Therefore $\operatorname{Im}\widehat{\rho} \subseteq \operatorname{Im}\widehat{\rho}_{\mathbf{w}}$ and we conclude that $\operatorname{Im}\widehat{\rho} = \operatorname{Im}\widehat{\rho}_{\mathbf{w}}$. □

Corollary 4.12. — *We have*

$$\left(\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_v}}\right)_{\mathrm{red}} = \bigcup_{\mathbf{w}\in\mathcal{W}_v} \mathcal{I}_{\mathbf{w}}.$$

Moreover, $\mathcal{I}_{\mathbf{w}}$ is irreducible of dimension $l(\mathbf{w})$, for $\mathbf{w}\in\mathcal{W}_v$. Therefore we have

$$\dim\widehat{\mathcal{O}_{X_\infty,P_v}} = \dim\widehat{\mathcal{O}_{(X_\infty)_{\mathrm{red}},P_v}} = \operatorname{tcht} P_v. \tag{4.19}$$

Proof. — The first assertion follows from the first assertion of Lemma 4.7 and Lemma 4.8 applied to 1-wedges on X. In fact, since the ring $\widehat{\mathcal{O}_{X_\infty,P_v}}$ is Noetherian ([30, Corollary 4.6]), given $P'\in\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_v}}$, there exists a k-morphism $\widehat{\varphi}:\operatorname{Spec}K[[\xi]]\to\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_v}}$, where K is a field extension of $\kappa(P_v)$, such that the image of the closed (resp. generic) point of $\operatorname{Spec}K[[\xi]]$ is the maximal ideal of $\widehat{\mathcal{O}_{X_\infty,P_v}}$ (resp. P'). Equivalently, $\widehat{\varphi}$ defines a wedge $\Phi:\operatorname{Spec}K[[\xi,t]]\to X$ with special arc P_v. By Lemma 4.8, there exists $\mathbf{w}\in\mathcal{W}_v$ such that Φ lifts to $\operatorname{Spec}\widehat{R}_{\mathbf{w}}$. Applying Lemma 4.7, this implies that P' lies in the image $\mathcal{I}_{\mathbf{w}}$ of $\widehat{\varrho}_{\mathbf{w}}$.

The second assertion follows from the second assertion of Lemma 4.7, since $\widehat{R}_{\mathbf{w}}$ is a domain and $\dim\widehat{R}_{\mathbf{w}}=l(\mathbf{w})$ (see (4.9)). From this and Corollary 3.6 we conclude the last assertion. □

Let us fix $\mathbf{w}=\{(w_j;n_j)\}_{1\leqslant j\leqslant s}$ in $\mathcal{W}_v$. We will next consider the image by $\widehat{\varrho}_{\mathbf{w}}:\operatorname{Spec}\widehat{R}_{\mathbf{w}}\to\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_v}}$ of chains of prime ideals of $\widehat{R}_{\mathbf{w}}$. First, given a prime ideal $\widehat{Q}$ of $\widehat{R}_{\mathbf{w}}$, we define

$$\nu_{\widehat{Q}}(y_j) := \begin{cases} \min\{n \;/\; \overline{Y}_{j,n}\notin\widehat{Q}\} & \text{if } \exists\, n<n_j \text{ such that } \overline{Y}_{j,n}\notin\widehat{Q},\\ n_j & \text{otherwise}\end{cases} \tag{4.20}$$

for $1\leqslant j\leqslant s$. Note that $\nu_{\widehat{Q}}(y_j)\leqslant n_j$ for $1\leqslant j\leqslant s$.

Lemma 4.13. — *Let $\widehat{Q}$ be a prime ideal of $\widehat{R}_{\mathbf{w}}$. Let $\widehat{P}:=\widehat{\varrho}_{\mathbf{w}}(\widehat{Q})$ and denote by P the contraction of $\widehat{P}$ by the morphism $\mathcal{O}_{X_\infty}\to\widehat{\mathcal{O}_{X_\infty,P_v}}$. Then P is a stable point of X_∞ and*

$$v_P=\sum_{j=1}^{s}\nu_{\widehat{Q}}(y_j)\,w_j.$$

Proof. — First, note that the prime ideal P of $\mathcal{O}_{X_\infty}$ is contained in P_v since the morphism $\mathcal{O}_{X_\infty}\to\widehat{\mathcal{O}_{X_\infty,P_v}}$ factors through $\mathcal{O}_{X_\infty,P_v}$. Therefore P is a stable point of X_∞ ([31, Proposition 3.7(vi)]). Now, recall Proposition 3.9 and set

$$\nu_{\widehat{P}}(x_i) := \begin{cases} \min\{n \;/\; \overline{X}_{i,n}\notin\widehat{P}\} & \text{if there exists } n<c_i \text{ such that } \overline{X}_{i,n}\notin\widehat{P},\\ c_i & \text{otherwise}\end{cases}$$

for $1 \leqslant i \leqslant m$. From (4.13) it follows that, for $1 \leqslant i \leqslant m$, we have

$$\nu_{\widehat{P}}(x_i) = \sum_{j=1}^{s} \nu_{\widehat{Q}}(y_j)\langle u_i, w_j\rangle = \left\langle u_i, \sum_{j=1}^{s} \nu_{\widehat{Q}}(y_j) w_j \right\rangle$$

Finally, the arc $\mathcal{O}_{X_\infty} \to \kappa(\widehat{P})[[t]]$ defines the point P of X_∞, therefore, applying the definition of ν_P and (4.11), we obtain

$$\langle u_i, v_P\rangle = \nu_P(x_i) = \nu_{\widehat{P}}(x_i) = \left\langle u_i, \sum_{j=1}^{s} \nu_{\widehat{Q}}(y_j) w_j \right\rangle$$

for $1 \leqslant j \leqslant m$ (recall (3.4)). Since $x_i = \aleph^{u_i}$, $1 \leqslant i \leqslant m$, and $\{u_1, \dots, u_d\}$ is a basis of M, from this the lemma follows. □

Given $\underline{\ell} := (\ell_1, \dots, \ell_s) \in (\mathbb{Z}_{\geqslant 0})^s$ with $\ell_j \leqslant n_j$ for $1 \leqslant j \leqslant s$, let $\widehat{Q}_{\underline{\ell}}$ be the ideal of $\widehat{R}_{\mathbf{w}}$ defined by

$$\widehat{Q}_{\underline{\ell}} := (\overline{Y}_{1,0}, \dots, \overline{Y}_{1,\ell_1-1}, \dots, \overline{Y}_{s,0}, \dots, \overline{Y}_{s,\ell_s-1}) \subset \widehat{R}_{\mathbf{w}}.$$

From the definition of $\widehat{R}_{\mathbf{w}}$ (see (4.9)) it follows that $\widehat{Q}_{\underline{\ell}}$ is a prime ideal of $\widehat{R}_{\mathbf{w}}$. Let $\widehat{Q}^e_{\underline{\ell}}$ be the extension of $\widehat{Q}_{\underline{\ell}}$ to $\widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty, Q^{\mathbf{w}}}}$. By (4.8) in Proposition 4.6 we have

$$\widehat{Q}^e_{\underline{\ell}} = (Y_{1,0}, \dots, Y_{1,\ell_1-1}, \dots, Y_{s,0}, \dots, Y_{s,\ell_s-1})\, \widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty, Q^{\mathbf{w}}}}.$$

From Proposition 4.6 it follows that $\widehat{Q}^e_{\underline{\ell}}$ is a prime ideal of $\widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty, Q^{\mathbf{w}}}}$. In addition we have $\widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{\underline{\ell}}) = \widehat{\rho}_{\mathbf{w}}(\widehat{Q}^e_{\underline{\ell}})$.

Now, let us consider the following saturated chain of prime ideals in $\widehat{R}_{\mathbf{w}}$:

$$\begin{aligned}(0) = \widehat{Q}_{(0,\dots,0)} &\subset \widehat{Q}_{(1,0,\dots,0)} \subset \widehat{Q}_{(2,0,\dots,0)} \subset \cdots \subset \widehat{Q}_{(n_1,0,\dots,0)} \\ &\subset \widehat{Q}_{(n_1,1,\dots,0)} \subset \cdots \subset \widehat{Q}_{(n_1,n_2,0,\dots,0)} \subset \cdots \subset \widehat{Q}_{(n_1,\dots,n_{s-1},0)} \\ &\subset \widehat{Q}_{(n_1,\dots,n_{s-1},1)} \subset \cdots \subset \widehat{Q}_{(n_1,\dots,n_s-1)} \subset \widehat{Q}_{(n_1,\dots,n_s)}.\end{aligned} \tag{4.21}$$

Next we will consider its image by $\widehat{\varrho}_{\mathbf{w}}$.

