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ALBANESE MAP OF SPECIAL MANIFOLDS: A CORRECTION

by Frédéric CAMPANA

Abstract. — We show that any fibration of a “special” compact Kähler manifold X onto an Abelian variety has no multiple fibre in codimension one. This statement strengthens and extends previous results of Kawamata and Viehweg when $\kappa(X) = 0$. This also corrects the proof given in the present journal in 2004, which was incomplete.

Résumé. — On montre qu’une fibration d’une variété compacte Kählérienne X sur une variété Abélienne n’a pas de fibre multiple en codimension 1. Ce résultat renforce et généralise des résultats précédents de Kawamata et Viehweg lorsque $\kappa(X) = 0$, et corrige la preuve incomplète donnée dans ce journal en 2004.

1. Introduction

The following statement is given in [3, Proposition 5.3]:

Theorem 1.1. — *Let $a_X : X \to A_X$ is the Albanese map of X, assumed to be special. Then a_X is onto, has connected fibres, and no multiple fibre in codimension one.*

This statement is used neither in [3], nor in [4].

Recall ([3, Definition 2.1 and Theorem 2.27]) that X is special if $\kappa(X, L) < p$ for any $p > 0$, and any rank-one coherent subsheaf of Ω^p_X. This implies that X has no surjective meromorphic $g : X \dashrightarrow Z$ map onto a manifold Z of general type and positive dimension p, and more generally exactly means that the “orbifold base” (Z, D_g) of such a g, constructed out of its multiple fibres, is never of “general type”. See Section 2 below for some more details.

While the proofs given there for the first two properties (which generalise earlier results by Y. Kawamata and Kawamata–Viehweg in the case

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when $\kappa(X) = 0$) are complete, the proof of the third property, even in the projective case, is not (as pointed out to me by K. Yamanoi and E. Rousseau[(1)]). The aim of this note is to correct it in the projective case (but only partially in the compact Kähler case) by using the main result of [4], itself based on [1] (or, alternatively, [5]).

THEOREM 1.2. — *Let $f : X \to A$ be a holomorphic map from the connected, compact, Kähler manifold to a compact complex torus A. If X is special, then:*

(1) *f is surjective,*
(2) *If f is the Albanese map of X, then f has connected fibres,*
(3) *f has no multiple fibre[(2)] in codimension one if the fibres of f are connected[(3)], and if A is an Abelian variety (equivalently: $D_f = 0$ if (A, D_f) is the "orbifold base" of f).*

Remarks 1.3.

(1) Although the assumption that A is an Abelian variety in (3) is certainly not necessary, our proof rests on [1] (or [5]) for A projective. Although the Kähler version is known, by [9], we cannot treat the Kähler case in general with the present arguments, because we use Poincaré reducibility.
(2) An easy case, not covered by Theorem 1.2 in which (3) can be proved, is when the algebraic dimension $a(A)$ of X vanishes, because then A does not contain any effective divisor. The argument of the second step of the proof applies, more generally, to show that the conclusion of Theorem 1.2(3) still holds true when $A = T \times V$ is a compact complex torus product of a subtorus T of algebraic dimension zero, with V, an Abelian variety, because every irreducible divisor on A is a product $T \times D$, where D is a divisor on V.
(3) (1) and (2), and their proofs, still apply when X is "weakly special", which means that no finite étale cover X' of X has a surjective meromorphic fibration onto a positive-dimensional projective manifold of

(1) The problem comes from the potential multiplicity-one exceptional divisors of a_X which are no longer f-exceptional if $g = q \circ f$, where $q : A \to B$ is a non-trivial torus quotient sending D_f to an ample divisor of B. We overcome this difficulty in the second step of the proof, by cutting the fibration by means of Poincaré reducibility. The first step is the same as in [3].

(2) We use here the "inf" multiplicities, not the more classical "gcd" version, which would give a weaker result.

(3) This can be realised, using the Stein factorisation of f. Indeed, since f factorises through the Albanese map a_X of X, which has connected fibres the Stein factorisation of f coincides with the one of a_X, which maps $Alb(X)$ to a torus, hence finite étale over A.

general type. Weak-specialness coincides with specialness for curves and surfaces ([3, Proposition 9.29]), but differs from dimension 3 on, by examples of Bogomolov–Tschinkel ([2], [3, Theorem 9.30]).

