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LOGARITHMIC *p*-BASES AND ARITHMETICAL DIFFERENTIAL MODULES

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ABSTRACT. — We introduce the notion of log *p*-smoothness which weakens that of log-smoothness and that of having locally *p*-bases. We extend Berthelot's theory of arithmetic \mathcal{D} -modules in this context, in particular on the construction of the sheaf of differential operators and its properties.

RÉSUMÉ. — Nous introduisons la notion de log *p*-lissité. Cette notion étend à la fois celle de lissité logarithmique et celle d'avoir localement des *p*-bases. Nous vérifions que la théorie des \mathcal{D} -modules arithmétiques de Berthelot se généralise bien dans ce cadre, notamment sur la construction du faisceau des opérateurs différentiels et de ses propriétés.

Introduction

Berthelot's theory of arithmetic \mathcal{D} -modules in the context of varieties over a perfect field k of characteristic p > 0 gives a p-adic cohomology satisfying properties analogous to that in Grothendieck's theory of l-adic étale cohomology. For instance, we have the stability under six operations of the overholonomicity with Frobenius structures (see [9]) and a theory of weights (see [1]) which are respectively the analogue of Grothendieck's stability of constructible l-adic sheaves and of Deligne's theory of weights of constructible l-adic complexes. However, we also need a theory of arithmetic \mathcal{D} -modules in a wider context. For instance, nearby cycles or vanishing cycles require to work over certain schemes which are not varieties over k (e.g. the spectrum of a henselian ring). In relation to the p-adic local monodromy theorem (see [2, 23, 26]), Crew studied the case of k[[t]]/k(e.g. he proved the holonomicity of F-isocrystals on the bounded Robba

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ring in [10]). More recently, D. Pigeon in [29] extended the construction of Berthelot's sheaf of differential operators in the context of schemes having locally *p*-bases. Roughly speaking, his generalization relies on replacing the notion of etaleness by relative perfectness. For instance, since k[t]/k[t]is relatively perfect (for details, see Example 1.39), Pigeon's context also "englobes" Crew's case k[t]/k. In fact, to be more precise, the objects in Crew's construction live over Spf $\mathcal{V}[t]$ endowed with the (p, t)-adic topology whereas in Pigeon's context (and in our work here) we only consider the padic topology. In other words, objects are living over the topological space of Spec k[t] in Pigeon or our context whereas in Crew's one they are living over that of Spec k. Taking global sections, one might consider some comparison theorems between both contexts; but this is not the purpose of the paper. On the other hand, it would also be useful (e.g. to be able to use Kedlaya's semistable reduction theorem ([24]) to work with log schemes as in C. Montagnon's thesis (see [27]). This is the goal of the present paper where we introduce the notion of $\log p$ -smoothness that generalizes at the same time the notions of having p-bases locally and of log smoothness. We extend in this work Berthelot's construction of arithmetic \mathcal{D} -modules in this context which generalizes Berthelot, Montagnon and Pigeon's one. Over Laurent series fields, it would be interesting to compare our constructions with the theory of rigid cohomology as developed by C. Lazda and A. Pál (see [25]), which would be a stimulating continuation of the present paper. Moreover, we might hope to check some finiteness properties by using the framework of arithmetic \mathcal{D} -modules.

Let us describe the content of the paper. We introduce the notion of weakly log smooth morphism $X \to S$ (see Definition 1.70). This notion is the most general one we could find so that the usual sheaf of differential operators $\mathcal{D}_{X/S}$ is locally free as \mathcal{O}_X -module (and with finite "differential variables") (see Definition 2.9). Similarly, we introduce the notion of weakly log smooth morphism of level m (see Definition 1.73) which is the most general one we find so that Berthelot's sheaf of differential operators of level m of X/S, denoted by $\mathcal{D}_{X/S}^{(m)}$, is locally free as \mathcal{O}_X -module and finitely generated as an \mathcal{O}_X -ring (see Point 2.29). This latter geometrical notion is a generalization of the notions of (log-)étaleness, (log) basis, (log) smoothness and relative perfectness see (Propositions 1.25 and 1.74), which yields a unification of theses notions. By definition, a morphism of log schemes $X \to S$ is weakly log smooth of level m if, étale locally on X, f has formal log bases of level m. This latter notion of formal log basis of level m is related with Tsuji's notion of p-basis defined in [31, 1.4] (see Remark 1.72). We also introduce the notion of a log *p*-smooth morphism of formal log schemes which is a more restrictive notion than that of weakly log smooth morphism of level *m*, for any integer *m* (see Point 3.30). This yields that, when $\mathfrak{X} \to \mathcal{S} := \operatorname{Spf} \mathcal{V}$ is a log *p*-smooth morphism of log formal \mathcal{V} -schemes, then Berthelot's sheaf of differential operators of finite level and infinite orders of \mathfrak{X}/\mathcal{S} , denoted by $\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathcal{S}}$, has the usual description (see Point 3.35).

Moreover, when $\mathfrak{X} \to S$ is a log *p*-smooth morphism of log formal \mathcal{V} schemes such that $\underline{\mathfrak{X}}$ has no *p*-torsion, we put a canonical structure of right $\mathcal{D}_{\mathfrak{X}/S}^{\dagger}$ -module on $\omega_{\mathfrak{X}/S}$ (see Proposition 3.39). A standard elegant proof of this fact needs Grothendieck's extraordinary pullback as defined in [19]. Since we do not know a suitable generalization of Grothendieck's extraordinary pullback in our context, we construct the canonical right $\mathcal{D}_{\mathfrak{X}/S}^{\dagger}$ -module structure on $\omega_{\mathfrak{X}/S}$ by glueing local computations using logarithmic transposition. We conclude by defining extraordinary pull-backs, pushforwards and duals similarly to Berthelot in his theory of arithmetic \mathcal{D} -modules.

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Convention, notation of the paper

Let \mathcal{V} be a complete discrete valued ring of mixed characteristic $(0, p), \pi$ be a uniformizer, K its field of fractions, k its residue field which is supposed to be perfect. If $X \to Y$ is a morphism of log schemes, we denote by $\underline{f}: \underline{X} \to \underline{Y}$ the underlying morphism of schemes. A scheme means a log schemes endowed with the trivial log structure. By convention, a fine log scheme X is noetherian means that \underline{X} is noetherian as a scheme. A fs log scheme means a fine saturated log scheme. The formal scheme Spf \mathcal{V} is meant for the *p*-adic topology. A formal \mathcal{V} -scheme means a *p*-adic formal scheme over Spf \mathcal{V} . When we say "etale locally" this means we use Definition IV.6.3 of [11] of the etale topology (see also [11, Proposition IV.6.3.1(iv)] for an alternative definition). In the category of fine log schemes, replacing the morphisms of schemes by the strict morphisms of fine log schemes, we get the similar notion of étale locality (see Remark 1.7(3)). Unless otherwise stated fiber products of fine log schemes (resp. fine formal log schemes) are always computed in the category of fine log schemes (resp. fine formal log schemes). The sheaves of monoids are by convention sheaves for the étale topology. Notice that since we are working with fine log schemes, from [28, III.1.4.1 and III.1.4.4], the reader who prefers to work with sheaves of monoids for the Zarisky topology can switch to it without any problem. Remark 2.17 should convince the reader why it is too pathological to work with coherent log structures instead of fine log structures.

1. Log *p*-étaleness, relative log perfection, *m*-PD envelope

Let *i* be an integer and *S* be a fine log scheme over the scheme $\operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$.

1.1. Formal log etaleness, nth infinitesimal neighborhood

In this subsection, the reader might notice that we could consider the case where $S = \operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ without loss of generality. But, we will need in Section 1.4 the case where S is any fine log scheme over $\operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$. In order to keep notation as homogenous as possible, S will remain any fine log scheme over $\operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$.

Notation 1.1. — If X is a log scheme over $\operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$, then for any integer $0 \leq k \leq i$ we put $X_k := X \times_{\operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})} \operatorname{Spec}(\mathbb{Z}/p^{k+1}\mathbb{Z})$. If $f : X \to Y$ is a morphism of $\operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ -log schemes, then for any integer $0 \leq k \leq i$ we set $f_k : X_k \to Y_k$ the morphism canonically induced by f.

Let \mathcal{I} be a quasi-coherent sheaf (for the Zariski topology) on a log scheme X. The preasheaf which associates $\mathcal{I}(U)^{(n)}$ (the subideal of $\mathcal{I}(U)$ generated by *n*th powers of elements of $\mathcal{I}(U)$) to an affine open set U of X is a quasi-coherent sheaf. We denote it by $\mathcal{I}^{(n)}$. Similarly, using [5, A.1.5.(ii)], when \mathcal{I} is endowed with an *m*-PD structure, we get a quasi-coherent sheaf $\mathcal{I}^{\{n\}_{(m)}}$ (which depends on the *m*-PD structure of \mathcal{I}) such that for an affine open set U of X, $\mathcal{I}^{\{n\}_{(m)}}(U) = \mathcal{I}(U)^{\{n\}_{(m)}}$.

If X is a fine log scheme over \mathbb{F}_p , we denote by $F_X \colon X \to X$ the absolute Frobenius of X as defined by Kato in [21, 4.7].

DEFINITION 1.2. — Let $u: Z \to X$ be a morphism of log-schemes.

- (1) According to [28, II.1.1.12], we say that u is an immersion (resp. closed immersion) if \underline{u} is an immersion (resp. a closed immersion) of schemes and if $u^*M_X \to M_Z$ is surjective (here u^* means the pullback of log structures [21, 1.4]). An immersion (resp. a closed immersion) is exact if and only if $u^*M_X \to M_Z$ is an isomorphism.
- (2) We say that u is an open immersion if u is an exact immersion such that <u>u</u> is an open immersion of schemes.
- (3) Let n be an integer. A "log thickening of order (n)" (resp. "log thickening of order n") is an exact closed immersion u: U → T such that I⁽ⁿ⁾ = 0 (resp. such that Iⁿ⁺¹ = 0), where I is the ideal associated with the closed immersion <u>u</u>. The convention of the respective case is that of [28, IV.2.1.1] and is convenient when we are dealing with n-infinitesimal neighborhood.
- (4) Let a ∈ N. A "(p)-nilpotent log thickening of order a" is a log thickening of order (p^{a+1}). A "(p)-nilpotent log thickening" is a (p)-nilpotent log thickening of order a for some a ∈ N large enough.

An S-immersion (resp. S-log thickening) is an immersion (resp. log thickening) which is an S-morphism.

Remark 1.3.

- (1) If $u: Z \to X$ and $f: X \to Y$ are two S-morphisms of log schemes such that $f \circ u$ is an S-immersion, then so is u (use [13, 5.3.13]).
- (2) If $u: Z \to X$ and $f: X \to Y$ are two S-morphisms of log schemes such that $f \circ u$ is a closed S-immersion and <u>f</u> is separated, then u is a closed immersion (use [13, 5.4.4]).
- (3) We can decompose a (strict) S-immersion u into $u = u_1 \circ u_2$, where u_1 is an open S-immersion and u_2 is a (strict) closed S-immersion.
- (4) Let $u: U \hookrightarrow T$ be an S-log thickening of order (p^a) for some integer a. Since p is nilpotent in \mathcal{O}_T , by applying finitely many times the functor $\mathcal{I} \mapsto \mathcal{I}^{(p)}$ to the ideal defined by <u>u</u> we obtain the zero ideal, which justifies the definition of "(p)-nilpotent log thickening". This also implies that u is the composition of several S-log thickenings of order (p).

Definition 1.4.

(1) We denote by \mathscr{C} the category whose objects are S-immersions of fine log-schemes and whose morphisms $u' \to u$ are commutative

diagrams of the form

(1.1) $\begin{array}{c} X' \xrightarrow{f} X \\ u' & u \\ Z' \longrightarrow Z. \end{array}$

We say that $u' \to u$ is strict (resp. flat, resp. cartesian) if f is strict (resp. f is flat, resp. the square (1.1) is cartesian).

- (2) Let n ∈ N. We denote by C_(n) (resp. C_n, resp. Thick_(p)) the full subcategory of C whose objects are S-log thickenings of order (n) (resp. S-log thickening of order n, resp. (p)-nilpotent S-log thickenings).
- (3) Let u be an object of C. A "log thickening of order (n) (resp. of order n) induced by u" is an object u' of C_(n) (resp. C_n) endowed with a morphism u' → u of C satisfying the following universal property: for any object u" of C_(n) (resp. C_n) endowed with a morphism f: u" → u of C there exists a unique morphism u" → u' of C_(n) (resp. C_n) whose composition with u' → u is f. The unicity up to canonical isomorphism of the log thickening of order (n) (resp. of order n) induced by u is obvious. We will denote by P⁽ⁿ⁾(u) (resp. Pⁿ(u)) the log thickening of order (n) (resp. of order n) induced by u. We also say that Pⁿ(u) is the "nth infinitesimal neighbourhood of u" (see [21, 5.8]). The existence is checked below (see Propositions 1.11 and 1.28).
- (4) We denote by C^{sat} (resp. C^{sat}_(n), resp. C^{sat}_n, resp. Thick^{sat}_(p)) the full subcategory of C (resp. C_(n), resp. C_n, resp. Thick_(p)) whose objects are also morphisms of fs log-schemes.

Remark 1.5.

- (1) If $u' \to u$ is a strict cartesian morphism of \mathscr{C} with $u \in \mathscr{C}_{(n)}$ (resp. $u \in \mathscr{C}_n$, resp. $u \in \mathscr{T}$ hick $_{(p)}$), then $u' \in \mathscr{C}_{(n)}$ (resp. $u' \in \mathscr{C}_n$, resp. $u' \in \mathscr{T}$ hick $_{(p)}$). Indeed, the corresponding square of the form (1.1) of $u' \to u$ remains cartesian after applying the forgetful functor from the category of fine log schemes to the category of schemes (to check this fact, we need a priori the strictness of $u' \to u$).
- (2) The category \mathscr{C} has fibered products. More precisely, let $u: Z \hookrightarrow X$, $u': Z' \hookrightarrow X', u'': Z'' \hookrightarrow X''$ be some objects of \mathscr{C} ; let $u' \to u$ and $u'' \to u$ be two morphisms of \mathscr{C} . Then $u' \times_u u''$ is the immersion $Z' \times_Z Z'' \hookrightarrow X' \times_X X''$. If $u' \to u$ is moreover cartesian, then so is the projection $u' \times_u u'' \to u''$.

In order to be precise, let us clarify the standard definitions.

DEFINITION 1.6. — Let $f: X \to Y$ be an S-morphism of fine log schemes.

(1) We say that f is "fine formally log étale" (resp. "fine formally log unramified") if it satisfies the following property: for any commutative diagram of fine log schemes of the form

(1.2)
$$U \xrightarrow{u_0} X \\ \int_{\iota} \bigvee_{v} \int_{V} f \\ T \xrightarrow{v} Y$$

such that ι is an object of \mathscr{C}_1 , there exists a unique morphism (resp. there exists at most one morphism) $u: T \to X$ such that $u \circ \iota = u_0$ and $f \circ u = v$.

- (2) Replacing C₁ by C₁^{sat} we get the notion of "fs formally log étale" morphism and of "fs formally log unramified") morphism.
- (3) We say that f is "log étale" if f is fine formally log étale and if \underline{f} is locally of finite presentation.
- (4) We say that f is "étale" if f is log étale and strict (which is equivalent to saying that f is étale and f is strict).

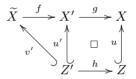
Remark 1.7.

- (1) The definitions appearing in Definition 1.6 (or Definition 1.14) do not depend on the choice of the fine log scheme S over Z/pⁱ⁺¹Z. More precisely, let f: X → Y be an S-morphism of fine log schemes. Then we can consider f as a Spec(Z/pⁱ⁺¹Z)-morphism of fine log schemes. We notice that the properties satisfied by f can be checked equivalently when S is equal to Spec(Z/pⁱ⁺¹Z) (i.e. we replace S by Spec(Z/pⁱ⁺¹Z) in the corresponding categories defined in Definition 1.4).
- (2) Let f: X → Y be an S-morphism of fine log schemes. The notion of etaleness of Kato appearing in [21, 3.3] is what we have defined as "log etaleness". We distinguish by definition "log etale" from "etale" morphisms in order to avoid confusion when we say for instance "etale locally".
- (3) There exists in the literature a notion of etale morphism of log schemes with coherent log structures (see in Ogus's book at [28, IV.3.1.1]). This notion is compatible with Kato's notion of etale morphism of fine log schemes. Indeed, both notions have the same characterization when we focus on morphisms of fine log schemes (see resp. [21, Theorem 3.5] and [28, Theorem IV.3.3.1]).

To avoid confusion with the etale notion in the classical sense, we will call such a morphism a "log etale" morphism of log schemes with coherent log structures (instead of "etale morphism").

Moreover, let $f: X \to Y$ be a morphism of log schemes with coherent log structures. From [28, IV.3.1.11], $X^{\text{int}} \to X$ and $Y^{\text{int}} \to Y$ are log etale (see [28, III.2.1.5.1] concerning the functor $X \mapsto X^{\text{int}}$). Hence, using Remark IV.3.1.2 of [28], we can check that f is log etale if and only if f^{int} is log etale.

Point 1.8. — We recall in the paragraph how we can exactify an immersion. Let $u: Z \hookrightarrow X$ be an S-immersion of fine log-schemes. Let \overline{z} be a geometric point of Z. Using the proof of [21, 4.10.1] and Proposition IV.18.1.1 of [18], we can check that there exists a commutative diagram of the form



such that the square is cartesian, f is log étale, \underline{f} is affine, g is étale, v' is an exact closed S-immersion and h is an étale neighborhood of \overline{z} in Z.

LEMMA 1.9. — Let $u' \to u$ be a strict cartesian morphism of \mathscr{C} . Suppose that the $P^{(n)}(u)$ (resp. $P^n(u)$), the log thickening of order (n) (resp. of order n) induced by u, exists. Then the log thickening of order (n) (resp. of order n) induced by u' exists and we have $P^{(n)}(u') = P^{(n)}(u) \times_u u'$ (resp. $P^n(u') = P^n(u) \times_u u'$).

Proof. — Using Remark 1.5, since $u' \to u$ is strict and cartesian, then so is the projection $P^{(n)}(u) \times_u u' \to P^{(n)}(u)$ and then $P^{(n)}(u) \times_u u' \in \mathscr{C}_{(n)}$. Hence, we can check easily that $P^{(n)}(u) \times_u u'$ endowed with the projection $P^{(n)}(u) \times_u u' \to u'$ satisfies the corresponding universal property of $P^{(n)}(u')$. We can check similarly the respective case.

LEMMA 1.10. — Let $n \in \mathbb{N}$, $f: X \to Y$ be a fine formally log étale morphism of fine log S-schemes, $u: Z \hookrightarrow X$ and $v: Z \hookrightarrow Y$ be two Simmersions of fine log schemes such that $v = f \circ u$. If $P^n(u)$ exists, then $P^n(v)$ exists and we have $P^n(u) = P^n(v)$.

Proof. — Abstract nonsense.

PROPOSITION 1.11. — For any integer n, the inclusion functor For_n : $\mathscr{C}_n \to \mathscr{C}$ has a right adjoint functor which we will denote by $P^n \colon \mathscr{C} \to \mathscr{C}_n$.

Let $u: Z \hookrightarrow X$ be an object of \mathscr{C} . Then Z is also the source of $P^n(u)$. Moreover, denoting abusively by $P^n(u)$ the target of the arrow $P^n(u)$, the underlying morphism of schemes of $P^n(u) \to X$ is affine. We denote by $\mathcal{P}^n(u)$ the quasi-coherent \mathcal{O}_X -algebra so that $\underline{P^n(u)} = \operatorname{Spec}(\mathcal{P}^n(u))$. If X is noetherian, then so is $P^n(u)$.

Proof. — The construction of P^n is given in [21, 5.8]. Since Proposition 1.11 is slightly more precise than the existence of P^n , for the reader convenience, let us give a detailled proof. Let $u: Z \hookrightarrow X$ be an S-immersion of fine log-schemes. Using Lemma 1.9, the existence of $P^n(u)$ (and then the whole proposition) is étale local on X (i.e. following our convention, this is local for the Zariski topology and we can proceed by descent of a finite covering with étale quasi-compact morphisms). Hence, by Point 1.8, we may thus assume that there exists a commutative diagram of the form



such that f is log étale, \underline{f} is affine and \widetilde{u} is an exact closed S-immersion. Let \mathcal{I} be the ideal defined by \widetilde{u} . Let $P^n \hookrightarrow \widetilde{X}$ be the exact closed immersion which is induced by \mathcal{I}^{n+1} . Using Lemma 1.10, we can check that $P^n(u)$ is the exact closed immersion $Z \hookrightarrow P^n$. When X is noetherian, then so are \widetilde{X} and P^n .

LEMMA 1.12. — Let $u \to v$ be a morphism of \mathscr{C} . Let $w := P^n(v) \times_v u$ (this is the product in \mathscr{C}). Then $P^n(w)$ and $P^n(u)$ are isomorphic in \mathscr{C}_n .

Proof. — We can easily check that the composition $P^n(w) \to w \to u$ satisfies the universal property of $P^n(u) \to u$. Hence, we are done.

PROPOSITION 1.13. — Let $f: X \to Y$ be an S-morphism of fine log schemes and $\Delta_{X/Y}: X \hookrightarrow X \times_Y X$ (as always the product is taken in the category of fine log schemes) be the diagonal S-immersion. The following assertions are equivalent:

- (1) the morphism $P^1(\Delta_{X/Y})$ is an isomorphism;
- (2) the morphism f is fine formally log unramified;
- (3) the morphism f is formally log unramified (this notion is defined at [28, IV.3.1.1]).

Proof. — Following [28, IV.3.1.3], the last two assertions are equivalent. Moreover, by definition, if f is formally log unramified then f is fine formally log unramified. Using [21, 5.8] and with Notation 2.5, the property $\Omega_{X/Y} = 0$ is equivalent to say that $P^1(\Delta_{X/Y})$ is an isomorphism. Copying the proof of "if f is formally log unramified then $\Omega_{X/Y} = 0$ " of [28, IV.3.1.3] we can check in the same way that if f is fine formally log unramified then $\Omega_{X/Y} = 0$ (indeed, since X fine, then the log scheme $T := X \oplus \Omega_{X/Y}$ is fine because its log structure is $M_X \oplus \Omega_{X/Y}$: see [28, IV.2.1.5]).

1.2. Log *p*-étaleness

DEFINITION 1.14. — Let $f: X \to Y$ be an S-morphism of fine log schemes.

(1) We say that f is "log p-étale" (resp. "log p-unramified") if it satisfies the following property: for any commutative diagram of fine log schemes of the form

(1.3)
$$U \xrightarrow{u_0} X$$
$$\int_{\iota} \bigvee_{\iota} \bigvee_{f} f$$
$$T \xrightarrow{v} Y$$

such that ι is an object of $\mathscr{C}_{(p)}$, there exists a unique morphism (resp. there exists at most one morphism) $u: T \to X$ such that $u \circ \iota = u_0$ and $f \circ u = v$.

- (2) Replacing $\mathscr{C}_{(p)}$ by $\mathscr{C}_{(p)}^{\text{sat}}$ we get the notion of "fs log *p*-étale" and of "fs log *p*-unramified".
- (3) Replacing "fine log S-schemes" by "S-schemes" in (1), we get the notion of "p-étale" (resp. "p-unramified") morphism of schemes.

Remark 1.15. — With Remark 1.3(4) in mind, we can replace $\mathscr{C}_{(p)}$ by \mathscr{T} hick_(p) (resp. $\mathscr{C}_{(p)}^{\text{sat}}$ by \mathscr{T} hick^{sat}_(p)) in the definition of log p-étale or log p-unramified (resp. fs log p-étale or fs log p-unramified).

Moreover, since $\mathscr{C}_{(p)}$ contains \mathscr{C}_1 , then a log *p*-étale (resp. a log *p*-unramified) morphism is fine formally étale (resp. fine formally unramified). Since $\mathscr{C}_{(p)}^{\text{sat}}$ contains $\mathscr{C}_1^{\text{sat}}$, then a fs formally log *p*-étale (resp. fs formally log *p*unramified) is fs formally log étale (resp. fs formally log unramified). We have similar remarks without logarithmic structures, e.g. a *p*-étale morphism is an étale morphism. LEMMA 1.16. — Let $f: X \to Y$ be an S-morphism of fine log-schemes. Then f is fs log p-étale (resp. fs formally log etale, resp. fs log p-unramified, resp. fs formally log unramified) if and only if so is f^{sat} .

Proof. — This is checked by using the fact that the functor $X \mapsto X^{\text{sat}}$ is a right adjoint of the inclusion functor from the category of fs log schemes to the category of fine log schemes (see [28, III.2.1.5]).

LEMMA 1.17. — Let $f: X \to Y$ be a strict S-morphism of fine log schemes, $\underline{f}: \underline{X} \to \underline{Y}$, $f^{\text{sat}}: X^{\text{sat}} \to Y^{\text{sat}} \underline{f}^{\text{sat}}: \underline{X}^{\text{sat}} \to \underline{Y}^{\text{sat}}$ be the induced morphisms (see [28, III.2.1.5] concerning the functor $X \mapsto X^{\text{sat}}$).

- (1) The morphism f is log p-étale if and only if f is p-étale.
- (2) The morphism f is fs log p-étale if and only if f^{sat} is p-étale.

Proof. — If f is log p-étale then this is straightforward that \underline{f} is p-étale (from a diagram of the form (1.3) in the category of schemes, use the diagram of fine log schemes with strict morphisms which is induced by base change with $Y \to \underline{Y}$). To check the converse, using the fact that any morphism $u: T \to Y$ of fine log schemes factorizes uniquely of the form $T \to T' \to Y$ where $T' \to Y$ is strict and $\underline{T} = \underline{T}'$, we reduce to check the universal property of lop p-étaleness in the case where the morphisms of the diagram (1.3) are strict, which is clear. The second part of the Lemma is proved similarly.

LEMMA 1.18. — Let $f: X \to Y$ be a strict S-morphism of fine log schemes. We suppose \underline{f} is locally of finite presentation. In that case, the morphism f is log p-étale if and only if f is étale.

Proof. — Since f is strict, following Lemma 1.17, then a morphism is log p-étale if and only if \underline{f} is p-étale. Moreover, since f is strict, then f is étale if and only if \underline{f} is p-étale. Hence, we reduce to check \underline{f} is p-étale and locally of finite presentation if and only if \underline{f} is étale. Since a p-étale morphism is formally étale (see Remark 1.15), since a formally étale morphism locally of finite presentation is étale we get one implication. Let us check the converse. Since this is Zariski local on Y, we can suppose Y affine. In that case, Y is a filtrant projective limit of schemes of finite type over \mathbb{Z} . Since f is locally of finite presentation, then using [17, 8.8.2.(ii)], we reduce to the case where Y is noetherian. Since this is Zariski local on X, we can also suppose X affine and f is of finite type. In particular X is also noetherian. In that case, we can easily check the f satisfies the universal property of p-étaleness if f satisfies that of étaleness.

LEMMA 1.19. — Let $f: X \to Y$ and $g: Y' \to Y$ be two S-morphisms of fine log-schemes. Set $X' := X \times_Y Y'$ in the category of fine log schemes and $f': X' \to Y'$ the projection. If f is log p-étale (resp. fine formally log étale, resp. log étale, resp. fine formally log unramified, resp. fs log p-étale, resp. fs formally log etale, resp. fs formally log unramified), then so is f'.