Lemma 4.14. — *The chain of prime ideals in $\widehat{\mathcal{O}}_{X_\infty, P_v}$*

$$\begin{aligned}\widehat{P}_{\mathbf{w}} = \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(0,\dots,0)}) &\subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(1,0,\dots,0)}) \subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(2,0,\dots,0)}) \subset \cdots \\ \subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(n_1,0,\dots,0)}) &\subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(n_1,1,\dots,0)}) \subset \cdots \subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(n_1,n_2,0,\dots,0)}) \\ \subset \cdots &\subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(n_1,\dots,n_s-1)}) \subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(n_1,\dots,n_s)}) = P_v \widehat{\mathcal{O}_{X_\infty, P_v}}\end{aligned} \tag{4.22}$$

contracts in $\mathcal{O}_{X_\infty, P_v}$ to the chain defined by

$$\begin{aligned}\sqrt{(0)} \subset P_{w_1} \subset P_{2w_1} \subset \cdots \subset P_{n_1 w_1} \subset P_{n_1 w_1 + w_2} \subset \cdots \subset P_{n_1 w_1 + n_2 w_2} \\ \subset \cdots \subset P_{n_1 w_1 + \cdots + (n_s - 1) w_s} \subset P_{n_1 w_1 + \cdots + n_s w_s} = P_v.\end{aligned}$$

Therefore the chain (4.22) *has length* $l(\mathbf{w})$ *and it is saturated.*

Proof. — Recall that, since $Y^{\mathbf{w}}$ is regular, the local ring $\mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}$ is also regular ([31, Proposition 4.2]). Hence we have a commutative diagram of morphisms:

$$\begin{array}{ccc} \mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}} & \hookrightarrow & \widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}} \\ \uparrow & & \uparrow \\ \mathcal{O}_{(X_\infty)_{\mathrm{red}},P_v} & \hookrightarrow & \widehat{\mathcal{O}_{X_\infty,P_v}} \end{array}.$$

where the vertical arrows are induced by $(\rho_{\mathbf{w}})_\infty : Y^{\mathbf{w}}_\infty \to X_\infty$ and the horizontal arrows are injective ([31, Corollary 4.3]). Let us show that the left hand side morphism $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P_v} \to \mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}$ is injective. If $\operatorname{char} k = 0$ this follows from [31, Proposition 4.5], since $\rho_{\mathbf{w}} : Y^{\mathbf{w}} \to X$ is a dominant morphism. For the general case, we need to apply the specific form of the morphism $\rho_{\mathbf{w}}$. In fact, after replacing X by X' (resp. $Y^{\mathbf{w}}$ by Y') where $X' \to X$ and $Y' \to Y^{\mathbf{w}}$ are birational proper equivariant morphisms, and applying [31, Proposition 4.1], it suffices to show the injectivity of $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P_v} \to \mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}$ in the case in which X is regular (and toric). In this case, $X = \operatorname{Spec} k[x_1,\ldots,x_d]$ and $\rho_{\mathbf{w}} : Y^{\mathbf{w}} \to X$ is given by

$$x_i = \aleph^{u_i} \mapsto z_i \cdot y_1^{\langle u_i,w_1\rangle} \cdots y_s^{\langle u_i,w_s\rangle} \quad \text{for } 1 \leqslant i \leqslant d.$$

Then, $X_\infty = \operatorname{Spec} = k[\underline{X}_0,\ldots,\underline{X}_n,\ldots]$, where $\underline{X}_n = (X_{1,n},\ldots,X_{d,n})$ and for the morphism $(\rho_{\mathbf{w}})^\sharp_\infty : \mathcal{O}_{X_\infty} \to \mathcal{O}_{Y^{\mathbf{w}}_\infty}$ we have

$$(\rho_{\mathbf{w}})^\sharp_\infty(X_{i,n}) = Z_{i,n} \cdot Y_{1,0}^{\langle u_i,w_1\rangle} \cdots Y_{s,0}^{\langle u_i,w_s\rangle} \mod (\{Y_{j,n'}\}_{1\leqslant j\leqslant s, 1\leqslant n'\leqslant n})$$

for $1 \leqslant i \leqslant d$ and $n \geqslant 0$. This implies that $(\rho_{\mathbf{w}})^\sharp_\infty : \mathcal{O}_{X_\infty} \to \mathcal{O}_{Y^{\mathbf{w}}_\infty}$ is injective in this regular case. Thus $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P_v} \to \mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}$ is injective.

Now, recall that $\widehat{P}_{\mathbf{w}}$ is the kernel of the right hand side morphism

$$\widehat{\mathcal{O}}_{X_\infty,P_v} \to \widehat{\mathcal{O}}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}.$$

Therefore we have

$$\widehat{P}_{\mathbf{w}} \cap \mathcal{O}_{(X_\infty)_{\mathrm{red}},P_v} = (0)$$

and hence the contraction of $\widehat{P}_{\mathbf{w}}$ by the morphism $\mathcal{O}_{X_\infty,P_v} \to \widehat{\mathcal{O}_{X_\infty,P_v}}$ is $\sqrt{(0)}$. Even more, for $\underline{\ell} \in (\mathbb{Z}_{\geqslant 0})^s$ with $\ell_j \leqslant n_j$, $1 \leqslant j \leqslant s$, the contraction of $\widehat{Q}^e_{\underline{\ell}} = \widehat{Q}_{\underline{\ell}}\widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}}$ by the morphism $\mathcal{O}_{Y^{\mathbf{w}}_\infty} \to \widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}}}$ is the prime ideal

$$Q_{\underline{\ell}} := (Y_{1,0},\ldots,Y_{1,\ell_1-1},\ldots,Y_{s,0},\ldots,Y_{s,\ell_s-1}) \subset \mathcal{O}_{Y^{\mathbf{w}}_\infty} = k[\{\underline{Y}_n,\underline{Z}_n\}_{n\geqslant 0}]$$

and the image of $Q_{\underline{\ell}}$ by $(\rho_{\mathbf{w}})_\infty : Y_\infty \to X_\infty$ is the toric stable prime ideal $P_{\sum_j \ell_j w_j}$. Since the image $\widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{\underline{\ell}})$ of $\widehat{Q}_{\underline{\ell}}$ by $\operatorname{Spec} \widehat{R}_{\mathbf{w}} \to \operatorname{Spec} \widehat{\mathcal{O}}_{X_\infty,P_v}$

is equal to the image of $\widehat{Q}^e_{\underline{\ell}}$ by $\operatorname{Spec}\widehat{\mathcal{O}}_{Y^{\mathbf{w}}_\infty,Q^{\mathbf{w}}} \to \operatorname{Spec}\widehat{\mathcal{O}}_{X_\infty,P_v}$, we have that $\widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{\underline{\ell}})$ contracts in $\mathcal{O}_{X_\infty,P_v}$ to $P_{\sum_j \ell_j w_j}\mathcal{O}_{X_\infty,P_v}$. We conclude then that the image of the chain (4.22) by the morphism $\widehat{\varrho}_{\mathbf{w}} : \operatorname{Spec}\widehat{R}_{\mathbf{w}} \to \operatorname{Spec}\mathcal{O}_{X_\infty,P_v}$ is defined by the following chain of prime ideals in $\mathcal{O}_{X_\infty}$

$$\begin{aligned}\sqrt{(0)} \subset P_{w_1} \subset P_{2w_1} \subset \cdots \subset P_{n_1w_1} \subset P_{n_1w_1+w_2} \subset \cdots \subset P_{n_1w_1+n_2w_2} \\ \subset \cdots \subset P_{n_1w_1+\cdots+(n_s-1)w_s} \subset P_{n_1w_1+\cdots+n_sw_s} = P_v.\end{aligned}$$

This chain is saturated by Corollary 4.3, and has length $n_1+\cdots+n_s = \mathrm{l}(\mathbf{w})$. But this chain also defines the image by $\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_v}} \to \operatorname{Spec}\mathcal{O}_{X_\infty,P_v}$ of the chain (4.22). Therefore (4.22) has length $\mathrm{l}(\mathbf{w}) = \dim\widehat{\mathcal{O}_{X_\infty,P_v}}/\widehat{P}_{\mathbf{w}}$ (Corollary 4.12). This concludes the proof. □

5. Irreducible components of $\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P}}$

In this section we will describe the irreducible components of $\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P}}$ and their respective dimensions, where X is a normal toric variety and P is a toric stable point in X_∞. For this, we will first deal with a going up theorem (Proposition 5.3). Although a going up theorem is consequence of the integral property in Lemma 4.7, we present an alternative proof, more adapted to a possible generalization for nontoric varieties (Proposition 5.2), applying local uniformization of valuations which are composition of discrete valuations, which is consequence of the reduction of local uniformization to the rank one case by Novacoski and Spivakovsky [28], and the finiteness property of the stable points of the space of arcs of any variety [30].

