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Lorenzo Brandolese & Sylvie Monniaux

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WELL-POSEDNESS FOR THE BOUSSINESQ SYSTEM IN CRITICAL SPACES VIA MAXIMAL REGULARITY

by Lorenzo BRANDOLESE & Sylvie MONNIAUX (*)

Abstract. — We establish the existence and the uniqueness for the Boussinesq system in $\mathbb{R}^3$ in the critical space $C([0,T], L^3(\mathbb{R}^3)^3) \times L^2(0,T; L^{3/2}(\mathbb{R}^3))$.

Résumé. — Nous prouvons des résultats d'existence et unicité pour le système de Boussinesq dans $\mathbb{R}^3$ dans $C([0,T], L^3(\mathbb{R}^3)^3) \times L^2(0,T; L^{3/2}(\mathbb{R}^3))$, espace critique pour ce système.

1. Introduction

We consider the Cauchy problem associated with the Boussinesq system in $\mathbb{R}^3$:

$$
\text{(B)} \qquad
\begin{aligned}
\partial_t u - \Delta u + \quad \pi + \quad \cdot (u \quad u) &= \theta e_3 &&\text{in} \quad (0,T) \times \mathbb{R}^3\\
\operatorname{div} u &= 0 &&\text{in} \quad (0,T) \times \mathbb{R}^3\\
\partial_t \theta - \Delta \theta + u \cdot \quad \theta &= 0 &&\text{in} \quad (0,T) \times \mathbb{R}^3,
\end{aligned}
$$

where u denotes the velocity of the fluid, π the pressure, θ the temperature and e_3 the third vector of the canonical basis in $\mathbb{R}$. The given initial velocity and temperature are denoted respectively u_0 and θ_0. The initial velocity will be always assumed to satisfy the condition $\operatorname{div} u_0 = 0$. The system (B) appears in the study of the motion of incompressible viscous flows when one takes into account buoyancy effects arising from temperature variations inside the fluid. When the latter are neglected ($\theta \equiv 0$), the system boils down to the classical Navier–Stokes equations.

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The natural scaling leaving the Boussinesq system invariant is $\lambda \to (u_\lambda, \theta_\lambda)$ with

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x) \qquad \text{and} \qquad \theta_\lambda(t,x) = \lambda^3\theta(\lambda^2 t, \lambda x).$$

This motivates the study of (B) in function spaces that are left invariant by the above scaling. Assuming $(u_0, \theta_0) \in L^3(\mathbb{R})^3 \times L^1(\mathbb{R})$, an adaptation of Kato's L^p-theory on strong solutions of Navier–Stokes [13], yields the local-in-time existence and the uniqueness in appropriate scale-invariant function spaces where the fixed point argument applies. But, as discussed in [3], the uniqueness problem in the natural space $C([0,T], L^3(\mathbb{R})^3) \times C([0,T], L^1(\mathbb{R}))$ seems to be out of reach, due to the lack of regularity results in this class, and to the difficulty of giving a meaning, in the distributional sense, to the nonlinearity $\nabla \cdot (u\theta)$ (and, of course, to $u \cdot \nabla\theta$) in the last equation in (B). To circumvent this difficulty, the uniqueness for the Cauchy problem, in [3], was established in a smaller space for the temperature, namely $C([0,T], L^1(\mathbb{R})) \cap L^\infty_{\mathrm{loc}}(0,T; L^{q,\infty}(\mathbb{R}))$, for some $q > 3/2$. This restriction on the temperature, however, is a bit artificial: the excluded borderline case, $q = 3/2$, is precisely the most interesting one, as it corresponds to the minimal regularity to be imposed on the temperature, when the velocity is in the natural space $L^3(\mathbb{R})^3$, to give a sense to the nonlinearity.

The scaling relations then lead us to consider solutions such that $t \to \|\theta(t)\|_{L^{3/2}}$ is in L^2. Therefore, it seems natural to address the uniqueness problem (and the existence) in

$$C([0,T], L^3(\mathbb{R})^3) \times L^2(0,T; L^{\frac{3}{2}}(\mathbb{R}^3)).$$

As the Lorentz-space approach of [5, 14, 17], applied in [3], fails when $q = 3/2$, we have to adopt a different strategy. Our main tools will be maximal regularity estimates. The idea of using the maximal regularity in uniqueness problems goes back to [18], where the second author gave a short proof of celebrated Furioli, Lemarié and Terraneo's uniqueness theorem [9] of mild solutions of the Navier–Stokes equations in $C([0,T], L^3(\mathbb{R}^3))$. In the present paper, we will need to use the maximal regularity in an original way, in order to make it applicable despite the product $u\theta$ a priori just belongs to $L^1(\mathbb{R}^3)^3$.

In fact, our approach allows us to obtain the uniqueness, and then the regularity as a byproduct of the existence theory, in a larger class, namely

$$\Big(C([0,T], L^3(\mathbb{R}^3)^3) + r\, L^\infty(0,T; L^3(\mathbb{R}^3)^3)\Big) \times L^2(0,T; L^{\frac{3}{2}}(\mathbb{R}^3)) \tag{1.1}$$

for some small enough $r > 0$.

One could speculate that the smallness condition on the parameter r may be unessential and that the uniqueness and the regularity could be true in the larger space $L^\infty(0,T;L^3(\mathbb{R}^3)^3)\times L^2(0,T;L^{\frac32}(\mathbb{R}^3))$. This would be a nontrivial generalization for the system (B) of the deep result of Escauriaza, Seregin and Šverák [8], about endpoint Serrin regularity criteria for the Navier–Stokes equations. Establishing such a result would probably require various ingredients (backward uniqueness, profile decompositions, [8, 10, 19], etc.). Whether or not such a stronger statement is true, we feel that our main theorem would remain of interest because of its attractive proof, entirely based on maximal regularity estimates.

2. Statement of the main results

Let $T>0$ and $r>0$. Let $\mathcal{S}'(\mathbb{R}^3)$ denote the dual of Schwartz space. In order to state our uniqueness result in the class

$$X_{T,r} := \Big(\mathcal{C}([0,T],L^3(\mathbb{R}^3)^3) + r\,L^\infty(0,T;L^3(\mathbb{R}^3)^3)\Big)\times L^2(0,T;L^{\frac32}(\mathbb{R}^3)),$$

we first clarify what we mean by solution of (B). By definition, a mild solution of (B) with inital data $(u_0,\theta_0)\in\mathcal{S}'(\mathbb{R}^3)^3\times\mathcal{S}'(\mathbb{R}^3)$, and $\operatorname{div}u_0=0$, is a couple $(u,\theta)\in X_{T,r}$ solving the Boussinesq system written in its integral form (2.1) below:

$$\begin{aligned} u &= a + B(u,u) + L(\theta)\\ \theta &= b + C(u,\theta)\end{aligned}\tag{2.1}$$

with

$$a(t) = e^{t\Delta}u_0,\qquad b(t)=e^{t\Delta}\theta_0.$$

The operators B, C and L are defined, for $t\in[0,T]$, and $(u,\theta)\in X_{T,r}$, by

$$B(u,v)(t) = -\int_0^t e^{(t-s)\Delta}\,\mathbb{P}\Big(\nabla\cdot\big(u(s)\otimes v(s)\big)\Big)\,ds,\tag{2.2}$$

$$C(u,\theta)(t) = -\int_0^t e^{(t-s)\Delta}\operatorname{div}\big(\theta(s)u(s)\big)\,ds,\tag{2.3}$$

and

$$L(\theta)(t) = \int_0^t e^{(t-s)\Delta}\,\mathbb{P}(\theta(s)e_3)\,ds.\tag{2.4}$$

Here $\mathbb{P}$ denotes Leray's projector onto divergence-free vector fields and $\big(e^{t\Delta}\big)_{t\geqslant0}$ is the heat semigroup.

Our main result then is stated as follows.

Theorem 2.1. — *There is an absolute constant* $r_0 > 0$ *such that, if* (u_1,θ_1) *and* (u_2,θ_2) *are two mild solutions to* (B) *in* $X_{T,r}$, *with the same initial data* $(u_0,\theta_0) \in \mathcal{S}(\mathbb{R}^3)^3 \times \mathcal{S}(\mathbb{R}^3)$, $\operatorname{div} u_0 = 0$ *and* $0 \leqslant r < r_0$, *then*

$$(u_1,\theta_1) = (u_2,\theta_2).$$

As we will see, any solution (u,θ) as in Theorem 2.1 must belong to $C([0,T],\mathcal{S}(\mathbb{R}^3)^3 \times \mathcal{S}(\mathbb{R}^3))$, and the initial data must belong more precisely to $L^3(\mathbb{R}^3)^3 \times B^{-1}_{3/2,2}(\mathbb{R}^3)$. The above theorem is then completed by the corresponding existence result:

Theorem 2.2.