 \square

Proof. — Abstract nonsense and standard.

LEMMA 1.20. — Let $f: X \to Y$ be an S-morphism of fine log-schemes and $f_0: X_0 \to Y_0$ be the induced S_0 -morphism. The morphism f is log p-étale (fs log p-étale) if and only if f is fine formally log etale (resp. fs formally log étale) and f_0 is log p-étale (resp. fs log p-étale). Similarly replacing everywhere "étale" by "unramified".

Proof. — If f is log p-étale then by definition f is fine formally log etale and by using Lemma 1.19 f_0 is log p-étale. Conversely, suppose that f is fine formally log etale and f_0 is log p-étale. Let

$$(1.4) \qquad \begin{array}{c} U \xrightarrow{u} X \\ & \swarrow \\ & \downarrow \\ T \xrightarrow{w} Y \end{array}$$

be a commutative diagram of fine log schemes such that ι is an object of $\mathscr{C}_{(p)}$. Since f_0 is log *p*-étale, there exists a unique morphism $v_0: T_0 \to X_0$ such that $v_0 \circ \iota_0 = u_0$ and $f_0 \circ v_0 = w_0$ (recall f_0, w_0, ι_0, u_0 mean the reduction modulo *p*). Let $\alpha_T: T_0 \to T, \alpha_X: X_0 \to X$, and $\alpha_U: U_0 \to U$ be the canonical nilpotent exact closed immersions. Since α_T is a nilpotent exact closed immersion, since *f* is in particular fine formally log etale then there exists a unique morphism $v: T \to X$ such that $v \circ \alpha_T = \alpha_X \circ v_0$ and $f \circ v = w$. Since $\alpha_U: U_0 \to U$ is a nilpotent exact closed immersion, since *f* is in particular fine formally unramified, since $(v \circ \iota) \circ \alpha_U = u \circ \alpha_U$, and $f \circ (v \circ \iota) = f \circ u$, then $v \circ \iota = u$. Hence, the morphism $v: T \to X$ is such that $f \circ v = w$ and $v \circ \iota = u$. To check the uniqueness of such morphism $v: T \to X$, since f_0 is log *p*-unramified then v_0 , the reduction of such v, is unique. Since *f* is in particular fine formally unramified, then such v is unique. The unramified or fs cases are checked similarly.

LEMMA 1.21. — Let $f: X \to Y$ and $g: Y \to Z$ be two S-morphisms of fine log schemes. The morphisms f and g are log p-étale (resp. fine formally log etale, resp. log étale, resp. fs log p-étale, resp. fs formally log etale) if and only if so are $g \circ f$ and g.

Proof. — Abstract nonsense and standard.

LEMMA 1.22. — Let $f: X \to Y$ and $g: X' \to X$ be two S-morphisms of fine log-schemes such that g is étale, quasi-compact and surjective. The morphism f is log p-étale (resp. fine formally log etale, resp. log étale, resp. fs log p-étale, resp. fs formally log etale) if and only so is $f \circ g$.

Proof. — First, let us prove the non respective case. From Lemma 1.21, since an étale morphism is log *p*-étale (see Lemma 1.18), we can check that if f is log *p*-étale then so is $f \circ g$. Conversely, suppose $f \circ g$ is log *p*-étale. Let

$$(1.5) \qquad \begin{array}{c} U \xrightarrow{u_0} X \\ & \swarrow \\ & \downarrow \\ T \xrightarrow{v} Y \end{array}$$

be a commutative diagram of fine log schemes such that ι is an object of $\mathscr{C}_{(p)}$. Put $U' := U \times_X X'$ and $u'_0 : U' \to X'$ the morphism induced from u_0 by base change by q. Since the projection $q': U' \to U$ is étale, using Theorem [18, IV.18.1.2], there exists a unique (up to isomorphisms) étale morphism $h: T' \to T$ such that we have an isomorphism of the form $U' \xrightarrow{\sim} T' \times_T U$. Let $\iota' \colon U' \hookrightarrow T'$ be the projection. Since $f \circ g$ is log pétale, there exists a unique morphism $u': T' \to X'$ such that $u' \circ \iota' = u'_0$ and $f \circ g \circ u' = v \circ h$. Set $T'' := T' \times_T T', U'' := U' \times_U U'$. Let $p_1 : T'' \to T'$, and $p_2: T'' \to T'$ (resp. $p_1: U'' \to U'$, and $p_2: U'' \to U'$) be respectively the left and right projections. Let $\iota'' := \iota' \times \iota' : U'' \hookrightarrow T''$. Since $f \circ q \circ (u' \circ p_1) =$ $f \circ g \circ (u' \circ p_2), (u' \circ p_1) \circ \iota'' = (u' \circ p_2) \circ \iota'', \text{ and since } f \circ g \text{ is log } p\text{-ramified},$ we get $u' \circ p_1 = u' \circ p_2$ and then $(g \circ u') \circ p_1 = (g \circ u') \circ p_2$. Since $T' \to T$ is a strict epimorphism, this yields that there exists a unique morphism $u: T \to X$ such that $q \circ u' = u \circ h$. Since $u \circ \iota \circ q' = u_0 \circ q'$, since q' is étale and surjective, then $u \circ \iota = u_0$. Since $f \circ u \circ h = v \circ h$, since h is étale and surjective, then $f \circ u = v$. We conclude that f is log p-étale. The respective cases are checked similarly. \square

LEMMA 1.23. — Let $f: X \to Y$ and $g: Y' \to Y$ be two S-morphisms of fine log-schemes such that g is étale, quasi-compact and surjective. Set $X' := X \times_Y Y'$ in the category of fine log schemes and $f': X' \to Y'$ the projection. The morphism f is log p-étale (resp. fine formally log etale, resp. log étale, resp. fs log p-étale, resp. fs formally log etale) if and only so is f'.

Proof. — Since the respective cases are similar (for both last respective cases, remark also that using Lemma 1.16 we can replace fiber products in the category of fine log schemes by fiber products the category of fs log

schemes), let us only prove the non respective case. From Lemma 1.19, if f is log p-étale then so is f'. Conversally, suppose that f' is log p-étale. Let

$$(1.6) \qquad \begin{array}{c} U \xrightarrow{u_0} X \\ & \swarrow \\ & \downarrow \\ T \xrightarrow{v} Y \end{array}$$

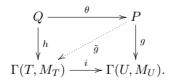
be a commutative diagram of fine log schemes such that ι is an object of $\mathscr{C}_{(p)}$. Put $T' := T \times_Y Y', U' := U \times_Y Y'$, and let $\iota' : U' \hookrightarrow T', u'_0 : U' \to U'$ $X', v': T' \to Y'$ be the morphism induced respectively from ι, u_0, v by base change by q. Since f' is log p-étale, there exists a unique morphism $w': T' \to X'$ such that $w' \circ \iota' = u'_0$ and $f' \circ w' = v'$. Set $Y'' := Y' \times_Y Y'$, $T'' := T \times_Y Y'', U'' := U \times_Y Y'',$ and let $\iota'': U' \hookrightarrow T', u_0': U'' \to X'',$ $v'': T'' \to Y'', f'': X'' \to Y''$ be the morphism induced respectively from ι, u_0, v, f by base change by $Y'' \to Y$. Let $p_1: Y'' \to Y', p_2: Y'' \to Y'$ (resp. $p_1: X'' \to X', p_2: X'' \to X'$, resp. $p_1: T'' \to T', p_2: T'' \to T'$) be the left and right projections. Since f'' is log *p*-étale, there exists a unique morphism $w'': T'' \to X''$ such that $w'' \circ \iota'' = u_0''$ and $f'' \circ w'' = v''$. Since f' is log p-étale, by using the cube whose front square is (1.6) with primes, whose back square is (1.6) with double primes, whose horizontal morphism from the back square toward the front square are the first projections p_1 , we can check $w' \circ p_1 = p_1 \circ w''$. Similarly, we can check $w' \circ p_2 = p_2 \circ w''$. Hence, putting $u' := q \circ w' \colon T' \to X$, we get $u' \circ p_1 = u' \circ p_2$. Since $T' \to T$ is a strict epimorphism, we conclude that there exists a unique morphism $u: T \to X$ such that u' is the composition of u with $T' \to T$. Since $U' \to U$ (resp. $T' \to T$) is étale and surjective then $u \circ \iota = u_0$ (resp. $f \circ u = v$). Hence, f is log p-étale. \square

LEMMA 1.24. — Let $u: \mathbb{Z} \hookrightarrow X$ be an object of $\mathscr{T}hick_{(p)}$ and \mathcal{I} be the ideal defined by the closed immersion \underline{u} . Then $1 + \mathcal{I}$ is a subgroup of \mathcal{O}_X^* . Let n be an integer prime to p. The homomorphism $1 + \mathcal{I} \to 1 + \mathcal{I}$ of groups defined by $x \mapsto x^n$ is an isomorphism. Moreover, if \underline{X} is affine then for any q > 0, we have the vanishing $H^q(X, 1 + \mathcal{I}) = 0$.

Proof. — For N large enough, $\mathcal{I}^{(p^N)} = 0$ and $p^N = 0$ in \mathcal{O}_S . Hence, for any local section y of \mathcal{I} , we compute $(1+y)^{p^N} = 1$. This yields that $1 + \mathcal{I}$ is a subgroup of \mathcal{O}_X^* . Since n be an integer prime to p, then there exists $m \in \mathbb{N}$ such that $nm \equiv 1 \mod p^N$. Hence, the homomorphism $1 + \mathcal{I} \to 1 + \mathcal{I}$ given by $x \mapsto x^n$ is an isomorphism whose inverse function is given by $x \mapsto x^m$. It remains to check the last statement. Suppose \underline{X} affine. The ideal $I := \Gamma(X, \mathcal{I})$ is the filtrant inductive limit of its subideal of finite type. Since $\mathcal{I}^{(p^N)} = 0$, then I is the filtrant inductive limit of nilpotent subideals. Since the canonical morphism $\mathcal{O}_X \otimes_{\Gamma(X,\mathcal{O}_X)} I \to \mathcal{I}$ is an isomorphism, this yields that \mathcal{I} is a filtrant inductive limit of nilpotent quasi-coherent ideals of \mathcal{O}_X . Hence, $1 + \mathcal{I}$ is a filtrant inductive limit of sheaves of the form $1 + \mathcal{J}$, where \mathcal{J} is a nilpotent quasi-coherent ideal of \mathcal{O}_X . Using [12, VI.5.1], to check the last statement, we can suppose that \mathcal{I} is nilpotent. Let $N \ge 2$ be so that $\mathcal{I}^N = 0$. By induction on N, we reduce to the case N = 2 as follows. Let $v: X' \hookrightarrow X$ be the exact closed immersion given by \mathcal{I}^{N-1} . Then $Z \hookrightarrow X'$ is given by $\overline{\mathcal{I}} := v^{-1}(\mathcal{I}/\mathcal{I}^{N-1})$. By induction hypothesis, for any q > 0, we have the vanishing $H^q(X', 1 + \overline{\mathcal{I}}) = 0$. Since $X' \hookrightarrow X$ is an homeomorphism, this yields for any q > 0, we have the vanishing $H^q(X, 1 + \mathcal{I}/\mathcal{I}^{N-1}) = 0$. Using again the induction hypothesis, for any q > 0, we have the vanishing $H^q(X, 1 + \mathcal{I}^{N-1}) = 0$. Using the exact sequence, $1 \to 1 + \mathcal{I}^{N-1} \to 1 + \mathcal{I} \to 1 + \mathcal{I}/\mathcal{I}^{N-1} \to 1$, for any q > 0, we get the vanishing $H^q(X, 1 + \mathcal{I}) = 0$. Hence, we can suppose $\mathcal{I}^2 = 0$. In that case, we have the isomorphism of groups $(\mathcal{I}, +) \to (1 + \mathcal{I}, \times)$ given by $y \mapsto 1 + y$. Since \mathcal{I} is quasi-coherent, using [14, 1.3.1], we are done.

PROPOSITION 1.25. — Let $f: X \to Y$ be a log étale S-morphism of fine log-schemes. Then f is log p-étale.

Proof. — Following Lemmas 1.22 and 1.23 the log *p*-étaleness is étale local in both X and Y. Hence, using [21, Theorem 3.5], we reduce to the case where $X = A_P$, $Y = A_Q$ and where there exists a chart of f subordinate to a morphism $\phi: Q \to P$ of fine monoids (see the definition [28, II.2.4.1]) such that the kernel and cokernel of ϕ^{gp} is finite of order prime to p (i.e. is invertible in $\mathbb{Z}/p^{i+1}\mathbb{Z}$). Let $\iota: U \hookrightarrow T$ be an object of $\mathscr{C}_{(p)}$. A morphism $\iota \to f$ can be thought of as commutative diagram



We need to check the existence and unicity of a map $\tilde{g}: P \to \Gamma(T, M_T)$ such that $i \circ \tilde{g} = g$ and $\tilde{g} \circ \theta = h$. Since this is locally étale, we can suppose T affine. Following [28, IV.2.1.2.4], the natural map $M_T \to M_T^{\text{gr}} \times_{M_U} M_U^{\text{gr}}$ is an isomorphism. Moreover, the morphism $M_T \hookrightarrow M_T^{\text{gr}}$ is injective. Hence, we reduce to check there exists a unique morphism $\tilde{g}: P^{\text{gr}} \to \Gamma(T, M_T^{\text{gr}})$

making commutative the diagram

where h', g' are the morphism canonically induced from h, g and where i' is induced from the map $M_T^{\rm gr} \to M_U^{\rm gr}$. By [28, IV.2.1.2.2], we have $1 + \mathcal{I} = \ker(M_T^{\rm gr} \to M_U^{\rm gr})$, where \mathcal{I} is the ideal defined by the closed immersion $\underline{\iota}$. Using Lemma 1.24, since T is affine, we get the exact sequence

(1.8)
$$1 \to \Gamma(T, 1 + \mathcal{I}) \to \Gamma(T, M_T^{\mathrm{gr}}) \to \Gamma(U, M_U^{\mathrm{gr}}) \to 1.$$

First, let us check the unicity. Let $\widetilde{g}'_1, \widetilde{g}'_2: P^{\operatorname{gr}} \to \Gamma(T, M^{\operatorname{gr}}_T)$ be two morphisms making the diagram (1.7) commutative. Let us denote by $\widetilde{g}'_1(\widetilde{g}'_2)^{-1}: P^{\operatorname{gr}} \to \Gamma(T, M^{\operatorname{gr}}_T)$ the morphism defined by $x \mapsto \widetilde{g}'_1(x)(\widetilde{g}'_2(x))^{-1}$. From the exactness (1.8), since $i' \circ \widetilde{g}'_1 = i' \circ \widetilde{g}'_2$, we get the factorization $\widetilde{g}'_1(\widetilde{g}'_2)^{-1}: P^{\operatorname{gr}} \to \Gamma(T, 1+\mathcal{I})$. Since $(\widetilde{g}'_1(\widetilde{g}'_2)^{-1}) \circ \theta^{\operatorname{gr}} = 1$, the morphism $\widetilde{g}'_1(\widetilde{g}'_2)^{-1}$ has the canonical factorization by a morphism of the form $\operatorname{coker}(\theta^{\operatorname{gr}}) \to \Gamma(T, 1+\mathcal{I})$. Let N be the cardinal of $\operatorname{coker}(\theta^{\operatorname{gr}})$. Since N is of order prime to p, then the homomorphism $\Gamma(T, 1+\mathcal{I}) \to \Gamma(T, 1+\mathcal{I})$ given by $x \mapsto x^N$ is a bijection (see Lemma 1.24). This yields that any homomorphism $\operatorname{coker}(\theta^{\operatorname{gr}}) \to \Gamma(T, 1+\mathcal{I})$ is 1. Hence, $\widetilde{g}'_1 = \widetilde{g}'_2$.

Now, let us prove the existence. It is sufficient to copy in the proof of [28, IV.3.1.8] the part corresponding to the implication $3.1.8(1) \Rightarrow 3.1.8(2)$. For the convenience of the reader, let us clarify it. Put $E := \Gamma(T, M_T^{\text{gr}}) \times_{\Gamma(U, M_U^{\text{gr}})} P^{\text{gr}}$ (the morphisms used to define the fiber product are those appearing in the diagram (1.7)). Taking the pullback of (1.8) by g', we get the exact sequence $1 \to \Gamma(T, 1 + \mathcal{I}) \to E \xrightarrow{\pi} P^{\text{gr}} \to 1$, where the first map is given by $x \mapsto (x, 1)$ and where the map π is the projection $(x, y) \to y$. We put $\phi := (h', \theta^{\text{gr}}) : Q^{\text{gr}} \to E$. We get the commutative diagram

whose top and middle rows are exact. Since the homomorphism $\Gamma(T, 1 + \mathcal{I}) \to \Gamma(T, 1 + \mathcal{I})$ given by $x \mapsto x^n$ is a bijection for any integer n prime to p, since ker (θ^{gr}) is finite of order prime to p, then the morphism ker $(\theta^{\text{gr}}) \to \Gamma(T, 1 + \mathcal{I})$ is equal to 1. Hence, using the snake lemma to the top and middle rows, we can check that the bottom row is also exact. Since the homomorphism $\Gamma(T, 1 + \mathcal{I}) \to \Gamma(T, 1 + \mathcal{I})$ given by $x \mapsto x^n$ is a bijection for n equal to the cardinal of $\operatorname{coker}(\theta^{\text{gr}})$, then $\operatorname{Ext}^1(\operatorname{coker}(\theta^{\text{gr}}), \Gamma(T, 1 + \mathcal{I})) = 1$. Hence, the bottom row splits (and even uniquely). Let $\tau : P^{\text{gr}} \to \operatorname{coker}(\phi)$ be the composition of $P^{\text{gr}} \to \operatorname{coker}(\theta^{\text{gr}})$ with a section $\operatorname{coker}(\theta^{\text{gr}}) \to \operatorname{coker}(\phi)$ of the surjection of the bottom surjective map.

We remark that the set of morphisms $\tilde{g}: P^{\mathrm{gr}} \to \Gamma(T, M_T^{\mathrm{gr}})$ making commutative the diagram (1.7) is equipotent with the set of sections $\sigma: P^{\mathrm{gr}} \to E$ of π such that $\sigma \circ \theta^{\mathrm{gr}} = \phi$ (the bijection is given by $\tilde{g} \mapsto (\tilde{g}, \mathrm{id})$). Since the middle and bottow rows are exact, we get that the square on the bottom right is exact. Hence, we get the morphism $\sigma := (\tau, \pi): P^{\mathrm{gr}} \to E$. We can check that this morphism is a section of π such that $\sigma \circ \theta^{\mathrm{gr}} = \phi$.

Example 1.26. — We set $k[\underline{T}] := k[T_1, \ldots, T_n]$ and $k[\underline{T}] = k[T_1, \ldots, T_n]$. Since k is perfect, then the morphism $\operatorname{Spec} k[\underline{T}] \to \operatorname{Spec} k[\underline{T}]$ is p-étale because it is relatively perfect (see Lemma 1.31 and Example 1.39). Beware that such a morphism is not of finite type and then not étale.

An important application of this fact is that a strictly semistable variety over Spec $k[\![t]\!]$ is *p*-smooth over Spec *k* (see Definition 1.85 and Example 1.89). Following de Jong desingularization theorem ([20, 6.5]), any variety over Spec $k[\![t]\!]$ is strictly semistable up to some alteration. Remark that a strictly semistable variety over Spec $k[\![t]\!]$ is not smooth over Spec $k[\![t]\!]$ but only log smooth over the log scheme (Spec $k[\![t]\!]$, M_t) where M_t is the log structure associated to the special fiber (see [21]). Hence, when we want to construct *p*-adic coefficients in the context of schemes of finite type over Spec $k[\![t]\!]$, on one hand we have to add logarithmic structures but on the other hand log smoothness is a sufficient notion. The situation in the absolute context is symmetric: when we want to study schemes of finite type over Spec $k[\![t]\!]$ absolutely, i.e. we consider these geometrical objects over *k*, on one hand we can avoid to bother with logarithmic structures but on the other hand we do need to work with the notion of *p*-smoothness.

LEMMA 1.27. — Let $n \in \mathbb{N}$, $f: X \to Y$ be a log *p*-étale morphism of fine log *S*-schemes, $u: Z \to X$ and $v: Z \to Y$ be two *S*-immersions of fine log schemes such that $v = f \circ u$. If $P^{(p^n)}(u)$ exists, then $P^{(p^n)}(v)$ exists and we have $P^{(p^n)}(u) = P^{(p^n)}(v)$.

Proof. — Abstract nonsense.

PROPOSITION 1.28. — For any integer n, the canonical functor $\mathscr{C}_{(p^n)} \to \mathscr{C}$ has a right adjoint functor which we will denote by $P^{(p^n)} \colon \mathscr{C} \to \mathscr{C}_{(p^n)}$. Let $u \colon Z \hookrightarrow X$ be an object of \mathscr{C} . Then Z is also the source of $P^{(p^n)}(u)$. Moreover, denoting abusively by $P^{(p^n)}(u)$ the target of the arrow $P^{(p^n)}(u)$, the underlying morphism of schemes of $P^{(p^n)}(u) \to X$ is affine. When X is noetherian, then so is $P^{(p^n)}(u)$.

Proof. — The proof is similar to that of Proposition 1.11: Let $u: Z \hookrightarrow X$ be an S-immersion of fine log-schemes. Using Lemma 1.9, the existence of $P^{(p^n)}(u)$ (and then the proposition) is étale local on X (i.e. following our convention, this is local for the Zariski topology and we can proceed by descent of a finite covering with étale quasi-compact morphisms). Hence, by Point 1.8, we may thus assume that there exists a commutative diagram of the form



such that f is log étale, \underline{f} is affine and \widetilde{u} is an exact closed S-immersion. Let \mathcal{I} be the ideal defined by \widetilde{u} . Let $P^{(p^n)} \hookrightarrow \widetilde{X}$ be the exact closed immersion which is induced by $\mathcal{I}^{(p^n)}$. Using Proposition 1.25 and Lemma 1.27, we can check that $P^{(p^n)}(u)$ is the exact closed immersion $Z \hookrightarrow P^{(p^n)}$.

1.3. Log relative perfectness

The following definition will be extended in Definition 1.32.

DEFINITION 1.29. — Let $f: X \to Y$ be an S_0 -morphism of fine logschemes. We say that f is "fine log relatively perfect" (resp. "fs log relatively perfect") if the diagram on the left (resp. on the right)

(1.9)
$$\begin{array}{cccc} X \xrightarrow{F_X} X & X^{\text{sat}} \xrightarrow{F_{X^{\text{sat}}}} X^{\text{sat}} \\ & & & & & \\ & & & & \\ f & & & & \\ Y \xrightarrow{F_Y} Y, & Y^{\text{sat}} \xrightarrow{F_{Y^{\text{sat}}}} Y^{\text{sat}} \\ & & & & \\ \end{array}$$

is cartesian in the category of fine log-schemes (resp. fs log-schemes). This definition does not depend on the choice of the fine log scheme S over $\mathbb{Z}/p^{i+1}\mathbb{Z}$. We remark that f is fs log relatively perfect if and only if so is f^{sat} .

Remark 1.30. — Let $\iota: U \hookrightarrow T$ be a log S_0 -thickening of order (p). Let $T \times_{F_T,T,\iota} U$ be the base change of F_T by ι . Let $p_1: T \times_{F_T,T,\iota} U \to T$, and $p_2: T \times_{F_T,T,\iota} U \to U$ be the projections. Since ι is of order (p) then p_1 is an isomorphism. Put $\varpi_{\iota} := p_2 \circ p_1^{-1}: T \to U$. We remark that the morphism ϖ_{ι} is the unique morphism $T \to U$ making commutative the diagram

(1.10)
$$U \xrightarrow{F_U} U$$
$$\iota \bigvee_{F_T} \bigcup_{F_T} \bigcup_{T} U$$
$$T \xrightarrow{\varpi_\iota} \int_{F_T} U$$

LEMMA 1.31. — Let $f: X \to Y$ be an S_0 -morphism of fine log-schemes. If f is fine log relatively perfect (resp. fs log relatively perfect) then f is log p-étale (resp. fs log p-étale).

Proof. — Let us check the fine version. Let

$$(1.11) \qquad \begin{array}{c} U \xrightarrow{u_0} X \\ & & \downarrow f \\ T \xrightarrow{v} Y \end{array}$$

be a commutative diagram of fs S_0 -log schemes such that i is a log S_0 -thickening of order (p). First, let us check the unicity. Let $u: T \to X$ be a morphism such that $u \circ \iota = u_0$ and $f \circ u = v$. With the notation of Remark 1.30, we get $F_X \circ u = u \circ F_T = u \circ \iota \circ \varpi_\iota = u_0 \circ \varpi_\iota$. Since we have also $f \circ u = v$, we obtain the uniqueness of u from the cartesianity of (1.9). Now, let us check the existence. We have $F_Y \circ v = v \circ F_T = v \circ \iota \circ \varpi_\iota =$ $f \circ u_0 \circ \varpi_\iota$. Hence, via the cartesianity of (1.9), we get the Y-morphism $u = (v, u_0 \circ \varpi_\iota): T \to X = Y \times_{F_Y,Y,f} X$. By definition, $f \circ u = v$. Moreover, $u \circ \iota = (v \circ \iota, u_0 \circ \varpi_\iota \circ \iota) = (f \circ u_0, u_0 \circ F_U) = (f \circ u_0, F_X \circ u_0) = u_0: T' \to$ $X = Y \times_{F_Y,Y,f} X$.

Suppose now that f is fs log relatively perfect. Using Lemma 1.16, we reduce to check that f^{sat} is fs log p-étale. Since f^{sat} is fs log relatively perfect, we can proceed in the same way by replacing "fine" by "fs". \Box

DEFINITION 1.32. — Let $f: X \to Y$ be an S-morphism of fine logschemes. We say that f is "fine log relatively perfect" (resp. "fs log relatively perfect") if f is fine formally log etale (resp. fs formally log etale) and if f_0 is fine log relatively perfect (resp. fs log relatively perfect).

This definition does not depend on the choice of the fine log scheme S over $\operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$. Let us clarify what it means. We can view f as a $\operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ -morphism of fine log-schemes. Then, we remark that f

is fine formally log etale (resp. fs formally log etale) as a $\operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ morphism if and only if so is f as an S-morphism (indeed, a log-scheme over Y is an S-log-scheme). Finally f_0 is fine log relatively perfect (resp. fs log relatively perfect) as $\operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})$ -morphism if and only if so is f_0 as an S_0 -morphism.

Remark 1.33. — From Lemma 1.31, this definition of log relative perfectness of Definition 1.32 agrees that of Definition 1.29 when i = 0, i.e. $S = S_0$.

LEMMA 1.34. — Let $f: X \to Y$ be an S-morphism of fine log-schemes. Then f is fs log relatively perfect if and only if so is f^{sat} .

Proof. — From Lemma 1.16, f is fs formally log étale if and only if so is f^{sat} . Moreover, since $(f^{\text{sat}})_0 = (f_0)^{\text{sat}}$, then f_0 is fs log relatively perfect if and only if so is $(f^{\text{sat}})_0$.

PROPOSITION 1.35. — A fine (resp. fs) log relatively perfect morphism is log p-étale (resp. fs log p-étale).