Let us first recall the concept of *composition of valuations*: Let $k \subseteq K$ be a field extension, and let ν_1 be a valuation on K. We denote by R_{ν_1} the valuation ring, $\mathcal{M}_{\nu_1}$ its maximal ideal and $k_{\nu_1} := R_{\nu_1}/\mathcal{M}_{\nu_1}$. Let ν_2 be a valuation of the residue field k_{ν_1}. Then the ring

$$R := \{g \in R_{\nu_1} \ / \ g \bmod \mathcal{M}_{\nu_1} \in R_{\nu_2}\}$$

is the valuation ring of a valuation ν, which is called composite of ν_1 with ν_2, and denoted by $\nu = \nu_1 \circ \nu_2$ (see [35, Chapter VI, Section 10]). That is, $R_\nu = R$, its maximal ideal is

$$\mathcal{M}_\nu := \{g \in R_{\nu_1} \ / \ g \bmod \mathcal{M}_{\nu_1} \in \mathcal{M}_{\nu_2}\}$$

and the ideal $\mathcal{P} := \mathcal{M}_{\nu_1} \cap R_\nu$, which is contained in $\mathcal{M}_{\nu_1}$, satisfies

$$(R_\nu)_{\mathcal{P}} = R_{\nu_1} \quad \text{and} \quad R_\nu \ / \ \mathcal{P} \cong R_{\nu_2}.$$

Lemma 5.1. — *Let $(A,\wp)$ be a Noetherian local domain of dimension $\geqslant 1$. Let*

$$(0)\subset P_1\subset\cdots\subset P_r=\wp$$

be a saturated chain of prime ideals of A. Then, there exists a valuation ring R_ν of the fraction field $Fr(A)$ of A dominating A, and a saturated chain of prime ideals in R_ν

$$(0)=Q_0\subset Q_1\subset\cdots\subset Q_r=\mathcal{M}_\nu$$

such that $Q_l\cap A=P_l$, $1\leqslant l\leqslant r$. Moreover, ν can be taken to be the composition of r discrete valuations, precisely $\nu=\nu_1\circ\ldots\circ\nu_r$ where $R_{\nu_l}\cong (R_\nu)_{Q_l}\ /\ Q_{l-1}$ is a discrete valuation ring for $1\leqslant l\leqslant r$.

Proof. — We argue by induction on r. For $r=1$ we have that $A_\wp$ is a Noetherian local domain of dimension 1, hence it is clear that the result holds: it suffices to consider the normalization of $A_\wp$.

Now suppose that $r\geqslant 2$ and we have proved the result for $r-1$. Let $(0)\subset P_1\subset\cdots\subset P_r$ be a saturated chain of prime ideals of A. Then $A_{P_{r-1}}$ is a Noetherian ring and

$$(0)\subset P_1A_{P_{r-1}}\subset\cdots\subset P_{r-1}A_{P_{r-1}}$$

is a saturated chain of prime ideals in $A_{P_{r-1}}$. Therefore, by the inductive hypothesis, there exists a valuation ν' of $Fr(A_{P_{r-1}})=Fr(A)$ which is composition of $r-1$ discrete valuations and there exists a saturated chain of prime ideals $(0)\subset Q'_1\subset\cdots\subset Q'_{r-1}=\mathcal{M}_{\nu'}$ in $R_{\nu'}$ such that

$$Q'_l\cap A_{P_{r-1}}=P_lA_{P_{r-1}}\quad\text{for } 1\leqslant l\leqslant r-1. \tag{5.1}$$

Precisely, we have $\nu'=\nu_1\circ\ldots\circ\nu_{r-1}$ where, for $1\leqslant l\leqslant r-1$,

$$R_{\nu_l}=(R_{\nu'})_{Q'_l}\ /\ Q'_{l-1} \tag{5.2}$$

is a discrete valuation ring.

Now, $A_\wp/P_{r-1}$ is a 1-dimensional Noetherian local domain whose fraction field is $k_{\nu'}$. Therefore there exists a discrete valuation ν_r of $k_{\nu'}$ dominating $A_\wp/P_{r-1}$. Let us consider the composite valuation $\nu=\nu'\circ\nu_r=\nu_1\circ\ldots\circ\nu_{r-1}\circ\nu_r$ and the chain of ideals $(0)\subset Q_1\subset\cdots\subset Q_{r-1}\subset Q_r:=\mathcal{M}_\nu$ where Q_l is the contraction of Q'_l to R_ν, $1\leqslant l\leqslant r-1$. Then $(R_\nu)_{Q_{r-1}}=R_{\nu'}$ and $R_\nu\ /\ Q_{r-1}\cong R_{\nu_r}$, hence from (5.1) and (5.2) we conclude that ν and the chain of the Q_l's satisfy the lemma. □

Proposition 5.2. — *Let X be a variety over a perfect field k and let P be a stable point of X_∞. Given a minimal prime ideal $\widehat{P}_0$ of $\widehat{\mathcal{O}_{X_\infty,P}}$ and* *a saturated chain of prime ideals in $\widehat{\mathcal{O}_{X_\infty,P}}$:*

$$\widehat{P}_0 \subset \widehat{P}_1 \subset \cdots \subset \widehat{P}_{r-1} \subset \widehat{P}_r = P\widehat{\mathcal{O}_{X_\infty,P}}$$

there exists a finite algebraic field extension K of $\kappa(P)$ and a K-r-wedge $\Phi : \operatorname{Spec} K[[\underline{\xi}, t]] \to X$ with special arc P such that, for $0 \leqslant l \leqslant r$, the image of $(\xi_1, \ldots, \xi_l)$ by the induced morphism $\widehat{\varphi} : \operatorname{Spec} K[[\underline{\xi}]] \to \operatorname{Spec} \widehat{\mathcal{O}_{X_\infty,P}}$ is $\widehat{P}_l$.

Proof. — We may suppose with no loss of generality that X is irreducible. If $r = 0$ then P is the generic point of $X_\infty \setminus (\operatorname{Sing} X)_\infty$ ([31, Theorem 2.9]). Therefore $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P}$ is a field, and also is $\widehat{\mathcal{O}_{X_\infty,P}}$, since it is isomorphic to $\widehat{\mathcal{O}_{(X_\infty)_{\mathrm{red}},P}}$ ([31, Theorem 3.13]). Hence the result is clear in this case.

Suppose that $r \geqslant 1$. The ring $\widehat{\mathcal{O}_{X_\infty,P}}$ is a Noetherian ring ([30, Corollary 4.6]), hence we may apply Lemma 5.1 to the Noetherian local domain $\widehat{\mathcal{O}_{X_\infty,P}}/\widehat{P}_0$. We obtain that there exists a valuation ν of $Fr(\widehat{\mathcal{O}_{X_\infty,P}}/\widehat{P}_0)$ dominating $\widehat{\mathcal{O}_{X_\infty,P}}/\widehat{P}_0$, which is the composition of r discrete valuations, and a saturated chain of prime ideals in R_ν,

$$(0) = Q_0 \subset Q_1 \subset \cdots \subset Q_r = \mathcal{M}_\nu, \tag{5.3}$$

such that the contraction of Q_i to $\widehat{\mathcal{O}_{X_\infty,P}}$ is $\widehat{P}_i$, $0 \leqslant i \leqslant r$. Since ν is composition of r discrete valuations and local uniformization holds for discrete valuations, by [28, Theorem 3.1], ν admits local uniformization. That is, there exists a finitely generated $\widehat{\mathcal{O}_{X_\infty,P}}$-algebra R contained in R_ν such that $\widetilde{R} := R_{\mathcal{M}_\nu \cap R}$ is a regular ring.

Even more ([28, Proof of Theorem 3.1]), if

$$(0) = \widetilde{Q}_0 \subset \widetilde{Q}_1 \subset \cdots \subset \widetilde{Q}_r = \widetilde{\mathcal{M}},$$

is the chain induced by (5.3) in $\widetilde{R}$, i.e. $\widetilde{Q}_i = Q_i \cap \widetilde{R}$, $0 \leqslant i \leqslant r$, and $\widetilde{\mathcal{M}}$ is the maximal ideal of $\widetilde{R}$, then a regular system of parameters $\{\xi_1, \ldots, \xi_r\}$ of $\mathcal{M}$ can be obtained with the following property: ξ_1 is a regular system of parameters of $\widetilde{Q}_1\widetilde{R}_{\widetilde{Q}_1}$ and, for $2 \leqslant i \leqslant r-1$, the class of ξ_i in $\widetilde{R}_{\widetilde{Q}_i}/\widetilde{Q}_{i-1}$ is a regular system of parameters of $\widetilde{Q}_i\widetilde{R}_{\widetilde{Q}_i}/\widetilde{Q}_{i-1}$ and, in addition, $\{\xi_1, \ldots, \xi_i\}$ is a regular system of parameters of $\widetilde{Q}_i\widetilde{R}_{\widetilde{Q}_i}$ for $1 \leqslant i \leqslant r$. Then the completion of $\widetilde{R}$ is $K[[\xi_1, \ldots, \xi_r]]$, where K is a finite algebraic extension of $\kappa(P)$, since k_ν is a finite algebraic extension of $\kappa(P)$ by Abhyankar's inequality ([1, Theorem 1]). Here note that $r = \dim \widehat{\mathcal{O}_{X_\infty,P}} \,/\, \widehat{P}_0$ because $\widehat{\mathcal{O}_{X_\infty,P}}$ is a catenary ring. Therefore, the inclusion of $\widehat{\mathcal{O}_{X_\infty,P}}$ in $\widehat{\widetilde{R}} = K[[\underline{\xi}]]$ induces the desired wedge Φ. □

