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CONGRUENCE RFRS TOWERS

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With an appendix by Mehmet Haluk ENGÜN

Abstract. — We describe a criterion for a real or complex hyperbolic lattice to admit a residually finite rational solvable (RFRS) tower that consists entirely of congruence subgroups. We use this to show that certain Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_d)$ are virtually fibered on congruence subgroups, and also exhibit the first examples of RFRS Kähler groups that are not a subgroup of a product of surface groups and abelian groups.

Résumé. — Nous donnons un critère pour qu'un réseau réel ou complexe hyperbolique admette une tour résiduellement finie rationnelle soluble (RFRS) qui se compose entièrement de sous-groupes de congruence. Nous l'utilisons pour montrer que certains groupes de Bianchi $\mathrm{PSL}_2(\mathcal{O}_d)$ sont virtuellement fibrés sur des sous-groupes de congruence, et donnons aussi les premiers exemples de groupes de Kähler RFRS qui ne sont pas des sous-groupes d'un produit de groupes de surface et de groupes abéliens.

1. Introduction

Let Γ be a finitely generated group. The first author introduced the notion of Γ being *virtually RFRS* to prove that certain hyperbolic 3-manifolds are virtually fibered [1], and eventually this was used to prove that all finite-volume hyperbolic 3-manifolds virtually fiber [2, 16, 29]. Finding such a cover effectively remains an open problem.

In this paper, we study finding RFRS towers arising from congruence covers of arithmetic manifolds. For example, we will prove:

Keywords: RFRS towers, Bianchi groups, congruence subgroups, real and complex hyperbolic lattices, virtual fibering, Kähler groups.

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Theorem 1.1. — *The Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_d)$ with $d \equiv -1 \pmod{8}$ and d square-free contain a RFRS tower consisting entirely of congruence subgroups. In particular, these Bianchi orbifolds virtually fiber on a congruence cover.*

We achieve this using the fact that these Bianchi groups virtually embed in the group $\mathrm{O}(4, 1; \mathbb{Z})$. We then apply a very general idea to the congruence subgroup of level 4 in $\mathrm{O}(4, 1; \mathbb{Z})$ to show that it is virtually RFRS with tower $\{ \Gamma_j \}$ for which each Γ_j contains the congruence subgroup of level 2^{n_j} for some n_j . This example also allows us to find infinitely many commensurability classes of cocompact arithmetic Kleinian groups that virtually fiber on a congruence cover; see [3, Lem. 4.6] for examples. We will also show:

Theorem 1.2. — *There is a torsion-free cocompact lattice in $\mathrm{PU}(2, 1)$ that is RFRS. Therefore, there is a RFRS Kähler group that is not isomorphic to a subgroup of the direct product of surface groups and abelian groups.*

This addresses a question raised by recent work of Friedl and Vidussi; see the discussion immediately following [15, Thm. E]. Our example is a congruence subgroup of a particular Deligne–Mostow lattice [12]. Note that nonuniform lattices in $\mathrm{PU}(n, 1)$ cannot be RFRS, since their cusp groups are two-step nilpotent groups, which themselves are not RFRS. In particular, the methods of this paper cannot apply to nonuniform complex hyperbolic lattices. See Remark 3.17.

We briefly describe the method of constructing these towers. Suppose that k is a number field and G is a k -algebraic group such that $G(k) \backslash \mathrm{O}(\mathbb{R})$ modulo compact factors is isomorphic to $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$. Let \mathcal{O}_k be the ring of integers of k and \mathfrak{p} a prime ideal of \mathcal{O}_k with residue characteristic p . Suppose that $\Gamma(\mathfrak{p})$ is the congruence subgroup of level \mathfrak{p} in the arithmetic lattice $G(\mathcal{O}_k)$, and that $\Gamma < \Gamma(\mathfrak{p})$ is a finite index subgroup such that $H^1(\Gamma; \mathbb{Z})$ has no p -torsion.

Using the fact that $G(k)$ is closely related to the commensurator of Γ , we find a sequence $\{g_n\}$ in $G(k)$ such that

$$\Gamma_n = g_n \Gamma g_n^{-1}$$

is a RFRS tower for Γ . The key is to find an initial $g_1 \in G(k)$ so that

$$\Gamma < \Gamma_1 = g_1 \Gamma g_1^{-1}$$

is an elementary abelian p -group. One then inductively defines each g_n in a manner most succinctly described using the p -adic Bruhat–Tits building for $G(k_{\mathfrak{p}})$, where $k_{\mathfrak{p}}$ is the local field associated with \mathfrak{p} . Note that the above implicitly assumes that $H_1(\ ; \mathbb{Z})$ is infinite, hence our results can only apply for lattices in $SO(n, 1)$ and $SU(n, 1)$.

We close by briefly recalling the connection between RFRS and various notions of fibering. When $\{ \Gamma_j \}$ is a RFRS tower with $\Gamma_j = \pi_1(M_j)$ the fundamental group of an irreducible 3-manifold, the first author proved that there is some j_0 so that M_j fibers over S^1 for all $j > j_0$ [1, Thm. 5.1] (the statement of the theorem does not explicitly say that the fibered manifold comes from the RFRS tower, but it is implicit in the proof). It is a famous theorem of Stallings that this is equivalent to Γ_j admitting a homomorphism onto \mathbb{Z} with finitely generated kernel for all $j > j_0$. Such a homomorphism is called an *algebraic fibration*, and recent work shows that being virtually RFRS is closely related to being virtually algebraically fibered.

For example, Friedl and Vidussi [15, Thm. E] showed that virtually RFRS Kähler groups are either virtually algebraically fibered or virtually surface groups, and Kielak proved that an infinite finitely generated virtually RFRS group is virtually algebraically fibered if and only if its first $l^{(2)}$ betti number is zero [18, Thm. 5.3]. Both proofs provide a group in the RFRS tower that algebraically fibers. See [26] and [17] for more about algebraic fibrations of lattices in Lie groups and Coxeter groups. In particular, in [26] the second author showed that all arithmetic subgroups of $SU(n, 1)$ of “simplest type” virtually algebraically fiber on a congruence subgroup; this includes the example used to prove Theorem 1.2 above. On the other hand, there are known examples of arithmetic lattices in $SU(n, 1)$ so that ab is finite for all congruence subgroups Γ [23, Thm. 15.3.1], hence Γ cannot virtually algebraically fiber on a congruence subgroup. It is unknown whether all lattices in $SU(n, 1)$ (arithmetic or not) virtually algebraically fiber. We rephrase our results in this language in Corollaries 3.6 and 3.16, and see Section 4 for further discussion.

In the Appendix, Mehmet Haluk Engün shows that the methods developed in this paper can be used to construct congruence RFRS towers for many prime-level congruence subgroups of Bianchi groups. Specifically, for $d = -1, -2, -3, -7, -11$ and \mathfrak{p} a prime ideal of residue characteristic p with relatively small norm, Engün shows that the abelianization of the level \mathfrak{p} congruence subgroup of $PSL(O_d)$ is p -torsion-free roughly twice as often as not. As we show in Section 3.1, having no p -torsion in the abelianization

of the level \mathfrak{p} congruence subgroup allows one to construct a RFRS tower in the tower of \mathfrak{p} -congruence subgroups of $\mathrm{PSL}(O_d)$.

This paper is organized as follows. In Section 2 we describe some basic preliminary results on RFRS towers and congruence towers. In Section 3 we give three examples that describe our general method for producing congruence towers that are RFRS. These examples suffice to prove the theorems stated above. Finally, in Section 4 we make closing comments and raise some questions.

Acknowledgments

We thank Alan Reid and Steven Tschantz for helpful conversations and the referee for pointing us to [30].

2. Preliminaries on towers

In this section, we discuss two types of towers of finite index subgroups of a group: RFRS towers, and \mathfrak{p} -congruence towers.