2. Multiple fibres of maps to complex tori

If $f : X \to Y$ is a fibration, that is: a holomorphic map with connected fibres onto a (smooth) compact complex manifold Y, we define its "orbifold base" D_f as follows: for every prime divisor $E \subset Y$, let $f^*(E) := \sum_{k\in K} t_k.D_k + R$, where the $D_k's$ are the pairwise distinct prime divisors of X surjectively mapped by f onto E, while R is an effective f-exceptional divisor of X. The multiplicity[(4)] $m_f(E)$ of the generic fibre of f over E is then defined as: $m_f(E) := \inf\{t_k, k \in K\}$, and then $D_f := \sum_{E\subset Y}(1 - \frac{1}{m_f(E)}).E$, which is an effective $\mathbb{Q}$-divisor of Y (note indeed that this sum is finite since $m_f(E) = 1$ for all, but finitely many of the $E's$).

Proof of Theorem 1.2.

(1). — Assume by contradiction that $f(X) := Z \neq A$. After Ueno's theorem ([8], p. 120), there is a quotient torus $q : A \to B$ such that $q(Z) \subset B$ is of dimension $p > 0$, and of general type. The composed map $g := q \circ f : X \to Z$ thus contradicts the specialness of X, since $\kappa(X, g^*(K_Z)) = p$ and $g^*(K_Z) \subset \Omega^p_X$.

(2). — Let Z be a smooth model of the Stein factorisation Z_0 of f. We replace X by Z, which is still special since dominated by X. We may thus assume that f is generically finite. By [7, Theorem 23], replacing Z by a suitable finite étale cover which is still special by [3, §5.5], we may assume that Z fibres over over a manifold of general type if Z is not birational to A. This again contradicts the specialness of Z and thus of X.

(3). — Assume that $D_f \neq 0$. We shall first treat the case when D_f is ample, and reduce to this case in a second step.

Let thus D_f be ample on A. By a flattening of f, followed by suitable blow-ups of X and A, we may assume (see [3, Lemma 1.3]) that $f = v \circ f'$, where $v : A' \to A$ is bimeromorphic with A' smooth, and $f' : X \to A'$ is a fibration such that its orbifold base $D' := D_{f'} = \overline{D} + E'$, with E' effective and v-exceptional, $\overline{D}$ is the strict transform of D_f in A'. This shows that

(4) We use here the "inf" multiplicity instead of the usual "gcd" multiplicity for reasons explained in [3].

in particular $f^*(K_{A'}+\overline{D}) \subset \Omega^p_X, p := \dim(A)$. The following lemma shows that the line bundle $K_{A'}+\overline{D}$ has Kodaira dimension $\dim(A) = p$, since D_f is ample on A. This contradicts the specialness of X if D_f is assumed to be ample.

Lemma 2.1 ([3, 1.14]). — *Let $v : Y' \to Y$ be a bimeromorphic map between compact connected complex manifolds. Let D be an effective $\mathbb{Q}$-divisor on Y, and let $\overline{D}$ be its strict transform on Y' . Then:*

(1) *$\kappa(Y, K_Y + D) \geqslant \kappa(Y', K_{Y'} + \overline{D})$.*
(2) *$K_{Y'} + \overline{D} = v^*(K_Y + \varepsilon.D) + E'_\varepsilon, \forall \varepsilon > 0$ small enough, E'_ε effective.*
(3) *If $\kappa(Y) \geqslant 0$, then $\kappa(Y', K_{Y'} + \overline{D}) = \kappa(Y, K_Y + D)$.*

Proof.

(1). — Let U be the largest Zariski open subset of Y over which v is isomorphic, and $U' := v^{-1}(U)$. We have an obvious equality, for each $m > 0$: $v^*(H^0(U, m.(K_Y + D))) = H^0(U', m.(K_{Y'} + \overline{D}))$. Since the complement of U in Y has complex codimension at least 2, Hartog's theorem shows that le left hand side of this last equality coincides with $v^*(H^0(Y, m.(K_Y + D)))$. On the other hand, the right hand side contains $H^0(Y', m.(K_{Y'} + \overline{D}))$, and so $\kappa(Y, K_Y + D) \geqslant \kappa(Y', K_{Y'} + \overline{D}))$