(i) *Let* $(u_0,\theta_0) \in L^3(\mathbb{R}^3)^3 \times B^{-1}_{3/2,2}(\mathbb{R}^3)$, *with* $\operatorname{div} u_0 = 0$. *Then there exists* $T > 0$ *and a solution of* (2.1) $(u,\theta) \in C([0,T],L^3(\mathbb{R}^3)^3) \times L^2(0,T;L^{\frac{3}{2}}(\mathbb{R}^3))$.

(ii) *If* θ_0 *belongs to the smaller homogeneous Besov space* $\dot B^{-1}_{3/2,2}(\mathbb{R}^3)$ *and if* $\|u_0\|_{L^3} + \|\theta_0\|_{\dot B^{-1}_{3/2,2}}$ *is small enough, than such solution is global and* $(u,\theta) \in C_b(0,\infty;L^3(\mathbb{R}^3)^3) \times L^2(0,\infty;L^{\frac{3}{2}}(\mathbb{R}^3))$.

For some other existence results for the Boussinesq system in different functional setting we refer, e.g., to [5, 6, 12].

3. Applications of the maximal regularity

The purpose of this section is to study the properties of the operators B, C and L, respectively defined by (2.2), (2.3) and (2.4), by means of the following maximal regularity result. The theorem below is classical. See [21] for the case $p \in (1,\infty)$ and $q = 2$ for the negative generator of an analytic semigroup. The general case $1 < p, q < \infty$ for the Laplacian in the whole space was first proved in [15, Chapter IV, §3]. See also [16, Theorem 7.3] for a modern proof of the case of the Laplacian in the whole space, or [11, Theorem 3.1] and [4, Theorem 1.2] for a slightly more general situation. The proof of the estimates for the mixed fractional space-time derivatives goes back to [22, Theorem 6]; see also [20, Proposition 2.4].

To begin with, let us define the fractional time derivative operator. Let $1 < p < \infty$ and X be a UMD Banach space (i.e., for which the Hilbert transform is bounded in $L^p(\mathbb{R};X)$; this is in particular the case if $X = L^q(\mathbb{R}^d)$ for $1 < q < \infty$). Denote by $\frac{\mathrm{d}}{\mathrm{d}t}$ the operator defined on $L^p(0,\infty;X)$ with domain

$$W_0^{1,p}(0,\infty;X) := \{f \in L^p(0,\infty;X); \partial_t f \in L^p(0,\infty;X) \text{ and } f(0,\cdot) = 0\}.$$

This operator is invertible, sectorial in $L^p(0,\infty;X)$ and admits bounded imaginary powers satisfying
$$\left\| \left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^{is} \right\|_{\mathcal{L}(L^p(0,\infty;X))} \leqslant C_p(X)(1+s^2)\,\mathrm{e}^{\frac{\pi}{2}|s|}, \quad \forall s \in \mathbb{R},$$
where $C_p(X)$ is a constant which depends only on p and X (see, e.g., [7, Theorem 3.1] or [1, ex. 4.7.3.c p. 160]). This implies in particular that for $\alpha \in [0,1]$, the fractional derivative operator $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha}$ is invertible and its domain is $W_0^{\alpha,p}(0,\infty;X)$.

Theorem 3.1 (Maximal regularity). — *Let* $1 < p, q < \infty$*. Let* R *be the operator defined for* $f \in L^1_{\mathrm{loc}}(0,\infty;\mathcal{S}'(\mathbb{R}^d))$, $d > 1$, *by*

$$Rf(t) = \int_0^t \mathrm{e}^{(t-s)\Delta} f(s)\,\mathrm{d}s, \quad t > 0. \tag{3.1}$$

Such operator R *is bounded from the space* $L^p\big(0,\infty;L^q(\mathbb{R}^d)\big)$ *to the space* $\dot{W}^{1,p}\big(0,\infty;L^q(\mathbb{R}^d)\big) \cap L^p\big(0,\infty;\dot{W}^{2,q}(\mathbb{R}^d)\big)$*. In other words,* $\frac{\mathrm{d}}{\mathrm{d}t}R$, ΔR, *and* $(-\Delta)^{\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{1-\alpha}R$, *for any* $0 < \alpha < 1$, *are bounded operators in the space* $L^p\big(0,\infty;L^q(\mathbb{R}^d)\big)$*. Moreover, there exists a constant* $C_{p,q}$ *such that*
$$\left\| \frac{\mathrm{d}}{\mathrm{d}t}Rf \right\|_{L^p(L^q)} + \left\| \Delta Rf \right\|_{L^p(L^q)} + \left\| (-\Delta)^{\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{1-\alpha} Rf \right\|_{L^p(L^q)} \leqslant C_{p,q} \|f\|_{L^p(L^q)},$$
for all $\alpha \in (0,1)$.

To establish Theorem 2.1, we assume that we have two mild solutions $(u_1,\theta_1) \in X_{T,r}$ and $(u_2,\theta_2) \in X_{T,r}$ of (B), arising from the same initial datum $(u_0,\theta_0) \in \mathcal{S}'(\mathbb{R}^3)^3 \times \mathcal{S}'(\mathbb{R}^3)$. Letting $u = u_1 - u_2$ and $\theta = \theta_1 - \theta_2$, we then obtain $(u,\theta) \in X_{T,r}$ and

$$\begin{aligned} u &= B(u,u_1) + B(u_2,u) + L(\theta), \\ \theta &= C(u_1,\theta) + C(u,\theta_2). \end{aligned} \tag{3.2}$$

The maximal regularity theorem allows us to obtain all the relevant estimates for the operators B and C and L.

Proposition 3.2. — *For all* $\varepsilon > 0$, *there exists* $r > 0$ *such that for all* $v, w \in C([0,T], L^3(\mathbb{R}^3)^3) + r\,L^{\infty}([0,T], L^3(\mathbb{R}^3)^3)$, *and for all* $1 < p < \infty$, *there exists* $\tau = \tau(\varepsilon,p,v,w) > 0$ *such that the linear operator*

$$B(\cdot,v) + B(w,\cdot)\colon L^4(0,\tau;L^6(\mathbb{R}^3)^3) \longrightarrow L^4(0,\tau;L^6(\mathbb{R}^3)^3), \tag{3.3}$$

$$B(\cdot,v) + B(w,\cdot)\colon L^p(0,\tau;L^3(\mathbb{R}^3)^3) \longrightarrow L^p(0,\tau;L^3(\mathbb{R}^3)^3) \tag{3.4}$$

is bounded, with operator norm less than ε.

Proof. — Let $r>0$, to be chosen later. For $v,w\in C([0,T],L^3(\mathbb{R}^3)^3)+rL^\infty(0,T;L^3(\mathbb{R}^3)^3)$, we can find $v_r,w_r\in C_c([0,T]\times\mathbb{R}^3)$ such that

$$\operatorname{ess\,sup}_{[0,T]}\|v-v_r\|_{L^3}+\operatorname{ess\,sup}_{[0,T]}\|w-w_r\|_{L^3}\le 3r. \tag{3.5}$$

Let us introduce the functions f and g defined for $s\in[0,T]$ by

$$f(s)=(-\Delta)^{-1}\mathbb{P}\Big(\nabla\cdot\big(u(s)\otimes(v-v_r)(s)+(w-w_r)(s)\otimes u(s)\big)\Big) \tag{3.6}$$

and

$$g(s)=-(-\Delta)^{-3/4}\mathbb{P}\Big(\nabla\cdot\big(u(s)\otimes v_r(s)+w_r(s)\otimes u(s)\big)\Big).$$

We have that

$$B(u,v)+B(w,u)=\Delta Rf+(-\Delta)^{3/4}Rg, \tag{3.7}$$

where R is the vector-valued analogue of the scalar operator defined in Theorem 3.1.