2.1. RFRS towers

Let Γ be a finitely generated group with commutator subgroup denoted by $\Gamma^{(1)} = [\Gamma, \Gamma]$ and abelianization

$$\Gamma^{ab} = \Gamma / \Gamma^{(1)} = H_1(\Gamma; \mathbb{Z}).$$

We then define the *rational abelianization* Γ^{rab} to be the image of Γ^{ab} in

$$\Gamma^{ab} \otimes_{\mathbb{Z}} \mathbb{Q} = H_1(\Gamma; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = H_1(\Gamma; \mathbb{Q})$$

under the natural homomorphism and the *rational commutator subgroup*

$$\Gamma_r^{(1)} = \ker \Gamma \rightarrow \Gamma^{rab}.$$

Clearly $\Gamma_r^{(1)} \triangleleft \Gamma^{(1)}$ is finite index and $\Gamma^{rab} = H_1(\Gamma; \mathbb{Z}) / \text{Torsion}$.

Given a group Γ , let $\{\Gamma_j\}$ be a cofinal tower of finite index subgroups of Γ with $\Gamma_0 = \Gamma$. In other words,

- (1) $\Gamma_j = \{1\}$;
- (2) Γ_j is a finite index subgroup of Γ ;
- (3) $\Gamma_{j+1} \triangleleft \Gamma_j$ for all j .

We say that $\{ \rho_j \}$ is a *RFRS tower* if, in addition,

$$(\rho_j)_r^{(1)} \trianglelefteq \rho_{j+1} \text{ for all } j > 0.$$

Remark 2.1. — The original definition of RFRS [1, Def. 2.1] also required that ρ_j be normal in ρ . However, it is also pointed out in [1] that if there is a RFRS tower, then there is also a normal RFRS tower by passing to core subgroups (i.e., the largest normal refinement).

We say that ρ is *RFRS* if it admits such a tower and that it is *virtually RFRS* if it contains a finite index subgroup that is RFRS. We note that RFRS is short for “residually finite \mathbb{Q} -solvable”, and refer to [1] for further details and examples.

We briefly recall that if G is a group and $\rho \trianglelefteq G$ a subgroup, the *commensurator* of ρ in G is the group consisting of those $g \in G$ such that $(g \rho g^{-1})$ has finite index in both ρ and $g \rho g^{-1}$. Our key technical lemma is the following:

Lemma 2.2. — *Let G be a group and $\rho \trianglelefteq G$ a finitely generated subgroup such that ρ^{ab} has no p -torsion. Suppose that $\{g_0 = \text{Id}, g_1, g_2, \dots\}$ is a sequence in G such that each g_i is in the commensurator of ρ in G . Define*

$$\rho_i = g_i \rho_i g_i^{-1}$$

$$\rho = \bigcap_{i=0}^{\infty} \rho_i.$$

Finally, suppose:

- (1) The sequence $\{ \rho_n \}$ is a cofinal tower of subgroups.
- (2) For each n , there exists some $0 \leq i \leq n-1$ such that $\rho_i / (\rho_i \cap \rho_n)$ is an abelian p -group.

Then $\{ \rho_n \}$ is a RFRS sequence for ρ .

Proof. — Note that $\rho_0 = \rho$ and

$$\rho_0 / (\rho_0 \cap \rho_1) = \rho_0 / \rho_1$$

is an abelian p -group. Since $\rho_0^{ab} = \rho^{ab}$ has no p -torsion, the projection from ρ_0 onto ρ_0 / ρ_1 must factor through ρ_0^{rab} , i.e., $(\rho_0)_r^{(1)} \trianglelefteq \rho_1$.

We now show that $(\rho_{n+1})_r^{(1)} \trianglelefteq \rho_{n+2}$ for all $n > 0$. Since ρ_{n+2} equals $\rho_{n+1} \cap \rho_{n+2}$, to prove that $\{ \rho_n \}$ is a RFRS sequence, we must show that $(\rho_{n+1})_r^{(1)} \trianglelefteq \rho_{n+2}$.

Fix $0 \leq i \leq n+1$ such that $\rho_i / (\rho_i \cap \rho_{n+2})$ is an abelian p -group. Since $\rho_i = \rho$, we see that ρ_i^{ab} has no p -torsion, and hence $(\rho_i)_r^{(1)} \trianglelefteq (\rho_i \cap \rho_{n+2})$.

Then $\Gamma_{n+1}^{ab} \trianglelefteq \Gamma_i^{ab}$ by construction, and the natural map $\Gamma_{n+1}^{ab} \rightarrow \Gamma_i^{ab}$ induced by the inclusion must factor through the map from Γ_{n+1}^{ab} to Γ_{n+1}^{ab} . Indeed, Γ_i^{ab} is torsion-free, so Γ_{n+2}^{ab} factors through Γ_{n+1}^{ab} . It follows that

$$\Gamma_{n+1}^{(1)} \trianglelefteq \Gamma_i^{(1)} \trianglelefteq \Gamma_{n+2}.$$

This gives that $\Gamma_{n+1}^{(1)} \trianglelefteq \Gamma_{n+2}$, as desired. Since $\{\Gamma_n\}$ satisfies the other hypotheses to be a RFRS sequence by assumption, this completes the proof of the lemma.

Our goal will be to apply Lemma 2.2 to certain \mathfrak{p} -congruence towers in arithmetic lattices. We now introduce these towers.

2.2. \mathfrak{p} -congruence towers

We refer the reader to [20, Ch. I and II] for terminology and results from algebraic number theory used in this section and elsewhere in the paper. Let k be a number field with integer ring O_k , $G \leq \mathrm{GL}_n(k)$ be a k -algebraic matrix group, and $\Gamma = G(O_k)$. Given a prime ideal \mathfrak{p} of O_k and $j > 1$, let $\Gamma(\mathfrak{p}^j)$ be the *level \mathfrak{p}^j congruence subgroup* of Γ , i.e., all those elements that are congruent to the identity modulo \mathfrak{p}^j . The collection $\{\Gamma(\mathfrak{p}^j)\}$ is the *\mathfrak{p} -congruence tower* for Γ . This is a cofinal tower of normal subgroups of Γ .

We record some elementary facts. Let p be a rational prime. Recall that in a *p -group* every element has order a power of p , and in an *elementary p -group* every element has order p .

Lemma 2.3. — *Suppose k is a number field with ring of integers O_k , $G \leq \mathrm{GL}_n(k)$ is a k -algebraic matrix group, and $\Gamma = G(O_k)$. Let \mathfrak{p} be a prime ideal of O_k and p the characteristic of the finite field O_k/\mathfrak{p} . Then:*

- (1) *For all $j > 1$, $\Gamma(\mathfrak{p}^j)/\Gamma(\mathfrak{p}^{j+1})$ is an elementary abelian p -group.*
- (2) *For all $k > j > 1$, $\Gamma(\mathfrak{p}^j)/\Gamma(\mathfrak{p}^k)$ is a p -group.*
- (3) *For all $j > 2$, $\Gamma(\mathfrak{p}^j)/\Gamma(\mathfrak{p}^k)$ is abelian for every $k \trianglelefteq 2j$. In particular, $\Gamma(\mathfrak{p}^j)/\Gamma(\mathfrak{p}^{j+2})$ is abelian.*

Proof. — Let $O_{\mathfrak{p}}$ be the integral closure of O_k in the completion $k_{\mathfrak{p}}$ of k with respect to its \mathfrak{p} -adic norm. Fix a uniformizing element π for $O_{\mathfrak{p}}$. Then we have that $O_k/\mathfrak{p} = O_{\mathfrak{p}}/(\pi)O_{\mathfrak{p}}$ and p is the characteristic of this finite field.

If $\gamma \in \Gamma(\mathfrak{p}^j)$, then we can write

$$\gamma = \mathrm{Id} + \pi^j M$$

for some $M \in M_n(\mathcal{O}_{\mathfrak{p}})$. Then

$$\rho = \sum_{k=0}^p \binom{p}{k} M^k,$$

which is visibly congruent to the identity modulo \mathfrak{p}^{j+1} . This proves that every element of $(\mathfrak{p}^j) \setminus (\mathfrak{p}^{j+1})$ has order p .