Proof. — This is a consequence of Lemmas 1.20 and 1.31. \Box

LEMMA 1.36. — Let $f: X \to Y$ and $g: Y' \to Y$ be two S-morphisms of fine log-schemes. Set $X' := X \times_Y Y'$ in the category of fine log schemes and $f': X' \to Y'$ the projection. If f is fine log relatively perfect (resp. fs log relatively perfect), then so is f'.

Proof. — Since the fs part is similar, let us only consider the fine part. Using Lemma 1.19, we reduce to the case where i = 0. Since the outline of both diagrams

$$(1.12) \qquad \begin{array}{c} X' \xrightarrow{F_{X'}} X' \longrightarrow X & X' \longrightarrow X \xrightarrow{F_X} X \\ \downarrow f' & \downarrow f' & \downarrow f & \downarrow f' & \downarrow f' & \downarrow f & \downarrow f & \downarrow f \\ Y' \xrightarrow{F_{Y'}} Y' \longrightarrow Y, & Y' \longrightarrow Y \xrightarrow{F_Y} Y \end{array}$$

are the same, we conclude.

Point 1.37 (Relative perfectnes in the sense of Kato). — Let $A \to B$ be an homomorphism of k-algebras. We denote by $A^{(p)}$ be the A-algebra given by the absolute Frobenius $F_A: A^{(p)} \to A$, and similarly for B. Let $b_1, \ldots, b_n \in B$ be some elements and $A[\underline{T}] = A[T_1, \ldots, T_n] \to B$ be the homomorphism given by $T_i \mapsto b_i$.

- (a) By definition, the homomorphism $A[\underline{T}] \to B$ is relatively perfect in the sense of Kato if and only if the canonical homomorphism $(A[\underline{T}])^{(p)} \otimes_{A[\underline{T}]} B \to B^{(p)}$ is an isomorphism (see [22, 1.1]).
- (b) We have the canonical isomorphisms

$$(A[\underline{T}])^{(p)} \otimes_{A[\underline{T}]} B \xrightarrow{\sim} (A[\underline{T}])^{(p)} \otimes_{A^{(p)}[\underline{T}]} (A^{(p)}[\underline{T}] \otimes_{A[\underline{T}]} B)$$
$$\xrightarrow{\sim} (A[\underline{T}])^{(p)} \otimes_{A^{(p)}[\underline{T}]} (A^{(p)} \otimes_A B).$$

This yields that the homomorphism $A[\underline{T}] \to B$ is relatively perfect in the sense of Kato if and only if the canonical homomorphism

(1.13)
$$(A[\underline{T}])^{(p)} \otimes_{A^{(p)}[\underline{T}]} (A^{(p)} \otimes_A B) \to B^{(p)}$$

is an isomorphism. Since $(A[\underline{T}])^{(p)}$ is a free $A^{(p)}[\underline{T}]$ -module with the basis $\prod_{i=1}^{d} T_{i}^{n_{i}}$, for $n_{i} < p$ for any *i*, then the homomorphism (1.13) is an isomorphism if and only if $B^{(p)}$ is a free $A^{(p)} \otimes_{A} B$ -module with the basis $\prod_{i=1}^{d} b_{i}^{n_{i}}$, for $n_{i} < p$ for any *i*. In that case, following Kato's terminology, b_{1}, \ldots, b_{n} form a *p*-basis of B/A.

LEMMA 1.38. — Let $f: X \to Y$ be a strict S-morphism of fine log schemes (resp. fs log schemes). Then f is fine log relatively perfect (resp. fs log relatively perfect) if and only if \underline{f} is relatively perfect as defined by Kato in [22, 1.1] (see Point 1.37 for the affine case).

Proof. — Since *f* is strict then *f* is the base change of *f* by *Y* → *Y* in the category of fine log schemes (resp. fs log schemes). This yields that if *f* is relatively perfect then *f* is fine log relatively perfect (resp. fs log relatively perfect). Conversely, suppose that the morphism *f* is fine (resp. fs) formally log étale. Then, similarly to the proof of Lemma 1.17, we can check that *f* is formally étale. Hence, we reduce to the case *i* = 0. Moreover, since *f* is strict then $\underline{Y} \times_{F_Y,Y,f} X = \underline{Y} \times_{F_Y,Y,f} \underline{X}$. Since *f* is fine log relatively perfect (resp. fs log relatively perfect), we get $Y = Y \times_{F_Y,Y,f} X$. This yields $\underline{Y} = \underline{Y} \times_{F_Y,Y,f} \underline{X}$ and we are done.

Example 1.39. — We set $k[\underline{T}] := k[T_1, \ldots, T_n]$ and $k[\underline{T}]] = k[T_1, \ldots, T_n]$. Since k is perfect (in particular, it has finite p-basis), then the morphism Spec $k[\underline{T}]] \rightarrow$ Spec $k[\underline{T}]$ is relatively perfect (but this is not étale). Indeed, since $k[\underline{T}]]$ is not a $k[\underline{T}]$ -algebra of finite type, then it can not be étale. It remains to check that $k[\underline{T}] \rightarrow k[\underline{T}]]$ is relatively perfect, i.e. following Point 1.37 (b) that the canonical homomorphism

$$(k[\underline{T}])^{(p)} \otimes_{k^{(p)}[\underline{T}]} \left(k^{(p)} \otimes_k k[\underline{T}]\right) \to (k[\underline{T}])^{(p)}$$

is an isomorphism, i.e. $(k[\underline{T}])^{(p)}$ is a free $k^{(p)} \otimes_k k[\underline{T}]$ -module with the basis $\prod_{i=1}^{d} T_i^{n_i}$, for $n_i < p$ for any *i*. Since $k \to k^{(p)}$ is finite, then $k^{(p)} \otimes_k k[\underline{T}] = k^{(p)}[\underline{T}]$. The induced map $k^{(p)}[\underline{T}] \to (k[\underline{T}])^{(p)}$ is given by $\sum a_k T^k \to \sum a_k T^{pk}$. Hence, we conclude by an easy computation.

LEMMA 1.40. — Let $f: X \to Y$ and $g: Y \to Z$ be two S-morphisms of fine log schemes. The morphisms f and g are fine log relatively perfect (resp. fs log relatively perfect) if and only if so are $g \circ f$ and g.

Proof. — Using Lemma 1.21, we reduce to the case where i = 0. Then, this is abstract nonsense.

LEMMA 1.41. — Let $f: X \to Y$ be an étale S-morphism of fine logschemes. Then f is fine log relatively perfect.

Proof. — Using Lemma 1.38, we reduce to check that an étale morphism of schemes is relatively perfect as defined by Kato in [22, 1.1], which is well known. \Box

LEMMA 1.42. — Let $f: X \to Y$ and $g: X' \to X$ be two S-morphisms of fine log-schemes such that g is étale, quasi-compact and surjective. The morphism f is fine log relatively perfect (resp. fs log relatively perfect) if and only so is $f \circ g$.

Proof. — From Lemmas 1.40 and 1.41, if f is fine log relatively perfect then so is $f \circ g$. Conversely, suppose $f \circ g$ is fine log relatively perfect. Using Lemma 1.22, we reduce to the case where i = 0. We have to check that the morphism $(F_X, f): X \to X \times_{f,Y,F_Y} Y$ is an isomorphism. Since is Zariski local, we can suppose X affine. Since $f \circ g$ is fine log relatively perfect, then $(F_{X'}, f \circ g): X' \to X' \times_{f \circ g, Y, F_Y} Y$, is an isomorphism. We notice that $(F_{X'}, f \circ g)$, is the base change of (F_X, f) by $g \times id: X' \times_{f \circ g, Y, F_Y} Y$ $Y \to X \times_{f,Y,F_Y} Y$ Since $g \times id$ is etale, quasi-compact and surjective, then using [16, IV.2.7.1.(viii)], we can conclude. The fs log relatively perfect case is checked similarly. □

LEMMA 1.43. — Let $f: X \to Y$ and $g: Y' \to Y$ be two S-morphisms of fine log-schemes such that g is étale, quasi-compact and surjective. Set $X' := X \times_Y Y'$ in the category of fine log schemes and $f': X' \to Y'$ the projection. The morphism f is fine log relatively perfect (resp. fs log relatively perfect) if and only so is f'. Proof. — From Lemma 1.36, f is fine log relatively perfect then so is f'. Let us check converse: suppose f' is fine log relatively perfect. Using Lemma 1.23, we reduce to the case where i = 0. We have to check that the morphism $(F_X, f): X \to X \times_{f,Y,F_Y} Y$ is an isomorphism. Since f' is fine log relatively perfect then the base change of (F_X, f) by id $\times g: X' \times_{f',Y',F_Y'} Y' = X \times_{f,Y,F_Y} \circ_g Y' \to X \times_{f,Y,F_Y} Y$ is an isomorphism (to check the equality, recall from Lemma 1.41 that g is relatively perfect). Hence, using [16, IV.2.7.1.(viii)], we can check that (F_X, f) is an isomorphism. The fs log relatively perfect case is checked similarly.

Notation 1.44. — Let P be a monoid. We denote by $A_P := (\operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z}[P]), M_P)$ the log formal scheme whose underlying scheme is $\operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z}[P])$ and whose log structure is the log structure associated with the pre-log structure induced canonically by $P \to \mathbb{Z}/p^{i+1}\mathbb{Z}[P]$.

Beware that the notation $A_P := (\operatorname{Spec}(\mathbb{Z}[P]), M_P)$ seems more common in the literature, but since we are always working over $\mathbb{Z}/p^{i+1}\mathbb{Z}$ this is much more convenient for us to put $A_P := (\operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z}[P]), M_P)$.

PROPOSITION 1.45. — Let $f: X \to Y$ be a log étale S-morphism of fine log-schemes. Then f is fs log relatively perfect.

Proof. — Since these notions do not depend on *S*, we can suppose *S* = Spec($\mathbb{Z}/p^{i+1}\mathbb{Z}$). From [28, IV.3.1.11], $X^{\text{sat}} \to X$ and $Y^{\text{sat}} \to Y$ are log etale (see [28, III.2.1.5] concerning the functor $X \mapsto X^{\text{sat}}$). Hence, using Remark [28, IV.3.1.2], $f^{\text{sat}}: X^{\text{sat}} \to Y^{\text{sat}}$ is also log étale. From Lemmas 1.16 and 1.34, we can suppose that $f = f^{\text{sat}}$. Since *f* is formally log étale, then we are reduced to the case i = 0. Next we observe that the case where *f* is strict is already known (see Lemmas 1.38 and 1.41). Following Lemmas 1.42 and 1.43, the fs relative perfectness of *f* is étale local on both *X* and *Y*. Hence, using [28, II.3.6] and [21, Theorem 3.5], we reduce to the case where $X = A_P$, $Y = A_Q$ (see Notation 1.44) and *f* is induced by a morphism $\phi: Q \to P$ of *fs* monoids such that the kernel and cokernel of ϕ^{gp} is finite of order prime to *p*. It remains to prove that the left square below is cartesian in the category of fs log schemes:

(1.14)
$$\begin{array}{ccc} A_P \xrightarrow{F_{A_P}} & P \xleftarrow{p} P \\ \downarrow f & \downarrow f & \phi \uparrow & \phi \uparrow \\ A_Q \xrightarrow{F_{A_Q}} & A_Q, & Q \xleftarrow{p} Q. \end{array}$$

Since the functor $P \mapsto A_P$ from the category of fine satured monoids to the category of fs log-schemes over $\operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ transforms cocartesian squares into cartesian squares, it is thus sufficient to check that the right square of (1.14) is cocartesian in the category of fine satured monoids. Let us check that P satisfies the universal property of the pushout.

Step 1. — First, we reduce to check the universal property in the case where $P = P^{\text{gp}}$, $Q = Q^{\text{gp}}$ and $O = O^{\text{gp}}$ as follows.

Let $\gamma_1, \gamma_2 \colon P \to O$ be two morphisms of monoids such that for i = 1, 2we have $\gamma_i \circ \phi = \alpha$ and $\gamma_i \circ p = \beta$. Then $\gamma_i^{\text{gp}} \colon P^{\text{gp}} \to O^{\text{gp}}$ is a morphism of groups such that $\gamma_i^{\text{gp}} \circ \phi^{\text{gp}} = \alpha^{\text{gp}}$ and $\gamma_i^{\text{gp}} \circ p = \beta^{\text{gp}}$. Hence, $\gamma_1^{\text{gp}} = \gamma_2^{\text{gp}}$. Since $O \to O^{\text{gp}}$ is injective, this yields $\gamma_1 = \gamma_2$.

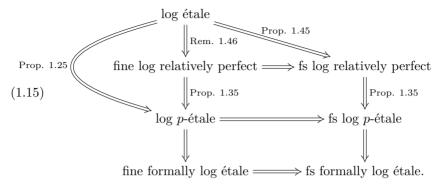
Suppose there exists a morphism of groups $\tilde{\gamma} \colon P^{\text{gp}} \to O^{\text{gp}}$ such that $\tilde{\gamma} \circ \phi^{\text{gp}} = \alpha^{\text{gp}}$ and $\tilde{\gamma} \circ p = \beta^{\text{gp}}$. Let $x \in P$. Then $p\tilde{\gamma}(x) = \tilde{\gamma}(px) = \beta^{\text{gp}}(px) = \beta(px) \in O$. Since O is a fine saturated monoid, then $\tilde{\gamma}(x) \in O$. Since $O \to O^{\text{gp}}$ is injective, this yields the morphism of monoids $\gamma \colon P \to O$ such that $\gamma^{\text{gr}} = \tilde{\gamma}, \gamma \circ \phi = \alpha$ and $\gamma \circ p = \beta$.

Let $\alpha: P \to O$ and $\beta: Q \to O$ be two morphisms of fine satured monoids such that $\alpha \circ \phi = \beta \circ p$. Since O is saturated, if z is an element of O^{gp} such that pz is in O then z is an element of O.

Step 2. — Let us conclude the proof. Let $x_0 \in P$. There exist $y \in Q$ and $x \in P$ such than $x_0 = \phi(y) + px$. (Indeed, let n be an integer prime to p such that $n \cdot \operatorname{coker}(\phi) = 0$. There exists $y_0 \in Q$ such that $nx_0 = \phi(y_0)$. Using Bezout lemma, we get the desired result.) We define a morphism $h: P \to O$ satisfying $\alpha = h \circ p$ and $\beta = h \circ \phi$ by putting $h(x_0) := \beta(y) + \alpha(x)$. The morphism h is well defined. Indeed, if $x_0 = \phi(y) + px = \phi(y') + px'$, then $\phi(y - y') = p \ (x' - x)$ and then we can suppose y' = 0 and x = 0, i.e. $x_0 = \phi(y) = px'$. Since p is an isomorphism on $\operatorname{coker}(\phi)$, there exists $z' \in Q$ such that $x' = \phi(z')$. Since p is an isomorphism on $\operatorname{ker}(\phi)$, there exists $z' \in Q$ such that y = p(z' + z''). We compute $\beta(y) = \beta(p(z' + z'')) = \alpha \circ \phi(z' + z'') = \alpha \circ \phi(z') = \alpha(x')$. The unicity of such h is also clear.

Remark 1.46. — The "fine" version of Proposition 1.45 is wrong, i.e. a log étale morphism is not necessarily fine log relatively perfect. Indeed, we have the following counter-example. Let $n \ge 2$ such that $p \nmid n$. In the category of fine (resp fine saturated) monoids the inductive limit of the diagram $\mathbb{N} \xleftarrow{p} \mathbb{N} \xrightarrow{n} \mathbb{N}$ is the submonoid of \mathbb{N} generated by p and n (resp. the associated saturated monoid, i.e. \mathbb{N} itself). Let $f: A_{\mathbb{N}} \to A_{\mathbb{N}}$ denote the morphism induced by $n: \mathbb{N} \to \mathbb{N}$. Hence, the morphism f is log étale but not fine relatively perfect.

Point 1.47. — The following diagram summarizes the relations between the first definitions we consider so far:



From now on we will work with fine (not necessarily saturated) log schemes. The reader who is only interested in the category of fs log schemes can replace log p-etaleness by fs log p-etaleness in the sequel.

1.4. Formally log-étaleness of level m, log p-étaleness of level m, m-PD-envelopes, nth infinitesimal neighborhood of level m

Let (I_S, J_S, γ) be a quasi-coherent *m*-PD-ideal of \mathcal{O}_S . Let us fix some definitions.

DEFINITION 1.48. — Let $n \ge 1$ be an integer.

(1) Let $\mathscr{C}_{\gamma}^{(m)}$ (resp. $\mathscr{C}_{\gamma,n}^{(m)}$) be the category whose objects are pairs (u, δ) where u is an exact closed S-immersion $Z \hookrightarrow X$ of fine log-schemes and δ is an m-PD-structure on the ideal \mathcal{I} of \mathcal{O}_X given by u which is compatible (see [4, Definition 1.3.2.(ii)]) with γ (resp. and such that $\mathcal{I}^{\{n+1\}_{(m)}} = 0$), where $\mathcal{I}^{\{n+1\}_{(m)}}$ is defined in the appendix of [5]); whose morphisms $(u', \delta') \to (u, \delta)$ are commutative diagrams of the form

(1.16)
$$\begin{array}{c} X' \xrightarrow{f} X \\ u' & \downarrow \\ Z' \xrightarrow{} Z \end{array}$$

such that f is an *m*-PD-morphism with respect to the *m*-PDstructures δ and δ' (i.e., denoting by \mathcal{I}' the sheaf of ideals of $\mathcal{O}_{X'}$ defined by u', for any affine open sets U' of X' and U of X such that $f(U') \subset U$, the morphism f induces the m-PD-morphism $(\mathcal{O}_X(U), \mathcal{I}(U), \delta) \to (\mathcal{O}_{X'}(U'), \mathcal{I}'(U'), \delta'))$. Beware that theses categories depend on S and also on the quasi-coherent m-PD-ideal (I_S, J_S, γ) . The objects of $\mathscr{C}_{\gamma}^{(m)}$ (resp. $\mathscr{C}_{\gamma,n}^{(m)}$) are called m-PD-S-immersions compatible with γ (resp. m-PD-S-immersions of order n compatible with γ). We remark that we have the inclusions $\mathscr{C}_{\gamma}^{(m)} \subset \mathscr{C}_{\gamma}^{(m')}$ for any integer $m' \ge m$ (recall an m-PD-structure is also an m'-PD-structure).

We say that a morphism $(u', \delta') \to (u, \delta)$ of $\mathscr{C}_{\gamma}^{(m)}$ (resp. $\mathscr{C}_{\gamma,n}^{(m)}$) is strict (resp. flat, resp. cartesian) if f is strict (resp. <u>f</u> is flat, resp. the square (1.16) is cartesian).

- (2) Let u be an object of \mathscr{C} (see Definition 1.4). An "m-PD-envelope compatible with γ of u" is an object (u', δ') of $\mathscr{C}_{\gamma}^{(m)}$ endowed with a morphism $u' \to u$ in \mathscr{C} satisfying the following universal property: for any object (u'', δ'') of $\mathscr{C}_{\gamma}^{(m)}$ endowed with a morphism $f: u'' \to u$ of \mathscr{C} there exists a unique morphism $(u'', \delta'') \to (u', \delta')$ of $\mathscr{C}_{\gamma}^{(m)}$ whose composition with $u' \to u$ is f. The unicity up to canonical isomorphism of the m-PD-envelope compatible with γ of u is obvious. We will denote by $P_{(m),\gamma}(u)$ the m-PD-envelope compatible with γ of u. By abuse of notation we also denote by $P_{(m),\gamma}(u)$ the underlying exact closed immersion or its target. The existence is checked below (see Proposition 1.63).
- (3) Let u be an object of C. An "m-PD-envelope of order n compatible with γ of u" is an object (u', δ') of C^(m)_{γ,n} endowed with a morphism u' → u in C satisfying the following universal property: for any object (u", δ") of C^(m)_{γ,n} endowed with a morphism f: u" → u of C there exists a unique morphism (u", δ") → (u', δ') of C^(m)_{γ,n} whose composition with u' → u is f. The unicity up to canonical isomorphism of the m-PD-envelope of order n compatible with γ of u is obvious. We will denote by Pⁿ_{(m),γ}(u) the m-PD-envelope of order n compatible with γ of u. By abuse of notation we also denote by Pⁿ_{(m),γ}(u) the underlying exact closed immersion or its target. The existence is checked below (see Proposition 1.63).
- (4) Since p is nilpotent in S, we get the forgetful functor $\operatorname{For}^{(m)} : \mathscr{C}_{\gamma}^{(m)} \to \mathscr{T}\operatorname{hick}_{(p)}$ (resp. $\operatorname{For}_{n}^{(m)} : \mathscr{C}_{\gamma,n}^{(m)} \to \mathscr{T}\operatorname{hick}_{(p)}$) given by $(u, \delta) \mapsto u$. We denote by $\mathscr{C}_{\gamma}^{\prime(m)}$ (resp. $\mathscr{C}_{\gamma,n}^{\prime(m)}$) the image of $\operatorname{For}^{(m)}$ (resp. $\operatorname{For}_{n}^{(m)}$).

Notation 1.49. — In this paragraph, suppose $J_S = p\mathcal{O}_S$. Then, there is a unique PD-structure on J_S which we will denote by γ_{\emptyset} . Let $u: Z \hookrightarrow X$ be an exact closed S-immersion of fine log-schemes and δ be an m-PDstructure on the ideal \mathcal{I} of \mathcal{O}_X defined by u. It follows from Lemma [4, 1.2.4] that the m-PD-structure δ of \mathcal{I} is always compatible with γ_{\emptyset} . Hence, in the description of $\mathscr{C}_{\gamma_{\emptyset}}^{(m)}$ (resp. $\mathscr{C}_{\gamma_{\emptyset},n}^{(m)}$) we can remove "compatible with γ_{\emptyset} " without changing the respective categories. For this reason, we put $\mathscr{C}^{(m)} := \mathscr{C}_{\gamma_{\emptyset}}^{(m)}$ (resp. $\mathscr{C}_{n}^{(m)} := \mathscr{C}_{\gamma_{\emptyset,n}}^{(m)}$). But, recall these categories depend on S even if this is not written in the notation. Finally, for any quasicoherent m-PD-ideal (I_S, J_S, γ) of \mathcal{O}_S , we have the inclusions

(1.17)
$$\mathscr{C}_{\gamma}^{(m)} \subset \mathscr{C}^{(m)}, \text{ and } \mathscr{C}_{\gamma,n}^{(m)} \subset \mathscr{C}_{n}^{(m)}.$$

DEFINITION 1.50. — Let $f: X \to Y$ be an S-morphism of fine log schemes.

 We say that f is "formally log étale of level m compatible with γ" (resp. "formally log unramified of level m compatible with γ") if it satisfies the following property: for any commutative diagram of fine log schemes of the form

$$(1.18) \qquad \begin{array}{c} U \xrightarrow{u_0} X \\ \downarrow & \downarrow f \\ T \xrightarrow{v} Y \end{array}$$

such that ι is an object of $\mathscr{C}_{\gamma,1}^{(m)}$, there exists a unique morphism (resp. there exists at most one morphism) $u: T \to X$ such that $u \circ \iota = u_0$ and $f \circ u = v$.

- (2) Replacing $\mathscr{C}'_{\gamma,1}^{(m)}$ by $\mathscr{C}'_{\gamma}^{(m)}$, we get the notion of "log *p*-étale of level *m* compatible with γ " (resp. "log *p*-unramified of level *m* compatible with γ ") morphism of fine log schemes. To justify the terminology, the reader might see Proposition 1.55.
- (3) Replacing "fine log S-schemes" by "S-schemes" in Definition 1.50(1), we get the notion of "formally étale of level m compatible with γ" (resp. "formally unramified of level m compatible with γ") morphism of schemes and of "p-étale of level m compatible with γ" (resp. "p-unramified of level m compatible with γ") morphism of schemes.
- (4) When $\gamma = \gamma_{\emptyset}$ (see Notation 1.49), we remove for simplicity "compatible with γ_{\emptyset} " in the terminology.

Remark 1.51. — Let (u, δ) be an object of $\mathscr{C}_{\gamma,n}^{(m)}$, i.e. $u: Z \hookrightarrow X$ is an exact closed S-immersion of fine log-schemes and δ is an m-PD-structure

on the ideal \mathcal{I} of \mathcal{O}_X defined by u which is compatible (see [4, Definition 1.3.2.(ii)]) with γ and such that $\mathcal{I}^{\{n+1\}_{(m)}} = 0$. Then, by applying finitely many times the functor $\mathcal{J} \mapsto \mathcal{J}^{\{2\}_{(m)}}$ we obtain the zero ideal. Hence, (u, δ) is a finite composition of objects of $\mathscr{C}_{\gamma,1}^{(m)}$. Hence, in Definition 1.50, we can replace $\mathscr{C}'_{\gamma,1}^{(m)}$ by $\mathscr{C}'_{\gamma,n}^{(m)}$.

LEMMA 1.52. — The collection of formally log étale (resp. log p-étale) of level m compatible with γ morphisms of fine S-log schemes is stable under base change and under composition. Similarly replacing "étale" by "unramified" or/and removing "log".

Proof. — This is checked similarly to Lemmas 1.19 and 1.21. \Box

Remark 1.53. — Let $f: X \to Y$ be an S-morphism of fine log schemes.

- (1) If the morphism f is log p-étale then f is log p-étale of level m compatible with γ for any $m \in \mathbb{N}$ (recall $\mathscr{C}'_{\gamma}^{(m)} \subset \mathscr{T}\mathrm{hick}_{(p)}$). If f is log p-étale of level m compatible with γ then f is formally log étale of level m compatible with γ (recall $\mathscr{C}'_{1,\gamma}^{(m)} \subset \mathscr{C}'_{\gamma}^{(m)}$). Similarly replacing "étale" by "unramified". Even if we do not have counterexamples, the converse seems false in general.
- (2) If f is log étale, then f is log p-étale (recall Proposition 1.25) and then f is log p-étale of level m compatible with γ for any $m \in \mathbb{N}$ and then f is formally log étale of level m compatible with γ for any $m \in \mathbb{N}$.

LEMMA 1.54. — Suppose that $J_S + p\mathcal{O}_S$ is locally principal.

- (1) We have the inclusion $\mathscr{C}_1 \subset \mathscr{C}'^{(m)}_{\gamma,1}$.
- (2) For any $n, m \in \mathbb{N}$ such that $n + 1 \leq p^m$, we have the inclusion $\mathscr{C}_n \subset \mathscr{C}'_{\gamma,n}^{(m)}$;
- (3) We have the equality $\bigcup_{m \in \mathbb{N}} \mathscr{C}'^{(m)}_{\gamma} = \mathscr{T}\mathrm{hick}_{(p)}.$

Proof. — Let us check the first two assertions. Let $u: U \to T$ a S-log thickening of order n, let \mathcal{I} be the ideal defined by the closed immersion \underline{u} . When $\mathcal{I}^2 = 0$, we get a PD-structure γ on \mathcal{I} defined by putting $\gamma_n = 0$ for any integer $n \ge 2$. Since $J_S + p\mathcal{O}_S$ is locally principal, then from [4, 1.3.2.(i).b)] γ extends to T. Hence, $\mathscr{C}_1 \subset \mathscr{C}'_{\gamma,1}^{(0)}$, which yields the first inclusion to prove. Suppose now $\mathcal{I}^{n+1} = 0$ and $n+1 \le p^m$. In that case, $\mathcal{I}^{(p^m)} = 0$. Hence, $(0, \delta)$ is an *m*-PD-structure of \mathcal{I} (where δ is the unique PD-structure on 0). Let us check that the *m*-PD structure $(0, \delta)$ of \mathcal{I} is compatible with γ . By definition, we have to check two properties (see [4, 1.3.2.(ii))]). Since γ extends to T, then the property [4, 1.3.2.1] is satisfied (see Definition [4, 1.2.2]). The second one [4, 1.3.2.2] is a straightforward consequence of Lemma [4, 1.2.4.(i)]. Hence, $(u, \delta) \in \mathscr{C}_{\gamma}^{(m)}$. Since $\mathcal{I}^{n+1} = 0$, we have in fact $(u, \delta) \in \mathscr{C}_{\gamma,n}^{(m)}$. By definition, this yields $u \in \mathscr{C}'_{\gamma,n}^{(m)}$.

