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ALBANESE MAP OF SPECIAL MANIFOLDS: A CORRECTION

by Frédéric CAMPANA

Abstract. — We show that any fibration of a “special” compact Kähler manifold X onto an Abelian variety has no multiple fibre in codimension one. This statement strengthens and extends previous results of Kawamata and Viehweg when $\kappa(X) = 0$. This also corrects the proof given in the present journal in 2004, which was incomplete.

Résumé. — On montre qu’une fibration d’une variété compacte Kählérienne X sur une variété Abélienne n’a pas de fibre multiple en codimension 1. Ce résultat renforce et généralise des résultats précédents de Kawamata et Viehweg lorsque $\kappa(X) = 0$, et corrige la preuve incomplète donnée dans ce journal en 2004.

1. Introduction

The following statement is given in [3, Proposition 5.3]:

Theorem 1.1. — *Let $a_X : X \rightarrow A_X$ is the Albanese map of X , assumed to be special. Then a_X is onto, has connected fibres, and no multiple fibre in codimension one.*

This statement is used neither in [3], nor in [4].

Recall ([3, Definition 2.1 and Theorem 2.27]) that X is special if $\kappa(X, L) < p$ for any $p > 0$, and any rank-one coherent subsheaf of $\mathcal{O}_X(p)$. This implies that X has no surjective meromorphic $g : X \dashrightarrow Z$ map onto a manifold Z of general type and positive dimension p , and more generally exactly means that the “orbifold base” (Z, D_g) of such a g , constructed out of its multiple fibres, is never of “general type”. See Section 2 below for some more details.

While the proofs given there for the first two properties (which generalise earlier results by Y. Kawamata and Kawamata–Viehweg in the case

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when $(X) = 0$) are complete, the proof of the third property, even in the projective case, is not (as pointed out to me by K. Yamanoi and E. Rousseau⁽¹⁾). The aim of this note is to correct it in the projective case (but only partially in the compact Kähler case) by using the main result of [4], itself based on [1] (or, alternatively, [5]).

Theorem 1.2. — *Let $f : X \rightarrow A$ be a holomorphic map from the connected, compact, Kähler manifold to a compact complex torus A . If X is special, then:*

- (1) f is surjective,
- (2) If f is the Albanese map of X , then f has connected fibres,
- (3) f has no multiple fibre⁽²⁾ in codimension one if the fibres of f are connected⁽³⁾, and if A is an Abelian variety (equivalently: $D_f = 0$ if (A, D_f) is the “orbifold base” of f).

Remarks 1.3.

- (1) Although the assumption that A is an Abelian variety in (3) is certainly not necessary, our proof rests on [1] (or [5]) for A projective. Although the Kähler version is known, by [9], we cannot treat the Kähler case in general with the present arguments, because we use Poincaré reducibility.
- (2) An easy case, not covered by Theorem 1.2 in which (3) can be proved, is when the algebraic dimension $a(A)$ of X vanishes, because then A does not contain any effective divisor. The argument of the second step of the proof applies, more generally, to show that the conclusion of Theorem 1.2(3) still holds true when $A = T \times V$ is a compact complex torus product of a subtorus T of algebraic dimension zero, with V , an Abelian variety, because every irreducible divisor on A is a product $T \times D$, where D is a divisor on V .
- (3) (1) and (2), and their proofs, still apply when X is “weakly special”, which means that no finite étale cover X' of X has a surjective meromorphic fibration onto a positive-dimensional projective manifold of

(1) The problem comes from the potential multiplicity-one exceptional divisors of a_X which are no longer f -exceptional if $g = q \circ f$, where $q : A \rightarrow B$ is a non-trivial torus quotient sending D_f to an ample divisor of B . We overcome this difficulty in the second step of the proof, by cutting the fibration by means of Poincaré reducibility. The first step is the same as in [3].

(2) We use here the “inf” multiplicities, not the more classical “gcd” version, which would give a weaker result.