Now, suppose that

$$\begin{aligned} &= \text{Id} + {}^j M \\ &= \text{Id} + {}^j N \end{aligned}$$

for $\gamma \in (\mathfrak{p}^j)$. Then:

$$\begin{aligned} &= (\text{Id} + {}^j M)(\text{Id} + {}^j N) \\ &= \text{Id} + {}^j(M + N) + {}^{2j} MN \\ &= (\text{Id} + {}^j N)(\text{Id} + {}^j M) \\ &= \text{Id} + {}^j(N + M) + {}^{2j} NM \end{aligned}$$

We see that γ and γ^{-1} commute modulo \mathfrak{p}^k for all $k \leq 2j$. Since $2j > j + 1$ for $j > 1$, w , this proves the first and third assertions of the lemma. The second statement is an immediate consequence of the first.

Remark 2.4. — Replacing \mathcal{O}_k with $\mathcal{O}_{\mathfrak{p}}$ in the proof of Lemma 2.3 is only necessary when \mathfrak{p} is not a principal ideal. When it is principal, one can implement the proof in \mathcal{O}_k instead with γ a generator for \mathfrak{p} .

3. Examples

We now describe the examples that suffice to prove the main results stated in the introduction. Our techniques work in much greater generality, and the reader will hopefully find these examples illustrative enough to apply our methods in other settings.

3.1. The magic manifold

It goes back to Thurston that the fundamental group of the magic manifold arises from the congruence subgroup $\Gamma_2(2)$ inside $\text{PGL}_2(\mathbb{Q}(\sqrt{-7}))$. It is homeomorphic to the complement in S^3 of the 3-chain link 6_3^3 (see Figure 3.1 and [27, Ex. 6.8.2]).

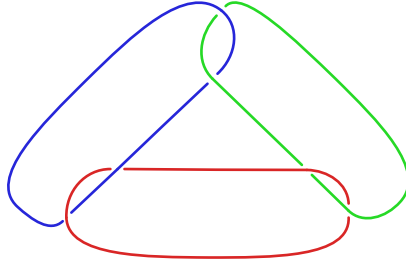


Figure 3.1. The magic manifold is the complement of the 3-chain link.

Note that $\mathfrak{p} = \left(\frac{1+\sqrt{-7}}{2}\right)$ is a prime ideal dividing 2. We will show that the magic manifold admits a 2-congruence tower that is RFRS. Note that the magic manifold is itself fibered, so the fact that $\mathrm{PSL}_2(\mathcal{O}_d)$ fibers on a congruence subgroup is not new in this case.

Since \mathfrak{p} has norm 2, the completion of $\mathbb{Q}(\sqrt{-7})$ at \mathfrak{p} is \mathbb{Q}_2 , and hence we obtain an embedding of $\mathbb{Q}(\sqrt{-7})$ into $\mathrm{PGL}_2(\mathbb{Q}_2)$. Consider the action of $\mathrm{PGL}_2(\mathbb{Q}_2)$ on its Bruhat–Tits tree T , which is a 3-regular tree (see Figure 3.2).

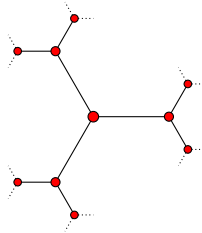


Figure 3.2. The Bruhat–Tits tree T for $\mathrm{PGL}_2(\mathbb{Q}_2)$.

We briefly recall that vertices of T are homothety classes of \mathbb{Z}_2 -lattices in \mathbb{Q}_2^2 , and two vertices $[L_1]$ and $[L_2]$ are adjacent if and only if there are representatives in the homothety classes such that $L_2 \subset L_1$ with L_1/L_2 isomorphic to the finite field \mathbb{F}_2 with two elements. See [25, §II.1] for details.

Then $\mathfrak{S} \subset \mathrm{PGL}_2(\mathbb{Z}_2)$ naturally stabilizes the vertex v_0 associated with the standard lattice \mathbb{Z}_2^2 . Notice that $\mathrm{PGL}_2(\mathbb{Q}(\sqrt{-7}))$ acts transitively on T , and the element

$$g_1 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

exchanges v_0 with a neighbor v_1 . One then checks that

$$(\mathfrak{p}^2) \subset \langle g_1, g_1^{-1} \rangle,$$

and it follows from Lemma 2.3 that $\langle g_1, g_1^{-1} \rangle$ is an elementary abelian 2-group.

We now define $g_n \in \text{PGL}_2(\mathbb{Q}(\sqrt{-7}))$ and $v_n = g_n(v_0) \in T$ by choosing some v_i for $0 \leq i \leq n-1$ for which not all neighbors of v_i are contained in $\{v_0, \dots, v_{n-1}\}$, letting v_n be one such neighbor of v_i , and taking g_n to be the conjugate of g_1 in $\text{PGL}_2(\mathbb{Q}(\sqrt{-7}))$ that swaps v_i and v_n . Define $\Gamma_n = \langle g_n, g_n^{-1} \rangle$. Then

$$\Gamma_n \langle \Gamma_n, \Gamma_i \rangle$$

is an elementary abelian 2-group by the same reasoning that we applied to $\langle \Gamma_1, \Gamma_0 \rangle$.

Let v_n range over all vertices of T . Defining

$$\Gamma_n = \langle \Gamma_i \rangle_{i=0}^n,$$

we have that Γ_n lies in the stabilizer in $\text{PGL}_2(\mathbb{Q}_2)$ of every homothety class of lattices in \mathbb{Q}_2^2 , which is clearly trivial. Therefore $\{\Gamma_n\}$ is cofinal. In particular, Lemma 2.2 applies to show that this tower is RFRS.

Remark 3.1. — This idea applies to any principal congruence arithmetic link. In [5] it is shown that there are principal congruence links for discriminant $d = 1, 2, 3, 5, 7, 11, 15, 19, 23, 31, 47, 71$. This includes discriminants $d = 7, 15, 23, 31, 47, 71$ that are congruent to $-1 \pmod 8$; these values of d are not handled by the next section. More generally, this construction works for congruence subgroups of arithmetic Kleinian groups with no p -torsion in their 1st homology for the appropriate p ; see Appendix A by Engün for further examples.

3.2. Bianchi groups and $\text{O}(4, 1; \mathbb{Z})$

Consider the quadratic form q_0 in 5 variables with matrix

$$Q_0 = \text{diag}(1, 1, 1, 1, -1),$$

and let $\text{O}(4, 1; \mathbb{Z})$ be the group of integral automorphisms of q_0 . Then $\text{O}(4, 1; \mathbb{Z})$ determines a nonuniform arithmetic lattice in $\text{O}(4, 1)$. For an integer $N > 1$, let $\Gamma(N)$ denote the congruence subgroup of $\text{O}(4, 1; \mathbb{Z})$ of level N .

It is known that $O(4, 1; \mathbb{Z})$ is the group generated by reflections in the simplex in hyperbolic 4-space with Coxeter diagram given in Figure 3.3. Moreover, the congruence subgroup $\Gamma(2)$ of level two is the right-angled Coxeter group generated by reflections in the sides of a polyhedron obtained from 120 copies of the simplex for $O(4, 1; \mathbb{Z})$. See [22].

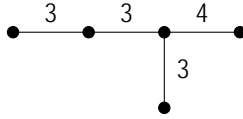


Figure 3.3. The Coxeter diagram for $O(4, 1; \mathbb{Z})$.

It will be convenient to change coordinates. The matrix

$$= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 \end{pmatrix} \in \text{SL}_5(\mathbb{Z})$$

conjugates $O(4, 1; \mathbb{Z})$ to $O(q; \mathbb{Z})$, where q is the quadratic form with matrix

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

i.e., $Q = {}^t Q_0$. Since Q is integral of determinant one, it preserves all congruence subgroups of $\text{SL}_5(\mathbb{Z})$. Thus $\Gamma(N)^{-1}$ is the level N congruence subgroup of $O(q; \mathbb{Z})$ for all $N > 1$, and we continue calling it just $\Gamma(N)$.