Let us check the last statement. The inclusion $\bigcup_{m\in\mathbb{N}} \mathscr{C}'_{\gamma}^{(m)} \subset \mathscr{T}$ hick $_{(p)}$ is tautologic. Conversely, let $u: U \hookrightarrow T$ a S-log thickening of order (p^m) , let \mathcal{I} be the ideal defined by the closed immersion \underline{u} . Since $\mathcal{I}^{(p^m)} = 0$, then following the first part of the proof, we get that the *m*-PD structure $(0, \delta)$ is compatible with γ of \mathcal{I} . Hence, $u \in \mathscr{C}'_{\gamma}^{(m)}$, which concludes the proof of the last statement.

PROPOSITION 1.55. — Suppose that $J_S + p\mathcal{O}_S$ is locally principal. Let $f: X \to Y$ be an S-morphism of fine log schemes.

- (1) The morphism f is log p-étale if and only if for any $m \in \mathbb{N}$, f is log p-étale of level m compatible with γ . Similarly replacing "étale" by "unramified".
- (2) If f is formally log étale of level m compatible with γ then f is fine formally log étale. Similarly replacing "étale" by "unramified".

Proof. — This is a consequence of Lemma 1.54.

LEMMA 1.56. — Suppose that $J_S + p\mathcal{O}_S$ is locally principal. Let $f: X \to Y$ be an S-morphism of fine log-schemes and $f_0: X_0 \to Y_0$ be the induced S_0 -morphism.

The morphism f is formally log étale of level m compatible with γ (log p-étale of level m compatible with γ) if and only if f is fine formally log etale and f_0 is formally log étale of level m compatible with γ (log p-étale of level m compatible with γ). Similarly replacing everywhere "étale" by "unramified".

Proof. — Using $\mathscr{C}_1 \subset \mathscr{C}'^{(m)}_{\gamma,1}$ (see Lemma 1.54), we proceed similarly to Lemma 1.20. □

Point 1.57. — Forgetting log structures, i.e. replacing in Definitions 1.4 and 1.48 fine log-schemes by schemes, we define similarly the categories $\underline{\mathscr{C}}$, $\underline{\mathscr{C}}_{\gamma}^{(m)}$ and $\underline{\mathscr{C}}_{\gamma,n}^{(m)}$. Following [4, 1.4.1 and 2.1.1], the forgetful functor $\underline{\operatorname{For}}^{(m)}$: $\underline{\mathscr{C}}_{\gamma}^{(m)} \to \underline{\mathscr{C}}$ defined by $(\underline{u}, \delta) \mapsto \underline{u}$ has a right adjoint that we will denote by $\underline{P}_{(m),\gamma}$. If \underline{u} is an object of $\underline{\mathscr{C}}$ then $\underline{P}_{(m),\gamma}(\underline{u})$ is called the "*m*-PD-envelope compatible with γ of \underline{u} ". Moreover, since p is nilpotent then the morphism of schemes induced by the targets of $\underline{P}_{(m),\gamma}(\underline{u}) \to \underline{u}$ is affine (see [4, 2.1.1]).

Point 1.58. — Let $u: Z \hookrightarrow X$ be an exact S-immersion of fine logschemes. Set $(\underline{v}, \delta) := \underline{P}_{(m),\gamma}(\underline{u})$ (see Point 1.57). Let (v, δ) be the object

of $\mathscr{C}_{\gamma}^{(m)}$ whose underlying object of $\underline{\mathscr{C}}_{\gamma}^{(m)}$ is (\underline{v}, δ) and v is defined so that the morphism $v \to u$ of \mathscr{C} is strict (see Definition 1.4). Then (v, δ) is the *m*-PD-envelope compatible with γ of u.

Remark 1.59. — Let $\alpha: (u', \delta') \to (u, \delta)$ be a strict cartesian morphism of $\mathscr{C}_{\gamma}^{(m)}$. Let (u'', δ'') be an object of $\mathscr{C}_{\gamma}^{(m)}$ and $\beta: u'' \to u'$ be a morphism of \mathscr{C} . We remark that if $\operatorname{For}^{(m)}(\alpha) \circ \beta$ is in the image of $\operatorname{For}^{(m)}$ then so is β . Indeed, the morphism α is defined by a cartesian diagram of the form (1.16). Since α is moreover strict, then we remark that $\underline{Z}' = \underline{Z} \times \underline{X} \underline{X}'$, i.e. the diagram (1.16) remains cartesian after applying the forgetful functor from the category of fine log schemes to the category of schemes. Hence, we can conclude.

Point 1.60. — Let $u' \to u$ be a strict, flat, cartesian morphism of \mathscr{C} , i.e. let



be a cartesian square whose morphism g is strict and \underline{g} is flat. Suppose that the *m*-PD-envelope compatible with γ of u exists (in fact, this existence will be proved later in Proposition 1.63). Let (v, δ) be this *m*-PD-envelope. Set $v' := v \times_u u'$ and let $g' : v' \to v$ be the projection. Since \underline{g} is flat and g is strict, then g' is strict and $\underline{g'}$ is flat. From [4, 1.3.2.(i)], there exists a canonical *m*-PD-structure δ' compatible with γ on the ideal defined by $v' := v \times_u u'$ such that the projection $g' : v' \to v$ induces a strict cartesian morphism of $\mathscr{C}_{\gamma}^{(m)}$ of the form $(v', \delta') \to (v, \delta)$. With Remark 1.59, we can check that (v', δ') is an *m*-PD-envelope compatible with γ of u'.

LEMMA 1.61. — Let $f: X \to Y$ be formally log étale of level m compatible with γ S-morphism, $u: Z \hookrightarrow X$ and $v: Z \hookrightarrow Y$ be two S-immersions of fine log schemes such that $v = f \circ u$.

- (1) If the *m*-PD envelope of order *n* compatible with γ of *u* exists then it is also an *m*-PD envelope of order *n* of *v*.
- (2) Suppose f is moreover log p-étale of level m compatible with γ. In that case, if the m-PD envelope compatible with γ of u exists then it is also an m-PD envelope of v.

Proof. — Let us check the second statement (resp. the first one). Let $(P(u), \delta)$ be the *m*-PD envelope (resp. of order *n*) compatible with γ of *u*. Let us check that the composition of the canonical morphism $P(u) \to u$

with the morphism $u \to v$ (induced by f) satisfies the universal property of the *m*-PD envelope (resp. of order *n*) compatible with γ of v. Let (v', δ') be an object of $\mathscr{C}_{\gamma}^{(m)}$ (resp. $\mathscr{C}_{\gamma,n}^{(m)}$) and $g \colon v' \to v$ be a morphism of \mathscr{C} . Using the universal property of log *p*-étaleness of level *m* compatible with γ (resp. formally log étaleness) of Definition 1.50, we get a unique morphism $h \colon v' \to u$ of \mathscr{C} whose composition with $u \to v$ gives g. Using the universal property of the *m*-PD-envelope of u compatible with γ that there exists a unique morphism $(v', \delta') \to (P(u), \delta)$ of $\mathscr{C}_{\gamma}^{(m)}$ (resp. $\mathscr{C}_{\gamma,n}^{(m)}$) such that the composition of $v' \to P(u)$ with $P(u) \to u$ is h.

LEMMA 1.62. — The inclusion functor $\operatorname{For}_n : \mathscr{C}_{\gamma,n}^{(m)} \to \mathscr{C}_{\gamma}^{(m)}$ has a right adjoint. We denote by $Q_{(m),\gamma}^n : \mathscr{C}_{\gamma}^{(m)} \to \mathscr{C}_{\gamma,n}^{(m)}$ this right adjoint functor. The functor $Q_{(m),\gamma}^n$ preserves the sources.

Proof. — Let (u, δ) be an object of $\mathscr{C}_{\gamma}^{(m)}$ and \mathcal{I} be the ideal defined by the exact closed immersion $u: Z \hookrightarrow X$. Let $Q^n \hookrightarrow X$ be the exact closed immersion which is defined by $\mathcal{I}^{\{n+1\}_{(m)}}$. It follows from [4, 1.3.8.(iii)] that $Q^n_{(m),\gamma}(u)$ exists and is equal to the exact closed immersion $Z \hookrightarrow Q^n$. \Box

PROPOSITION 1.63. — Let $u: Z \hookrightarrow X$ be an object of \mathscr{C} .

- The m-PD-envelope compatible with γ of u exists. In other words, the canonical functor For^(m): C_γ^(m) → C has a right adjoint. We denote by P_{(m),γ}: C → C_γ^(m) this right adjoint functor. Similarly, the m-PD-envelope of order n compatible with γ of u exists, i.e. we get the right adjoint functor Pⁿ_{(m),γ}: C → C_γ^(m) of the canonical functor For^(m)_n: C_{γn}^(m) → C. We have the relation Pⁿ_{(m),γ} = Qⁿ_{(m),γ} ∘ P_{(m),γ}.
- (2) If γ extends to Z then the source of $P_{(m),\gamma}(u)$ is Z.
- (3) By denoting abusively by $P_{(m),\gamma}(u)$ (resp. $P_{(m),\gamma}^n(u)$) the target of the arrow $P_{(m),\gamma}(u)$ (resp. $P_{(m),\gamma}^n(u)$), the underlying morphism of schemes of $P_{(m),\gamma}(u) \to X$ (resp. $P_{(m),\gamma}^n(u) \to X$) is affine. We denote by $\mathcal{P}_{(m),\gamma}(u)$ (resp. $\mathcal{P}_{(m),\gamma}^n(u)$) the quasi-coherent \mathcal{O}_X algebra so that $\underline{P_{(m),\gamma}(u)} = \operatorname{Spec}(\mathcal{P}_{(m),\gamma}(u))$ (resp. $\underline{P}_{(m),\gamma}^n(u) =$ $\operatorname{Spec}(\mathcal{P}_{(m),\gamma}^n(u))$). The m-PD structure of $\mathcal{P}_{(m),\gamma}(u)$ will be denoted by $(\mathscr{I}_{(m),\gamma}(u), \mathscr{J}_{(m),\gamma}(u), [])$.
- (4) Suppose that $J_S + p\mathcal{O}_S$ is locally principal and that X is noetherian (i.e. \underline{X} is noetherian). Then $P^n_{(m),\gamma}(u)$ a noetherian scheme.

Proof.

Step 1. — First, let us check the proposition concerning the existence of $P_{(m),\gamma}(u)$ and its properties (i.e. the second part of the proposition and the

affinity of the morphism $P_{(m),\gamma}(u) \to X$). Using Point 1.60, the existence of $P_{(m),\gamma}(u)$ and its properties are étale local on X. Hence, by Point 1.8, we may thus assume that there exists a commutative diagram of the form



such that f is log étale, \underline{f} is affine and \widetilde{u} is an exact closed S-immersion. In that case, following Point 1.58 the m-PD-envelope compatible with γ of \widetilde{u} exists and the induced object of $\underline{\mathscr{C}}_{\gamma}^{(m)}$ is $\underline{P}_{(m),\gamma}(\underline{\widetilde{u}})$. Following Lemma 1.61, the m-PD-envelope compatible with γ of u exists and is isomorphic to that of \widetilde{u} . Concerning the second statement, when γ extends to Z, following [4, 2.1.1] (or [4, 1.4.5] for the affine version), the source of the immersion $\underline{P}_{(m),\gamma}(\underline{\widetilde{u}})$ is \underline{Z} . Since $P_{(m),\gamma}(\overline{\widetilde{u}})$, \widetilde{u} are exact closed immersion, since the morphism $P_{(m),\gamma}(\overline{\widetilde{u}}) \to \widetilde{u}$ is strict (see Point 1.58), then so is the morphism of sources induced by $P_{(m),\gamma}(\overline{\widetilde{u}}) \to \widetilde{u}$. Hence, we get the second statement. We can check the third statement by recalling that the target of $\underline{P}_{(m),\gamma}(\underline{\widetilde{u}})$ is affine over $\underline{\widetilde{X}}$ (see Point 1.57) and that $\underline{P}_{(m),\gamma}(\underline{\widetilde{u}}) \to \widetilde{u}$ is strict. Concerning the noetherianity, if X is noetherian then so is \widetilde{X} . Hence, using [4, 1.4.4] and the description of the m-PD filtration given in the proof of [4, A.2], we get that $\underline{P}_{(m),\gamma}(\underline{\widetilde{u}})$ is noetherian (but not $\underline{P}_{(m),\gamma}(\underline{\widetilde{u}})$).

Step 2. — From Lemma 1.62, we can check that the functor $Q_{(m),\gamma}^n \circ P_{(m),\gamma}$ is a right adjoint of $\operatorname{For}_n^{(m)} \colon \mathscr{C}_{\gamma,n}^{(m)} \to \mathscr{C}$. Moreover, with the description of the functor $Q_{(m),\gamma}^n$ given in the proof of Lemma 1.62, we can check the other properties concerning $P_{(m),\gamma}^n$ from that of $P_{(m),\gamma}$.

DEFINITION 1.64. — Let u be an object of \mathscr{C} . We say that $P_{(m),\gamma}^n(u)$ is the "nth infinitesimal neighborhood of level m compatible with γ of u".

Remark 1.65. — Let (u, δ) be an object of $\mathscr{C}_{\gamma}^{(m)}$. Then $P_{(m),\delta}(u) = (u, \delta)$. But, beware that $P_{(m),\gamma}(u) \neq (u, \delta)$ in general.

Point 1.66 (The case of an exact closed immersion). — Let $u: \mathbb{Z} \hookrightarrow \mathbb{X}$ be an exact closed S-immersion of fine log-schemes and \mathcal{I} be the ideal defined by u. We denote by $u^{(m)}: \mathbb{Z}^{(m)} \hookrightarrow \mathbb{X}$ the exact closed S-immersion of fine log-schemes so that $\mathcal{I}^{(p^m)}$ is the ideal defined by $u^{(m)}$. Since the closed immersion u is exact, in the proof of Proposition 1.63, we can skip the part concerning the exactification of u (i.e. we can suppose f = id or equivalently $\tilde{u} = u$). Hence, we remark that, as in the proof of [4, 1.4.1], we get the equality

(1.19)
$$P_{(m),\gamma}(u) = P_{(0),\gamma}(u^{(m)}).$$

We have also the same construction as in the proof of [4, 1.4.1] (too technical to be described here in few words) of the *m*-PD ideal $(\mathscr{I}_{(m),\gamma}(u), \mathscr{I}_{(m),\gamma}(u), [])$ of $\mathcal{P}_{(m),\gamma}(u)$ directly from the level 0 case. For the detailed descriptions, see the proof of [4, 1.4.1]. These descriptions, in particular (1.19), are useful to check the Frobenius descent for arithmetic \mathcal{D} -modules (see [5, 2.3.6]).

LEMMA 1.67. — We have the equality $P_{(m),\gamma}^n \circ \operatorname{For}_n \circ P^n = P_{(m),\gamma}^n$, where $\operatorname{For}_n \colon \mathscr{C}_n \to \mathscr{C}$ is the canonical functor and $P^n \colon \mathscr{C} \to \mathscr{C}_n$ is its right adjoint (see Proposition 1.11).

Proof. — Let $u: Z \hookrightarrow X$ be an object of \mathscr{C} . Looking at the construction of P^n and $P^n_{(m),\gamma}$ (see the proof of Proposition 1.11 and Proposition 1.63), we reduce to the case where u is an exact closed immersion. In that case, the Lemma is a reformulation of [4, 1.4.3.2].

The following proposition will not be useful later but it allows us to extend Proposition 1.13 is some particular case.

PROPOSITION 1.68. — Suppose that $J_S + p\mathcal{O}_S$ is locally principal. Let $f: X \to Y$ be an S-morphism of fine log schemes and $\Delta_{X/Y}: X \hookrightarrow X \times_Y X$ (as always the product is taken in the category of fine log schemes) be the diagonal S-immersion. The following assertions are equivalent:

- (1) the morphism f is fine formally log unramified;
- (2) the morphism $P^1(\Delta_{X/Y})$ is an isomorphism;
- (3) the morphism f is formally log unramified of level m compatible with γ;
- (4) the morphism $P^1_{(m),\gamma}(\Delta_{X/Y})$ is an isomorphism.

Proof. — The equivalence between (1) and (2) has already been checked (see Proposition 1.13). Following Lemma 1.54, since $J_S + p\mathcal{O}_S$ is locally principal, then $\mathscr{C}_1 \subset \mathscr{C}'^{(m)}_{\gamma,1}$. Hence, we have (3) \Rightarrow (1). It follows from Lemma 1.67 that $P^1_{(m),\gamma}(P^1(\Delta_{X/Y})) = P^1_{(m),\gamma}(\Delta_{X/Y})$. If $P^1(\Delta_{X/Y})$ is an isomorphism, then $P^1_{(m),\gamma}(P^1(\Delta_{X/Y})) = P^1(\Delta_{X/Y})$. Hence, we get the implication (2) \Rightarrow (4). It remains to check (4) \Rightarrow (3). Suppose $P^1_{(m),\gamma}(\Delta_{X/Y})$ is an isomorphism and let $(\iota, \delta) \in \mathscr{C}_{\gamma, 1}^{(m)}$ and let



be a commutative diagram of fine log schemes. Suppose there exist a morphism $u: T \to X$ such that $u \circ \iota = u_0$ and $f \circ u = v$, and a morphism $u': T \to X$ such that $u' \circ \iota = u_0$ and $f \circ u' = v$. We get the morphism $(u, u'): T \to X \times_Y X$. We denote by and $\phi: \iota \to \Delta_{X/Y}$ be a morphism of \mathscr{C} induced by (u', u) and u_0 . Using the universal property of the *m*-PD envelope of order 1, there exists a unique morphism $\psi: (\iota, \delta) \to P^1_{(m),\gamma}(\Delta_{X/Y})$ of $\mathscr{C}^{(m)}_{\gamma,1}$ such that the composition of $\operatorname{For}^{(m)}_n(\psi)$ with the canonical map $P^1_{(m),\gamma}(\Delta_{X/Y}) \to \Delta_{X/Y}$ is ϕ . Since $P^1_{(m),\gamma}(\Delta_{X/Y})$ is an isomorphism, this yields that $(u, u'): T \to X \times_Y X$ is the composition of a morphism of the form $T \to X$ with $\Delta_{X/Y}$. Hence, u = u'.

LEMMA 1.69. — Let $u \to v$ be a morphism of \mathscr{C} . Let δ be the *m*-PDstructure of $P_{(m),\gamma}(v)$ and $w := P_{(m),\gamma}(v) \times_v u$ (this is the product in \mathscr{C}). We denote by $P_{(m),\delta}(w)$ the *m*-PD-envelope of *w* compatible with δ . Then $P_{(m),\delta}(w)$ and $P_{(m),\gamma}(u)$ are isomorphic in $\mathscr{C}_{\gamma}^{(m)}$. Moreover, $P_{(m),\delta}^n(w)$ and $P_{(m),\gamma}^n(u)$ are isomorphic in $\mathscr{C}_{\gamma,n}^{(m)}$.

Proof. — Since the second assertion is checked in the same way, let us prove the first one. Let us check that the composition $P_{(m),\delta}(w) \to w \to u$ satisfies the universal property of $P_{(m),\gamma}(u) \to u$. Let $(u', \delta') \in \mathscr{C}_{\gamma}^{(m)}$ and $f: u' \to u$ be a morphism of \mathscr{C} . First let us check the existence. Composing f with $u \to v$ we get a morphism $g: u' \to v$. Using the universal property of the m-PD envelope, there exists a morphism $\phi: (u', \delta') \to (P_{(m),\gamma}(v), \delta)$ of $\mathscr{C}_{\gamma}^{(m)}$ such that the composition $u' \to P_{(m),\gamma}(v) \to v$ is g. Hence, we get the morphism $(\phi, f): u' \to w$. Using the universal property of $P_{(m),\delta}(w)$, we get a morphism $u' \to P_{(m),\delta}(w)$ of $\mathscr{C}_{\delta}^{(m)}$ (and then of $\mathscr{C}_{\gamma}^{(m)}$) whose composition with $P_{(m),\delta}(w) \to w \to u$ is f. Let us check the unicity. Let $\alpha: u' \to P_{(m),\delta}(w)$ be a morphism of $\mathscr{C}_{\gamma}^{(m)}$ whose composition with $P_{(m),\delta}(w) \to w \to u$ is f. This implies that the composition of α with $P_{(m),\delta}(w) \to w \to P_{(m),\gamma}(v) \to v$ is g. Since the composition $P_{(m),\delta}(w) \to w \to P_{(m),\gamma}(v)$ is a morphism of $\mathscr{C}_{\delta}^{(m)}$, then so is the composition of α with $P_{(m),\delta}(w) \to w \to P_{(m),\gamma}(v)$ (in particular, this implies that $\alpha \in \mathscr{C}_{\delta}^{(m)}$). Using the universal property of $P_{(m),\gamma}(v)$, this latter composition morphism is uniquely determined by g. Hence, the composition of α with $P_{(m),\delta}(w) \to w$ is a morphism of \mathscr{C} uniquely determined by f. Since α is a morphism of $\mathscr{C}_{\delta}^{(m)}$, we conclude using the universal property of $P_{(m),\delta}(w)$.

1.5. Formal log-bases, weak log-smoothness, log p-bases, log p-smoothness (of level m)

Let (I_S, J_S, γ) be a quasi-coherent *m*-PD-ideal of \mathcal{O}_S .

DEFINITION 1.70. — Let $f: X \to Y$ be an morphism of fine S-log schemes.

- (1) We say that a set $(b_{\lambda})_{\lambda \in \Lambda}$ of elements of $\Gamma(X, M_X)$ is a "formal log basis of f" if the corresponding Y-morphism $X \to Y \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{N}^{(\Lambda)}}$ is fine formally log étale (see Notation 1.44). When Λ is a finite set, we say that $(b_{\lambda})_{\lambda \in \Lambda}$ is a "finite formal log basis of f".
- (2) We say that f is "weakly log smooth" if, étale locally on X, f has finite formal log bases. Notice that this notion of weak log smoothness is étale local on Y.

DEFINITION 1.71. — Let $f: X \to Y$ be an morphism of fine S-log schemes.

- (1) We say that a set $(b_{\lambda})_{\lambda \in \Lambda}$ of elements of $\Gamma(X, M_X)$ is a "log *p*-basis of *f*" if the corresponding *Y*-morphism $X \to Y \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{N}^{(\Lambda)}}$ is log *p*-étale. When Λ is a finite set, we say that they $(b_{\lambda})_{\lambda \in \Lambda}$ is a "finite log *p*-basis of *f*"
- (2) We say that f "has log p-bases locally" if, étale locally on X, f has log p-bases. Notice that this notion of "having log p-bases locally" is étale local on Y.
- (3) We say that f is "log p-smooth" if, étale locally on X, f has finite log p-bases. Notice that this notion of log p-smoothness is étale local on Y.

Remark 1.72. — Tsuji defined in [31, 1.4.1)] (resp. [31, 1.4.2)]) the notion of "a morphism of log-schemes having *p*-basis" (resp. "a morphism of logschemes having *p*-bases locally"). Our notion of having log *p*-bases locally (see Definition 1.71) is more general that Tsuji's notion of having *p*-bases locally. Indeed, this fact is a consequence of Lemmas 1.35 and 1.38 and [31, Lemma 1.5]. DEFINITION 1.73. — Let $f: X \to Y$ be an morphism of fine S-log schemes.

- (1) We say that a finite set $(b_{\lambda})_{\lambda=1,\dots,r}$ of elements of $\Gamma(X, M_X)$ is a "formal log basis of level m (compatible with γ) of f" if the corresponding Y-morphism $X \to Y \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{N}^r}$ is formally log étale of level m (compatible with γ).
- (2) We say that f is "weakly log smooth of level m (compatible with γ)" if, étale locally on X, f has formal log bases of level m (compatible with γ). Notice that this notion is étale local on Y (use the last remark of Remark 1.53).
- (3) We say that a finite set (b_λ)_{λ=1,...,r} of elements of Γ(X, M_X) is a "log p-basis of level m (compatible with γ) of f" if the corresponding Ymorphism X → Y ×_{Z/pⁱ⁺¹Z} A_N^r is log p-étale of level m (compatible with γ).
- (4) We say that f is "log p-smooth of level m (compatible with γ)" if, étale locally on X, f has finite log p-bases of level m (compatible with γ). Notice that this notion of log p-smoothness of level m (compatible with γ) is étale local on Y (use the last remark of Remark 1.53).

The following Proposition indicates the link between our notions related to log smoothness.

PROPOSITION 1.74. — Let $f: X \to Y$ be an S-morphism of fine log-schemes.

- (1) If f is log smooth then f is log p-smooth.
- (2) If f is log p-smooth then f is log p-smooth of level m compatible with γ .
- (3) If f is log p-smooth of level m compatible with γ then f is weakly log smooth of level m compatible with γ.
- (4) If f is weakly log smooth of level m compatible with γ then f is weakly log smooth.
- (5) If f is log p-smooth (resp. weakly log smooth) of level m + 1 compatible with γ then f is log p-smooth (resp. weakly log smooth) of level m compatible with γ.

Proof. — The first statement is a straightforward consequence of Theorem [28, IV.3.2.6] and Proposition 1.25. Using Proposition 1.55.2, we get the last assertion. The other implications are consequences of Remark 1.53. \Box

PROPOSITION 1.75. — The collection of formally log étale of level m compatible with γ (resp. log p-étale of level m compatible with γ , resp.

log p-étale, resp. log smooth, resp. log p-smooth, resp. having log p-bases locally, resp. weakly log smooth, resp. weakly log smooth of level m compatible with γ , resp. log p-smooth of level m compatible with γ) morphisms of fine S-log schemes is stable under base change and under composition.

Proof. — The etale cases are already known (see Lemmas 1.19, 1.21 and 1.52). Let us check the stability of the collection of morphisms having log *p*-bases locally. Using the étale case, the stability under base change is obvious. Moreover, from the non respective case, if $X \to Y \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{N}^{(\Lambda)}}$ and $Y \to Z \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{N}^{(\Lambda')}}$ are log *p*-étale, then so is the composition $X \to Z \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{N}^{(\Lambda')}}$. This implies the stability under composition. The other cases are checked similarly. □

The following definition will not be useful but we add it for completeness.

DEFINITION 1.76. — Let $f: X \to Y$ be an S-morphism of fine log schemes. We say that f is formally log p-smooth if it satisfies the following property: for any commutative diagram of fine log schemes of the form



such that ι is an object of $\mathscr{C}_{(p)}$, there exists etale locally on X a morphism $u: T \to X$ such that $u \circ \iota = u_0$ and $f \circ u = v$.