(i) Let us first consider (3.3). We easily see that the norm of f in $L^4(0,\tau;L^6(\mathbb{R}^3)^3)$ is bounded by the norm of $u\otimes(v-v_r)+(w-w_r)\otimes u$ in $L^4(0,\tau;L^2(\mathbb{R}^3)^{3\times3})$. Indeed, the operator $(-\Delta)^{-1}\mathbb{P}\nabla\cdot$ is bounded from $L^2(\mathbb{R}^3)^{3\times3}$ to $L^6(\mathbb{R}^3)^3$. Hence,

$$\begin{aligned}\|\Delta Rf\|_{L^4(0,\tau;L^6(\mathbb{R}^3)^3)}&\\ &\le C_{4,6}\|u\otimes(v-v_r)+(w-w_r)\otimes u\|_{L^4(0,\tau;L^2(\mathbb{R}^3)^{3\times3})}\\ &\le 3rC_{4,6}\|u\|_{L^4(0,\tau;L^6(\mathbb{R}^3)^3)}.\end{aligned}$$

The norm of g in $L^4(0,\tau;L^6(\mathbb{R}^3)^3)$ is bounded by the norm of $u\otimes v_r+w_r\otimes u$ in $L^4(0,\tau;L^3(\mathbb{R}^3)^{3\times3})$. To see this, first observe that the operator $(-\Delta)^{-3/4}\mathbb{P}\nabla\cdot$ is bounded from $L^3(\mathbb{R}^3)^{3\times3}$ to $L^6(\mathbb{R}^3)^3$. Moreover, $\|(-\Delta)^{3/4}e^{t\Delta}\|_{\mathcal{L}(L^6(\mathbb{R}^3)^3)}\lesssim t^{-3/4}$. As $(-\Delta)^{3/4}R$ is a convolution operator, we have

$$\begin{aligned}\|(-\Delta)^{3/4}Rg\|_{L^4(0,\tau;L^6(\mathbb{R}^3)^3)}&\\ &\le c\,\|t\mapsto(-\Delta)^{3/4}e^{t\Delta}\|_{L^1(0,\tau;\mathcal{L}(L^6(\mathbb{R}^3)^3))}\|g\|_{L^4(0,\tau;L^6(\mathbb{R}^3)^3)}\\ &\le c\,\tau^{\frac14}\|u\otimes v_r+w_r\otimes u\|_{L^4(0,\tau;L^3(\mathbb{R}^3)^{3\times3})}\\ &\le c\,\tau^{\frac14}\Big(\|v_r\|_{L^\infty((0,\tau)\times\mathbb{R}^3)^3}+\|w_r\|_{L^\infty((0,\tau)\times\mathbb{R}^3)^3}\Big)\|u\|_{L^4(0,\tau;L^6(\mathbb{R}^3)^3)}.\end{aligned}$$

We first choose $r>0$, such that $3rC_{4,6}\le\frac{\varepsilon}{2}$, next v_r and w_r in $C_c([0,T]\times\mathbb{R}^3)^3$, satisfying (3.5) and last $\tau>0$ such that

$$c\,\tau^{\frac14}\Big(\|v_r\|_{L^\infty((0,\tau)\times\mathbb{R}^3)^3}+\|w_r\|_{L^\infty((0,\tau)\times\mathbb{R}^3)^3}\Big)\le\frac{\varepsilon}{2}.$$

This finally establishes (3.3).

(ii) Let us now consider assertion (3.4). Let $1 < p < \infty$. We slightly modify the expression of $B(\cdot, v) + B(w, \cdot)$ given by (3.7):

$$B(u, v) + B(w, u) = \Delta R f + (-\Delta)^{1/2} R g, \tag{3.8}$$

where we set, for $s \in [0, T]$,

$$g(s) = -(-\Delta)^{-1/2}\mathbb{P}\Big(\nabla\cdot\big(u(s)\otimes v_r(s) + w_r(s)\otimes u(s)\big)\Big). \tag{3.9}$$

The function f defined by (3.6) is bounded in $L^p(0,\tau; L^3(\mathbb{R}^3)^3)$ by $3r\|u\|_{L^p(0,\tau;L^3(\mathbb{R}^3)^3)}$, up to a multiplicative constant involving $C_{p,3}$ and the norm of bounded operator $(-\Delta)^{-1}\mathbb{P}\nabla\cdot$ from $L^{\frac{3}{2}}(\mathbb{R}^3)^{3\times 3}$ to $L^3(\mathbb{R}^3)^3$. The norm of g in $L^p(0,\tau; L^3(\mathbb{R}^3)^3)$ is bounded by the norm of $u\otimes v_r + w_r\otimes u$ in $L^p(0,\tau;L^3(\mathbb{R}^3)^{3\times 3})$. Indeed, the operator $(-\Delta)^{-\frac{1}{2}}\mathbb{P}\nabla\cdot$ is bounded from $L^3(\mathbb{R}^3)^{3\times 3}$ in $L^3(\mathbb{R}^3)^3$ and so

$$\begin{aligned}\|(-\Delta)^{1/2}Rg\|_{L^p(0,\tau;L^3)} &\leqslant c\,\|t\mapsto(-\Delta)^{1/2}e^{t\Delta}\|_{L^1(0,\tau;\mathcal{L}(L^3))}\,\|g\|_{L^p(0,\tau;L^3)}\\ &\lesssim \sqrt{\tau}\Big(\|v_r\|_{L^\infty((0,\tau)\times\mathbb{R}^3)^3} + \|w_r\|_{L^\infty((0,\tau)\times\mathbb{R}^3)^3}\Big)\|u\|_{L^p(0,\tau;L^3)}.\end{aligned}$$

Proceeding as in item (i) settles (3.4).

This establishes Proposition 3.2.

Remark 3.3. — Notice that if one assumes that v and w belong to the larger space $L^\infty(0,T;L^3(\mathbb{R}^3)^3)$, and if $r > 0$ is fixed, then in general one cannot ensure the existence of v_r and w_r in $L^\infty((0,T)\times\mathbb{R}^3)$ such that $\operatorname{ess\,sup}_{t\in(0,T)}\|v(t)-v_r(t)\|_{L^3} + \operatorname{ess\,sup}_{t\in(0,T)}\|w(t)-w_r(t)\|_{L^3} < 3r$. This is the case if, for example, v or w are of the form $t^{-1}\varphi(\cdot/t)$ with $\varphi\in L^3(\mathbb{R}^3)$ and r is small with respect to $\|\varphi\|_{L^3}$.

Proposition 3.4.

(1) *For all $\varepsilon > 0$, there exists $r > 0$ such that all v belonging to the space $C([0,T], L^3(\mathbb{R}^3)^3) + r\,L^\infty(0,T;L^3(\mathbb{R}^3)^3)$, there exists a positive time $\tau = \tau(\varepsilon, v) > 0$ such that*

$$C(v,\cdot) : L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3)) \longrightarrow L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))$$

is bounded with norm less than ε.

(2) *For all $\varepsilon > 0$ there exists $r > 0$ such that, for all v belonging to the space $C([0,T], L^3(\mathbb{R}^3)^3) + r\,L^\infty(0,T;L^3(\mathbb{R}^3)^3)$, there exists a positive time $\tau = \tau(\varepsilon, v) > 0$ such that*

$$C(v,\cdot) : L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3)) \cap L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3)) \longrightarrow L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3))$$

is bounded with norm less than ε.

(3) *For all* $\varepsilon > 0$ *all* $v \in L^4(0,T;L^6(\mathbb{R}^3))^3$, *there exists* $\tau = \tau(\varepsilon, v) > 0$ *such that*

$$C(v,\cdot) : L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3)) \longrightarrow L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3))$$

is bounded with norm less than ε.

(4) *For all* $\varepsilon > 0$, *all* $\vartheta \in L^2(0,T;L^{\frac{3}{2}}(\mathbb{R}^3))$, *there exists* $\tau = \tau(\varepsilon, \vartheta) > 0$ *such that*

$$C(\cdot,\vartheta) : L^4(0,\tau;L^6(\mathbb{R}^3)^3) \longrightarrow L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))$$

is bounded with norm less than ε.

Proof. — We proceed as in the previous proposition. For $r > 0$, let us choose $v_r \in C_c([0,T]\times\mathbb{R}^3)^3$ such that

$$\operatorname{ess\,sup}_{[0,T]} \|v - v_r\|_{L^3} \leq 2r. \tag{3.10}$$

Then we have, for all $\theta \in L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))$,

$$C(v,\theta) = C(v - v_r,\theta) + C(v_r,\theta) = \Delta R f + (-\Delta)^{1/2} R g$$

where, now, we set $f(s) = (-\Delta)^{-1}\operatorname{div}\big(\theta(s)(v(s) - v_r(s))\big)$ and $g(s) = -(-\Delta)^{-1/2}\operatorname{div}\big(\theta(s)v_r(s)\big)$. As in the proof of previous proposition we see that the norm of f in $L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))$ is bounded by the norm of $\big(\theta(v-v_r)\big)$ in $L^{\frac{4}{3}}(0,\tau;L^{\frac{6}{5}}(\mathbb{R}^3)^3)$ (because of the Sobolev embedding $\dot W^{1,\frac{6}{5}} \hookrightarrow L^2$ in dimension 3). As ΔR is a bounded operator in $L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))$, we have

$$\begin{aligned}
\|(-\Delta)Rf\|_{L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))} &\leq C_{\frac{4}{3},2}\, C\, \|\theta(v-v_r)\|_{L^{\frac{4}{3}}(0,\tau;L^{\frac{6}{5}}(\mathbb{R}^3)^3)}\\
&\leq 2rC_{\frac{4}{3},2}\, C\, \|\theta\|_{L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))}
\end{aligned}$$

where C is the constant arising from the Sobolev embedding $\dot W^{1,\frac{6}{5}} \hookrightarrow L^2$ in dimension 3 and r comes from the choice of v_r.