We will also need the Bruhat–Tits building associated with $O(q; \mathbb{Q}_2)$, which is a $(5, 3)$ -regular tree T . See [28, §2.7] and Figure 3.4. Considering $O(q; \mathbb{Q}_2)$ as a subgroup of $\text{GL}_5(\mathbb{Q}_2)$, we obtain an injection of buildings $T \hookrightarrow X$, where X is the building associated with $\text{PGL}_5(\mathbb{Q}_2)$. We briefly describe T using this embedding. While we do not need details of the finer structure of T , for the reader’s convenience in what follows we give a complete argument that T is a $(3, 5)$ -regular tree.

As in the 2-dimensional case, vertices of X are in one-to-one correspondence with homothety classes of \mathbb{Z}_2 -lattices in \mathbb{Q}_2^5 , where vertices x and y are adjacent if there are representatives L_x and L_y for the two homothety

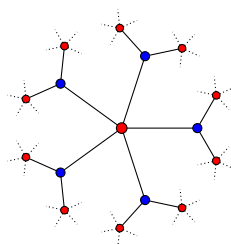


Figure 3.4. The Bruhat-Tits tree T for $O(q; \mathbb{Q}_2)$.

classes so that $2L_x = L_y = L_x$. See [25, Exer. II.1.4]. We fix the base vertex

$$x_0 = [e_1, \dots, e_5],$$

where $\{e_i\}$ is the basis for which q has the given matrix and $-$ denotes the \mathbb{Z}_2 -span. The stabilizer of x_0 in $GL_5(\mathbb{Q}_2)$ is generated by $GL_5(\mathbb{Z}_2)$ and the scalar matrices. Then $O(q; \mathbb{Z}_2)$ stabilizes x_0 , which implies that we can realize T as the convex hull of the $O(q; \mathbb{Q}_2)$ -orbit in X of x_0 .

The apartment A of X associated with the \mathbb{Q}_2 -split torus of diagonal matrices in $GL_5(\mathbb{Q}_2)$ (e.g., see [28, §1]) can be identified with the set of homothety classes

$$\{ [2^{r_1} e_1, \dots, 2^{r_5} e_5] \}_{r_i \in \mathbb{Z}},$$

i.e., the orbit of x_0 under the diagonal subgroup. The \mathbb{Q}_2 -split torus of diagonal matrices in the rank one group $O(q; \mathbb{Q}_2)$ is

$$S = \begin{pmatrix} 0 & 0 \\ 0 & \text{Id} & 0 \\ 0 & 0 & -1 \end{pmatrix} : \mathbb{Q}_2, \quad ,$$

where Id is the 3×3 identity matrix, and the convex hull of the S -orbit of x_0 is then the apartment A_0 of T associated with S . The S -orbit of x_0 is

$$x_r = [2^r e_1, e_2, \dots, e_4, 2^{-r} e_5]$$

for $r \in \mathbb{Z}$, and its convex hull also includes the vertices

$$x_{r+\frac{1}{2}} = [2^{r+1} e_1, e_2, \dots, e_4, 2^{-r} e_5].$$

We see that A_0 is a line with vertex set $\{x : [2^x e_1, e_2, \dots, e_4, 2^{-x} e_5]\}$, where x is adjacent to x' if and only if $|x - x'| = \frac{1}{2}$.

Since $O(q; \mathbb{Q}_2)$ acts transitively on apartments of T [28, §2], the vertex set of T is the $O(q; \mathbb{Q}_2)$ -orbit of $\{x_0, x_{\frac{1}{2}}\}$. In particular, to prove that T is a $(5, 3)$ -regular tree we need to prove the following two lemmas.

Lemma 3.2. — *The vertex x_0 of T has valence 5.*

Lemma 3.3. — *The vertex $x_{\frac{1}{2}}$ in T has valence 3.*

Proof of Lemma 3.2. — We must compute the $O(q; \mathbb{Z}_2)$ -orbit of $x_{\frac{1}{2}}$. We define $L_0 = \langle e_1, \dots, e_5 \rangle$ and $L_{\frac{1}{2}} = \langle 2e_1, e_2, \dots, e_5 \rangle$. Neighbors of x_0 in X are in one-to-one correspondence with proper nonzero subspaces of

$$V_0 = L_0/2L_0 = \mathbb{F}_2^5.$$

Let $\{\bar{e}_i\}$ be the basis for V_0 induced by $\{e_i\}$.

If \bar{q}_0 denotes the quadratic form on V_0 induced by the restriction of q to L_0 , then we see that the image of $L_{\frac{1}{2}}$ in V_0 is the \bar{q}_0 -orthogonal complement \bar{e}_5 of \bar{e}_5 , which we note is a codimension one subspace that contains the isotropic vector \bar{e}_5 . To prove the lemma it then suffices to compute the orbit of \bar{e}_5 under the image G_0 of $O(q; \mathbb{Z}_2)$ under reduction modulo 2 (e.g., see [28, §3.5.4]).

Since $v = \bar{e}_5$ if and only if $v = \bar{e}_5$, it moreover suffices to compute the G_0 -orbit of \bar{e}_5 . One checks that this orbit is

$$\{\bar{e}_5, \bar{e}_1, \bar{e}_1 + \bar{e}_2 + \bar{e}_3 + \bar{e}_5, \bar{e}_1 + \bar{e}_2 + \bar{e}_4 + \bar{e}_5, \bar{e}_1 + \bar{e}_3 + \bar{e}_4 + \bar{e}_5\}.$$

This proves the lemma.

Proof of Lemma 3.3. — The proof is very similar to the proof of Lemma 3.2, so we sketch the argument and leave it to the reader to verify the details. With notation as in that proof, we consider

$$2L_{\frac{1}{2}} \cap 2L_0 \cap L_{\frac{1}{2}} = \langle \bar{f}_1, \dots, \bar{f}_5 \rangle.$$

Then $V_{\frac{1}{2}} = L_{\frac{1}{2}}/2L_{\frac{1}{2}}$ is a vector space with basis $\{\bar{f}_i\}$ with respect to which the quadratic form $\bar{q}_{\frac{1}{2}}$ contains a two-dimensional totally degenerate subspace spanned by \bar{f}_1 and \bar{f}_5 .

Since the image of $2L_0$ in $V_{\frac{1}{2}}$ is the line spanned by \bar{f}_1 , we must compute its orbit under the reduction modulo 2 of the stabilizer in $O(q; \mathbb{Q}_2)$ of $L_{\frac{1}{2}}$. One shows that this orbit consists of the lines spanned by \bar{f}_1 , \bar{f}_5 , and $\bar{f}_1 + \bar{f}_5$, and the lemma follows.

Remark 3.4. — The neighbors of x_0 in A_0 are $x_{\frac{1}{2}}$ and $x_{-\frac{1}{2}}$. The other neighbors have representatives:

$$\begin{aligned} &2e_1, e_1 + e_2, e_1 + e_3, e_4, e_1 + e_2 + e_3 + e_5 \\ &2e_1, e_1 + e_2, e_3, e_1 + e_4, e_1 + e_2 + e_4 + e_5 \\ &2e_1, e_2, e_1 + e_3, e_1 + e_4, e_1 + e_3 + e_4 + e_5 \end{aligned}$$

Similarly, $x_{\frac{1}{2}}$ has neighbors x_0 and x_1 along with the vertex with representative

$$e_1 + \frac{1}{2}e_5, e_2, e_3, e_4, e_5,$$

which is the image of L_0 under the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{2} & -1 & 0 & 0 & 1 \end{pmatrix} \quad SO(q; \mathbb{Q}_2)$$

that stabilizes $L_{\frac{1}{2}}$.

We now prove the main result of this section. Recall that the Bianchi groups $PSL_2(\mathcal{O}_d)$ with $d \equiv -1 \pmod{8}$ are all commensurable with subgroups of $O(4, 1; \mathbb{Z})$, and one can choose the subgroup of $PSL_2(\mathcal{O}_d)$ contained in $O(4, 1; \mathbb{Z})$ to be a congruence subgroup of each. We can then use the following to prove Theorem 1.1.