Remark 1.77. — Let $f: X \to Y$ be an S-morphism of fine log schemes. Using Theorem [18, IV.18.1.2], we can check that if f has log p-bases locally then f is formally log p-smooth. (The converse seems to be false even if we do not have a counter example.) In particular, if f is log p-smooth then fis formally log p-smooth.

Finally, we remark that f is log p-étale if and only if f is log p-unramified and formally log p-smooth.

Notation 1.78. — Let D be a fine log scheme over S. We denote by $\mathcal{O}_D\langle T_1, \ldots, T_r \rangle_{(m)}$ the *m*-PD-polynomial ring (see the definition just after [4, 1.5.1]). We denote by $(\mathscr{I}_{D,(m),r}, \mathscr{J}_{D,(m),r}, [\cdot])$ its canonical *m*-PD-structure.

LEMMA 1.79. — Let $r \ge 0$ be an integer, $(v, \delta) \in \mathscr{C}_{\gamma}^{(m)}$ where $v: T \hookrightarrow D$ is an exact closed S-immersion of fine log-schemes and $(\widetilde{\mathcal{K}}, \delta)$ is an m-PD-structure compatible with γ on the ideal \mathcal{K} of \mathcal{O}_D defined by v. Let $(e_{\lambda})_{\lambda=1,\ldots,r}$ be the canonical basis of \mathbb{Z}^r . Let $i: D \to D \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{Z}^r}$ be the exact closed D-immersion defined by $e_{\lambda} \mapsto 1 \in \Gamma(D, M_D)$. With Proposition 1.63 and Notation 1.78, we have the following properties.

(1) The homomorphism of rings

$$\mathcal{O}_D(T_1,\ldots,T)_{(m)}\to\mathcal{P}_{(m),\delta}(i\circ v)$$

given by $T_{\lambda} \mapsto e_{\lambda} - 1$ is an isomorphism.

(2) The structural *m*-PD structure $(\mathscr{I}_{(m),\gamma}(i \circ v), \mathscr{I}_{(m),\gamma}(i \circ v), [])$ of $\mathcal{P}_{(m),\delta}(i \circ v)$ is given by

$$\begin{aligned} \mathscr{I}_{(m),\gamma}(i \circ v) &= \mathscr{I}_{D,(m),r} + \mathcal{KP}_{(m),\delta}(i \circ v), \\ \mathscr{J}_{(m),\gamma}(i \circ v) &= \mathscr{I}_{D,(m),r} + \widetilde{\mathcal{KP}}_{(m),\delta}(i \circ v). \end{aligned}$$

Proof. — By using Remark 1.4.3.(iii) of [4] and Point 1.58, we can suppose that v = id. Since the ideal of the exact closed immersion *i* is generated by the regular sequence $(e_{\lambda} - 1)_{\lambda=1,\ldots,r}$, using [4, 1.5.3] and Point 1.58 we can check that the morphism of \mathcal{O}_D -algebras $\mathcal{O}_D\langle T_1,\ldots,T_r\rangle_{(m)} \to$ $\mathcal{P}_{(m),\delta}(i)$ given by $T_{\lambda} \mapsto e_{\lambda} - 1$ is an isomorphism. \square

Notation 1.80. — With Lemma 1.79, we set $\mathcal{O}_{(v,\delta)}(T_1,\ldots,T_r)_{(m)}$:= $\mathcal{P}_{(m),\delta}(i \circ v)$ and $\mathcal{O}_{(v,\delta)}\langle T_1, \dots, T_r \rangle_{(m),n} := \mathcal{P}^n_{(m),\delta}(i \circ v).$

LEMMA 1.81. — Let $i: X \hookrightarrow P$ be an exact closed S-immersion of fine log schemes. Then $\ker(i^{-1}\mathcal{O}_P^* \to \mathcal{O}_X^*) = \ker(i^{-1}M_P^{\mathrm{gr}} \to M_X^{\mathrm{gr}})$. In particular, $\ker(\mathcal{O}_{P\bar{x}}^* \to \mathcal{O}_{X\bar{x}}^*) = \ker(M_{P\bar{x}}^{\mathrm{gr}} \to M_{X\bar{x}}^{\mathrm{gr}}) \text{ for any geometric point } \bar{x} \text{ of } X.$

Proof. — Let \bar{x} be a geometric point of X. Since i is an exact closed immersion, we have $(M_P/\mathcal{O}_P^*)_{\overline{x}} = (M_X/\mathcal{O}_X^*)_{\overline{x}}$ (use [21, 1.4.1]) and thus $(M_P^{\rm gp}/\mathcal{O}_P^*)_{\bar{x}} = (M_X^{\rm gp}/\mathcal{O}_X^*)_{\bar{x}}$ (because the functor $M \mapsto M^{\rm gr}$ commutes with inductive limits). Hence, the inclusion $\ker(\mathcal{O}_{P,\bar{x}}^* \to \mathcal{O}_{X,\bar{x}}^*) \subset$ $\ker(M_{P,\bar{x}}^{\mathrm{gr}} \to M_{X,\bar{x}}^{\mathrm{gr}})$ is in fact an equality. Since we have the canonical inclusion $\ker(i^{-1}\mathcal{O}_P^* \to \mathcal{O}_X^*) \subset \ker(i^{-1}M_P^{\operatorname{gr}} \to M_X^{\operatorname{gr}})$, this yields that this latter inclusion is an equality. \square

The next result is our main motivation for defining weak log smoothness of level m (compatible with γ) and log p-smoothness of level m (see Definition 1.73).

PROPOSITION 1.82. — Let $f: X \to Y$ be an S-morphism of fine logschemes, $(b_{\lambda})_{\lambda=1,\ldots,r}$ be some elements of $\Gamma(X, M_X)$ such that $(b_{\lambda})_{\lambda=1,\ldots,r}$ is a formal log basis of level m compatible with γ of f.

Let $u: Z \hookrightarrow X$ and $v: Z \hookrightarrow Y$ be two S-immersions of fine log schemes such that $v = f \circ u$. Suppose given $y_{\lambda} \in \Gamma(Y, M_Y)$ whose images in $\Gamma(Z, M_Z)$ coincide with the images of b_{λ} .

Let $P_{(m),\gamma}(u) = (T' \hookrightarrow D', \delta'), P_{(m),\gamma}(v) = (T \hookrightarrow D, \delta)$, and $\alpha \colon D' \to X$ be the canonical morphism. Using multiplicative notation, put $u_{\lambda} := \frac{\alpha^*(b_{\lambda})}{\alpha^*(f^*(y_{\lambda}))} \in \ker(\Gamma(D', M_{D'}^{\mathrm{gr}}) \to \Gamma(T', M_{T'}^{\mathrm{gr}})) = \ker(\Gamma(D', \mathcal{O}_{D'}^*) \to \Gamma(T', \mathcal{O}_{T'}^*))$ (see Lemma 1.81). Let $P_{(m),\gamma}^n(u) = (T' \hookrightarrow D'_n, \delta'_n)$, and $u_{\lambda,n}$ be the image of u_{λ} in $\ker(\Gamma(D'_n, \mathcal{O}_{D'_n}^*) \to \Gamma(T', \mathcal{O}_{T'}^*))$.

(1) By using Notation 1.80, we have the isomorphism of m-PD- \mathcal{O}_D -algebras

(1.21)
$$\mathcal{O}_{P_{(m),\gamma}(v)}\langle T_1, \dots, T_r \rangle_{(m),n} \xrightarrow{\sim} \mathcal{P}^n_{(m),\gamma}(u) \\ T_{\lambda} \longmapsto u_{\lambda,n} - 1.$$

(2) If $(b_{\lambda})_{\lambda=1,\dots,r}$ is moreover a log p-basis of level m compatible with γ of f, then, we have the isomorphism of m-PD- \mathcal{O}_D -algebras

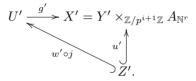
(1.22)
$$\mathcal{O}_{P_{(m),\gamma}(v)}\langle T_1, \dots, T_r \rangle_{(m)} \xrightarrow{\sim} \mathcal{P}_{(m),\gamma}(u) \\ T_{\lambda} \longmapsto u_{\lambda} - 1.$$

Proof. — In order to check (1.21) (resp. (1.22)), using the first part of Lemma 1.61 (resp. the second part of Lemma 1.61) and using the first remark of Remark 1.3, we may assume that $X = Y \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{N}^r}$, $f: X \to Y$ is the first projection, and that the family $(b_\lambda)_{\lambda=1,\dots,r}$ are the elements of $\Gamma(X, M_X)$ corresponding to the canonical basis $(e_\lambda)_{\lambda=1,\dots,r}$ of \mathbb{N}^r . Using Lemma 1.69, we may furthermore assume that Y = S, $Z \hookrightarrow Y$ is the exact closed immersion whose ideal of definition is I_S . In particular, we get D = Y and γ is the canonical *m*-PD structure of *D*.

Let \overline{z} be a geometric point of Z. From Point 1.8, there exists a commutative diagram of the form

where g is log étale, h is an étale neighborhood of \overline{z} in Z, and w is an exact closed S-immersion. We set $v_{\lambda} := \frac{g^*(b_{\lambda})}{(f \circ g)^*(y_{\lambda})} \in \operatorname{Ker}(\Gamma(U, M_U^{\mathrm{gp}}) \to \Gamma(W, M_W^{\mathrm{gp}}))$. Since w is an exact closed immersion, using Lemma 1.81, shrinking U if necessary we may thus assume that $v_{\lambda} \in \Gamma(U, \mathcal{O}_U^*)$.

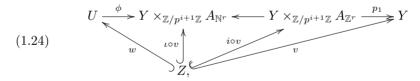
Step 1. — In this step, we reduce to the case where h = id. According to [18, IV.18.1.1], there exist an étale neighborhood $Y' \to Y$ of \overline{z} in Y and an open Z-immersion (see Definition 1.2) $\rho: Z' := Z \times_Y Y' \to W$ which is a morphism of étale neighborhoods of \overline{z} in Z (in particular $h \circ \rho: Z \times_Y Y' \to Z$ is the canonical projection). Let us use the prime symbol to denote the base change by $Y' \to Y$ of a Y-log scheme or a morphism of Y-log schemes. Set $j := (\rho, v') \colon Z' \to W \times_Y Y' = W'$. Since $h \circ \rho \colon Z' \to Z$ is the canonical projection, then we compute that $h' \circ j = \text{id}$. Since id is an immersion, then j is an immersion (see the first remark of Remark 1.3). Since h' and id are etale then so is j. Hence, j is an open Y-immersion. Using $h' \circ j = \text{id}$, we get the commutative diagram over Y'



Using Remark 1.3(3) we may assume (shrinking U if necessary) that the exact Y-immersion $w' \circ j$ is closed. Since the proposition is étale local on Y, we can drop the primes, i.e. we can suppose h = id.

Step 2. — Consider the Y-morphism $\phi: U \to Y \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{N}^r}$ defined by the v_{λ} 's. Since the $(\operatorname{dlog} g^*(b_1), \ldots, \operatorname{dlog} g^*(b_r))$ forms a basis of $\Omega_{U/Y}$ (because g is log étale), then so does $(\operatorname{dlog} v_1, \ldots, \operatorname{dlog} v_r)$. This implies that the canonical map $\phi^*\Omega_{X/Y} \to \Omega_{U/Y}$ induced by ϕ is an isomorphism. Since U/Y is log smooth we get that ϕ is log étale (use [21, 3.12]).

Let $\iota: Y \to Y \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{N}^r}$ be the Y-morphism defined by $e_{\lambda} \mapsto 1 \in \Gamma(Y, M_Y)$, and $i: Y \to Y \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{Z}^r}$ be the exact closed Y-immersion defined by $e_{\lambda} \mapsto 1 \in \Gamma(Y, M_Y)$. We compute that the diagram of morphisms of log schemes



where p_1 is the first projection, is commutative.

Step 3. — In this step, we reduce to the case where $u = i \circ v$, $X = Y \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{Z}^r}$, $(b_{\lambda})_{\lambda=1,\ldots,r}$ are the elements of $\Gamma(X, M_X)$ corresponding to the canonical basis of \mathbb{Z}^r , and $(y_{\lambda})_{\lambda=1,\ldots,r}$ are equal to 1.

By using the commutativity of (1.23) (in the case where h = id thanks to Step 1, and using Lemma 1.61, since g is log etale, then the *m*-PD envelope compatible with γ of w is equal to $T' \hookrightarrow D'$. Again, by using the commutativity of (1.24), and using Lemma 1.61, since ϕ is log etale, then the *m*-PD envelope compatible with γ of w (equal to $T' \hookrightarrow D'$) is equal to the *m*-PD envelope compatible with γ of $\iota \circ v$. More precisely, following the proof of Lemma 1.61, the composition of the structural morphism $(T' \hookrightarrow D') \to w$ with ϕ is equal to the structural morphism $(T' \hookrightarrow D') \to \iota \circ v)$. Hence, we compute the image of b_{λ} via the structural morphism $(T' \hookrightarrow D') \to \iota \circ v)$ is u_{λ} . Again, since $Y \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{Z}^r} \to Y \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{N}^r}$ is log etale, then the *m*-PD envelope compatible with γ of $i \circ v$ is equal to $T' \hookrightarrow D'$. Hence, the image of b_{λ} via the structural morphism $(T' \hookrightarrow D') \to i \circ v)$ is still u_{λ} .

Step 4. — By using Lemma 1.79, we conclude.

Remark 1.83. — Suppose we are in the situation of Proposition 1.82: let $f: X \to Y$ be an S-morphism of fine log-schemes, $(b_{\lambda})_{\lambda=1,\ldots,r}$ be some elements of $\Gamma(X, M_X)$. Let $u: Z \hookrightarrow X$ and $v: Z \hookrightarrow Y$ be two S-immersions of fine log schemes such that $v = f \circ u$. Then, since $v^*M_Y \to M_Z$ is surjective (for the étale topology), then étale locally on Y, there exist $y_{\lambda} \in$ $\Gamma(Y, M_Y)$ whose images in $\Gamma(Z, M_Z)$ coincide with the images of b_{λ} .

The next result is our main motivation for defining weak log smoothness (see Definition 1.70).

PROPOSITION 1.84. — Let $f: X \to Y$ be an S-morphism of fine logschemes endowed with a formal log basis $(b_{\lambda})_{\lambda=1,\ldots,r}$ (see Definition 1.70). Let $u: Z \hookrightarrow X$ and $v: Z \hookrightarrow Y$ be two S-immersions of fine log schemes such that $v = f \circ u$. Suppose given $y_{\lambda} \in \Gamma(Y, M_Y)$ whose images in $\Gamma(Z, M_Z)$ coincide with the images of b_{λ} . Let $Z \hookrightarrow D'_n$ and $Z \hookrightarrow D_n$ be the *n*th infinitesimal neighborhood of order *n* of *u* and *v* respectively (see Definition 1.4(3) and recall following Proposition 1.11, the source is indeed Z). Let $\alpha: D'_n \to X$ be the canonical morphism. Using multiplicative notation, put $u_{\lambda} := \frac{\alpha^*(b_{\lambda})}{\alpha^*(f^*(y_{\lambda}))} \in \ker(\Gamma(D'_n, M_{D'_n}^{\mathrm{gr}}) \to \Gamma(Z, M_Z^{\mathrm{gr}})) =$ $\ker(\Gamma(D'_n, \mathcal{O}_{D'_n}^r) \to \Gamma(Z, \mathcal{O}_Z^r))$ (see Lemma 1.81). We set

$$\mathcal{O}_{D_n}[T_1,\ldots,T_r]_n := \mathcal{O}_{D_n}[T_1,\ldots,T_r]/(I_{D_n}+(T_1,\ldots,T_r))^{n+1}$$

where I_{D_n} is the ideal defined by the closed immersion $Z \hookrightarrow D_n$. Then, we have the isomorphism of \mathcal{O}_{D_n} -algebras

(1.25)
$$\mathcal{O}_{D_n}[T_1, \dots, T_r]_n \xrightarrow{\sim} \mathcal{O}_{D'_n} \\ T_{\lambda} \longmapsto u_{\lambda} - 1.$$

Proof. — By using Lemma 1.10, we reduce to the case where $X = Y \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{N}^r}$, $f: X \to Y$ is the first projection, and that the family $(b_{\lambda})_{\lambda=1,\ldots,r}$ of elements of $\Gamma(X, M_X)$ is given by the canonical basis $(e_{\lambda})_{\lambda=1,\ldots,r}$ of \mathbb{N}^r . Using Lemma 1.12, we may furthermore assume that

 \square

 $Y = S, Z \hookrightarrow Y$ is an exact closed immersion of order *n*. In particular, we get $D_n = Y$.

Let $i: Y \to Y \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{Z}^r}$ be the exact closed Y-immersion defined by $e_{\lambda} \mapsto 1 \in \Gamma(Y, M_Y)$. By copying the parts 1–3 of the proof of Proposition 1.82 (we replace the use of Lemma 1.61 by the use of Lemma 1.10), we reduce to the case where $u = i \circ v$, $X = Y \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{Z}^r}$, $(b_{\lambda})_{\lambda=1,\ldots,r}$ are the elements of $\Gamma(X, M_X)$ corresponding to the canonical basis of \mathbb{Z}^r , and $(y_{\lambda})_{\lambda=1,\ldots,r}$ are equal to 1. This case is obvious.

1.6. The case of schemes: local coordinates

DEFINITION 1.85. — Let $f: X \to Y$ be an S-morphism of fine log-schemes.

 We say that a finite set (t_λ)_{λ=1,...,r} of elements of Γ(X, O_X) are "log p-étale coordinates of f" (resp. "formal log étale coordinates of f", resp. "formal log étale coordinates of level m of f", resp. "log p-étale coordinates of level m of f"), if the corresponding Ymorphism X → Y ×_Z A^r, where A^r is the rth affine space over Z endowed with the trivial logarithmic structure, is log p-étale (resp. formally log étale, resp. formally log étale of level m, log p-étale of level m).

When f is strict (this is equivalent to say that the Y-morphism $X \to Y \times_{\mathbb{Z}} \mathbb{A}^r$ is strict) we remove "log" in the terminology, e.g. we get the notion of "p-étale coordinates".

(2) We say that f is "p-smooth" (resp. "weakly smooth", resp. "weakly smooth of level m", resp. "p-smooth of level m"), if f is strict and if, étale locally on X, f has p-étale coordinates" (resp. "formal étale coordinates", resp. "formal étale coordinates of level m", resp. "p-étale coordinates of level m"). Notice that these notions are étale local on Y.

LEMMA 1.86. — Let $\star \in \{\log p \text{-}\acute{e}tale, \text{ formal } \log \acute{e}tale, \text{ formal } \log \acute{e}tale \text{ of } level m, \log p \text{-}\acute{e}tale \text{ of } level m\}$. Let $f \colon X \to Y$ be an S-morphism of fine log-schemes.

- (1) Let $(t_{\lambda})_{\lambda=1,\ldots,r}$ in $\Gamma(X, \mathcal{O}_X^*)$. Then $(t_{\lambda})_{\lambda=1,\ldots,r}$ form a \star -basis if and only if they are \star -coordinates.
- (2) If the morphism f has \star -coordinates with r elements then, Zariski locally on X, there exist r elements of $\Gamma(X, \mathcal{O}_X^*)$ which are \star -coordinates of f.

Proof. — Since the other cases are treated similarly, let us consider the case where $\star = \log p$ -étale. Let us check the first assertion. Using Lemma 1.21 we can check that in both cases the elements $(t_{\lambda})_{\lambda=1,\ldots,r}$ induce a log *p*-étale morphism of the form $X \to Y \times_{\mathbb{Z}} \mathbb{G}_{\mathrm{m}}^{r}$. We conclude by using Lemma 1.21.

Let us check the second assertion. By hypothesis, we have a log *p*-étale morphism of the form $\phi: X \to Y \times_{\mathbb{Z}} \mathbb{A}^r$. Let (t_1, \ldots, t_r) be the elements of $\Gamma(X, \mathcal{O}_X)$ defining ϕ . We have a covering of X by open subsets of the form $D(u_1 \cdots u_r)$ where u_λ is either t_λ or $t_\lambda - 1$. Using a obvious change of coordinates, we can check that the morphism $D(u_1 \cdots u_r) \to Y \times_{\mathbb{Z}} \mathbb{A}^r$ given by the restriction of (u_1, \ldots, u_r) on $D(u_1 \cdots u_r)$ is log *p*-étale. \Box

LEMMA 1.87. — Let $\star \in \{p\text{-smooth, weakly smooth, weakly smooth of level } m, p\text{-smooth of level } m\}$. Let $f: X \to Y$ be an S-morphism of fine log-schemes.

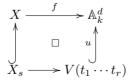
- (1) If f is \star then f is log \star .
- (2) Suppose that f: X → Y is in fact an S-morphism of schemes. The morphism f is log * if and only if f is *.

Proof. — Since the other cases are treated similarly, let us consider the case where $\star = p$ -smooth. Let us check the first assertion. Suppose that f is p-smooth. Then \underline{f} is p-smooth by definition. Hence, from (1) and (2), we get that \underline{f} is log p-smooth. Since f is strict, then f is the base change of \underline{f} by $Y \to \overline{Y}$. Hence, using Proposition 1.75, we get that f is log p-smooth.

Let us check the last assertion. Suppose that f is an S-morphism of schemes which is log p-smooth. By definition, etale locally on X, there exists a log p-étale of the form $X \to Y \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{N}^r}$ given by some elements (b_1, \ldots, b_r) of $\Gamma(X, M_X)$. Since X is a scheme, $M_X = \mathcal{O}_X^*$. Hence, using (1), we get that (b_1, \ldots, b_r) form also a finite p-basis. The converse is already known from (??).

Remark 1.88. — It is not clear that a strict log p-smooth morphism is p-smooth.

Example 1.89. — Let X be a strictly semistable variety over $\operatorname{Spec} k[\![t]\!]$ (see Definition [20, 2.16]). We denote by $X_s = V(t)$ the special fiber of X. Then, Zariski locally on X, there exists a cartesian diagram of the form



where $r \leq d$ are two integers, f is a p-smooth morphism, $\mathbb{A}_k^d = \operatorname{Spec} k[t_1, \ldots, t_d]$, and u is the closed immersion induced by $t_1 \cdots t_r$. Indeed, the fact that X is a strictly semistable variety over $\operatorname{Spec} k[t]$ means that, locally on X, there exists an integer r and a smooth morphism of the form $X \to \operatorname{Spec} k[t][T_0, \ldots, T_r]/(T_0 \cdots T_r - t)$. We have the canonical cartesian diagram

where g is the morphism given by $t_i \mapsto T_i \mod T_0 \cdots T_r - t$, $\mathbb{A}_k^{r+1} = \operatorname{Spec} k[t_0, \ldots, t_r]$, and u is the closed immersion induced by $t_0 \cdots t_r$. Using Example 1.39, since relative perfectness is stable under base change, then we can check that g is a relatively perfect morphism.

PROPOSITION 1.90. — The collection of p-smooth (resp. weakly smooth, resp. weakly smooth of level m, resp. p-smooth of level m) is stable under base change and under composition.

Proof. — This is similar to Proposition 1.75.

PROPOSITION 1.91. — Let $f: X \to Y$ be an S-morphism of fine logschemes, $(t_{\lambda})_{\lambda=1,\ldots,r}$ be some elements of $\Gamma(X, \mathcal{O}_X)$. Let $u: Z \hookrightarrow X$ and $v: Z \hookrightarrow Y$ be two S-immersions of fine log schemes such that $v = f \circ u$. Suppose that there exist $y_{\lambda} \in \Gamma(Y, \mathcal{O}_Y)$ whose images in $\Gamma(Z, \mathcal{O}_Z)$ coincide with the images of t_{λ} .

(1) If $(t_{\lambda})_{\lambda=1,...,r}$ are formal log étale coordinates of level m of f, then, we have the following isomorphism of m-PD- $\mathcal{O}_{P_{(m)}^n \sim (v)}$ -algebras

(1.26)
$$\mathcal{O}_{P^n_{(m),\gamma}(v)}\langle T_1,\ldots,T_r\rangle_{(m),n} \xrightarrow{\sim} \mathcal{O}_{P^n_{(m),\gamma}(u)} \\ T_\lambda \longmapsto t_\lambda - f^*(y_\lambda).$$

where by abuse of notation we denote by t_{λ} and $f^*(y_{\lambda})$ the canonical image in $\mathcal{O}_{P^n_{(m),\gamma}(u)}$.

(2) If $(t_{\lambda})_{\lambda=1,\dots,r}$ are log p-étale coordinates of level m of f, then, we have the following isomorphism of m-PD- $\mathcal{O}_{P(m),\gamma}(v)$ -algebras

(1.27)
$$\mathcal{O}_{P_{(m),\gamma}(v)}\langle T_1,\ldots,T_r\rangle_{(m)} \xrightarrow{\sim} \mathcal{O}_{P_{(m),\gamma}(u)} \\ T_{\lambda} \longmapsto t_{\lambda} - f^*(y_{\lambda}).$$

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$$\square$$

(3) If $(t_{\lambda})_{\lambda=1,\ldots,r}$ are formal log étale coordinates of f, then, we have the following isomorphism of $\mathcal{O}_{P^n(v)}$ -algebras

(1.28)
$$\mathcal{O}_{P^n(v)}[T_1, \dots, T_r]_n \xrightarrow{\sim} \mathcal{O}_{P^n(u)} \\ T_{\lambda} \longmapsto t_{\lambda} - f^*(y_{\lambda}),$$

where $\mathcal{O}_{P^n(v)}[T_1,\ldots,T_r]/(I_n+(T_1,\ldots,T_r))^{n+1}$ is denoted $\mathcal{O}_{P^n(v)}[T_1,\ldots,T_r]_n$, with I_n is the ideal defined by the closed immersion $P^n(v)$.

Proof.

Step 1. — Using Lemma 1.61 for the first two or Lemma 1.10 for the last one (and the first remark of Remark 1.3), we may assume that X = $Y \times_{\mathbb{Z}} \mathbb{A}^r$, $f: X \to Y$ is the first projection, and that the family $(t_{\lambda})_{\lambda=1,\ldots,r}$ are the elements of $\Gamma(X, \mathcal{O}_X)$ corresponding to the coordinates of \mathbb{A}^r . Using Lemma 1.69 for the first two and Lemma 1.12 for the last one, we may furthermore assume that $Y = S, Z \to Y$ is the exact closed immersion whose ideal of definition is I_S .

Let $\phi: Y \times_{\mathbb{Z}} \mathbb{A}^r \to Y \times_{\mathbb{Z}} \mathbb{A}^r$ be the Y-morphism given by $t_1 - f^*(y_1), \ldots, t_r - f^*(y_r)$. Let $i: Y \hookrightarrow Y \times_{\mathbb{Z}} \mathbb{A}^r$ be the exact closed immersion defined by $t_{\lambda} \mapsto 0$. Since ϕ is etale, since $\phi \circ u = i$, using Lemma 1.61 for the first two or Lemma 1.10 for the last one, we reduce to the case where u is equal to i, and $(y_{\lambda})_{\lambda=1,\ldots,r}$ are equal to 0.

Step 2.

- (a) Let us check (1.27). By using the remark [4, 1.4.3.(iii)] and Point 1.58, we can suppose that v = id. Since the ideal of the exact closed immersion *i* is generated by the regular sequence $(t_{\lambda})_{\lambda=1,\ldots,r}$, using [4, 1.5.3] and Point 1.58 we can check that the morphism $\mathcal{O}_Y \langle T_1, \ldots, T_r \rangle_{(m)} \to \mathcal{O}_{P_{(m),\gamma}(\iota)} = \mathcal{O}_{P_{(m),\gamma}(\iota)}$ given by $T_{\lambda} \mapsto t_{\lambda}$ is an isomorphism.
- (b) The case (a) implies (1.26).
- (c) The check of (1.28) is obvious.