(3) This can be realised, using the Stein factorisation of f . Indeed, since f factorises through the Albanese map a_X of X , which has connected fibres the Stein factorisation of f coincides with the one of a_X , which maps $\text{Alb}(X)$ to a torus, hence finite étale over A .

general type. Weak-specialness coincides with specialness for curves and surfaces ([3, Proposition 9.29]), but differs from dimension 3 on, by examples of Bogomolov–Tschinkel ([2], [3, Theorem 9.30]).

2. Multiple fibres of maps to complex tori

If $f : X \rightarrow Y$ is a fibration, that is: a holomorphic map with connected fibres onto a (smooth) compact complex manifold Y , we define its “orbifold base” D_f as follows: for every prime divisor $E \subset Y$, let $f^{-1}(E) := \sum_k t_k \cdot D_k + R$, where the D_k s are the pairwise distinct prime divisors of X surjectively mapped by f onto E , while R is an effective f -exceptional divisor of X . The multiplicity⁽⁴⁾ $m_f(E)$ of the generic fibre of f over E is then defined as: $m_f(E) := \inf\{t_k, k \in K\}$, and then $D_f := \sum_{E \subset Y} (1 - \frac{1}{m_f(E)}) \cdot E$, which is an effective \mathbb{Q} -divisor of Y (note indeed that this sum is finite since $m_f(E) = 1$ for all, but finitely many of the E s).

Proof of Theorem 1.2.

(1). — Assume by contradiction that $f(X) := Z = A$. After Ueno’s theorem ([8], p. 120), there is a quotient torus $q : A \rightarrow B$ such that $q(Z) \subset B$ is of dimension $p > 0$, and of general type. The composed map $g := q \circ f : X \rightarrow Z$ thus contradicts the specialness of X , since $(X, g^*(K_Z)) = p$ and $g^*(K_Z) \sim \frac{p}{X}$.

(2). — Let Z be a smooth model of the Stein factorisation Z_0 of f . We replace X by Z , which is still special since dominated by X . We may thus assume that f is generically finite. By [7, Theorem 23], replacing Z by a suitable finite étale cover which is still special by [3, §5.5], we may assume that Z fibres over over a manifold of general type if Z is not birational to A . This again contradicts the specialness of Z and thus of X .

(3). — Assume that $D_f = 0$. We shall first treat the case when D_f is ample, and reduce to this case in a second step.

Let thus D_f be ample on A . By a flattening of f , followed by suitable blow-ups of X and A , we may assume (see [3, Lemma 1.3]) that $f = v \circ f'$, where $v : A \rightarrow A$ is bimeromorphic with A smooth, and $f' : X \rightarrow A$ is a fibration such that its orbifold base $D := D_f = \bar{D} + E$, with E effective and v -exceptional, \bar{D} is the strict transform of D_f in A . This shows that

⁽⁴⁾ We use here the “inf” multiplicity instead of the usual “gcd” multiplicity for reasons explained in [3].

in particular $f^*(K_A + \bar{D}) \cong \mathcal{O}_X(\rho)$, $\rho := \dim(A)$. The following lemma shows that the line bundle $K_A + \bar{D}$ has Kodaira dimension $\dim(A) = \rho$, since D_f is ample on A . This contradicts the specialness of X if D_f is assumed to be ample.

Lemma 2.1 ([3, 1.14]). — *Let $v : Y \dashrightarrow Y$ be a bimeromorphic map between compact connected complex manifolds. Let D be an effective \mathbb{Q} -divisor on Y , and let \bar{D} be its strict transform on Y . Then:*

- (1) $\kappa(Y, K_Y + D) > \kappa(Y, K_Y + \bar{D})$.
- (2) $K_Y + \bar{D} = v^*(K_Y + D) + E$, E effective, $\epsilon > 0$ small enough.
- (3) If $\kappa(Y) > 0$, then $\kappa(Y, K_Y + \bar{D}) = \kappa(Y, K_Y + D)$.

Proof.

(1). — Let U be the largest Zariski open subset of Y over which v is isomorphic, and $U := v^{-1}(U)$. We have an obvious equality, for each $m > 0$: $v^*(H^0(U, m.(K_Y + D))) = H^0(U, m.(K_Y + \bar{D}))$. Since the complement of U in Y has complex codimension at least 2, Hartog's theorem shows that the left hand side of this last equality coincides with $v^*(H^0(Y, m.(K_Y + D)))$. On the other hand, the right hand side contains $H^0(Y, m.(K_Y + \bar{D}))$, and so $\kappa(Y, K_Y + D) > \kappa(Y, K_Y + \bar{D})$.