Proposition 3.5. — *The congruence subgroup (4) of level 4 in $O(4, 1; \mathbb{Z})$ admits a congruence RFRS tower.*

Proof. — Recall that the congruence subgroup (2) of level 2 in $O(n, 1; \mathbb{Z})$ is a right-angled Coxeter group [22, Thm. 7]. One computes that (2)/(4) and (2)/(2)⁽¹⁾ are elementary abelian 2-groups of the same order, hence (4) = (2)⁽¹⁾. We then see that (4) has torsion-free abelianization by [10, §4.5], as the abelianization of the commutator subgroup of any right-angled Coxeter group is isomorphic to the reduced degree zero homology of a certain complex, hence it is necessarily torsion-free. Alternately, using the presentation for $O(4, 1; \mathbb{Z})$ as a Coxeter group with diagram as in Figure 3.3, one can easily check using a computer algebra program like Magma [9] that in fact (4)^{ab} = \mathbb{Z}^{55} . In particular, (4)^{ab} has no 2-torsion.

Consider the matrix

$$g_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix} \quad SO(q; \mathbb{Q}) < SO(q; \mathbb{Q}_2)$$

that exchanges the vertices $x_0, x_1 \in T$ and fixes the intermediate vertex $x_{\frac{1}{2}}$.

We now set $\rho_0 = (4)$ and $\rho_1 = g_1 \rho_0 g_1^{-1}$. We claim that

$$(16) \subset \rho_0 \subset \rho_1 \subset \rho_0.$$

To see this, one first notices that

$$\begin{array}{cccccc}
 & 1 + 16a_5 & -32e_2 & -32e_3 & -32e_4 & 64e_1 \\
 & -8b_5 & 1 + 16b_2 & 16b_3 & 16b_4 & -32b_1 \\
 g_1 & -8c_5 & 16c_2 & 1 + 16c_3 & 16c_4 & -32c_1 \\
 & -8d_5 & 16d_2 & 16d_3 & 1 + 16d_4 & -32d_1 \\
 & 4a_1 & -8a_2 & -8a_3 & -8a_4 & 1 + 16a_1 \\
 & & 1 + 16a_1 & 16a_2 & 16a_3 & 16a_4 & 16a_5 \\
 & & 16b_1 & 1 + 16b_2 & 16b_3 & 16b_4 & 16b_5 \\
 = & & 16c_1 & 16c_2 & 1 + 16c_3 & 16c_4 & 16c_5 \\
 & & 16d_5 & 16d_2 & 16d_3 & 1 + 16d_4 & 16d_5 \\
 & & 16e_1 & 16e_2 & 16e_3 & 16e_4 & 1 + 16e_5
 \end{array} g_1^{-1}$$

for $a_1, \dots, e_5 \in \mathbb{Z}$. Also, since $g_1 \in \text{SO}(q, \mathbb{Q})$, the matrix on the right-hand side preserves q if and only if the matrix being conjugated on the left-hand side does. This proves the claim.

Lemma 2.3 implies that $(4) \setminus (16)$ is an abelian 2-group, hence so is $\rho_0 \setminus \rho_1$. We now define g_n inductively as follows. Let x_n be a vertex of T in the $\text{SO}(q; \mathbb{Q}_2)$ -orbit of x_0 that is distance 2 in T from some vertex x_i in $\{x_0, \dots, x_{n-1}\}$. Since $\text{SO}(q; \mathbb{Q})$ is dense in $\text{SO}(q; \mathbb{Q}_2)$, there exists an h_n in $\text{SO}(q; \mathbb{Q})$ so that $h_n(x_0) = x_i$ and $h_n(x_1) = x_n$. We define $g_n = h_n g_1 h_n^{-1}$.

Choose the sequence $\{x_n\}$ to exhaust the $\text{SO}(q; \mathbb{Q}_2)$ -orbit of x_0 . Then it is easy to see that the sequence $\{g_n\}$ satisfies all the conditions of Lemma 2.2. In particular, if $\rho_n = g_n \rho_0 g_n^{-1}$ and

$$\rho_n = \prod_{i=0}^{n-1} \rho_i,$$

then $\{\rho_n\}$ is cofinal, and the elements g_n satisfy the requisite assumptions by construction. Therefore there is a RFRS tower for (4) .

Corollary 3.6. — *The group $\text{SO}(4, 1; \mathbb{Z})$ has a congruence subgroup that is algebraically fibered.*

Proof. — Since $\text{SO}(4, 1; \mathbb{Z})$ has $b_1^{(2)} = 0$ (see [21, Lem. 1]), a result of Dawid Kielak [18, Thm. 5.3] implies that some level 2^k congruence subgroup is algebraically fibered.

Remark 3.7. — The proof of Proposition 3.5 would work without alteration for the level 4 congruence subgroup of $SO(n, 1; \mathbb{Z})$ for any $n > 2$, as long as it has no 2-torsion in its abelianization. This holds for $n = 2, 3, 4$, however Steven Tschantz computed that $H_1(\Gamma(4); \mathbb{Z}) = \mathbb{Z}^{256} \times \mathbb{Z}/2$ for $\Gamma = SO(5, 1; \mathbb{Z})$.

Remark 3.8. — For $n = 2, \dots, 7$, the congruence subgroup $\Gamma(2)$ of level 2 in $O(n, 1; \mathbb{Z})$ is a right-angled Coxeter group [22, Thm. 7]. As noted in the proof of Proposition 3.5, the commutator subgroup $\Gamma(2)^{(1)}$ then has torsion-free abelianization by [10, §4.5]. For $n \leq 4$, $\Gamma(2)^{(1)}$ equals $\Gamma(4)$. For $n > 4$, we have that $\Gamma(2)^{(1)}$ is a proper finite index subgroup of $\Gamma(4)$. It is possible that $H_1(\Gamma(4))$ is torsion-free for $n > 5$ in spite of Tschantz's computation for $n = 5$.

Proof of Theorem 1.1. — For $d \equiv -1 \pmod{8}$, there is a finite index subgroup Γ of $PSL_2(O_d)$ that is isomorphic to a subgroup of the group $\Gamma(4)$ in Proposition 3.5. For d square-free, the quadratic form $q_d = (1, 1, 1, -d)$ is isotropic if and only if $d \equiv -1 \pmod{8}$; [3, Thm. 6.2] and the subsequent discussion. In this case, $PO(q_d; \mathbb{Z})$ is commensurable with $PSL_2(O_d)$ [3, Thm. 2.3], hence one can embed $PO(q_d; \mathbb{Z})$ into $PO(4, 1; \mathbb{Z})$ up to commensurability by [3, Lem. 6.3] and as in the proof of [3, Lem. 4.6(i)]. The proof that such a Γ exists in fact produces a congruence subgroup of $PSL_2(O_d)$. Intersecting this with the RFRS tower given in Proposition 3.5 produces the desired RFRS tower for Γ . This proves the theorem.

Remark 3.9. — The above gives an explicit congruence subgroup of $PSL_2(O_d)$ that begins a congruence RFRS tower. We sketch the argument bounding the index of this subgroup when $d \equiv 1 \pmod{4}$ is a sum of two squares (equivalently, no prime dividing d is congruent to 3 modulo 4). In this case, $PSL_2(O_d)$ can be realized as the subgroup of $SO(q_d; \mathbb{Z})$ for

$$q_d = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & d \end{pmatrix}$$

that preserves the upper left 4×4 block (e.g., see [11, §3.1]). Then q_d is equivalent to the standard quadratic form $(-1, 1, 1, 1, 1)$ under the change

of variables with matrix

$$h_d = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{a}{d} & \frac{b}{d} \\ 0 & 0 & 0 & -\frac{b}{d} & \frac{a}{d} \end{pmatrix}$$

where $d = a^2 + b^2$.

Thus if $\text{PSL}_2(O_d) \subset \text{SO}(q_d; \mathbb{Z})$, then $h_d^{-1} h_d$ preserves the standard quadratic form. If $\text{SO}(q_d; \mathbb{Z})$ is congruent to the identity modulo $8d$, then $h_d^{-1} h_d \in \text{SO}(4, 1; \mathbb{Z})$ is congruent to the identity modulo 4. Following [11, Eq. (3.1)], one sees that the congruence subgroup of $\text{PSL}_2(O_d)$ of level $8d$ maps into the congruence subgroup of $\text{SO}(q_d; \mathbb{Z})$ of level $8d$. Computing the index of this congruence subgroup, we see that $\text{PSL}_2(O_d)$ contains a RFRS subgroup of index at most

$$[\text{PSL}_2(O_d) / (8d)O_d : \text{p} / (8d)O_d] = (8d)^6 \left(1 - \frac{1}{N(\mathfrak{p})^2} \right).$$

However, one can likely improve upon this.