The following result is a consequence of Lemma 4.7. It can also be obtained as a direct consequence of Proposition 5.2 and Lemma 4.8:

PROPOSITION 5.3. — *Let X be a normal toric variety over a perfect field k and let P be a toric stable point of X_∞. There exist a finite set $\mathcal{W}$ and, for each $\mathbf{w} \in \mathcal{W}$, a morphism $\varrho_{\mathbf{w}} : \operatorname{Spec} \widehat{R}_{\mathbf{w}} \to \operatorname{Spec} \widehat{\mathcal{O}_{X_\infty,P}}$, where $\widehat{R}_{\mathbf{w}}$ is a regular local ring with residue field $\kappa(P)$, satisfying the following property: For every saturated chain of prime ideals in $\widehat{\mathcal{O}_{X_\infty,P}}$:*

$$\widehat{P}_0 \subset \widehat{P}_1 \subset \cdots \subset \widehat{P}_{r-1} \subset \widehat{P}_r = P\widehat{\mathcal{O}_{X_\infty,P}} \tag{5.4}$$

where $\widehat{P}_0$ is a minimal prime ideal of $\widehat{\mathcal{O}_{X_\infty,P}}$, there exists $\mathbf{w} \in \mathcal{W}$ and there exists a chain of prime ideals in $\widehat{R}_{\mathbf{w}}$:

$$(0) = \widehat{Q}_0 \subset \widehat{Q}_1 \subset \cdots \subset \widehat{Q}_{r-1} \subset \widehat{Q}_r \tag{5.5}$$

such that the image of $\widehat{Q}_l$ by $\widehat{\varrho}_{\mathbf{w}}$ is $\widehat{P}_l$ for $0 \leqslant l \leqslant r$.

Proof. — We may suppose that X is affine and defined by a strongly convex cone σ, and $P = P_v$ where $v \in \sigma \cap M$. Then, taking $\mathcal{W} := \mathcal{W}_v$, the result is satisfied. In fact, given a chain (5.4) of prime ideals of $\widehat{\mathcal{O}_{X_\infty,P}}$, since $\widehat{P}_0$ is a minimal prime ideal of $\widehat{\mathcal{O}_{X_\infty,P}}$, by Corollary 4.12 there exists $\mathbf{w} \in \mathcal{W}_v$ such that $\widehat{P}_0 = \widehat{P}_{\mathbf{w}}$. Then, the assertion follows from the going up theorem applied to the integral extension of rings $\widehat{\mathcal{O}_{X_\infty,P_v}}/\widehat{P}_{\mathbf{w}} \hookrightarrow \widehat{R}_{\mathbf{w}}$.

An alternative proof follows applying Proposition 5.2: there exists a K-r-wedge $\Phi : \operatorname{Spec} K[[\underline{\xi}, t]] \to X$, with special arc P such that, for $0 \leqslant l \leqslant r$, the image of $(\xi_1, \ldots, \xi_l)$ by the induced morphism $\widehat{\varphi} : \operatorname{Spec} K[[\underline{\xi}]] \to \operatorname{Spec} \widehat{\mathcal{O}_{X_\infty,P}}$ is $\widehat{P}_l$. Then, by Lemma 4.8, there exists $\mathbf{w} \in \mathcal{W}_v$ such that $\widehat{\varphi}$ lifts to a morphism $\widehat{\psi}_{\mathbf{w}} : \operatorname{Spec} K[[\underline{\xi}]] \to \operatorname{Spec} \widehat{R}_{\mathbf{w}}$, i.e. we have $\widehat{\varrho}_{\mathbf{w}} \circ \widehat{\psi}_{\mathbf{w}} = \widehat{\varphi}$. Therefore, if we define $\widehat{Q}_l$ to be the image of $(\xi_1, \ldots, \xi_l)$ by $\widehat{\psi}_{\mathbf{w}}$, then we obtain a chain (5.5) with $\widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_l) = \widehat{P}_l$ for $0 \leqslant l \leqslant r$. □

The following can be said about the uniqueness of $\mathbf{w}$ in Proposition 5.3:

LEMMA 5.4. — *Let $X = X_\sigma$ be an affine normal toric variety over a perfect field k, let $v \in \sigma \cap N$ and $\mathbf{w} = \{(w_j; n_j)\}_{j=1}^s$ an element of $\mathcal{W}_v$. Let us consider the chain of prime ideals in $\widehat{\mathcal{O}_{X_\infty,P_v}}$*

$$\begin{aligned} \widehat{P}_{\mathbf{w}} = \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(0,\ldots,0)}) \subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(1,0,\ldots,0)}) \subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(2,0,\ldots,0)}) \subset \cdots \\ \subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(n_1,0,\ldots,0)}) \subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(n_1,1,\ldots,0)}) \subset \cdots \subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(n_1,n_2,0,\ldots,0)}) \\ \subset \cdots \subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(n_1,\ldots,n_s-1)}) \subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(n_1,\ldots,n_s)}) = P_v\widehat{\mathcal{O}_{X_\infty,P_v}} \end{aligned} \tag{4.22}$$

Then $\mathbf{w}$ *is the unique element in* $\mathcal{W}_v$ *satisfying that the chain* (4.22) *lifts to a chain of prime ideals in* $\widehat{R}_{\mathbf{w}}$.

Proof. — Let us denote $\widehat{P}_0 \subset \widehat{P}_1 \subset \cdots \subset \widehat{P}_{l(\mathbf{w})-1} \subset \widehat{P}_{l(\mathbf{w})} = P\widehat{\mathcal{O}_{X_\infty,P_v}}$ the chain (4.22), recall that it has length $l(\mathbf{w})$ (Lemma 4.14). Let us denote the integers $\{1,2,\ldots,l(\mathbf{w})\}$ by $1 = l_{1,1} < l_{1,2} < \cdots < l_{1,n_1} < l_{2,1} < \cdots < l_{2,n_2} < \cdots < l_{s,n_s-1} < l_{s,n_s} = l(\mathbf{w})$, in such a way that, for $1 \leqslant j \leqslant s$, $1 \leqslant n \leqslant n_j$,

$$\widehat{P}_{l_{j,n}} = \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(n_1,\ldots,n_{j-1},n,0,\ldots,0)}).$$

By Lemma 4.14, the contraction of $\widehat{P}_{l_{j,n}}$ to $\mathcal{O}_{X_\infty}$ is $P_{v_{j,l}}$ where

$$v_{j,l} = n_1 w_1 + \cdots + n_{j-1} w_{j-1} + n w_j.$$

Suppose that there exist $\mathbf{w}' \in \mathcal{W}_v$ and a chain of prime ideals in $\widehat{R}_{\mathbf{w}'}$:

$$\widehat{Q}_0 \subset \widehat{Q}_1 \subset \cdots \subset \widehat{Q}_{r-1} \subset \widehat{Q}_{l(\mathbf{w})}$$

such that the image of $\widehat{Q}_l$ by $\widehat{\varrho}_{\mathbf{w}'} : \operatorname{Spec} \widehat{R}_{\mathbf{w}'} \to \operatorname{Spec} \widehat{\mathcal{O}_{X_\infty,P_v}}$ is $\widehat{P}_l$, for $0 \leqslant l \leqslant l(\mathbf{w})$. Set $\mathbf{w}' = \{(w'_i, n'_i)\}_{i=1}^{s'}$ where $w'_1,\ldots,w'_{s'}$ are minimal elements of $\sigma \cap N$ and $n'_1,\ldots,n'_{s'} \in \mathbb{N}\setminus\{0\}$ are such that $v = n'_1 w'_1 + \cdots + n'_{s'} w'_{s'}$.