The norm of g in $L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))$ is bounded by the norm of θv_r dans $L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3)^3)$, because $(-\Delta)^{-\frac{1}{2}}\operatorname{div}$ is a bounded operator in $L^2(\mathbb{R}^3)^3$. Moreover, $\|(-\Delta)^{1/2}e^{t\Delta}\|_{\mathcal{L}(L^2(\mathbb{R}^3))} \lesssim t^{-1/2}$. Then, viewing $(-\Delta)^{1/2}R$ as before as a convolution operator, we get

$$\begin{aligned}
&\|(-\Delta)^{1/2}Rg\|_{L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))}\\
&\qquad\leq c\,\|t\mapsto(-\Delta)^{1/2}e^{t\Delta}\|_{L^1(0,\tau;\mathcal{L}(L^2(\mathbb{R}^3)))}\,\|g\|_{L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))}\\
&\qquad\leq c\,\tau^{\frac{1}{2}}\,\|\theta v_r\|_{L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3)^3)}\\
&\qquad\leq c\,\tau^{\frac{1}{2}}\,\|v_r\|_{L^\infty((0,\tau)\times\mathbb{R}^3)^3)}\,\|\theta\|_{L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))}.
\end{aligned}$$

We then choose $r > 0$ such that $2rC_{\frac{4}{3},2}C \leqslant \frac{\varepsilon}{2}$, next $v_r \in C_c([0,T]\times \mathbb{R}^3)^3$ satisfying (3.10) and last $\theta > 0$ such that $c\,\tau^{\frac{1}{2}}\|v_r\|_{L^\infty(((0,\tau)\times\mathbb{R}^3)^3)} \leqslant \frac{\varepsilon}{2}$. This establishes the first assertion of the proposition.

To prove the second assertion we proceed as before: for $r > 0$, we choose $v_r \in C([0,T]\times\mathbb{R}^3)^3$ such that (3.10) holds. Then for all $\theta \in L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3))$,

$$C(v,\theta) = C(v - v_r,\theta) + C(v_r,\theta) = (-\Delta)^{\frac{3}{4}}Rf + (-\Delta)^{1/2}Rg \tag{3.11}$$

where for $s \in [0,\tau]$,

$$f(s) = -(-\Delta)^{-\frac{3}{4}}\operatorname{div}\big(\theta(s)(v(s) - v_r(s))\big)$$

and

$$g(s) = -(-\Delta)^{-1/2}\operatorname{div}\big(\theta(s)v_r(s)\big).$$

We easily see that the norm of f in $L^{\frac{4}{3}}(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3))$ is controlled by the norm of $\big(\theta(v - v_r)\big)$ in $L^{\frac{4}{3}}(0,\tau;L^{\frac{6}{5}}(\mathbb{R}^3)^3)$ (owing to the Sobolev embedding $\dot{W}^{\frac{1}{2},\frac{6}{5}} \hookrightarrow L^{\frac{3}{2}}$ in dimension 3). As $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\frac{1}{4}}(-\Delta)^{\frac{3}{4}}R$ is a bounded operator in $L^{\frac{4}{3}}(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3))$, we have that

$$\begin{aligned}\|(-\Delta)^{\frac{3}{4}}Rf\|_{L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3))} &\leqslant \tilde{C}\left\|\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\frac{1}{4}}(-\Delta)^{\frac{3}{4}}Rf\right\|_{L^{\frac{4}{3}}(0,\tau;L^{\frac{3}{2}})}\\ &\leqslant \tilde{C}C_{\frac{4}{3},\frac{3}{2}}C\,\|\theta(v - v_r)\|_{L^{\frac{4}{3}}(0,\tau;L^{\frac{6}{5}}(\mathbb{R}^3)^3)}\\ &\leqslant \tilde{C}C_{\frac{4}{3},\frac{3}{2}}C\,2r\,\|\theta\|_{L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))}.\end{aligned} \tag{3.12}$$

Here $\tilde{C}$ is the constant coming from the Sobolev embedding $\dot{W}^{\frac{1}{4},\frac{4}{3}} \hookrightarrow L^2$ in dimension 1, C is the constant of the Sobolev embedding $\dot{W}^{\frac{1}{2},\frac{6}{5}} \hookrightarrow L^{\frac{3}{2}}$ in dimension 3 and r comes from the choice of v_r. The norm of g in $L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3))$ is controlled by the norm of θv_r in $L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3)^3)$, because $(-\Delta)^{-\frac{1}{2}}\operatorname{div}$ is a bounded operator from $L^{\frac{3}{2}}(\mathbb{R}^3)^3$ to $L^{\frac{3}{2}}(\mathbb{R}^3)$. As $(-\Delta)^{\frac{1}{2}}R$ is a convolution operator, we can write

$$\begin{aligned}\|(-\Delta)^{\frac{1}{2}}Rg\|_{L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3))} &\leqslant c\,\|t \mapsto (-\Delta)^{\frac{1}{2}}e^{t\Delta}\|_{L^1(0,\tau;\mathcal{L}(L^{\frac{3}{2}}(\mathbb{R}^3)))}\|g\|_{L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3))}\\ &\leqslant c\,\tau^{\frac{1}{2}}\|\theta v_r\|_{L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3)^3)}\\ &\leqslant c\,\tau^{\frac{1}{2}}\|v_r\|_{L^\infty((0,\tau)\times\mathbb{R}^3)^3)}\|\theta\|_{L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3))}.\end{aligned} \tag{3.13}$$

It just remains to choose $r > 0$ such that $2r\tilde{C}C_{\frac{4}{3},\frac{3}{2}}C \leqslant \frac{\varepsilon}{2}$, next $v_r \in L^\infty((0,T)\times\mathbb{R}^3)^3$ such that (3.10) holds and finally $\theta > 0$ such that

$c\,\tau^{\frac{1}{2}}\,\|v_r\|_{L^\infty((0,\tau)\times\mathbb{R}^3)^3)} \leqslant \frac{\varepsilon}{2}$. This establishes the second assertion of the proposition.

Let us prove the third assertion. As $v \in L^4(0,T;L^6(\mathbb{R}^3)^3)$, for an arbitrary $r>0$ we can choose now $v_r \in L^\infty((0,T)\times\mathbb{R}^3)^3$ such that

$$\|v - v_r\|_{L^4(0,T;L^6(\mathbb{R}^3)^3)} < r. \tag{3.14}$$

If $\tau > 0$ and $\theta \in L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3))$, then $(v-v_r)\theta \in L^{\frac{4}{3}}(0,\tau;L^{\frac{6}{5}}(\mathbb{R}^3)^3)$ by Hölder inequality. Therefore, splitting $C(v,\theta)$ as in (3.11), the above computations (3.12)–(3.13) can be reproduced: the only change that needs to be done is the application of (3.14) instead of (3.10). We get in this way

$$\|(-\Delta)^{\frac{3}{4}}Rf\|_{L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3))} \leqslant \tilde{C}C_{\frac{4}{3},\frac{3}{2}}\,C\,r\,\|\theta\|_{L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3))}.$$

This, combined with (3.13) proves our third assertion.