(2). — Let the rational numbers a_i, b_i be defined as follows: $v^*(D) = \overline{D} + \sum_i b_i.E_i$, $K_{Y'} = v^*(K_Y) + \sum_i a_i.E_i$, where the E'_is are the exceptional divisors of v. Thus: $a_i > 0, b_i \geqslant 0, \forall i$. Then:

$$\begin{aligned} K_{Y'} + \overline{D} &= v^*(K_Y) + \sum a_i.E_i + (1-\varepsilon).\overline{D} + v^*(\varepsilon.D) - \Big(\varepsilon.\sum b_i.E_i\Big) \\ &= v^*(K_Y + \varepsilon.D) + (1-\varepsilon).\overline{D} + \sum (a_i - \varepsilon.b_i).E_i. \end{aligned}$$

We thus get the claim when $0 < \varepsilon \leqslant \min\{1, \frac{a_i}{b_i}, \forall i\}$, with $E'_\varepsilon := (1-\varepsilon).\overline{D} + \sum (a_i - \varepsilon.b_i).E_i$.

(3). — Let $\varepsilon > 0$ be sufficiently small, as above. Since K_Y is assumed to be $\mathbb{Q}$-effective, so is $(1-\varepsilon).v^*(K_Y)$, and so $K_{Y'} + \overline{D} = \varepsilon.v^*(K_Y + D) + E''_\varepsilon$, with: $E''_\varepsilon := E'_\varepsilon + (1-\varepsilon).v^*(K_Y)$, which is $\mathbb{Q}$-effective. Thus $\kappa(Y', K_{Y'} + \overline{D}) \geqslant \kappa(Y, K_Y + D)$. □

Remark 2.2. — The main property used in Lemma 2.1 is that $a_i > 0, \forall i$, i.e: the lift of the canonical sheaf under a modification of a smooth manifold vanishes on the exceptional divisor. This property does not hold true for sheaves of forms of degree less than the dimension. We thus need to use a less direct route, which requires deeper ingredients.

We shall now reduce to the case when D_f is ample.

Let $L := \mathcal{O}_A(m.D_f)$ be the line bundle on A with a section vanishing on some integral multiple $m.D_f$ of the orbifold base of f. Let $T \subset \mathrm{Aut}^0(A)$ be the connected component of the group of translations of A preserving L. Let $q : A \to B := A/T$ be the quotient map. Then $D_f = \frac{1}{m}.q^*(D)$ for some *ample* effective divisor D on B (see, for example [6, Théorème 5.1]). Since $D_f \neq 0$, $p := \dim(B) > 0$.

We use Poincaré reducibility to reduce to the case that $A = B \times B'$, where $B' \to T$ is a suitable finite étale cover of T. This indeed amounts to replace X and D_f by the corresponding finite étale covers, preserving all of our hypothesis, in particular the specialness of X (by [3, §5.5], again).

We now consider the composed fibrations $g := q \circ f : X \to B$, and $h := q' \circ f : X \to B'$, where $q' : A \to B'$ is the second projection. By [4, 2.4] the general fibre $X_{b'} := h^{-1}(b'), b' \in B'$ of g', is special[(5)]. Let $f_{b'} : X_{b'} \to A_{b'} := B \times \{b'\}$ be the restriction of f to $X_{b'}$. The orbifold base $(A_{b'}, D_{f_{b'}})$ of $f_{b'}$ is then nothing, but $(A, D_f) \cap A_{b'}$, and is thus ample on $A_{b'} \cong B$. By the first part of the proof, this contradicts the specialness of $X_{b'}$ if $D_{f_{b'}} \neq 0$. Thus $D_f \cap A_{b'} = 0$, and so $D_f = 0$. □

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[(5)] The statement of [4] is only given for X projective, and f the Albanese map of X, but it is easy to check that the proof applies for X compact Kähler, and f any map with connected fibres to an Abelian variety. Said otherwise, [4, 2.6] applies if X is compact Kähler and if (Z, D) is the relative orbifold base of f. Indeed, since the generic orbifold fibres (Z, D) are supposed, by contradiction, to be of general type, we deduce that Z is Moishezon over A, hence Moishezon, since so is A. So Z becomes projective after a suitable modification of X, and [4, 2.6] applies.

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