We will prove, by induction on (j,n), $1 \leqslant j \leqslant s$, $1 \leqslant n \leqslant n_j$, with the lexicographic order, that after a possible reordering of the w'_i's, we have

$$w'_i = w_i \quad \text{for } 1 \leqslant i \leqslant j \quad \text{and} \quad \nu_{\widehat{Q}_{l_{j,n}}}(y_i) = \begin{cases} n_i & \text{if } i < j \\ n & \text{if } i = j, \end{cases} \tag{5.6}$$

(see definition in (4.20)). In fact, for $(j,n) = (1,1)$, since $v_{1,1} = w_1$ and w_1 is a minimal element of $\sigma \cap N$, from Lemma 4.13 applied to the ideal $\widehat{Q}_{l_{1,1}}$ of $\widehat{R}_{\mathbf{w}'}$ it follows that there exists i, $1 \leqslant i \leqslant s'$, such that $w'_i = w_1$ and $\nu_{\widehat{Q}_{l_{1,1}}}(y_i) = 1$. We may suppose that $i = 1$. Now fix (j,n) and suppose that (5.6) holds for $(j',n') < (j,n)$. If $n = 1$ then, by the inductive hypothesis and since $\widehat{Q}_{l_{j,n-1}} \subset \widehat{Q}_{l_{j,n}}$, we have $w'_i = w_i$ for $1 \leqslant i \leqslant j-1$, and $\nu_{\widehat{Q}_{l_{j,n}}}(y_i) \geqslant \nu_{\widehat{Q}_{l_{j,n-1}}}(y_i) = n_i$ for $1 \leqslant i \leqslant j-1$. But now, $v_{l_{j,1}} = \sum_{i=1}^{j-1} n_i w_i + w_j$ and w_j is a minimal element of $\sigma \cap N$, imply that there exists i', $1 \leqslant i' \leqslant s'$, such that $w'_{i'} = w_j$ and $\nu_{\widehat{Q}_{l_{j,1}}}(y_{i'}) = 1$ by Lemma 4.13 applied to the ideal $\widehat{Q}_{l_{j,1}}$ of $\widehat{R}_{\mathbf{w}'}$. We may suppose that $i' = j$. Hence (5.6) holds for $(j,1)$. Finally, if $n > 1$ then, by the inductive hypothesis, we have $w'_i = w_i$ for $1 \leqslant i \leqslant j$ and

$$\nu_{\widehat{Q}_{l_{j,n}}}(y_i) \geqslant \nu_{\widehat{Q}_{l_{j,n-1}}}(y_i) = n_i \text{ for } 1 \leqslant i \leqslant j-1,$$
$$\nu_{\widehat{Q}_{l_{j,n}}}(y_j) \geqslant \nu_{\widehat{Q}_{l_{j,n-1}}}(y_j) = n-1.$$

Then, $v_{l_{j,n}} = \sum_{i=1}^{j-1} n_i w_i + n w_j$ and w_j is a minimal element of $\sigma \cap N$, imply that $\nu_{\widehat{Q}_{l_{j,n}}}(y_j) = \nu_{\widehat{Q}_{l_{j,n-1}}}(y_j) + 1$ by Lemma 4.13. This proves (5.6).

Now, from (5.6) it follows that $s' \geqslant s$ and that, after a possible reordering of the w'_i's, we have $w'_j = w_j$ and $n_j \leqslant n'_j$ for $1 \leqslant j \leqslant s$. Then, since $\sum_{j=1}^{s} n_j w_j = v = \sum_{l=1}^{s'} n'_l w'_l$, we conclude that $\mathbf{w}' = \{(w_j; n_j)\}_{j=1}^{s}$, i.e. $\mathbf{w}' = \mathbf{w}$. □

PROPOSITION 5.5. — *Let $X = X_\sigma$ be an affine normal toric variety over a perfect field k, let $v \in \sigma \cap N$ and $\mathbf{w} \in \mathcal{W}_v$. Then $\mathcal{I}_{\mathbf{w}}$ is an irreducible component of* $\operatorname{Spec} \widehat{\mathcal{O}_{X_\infty, P_v}}$ *and* $\dim \mathcal{I}_{\mathbf{w}} = \mathrm{l}(\mathbf{w})$.

Moreover, for $\mathbf{w}, \mathbf{w}' \in \mathcal{W}_v$, $\mathbf{w} \neq \mathbf{w}'$, we have $\mathcal{I}_{\mathbf{w}} \neq \mathcal{I}_{\mathbf{w}'}$.

Proof. — The chain (4.22) of prime ideals in $\widehat{\mathcal{O}_{X_\infty, P_v}}$

$$\widehat{P}_{\mathbf{w}} = \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(0,\ldots,0)}) \subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(1,0,\ldots,0)}) \subset \cdots \subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(n_1,0,\ldots,0)}) \\ \subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(n_1,1,\ldots,0)}) \subset \cdots \subset \widehat{\varrho}_{\mathbf{w}}(\widehat{Q}_{(n_1,\ldots,n_s)}) = P_v \widehat{\mathcal{O}_{X_\infty, P_v}}$$

is saturated (Lemma 4.14). Therefore, $\widehat{P}_{\mathbf{w}}$ is a minimal prime ideal of $\widehat{\mathcal{O}_{X_\infty, P_v}}$. This proves the first assertion (see Corollary 4.12).

For the second assertion, suppose that $\mathbf{w}, \mathbf{w}' \in \mathcal{W}_v$ satisfy $\mathcal{I}_{\mathbf{w}} = \mathcal{I}_{\mathbf{w}'}$. Then $\widehat{\mathcal{O}_{X_\infty, P_v}}/\widehat{P}_{\mathbf{w}} = \widehat{\mathcal{O}_{X_\infty, P_v}}/\widehat{P}_{\mathbf{w}'} \hookrightarrow \widehat{R}_{\mathbf{w}'}$ is an integral extensions of rings and (4.22) defines a chain of prime ideals in $\widehat{\mathcal{O}_{X_\infty, P_v}}/\widehat{P}_{\mathbf{w}'}$. By the going up Theorem, this chain lifts to $\widehat{R}_{\mathbf{w}'}$. But then $\mathbf{w}' = \mathbf{w}$ by Lemma 5.4. This concludes the proof. □

THEOREM 5.6. — *Let $X = X_\Sigma$ be a normal toric variety over a perfect field k and let P be a toric stable point of X_∞. There exist a finite set $\mathcal{W}$ and, for each $\mathbf{w} \in \mathcal{W}$, a morphism $\rho_{\mathbf{w}} : Y^{\mathbf{w}} \to X$, where $Y^{\mathbf{w}}$ is a smooth variety, and a stable point $Q^{\mathbf{w}}$ of $Y^{\mathbf{w}}_\infty$ whose image by $(\rho_{\mathbf{w}})_\infty : Y^{\mathbf{w}}_\infty \to X_\infty$ is P, such that the following holds:*

There is a one to one correspondence between elements $\mathbf{w} \in \mathcal{W}$ and irreducible components of $\operatorname{Spec} \widehat{\mathcal{O}_{X_\infty, P_v}}$. *Moreover, the irreducible component corresponding to an element $\mathbf{w}$ is the image $\mathcal{I}_{\mathbf{w}}$ of the morphism* $\widehat{\rho}_{\mathbf{w}} : \operatorname{Spec} \widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty, Q^{\mathbf{w}}}} \to \operatorname{Spec} \widehat{\mathcal{O}_{X_\infty, P_v}}$, *and $\mathcal{I}_{\mathbf{w}}$ has dimension* $\dim \widehat{\mathcal{O}_{Y^{\mathbf{w}}_\infty, Q^{\mathbf{w}}}}$.

Precisely, if $\sigma \in \Sigma$ is the d-dimensional cone ($d = \dim X$) such that $P \in (X_\sigma)_\infty$ and $v \in \sigma \cap N$ is such that $P = P_v$, then $\mathcal{W} = \mathcal{W}_v$ is the set of partitions of v and $\dim \mathcal{I}_{\mathbf{w}} = \mathrm{l}(\mathbf{w})$ *for $\mathbf{w} \in \mathcal{W}_v$. Therefore* $\dim \widehat{\mathcal{O}_{X_\infty, P_v}} = \operatorname{tcht} P_v$.

Proof. — It follows from Proposition 5.5, Corollary 4.12 and Lemma 4.7. (see also Remark 3.4). □

Remark 5.7. — The integer $\operatorname{tcht} P$ has appeared in [4] dealing with the dimension of the minimal formal model of local rings $\widehat{\mathcal{O}_{X_\infty,\gamma}}$, γ a k-point in $Z(P) \subset X_\infty$.

Precisely, given a primitive element $v \in \sigma \cap M$, in [4, Theorem 6.3], the embedding dimension and the dimension of a finite dimensional formal model of γ are computed for a general k-point $\gamma \in Z(P_v)$ (see Remark 3.10). In (3) of the proof of Theorem 6.3 in [4], partitions of v (called *decompositions* there) are used to determine the irreducible components of a finite dimensional formal model of γ and their dimensions.

Corollary 5.8. — *Let X be the 3-dimensional toric variety defined by the cone $\sigma = \langle(1,1,0),(1,0,1),(0,1,1)\rangle$ in $\mathbb{R}^3$, as in Corollary 4.4. Let $P = P_{(2,2,2)}$ be the stable point of X_∞ defined by the element $(2,2,2)$ of $\sigma \cap \mathbb{Z}^3$. Then the ring $\mathcal{O}_{X_\infty,P}$ is irreducible but it is not analytically irreducible. Moreover, $\widehat{\mathcal{O}_{X_\infty,P}} \cong \widehat{\mathcal{O}_{(X_\infty)_{\mathrm{red}},P}}$ is not equidimensional.*

Proof. — There are two different of partitions of $(2,2,2)$ in $\sigma \cap N$:

$$(2,2,2) = 2(1,1,1)$$
$$(2,2,2) = (1,1,0) + (1,0,1) + (0,1,1)$$

(see also Corollary 4.4). Thus, applying Theorem 5.6 it follows that $\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P}}$ has two irreducible components: one of dimension 2 and the other of dimension 3. □

6. Relation with discrepancies

In this section we will discuss the relation of $\dim \widehat{\mathcal{O}_{X_\infty,P_eE}}$, with the log discrepancy and with the Mather–Jacobian log discrepancy of X with respect to E. After studying the toric case and some other examples, we will propose some questions.