(2). — Let the rational numbers a_i, b_i be defined as follows: $v^*(D) = \bar{D} + \sum_i b_i.E_i$, $K_Y = v^*(K_Y) + \sum_i a_i.E_i$, where the E_i 's are the exceptional divisors of v . Thus: $a_i > 0, b_i > 0, \forall i$. Then:

$$\begin{aligned} K_Y + \bar{D} &= v^*(K_Y) + \sum_i a_i.E_i + (1 - \epsilon).\bar{D} + v^*(D) - \sum_i b_i.E_i \\ &= v^*(K_Y + D) + (1 - \epsilon).\bar{D} + (a_i - b_i).E_i. \end{aligned}$$

We thus get the claim when $0 < \epsilon < \min\{1, \frac{a_i}{b_i}, \forall i\}$, with $E := (1 - \epsilon).\bar{D} + \sum_i (a_i - b_i).E_i$.

(3). — Let $\epsilon > 0$ be sufficiently small, as above. Since K_Y is assumed to be \mathbb{Q} -effective, so is $(1 - \epsilon).v^*(K_Y)$, and so $K_Y + \bar{D} = \epsilon.v^*(K_Y + D) + E$, with: $E := E + (1 - \epsilon).v^*(K_Y)$, which is \mathbb{Q} -effective. Thus $\kappa(Y, K_Y + \bar{D}) > \kappa(Y, K_Y + D)$.

Remark 2.2. — The main property used in Lemma 2.1 is that $a_i > 0, \forall i$, i.e: the lift of the canonical sheaf under a modification of a smooth manifold vanishes on the exceptional divisor. This property does not hold true for sheaves of forms of degree less than the dimension. We thus need to use a less direct route, which requires deeper ingredients.

We shall now reduce to the case when D_f is ample.

Let $L := \mathcal{O}_A(m.D_f)$ be the line bundle on A with a section vanishing on some integral multiple $m.D_f$ of the orbifold base of f . Let $T = \text{Aut}^0(A)$ be the connected component of the group of translations of A preserving L . Let $q : A \rightarrow B := A/T$ be the quotient map. Then $D_f = \frac{1}{m}.q^*(D)$ for some ample effective divisor D on B (see, for example [6, Théorème 5.1]). Since $D_f = 0$, $\rho := \dim(B) > 0$.

We use Poincaré reducibility to reduce to the case that $A = B \times T$, where $B \rightarrow T$ is a suitable finite étale cover of T . This indeed amounts to replace X and D_f by the corresponding finite étale covers, preserving all of our hypothesis, in particular the specialness of X (by [3, §5.5], again).

We now consider the composed fibrations $g := q \circ f : X \rightarrow B$, and $h := q \circ f : X \rightarrow B$, where $q : A \rightarrow B$ is the second projection. By [4, 2.4] the general fibre $X_b := h^{-1}(b)$, $b \in B$ of g , is special⁽⁵⁾. Let $f_b : X_b \rightarrow A_b := B \times \{b\}$ be the restriction of f to X_b . The orbifold base (A_b, D_{f_b}) of f_b is then nothing, but $(A, D_f) \rightarrow A_b$, and is thus ample on $A_b = B$. By the first part of the proof, this contradicts the specialness of X_b if $D_{f_b} = 0$. Thus $D_f \rightarrow A_b = 0$, and so $D_f = 0$.

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⁽⁵⁾ The statement of [4] is only given for X projective, and f the Albanese map of X , but it is easy to check that the proof applies for X compact Kähler, and f any map with connected fibres to an Abelian variety. Said otherwise, [4, 2.6] applies if X is compact Kähler and if (Z, D) is the relative orbifold base of f . Indeed, since the generic orbifold fibres (Z, D) are supposed, by contradiction, to be of general type, we deduce that Z is Moishezon over A , hence Moishezon, since so is A . So Z becomes projective after a suitable modification of X , and [4, 2.6] applies.

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