The proof of the fourth assertion follows the same scheme. For $r>0$, choose $\vartheta_r \in C_c([0,T]\times\mathbb{R}^3)$ such that

$$\|\vartheta - \vartheta_r\|_{L^2(0,T;L^{\frac{3}{2}}(\mathbb{R}^3))} \leqslant r. \tag{3.15}$$

Then, for all $v \in L^4(0,\tau;L^6(\mathbb{R}^3)^3)$, we have

$$C(v,\vartheta) = C(v,\vartheta-\vartheta_r) + C(v,\vartheta_r) = \Delta Rf + (-\Delta)^{1/2}Rg$$

where for $s \in [0,\tau]$,

$$f(s) = (-\Delta)^{-1}\operatorname{div}\big((\vartheta(s)-\vartheta_r(s))v(s)\big)$$

and

$$g(s) = -(-\Delta)^{-\frac{1}{2}}\operatorname{div}\big(\vartheta_r(s)v(s)\big).$$

One easily shows that the norm of f in $L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))$ is bounded by $Cr\,\|v\|_{L^4(0,\tau;L^6(\mathbb{R}^3)^3)}$, where C is the norm of $(-\Delta)^{-1}\operatorname{div}$ as an operator from $L^{6/5}(\mathbb{R}^3)^3$ to $L^2(\mathbb{R}^3)$. Thus, as the operateur ΔR is bounded on $L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))$ by $C'_{\frac{4}{3},2} = C\,C_{\frac{4}{3},2}$, we have that

$$\|\Delta Rf\|_{L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))} \leqslant C'_{\frac{4}{3},2}\,r\,\|v\|_{L^4(0,\tau;L^6(\mathbb{R}^3)^3)}.$$

The norm of g in $L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))$ is bounded by the norm of ϑ_r in the space $L^2(0,\tau;L^3(\mathbb{R}^3))$ and the norm of v in $L^4(0,\tau;L^6(\mathbb{R}^3)^3)$. As $(-\Delta)^{1/2}R$ is a convolution operator, we deduce that

$$\begin{aligned}
\|(-\Delta)^{\frac{1}{2}}Rg\|_{L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))}
&\leqslant \|t\mapsto(-\Delta)^{1/2}e^{t\Delta}\|_{L^1(0,\tau;\mathcal{L}(L^2(\mathbb{R}^3)))}\,\|g\|_{L^{\frac{4}{3}}(0,\tau;L^2(\mathbb{R}^3))}\\
&\leqslant c\,\tau^{\frac{1}{2}}\,\|\vartheta_r\|_{L^2(0,\tau;L^3(\mathbb{R}^3))}\,\|v\|_{L^4(0,\tau;L^6(\mathbb{R}^3)^3)}.
\end{aligned}$$

We then choose $r>0$ such that $C_{\frac{4}{3},2}\, r \leqslant \frac{\varepsilon}{2}$, next $\vartheta_r \in C_c([0,T]\times\mathbb{R}^3)$ satisfying (3.15) and last $\tau>0$ such that $c\tau^{\frac12}\|\vartheta_r\|_{L^2(0,\tau;L^3(\mathbb{R}^3))} \leqslant \frac{\varepsilon}{2}$. We thus get the last assertion of the proposition.

Proposition 3.5. — *For all $\tau>0$, the operator L defined by* (2.4) *is linear and bounded from $L^2(0,\tau;L^{\frac32}(\mathbb{R}^3))$ to $L^4(0,\tau;L^6(\mathbb{R}^3)^3$ and from $L^{\frac43}(0,\tau;L^2(\mathbb{R}^3))$ to $L^4(0,\tau;L^6(\mathbb{R}^3)^3)$, with operator norms independent on τ. Moreover, for all $p\in[1,\infty)$, L is bounded from $L^2(0,\tau;L^{\frac32}(\mathbb{R}^3))$ to $L^p(0,\tau;L^3(\mathbb{R}^3))$, with norm of order $\tau^{1/p}$.*

Proof. — For $\theta\in L^2(0,\tau;L^{\frac32}(\mathbb{R}^3))$, we write

$$L(\theta) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{-\frac14}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\frac14}(-\Delta)^{\frac34}R\phi\right],$$

where $\phi(s) = (-\Delta)^{-\frac34}\mathbb{P}\left(\theta(s)e_3\right)$. Observe that $\phi\in L^2(0,\tau;L^6(\mathbb{R}^3)^3)$, because of the Sobolev embedding $(-\Delta)^{-\frac34}(L^{\frac32})\hookrightarrow L^6$ (in dimension 3), with norm bounded by the norm of θ in $L^2(0,\tau;L^{\frac32}(\mathbb{R}^3))$. By Theorem 3.1, we deduce that $L(\theta)\in\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{-\frac14}\left(L^2(0,\tau;L^6(\mathbb{R}^3)^3)\right)\hookrightarrow L^4(0,\tau;L^6(\mathbb{R}^3)^3)$, the last inclusion arising from the Sobolev embedding $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{-\frac14}(L^2)\hookrightarrow L^4$ (in dimension 1). This establishes the first assertion of the proposition.

When $\theta\in L^{\frac43}(0,\tau;L^2(\mathbb{R}^3))$, we write

$$L(\theta) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{-\frac12}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\frac12}(-\Delta)^{\frac12}R\psi\right], \tag{3.16}$$

with $\psi(s) = (-\Delta)^{-\frac12}\mathbb{P}\left(\theta(s)e_3\right)$. Notice that $\psi\in L^{\frac43}(0,\tau;L^6(\mathbb{R}^3)^3)$, because of the Sobolev embedding $(-\Delta)^{-\frac12}L^2\hookrightarrow L^6$ (in dimension 3), with norm bounded by the norm of θ in $L^{\frac43}(0,\tau;L^2(\mathbb{R}^3))$. Applying Theorem 3.1 with $\alpha=\frac12$, we get $L(\theta)\in\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{-\frac12}\left(L^{\frac43}(0,\tau;L^6(\mathbb{R}^3)^3)\right)\hookrightarrow L^4(0,\tau;L^6(\mathbb{R}^3)^3)$. The last inclusion comes from the Sobolev embedding $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{-\frac12}(L^{\frac43})\hookrightarrow L^4$ (in dimension 1). The second assertion of the proposition follows.

Next, for $\theta\in L^2(0,\tau;L^{\frac32}(\mathbb{R}^3))$, let us write $L(\theta)$ as before in (3.16). By Sobolev embedding $(-\Delta)^{-\frac12}(L^{\frac32})\hookrightarrow L^3$ in dimension 3, we have $\psi\in L^2(0,\tau;L^3)$ with norm bounded by $\|\theta\|_{L^2(0,\tau;L^{\frac32})}$. By Theorem 3.1 with $\alpha=\frac12$, we deduce that $L(\theta)\in\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{-\frac12}\left(L^2(0,\tau;L^3)\right)\hookrightarrow L^p(0,\tau;L^3)$ for all $1\leqslant p<\infty$. The last inclusion follows, for $2<p<\infty$, from the Hölder injection $L^2((0,\tau))\hookrightarrow L^q(0,\tau)$ for $q\in[1,2]$ (with norm $\tau^{1/q-1/2}$) and Hardy–Littlewood–Sobolev inequality $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{-\frac12}(L^q)\to L^p$ (in dimension 1), for all $p \in (2,\infty)$ and $\frac{1}{p}=\frac{1}{q}-\frac{1}{2}$. For $1 \leqslant p \leqslant 2$ it is sufficient to apply once more Hölder inequality.

4. The proof of the uniqueness

We need a few lemmas before proving Theorem 2.1.

Lemma 4.1. — *Let $(u_0,\theta_0) \in \mathcal{S}'(\mathbb{R}^3)^3 \times \mathcal{S}'(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$, and let $(u,\theta) \in L^\infty(0,T;L^3(\mathbb{R}^3)^3) \times L^2(0,T;L^{\frac{3}{2}}(\mathbb{R}^3))$ be a mild solution of (2.1) with initial data (u_0,θ_0). Then*

$$(u,\theta) \in C([0,T],\mathcal{S}'(\mathbb{R}^3)^3 \times \mathcal{S}'(\mathbb{R}^3)).$$

Moreover, we have $u_0 \in L^3(\mathbb{R}^3)^3$ and for every $t \in [0,T]$, $u(t)$ does also belong to $L^3(\mathbb{R}^3)^3$.

Proof. — Let us denote by $F(t,x)$ the kernel of the operator $e^{t\Delta}\mathbb{P}\nabla\cdot$. It is well known, and easy to check, that F satisfies the scaling relations $F(t,x) = t^{-2}F(1,x/\sqrt{t})$, with $F(1,\cdot) \in (L^1(\mathbb{R}^3)\cap C_0(\mathbb{R}^3))^{3\times3}$, where $C_0(\mathbb{R}^3)$ is the notation for continuous functions on $\mathbb{R}^3$ which go to 0 at infinity. From these properties and the dominated convergence theorem one deduces that, for all $1 \leqslant p \leqslant \infty$, that $F \in C(0,\infty;L^p(\mathbb{R}^3))$. Moreover, $\|F(t,\cdot)\|_1 = t^{-\frac{1}{2}}\|F(1,\cdot)\|_1$.

Now, if $(u,\theta) \in X_{T,r}$, then $u\otimes u \in L^\infty(0,T;L^{3/2}(\mathbb{R}^3)^{3\times3})$. Then, recalling the definition of the bilinear operator B and applying the above properties of F with $p=1$, next applying the L^1-$L^{3/2}$ convolution inequality, shows that the map $t \mapsto B(u,u)(t)$ is continuous from $(0,T]$ to $L^{3/2}(\mathbb{R}^3)^3$. Moreover, $\|B(u,u)(t)\|_{L^{3/2}} \to 0$ as $t \to 0$. Hence, the map $t \mapsto B(u,u)(t)$ is continuous from $[0,T]$ to $L^{3/2}(\mathbb{R}^3)^3$ with value 0 at $t=0$.