Remark 3.10. — Methods analogous to work of Michelle Chu [11] on effectively embedding subgroups of Bianchi groups in $\text{SO}(6, 1; \mathbb{Z})$ could allow one to prove that the above Bianchi groups contain a congruence RFRS tower of uniformly bounded index.

Remark 3.11. — If the congruence subgroup of level 4 in $\text{SO}(6, 1; \mathbb{Z})$ has no 2-torsion in its first homology, then Theorem 1.1 holds for all Bianchi groups. See [3, Lem. 4.4]. More generally, one only needs to find a prime p so that the congruence subgroup of level p in $\text{SO}(6, 1; \mathbb{Z})$ has no p -torsion in its abelianization, which seems likely but very difficult to verify computationally. If this holds, then all Bianchi groups contain a congruence RFRS tower and hence fiber on a congruence subgroup.

3.3. A complex hyperbolic example

Our example will come from a congruence cover of a Deligne–Mostow orbifold [12]. We recall that for certain $(n + 3)$ -tuples μ of integers (called *weights*) satisfying a condition called INT, Deligne and Mostow constructed lattices $\Gamma_\mu \subset \text{PU}(n, 1)$, where $\text{PU}(n, 1)$ is the symmetry group of the

weights and $\mu/\mu = \dots$. Let B^2 denote complex hyperbolic 2-space in what follows.

The example we consider here is $\mu = (2, 2, 2, 2, 2)$, hence $\dots = S_5$. Following [19], the underlying analytic space for the orbifold B^2/μ is the blowup of the complex projective plane P^2 at the four vertices of the complete quadrangle, and each divisor has orbifold weight 5. See Figure 3.5. (Note that the convention in [19] is to divide the elements of μ by their gcd, so μ is listed as $(1, 1, 1, 1, 1)$.) Then S_5 acts on this blowup of P^2 in a natural way with quotient the underlying analytic space for B^2/μ .

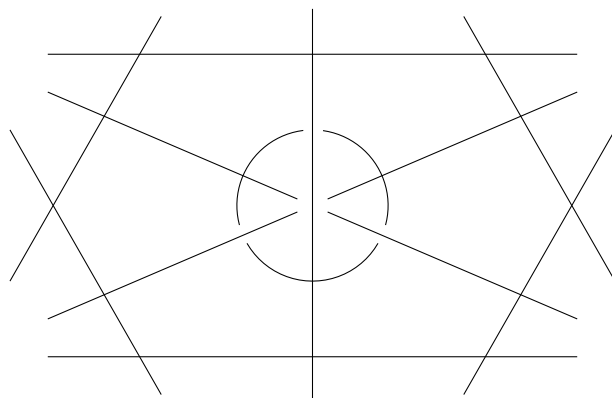


Figure 3.5. The orbifold B^2/μ . Each line or circle represents a P^1 in the orbifold locus, and each has orbifold weight 5. Local orbifold groups at intersection points are all $(Z/5)^2$.

It is known that these lattices are arithmetic. More specifically, let E be $\mathbb{Q}(\zeta_5)$, where ζ_5 is a primitive 5th root of unity, and $F = \mathbb{Q}(\sqrt{5})$ with $\sqrt{5}$ be its totally real quadratic subfield. Define $\alpha = \frac{1-\zeta_5}{2}$ and consider the hermitian form on E^3 with matrix

$$h = \begin{pmatrix} & & 1 & 0 \\ & & & 1 \\ 1 & & & \\ 0 & 1 & & \end{pmatrix}.$$

Then h has signature $(1, 2)$ at one complex place of E and signature $(3, 0)$ at the other complex place. Since $-h$ then has signature $(2, 1)$ the appropriate place, and because similar hermitian forms have isomorphic unitary groups, it follows that $PU(h, O_E)$ is a cocompact arithmetic lattice in $PU(2, 1)$, where $O_E = \mathbb{Z}[\zeta_5]$ is the ring of integers of E . Let $\mathfrak{p}_5 = \alpha O_E$ be the unique prime ideal of O_E dividing $5O_E$. Note that $O_E/\mathfrak{p}_5 = \mathbb{F}_5$,

$\mathfrak{p}_5^2 = \mathcal{O}_E$, and $\mathfrak{p}_5^4 = 5\mathcal{O}_E$. We then have the following, which was proved by Yamazaki and Yoshida.

Proposition 3.12 ([30, Prop. 4.3]). — *With notation as above, we have $\mu = \text{PU}(h, \mathcal{O}_E)$ and μ is the congruence subgroup $\mu(\mathfrak{p}_5)$ of level \mathfrak{p}_5 in μ .*

The lattice of interest to us is supplied by the following lemma.

Lemma 3.13. — *With notation as above, the congruence subgroup of level $\mathfrak{p}_5^2 = \mathcal{O}_E$ in μ is the commutator subgroup of μ and its abelianization is isomorphic to Z^{60} .*

Proof. — The first statement was proved by Yamazaki and Yoshida [30, Thm. 1]. Using a presentation for μ (e.g., see [30, Prop. 2.1]) one then computes the abelianization of the commutator subgroup of μ in Magma [9] to complete the proof of the lemma.

Remark 3.14. — We note the following analogy between $\text{O}(4, 1; Z)$ and μ . Recall that $\text{O}(4, 1; Z)$ is a Coxeter group whose congruence subgroup of level 2 is a right-angled Coxeter group, and the commutator subgroup of the right-angled group is the congruence subgroup of level 4 in $\text{O}(4, 1; Z)$. Analogously, μ is a complex hyperbolic reflection group whose congruence subgroup of level \mathfrak{p}_5 is the complex hyperbolic reflection group μ , and the commutator subgroup of μ is the congruence subgroup of μ of level \mathfrak{p}_5^2 .

We now describe the building used to apply our methods to prove that $\mu(\mathfrak{p}_5^2)$ admits a congruence RFRS tower. Let E_5 be the completion of E with respect to the valuation associated with \mathfrak{p}_5 . Then $E_5 = \mathbb{Q}_5(\sqrt{5})$ is a degree four totally ramified extension of \mathbb{Q}_5 with intermediate quadratic subfield $F_5 = \mathbb{Q}_5(\sqrt{-5})$ and π_5 is a uniformizer for E_5 . The group $\text{SU}(h, E_3)$ is the unique special unitary group in 3 variables with respect to E_5/F_5 , and the associated Bruhat–Tits building is a tree [28, §2.10].

As in Section 3.2, a change of coordinates will be convenient for describing this building. One can find a change of coordinates with entries in the ring of integers \mathcal{O}_5 of E_5 so that h has matrix $-h_0$ for

$$h_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

For example,

$$c = \begin{pmatrix} 1 & -1 & \frac{-1+\sqrt{5}}{8} \\ -1 & -1 & \frac{(1-\sqrt{5})(1+\sqrt{5})}{8} \\ -1 & -1 & 0 \\ & & -\frac{2+\sqrt{5}}{4} \end{pmatrix}$$

such that $\sqrt{5}$ is a square root of $1 + \sqrt{5}$ and $\sqrt{5}$ is a square root of $4 + 2\sqrt{5}$. Critically, $\sqrt{5} \in O_5$ (one can see this by showing that the prime O_F of O_F dividing 5 splits in both $F(\sqrt{5})$ and $F(\sqrt{5})$, and $\sqrt{5}, \sqrt{5}$ are invertible in O_5 since $1 + \sqrt{5}$ and $4 + 2\sqrt{5}$ have norm -4). Since this conjugation is integral over E_5 with determinant 1, and because similar hermitian forms have the same unitary group, we have that μ is isomorphic to the intersection of the F -points of $SU(h_0)$ with $SU(h_0, O_5)$.