2. Differential operators of level m over log p-smooth log schemes

Let *i* be an integer and *S* be a fine log scheme over the scheme $\operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$.

2.1. Sheaf of principal parts, sheaf of differential operators

Let $f: X \to S$ be a weakly log smooth morphism of fine log-schemes.

Notation 2.1. — Let $\Delta_{X/S}: X \to X \times_S X$ be the diagonal morphism, $\Delta_{X/S}^n := P^n(\Delta_{X/S}), \mathcal{P}_{X/S}^n := \mathcal{P}^n(\Delta_{X/S})$ (see Proposition 1.11). We denote by $M_{X/S}^n$ the log structure of $\Delta_{X/S}^n$. We denote abusively the target of $\Delta_{X/S}^n$ by $\Delta_{X/S}^n$.

We denote by respectively $p_1^n, p_0^n \colon \Delta_{X/S,(m)}^n \to X$ the composition of the canonical morphism $\Delta_{X/S}^n \to X \times_S X$ with the right and left projection $X \times_S X \to X$.

If $a \in M_X$, we denote by $\mu(a)$ the unique section of $\ker(\mathcal{O}^*_{\Delta^n_{X/S}} \to \mathcal{O}^*_X)$ such that we get in $M^n_{X/S}$ the equality $p_1^{n*}(a) = p_0^{n*}(a)\mu^n(a)$ (see Lemma 1.81). We get $\mu^n \colon M_X \to \ker(\mathcal{O}^*_{\Delta^n_{X/S}} \to \mathcal{O}^*_X)$ given by $a \mapsto \mu^n(a)$.

LEMMA 2.2. — The morphisms p_1^n and p_0^n are strict.

Proof. — This is similar to [27, 2.2.1]: let $\iota^n \colon X \hookrightarrow \Delta^n_{X/S}$ be the structural morphism. Since $\iota^{-1} = \text{id}$, then from [21, 1.4.1] we get the isomorphisms

$$p_i^{n*}(M_X)/\mathcal{P}_{X/S}^{n*} \xrightarrow{\sim} M/\mathcal{O}_X^*$$

and

$$M_{X/S}^n/\mathcal{P}_{X/S}^{n*} \xrightarrow{\sim} \iota^{n*}(M_{X/S}^n)/\mathcal{O}_X^* \xrightarrow{\sim} M_X/\mathcal{O}_X^*$$

(the last isomorphism is a consequence of the exactness of ι^n). Hence, $p_i^{n*}(M_X)/\mathcal{P}_{X/S}^{n*} \xrightarrow{\sim} M_{X/S}^n/\mathcal{P}_{X/S}^{n*}$. This implies that the canonical morphism $p_i^{n*}(M_X) \to M_{X/S}^n$ is an isomorphism.

PROPOSITION 2.3 (Local description of $\mathcal{P}_{X/S}^n$). — Let $(a_{\lambda})_{\lambda=1,...,r}$ be a formal log basis of f. Put $\eta_{\lambda,n} := \mu^n(a_{\lambda}) - 1$. We have the following isomorphism of \mathcal{O}_X -algebras:

(2.1)
$$\mathcal{O}_X[T_1, \dots, T_r]_n \xrightarrow{\sim} \mathcal{P}^n_{X/S} \\ T_{\lambda} \longmapsto \eta_{\lambda, n},$$

where the structure of \mathcal{O}_X -module of $\mathcal{P}^n_{X/S}$ is given by p_1^n or p_0^n .

Proof. — Since the case of p_1^n is checked symmetrically, let us compute the case where the \mathcal{O}_X -module of $\mathcal{P}_{X/S}^n$ is given by p_0^n . Consider the commutative diagram

(2.2)
$$\begin{array}{c} A_{\mathbb{N}^r} \underbrace{\leftarrow}{p_1} X \times_S A_{\mathbb{N}^r} \\ a \\ & \square \\ X \underbrace{\leftarrow}{p_1} X \times_S X \underbrace{\leftarrow}{p_0} X, \end{array}$$

where p_0 , p_1 means respectively the left and right projection, where a is the S-morphism induced by a_1, \ldots, a_r , where b is the X-morphism induced by b_1, \ldots, b_r with $b_{\lambda} := p_1^*(a_{\lambda})$. Since $(b_{\lambda})_{\lambda=1,\ldots,r}$ is a formal log basis of p_0 (because the square of the diagram (2.2) is cartesian), we can apply Proposition 1.84 in the case where f is p_0 , u is $\Delta_{X/S}$, b_{λ} is as above, and u_{λ} is a_{λ} .

Remark 2.4. — From the local description of Proposition 2.3, we get that the morphisms p_1^n and p_0^n are finite (i.e. the underlying morphism of schemes is finite).

Notation 2.5. — We denote by $\mathcal{I}_{X/S}^1$ the ideal of the closed immersion $\Delta_{X/S}^1$ and by $\Omega_{X/S}^1 := (\Delta_{X/S}^1)^{-1}(\mathcal{I}_{X/S}^1)$ the corresponding \mathcal{O}_X -module (recall $\Delta_{X/S*}^1$ is an homeomorphism). To justify the notation, we refer to the isomorphism [21, 5.8.1].

Remark 2.6. — Following the local description (2.1), since f has formal log bases locally, then $\Omega^1_{X/S}$ is a locally free \mathcal{O}_X -module of finite rank and the rank is equal to the cardinal of the formal log basis (a basis is given by η_1, \ldots, η_r).

Point 2.7. — The exact closed immersions $\Delta_{X/S}^n$ and $\Delta_{X/S}^{n'}$ induce $\Delta_{X/S}^{n,n'} := (\Delta_{X/S}^n, \Delta_{X/S}^{n'}): X \hookrightarrow \Delta_{X/S}^n \times_X \Delta_{X/S}^{n'}$. Since the morphisms p_1^n and p_0^n are strict (see Lemma 2.2), then $\Delta_{X/S}^{n,n'}$ is also an exact closed immersion. We get $\Delta_{X/S}^{n,n'} \in \mathscr{C}_{n+n'}$. Using the universal property of the n + n' infinitesimal neighborhood of $\Delta_{X/S}$, we get a unique morphism $\Delta_{X/S}^n \times_X \Delta_{X/S}^{n'} \to \Delta_{X/S}^{n+n'}$ of $\mathscr{C}_{n+n'}$ inducing the commutative diagram

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We denote by $\delta^{n,n'}: \mathcal{P}_{X/S}^{n+n'} \to \mathcal{P}_{X/S}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S}^{n'}$ the corresponding morphism.

By replacing p_{02} by p_{01} (resp. p_{12}), we get a unique morphism $\Delta_{X/S}^n \times_X \Delta_{X/S}^{n'} \to \Delta_{X/S}^{n+n'}$ making commutative the diagram (2.7). We denote by $q_0^{n,n'}: \mathcal{P}_{X/S}^{n+n'} \to \mathcal{P}_{X/S}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S}^{n'}$ (resp. $q_1^{n,n'}: \mathcal{P}_{X/S}^{n+n'} \to \mathcal{P}_{X/S}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S}^{n'}$) the corresponding morphism (or simply $q_0^{n,n'}$ or q_0). We notice that $q_0^{n,n'} = \pi_{X/S}^{n+n',n} \otimes 1$ and $q_1^{n,n'} = 1 \otimes \pi_{X/S}^{n+n',n'}$, where $\pi_{X/S}^{n_1,n_2}$ is the projection $\mathcal{P}_{X/S}^{n_1} \to \mathcal{P}_{X/S}^n$ for any integers $n_1 \ge n_2$.

LEMMA 2.8. — For any $a \in M_X$, for any integers $n, n' \in \mathbb{N}$, we have $\delta^{n,n'}(\mu^{n+n'}(a)) = \mu^n(a) \otimes \mu^{n'}(a)$.

Proof. — We copy word by word the proof of Montagnon of Lemma [27, 2.3.1]. $\hfill \Box$

DEFINITION 2.9. — The sheaf of differential operators of order $\leq n$ of f is defined by putting $\mathcal{D}_{X/S,n} := \mathscr{H} \operatorname{om}_{\mathcal{O}_X}(p_{0*}^n \mathcal{P}_{X/S}^n, \mathcal{O}_X)$. The sheaf of differential operators of f is defined by putting $\mathcal{D}_{X/S} := \bigcup_{n \in \mathbb{N}} \mathcal{D}_{X/S,n}$.

Let $P \in \mathcal{D}_{X/S,n}$, $P' \in \mathcal{D}_{X/S,n'}$. We define the product $PP' \in \mathcal{D}_{X/S,n+n'}$ to be the composition

(2.4)
$$PP': \mathcal{P}_{X/S}^{n+n'} \xrightarrow{\delta^{n,n'}} \mathcal{P}_{X/S}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S}^{n'} \xrightarrow{\mathrm{Id} \otimes P'} \mathcal{P}_{X/S}^n \xrightarrow{P} \mathcal{O}_X.$$

Suppose f has the formal log basis $(a_{\lambda})_{\lambda=1,\dots,r}$. Put $\eta_{\lambda} := \mu^n(a_{\lambda}) - 1$. Following Proposition 2.3, the elements $\{\underline{\eta}^k\}_{|\underline{k}| \leq n}$ form a basis of $\mathcal{P}^n_{X/S}$. The corresponding dual basis of $\mathcal{D}^{(m)}_{X/S,n}$ will be denoted by $\{\underline{\partial}^{[\underline{k}]}, |\underline{k}| \leq n\}$. Hence, $\mathcal{D}^{(m)}_{X/S}$ is a free \mathcal{O}_X -module (for both structures) with the basis $\{\underline{\partial}^{[\underline{k}]}, \underline{k} \in \mathbb{N}^r\}$.

PROPOSITION 2.10. — The sheaf $\mathcal{D}_{X/S}$ is a sheaf of rings with the product as defined in (2.4).

Proof. — Using Lemma 2.8 (instead of Lemma 2.27), we can check Proposition 2.10 similarly to Proposition 2.28. \Box

2.2. Sheaf of principal parts of level m

Let $f: X \to S$ be a weakly log smooth of level m morphism of fine logschemes. Let (I_S, J_S, γ) be a quasi-coherent m-PD-ideal of \mathcal{O}_S that such that γ extends to X (e.g. from [4, 1.3.2.(i).c)] when $J_S + p\mathcal{O}_S$ is locally principal). Recall that f is also a weakly log smooth of level m compatible with γ (this is a consequence of (1.17)). Remark 2.11. — Since γ extends to X, then the *m*-PD envelope compatible with γ (of order *n*) of the identity of X is the identity of X. Indeed, using the arguments given in the proof of Lemma 1.54, we can check that the ideal 0 of \mathcal{O}_X is endowed with a (unique) *m*-PD structure compatible with γ .

Notation 2.12. — Let $\Delta_{X/S}: X \to X \times_S X$ be the diagonal morphism, $\Delta_{X/S,(m),\gamma} := P_{(m),\gamma}(\Delta_{X/S}), \ \Delta_{X/S,(m),\gamma}^n := P_{(m),\gamma}^n(\Delta_{X/S}), \ \mathcal{P}_{X/S,(m),\gamma}^n := \mathcal{P}_{(m),\gamma}^n(\Delta_{X/S})$ (see Proposition 1.63). We denote by $M_{X/S,(m),\gamma}$ (resp. $M_{X/S,(m),\gamma}^n$) the log structure of $\Delta_{X/S,(m),\gamma}$ (resp. $\Delta_{X/S,(m),\gamma}^n)$. We denote abusively the target of $\Delta_{X/S,(m),\gamma}$ by $\Delta_{X/S,(m),\gamma}$. Since γ extends to X, the source of $\Delta_{X/S,(m),\gamma}^n$ is X, i.e. $\Delta_{X/S,(m),\gamma}^n$ is a closed immersion of the form $X \hookrightarrow (\operatorname{Spec} \mathcal{P}_{X/S,(m),\gamma}^n, M_{X/S,(m),\gamma}^n)$.

Let $p_1, p_0: \Delta_{X/S,(m),\gamma} \to X$ be respectively the composition of the canonical morphism $\Delta_{X/S,(m),\gamma} \to X \times_S X$ with the right and left projection $X \times_S X \to X$. Similarly we get $p_1^n, p_0^n: \Delta_{X/S,(m),\gamma}^n \to X$. As in Lemma 2.2, we can check that p_1 and p_0 are strict morphisms.

If $a \in M_X$, we denote by $\mu_{(m),\gamma}(a)$ the unique section of ker $(\mathcal{O}^*_{\Delta_{X/S,(m),\gamma}} \to \mathcal{O}^*_X)$ such that we get in $M_{X/S,(m),\gamma}$ the equality $p_1^*(a) = p_0^*(a)\mu_{(m),\gamma}(a)$ (see Lemma 1.81). We get $\mu_{(m),\gamma} \colon M_{X/S,(m),\gamma} \to \ker(\mathcal{O}^*_{\Delta_{X/S,(m),\gamma}} \to \mathcal{O}^*_X)$ given by $a \mapsto \mu_{(m),\gamma}(a)$. Similarly we define $\mu^n_{(m),\gamma} \colon M^n_{X/S,(m),\gamma} \to \ker(\mathcal{O}^*_{\Delta^n_{X/S,(m),\gamma}} \to \mathcal{O}^*_X)$.

PROPOSITION 2.13 (Local description of $\mathcal{P}_{X/S,(m),\gamma}$). — Let $(a_{\lambda})_{\lambda=1,\dots,r}$ be a finite set of $\Gamma(X, M_X)$ such that $(a_{\lambda})_{\lambda=1,\dots,r}$ is a formal log basis of level *m* compatible with γ of *f*. Put $\eta_{\lambda(m),\gamma} := \mu_{(m),\gamma}(a_{\lambda}) - 1$, and $\eta_{\lambda(m),\gamma,n} := \mu_{(m),\gamma}^n(a_{\lambda}) - 1$.

(1) We have he following \mathcal{O}_X -m-PD isomorphism

(2.5)
$$\mathcal{O}_X \langle T_1, \dots, T_r \rangle_{(m),n} \xrightarrow{\sim} \mathcal{P}^n_{X/S,(m),\gamma} \\ T_\lambda \longmapsto \eta_{\lambda,(m),\gamma,n},$$

where the structure of \mathcal{O}_X -module of $\mathcal{P}^n_{X/S,(m),\gamma}$ is given by p_1^n or p_0^n .

(2) If $(a_{\lambda})_{\lambda=1,\dots,r}$ is moreover a log *p*-basis of level *m* compatible with γ of *f* then we have he following \mathcal{O}_X -*m*-PD isomorphism

(2.6)
$$\mathcal{O}_X \langle T_1, \dots, T_r \rangle_{(m)} \xrightarrow{\sim} \mathcal{P}_{X/S, (m), \gamma} \\ T_\lambda \longmapsto \eta_{\lambda, (m), \gamma},$$

where the structure of \mathcal{O}_X -module of $\mathcal{P}_{X/S,(m),\gamma}$ is given by p_1 or p_0 .

Proof. — By symmetry, we can focus on the case where the structure of \mathcal{O}_X -module of $\mathcal{P}^n_{X/S,(m),\gamma}$ (resp. $\mathcal{P}_{X/S,(m),\gamma}$) is given by p_0^n (resp. p_0). In the first assertion (resp. the second one), we are in the situation to use formula (1.21) (resp. (1.22)) in the case where $u = \Delta$ and f is the left projection $p_0: X \times_S X \to X$. Indeed, we first remark that $(p_1^*(a_\lambda))_{\lambda=1,\dots,r}$ is a formal log basis of level m compatible with γ of p_0 (resp. $(p_1^*(a_\lambda))_{\lambda=1,\dots,r}$ is a log p-basis of level m compatible with γ of p_0). Indeed, the formal p-étaleness of level m property (resp. log p-étaleness of level m property) is stable under base change. Since the m-PD envelope compatible with γ of order n (resp. m-PD envelope compatible with γ) of the identity of X is X (see Remark 2.11), Proposition 1.82 yields the result.

Remark 2.14.

- (1) From the local description (2.5), we get that $\mathcal{P}^{n}_{X/S,(m),\gamma}$ does not depend on the *m*-PD-structure (satisfying the conditions of the subsection). Hence, from now, we reduce to the case where $\gamma = \gamma_{\emptyset}$ (see Notation 1.49) and we remove γ in the notation: we simply write $\mathcal{P}^{n}_{X/S,(m)}, \Delta^{n}_{X/S,(m)}, M^{n}_{X/S,(m)}, \mu^{n}_{(m)}$, and $\eta_{\lambda(m),n}$.
- (2) When f is log p-smooth of level m, from (2.6), $\mathcal{P}_{X/S,(m),\gamma}$ does not depend on the m-PD-structure (satisfying the conditions of the subsection). Hence, we can remove γ in the corresponding notation.

Remark 2.15. — From the local description of Proposition 2.13, we get that the morphisms p_1^n and p_0^n are finite (i.e. the underlying morphism of schemes is finite).

Remark 2.16. — For any integer $m' \ge m$, we remark that the canonical map $\mathcal{P}^n_{X/S,(m')} \to \mathcal{P}^n_{X/S,(m)}$ sends $\eta_{\lambda(m')}$ to $\eta_{\lambda(m)}$.

Remark 2.17. — Noticing that the main Theorem [28, IV.3.2.6] on log smoothness (this is the Theorem that leads us to our definition of log psmoothness) is valid for coherent log structures and not only fine log structures, one might wonder why we are focusing on fine log structures. The first reason we have in mind is that the important tool consisting of exactifying closed immersions (see Point 1.8) needs fine log structures. One might refute that in the first chapter we might replace in the definition of \mathscr{C} (see Definition 1.4) the word fine by the word coherent (but in the other categories, e.g. $\mathscr{C}_{\gamma}^{(m)}$ we keep fine log structures). But, if we replace in Proposition 2.13 fine log structures by coherent log structures, the isomorphism (2.5) is not any more true: instead we have $\mathcal{O}_{X^{\text{int}}}\langle T_1, \ldots, T_r \rangle_{(m),n} \xrightarrow{\sim} \mathcal{P}_{X/S,(m)}^n$. Recall that since X is only coherent and not fine then we have in general $\mathcal{O}_{X^{\text{int}}} \neq \mathcal{O}_X$. Point 2.18. — Let $g: S' \to S$ be a morphism of fine log schemes over $\mathbb{Z}/p^{i+1}\mathbb{Z}$, let $(I_{S'}, J_{S'}, \gamma')$ be a quasi-coherent *m*-PD-ideal of $\mathcal{O}_{S'}$ such that g becomes an *m*-PD-morphism. Put $X' := X \times_S S'$. We suppose that γ' extends to X'. Then, the *m*-PD-morphism $\Delta_{X'/S',(m)} \to \Delta_{X/S,(m)}$ induces the isomorphism $\Delta_{X'/S',(m)} \xrightarrow{\sim} \Delta_{X/S,(m)} \times_S S'$. Indeed, since the morphisms $p_0: \Delta_{X/S,(m)} \to X$ and $p_0: \Delta_{X'/S',(m)} \to X'$ are strict, then the morphism $\Delta_{X'/S',(m)} \to \Delta_{X/S,(m)} \times_S S'$ is strict. Hence, this is sufficient to check that the morphism $g^* \mathcal{P}_{X/S,(m)} \to \mathcal{P}_{X'/S',(m)}$ is an isomorphism. This can be checked by using the local description of (2.5).

Point 2.19. — Let $m' \ge m$ be two integers. Since $\mathscr{C}'_n^{(m)} \subset \mathscr{C}'_n^{(m')}$, then by using the universal property defining $\Delta^n_{X/S,(m')}$ we get a morphism $\psi^n_{m,m'}$: $\Delta^n_{X/S,(m)} \to \Delta^n_{X/S,(m')}$ and then the homomorphism $\psi^{n*}_{m,m'} \colon \mathcal{P}^n_{X/S,(m')} \to \mathcal{P}^n_{X/S,(m)}$.

From Lemma 1.67, we get $P_{(m)}^n(P^n(\Delta_{X/S})) = P_{(m)}^n(\Delta_{X/S})$. Hence, we get a canonical map $\psi_{m,m'}^n \colon \Delta_{X/S,(m)}^n \to \Delta_{X/S}^n$ and then the homomorphism $\psi_m^{n*} \colon \mathcal{P}_{X/S}^n \to \mathcal{P}_{X/S,(m)}^n$. Now, suppose that $X \to S$ is endowed with a formal log basis $(b_\lambda)_{\lambda=1,\dots,r}$

Now, suppose that $X \to S$ is endowed with a formal log basis $(b_{\lambda})_{\lambda=1,\ldots,r}$ of level m'. With the notation of Proposition 2.13, we have $\psi_{m,m'}^{n*}(\underline{\eta}_{(m')}^{\{k\}(m')}) = \frac{q!}{\underline{q'}!}\underline{\eta}_{(m)}^{\{k\}(m)}$, where $k_{\lambda} = p^m q_{\lambda} + r_{\lambda}$ and $k'_{\lambda} = p^{m'}q'_{\lambda} + r'_{\lambda}$ is the Euclidian division of k_{λ} by respectively p^m and $p^{m'}$, $\underline{\eta}_{(m)}^{\{k\}(m)} := \prod_{\lambda=1}^r \eta_{\lambda,(m)}^{\{k_{\lambda}\}(m)}$, $\underline{q} := \prod_{\lambda=1}^r q_{\lambda}$ and similarly with some primes. Moreover, we compute $\psi_m^{n*}(\underline{\eta}_m^{\underline{k}}) = \underline{q}! \underline{\eta}_{(m)}^{\{\underline{k}\}(m)}$.

Notation 2.20. — Let $\mathcal{I}_{X/S,(m)}^1$ be the ideal of the closed immersion $\Delta^1_{X/S,(m)}$ and $\Omega^1_{X/S,(m)} := (\Delta^1_{X/S,(m)})^{-1}(\mathcal{I}^1_{X/S,(m)})$. Thanks to the local description (2.1) (recall from Proposition 1.74 that since f is weakly log smooth of level m then f is weakly log smooth) and (2.5) and the local computation of Point 2.19, we can check that the homomorphism $\psi^{1*}_m : \mathcal{P}^1_{X/S,(m)}$, and that the homomorphism $\psi^{1*}_m : \mathcal{P}^1_{X/S,(m)}$ induces the isomorphism $\psi^{1*}_m : \Omega^1_{X/S,(m)} \xrightarrow{\sim} \Omega^1_{X/S,(m)}$, and that the homomorphism $\psi^{1*}_{m,m'} : \mathcal{P}^1_{X/S,(m')} \to \mathcal{P}^1_{X/S,(m)}$ induces the isomorphism $\psi^{1*}_{m,m'} : \Omega^1_{X/S,(m)}$. Hence, we can simply write $\Omega^1_{X/S}$ instead of $\Omega^1_{X/S,(m)}$.

Remark 2.21. — If X/S has a formal log basis $(b_{\lambda})_{\lambda=1,\ldots,r}$ of level m of f then (2.5) implies that $\Omega^1_{X/S}$ is free of rank r, a basis being given by the images dlog b_{λ} of the $\eta_{\lambda,(m)}$'s. This implies in particular that all formal log bases of level m of f have the same cardinality. We put $\omega_{X/S} := \wedge^r \Omega^1_{X/S}$.

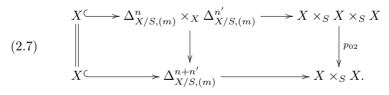
More generally (i.e. we do not any more assume that X/S has a formal log basis), we define $\omega_{X/S}$ in the same way.

Notation 2.22. — Let \mathcal{E} be an \mathcal{O}_X -module. By convention, $\mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{E}$ \mathcal{E} means $p_{1*}^n(\mathcal{P}_{X/S,(m)}^n) \otimes_{\mathcal{O}_X} \mathcal{E}$ and $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^n$ means $\mathcal{E} \otimes_{\mathcal{O}_X} p_{0*}^n(\mathcal{P}_{X/S,(m)}^n)$. For instance, $\mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'}$ is $p_{1*}^n(\mathcal{P}_{X/S,(m)}^n) \otimes_{\mathcal{O}_X} p_{0*}^{n'}(\mathcal{P}_{X/S,(m)}^{n'})$.

LEMMA 2.23. — We simply denote by $\Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'}$ the base change of $p_0^{n'} : \Delta_{X/S,(m)}^{n'} \to X$ by $p_1^n : \Delta_{X/S,(m)}^n \to X$. The immersion $X \hookrightarrow \Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'}$ induced by $X \hookrightarrow \Delta_{X/S,(m)}^n$ and $X \hookrightarrow \Delta_{X/S,(m)}^{n'}$ is an exact closed immersion endowed with a canonical m-PD structure of order n+n'. By abuse of notation, we denote by $\Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'}$ this object of $\mathscr{C}_{n+n'}^{(m)}$. This m-PD structure on $\Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'}$ is characterized by the following property: the projections $q_0^{n,n'} : \Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'} \to \Delta_{X/S,(m)}^n$ and $q_1^{n,n'} : \Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'} \to \Delta_{X/S,(m)}^{n'}$ are morphisms of $\mathscr{C}_{n+n'}^{(m)}$.

Proof. — Since $p_1^n, p_0^n: \Delta_{X/S,(m),\gamma}^n \to X$ are strict, we can check that $X \hookrightarrow \Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'}$ is an exact closed immersion. Using the local description (2.5), to check the uniqueness and existence of the *m*-PD-structure of order n + n', we proceed similarly to [4, 2.1.3.(i)] or [27, 2.3.2].

Point 2.24. — Using the universal property of the *m*-PD-envelope of order *n*, we get a unique morphism $\Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'} \to \Delta_{X/S,(m)}^{n+n'}$ of $\mathscr{C}_{n+n'}^{(m)}$ inducing the commutative diagram



We denote by $\delta_{(m)}^{n,n'} \colon \mathcal{P}_{X/S,(m)}^{n+n'} \to \mathcal{P}_{X/S,(m)}^{n} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'}$ the corresponding morphism.

By replacing p_{02} by p_{01} (resp. p_{12}), we get a unique morphism $\Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'} \to \Delta_{X/S,(m)}^{n+n'}$ making commutative the diagram (2.7). We denote by $q_{0(m)}^{n,n'} \colon \mathcal{P}_{X/S,(m)}^{n+n'} \to \mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'}$ (resp. $q_{1(m)}^{n,n'} \colon \mathcal{P}_{X/S,(m)}^{n+n'} \to \mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'}$ (resp. $q_{1(m)}^{n,n'} \colon \mathcal{P}_{X/S,(m)}^{n+n'} \to \mathcal{P}_{X/S,(m)}^n$) the corresponding morphism (or simply $q_0^{n,n'}$ or

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 q_0). We notice that $q_{0(m)}^{n,n'} = \pi_{X/S,(m)}^{n+n',n} \otimes 1$ and $q_{1(m)}^{n,n'} = 1 \otimes \pi_{X/S,(m)}^{n+n',n'}$, where $\pi_{X/S,(m)}^{n_1,n_2}$ is the projection $\mathcal{P}_{X/S,(m)}^{n_1} \to \mathcal{P}_{X/S,(m)}^{n_2}$ for any integers $n_1 \ge n_2$.