Let us now consider $L(\theta)$. Using the fact that the heat kernel

$$(t,x) \mapsto \frac{1}{(4\pi t)^{3/2}}\, e^{-\frac{|x|^2}{4t}}$$

is in $C_b(0,\infty;L^1(\mathbb{R}^3))$, we readily see that $L(\theta) \in C((0,T];L^{3/2}(\mathbb{R}^3))$. To study the behavior of $L(\theta)$ near $t=0$ we consider $\phi \in \mathcal{S}(\mathbb{R}^3)$ and observe, computing the Fourier transform of $\mathbb{P}\theta e_3$ with respect to the space variable, that $t \mapsto \hat{h}(t,\cdot) = \widehat{\mathbb{P}\theta e_3}(t,\cdot)$ belongs to $L^2(0,T;L^3(\mathbb{R}^3))$ by the Hausdorff–Young theorem. Then we have

$$\begin{aligned}
|\langle L(\theta)(t),\phi\rangle| &\leqslant \int_0^t |\langle \hat{h}(s), e^{-(t-s)|\cdot|^2}\hat{\phi}\rangle|\,ds \leqslant \int_0^t \|\hat{h}(s)\|_{L^3}\|\hat{\phi}\|_{L^{3/2}}\,ds \\
&\leqslant \|\hat{\phi}\|_{L^{3/2}}\int_0^t \|\theta(s)\|_{L^{3/2}}\,ds \leqslant C_\phi\|\theta\|_{L^2(0,T;L^{3/2}(\mathbb{R}^3))}\sqrt{t}.
\end{aligned}$$

Therefore, $L(\theta)(t) \to 0$ as $t \to 0$ in $\mathcal{S}'(\mathbb{R}^3)$ and we deduce that $L(\theta) \in C([0,T],\mathcal{S}'(\mathbb{R}^3)^3)$, with value 0 at $t = 0$.

Let us now consider $C(u,\theta)$. We have $u\theta \in L^2(0,T;L^1(\mathbb{R}^3)^3)$. Moreover, the kernel of the operator $e^{t\Delta}\mathbb{P}\nabla\cdot$ has the same scaling properties as F. Therefore, proceeding as for $B(u,u)$ we see on the one hand that $C(u,\theta) \in C((0,T];L^1(\mathbb{R}^3)^3)$. On the other hand, we can also write

$$C(u,\theta)(t) = \operatorname{div} \int_0^t e^{(t-s)\Delta}(u\theta)\mathrm{d}s.$$

But the $L^1(\mathbb{R}^3)$-norm of $\int_0^t e^{(t-s)\Delta}(u\theta)\mathrm{d}s$ is obviously bounded by the quantity $\sqrt{t}\,\|u\theta\|_{L^2(0,T;L^1(\mathbb{R}^3))}$ that goes to zero as $t \to 0$. Hence,

$$C(u,\theta)(t) \xrightarrow[t\to 0]{} 0 \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^3)^3,$$

by the continuity of the divergence operator from L^1 to $\mathcal{S}'$.

For the linear terms a and b it is obvious that they both belong to the space $C([0,T],\mathcal{S}'(\mathbb{R}^3))$, with values at $t = 0$ given by u_0 and θ_0, respectively.

Summarising, from the equation (2.1) we see that

$$(u,\theta) \in C([0,T],\mathcal{S}'(\mathbb{R}^3)^3 \times \mathcal{S}'(\mathbb{R}^3)),$$

with values at $t = 0$ given by (u_0,θ_0). But $u \in L^\infty(0,T;L^3(\mathbb{R}^3)^3)$, hence, for all $0 \leqslant t \leqslant T$, we can find a sequence $t_n \xrightarrow[n\to\infty]{} t$, contained in $[0,T]$, such that $u(t_n) \in L^3(\mathbb{R}^3)^3$ for all $n \in \mathbb{N}$, with L^3-norm uniformly bounded by $\|u\|_{L^\infty(0,T;L^3(\mathbb{R}^3)^3)}$, and $u(t_n) \xrightarrow[n\to\infty]{} u(t)$ in $\mathcal{S}'(\mathbb{R}^3)^3$. By duality we deduce that $u(t) \in L^3(\mathbb{R}^3)^3$ for every $t \in [0,T]$. In particular, the initial velocity u_0 must belong to $L^3(\mathbb{R}^3)^3$.

Lemma 4.2. — *There exists an absolute constant $r_0 > 0$ such that if $0 \leqslant r < r_0$ and $(u,\theta) \in X_{T,r}$ is a solution of* (2.1)*, with $(u_0,\theta_0) \in \mathcal{S}'(\mathbb{R}^3)^3 \times \mathcal{S}'(\mathbb{R}^3)$ and $\operatorname{div} u_0 = 0$, then there exists $\tau > 0$ such that $u \in L^4(0,\tau;L^6(\mathbb{R}^3)^3)$.*

Proof. — Let us take $p = 2$ throughout this proof (any other choice $1 < p < \infty$ would do: a different choice of p would just affect the value of r_0 and τ). We know, by Proposition 3.2, that there exists r_0 and $\tau > 0$ such that if $(u,\theta) \in X_{T,r}$ with $0 \leqslant r < r_0$, then the norm of the linear operator $B(\cdot,u)$ from $L^4(0,\tau;L^6(\mathbb{R}^3)^3) \cap L^p(0,\tau;L^3(\mathbb{R}^3)^3)$ to itself is bounded, with norm smaller than $\frac{1}{2}$. This shows that $\mathrm{Id} - B(\cdot,u)$ is invertible in $L^4(0,\tau;L^6(\mathbb{R}^3)^3) \cap L^p(0,\tau;L^3(\mathbb{R}^3)^3)$.

Moreover, as $\theta \in L^2(0,\tau;L^{\frac{3}{2}}(\mathbb{R}^3))$ by our assumption, we get from Proposition 3.5 that $L(\theta) \in L^4(0,\tau;L^6(\mathbb{R}^3)^3) \cap L^p(0,\tau;L^3(\mathbb{R}^3)^3)$.

As observed in the previous Lemma, we have $u_0 \in L^3(\mathbb{R}^3)^3$. Moreover, $L^3(\mathbb{R}^3) \subset \dot B^0_{3,3}(\mathbb{R}^3) \subset \dot B^{-1/2}_{6,3}(\mathbb{R}^3) \subset \dot B^{-1/2}_{6,4}(\mathbb{R}^3)$. See [2, Chap. 2] for generalities on Besov spaces. The characterisation of Besov spaces through the heat kernel (see [2, Theorem 2.34]) then implies that $t \mapsto e^{t\Delta} u_0 \in L^4(0,\tau; L^6(\mathbb{R}^3)^3)$. Since we have also that $t \mapsto e^{t\Delta} u_0 \in \mathcal{C}([0,\tau]; L^3(\mathbb{R}^3)^3)$, we obtain

$$a \in L^4(0,\tau; L^6(\mathbb{R}^3)^3) \cap L^p(0,\tau; L^3(\mathbb{R}^3)^3), \qquad \text{for all } 1 < p < \infty.$$

These considerations allow us to define

$$\tilde u = \big(\mathrm{Id} - B(\cdot, u)\big)^{-1}\big(a + L(\theta)\big).$$

We would like to show that $u = \tilde u$. By the assumption on u and the construction of $\tilde u$, these two functions satisfy

$$u = B(u,u) + a + L(\theta) \quad \text{and} \quad \tilde u = B(\tilde u, u) + a + L(\theta).$$

Moreover, $\tilde u \in L^4(0,\tau; L^6(\mathbb{R}^3)^3) \cap L^p(0,\tau; L^3(\mathbb{R}^3)^3)$. Their difference $v := u - \tilde u$ satisfies

$$v \in L^p(0,\tau; L^3(\mathbb{R}^3)^3) \quad \text{and} \quad v = B(v,u).$$

Reducing (if necessary) the value of τ, we deduce from the last point of Proposition 3.2 that

$$\|v\|_{L^p(0,\tau,L^3)} \leqslant \frac{1}{2}\|v\|_{L^p(0,\tau;L^3(\mathbb{R}^3)^3)}.$$

This implies that $v = 0$ in $L^p(0,\tau; L^3(\mathbb{R}^3)^3)$. In particular, $u = \tilde u \in L^4(0,\tau; L^6(\mathbb{R}^3)^3)$.