Following [28, §2.10], the building for $SU(h_0, E_5)$ has vertices the set of additive norms on E_5^3 so that

$$(h_0(x, y)) > (x) + (y)$$

for all pairs $x, y \in E_5^3$, where (\cdot) is the extension to E_5 of the normalized valuation on F_5 (i.e., with value group $\frac{1}{2}\mathbb{Z}$). There is an obvious action of $SU(h_0)$ on the set of norms, and the norm stabilized by $SU(h_0, O_5)$ is the vertex v_0 associated with the norm

$$v_0(x_1, x_2, x_3) = \inf \{ (x_j) : 1 \leq j \leq 3 \}.$$

In particular, μ stabilizes this vertex.

The matrix

$$g_0 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{4}{5} & 0 \\ -1 & 0 & 0 \end{pmatrix} \in SU(h)$$

(where $\bar{\cdot}$ is the conjugate of \cdot for the $\text{Gal}(E_5/F_5)$ -action) acts on the tree by sending v_0 to the vertex v_1 associated with the norm

$$v_1(x_1, x_2, x_3) = \inf \left((x_1) - \frac{1}{2}, (x_2), (x_3) + \frac{1}{2} \right),$$

since $(\sqrt{5}) = \frac{1}{2}$. It also fixes the intermediate vertex associated with

$$v_{1/2}(x_1, x_2, x_3) = \inf \left((x_1) - \frac{1}{4}, (x_2), (x_3) + \frac{1}{4} \right).$$

Then one checks by a direct matrix computation that the intersection of $g_0 \mu(\mathfrak{p}_5^2) g_0^{-1}$ with $\mu(\mathfrak{p}_5^2)$ contains (\mathfrak{p}_5^4) . Indeed, note that $\mathfrak{p}_5^4 = 5O_E$ and if

$$= \begin{pmatrix} 1 + 5c_3 & -(\frac{2}{5} + 2\sqrt{5} + 1)^5 c_2 & -(\frac{3}{5} + 2\sqrt{2} + \sqrt{5})^6 c_1 \\ (\frac{3}{5} + \frac{2}{5} - 1)^3 b_3 & 1 + 5b_2 & -(\frac{3}{5} + 2\sqrt{2} + 2\sqrt{5} + 1)^5 b_1 \\ (\frac{3}{5} + 2\sqrt{2} + 2\sqrt{5} + 1)^2 a_3 & (\frac{3}{5} + 2\sqrt{2} + \sqrt{5})^3 a_2 & 1 + 5a_1 \end{pmatrix},$$

then $\mu(\mathfrak{p}_5^2)$ and

$$g_0 g_0^{-1} = \begin{pmatrix} 1 + 5a_1 & 5a_2 & 5a_3 \\ 5b_1 & 1 + 5b_2 & 5b_3 \\ 5c_1 & 5c_2 & 1 + 5c_3 \end{pmatrix} \mu(\mathfrak{p}_5^4).$$

From here, one applies the techniques developed in the previous examples to prove Theorem 1.2.

Remark 3.15. — Analogous to our realization of the building for $\text{PO}(q; \mathbb{Q}_2)$ inside the building for $\text{PGL}_5(\mathbb{Q}_2)$ in Section 3.2, we can realize the building for $\text{PU}(h, E_5)$ inside the building for $\text{PGL}_3(E_5)$ by taking the vertex associated with an additive norm to be the homothety class of the O_E lattice on which takes nonnegative values. In the above notation and recalling that is normalized to have value group $\frac{1}{2}\mathbb{Z}$, this gives:

$$\begin{array}{l} 0 \quad [e_1, e_2, e_3] \\ 1 \quad [e_1, e_2, -^1e_3] \\ \quad [e_1, e_2, e_3] \end{array}$$

One can proceed as in Section 3.2 to compute the fundamental apartment associated with the standard \mathbb{Q}_5 -split torus and compute the valence of each vertex of the tree.

Combining Theorem 1.2 and [15, Thm. E], one obtains a new proof of the following (which was known by [26, Thm. 3] without knowing which congruence tower contains the fibration).

Corollary 3.16. — *The group μ virtually algebraically fibers on a congruence subgroup of level dividing 5.*

Remark 3.17. — We note that nonuniform lattices in $\text{PU}(n, 1)$ cannot be virtually RFRS for $n > 2$. This is because their cusp subgroups are virtually two-step nilpotent groups, but two-step nilpotent groups are not virtually RFRS and being RFRS descends to subgroups. However, if $\Gamma < \text{PU}(n, 1)$ was a nonuniform arithmetic lattice contained in the congruence subgroup of level \mathfrak{p} for which Γ^{ab} contains no p -torsion, where \mathfrak{p} is a prime of residue characteristic p , then the methods of this paper would produce a congruence RFRS tower, which is impossible. In particular, we conclude that Γ^{ab} must have p -torsion.

One way to find this p -torsion is as follows. Since Γ is contained in a congruence subgroup, away from some small exceptions the associated complex hyperbolic manifold B^n/Γ admits a *smooth toroidal compactification*

in the sense of [4]. One often sees that the cusp cross-sections of B^n/Γ are nil-manifolds with p -torsion in their homology. For example, for Γ the center of any peripheral subgroup generates p -torsion in the homology of the associated nil-manifold. Careful consideration of the standard Mayer–Vietoris sequence for the toroidal compactification (cf. [13, §4]) allows one to then conclude that this p -torsion in the homology of the cusp cross-section must in fact induce p -torsion in the homology of B^n/Γ .

In particular, peripheral subgroups of Γ can force B^n/Γ to have p -torsion when Γ is contained in the congruence subgroup of level \mathfrak{p} . Thus the obstruction to B^n/Γ containing a RFRS tower is also an obstruction to B^n/Γ having no p -torsion.

4. Conclusion

There are many natural questions that arise from the results and methods of this paper.

We recall that a group Γ is said to *algebraically fiber* if it has a homomorphism onto \mathbb{Z} with finitely generated kernel. This is an algebraic generalization of the well-known Stallings criterion for a compact 3-manifold to fiber over S^1 .

Question 4.1. — *Which commensurability classes of rank 1 arithmetic lattices contain a congruence subgroup that is algebraically fibered?*

This question was originally posed by Baker and Reid in personal communication. A 4-dimensional lattice that virtually algebraically fibers was given in [17, Rem. 5.3], though we do not know if the example fibers on a congruence subgroup. An obvious obstruction to having a virtual algebraic fibration on a congruence subgroup is if every congruence lattice in the commensurability class has trivial 1st betti number. For example, Bergeron and Clozel proved that the first betti number vanishes for all congruence arithmetic lattices in $\mathrm{PO}(7, 1)$ defined via triality [8, Thm. 1.1]. For all other arithmetic lattices in $\mathrm{PO}(n, 1)$, $n = 3$, one can find a congruence subgroup with nontrivial 1st betti number [7, Cor. 1.8] (the $n = 3$ case is open – see [24] for a discussion of what is known). There are also classes of arithmetic lattices in $\mathrm{PU}(n, 1)$ where each kind of behavior occurs. See [6] for more on what is known for cohomological vanishing for congruence arithmetic lattices in $\mathrm{PU}(n, 1)$ and [26] for more on algebraic fibrations in that setting.

Question 4.2. — *Given a congruence arithmetic group, how often does a principal congruence subgroup at a prime ideal \mathfrak{p} have no p -torsion in H_1 , where \mathfrak{p}/p ? Is there some arithmetic significance to this phenomenon?*

See Appendix A by Engün for data indicating that vanishing of p -torsion is quite frequent for congruence subgroups of Bianchi groups, but by no means ubiquitous.

Question 4.3. — *For each $n > 1$, is there a prime p so that the congruence subgroup $\Gamma(p)$ of level p in $\mathrm{SO}(n, 1; \mathbb{Z})$ has no p -torsion in its abelianization?*

If true, this would give a positive answer to Question 4.1 for arithmetic hyperbolic groups of simplest type (i.e., those defined by a quadratic form), since one can embed a congruence subgroup of these groups into $\mathrm{SO}(n, 1; \mathbb{Z})$ by restriction of scalars.