Given a variety X over a perfect field k and a divisorial valuation ν on X, there exists a proper and birational morphism $\pi : Y \to X$, with Y normal, such that the center of ν on Y is a divisor E of Y. We also denote by ν_E the valuation ν. Then, the image of the canonical homomorphism $d\pi : \pi^*(\wedge^d\Omega_X) \to \wedge^d\Omega_Y$ is an invertible sheaf at the generic point of E, i.e. there exists a nonnegative integer $\widehat{k}_E$ such that the fibre at E of the sheaf $d\pi(\pi^*(\wedge^d\Omega_X))$ is isomorphic to the fibre at E of $\mathcal{O}_Y(-\widehat{k}_E E)$. We call $\widehat{k}_E$ the *Mather discrepancy* of X with respect to the prime divisor E.

Note that $\widehat{k}_E$ only depends on the divisorial valuation $\nu = \nu_E$. Then, the *Mather–Jacobian log discrepancy* of X with respect to E is

$$a_{MJ}(E;X) \coloneqq \widehat{k}_E - \nu_E(\mathrm{Jac}_X) + 1$$

where Jac_X is the Jacobian ideal of X (see [17], [9]). In [26] we proved that, for any variety X over a field k of characteristic zero, a divisorial valuation ν_E and a positive integer e, we have

(i) $\operatorname{embdim} \widehat{\mathcal{O}_{(X_\infty)_{\mathrm{red}},P_{eE}}} = e(\widehat{k}_E + 1)$ ([26, Theorem 3.4]),

(ii) $\dim \widehat{\mathcal{O}_{X_\infty,P_{eE}}} \geqslant e\, a_{MJ}(E;X)$ ([26, Theorem 4.1]).

The result (i) has been extended to positive characteristic in [13].

Now let $X = X_\Sigma$ be a toric variety and let us consider a toric divisorial valuation, hence defined by a minimal lattice element v of some cone of Σ (see the beginning of Section 3). Recall that $D_v \coloneqq \overline{O_{\langle v\rangle}}$ is an irreducible Weil divisor on some resolution of singularities $X_{\Sigma'}$ of X_Σ.

Corollary 6.1. — *Let $X = X_\Sigma$ be a normal toric variety over a perfect field k and let us consider a toric divisorial valuation, hence defined by a minimal element v of $\sigma \cap N$ for some cone σ of Σ, and a positive integer e. Then we have*

$$e\, a_{MJ}(D_v;X) \leqslant \operatorname{tcht} P_{ev} \leqslant e(\widehat{k}_{D_v} + 1).$$

Proof. — If $\operatorname{char} k = 0$, then the corollary is a direct consequence of the results in [26]: see (i) and (ii) above. For k perfect of positive characteristic, we also obtain the second inequality by the extension of (i) in [13].

Moreover, the proof in [26] only uses the hypothesis $\operatorname{char} k = 0$ to determine a minimal system of generators of P_{eE}/P_{eE}^2 (recall the finiteness property of the stable points [30, Theorem 4.1]). But, if $X = X_\Sigma$ is a normal toric variety and $E = D_v$, then, for $\operatorname{char} k \geqslant 0$, a minimal system of generators of P_{ev}/P_{ev}^2 is defined as follows: Let $\{u_1, \ldots, u_d\} \subset \sigma^\vee \cap M$ which is a basis of the free $\mathbb{Z}$-module M, as in Section 3, and such that $\operatorname{ord}_{D_v} \pi^*(dx_1 \wedge \cdots \wedge dx_d)$ is minimal, hence equal to $\widehat{k}_{D_v}$. Here recall that $x_i = \aleph^{u_i}$, $1 \leqslant i \leqslant d$, hence

$$\widehat{k}_{D_v} + 1 = \sum_{i=1}^{d} \langle u_i, v\rangle. \tag{6.1}$$

From (3.14) in Proposition 3.8 it follows that the classes of $\{X_{i,0}, \ldots, X_{i,c_i-1}\}_{i=1}^d$ define a minimal system of generators of P_{ev}/P_{ev}^2, where $c_i = e\langle u_i, v\rangle$ for $1 \leqslant i \leqslant m$. In fact, from the definition of f_j, $d+1 \leqslant j \leqslant m$, (see (3.10)) we obtain

$$\widetilde{F}_{j,n} \in \left(\{X_{i,0}, \ldots, X_{i,c_i-1}\}_{i=1}^m\right)^2 \quad \text{for } d+1 \leqslant j \leqslant m,\ 0 \leqslant n < \epsilon_j.$$

Moreover, from the first equality in (3.11) it follows that, for $d+1 \leqslant j \leqslant m$ and $0 \leqslant n \leqslant c_j - 1$ (hence $\epsilon_j \leqslant n + \epsilon_j \leqslant \epsilon_j + c_j - 1$), $X_{j,n}$ appears with nonzero coefficient in the linear part of $\widetilde{F}_{j,\epsilon_j+n}$, thus we can eliminate $X_{j,n}$ from $\widetilde{F}_{j,\epsilon_j+n}$ in (3.14) and conclude the assertion.

Note that we have proved that embdim $\widehat{\mathcal{O}_{X_\infty,P_{ev}}} = \sum_{i=1}^{d} c_i = e\,(\widehat{k}_{D_v}+1)$ by (6.1), thus we recover (i). Moreover, since we have obtained a minimal system of generators of P_{ev}/P_{ev}^2, Theorem 4.1 in [26] can be applied, hence (ii) holds in this case for any $\operatorname{char} k \geqslant 0$. From this and (4.19) in Corollary 4.12 the result follows. □

Suppose now that X is a normal $\mathbb{Q}$-Gorenstein variety, thus K_X is a $\mathbb{Q}$-Cartier divisor, i.e. there exists a positive integer r such that rK_X is Cartier. Here $\mathcal{O}(K_X) \cong i_*\Omega^d_{X_{\mathrm{reg}}}$ where $i : X_{\mathrm{reg}} := X \setminus \operatorname{Sing} X \hookrightarrow X$ is the inclusion. Let ν be a divisorial valuation and let $\pi : Y \to X$ be a proper birational morphism with Y normal such that the center of ν on Y is a Weil divisor E of Y. The discrepancy $\mathbb{Q}$-divisor $K_{Y/X} := K_Y - \frac{1}{r}\pi^*(rK_X)$ is well defined. The *log discrepancy* of X with respect to E is

$$a(E;X) := k_E + 1$$

where $k_E = \operatorname{ord}_E(K_{Y/X})$ only depends on the divisorial valuation $\nu = \nu_E$. We have

$$a_{MJ}(E;X) \leqslant a(E;X)$$

and equality holds if X is a normal complete intersection ([9, Proposition 2.20]).

Now let $X = X_\Sigma$ be a toric variety. Recall that $K_X = \sum_{\alpha\in\Sigma(1)} D_\alpha$ where $\Sigma(1)$ is the set of 1-dimensional cones of Σ. Then, an affine normal toric variety X_σ is $\mathbb{Q}$-Gorenstein if and only if there exists $u_\sigma \in M_{\mathbb{Q}}$ such that $\langle u_\sigma, v_i\rangle = 1$ for all extremal vectors v_i, $1 \leqslant i \leqslant r$ of σ ([7, Proposition 11.4.12]). Here recall that the extremal vectors of σ are the primitive vectors of the 1-dimensional faces of σ, thus for such a u_σ we have $\operatorname{div}(\aleph^{-u_\sigma}) = K_X$.