Remark 4.3. — Under the conditions of Lemma 4.1, the initial temperature θ_0 must belong to the inhomogeneous Besov space $B^{-1}_{3/2,2}(\mathbb{R}^3)$. Indeed, $\theta \in L^2(0,\tau; L^{\frac32}(\mathbb{R}^3))$, and $C(u,\theta)$ then belongs to this same space by the third claim of Proposition 3.4 and the previous lemma, for $\tau > 0$ small enough. Then, by the second equation of (2.1), we obtain $b \in L^2(0,\tau; L^{\frac32}(\mathbb{R}^3))$. The characterisation of inhomogeneous Besov spaces with negative regularity (see [16, Theorem 5.3]) then immediately gives $\theta_0 \in B^{-1}_{3/2,2}(\mathbb{R}^3)$.

Lemma 4.4. — *Let $0 \leqslant r < r_0$ and (u_1,θ_1) and (u_2,θ_2) be two mild solutions of* (B) *in $X_{T,r}$ arising from $(u_0,\theta_0) \in \mathcal{S}'(\mathbb{R}^3)^3 \times \mathcal{S}'(\mathbb{R}^3)$, with* $\operatorname{div} u_0 = 0$. *Let also $\theta = \theta_1 - \theta_2$. Then there exists $\tau > 0$ such that $\theta \in L^{\frac43}(0,\tau; L^2(\mathbb{R}^3))$.*

Proof. — Let $u = u_1 - u_2$. Then $(u, \theta) \in X_{T,r}$ satisfies (3.2). We know by Lemma 4.2 that there exists $\tau_0 > 0$ such that $u_1, u_2 \in L^4(0, \tau_0; L^6(\mathbb{R}^3)^3)$. Applying the last two assertions of Proposition 3.4 we get $C(u, \theta_2) \in L^{\frac{4}{3}}(0, \tau_0; L^2(\mathbb{R}^3)) \cap L^2(0, \tau_0; L^{\frac{3}{2}}(\mathbb{R}^3))$. The first and the second assertions of Proposition 3.4 ensure the existence of $\tau > 0$ (we can assume $\tau \leq \tau_0$) such that $C(u_1, \cdot)$ is bounded from $L^{\frac{4}{3}}(0, \tau; L^2(\mathbb{R}^3)) \cap L^2(0, \tau; L^{\frac{3}{2}}(\mathbb{R}^3))$ to itself, with norm less than $\frac{1}{2}$. Therefore we can define

$$\widetilde{\theta} = \left(\mathrm{Id} - C(u_1, \cdot)\right)^{-1}(C(u, \theta_2)).$$

We see that $\widetilde{\theta} \in L^{\frac{4}{3}}(0, \tau; L^2(\mathbb{R}^3)) \cap L^2(0, \tau; L^{\frac{3}{2}}(\mathbb{R}^3))$, and moreover $\widetilde{\theta} = C(u_1, \widetilde{\theta}) + C(u, \theta_2)$. Let $\psi = \theta - \widetilde{\theta}$. Then, subtracting the second equation in (3.2), we obtain

$$\psi \in L^2(0, \tau; L^{\frac{3}{2}}(\mathbb{R}^3)) \quad \text{and} \quad \psi = C(u_1, \psi).$$

But $u_1 \in L^4(0, \tau; L^6(\mathbb{R}^3)^3)$ by Lemma 4.2. Hence, applying the third assertion of Proposition 3.4 we get $\psi = 0$ and so $\theta = \widetilde{\theta}$. The latter equality implies that $\theta \in L^{\frac{4}{3}}(0, \tau; L^2(\mathbb{R}^3))$.

Proof of Theorem 2.1. Let $r_0 > 0$ be the absolute constant determined in Lemma 4.2. Assume that (u_1, θ_1) and (u_2, θ_2) are two mild solutions in $X_{T,r}$ of (B), with $0 \leq r < r_0$, starting from $(u_0, \theta_0) \in \mathcal{S}'(\mathbb{R}^3)^3 \times \mathcal{S}'(\mathbb{R}^3)$. In fact, by Lemma 4.1, there is no restriction in assuming that $u_0 \in L^3(\mathbb{R}^3)^3$.

Then, setting $u = u_1 - u_2$ and $\theta = \theta_1 - \theta_2$, the couple (u, θ) satisfies (3.2). As $\theta_1, \theta_2 \in L^2(0, T; L^{\frac{3}{2}}(\mathbb{R}^3))$ by our assumption and the first item of Proposition 3.5, we know that $L(\theta) \in L^4(0, T; L^6(\mathbb{R}^3)^3)$. Applying Proposition 3.2, we know that there exists $\tau > 0$ such that $\|B(u, u_1) + B(u_2, u)\|_{L^4(0,\tau;L^6(\mathbb{R}^3)^3)} \leq \frac{1}{2} \|u\|_{L^4(0,\tau;L^6(\mathbb{R}^3)^3)}$. This allows us to show, applying Lemma 4.2, next using the first equation in (3.2), that

$$\|u\|_{L^4(0,\tau;L^6(\mathbb{R}^3)^3)} \leq 2 \|L(\theta)\|_{L^4(0,\tau;L^6(\mathbb{R}^3)^3)}.$$

Applying the first assertion of Proposition 3.4, with $v = u_1$ and the last assertion of Proposition 3.4 with $\vartheta = \theta_2$, we deduce from the second equality in (3.2) that, for all $\varepsilon > 0$, there exists $0 < \tau' \leq \tau$ such that

$$\|\theta\|_{L^{\frac{4}{3}}(0,\tau';L^2(\mathbb{R}^3))} \leq \varepsilon \left(\|\theta\|_{L^{\frac{4}{3}}(0,\tau';L^2(\mathbb{R}^3))} + \|u\|_{L^4(0,\tau';L^6(\mathbb{R}^3)^3)}\right).$$

But the $L^{\frac{4}{3}}(0, \tau'; L^2(\mathbb{R}^3))$-norm of θ is finite by Lemma 4.4, so

$$\|\theta\|_{L^{\frac{4}{3}}(0,\tau';L^2(\mathbb{R}^3))} \leq \frac{\varepsilon}{1 - \varepsilon} \|u\|_{L^4(0,\tau';L^6(\mathbb{R}^3)^3)}.$$

The second item of Proposition 3.5 allows us to take $\varepsilon > 0$ such that

$$2 \frac{\varepsilon}{1 - \varepsilon} \|L\|_{\mathcal{L}(L^{4/3}(0,\tau';L^2(\mathbb{R}^3)), L^4(0,\tau';L^6(\mathbb{R}^3)^3))} < 1.$$

We conclude that $u = 0$ in $L^4(0,\tau;L^6(\mathbb{R}^3)^3)$. This implies that $\theta = 0$ *a.e.* on $(0,\tau)$. The uniqueness is thus established at least during a short time interval $[0,\tau)$, for a suitable $0 < \tau \leqslant T$.

A standard argument now allows us to deduce that the uniqueness holds, in fact, in the whole interval $[0,T]$: let τ^* be the supremum of the times $t_0 \in [0,T]$ such that $(u_1,\theta_1) = (u_2,\theta_2)$ in $X_{t_0,r}$. Let us show that $\tau^* = T$. Indeed, otherwise, by the continuity of (u_1,θ_1) and (u_2,θ_2) from $[0,T]$ to $C([0,T],\mathcal{S}'(\mathbb{R}^3)^3 \times \mathcal{S}'(\mathbb{R}^3))$, established in Lemma 4.1, we deduce that

$$(u_1(\tau^*),\theta_1(\tau^*)) = (u_2(\tau^*),\theta_2(\tau^*)) \in \mathcal{S}'(\mathbb{R}^3)^3 \times \mathcal{S}'(\mathbb{R}^3). \tag{4.1}$$

But $(u_1,\theta_1)(\cdot+\tau^*)$ and $(u_2,\theta_2)(\cdot+\tau^*)$ are mild solutions of (B) in $X_{T-\tau^*,r}$, with initial data given by (4.1). Therefore, applying the uniqueness result in short-time intervals established before, we see that there exists τ^{**}, such that $0 < \tau^{**} < T - \tau^*$, and $(u_1,\theta_1)(\cdot+\tau^*) = (u_2,\theta_2)(\cdot+\tau^*)$ in $X_{\tau^{**},r}$. Then $(u_1,\theta_1) = (u_2,\theta_2)$ in $X_{\tau^*+\tau^{**},r}$ and this would contradict the definition of τ^*. The uniqueness is thus granted in the whole interval $[0,T]$.

5. Existence

Let us prove Theorem 2.2, that ensures the existence of solution in the space where we obtained the uniqueness. In fact, an existence theorem for solutions to the Boussinesq system was established in [3], under assumptions more general than that of Theorem 2.2. However, the solution constructed in [3] a priori does not satisfy the required condition on the temperature, $\theta \in L^2(0,T;L^{\frac32}(\mathbb{R}^3))$. Therefore, what remains to do in order to establish Theorem 2.2, is to show that the solution constructed in [3] does satisfy such condition, as soon as the initial temperature does belong to $\dot B^{-1}_{3/2,2}(\mathbb{R}^3)$.