The following Lemma will be useful to check the associativity of the product law of the sheaf of differential operator:

LEMMA 2.25. — We denote by $\Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'} \times_X \Delta_{X/S,(m)}^{n''}$ the base change of $p_0^{n'} \circ q_0^{n',n''} : \Delta_{X/S,(m)}^{n'} \times_X \Delta_{X/S,(m)}^{n''} \to X$ by $p_1^n : \Delta_{X/S,(m)}^n \to X$. The exact closed immersion $X \hookrightarrow \Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n''} \times_X \Delta_{X/S,(m)}^{n''}$ induced by $X \hookrightarrow \Delta_{X/S,(m)}^n, X \hookrightarrow \Delta_{X/S,(m)}^{n'}$ and $X \hookrightarrow \Delta_{X/S,(m)}^{n''}$ is endowed with a canonical m-PD structure. By abuse of notation, we denote by $\Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n'} \times_X \Delta_{X/S,(m)}^{n''} \times_X \Delta_{X/S,(m)}^{n''}$ this object of $\mathscr{C}_{n+n'+n''}^{(m)}$. This m-PD structure on $\Delta_{X/S,(m)}^n \times_X \Delta_{X/S,(m)}^{n''} \times_X \Delta_{X/S,(m)}^{n''}$

Proof. — This is checked similarly to Lemma 2.23.

2.3. Sheaf of differential operators of level m

We keep Section 2.2.

DEFINITION 2.26. — The sheaf of differential operators of level m and order $\leq n$ of f is defined by putting

$$\mathcal{D}_{X/S,n}^{(m)} := \mathscr{H} \mathrm{om}_{\mathcal{O}_X}(p_{0,(m)*}^n \mathcal{P}_{X/S,(m)}^n, \mathcal{O}_X).$$

The sheaf of differential operators of level m of f is defined by putting

$$\mathcal{D}_{X/S}^{(m)} := igcup_{n\in\mathbb{N}} \mathcal{D}_{X/S,n}^{(m)}.$$

Let $P \in \mathcal{D}_{X/S,n}^{(m)}$, $P' \in \mathcal{D}_{X/S,n'}^{(m)}$. We define the product $PP' \in \mathcal{D}_{X/S,n+n'}^{(m)}$ to be the composition

(2.8)
$$PP': \mathcal{P}_{X/S,(m)}^{n+n'} \xrightarrow{\delta_{(m)}^{n,n'}} \mathcal{P}_{X/S,(m)}^{n} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'} \xrightarrow{\mathrm{Id} \otimes P'} \mathcal{P}_{X/S,(m)}^{n} \xrightarrow{P} \mathcal{O}_X.$$

LEMMA 2.27. — For any $a \in M_X$, for any integers $n, n' \in \mathbb{N}$, we have $\delta_{(m)}^{n,n'}(\mu_{(m)}^{n+n'}(a)) = \mu^n(a)_{(m)} \otimes \mu_{(m)}^{n'}(a).$

Proof. — We copy word by word the proof of Montagnon of Lemma [27, 2.3.1]. $\hfill \Box$

PROPOSITION 2.28. — The sheaf $\mathcal{D}_{X/S}^{(m)}$ is a sheaf of rings with the product as defined in (2.8).

Proof. — We have to check the product as defined in (2.8) is associative. One checks the commutativity of the diagram

$$\begin{array}{c} \mathcal{P}^{n+n'+n''} & \xrightarrow{\mathcal{P}^{n+n'+n''}} \xrightarrow{\delta_{(m)}^{n,n'+n''}} \mathcal{P}^n \otimes_{\mathcal{O}} \mathcal{P}^{n'+n''} & \xrightarrow{\mathcal{P}^n} \otimes_{\mathcal{O}} \mathcal{P}^{n'+n''} & \xrightarrow{\mathcal{P}^n} \otimes_{\mathcal{O}} \mathcal{P}^{n'+n''} & \xrightarrow{\mathcal{P}^n} \otimes_{\mathcal{O}} \mathcal{P}^{n'} \otimes_{(m)} \mathcal{P}^n & \xrightarrow{\mathcal{P}^n} \mathcal{P}^n \otimes_{\mathcal{O}} \mathcal{P}^{n''} & \xrightarrow{\mathcal{P}^n} & \xrightarrow{\mathcal{P}^n} \otimes_{\mathcal{O}} \mathcal{P}^{n''} & \xrightarrow{\mathcal{P}^n} & \xrightarrow{\mathcal{P}^n} \otimes_{\mathcal{O}} \mathcal{P}^{n''} & \xrightarrow{\mathcal{P}^n} &$$

where $\mathcal{P}^n := \mathcal{P}^n_{X/S,(m)}$, i.e. we have removed the indication X/S, (m) and where $\mathcal{O} := \mathcal{O}_X$. Indeed, let us check the commutativity of the top square of the middle. Since this is local, we can suppose that f has a formal log basis $(a_\lambda)_{\lambda=1,\dots,r}$ of level m. Using Lemma 2.27 and the notation of Proposition 2.13, we compute that the images of $\eta_{1,(m),n+n'}, \dots, \eta_{r,(m),n+n'}$ by both maps $\mathcal{P}^{n+n'+n''}_{X/S,(m)} \to \mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n''}_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n''}_{X/S,(m)}$ are the same. Using Proposition 2.13, since both maps are m-PD-morphisms (see Lemma 2.25 for the m-PD-structure), we get the desired commutativity. Since the commutativity of the other squares are obvious, we conclude the proof.

Point 2.29 (Description in local coordinates). — Suppose that $X \to S$ is endowed with a formal log basis $(b_{\lambda})_{\lambda=1,...,r}$ of level m. With the notion of Point 2.19, the elements $\{\underline{\eta}_{(m)}^{\{\underline{k}\}_{(m)}}, |\underline{k}| \leq n\}$ form a basis of $\mathcal{P}_{X/S,(m)}^{n}$. The corresponding dual basis of $\mathcal{D}_{X/S,n}^{(m)}$ will be denoted by $\{\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}, |\underline{k}| \leq n\}$. This yields the basis (as \mathcal{O}_X -module for both structures) $\{\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}, \underline{k} \in \mathbb{N}^r\}$ of $\mathcal{D}_{X/S}^{(m)}$.

Let $\epsilon_1, \ldots, \epsilon_r$ be the canonical basis of \mathbb{N}^r , i.e. the coordinates of ϵ_λ are 0 except for the *i*th term which is 1. We put $\partial_\lambda := \underline{\partial}^{\langle \epsilon_\lambda \rangle_{(m)}}$. We have the same formulas than in [27, 2.3.3]. For instance, for any section $a \in \mathcal{O}_X$, for any $\underline{k}, \underline{k'}, \underline{k''} \in \mathbb{N}^n$,

(2.10)
$$\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} a = \sum_{\underline{i} \leq \underline{k}} \left\{ \frac{\underline{k}}{\underline{i}} \right\} \underline{\partial}^{\langle \underline{k} - \underline{i} \rangle_{(m)}}(a) \underline{\partial}^{\langle \underline{i} \rangle_{(m)}};$$

(2.11)
$$\underline{\partial}^{\langle \underline{k}' \rangle_{(m)}} \underline{\partial}^{\langle \underline{k}'' \rangle_{(m)}} = \sum_{\underline{k}=\max\{\underline{k}', \underline{k}''\}}^{\underline{k}' + \underline{k}''} C_{\underline{k}', \underline{k}''}^{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}},$$

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where $\underline{q}_{\underline{k}}$ means the quotient of the Euclidian division of \underline{k} by p^m and similarly with some primes, and where $C_{\underline{k}',\underline{k}''}^{\underline{k}} := \frac{\underline{k}!}{(\underline{k}' + \underline{k}'' - \underline{k})!(\underline{k} - \underline{k}')!(\underline{k} - \underline{k}'')!} \frac{\underline{q}_{\underline{k}'}!\underline{q}_{\underline{k}''}!}{\underline{q}_{\underline{k}}!}$. The calculations of Montagnon in [27, 2.3.1] work here as well and show that the \mathcal{O}_X -algebra $\mathcal{D}_{X/S}^{(m)}$ (for its left or right structure) is generated by the operators $\partial_{\lambda}^{\langle p^j \rangle_{(m)}}$ with $1 \leq \lambda \leq r, \ 0 \leq j \leq m$. These formulas yield that $\operatorname{gr} \mathcal{D}_{X/S}^{(m)}$ is commutative and that, when \underline{X} is affine and noetherian, the ring $\Gamma(X, \mathcal{D}_{X/S}^{(m)})$ is left and right noetherian.

Point 2.30 (Comparison of the local description of differential operators with or without logarithmic structure). — Suppose given formal étale coordinates $(t_{\lambda})_{\lambda=1,\ldots,r}$ of level m of X/S (see Definition 1.85).

(1) By Proposition 1.91, we get the following isomorphism of m-PD- \mathcal{O}_X -algebras

$$\mathcal{O}_X \langle T_1, \dots, T_r \rangle_{(m),n} \xrightarrow{\sim} \mathcal{P}^n_{X/S,(m)}$$
$$T_\lambda \longmapsto \tau_\lambda,$$

where $\tau_{\lambda} := p_1^*(t_{\lambda}) - p_0^*(t_{\lambda})$. The elements $\{\underline{\tau}^{\{\underline{k}\}_{(m)}}\}_{|\underline{k}| \leq n}$ form a basis of $\mathcal{P}^n_{X/S,(m)}$. The corresponding dual basis of $\mathcal{D}^{(m)}_{X/S,n}$ will be denoted by $\{\underline{\partial}_{\flat}^{\langle \underline{k} \rangle_{(m)}}\}_{|k| \leq n}$.

(2) Consider the following diagram

(2.12)
$$\begin{array}{c} Y \xrightarrow{o} A_{\mathbb{N}^r} \times T \\ \downarrow_f & \downarrow \\ X \xrightarrow{t} \mathbb{A}^r \times S \end{array}$$

where the right arrow is induced by a morphism of fine log schemess of the form $T \to S$ and by the canonical morphism $A_{\mathbb{N}^r} \to \mathbb{A}^r$, the bottom arrow is induced by the formal log étale coordinates $(t_{\lambda})_{\lambda=1,...,r}$ of level m and where the top arrow is induced by a formal log basis $(b_{\lambda})_{\lambda=1,...,r}$ of level m. Let $\underline{\eta}_{(m)}$ (resp. $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$) be the element constructed from $(b_{\lambda})_{\lambda=1,...,r}$ as defined in Proposition 2.13 (resp. Point 2.29). Then the functorial morphism $f^*\mathcal{P}^n_{X/S,(m)} \to$ $\mathcal{P}^n_{Y/T,(m)}$ and $\mathcal{D}^{(m)}_{Y/T} \to f^*\mathcal{D}^{(m)}_{X/S}$ (see Point 2.32) are explicitly described by

(2.13)
$$\underline{\tau}^{\{\underline{k}\}_{(m)}} \longmapsto \underline{t}^{\underline{k}} \underline{\eta}^{\{\underline{k}\}_{(m)}}_{(m)}$$

where the action of $\underline{t}^{\underline{k}}$ is induced by the left structure of \mathcal{O}_Y -algebra of $\mathcal{P}^n_{Y/T,(m)}$. Indeed, since $f^*\mathcal{P}^n_{X/S,(m)} \to \mathcal{P}^n_{Y/T,(m)}$ is an *m*-PDmorphism, we reduce to compute the image of τ_i . We compute the image of τ_i is $1 \otimes b_i - b_i \otimes 1 = b_i \eta_{i,(m)} = t_i \eta_{i,(m)}$ (recall Proposition 2.13 of $\eta_{i,(m)}$). Moreover, by duality, we get $\mathcal{D}^{(m)}_{Y/T} \to f^*\mathcal{D}^{(m)}_{X/S}$ (see Point 2.32) is explicitly described by

(2.14)
$$\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \mapsto \underline{t}^{\underline{k}} \underline{\partial}_{\mathrm{b}}^{\langle \underline{k} \rangle_{(m)}}.$$

(3) Suppose now that the t_{λ} 's lie in $\Gamma(X, \mathcal{O}_X^*)$. Then, from Lemma 1.86(1) they are also a formal log basis of level m. We have

(2.15)
$$\underline{\tau}^{\{\underline{k}\}_{(m)}} = p_0^*(\underline{t}^{\underline{k}})\underline{\eta}_{(m)}^{\{\underline{k}\}_{(m)}} \quad \text{and} \quad \underline{\partial}^{\langle\underline{k}\rangle_{(m)}} = \underline{t}^{\underline{k}}\underline{\partial}_{\flat}^{\langle\underline{k}\rangle_{(m)}}$$

where $\underline{\eta}_{(m)}$ (resp. $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$) is defined in Proposition 2.13 (resp. Point 2.29).

(4) Suppose now that the t_{λ} 's lie in $\Gamma(X, \mathcal{O}_X^*)$ and that $X \to S$ is strict. Since $X \to S$ is strict, then $\mathcal{P}_{\underline{X}/\underline{S},(m)}^n = \mathcal{P}_{X/S,(m)}^n$. This yields $\mathcal{D}_{\underline{X}/\underline{S}}^{(m)} = \mathcal{D}_{X/S}^{(m)}$, where $\mathcal{D}_{\underline{X}/\underline{S}}^{(m)}$ is the sheaf of differential operators defined by Berthelot in [4]. Then the local description of (2.12) extends that given in [4] when $\underline{X}/\underline{S}$ has étale coordinates.

Point 2.31. — For any $m' \ge m$, from the homomorphisms $\psi_{m,m'}^{n*}$: $\mathcal{P}_{X/S,(m')}^n \to \mathcal{P}_{X/S,(m)}^n$ of Point 2.19, we get by duality, the homomorphisms $\rho_{m',m} \colon \mathcal{D}_{X/S}^{(m)} \to \mathcal{D}_{X/S}^{(m')}$. Let $k_{\lambda} = p^m q_{\lambda} + r_{\lambda}$ and $k'_{\lambda} = p^{m'} q'_{\lambda} + r'_{\lambda}$ be the Euclidian division of k_{λ} and k'_{λ} by respectively p^m and $p^{m'}$. Now, suppose that $X \to S$ is endowed with a log basis $(b_{\lambda})_{\lambda=1,\dots,r}$ of level m'. With its notation, we get from Point 2.19 the equality $\rho_{m',m}(\underline{\partial}^{\langle \underline{k} \rangle (m)}) = \frac{q!}{q'!} \underline{\partial}^{\langle \underline{k} \rangle (m')}$.

Point 2.32. — Let $g: S' \to S$ be a morphism of fine log schemes over $\mathbb{Z}/p^{i+1}\mathbb{Z}$. Consider the commutative diagram

(2.16)
$$\begin{array}{c} X' \xrightarrow{f} X \\ \downarrow \pi_{X'} & \downarrow \pi_{X} \\ S' \xrightarrow{g} S \end{array}$$

such that π_X and $\pi_{X'}$ are formall log smooth of level m. Using the universal property of the *m*-PD envelope, we get the *m*-PD-morphism $f^*\mathcal{P}^n_{X/S,(m)} \to \mathcal{P}^n_{X'/S',(m)}$. This yields the morphism $\mathcal{D}^{(m)}_{X'/S',n} \to f^*\mathcal{D}^{(m)}_{X/S,n}$ and then $\mathcal{D}^{(m)}_{X'/S'} \to f^*\mathcal{D}^{(m)}_{X/S}$.

When the diagram (2.16) is cartesian (in the category of fine log schemes), the morphism $f^* \mathcal{P}^n_{X/S,(m)} \to \mathcal{P}^n_{X'/S',(m)}$ is in fact an isomorphism of rings and so is $\mathcal{D}^{(m)}_{X'/S'} \to f^* \mathcal{D}^{(m)}_{X/S}$. When g = id and f is formally log étale of level m, then the morphism

When g = id and f is formally log étale of level m, then the morphism $f^* \mathcal{P}^n_{X/S,(m)} \to \mathcal{P}^n_{X'/S,(m)}$ is in fact an isomorphism and so is $\mathcal{D}^{(m)}_{X'/S} \to f^* \mathcal{D}^{(m)}_{X/S}$.

2.4. Logarithmic PD stratification of level m

We keep Section 2.2. One can follow Berthelot's construction of PD stratifications of level m and check properties analogous to those of [6] or [4, 2.3] (or Montagnon logarithmic version in [27, 2.6]). Let us give a quick exposition. Even if one might consider the étale topology, an \mathcal{O}_X -module will mean an \mathcal{O}_X -module for the Zariski topology.

DEFINITION 2.33. — Let \mathcal{E} be an \mathcal{O}_X -module. An *m*-PD-stratification (or a PD-stratification of level *m*) is the data of a family of compatible (with respect to the projections $\pi_{X/S,(m)}^{n+1,n}$) $\mathcal{P}_{X/S,(m)}^n$ -linear isomorphisms

$$\epsilon_n^{\mathcal{E}} \colon \mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)}$$

satisfying the following conditions:

(1) $\epsilon_0^{\mathcal{E}} = \operatorname{Id}_{\mathcal{E}};$ (2) for any n, n', the diagram

$$\mathcal{P}_{X/S,(m)}^{n} \otimes_{\mathcal{O}_{X}} \mathcal{P}_{X/S,(m)}^{n'} \otimes_{\mathcal{O}_{X}} \mathcal{E}$$

$$\sim \left| \begin{array}{c} \downarrow \\ q_{1(m)}^{n,n'*}(\epsilon_{n+n'}^{\mathcal{E}}) & P_{X/S,(m)}^{n} \otimes_{\mathcal{O}_{X}} \mathcal{E} \otimes_{\mathcal{O}_{X}} \mathcal{P}_{X/S,(m)}^{n'} \\ \downarrow \\ q_{0(m)}^{n,n'*}(\epsilon_{n+n'}^{\mathcal{E}}) & P_{X/S,(m)}^{n} \otimes_{\mathcal{O}_{X}} \mathcal{P}_{X/S,(m)}^{n'} \\ \mathcal{E} \otimes_{\mathcal{O}_{X}} \mathcal{P}_{X/S,(m)}^{n} \otimes_{\mathcal{O}_{X}} \mathcal{P}_{X/S,(m)}^{n'} \\ \end{array} \right|$$

is commutative

PROPOSITION 2.34. — Let \mathcal{E} be an \mathcal{O}_X -module. The following datas are equivalent:

(1) A structure of left $\mathcal{D}_{X/S}^{(m)}$ -module on \mathcal{E} extending its structure of \mathcal{O}_X -module.

(2) A family of compatible \mathcal{O}_X -linear homomorphisms $\theta_n^{\mathcal{E}} \colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^n$ such that $\theta_0^{\mathcal{E}} = \mathrm{Id}_{\mathcal{E}}$ and for any integers n, n' the diagram

$$(2.17) \qquad \begin{array}{c} \mathcal{E} \otimes_{\mathcal{O}_{X}} \mathcal{P}_{X/S,(m)}^{n} \xrightarrow{\mathrm{Id} \otimes \delta_{(m)}^{n,n'}} \mathcal{E} \otimes_{\mathcal{O}_{X}} \mathcal{P}_{X/S,(m)}^{n} \otimes_{\mathcal{O}_{X}} \mathcal{P}_{X/S,(m)}^{n'} \\ \\ \theta_{n+n'}^{\varepsilon} \uparrow & \theta_{n'}^{\varepsilon} \\ \mathcal{E} \xrightarrow{\theta_{n'}}^{\varepsilon} \xrightarrow{\mathcal{E}} \otimes_{\mathcal{O}_{X}} \mathcal{P}_{X/S,(m)}^{n'} \end{array}$$

is commutative.

(3) An *m*-PD-stratification on \mathcal{E} .

An \mathcal{O}_X -linear morphism $\phi: \mathcal{E} \to \mathcal{F}$ between two left $\mathcal{D}_{X/S}^{(m)}$ -modules is $\mathcal{D}_{X/S}^{(m)}$ -linear if and only if it commutes with the homomorphisms θ_n (resp. ϵ_n).

Proof. — The proof is identical to that of [27, 2.6.1] or [4, 2.3.2].

Point 2.35. — If $X \to S$ is endowed with a formal log basis $(b_{\lambda})_{\lambda=1,\ldots,n}$ of level *m* then for any $x \in \mathcal{E}$ we have the Taylor development

(2.18)
$$\theta_n^{\mathcal{E}}(x) = \sum_{|\underline{k}| \leqslant n} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \cdot x \otimes \underline{\eta}_{(m)}^{\{\underline{k}\}}.$$

In order to define overconvergent isocrystals in our context (see Point 3.37), we will need the following definition and proposition.

DEFINITION 2.36. — Let \mathcal{B} be a commutative \mathcal{O}_X -algebra endowed with a structure of left $\mathcal{D}_{X/S}^{(m)}$ -module. We say that the structure of left $\mathcal{D}_{X/S}^{(m)}$ module on \mathcal{B} is compatible with its structure of \mathcal{O}_X -algebra if the isomorphisms $\epsilon_n^{\mathcal{B}}$ are isomorphisms of $\mathcal{P}_{X/S,(m)}^n$ -algebras. This compatibility is equivalent to the following condition: for any $f, g \in \mathcal{B}$ and $\underline{k} \in \mathbb{N}^d$,

$$\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}(fg) = \sum_{\underline{i} \leq \underline{k}} \left\{ \underline{k} \\ \underline{i} \right\} \underline{\partial}^{\langle \underline{i} \rangle_{(m)}}(f) \underline{\partial}^{\langle \underline{k} - \underline{i} \rangle_{(m)}}(g).$$

PROPOSITION 2.37. — Let \mathcal{B} be a commutative \mathcal{O}_X -algebra endowed with a compatible structure of left $\mathcal{D}_{X/S}^{(m)}$ -module. Then there exists on the tensor product $\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}^{(m)}$ a unique structure of rings satisfying the following conditions

(1) the canonical morphisms $\mathcal{B} \to \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}^{(m)}$ and $\mathcal{D}_{X/S}^{(m)} \to \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}^{(m)}$ are homomorphisms of sheaf of rings,

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(2) if $X \to S$ is endowed with a formal log basis $(b_{\lambda})_{\lambda=1,\dots,n}$ of level m, then, for any $b \in \mathcal{B}$ and $\underline{k} \in \mathbb{N}^n$, we have $(b \otimes 1)(1 \otimes P) = b \otimes P$ and

$$(1 \otimes \underline{\partial}^{\langle \underline{k} \rangle_{(m)}})(b \otimes 1) = \sum_{\underline{i} \leq \underline{k}} \left\{ \underline{k} \\ \underline{i} \right\} \underline{\partial}^{\langle \underline{i} \rangle_{(m)}}(b) \otimes \underline{\partial}^{\langle \underline{k} - \underline{i} \rangle_{(m)}}$$

If $\mathcal{B} \to \mathcal{B}'$ is a morphism of \mathcal{O}_X -algebras with compatible structure of left $\mathcal{D}_{X/S}^{(m)}$ -modules, then the induced morphism $\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}^{(m)} \to \mathcal{B}' \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}^{(m)}$ is a homomorphism of rings.

Proof. — We copy [4, 2.3.5].

2.5. Logarithmic transposition

We suppose that $X \to S$ is endowed with a formal log basis $(b_{\lambda})_{\lambda=1,\ldots,n}$ of level m.

Notation 2.38.

(1) We set $\alpha_{0,0} := 1$. Let $j \ge 1$ be an integer. We set $\alpha_{0,j} = 0$. For any $1 \le i \le j$, we set

$$\alpha_{i,j} := (-1)^j \left\{ {}^{j}_{i} \right\}_{(m)} q_{j-i}^{(m)!} \left({}^{j-1}_{j-i} \right)$$

where $q_{j-i}^{(m)}$ means the quotient of the Euclidian division of j-i by p^m .

For any integer $\lambda = 1, \ldots, n$, for any $k \ge 0$, we set

(2.19)
$$\widetilde{\partial}_{\lambda}^{\langle k \rangle_{(m)}} := \sum_{0 \leqslant i \leqslant k} \alpha_{i,k} \partial_{\lambda}^{\langle i \rangle_{(m)}}.$$

In particular, $\widetilde{\partial}_{\lambda}^{\langle 0 \rangle_{(m)}} = 1$. Remark that when $k \ge 1$, we have in fact, $\widetilde{\partial}_{\lambda}^{\langle k \rangle_{(m)}} = \sum_{1 \le i \le k} \alpha_{i,k} \partial_{\lambda}^{\langle i \rangle_{(m)}}$.

(2) Let $\underline{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$. We set $\underline{\widetilde{\partial}}^{\langle \underline{k} \rangle_{(m)}} := \prod_{\lambda=1}^n \widetilde{\partial}_{\lambda}^{\langle k_\lambda \rangle_{(m)}}$. For any $\underline{i} \leq \underline{k}$, we put $\alpha_{\underline{i},\underline{k}} := \prod_{\lambda=1}^n \alpha_{i_\lambda,k_\lambda} \in \mathbb{Z}$.

(2.20)
$$\underline{\widetilde{\partial}}^{\langle \underline{k} \rangle_{(m)}} := \prod_{\lambda=1}^{n} \widetilde{\partial}_{\lambda}^{\langle k_{\lambda} \rangle_{(m)}} = \sum_{\underline{i} \leq \underline{k}} \alpha_{\underline{i},\underline{k}} \partial^{\langle \underline{i} \rangle_{(m)}}.$$

(3) Let $P \in \Gamma(X, \mathcal{D}_{X/S}^{(m)})$ be differential operator. We can uniquely write P of the form $P = \sum_{\underline{k}} a_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$ with $a_{\underline{k}} \in \Gamma(X, \mathcal{O}_X)$. We set

(2.21)
$$\widetilde{P} := \sum_{\underline{k}} \underline{\widetilde{\partial}}^{\langle \underline{k} \rangle_{(m)}} a_{\underline{k}}$$

We say that \widetilde{P} is the "logarithmic transposition" of P.

(4) For any differential operators $P, Q \in \Gamma(X, \mathcal{D}_{X/S}^{(m)})$, for any $a \in \Gamma(X, \mathcal{O}_X)$, we can easily check $\tilde{a} = a, \widetilde{P+Q} = \widetilde{P} + \widetilde{Q}$ and $\widetilde{aP} = \widetilde{P}a$.

PROPOSITION 2.39 (Comparison between transposition with or without logarithmic structure). — We suppose that f is weakly smooth of level mand that $b_1, \ldots, b_n \in \mathcal{O}_X^*$ (and following Lemma 1.86 then they form also some formal log-étale coordinates of level m). In that case, we prefer to write $t_{\lambda} := b_{\lambda}$. With the notation of Point 2.30, any differential operator P of $\Gamma(X, \mathcal{D}_{X/S}^{(m)})$ can be written of the form $P = \sum_{\underline{k}} a_{\underline{k}} \underline{\partial}_{\flat}^{\langle \underline{k} \rangle (m)}$, with $a_{\underline{k}} \in$ $\Gamma(X, \mathcal{O}_X)$. By extending the non logarithmic transposition as defined by Berthelot (see [5, 1.3]) to our context by putting ${}^{\mathrm{t}}P := \sum_{\underline{k}} (-1)^{|\underline{k}|} \underline{\partial}_{\flat}^{\langle \underline{k} \rangle (m)} a_{\underline{k}}$, we have the equality

(2.22)
$$\widetilde{P} = \underline{t} \,{}^{\mathrm{t}}P \,\, \frac{1}{t},$$

where $\underline{t} = \underline{t}^{\underline{1}} = t_1 \cdots t_n$.