Theorem 6.2. — *Let $X = X_\Sigma$ be a normal toric $\mathbb{Q}$-Gorenstein variety over a perfect field k and let us consider a toric divisorial valuation, hence defined by a minimal element v of $\sigma \cap N$ for some cone σ of Σ. Then we have*

$$a(D_v;X) \leqslant \sup_e \frac{\operatorname{tcht} P_{ev}}{e} = \sup_e \frac{\dim \widehat{\mathcal{O}_{X_\infty,P_{ev}}}}{e}.$$

Moreover, there exist a positive integer e and an irreducible component $\mathcal{I}$ of $\operatorname{Spec} \widehat{\mathcal{O}_{X_\infty,P_{ev}}}$ *whose dimension is $e\,a(D_v;X)$.*

More precisely, if $\operatorname{char} k = 0$ *then* $\mathcal{I}$ *can be obtained as the image of an irreducible component of* $\operatorname{Spec}\widehat{\mathcal{O}_{Y_\infty,Q}}$ *where* $Y \to X$ *is the universal cover of* $X \setminus \operatorname{Sing} X$*, and* Q *is a stable point of* Y_∞ *whose image in* X_∞ *is* P_{ev}*.*

Proof. — We may suppose that X is affine, i.e. $X = X_\sigma$ where σ is a strongly convex cone. Let $v_1, \dots, v_r$ be the extremal vectors of $\sigma \cap N$. Let $u_\sigma \in \sigma^\vee \cap M_\mathbb{Q}$ be such that $\langle u_\sigma, v_i\rangle = 1$ for $1 \leqslant i \leqslant r$ (recall that X_σ is $\mathbb{Q}$-Gorenstein). Since $d = \dim X_\sigma$, we have $d \leqslant r$ and there exists some expression of the form

$$v = q_1 v_1 + \dots + q_r v_r \tag{6.2}$$

where $q_i \in \mathbb{Q}_{\geqslant 0}$. This implies that

$$a(D_v; X) = 1 + k_v = \langle u_\sigma, v\rangle = q_1 + \dots + q_r.$$

On the other hand, let $e \in \mathbb{N}$ be such that $eq_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant r$. Then (6.2) induces a partition $\mathbf{w} = \{(v_i, eq_i)\}_{i=1}^r$ of ev. By Theorem 5.6, $\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_{ev}}}$ has an irreducible component $\mathcal{I}_\mathbf{w}$ of dimension

$$l(\mathbf{w}) = \sum_{i=1}^{r} e\, q_i = e\, a(D_v; X).$$

From this and Theorem 5.6, the first part of the theorem follows.

For the last part, suppose that $\operatorname{char} k = 0$. There exist d extremal vectors which are $\mathbb{Q}$-linearly independent, let us suppose they are $v_1, \dots, v_d$. Then we may consider an expression (6.2) where only $v_1, \dots, v_d$ appear, i.e. $v = q_1 v_1 + \dots + q_d v_d$. The corresponding partition of ev is $\mathbf{w} = \{(v_i, eq_i)\}_{i=1}^d$ and the irreducible component $\mathcal{I} = \mathcal{I}_\mathbf{w}$ of $\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_{ev}}}$ is obtained as follows: Let $\rho : Y \coloneqq \operatorname{Spec} k[y_1, \dots, y_d] \to X$ be given by

$$\aleph^u \mapsto y_1^{\langle u, v_1\rangle} \cdots y_d^{\langle u, v_d\rangle} \quad \text{for } u \in \sigma^\vee \cap M.$$

and let $Q = (Y_{1,0}, \dots, Y_{1,eq_1-1}, \dots, Y_{d,0}, \dots, Y_{d,eq_d-1})$, a stable point of Y_∞ whose image by $\rho_\infty : Y_\infty \to X_\infty$ is P_{ev}. Then $\mathcal{I}_\mathbf{w}$ is the image of the morphism $\widehat{\rho} : \operatorname{Spec}\widehat{\mathcal{O}_{Y_\infty,Q}} \to \operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_{ev}}}$ (Lemma 4.11).

Note first that $\operatorname{codim}(X, \operatorname{Sing} X) \geqslant 2$ and, since $K_X = -\operatorname{div}(\aleph^{u_\sigma})$, we have

$$\rho^*(K_X) = -\operatorname{div}(y_1^{\langle u_\sigma, v_1\rangle} \cdots y_d^{\langle u_\sigma, v_d\rangle}) = -\operatorname{div}(y_1 \cdots y_d) = K_Y. \tag{6.3}$$

If $\sigma = \langle v_1, \dots, v_d\rangle$, i.e. σ is a simplicial cone, then $\rho : Y \to X$ is a finite morphism since $\rho^\sharp(\aleph^{v_i^+}) = y_i^{d_i}$, $1 \leqslant i \leqslant d$, with the notation of Remark 4.10. This implies that ρ is a finite étale morphism over $X \setminus \operatorname{Sing} X$.

In general ρ is not finite. Let $\sigma_0 \coloneqq \langle v_1, \dots, v_d\rangle$, then ρ factors through X_{σ_0}, i.e. $\rho = \pi \circ \rho_0$ where $\rho_0 : Y \to X_{\sigma_0}$ is given by $\aleph^u \mapsto y_1^{\langle u, v_1\rangle} \cdots y_d^{\langle u, v_d\rangle}$

for $u \in \sigma_0^\vee \cap M$, and $\pi : X_{\sigma_0} \to X_\sigma$ by the inclusion $k[\sigma^\vee \cap M] \subset k[\sigma_0^\vee \cap M]$. We have that ρ_0 is a finite morphism and π is an equivariant morphism which contracts the subvarieties of X_{σ_0} defined by the faces of σ_0 which are not faces of σ, thus ρ is not finite. However, if τ is a face of σ_0 which is not a face of σ, then $\tau^\circ \subset \sigma^\circ$, hence the subvariety $\overline{O_\tau}$ defined by τ contracts to the origin, which is contained in $\operatorname{Sing} X$. Therefore ρ_0, and also ρ, is finite and étale over $X \setminus \operatorname{Sing} X$.

Finally, let us show that $Y \backslash \rho^{-1}(\operatorname{Sing} X)$ is simply connected. Let $\widetilde{N} := \mathbb{Z}^d$ and let Δ be the cone $(\mathbb{R}_{\geqslant 0})^d$ in $\widetilde{N}_\mathbb{R}$, so that Y is the toric affine variety defined by Δ. Let $\widetilde{\rho} : \widetilde{N} \to N$ be the morphism of lattices induced by ρ. Hence, if $\widetilde{v}_i = (0, \dots, 0, 1, 0, \dots, 0)$, 1 in the i-th position, then $\widetilde{\rho}(\widetilde{v}_i) = v_i$ for $1 \leqslant i \leqslant d$. Let Σ (resp. $\widetilde{\Sigma}$) be the fan in $N_\mathbb{R}$ (resp. $\widetilde{N}_\mathbb{R}$) defining $X \setminus \operatorname{Sing} X$ (resp. $Y \backslash \rho^{-1}(\operatorname{Sing} X)$). Then, $\operatorname{codim}(X, \operatorname{Sing} X) \geqslant 2$ implies that $v_i \in \Sigma(1)$, $1 \leqslant i \leqslant d$, and hence $\widetilde{v}_1 = (1, 0, \dots, 0), \widetilde{v}_2 = (0, 1, 0, \dots, 0), \dots, \widetilde{v}_d = (0, \dots, 0, 1) \in \widetilde{\Sigma}(1)$. We conclude that $Y \setminus \rho^{-1}(\operatorname{Sing} X)$ is simply connected (see [29, Proposition 1.9]). Therefore $\rho : Y \to X$ is the universal cover of $X \setminus \operatorname{Sing} X$. □

Example 6.3. — As in Corollary 4.4 and Corollary 5.8, let X be the toric 3-dimensional variety defined by the cone $\sigma = \langle (1,1,0), (1,0,1), (0,1,1) \rangle$ in $\mathbb{R}^3$. It has an isolated singularity at the origin O. In addition, $u_\sigma = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is in $\sigma^\vee \cap M_\mathbb{Q}$ and satisfies $\langle u_\sigma, v_i \rangle = 1$ for all extremal vectors v_i, $1 \leqslant i \leqslant 3$ of σ. Therefore X is a normal $\mathbb{Q}$-Gorenstein singularity. The blowing up of X at O defines an equivariant resolution of singularities Y of X. In fact, Y is the toric variety defined by the elementary subdivision of σ by $v\mathbb{R}_{\geqslant 0}$, where $v = (1,1,1)$, which is a minimal element of $\sigma \cap N$. We have

$$a(D_v; X) = 1 + k_v = \langle u_\sigma, v \rangle = \frac{3}{2}$$

hence $k_v = \frac{1}{2} > 0$ and we conclude that (X, O) is a terminal singularity.

Now, we have $\dim \widehat{\mathcal{O}_{X_\infty, P_v}} = 1$ (Corollary 4.2) and $a_{MJ}(D_v; X) = \widehat{k}_v - \nu_v(\operatorname{Jac}_X) + 1 = 2 - 3 + 1 = 0$, hence

$$\frac{3}{2} = a(D_v; X) > \dim \widehat{\mathcal{O}_{X_\infty, P_v}} = 1 > a_{MJ}(D_v; X) = 0.$$

Let us next consider the stable point P_{2v}. In Corollary 5.8 we showed that $\widehat{\mathcal{O}_{X_\infty, P_{2v}}}$ has two irreducible components: one of dimension 2 and another of dimension $3 = 2(1 + k_v)$. In this case $2(1 + k_v) = \operatorname{tcht} P_{2v}$, that is,

$$a(D_v; X) = \frac{\dim \widehat{\mathcal{O}_{X_\infty, P_{2v}}}}{2} \leqslant \sup_e \frac{\dim \widehat{\mathcal{O}_{X_\infty, P_{ev}}}}{e}.$$

Example 6.4. — Let us show in the next example that the inequality in Theorem 6.2 may be strict. Let X be the toric surface defined by the cone $\sigma = \langle (1,0),(3,4)\rangle$ and let $v=(1,1)$, a primitive element of σ. Then $1+k_v = \left\langle \left(1,-\frac{1}{2}\right), v\right\rangle = \frac{1}{2}$. On the other hand, there are two partitions of $4(1,1)$:

$$(4,4) = (1,0)+(3,4), \quad (4,4) = 4(1,1).$$

Thus, by Theorem 5.6, $\widehat{\mathcal{O}_{X_\infty,P_{4v}}}$ has two irreducible components: one of dimension 2 and another of dimension 4. In this case, $4(1+k_v) = 2 < 4 = \operatorname{tcht} P_{4v}$. Therefore, the inequality $a(D_v;X) < \sup_e \frac{\operatorname{tcht} P_{ev}}{e}$ is strict.