For this, let us introduce some useful function spaces: For $1 \leqslant p \leqslant \infty$ and $0 < T \leqslant \infty$, we define $Z_{p,T}$ to be the subspace of all vector fields $u \in L^1_{\rm loc}(0,T;L^p(\mathbb{R}^3)^3)$ such that

$$\|u\|_{Z_{p,T}} = \operatorname{ess\,sup}_{t\in(0,T)} t^{\frac12(1-\frac3p)} \|u(t)\|_p < \infty.$$

In the same way, let $Y_{q,T}$ be the subspace of all the functions θ belonging to $L^1_{\rm loc}(0,T;L^q(\mathbb{R}^3))$ such that

$$\|\theta\|_{Y_{q,T}} = \operatorname{ess\,sup}_{t\in(0,T)} t^{\frac32(1-\frac1q)} \|\theta(t)\|_q < \infty.$$

We will need the following bilinear estimate:

Proposition 5.1. — *For all $u \in Z_{6,T}$ and $\theta \in L^2(0,T;L^{\frac{3}{2}}(\mathbb{R}^3))$,*

$$\|C(u,\theta)\|_{L^2(0,T;L^{\frac{3}{2}}(\mathbb{R}^3))} \leqslant \kappa\|u\|_{Z_{6,T}}\|\theta\|_{L^2(0,T;L^{\frac{3}{2}}(\mathbb{R}^3))},$$

where $\kappa > 0$ is some constant independent on T, u and θ.

Proof. — Using that $\|u(s)\|_{L^6} \leqslant s^{-1/4}\|u\|_{Z_{6,T}}$ and letting

$$f(s) = s^{-1/4}\|\theta(s)\|_{L^{3/2}}\mathbf{1}_{\mathbb{R}^+}(s), \quad s \in [0,T],$$

we can estimate

$$\|C(u,\theta)(t)\|_{L^{3/2}} \leqslant c\|u\|_{Z_{6,T}}\int_0^t (t-s)^{-3/4}f(s)\mathrm{d}s.$$

Here c is the $L^{\frac{6}{5}}(\mathbb{R}^3)$-norm of the kernel of $\mathrm{e}^{\Delta}\,\mathrm{div}$. But $f \in L^{\frac{4}{3},2}(\mathbb{R})$ by Hölder inequality in Lorentz spaces, with norm controlled by the norm of θ in $L^2(0,T;L^{\frac{3}{2}}(\mathbb{R}^3))$, independently of T. Moreover, $|\cdot|^{-3/4} \in L^{\frac{4}{3},\infty}(\mathbb{R})$, hence $t \mapsto \|C(u,\theta)(t)\|_{L^{3/2}}\mathbf{1}_{\mathbb{R}^+}(t)$ belongs to $L^{2,2}(\mathbb{R}) = L^2(\mathbb{R})$ by Young–O'Neil inequality (see [16, Theorem 2.3]).

Let us now recall the local existence theorem in [3, Theorem 2.4].

Theorem 5.2 (See [3]). — *If $3 < p < \infty$, $\frac{3}{2} < q < 3$ and $\frac{2}{3} < \frac{1}{p} + \frac{1}{q}$, and if (u_0, θ_0) belongs to the closure of the Schwartz class $\mathcal{S}(\mathbb{R}^3)^3 \times \mathcal{S}(\mathbb{R}^3)$ in the space $B^{-(1-3/p)}_{p,\infty}(\mathbb{R}^3)^3 \times B^{-3(1-1/q)}_{q,\infty}(\mathbb{R}^3)$, with $\mathrm{div}\, u_0 = 0$, then there exists $T > 0$ and a solution (u,θ) to (B) such that*

$$(u,\theta) \in \big(Z_{p,T} \cap Z_{\infty,T}\big) \times \big(Y_{q,T} \cap Y_{\infty,T}\big).$$

Moreover,

$$\|u\|_{Z_{p,T}\cap Z_{\infty,T}} + \|\theta\|_{Y_{q,T}\cap Y_{\infty,T}} \xrightarrow[T\to 0]{} 0.$$

Furthermore, if $u_0 \in L^3(\mathbb{R}^3)^3 \cap B^{-(1-3/p)}_{p,\infty}$, then $u \in C([0,T], L^3(\mathbb{R}^3)^3)$ and if $\theta_0 \in L^1(\mathbb{R}^3) \cap B^{-3(1-1/q)}_{q,\infty}$ then $\theta_0 \in C([0,T], L^1(\mathbb{R}^3))$.

Let us observe that the perturbation method used in [3] to establish Theorem 5.2 provides the well-posedness only in the space where the solution is constructed.

Proof of Theorem 2.2. — Under the assumptions of the first item of Theorem 2.2, we have $u_0 \in L^3(\mathbb{R}^3)^3$ and $\theta_0 \in B^{-1}_{\frac{3}{2},2}(\mathbb{R}^3)$, which is continuously embedded in $B^{-3(1-1/q)}_{q,\infty}$, for all $q > 3/2$. Moreover, the Schwartz class is dense both in L^3 and in Besov spaces with finite third index. Therefore we may apply Theorem 5.2. Choosing, for example, $p = 6$ and $q = 2$ we obtain the existence, for some $T > 0$, of a solution $(u, \theta) \in Z_{6,T} \times Y_{2,T}$, such that $u \in C([0, T], L^3(\mathbb{R}^3)^3)$ and

$$\|u\|_{Z_{6,T}} + \|\theta\|_{Y_{2,T}} \xrightarrow[T \to 0]{} 0. \tag{5.1}$$

By the Boussinesq equation (2.1), we have $\theta = b + C(u, \theta)$. Moreover, by the heat kernel characterisation of Besov spaces, we deduce from the condition $\theta_0 \in B^{-1}_{\frac{3}{2},2}(\mathbb{R}^3)$, that $b = e^{t\Delta}\theta_0 \in L^2(0, T; L^{\frac{3}{2}}(\mathbb{R}^3)^3)$. Now, reducing if necessary the length of time interval where the solution is considered, we can assume that T is such that $\kappa\|u\|_{Z_{6,T}} < 1$. Hence, by Proposition 5.1, we see that the linear operator $C(\cdot, u)\colon L^2(0, T; L^{\frac{3}{2}}(\mathbb{R}^3)^3) \to L^2(0, T; L^{\frac{3}{2}}(\mathbb{R}^3)^3)$ is bounded with norm less than 1. Therefore, the operator $\mathcal{T} := I - C(\cdot, u)$ is invertible in such space. But $\mathcal{T}(\theta) = b$, hence $\theta = \mathcal{T}^{-1}(b) \in L^2(0, T; L^{\frac{3}{2}}(\mathbb{R}^3)^3)$.

Let us prove the second assertion of Theorem 2.2. If θ_0 belongs to the homogeneous Besov space $\dot{B}^{-1}_{3/2,2}(\mathbb{R}^3)$, then $b \in L^2(0, \infty; L^{\frac{3}{2}}(\mathbb{R}^3))$. Moreover, the norm of b is controlled by $\|\theta_0\|_{\dot{B}^{-1}_{3/2,2}(\mathbb{R}^3)}$ (and conversely). If $\|u_0\|_{L^3} + \|\theta_0\|_{\dot{B}^{-1}_{3/2,2}(\mathbb{R}^3)}$ is smaller than a suitable absolute constant (or, more in general, if $\|u_0\|_{\dot{B}^{-(1-3/p)}_{p,\infty}(\mathbb{R}^3)} + \|\theta_0\|_{\dot{B}^{-3(1-1/q)}_{q,\infty}(\mathbb{R}^3)}$ is smaller than a constant depending only on p and q, where p and q are as in Theorem 5.2), then the estimates in [3] provide the global existence of the solution, with $u \in C_b(0, \infty; L^3(\mathbb{R}^3))$. Moreover, $\|u\|_{Z_{6,\infty}}$ is controlled by the size of the initial data (u_0, θ_0) in $L^3(\mathbb{R}^3)^3 \times \dot{B}^{-1}_{3/2,2}(\mathbb{R}^3)$. Therefore, $\kappa\|u\|_{Z_{6,\infty}}$ can be assumed to be smaller than 1. Then the above argument applies with $T = +\infty$. This completely establishes Theorem 2.2.

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Lorenzo BRANDOLESE
Institut Camille Jordan
Université de Lyon, Université Lyon 1
43 bd. du 11 Novembre
69622 Villeurbanne Cedex (France)
brandolese@math.univ-lyon1.fr

Sylvie MONNIAUX
Aix Marseille Univ, CNRS, I2M
Marseille (France)
sylvie.monniaux@univ-amu.fr