Question 4.4. — *When does $(\Gamma(\mathfrak{p}^n))_{n \in \mathbb{N}}$ form a RFRS sequence? Is the sequence RFRS whenever it is at the first stage, i.e., $(\Gamma(\mathfrak{p})^{(1)}) \triangleleft (\Gamma(\mathfrak{p}^2))$?*

This is roughly a version of another question posed by Baker and Reid in private communication. In this paper, we only show that an interlacing of this sequence is RFRS if $H_1(\Gamma(\mathfrak{p}); \mathbb{Z})$ has no p -torsion. Also recall Remark 3.17.

Question 4.5. — *When does this strategy work for nonarithmetic hyperbolic lattices? When is there a congruence subgroup that fibers, or a congruence RFRS tower?*

Note that any lattice in $\mathrm{PO}(n, 1)$, $n > 3$, or $\mathrm{PU}(n, 1)$, $n > 2$, is a subgroup of an S -arithmetic group by local rigidity. Indeed, the lattice can be embedded in $\mathrm{GL}_n(K)$ for K a number field, and hence lies in $\mathrm{GL}_n(\mathcal{O})$ for \mathcal{O} some finitely generated subring of K . Therefore, the notion of congruence subgroup makes sense when one avoids the primes in S , where S denotes the primes that are inverted in \mathcal{O} . Despite the fact that methods of this paper cannot apply to a nonarithmetic lattice (since its commensurator is discrete) and moreover the ambient S -arithmetic group containing it cannot admit a RFRS tower (since it has trivial virtual betti number), this does not preclude a nonarithmetic lattice from nevertheless admitting a congruence RFRS tower.

Appendix A. Torsion in the homology of principal congruence subgroups of Bianchi groups

A.1. Introduction

Let K be an imaginary quadratic field with ring of integers Z_K . An ideal \mathfrak{a} of Z_K determines a finite-index normal subgroup $\Gamma(\mathfrak{a})$ of the Bianchi group $SL_2(Z_K)$, called the *principal congruence subgroup of level \mathfrak{a}* . If \mathfrak{p} is a prime ideal of Z_K over the rational prime p , the question of whether the abelian group $H_1(\Gamma(\mathfrak{p}), Z)$ has p -torsion arises naturally in the current work of Ian Agol and Matthew Stover. In this appendix, we try to gain insight into this question by producing numerical data.

A.2. Methodology

Let K be one of the five imaginary quadratic fields for which Z_K is Euclidean, namely $K = \mathbb{Q}(\sqrt{-d})$ with $d = 1, 2, 3, 7, 11$. Let \mathfrak{p} be a prime ideal of Z_K . Our starting point is the basic fact that $H_1(\Gamma(\mathfrak{p}), Z) \cong H_1(\mathfrak{p})^{ab}$ where \mathfrak{p}^{ab} is the abelianization of \mathfrak{p} . To compute the abelianization of \mathfrak{p} , we will need a presentation. We will obtain this presentation from a presentation of $SL_2(Z_K)$ using the standard functions in the Finitely Presented Groups package of the computer algebra system Magma.

Presentations for Bianchi groups go back to the late 19th century. We prefer to use those given in [14, p. 37]. The presentations given there are for the projective Bianchi groups $PSL_2(Z_K)$. To obtain a presentation for $SL_2(Z_K)$, we simply introduce another generator $j = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, modify the existing relations accordingly and add new relations to ensure that j is central. We present here the result for the case $K = \mathbb{Q}(\sqrt{-1})$:

$$SL_2(Z_K) = \langle a, b, u, j \mid (ab)^3 = j, b^2 = j, j^2 = 1, [a, u] = 1, (bub u^{-1})^3 = 1, j = (bu^2bu^{-1})^2, j = (aubau^{-1}b)^2, [a, j] = 1, [u, j] = 1 \rangle.$$

We have the matrix realizations $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $u = \begin{pmatrix} 1 & \bar{1} \\ 0 & 1 \end{pmatrix}$.

The principal congruence subgroup $\Gamma(\mathfrak{p})$ is the kernel of the surjective homomorphism

$$SL_2(Z_K) \twoheadrightarrow SL_2(Z_K/\mathfrak{p}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$$

where \bar{x} is the reduction map $Z_K \rightarrow Z_K/\mathfrak{p}$. We implement this homomorphism in Magma and ask Magma to compute its kernel. Given the presentation of $SL_2(Z_K)$, Magma then can compute a presentation for

(\mathfrak{p}) using Reidemeister–Schreier type algorithms. Finally we ask Magma to compute the abelianization. For the readers’ convenience, we make our code public on our homepage.⁽¹⁾

A.3. Results

As mentioned above, we compute with prime ideals. As $H_1(\mathfrak{p}, Z)$ $H_1(\overline{\mathfrak{p}}, Z)$, for prime ideals with $\mathfrak{p} = \overline{\mathfrak{p}}$ (here), we computed with only one of them. We list the norm of the prime ideal \mathfrak{p} , the rank of $H_1(\mathfrak{p}, Z)$ and the size of the torsion subgroup of $H_1(\mathfrak{p}, Z)$. The size is given in its prime factorisation.

Norm(\mathfrak{p})	rank	size of torsion
$K = \mathbb{Q}(\sqrt{-1})$		
2	0	2^5
5	6	1
9	20	1
13	42	1
17	72	1
29	238	3^1
37	342	1
41	420	2^{62}
49	825	7^6
53	702	3^{104}
61	930	$2^{124}29^{62}$
73	1332	$3^{37}5^{74}19^{74}$
$K = \mathbb{Q}(\sqrt{-2})$		
2	3	2^2
3	4	1
11	60	1
17	144	2^9
19	180	1
25	403	5^7
41	881	$2^{85}5^17^{40}127^{40}$
43	924	$2^{88}3^{42}67^{42}127^{44}$
49	1724	7^{133}
59	1740	$2^{290}3^{58}11^{116}31^{58}59^{236}5743^{60}$
67	2244	$2^{200}3^{135}239^{66}271^{66}647^{66}727^{68}3801^{66}47917^{68}$
73	2738	$2^{369}3^{296}19^{73}73^{11}1511^{74}2089^{74}22051^{72}150959^{72}$

(1) <https://sites.google.com/site/mhalksengun/research>

Norm(p)	rank	size of torsion
$K = \mathbb{Q}(\sqrt{-3})$		
3	0	3^3
4	5	2^1
7	8	1
13	28	1
19	60	1
25	117	1
31	160	1
37	228	3^{19}
43	308	2^{44}
61	620	3^{62}
67	748	$2^{68}5^{67}$
79	1040	$2^{80}41^{78}$
$K = \mathbb{Q}(\sqrt{-7})$		
2	3	2^1
7	24	1
9	40	3^1
11	60	1
23	264	2^{22}
25	376	5^7
29	420	5^{15}
37	684	$3^{19}19^{38}$
43	924	$5^{87}41^{44}67^{42}$
53	1404	$2^{320}37^{52}59^{35}2857^{54}$
67	2244	$2^{1056}3^{67}11^{67}89^{68}131^{68}137^{66}463^{68}$
71	2520	$2^{288}5^{140}29^{72}59^{70}89^{70}311^{70}937^{70}19319^{72}$
$K = \mathbb{Q}(\sqrt{-11})$		
3	4	1
4	15	2^2
5	12	1
11	81	1
23	264	2^{112}
31	480	$2^{96}5^{30}29^{32}31^{30}$
37	722	$17^{38}19^{36}37^5 683^{36}$
47	1151	$3^{138}5^{47}17^{46}23^{137}4697^{46}191^{46}1609^{48}$
53	1404	$2^{320}5^{160}11^{27}19^{104}53^{52}431^{52}683^{52}859^{54}$
59	1740	$2^{348}3^{408}5^{58}7^{116}11^{58}17^{120}19^{58}31^{58}59^{58}199^{58}233^{60}5279^{60}20341^{58}$
67	2244	$2^{136}3^{132}17^{133}31^{68}67^{66}197^{68}331^{68}613^{66}2309^{68}5807^{68}67829^{66}256189^{66}$

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