Theorem 6.2 above motivates Questions 6.5 and 6.6 below. Question 6.5, which is weaker than Question 6.6, follows the line in [26, Theorem 4.1] (see (ii) at the beginning of this section).

QUESTION 6.5. — *Let X be a normal $\mathbb{Q}$-Gorenstein variety and let $\nu = \nu_E$ be a divisorial valuation, i.e. the center of ν on Y, $\pi : Y \to X$ a proper birational morphism with Y normal, is a Weil divisor E of Y.*

Do we have

$$a(E;X) \leqslant \sup_e \frac{\dim \widehat{\mathcal{O}_{X_\infty,P_{eE}}}}{e}?$$

QUESTION 6.6. — *Suppose that $a(E;X)>0$. Does there exist a positive integer e and an irreducible component $\mathcal{I}$ of* $\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_{eE}}}$ *whose dimension is $e\,a(E;X)$?*

Even more, in case that e and an irreducible component $\mathcal{I}$ of $\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_{eE}}}$ exist as before, we would like to understand the geometric sense of $\mathcal{I}$.

Remark 6.7. — Note first that from [26, Theorem 4.1] it follows that Question 6.5 has an affirmative answer if X is normal and complete intersection, since in this case $a(E;X) = a_{MJ}(E;X)$. But Question 6.6 is unknown in this case.

If X is nonsingular at the center P_0 of ν_E, then the ring $\mathcal{O}_{X_\infty,P_{eE}}$ is regular and essentially of finite type over a field, and $\dim \mathcal{O}_{X_\infty,P_{eE}} = e\,a(E;X)$ ([31, Proposition 4.2] and [32, Corollary 2.9]). Therefore, $\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_{eE}}}$ is irreducible and $\dim \widehat{\mathcal{O}_{X_\infty,P_{eE}}} = e\,a(E;X)$. Hence Questions 6.5 and 6.6 have affirmative answer in this case. Moreover, since (X,P_0) is nonsingular, its universal cover is trivial. Therefore, as in Theorem 6.2, the irreducible component $\mathcal{I}$ of $\operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_{eE}}}$ whose dimension is $ea(E;X)$ is obtained from the space of arcs of the universal cover of X (in this case $\mathcal{I} = \operatorname{Spec}\widehat{\mathcal{O}_{X_\infty,P_{eE}}}$).

Now, given a normal $\mathbb{Q}$-Gorenstein variety and a divisorial valuation $\nu = \nu_E$, keep the notation in Question 6.5. Since $\pi : Y \to X$ is proper and birational, there exists a stable point Q_{eE} of Y_∞ whose image by π_∞ is P_{eE} and we have that the induced morphism $\widehat{\mathcal{O}_{X_\infty,P_{eE}}} \to \widehat{\mathcal{O}_{Y_\infty,Q_{eE}}}$ is surjective ([31, Proposition 4.1]). Since Y is normal and E a divisor of Y, Y is regular at the generic point of E, hence $\dim \widehat{\mathcal{O}_{Y_\infty,Q_{eE}}} = e$ and we conclude that $\dim \widehat{\mathcal{O}_{X_\infty,P_{eE}}} \geqslant e$. Therefore

$$\sup_e \frac{\dim \widehat{\mathcal{O}_{X_\infty,P_{eE}}}}{e} \geqslant 1$$

and Question 6.5 has an affirmative answer whenever $a(E;X) \leqslant 1$.

A divisorial valuation ν over a variety X is called a terminal valuation if there exists a prime exceptional divisor E on a minimal model $Y \to X$ such that $\nu = \nu_E$ (see [12]). In this case, from T. de Fernex and R. Docampo's work [12], and [31, Corollary 5.12], it follows that $\dim \widehat{\mathcal{O}_{X_\infty,P_E}} = 1$. On the other hand, if X is normal and $\mathbb{Q}$-Gorenstein and ν_E a terminal valuation then $a(E;X) \leqslant 1$ ([18, Theorem 8.2.12]), hence Question 6.5 has an affirmative answer in this case. In particular, this implies that Question 6.5 has an affirmative answer for essential valuations over normal $\mathbb{Q}$-Gorenstein surfaces. Essential valuations on a variety X are those divisorial valuations on X whose center on any resolution of singularities $\widetilde{\pi} : \widetilde{X} \to X$ is an irreducible component of the exceptional locus of $\widetilde{\pi}$.

Next, we will study Questions 6.5 and 6.6 in a family of 3-dimensional varieties with isolated terminal singularities. This family was given by J. Johnson and J. Kollár [20] to illustrate examples of essential valuations ν_E which do not belong to the image of the Nash map, i.e. N_E is not an irreducible component of X_∞^{Sing}.

Example 6.8. — Let $m \geqslant 2$ and let $X = X(m)$ be the hypersurface defined by $xy = z^2 - w^m$ in $\mathbb{A}_k^4$, where k is a field of characteristic zero. It has an isolated singularity at the origin O which is a cA_1-type singularity. If we blow up O, the variety obtained has a unique singular point and it is locally $X(m-2)$. After $[\frac{m}{2}]$ blowing ups of closed points we obtain a resolution of singularities of X. Its exceptional locus consists on $[\frac{m}{2}]$ irreducible divisors $E_1, \dots, E_{[\frac{m}{2}]}$, where E_i is the strict transform of the exceptional locus of the i-th blow up. If m is odd and $m \geqslant 5$ (resp. m even or $m = 3$) then ν_{E_1} and ν_{E_2} (resp. ν_{E_1}) are the essential valuations ([20, Lemmas 12 to 17]) and, for all m, we have $N_{E_1} = X_\infty^{\mathrm{Sing}}$, i.e. N_{E_1} is the unique irreducible component of X_∞^{Sing} ([20, Theorem 1]).

Fix i, $1 \leqslant i \leqslant [\frac{m}{2}]$, and let us consider the divisorial valuation ν_{E_i}. We have $\nu_{E_i}(x) = \nu_{E_i}(y) = \nu_{E_i}(z) = i$, $\nu_{E_i}(w) = 1$ and $a(E_i; X) = k_{E_i} + 1 = i + 1$. Recall that X is a normal hypersurface, therefore from [26, Theorem 4.1] it follows that $\dim \widehat{\mathcal{O}_{X_\infty, P_{eE_i}}} \geqslant e(k_{E_i} + 1) = e(i+1)$ for $e \in \mathbb{Z}_{>0}$. Moreover, let Y be the A_1-singularity $xy = z^2$ in $\mathbb{A}^3_k$, obtained by intersecting X with $(w = 0)$, and let ν_F be its essential valuation. Following the ideas in Proposition 3.8, or more precisely in [31, Corollary 5.6], we may describe the ring $\widehat{\mathcal{O}_{X_\infty, P_{eE_i}}}$ and obtain that

$$\widehat{\mathcal{O}_{X_\infty, P_{eE_i}}} \,/\, (W_0, \ldots, W_{e-1}) \cong \widehat{\mathcal{O}_{Y_\infty, P_{eiF}}} \otimes_{\kappa(P_{eiF})} \kappa(P_{eE_i}).$$

Since $\widehat{\mathcal{O}_{X_\infty, P_{eE_i}}}$ is a catenary ring and $\dim \widehat{\mathcal{O}_{Y_\infty, P_{eiF}}} = ei$, we conclude that $\dim \widehat{\mathcal{O}_{X_\infty, P_{eE_i}}} \leqslant ei + e$ and hence $\dim \widehat{\mathcal{O}_{X_\infty, P_{eE_i}}} = e(k_{E_i} + 1) = e(i+1)$. Equivalently,

$$\frac{\dim \widehat{\mathcal{O}_{X_\infty, P_{eE_i}}}}{e} = a(E_i; X) \qquad \text{for every } e \geqslant 1.$$

In addition, $\widehat{\mathcal{O}_{X_\infty, P_{eE_i}}}$ is a complete intersection ring, hence every irreducible component of $\operatorname{Spec} \widehat{\mathcal{O}_{X_\infty, P_{eE_i}}}$ has dimension $e(k_{E_i}+1)$. This answers affirmatively Questions 6.5 and 6.6 in this case.

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