Proof. — Following (2.15), we have $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} = \underline{t}^{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}_{\flat}$. By additivity of the functors $P \mapsto \widetilde{P}$ and of $P \mapsto {}^{\mathrm{t}}P$, since for any $a \in \Gamma(X, \mathcal{O}_X)$ and differential operator $P \in \Gamma(X, \mathcal{D}_{X/S}^{(m)})$, we have $\widetilde{aP} = \widetilde{P}a$ and ${}^{t}(aP) = {}^{t}(P)a$, then we reduce to check the formula (2.22) to the case where $P = \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$. We compute

$$\underline{t}^{\mathsf{t}}(\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}) \ \underline{\frac{1}{\underline{t}}} = \underline{t}^{\mathsf{t}}(\underline{t}^{\underline{k}}\underline{\partial}_{\flat}^{\langle \underline{k} \rangle_{(m)}}) \ \underline{\frac{1}{\underline{t}}}$$

$$= \underline{t}(-1)^{|\underline{k}|}\underline{\partial}_{\flat}^{\langle \underline{k} \rangle_{(m)}}\underline{t}^{\underline{k}-\underline{1}}$$

$$= (-1)^{|\underline{k}|}\underline{t}^{\underline{1}-\underline{k}}\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}\underline{t}^{\underline{k}-\underline{1}}$$

$$= \prod_{\lambda=1}^{n} (-1)^{k_{\lambda}}t_{\lambda}^{1-k_{\lambda}}\partial_{\lambda}^{\langle k_{\lambda} \rangle_{(m)}}t_{\lambda}^{k_{\lambda}-1},$$

where $\underline{1} := (1, 1, \dots, 1) \in \mathbb{N}^n$. Hence, we reduce to check $\widetilde{\partial}_{\lambda}^{\langle k \rangle_{(m)}} = (-1)^k t_{\lambda}^{1-k} \partial_{\lambda}^{\langle k \rangle_{(m)}} t_{\lambda}^{k-1}$, for any integer $k \ge 0$. When k = 0, the equality is obvious since $\partial_{\lambda}^{\langle k \rangle_{(m)}} = 1$ and $\widetilde{\partial}_{\lambda}^{\langle k \rangle_{(m)}} = 1$. Let us suppose $k \ge 1$. From (2.10), we have $\partial_{\lambda}^{\langle k \rangle_{(m)}} t_{\lambda}^{k-1} = \sum_{i \le k} \{ {}^k_i \} \partial_{\lambda}^{\langle k-i \rangle_{(m)}} (t_{\lambda}^{k-1}) \partial_{\lambda}^{\langle i \rangle_{(m)}}$. Following the formula [27, 2.3.3.b)] (or we can use [4, 2.2.4.(ii)] and (2.15)), we have $\partial_{\lambda}^{\langle k-i \rangle_{(m)}} (t_{\lambda}^{k-1}) = t_{\lambda}^{k-1} q_{k-i}! {k-1 \choose k-i}$ if $i \ge 1$ and 0 if i = 0. Hence,

$$(-1)^{k} t_{\lambda}^{1-k} \partial_{\lambda}^{\langle k \rangle_{(m)}} t_{\lambda}^{k-1} = (-1)^{k} \sum_{1 \leqslant i \leqslant k} \left\{ \begin{smallmatrix} k \\ i \end{smallmatrix} \right\} q_{k-i}! \left(\begin{smallmatrix} k-1 \\ k-i \end{smallmatrix} \right) \partial_{\lambda}^{\langle i \rangle_{(m)}} = \widetilde{\partial}_{\lambda}^{\langle k \rangle_{(m)}}.$$

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Remark 2.40. — With Remark 2.40, the definition of (2.21) was precisely introduced to get $\underline{\widetilde{\partial}}^{\langle \underline{k} \rangle_{(m)}} = \underline{t} \, \underline{t} \, \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \, \underline{1}_{\underline{t}}$. One reason to introduce the logarithmic transposition is the formula (3.7).

PROPOSITION 2.41. — For any differential operators P and Q, we have $\widetilde{PQ} = \widetilde{Q}\widetilde{P}$.

Proof.

Step 0. — When $P \in \mathcal{O}_X$, the proposition is obvious.

Step 1. — Suppose that $P = \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$ and $Q = a \in \mathcal{O}_X$.

(i). — On one side, from the formula (2.20), we get

$$a\underline{\widetilde{\partial}}^{\langle\underline{k}\rangle_{(m)}} = \sum_{\underline{h} \leqslant \underline{k}} \alpha_{\underline{h},\underline{k}} a\underline{\partial}^{\langle\underline{h}\rangle_{(m)}}.$$

(ii). — On the other side, from the formula

$$\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} a = \sum_{\underline{j} \leq \underline{k}} \left\{ \underline{k}_{\underline{j}} \right\} \underline{\partial}^{\langle \underline{k} - \underline{j} \rangle_{(m)}}(a) \underline{\partial}^{\langle \underline{j} \rangle_{(m)}}$$

(see (2.10)), we get

$$(\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} a)^{\sim} = \sum_{\underline{j} \leqslant \underline{k}} \left\{ \frac{\underline{k}}{\underline{j}} \right\} \underline{\widetilde{\partial}}^{\langle \underline{j} \rangle_{(m)}} \underline{\partial}^{\langle \underline{k} - \underline{j} \rangle_{(m)}} (a).$$

From (2.20), we have $\underline{\widetilde{\partial}}^{\langle \underline{j} \rangle_{(m)}} = \sum_{\underline{i} \leq \underline{j}} \alpha_{\underline{i},\underline{j}} \partial^{\langle \underline{i} \rangle_{(m)}}$. Hence,

$$(\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} a)^{\sim} = \sum_{\underline{j} \leq \underline{k}} \sum_{\underline{i} \leq \underline{j}} \left\{ \frac{\underline{k}}{\underline{j}} \right\} \alpha_{\underline{i}, \underline{j}} \partial^{\langle \underline{i} \rangle_{(m)}} \underline{\partial}^{\langle \underline{k} - \underline{j} \rangle_{(m)}} (a).$$

Using again the formula (2.10), we get

$$\partial^{\langle \underline{i} \rangle_{(m)}} \underline{\partial}^{\langle \underline{k} - \underline{j} \rangle_{(m)}}(a) = \sum_{\underline{h} \leqslant \underline{i}} \left\{ \frac{\underline{i}}{\underline{h}} \right\} \underline{\partial}^{\langle \underline{i} - \underline{h} \rangle_{(m)}} \underline{\partial}^{\langle \underline{k} - \underline{j} \rangle_{(m)}}(a) \underline{\partial}^{\langle \underline{h} \rangle_{(m)}}.$$

Hence,

$$\begin{split} (\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} a)^{\sim} &= \sum_{\underline{j} \leq \underline{k}} \sum_{i \leq \underline{j}} \sum_{\underline{h} \leq \underline{i}} \left\{ \frac{\underline{k}}{\underline{j}} \right\} \alpha_{\underline{i}, \underline{j}} \left\{ \frac{\underline{i}}{\underline{h}} \right\} \underline{\partial}^{\langle \underline{i} - \underline{h} \rangle_{(m)}} \underline{\partial}^{\langle \underline{k} - \underline{j} \rangle_{(m)}} (a) \underline{\partial}^{\langle \underline{h} \rangle_{(m)}} \\ &= \sum_{\underline{h} \leq \underline{k}} \sum_{\underline{h} \leq \underline{i} \leq \underline{j} \leq \underline{k}} P_{\underline{h}, \underline{i}, \underline{j}, \underline{k}} (a) \underline{\partial}^{\langle \underline{h} \rangle_{(m)}}, \end{split}$$

where for any $\underline{h}, \underline{i}, \underline{j}, \underline{k} \in \mathbb{N}^n$ so that $\underline{h} \leq \underline{i} \leq \underline{j} \leq \underline{k}$, we put $P_{\underline{h}, \underline{i}, \underline{j}, \underline{k}} := \alpha_{\underline{i}, \underline{j}} \left\{ \frac{\underline{i}}{\underline{h}} \right\} \left\{ \frac{\underline{k}}{\underline{j}} \right\} \underline{\partial}^{(\underline{i} - \underline{h})_{(m)}} \underline{\partial}^{\langle \underline{k} - \underline{j} \rangle_{(m)}}$. Hence, when $\underline{h} \leq \underline{k}$ are fixed, this is sufficient to check $\alpha_{\underline{h}, \underline{k}} = \sum_{\underline{h} \leq \underline{i} \leq \underline{j} \leq \underline{k}} P_{\underline{h}, \underline{i}, \underline{j}, \underline{k}}$.

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(iii). — Let $A := \operatorname{Spec}(\mathbb{Z}[\mathbb{N}^r]), M_{\mathbb{N}^r}$ and $T := \operatorname{Spec}\mathbb{Z}$. Let U be an open dense subset of A where the log structure is trivial. Since the map $\Gamma(A, \mathcal{D}_{A/T}^{(m)}) \to \Gamma(U, \mathcal{D}_{U/T}^{(m)})$ is injective, then to check the equality $\alpha_{\underline{h},\underline{k}} = \sum_{\underline{h} \leq \underline{i} \leq \underline{j} \leq \underline{k}} P_{\underline{h},\underline{i},\underline{j},\underline{k}}$ in $\Gamma(A, \mathcal{D}_{A/T}^{(m)})$ (remark the coefficients of the differential operators $P_{\underline{h},\underline{i},\underline{j},\underline{k}}$ are integers, i.e. $P_{\underline{h},\underline{i},\underline{j},\underline{k}} \in \Gamma(A, \mathcal{D}_{A/T}^{(m)})$), we reduce to check it in $\Gamma(U, \mathcal{D}_{U/T}^{(m)})$. We can define the (resp. logarithmic) transposition in $\Gamma(U, \mathcal{D}_{U/T}^{(m)})$. The formula (2.22) is still valid (the computation is identical). Since ${}^{\mathrm{t}}(\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}b) = b^{\mathrm{t}}(\underline{\partial}^{\langle \underline{k} \rangle_{(m)}})$ (see [5]), then we get $b\underline{\widetilde{\partial}}^{\langle \underline{k} \rangle_{(m)}} = (\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}b)^{\sim}$ for any $b \in \Gamma(U, \mathcal{O}_U)$. Hence, following the computation of (ii), we get $\alpha_{\underline{h},\underline{k}} = \sum_{\underline{h} \leq \underline{i} \leq \underline{j} \leq \underline{k}} P_{\underline{h},\underline{i},\underline{j},\underline{k}}(b)$, for any $b \in \Gamma(U, \mathcal{O}_U)$. Since $\mathcal{D}_{U/T}^{(m)} \to \mathscr{E}\mathrm{nd}_{\mathcal{O}_T}(\mathcal{O}_A)$ is injective, this yields the equality in $\Gamma(A, \mathcal{D}_{A/T}^{(m)})$:

(2.23)
$$\alpha_{\underline{h},\underline{k}} = \sum_{\underline{h} \leq \underline{i} \leq \underline{j} \leq \underline{k}} P_{\underline{h},\underline{i},\underline{j},\underline{k}}.$$

According to Notation 1.44, $A \times_T \operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z}) = A_{\mathbb{N}^r}$. Hence, the equality (2.23) is also true in $\Gamma(A_{\mathbb{N}^r}, \mathcal{D}_{A_{\mathbb{N}^r}/S}^{(m)})$. Following Definition 1.73, the formal log basis $(b_\lambda)_{\lambda=1,\dots,n}$ of level m induces a formally log étale of level m (compatible with γ) Y-morphism of the form $g \colon X \to S \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{N}^r}$. By considering the image via the homomorphism of rings $g^{-1}\mathcal{D}_{S \times_{\mathbb{Z}/p^{i+1}\mathbb{Z}} A_{\mathbb{N}^r}/S}^{(m)} \to \mathcal{D}_{X/S}^{(m)}$, the equality (2.23) remains true in $\Gamma(X, \mathcal{D}_{X/S}^{(m)})$.

Step 2. — When $P = \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$ and $Q = \underline{\partial}^{\langle \underline{k}' \rangle_{(m)}}$, using the formulas (2.11) and (2.20), we can check that both $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \underline{\partial}^{\langle \underline{k}' \rangle_{(m)}}$ and $(\underline{\partial}^{\langle \underline{k}' \rangle_{(m)}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}})^{\sim}$ are of the form $\sum_{\underline{i} \leq \underline{k} + \underline{k'}} \alpha_{\underline{i}} \partial^{\langle \underline{i} \rangle_{(m)}}$, with $\alpha_{\underline{i}} \in \mathbb{Z}$. Hence, using the same arguments of Step 1 (iii) of the proof, we reduce to the case where $b_1, \ldots, b_n \in \mathcal{O}_X^*$, i.e., we can use the formula (2.22). Since $(\underline{t} \underline{\partial}^{\langle \underline{k}' \rangle_{(m)}})^{(\underline{t}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}) = \underline{t} (\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \underline{\partial}^{\langle \underline{k}' \rangle_{(m)}})$, this yields the equality $(\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \underline{\partial}^{\langle \underline{k}' \rangle_{(m)}})^{\sim} = \underline{\widetilde{\partial}}^{\langle \underline{k}' \rangle_{(m)}} \underline{\widetilde{\partial}}^{\langle \underline{k} \rangle_{(m)}}.$

Step 3. — Suppose $P = \underline{\partial}^{\langle \underline{k'} \rangle_{(m)}}$ and $Q = \underline{\partial}^{\langle \underline{k''} \rangle_{(m)}} a$, with $a \in \mathcal{O}_X$. From (2.10), we have the equality

$$\underline{\partial}^{\langle \underline{k'}\rangle_{(m)}}\underline{\partial}^{\langle \underline{k''}\rangle_{(m)}}a = \sum_{\underline{k}=\max\{\underline{k'},\underline{k''}\}}^{\underline{k'}+\underline{k''}}\beta_{\underline{k},\underline{k'},\underline{k''}}\underline{\partial}^{\langle \underline{k}\rangle_{(m)}}a,$$

where $\beta_{\underline{k},\underline{k}',\underline{k}''} := \frac{\underline{k}!}{(\underline{k}'+\underline{k}''-\underline{k})!(\underline{k}-\underline{k}')!(\underline{k}-\underline{k}'')!} \frac{\underline{q}_{\underline{k}'}!\underline{q}_{\underline{k}''}!}{\underline{q}_{\underline{k}}!} \in \mathbb{Z}$. Hence, from Step 1, by additivity we get

$$(\underline{\partial}^{\langle \underline{k}' \rangle_{(m)}} \underline{\partial}^{\langle \underline{k}'' \rangle_{(m)}} a)^{\sim} = a \sum_{\underline{k} = \max\{\underline{k}', \underline{k}''\}}^{\underline{k}' + \underline{k}''} \beta_{\underline{k}, \underline{k}', \underline{k}''} \underline{\widetilde{\partial}}^{\langle \underline{k} \rangle_{(m)}}$$

From Step 2, the case a = 1 is already checked, i.e., we have

$$(\underline{\partial}^{\langle \underline{k'} \rangle_{(m)}} \underline{\partial}^{\langle \underline{k''} \rangle_{(m)}})^{\sim} = \underline{\widetilde{\partial}}^{\langle \underline{k''} \rangle_{(m)}} \underline{\widetilde{\partial}}^{\langle \underline{k'} \rangle_{(m)}}.$$

Hence,

$$\sum_{\underline{k}=\max\{\underline{k}',\underline{k}''\}}^{\underline{k}'+\underline{k}''} \beta_{\underline{k},\underline{k}',\underline{k}''} \widetilde{\underline{\partial}}^{\langle\underline{k}\rangle_{(m)}} = \widetilde{\underline{\partial}}^{\langle\underline{k}''\rangle_{(m)}} \widetilde{\underline{\partial}}^{\langle\underline{k}'\rangle_{(m)}}.$$

Multiplying by a at the left, this yields, $a \sum_{\underline{k}=\max\{\underline{k}',\underline{k}''\}}^{\underline{k}'+\underline{k}''} \beta_{\underline{k},\underline{k}',\underline{k}''} \widetilde{\underline{\partial}}^{\langle\underline{k}\rangle(m)} = a \underline{\widetilde{\partial}}^{\langle\underline{k}'\rangle(m)} \underline{\widetilde{\partial}}^{\langle\underline{k}'\rangle(m)}$. Since from Step 1 we have $a \underline{\widetilde{\partial}}^{\langle\underline{k}''\rangle(m)} = (\underline{\partial}^{\langle\underline{k}''\rangle(m)}a)^{\sim}$, then $a \underline{\widetilde{\partial}}^{\langle\underline{k}''\rangle(m)} \underline{\widetilde{\partial}}^{\langle\underline{k}'\rangle(m)} = (\underline{\partial}^{\langle\underline{k}''\rangle(m)}a)^{\sim} \underline{\widetilde{\partial}}^{\langle\underline{k}'\rangle(m)}$. Hence,

$$\begin{aligned} (\underline{\partial}^{\langle \underline{k}' \rangle_{(m)}} \underline{\partial}^{\langle \underline{k}'' \rangle_{(m)}} a)^{\sim} &= a \sum_{\underline{k}=\max\{\underline{k}', \underline{k}''\}}^{\underline{k}' + \underline{k}''} \beta_{\underline{k}, \underline{k}', \underline{k}''} \underline{\widetilde{\partial}}^{\langle \underline{k} \rangle_{(m)}} \\ &= (\underline{\partial}^{\langle \underline{k}'' \rangle_{(m)}} a)^{\sim} \underline{\widetilde{\partial}}^{\langle \underline{k}' \rangle_{(m)}}. \end{aligned}$$

Step 4. — We can write $P = \sum_{\underline{k}} a_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$ and $Q = \sum_{\underline{k}} \underline{\partial}^{\langle \underline{k}' \rangle_{(m)}} a'_{\underline{k}}$, with $a_{\underline{k}}, a'_{\underline{k}} \in \Gamma(X, \mathcal{O}_X)$. By additivity, we reduce to the case $P = a \underline{\partial}^{\langle \underline{k}' \rangle_{(m)}}$ and $Q = \underline{\partial}^{\langle \underline{k}' \rangle_{(m)}} a'$, with $a, a' \in \Gamma(X, \mathcal{O}_X)$. Using Step 0, $(a \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \underline{\partial}^{\langle \underline{k}' \rangle_{(m)}} a')^{\sim} = (\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \underline{\partial}^{\langle \underline{k}' \rangle_{(m)}} a')^{\sim} a$. From Step 3 of the proof, $(\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \underline{\partial}^{\langle \underline{k}' \rangle_{(m)}} a')^{\sim} = (\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} a')^{\sim} \underline{\widetilde{\partial}}^{\langle \underline{k} \rangle_{(m)}} a')^{\sim} Brook equation (Additional Addition (Additional Additional A$

$$(a\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \underline{\partial}^{\langle \underline{k}' \rangle_{(m)}} a')^{\sim} = (\underline{\partial}^{\langle \underline{k}' \rangle_{(m)}} a')^{\sim} \underline{\widetilde{\partial}}^{\langle \underline{k} \rangle_{(m)}} a$$
$$= (\underline{\partial}^{\langle \underline{k}' \rangle_{(m)}} a')^{\sim} (a\underline{\partial}^{\langle \underline{k} \rangle_{(m)}})^{\sim}.$$

Remark 2.42 (Logarithmic transposition at the level 0). — The definition of $\underline{\widetilde{\partial}}^{\langle \underline{k} \rangle (m)}$ given at (2.20) looks very complicated compared to the non logarithmic transposition. At the level 0, this is possible to get a definition as simple as in the non logarithmic case as follows. Any differential operator of $\mathcal{D}_{X/S}^{(0)}$ can be written uniquely in the form $P = \sum_{\underline{k}} a_{\underline{k}} \underline{\partial}^{\underline{k}}$, where $\underline{\partial}^{\underline{k}} = \prod_{\lambda=1}^{n} \partial_{\lambda}^{k_{\lambda}}$ with $\partial_{\lambda} = \partial_{\lambda}^{\langle 1 \rangle_{(0)}}$. Beware that $\underline{\partial}^{\underline{k}} \neq \underline{\partial}^{\langle \underline{k} \rangle_{(0)}}$. Since $\underline{\widetilde{\partial}} = -\underline{\partial}$, we get $\widetilde{P} = \sum_{\underline{k}} (-1)^{|\underline{k}|} \underline{\partial}^{\underline{k}} a_{\underline{k}}$, which looks like the non logarithmic transposition definition.

PROPOSITION 2.43. — For any differential operator P, we have $\tilde{\tilde{P}} = P$.

Proof. — By additivity of the logarithmic transposition, we reduce to check the case where $P = (a\underline{\partial}^{\langle \underline{k} \rangle_{(m)}})^{\sim}$, with $a \in \Gamma(X, \mathcal{O}_X)$ and $\underline{k} \in \mathbb{N}^d$. Using Proposition 2.41, we come down to the case where $P = \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$, with $\underline{k} \in \mathbb{N}^d$. Using (2.20) twice, we can write $((\underline{\partial}^{\langle \underline{k} \rangle_{(m)}})^{\sim})^{\sim} = \sum_{\underline{i} \leq \underline{k}} \beta_{\underline{i},\underline{k}} \partial^{\langle \underline{i} \rangle_{(m)}}$, with $\beta_{\underline{i},\underline{k}} \in \mathbb{Z}$. Hence, using the same arguments of Step 1 (iii) of the proof of Proposition 2.41, we reduce to the case where $b_1, \ldots, b_n \in \mathcal{O}_X^*$, i.e., we can use the formula (2.22). Via the formula (2.22), we reduce to check $t(\underline{i}(\underline{\partial}_{\underline{k}}^{\langle \underline{k} \rangle_{(m)}})) = \underline{\partial}_{\underline{b}}^{\langle \underline{k} \rangle_{(m)}}$, which is obvious.

Remark 2.44. — Recall that from (2.10), we have

$$\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} a = \sum_{\underline{i} \leqslant \underline{k}} \left\{ \underline{k}_{\underline{i}} \right\} \underline{\partial}^{\langle \underline{k} - \underline{i} \rangle_{(m)}}(a) \underline{\partial}^{\langle \underline{i} \rangle_{(m)}}$$

Using Proposition 2.41, this yields

(2.24)
$$a\underline{\widetilde{\partial}}^{\langle \underline{k} \rangle_{(m)}} = \sum_{\underline{i} \leq \underline{k}} \underline{\widetilde{\partial}}^{\langle \underline{i} \rangle_{(m)}} \left\{ \frac{\underline{k}}{\underline{i}} \right\} \underline{\partial}^{\langle \underline{k} - \underline{i} \rangle_{(m)}}(a)$$

In the formula (2.24), beware that we can not replace $\underline{\widetilde{\partial}}$ by $\underline{\partial}$.

Point 2.45. — The logarithmic transposition commutes with the canonical morphism $\rho_{m+1,m} \colon \mathcal{D}_{X/S}^{(m)} \to \mathcal{D}_{X/S}^{(m+1)}$, i.e., for any $P \in \Gamma(X, \mathcal{D}_{X/S}^{(m)})$, we have the formula $\rho_{m+1,m}(\tilde{P}) = (\rho_{m+1,m}(P))^{\sim}$. Indeed, by additivity of the logarithmic transposition, we reduce to check the case where $P = a\underline{\partial}^{\langle \underline{k} \rangle \langle m \rangle}$, with $a \in \Gamma(X, \mathcal{O}_X)$ and $\underline{k} \in \mathbb{N}^d$. By \mathcal{O}_X -bilinearity of $\rho_{m+1,m}$, we come down to the case where $P = \underline{\partial}^{\langle \underline{k} \rangle \langle m \rangle}$. For any $\underline{i} \in \mathbb{N}^d$, we have $\rho_{m+1,m}(\underline{\partial}^{\langle \underline{i} \rangle \langle m \rangle}) = \gamma_{\underline{i}} \underline{\partial}^{\langle \underline{i} \rangle \langle m+1 \rangle}$, with $\gamma_{\underline{i}} \in \mathbb{Z}$. Hence, using (2.20), we can check that both terms $\rho_{m+1,m}(\underline{\widetilde{\partial}}^{\langle \underline{k} \rangle \langle m \rangle})$ and $(\rho_{m+1,m}(\underline{\partial}^{\langle \underline{k} \rangle \langle m \rangle})) \sim$ are of the form $\sum_{\underline{i} \leq \underline{k}} \alpha_{\underline{i}} \partial^{\langle \underline{i} \rangle \langle m \rangle}$, with $\alpha_{\underline{i}} \in \mathbb{Z}$. Hence, using the same arguments of Step 1 (iii) of the proof, we reduce to the case where $b_1, \ldots, b_n \in \mathcal{O}_X^*$, i.e., we can use the formula (2.22). Via the formula (2.22), we reduce to check $\rho_{m+1,m}(^{\underline{i}} \underline{\partial}_{\underline{k}}^{\langle \underline{k} \rangle \langle m \rangle}) = {}^{\mathrm{t}}(\rho_{m+1,m}(\underline{\partial}_{\underline{k}}^{\langle \underline{k} \rangle \langle m \rangle}))$, which is obvious.

3. Differential operators over fine log formal schemes

We recall that Shiho introduced the notion of log formal \mathcal{V} -schemes (see [30, 2.1.1.(4)]) as follows: A log formal \mathcal{V} -scheme \mathfrak{X} is a formal \mathcal{V} -scheme $\underline{\mathfrak{X}}$ endowed with a logarithmic structure $\alpha \colon M_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$, where $\mathcal{O}_{\mathfrak{X}} \coloneqq \mathcal{O}_{\mathfrak{X}}$ (this means that α is a logarithmic morphism of sheaves of monoids for the étale topology over \mathfrak{X} , i.e. α is such that $\alpha^{-1}(\mathcal{O}_{\mathfrak{X}}^*) \to \mathcal{O}_{\mathfrak{X}}^*$ is an isomorphism). When $M_{\mathfrak{X}}$ is fine as sheave for the étale topology over the special fiber of \mathfrak{X} (i.e. when $M_{\mathfrak{X}}$ is integral and $M_{\mathfrak{X}}$ is coherent in the sense defined at the end of the remark [28, II.2.1.5]), we say that the logarithmic structure $M_{\mathfrak{X}}$ is fine. We say that \mathfrak{X} is a fine log formal \mathcal{V} -scheme if $M_{\mathfrak{X}}$ is fine. Let $\mathcal{S} := \operatorname{Spf} \mathcal{V}$ be the formal \mathcal{V} -scheme (endowed with the trivial log structure). A fine \mathcal{S} -log formal scheme \mathfrak{X} will be a morphism of fine log formal \mathcal{V} -schemes of the form $\mathfrak{X} \to \mathcal{S}$.

If \mathfrak{X} is a fine log formal \mathcal{V} -scheme and $i \in \mathbb{N}$, then we denote by X_i the fine log $\mathcal{V}/\pi^{i+1}\mathcal{V}$ -scheme so that $\underline{X}_i := \mathfrak{X} \times_{\mathrm{Spf}(\mathcal{V})} \mathrm{Spec}(\mathcal{V}/\pi^{i+1}\mathcal{V})$ and the morphism $X_i \to \mathfrak{X}$ is strict. For i = 0, we can simply denote X_0 by X. If $f: \mathfrak{X} \to \mathcal{Y}$ is a morphism of fine log formal \mathcal{V} -schemes, then we denote by $f_i: X_i \to Y_i$ the induced morphism of fine log-schemes over $\mathcal{V}/\pi^{i+1}\mathcal{V}$. We remark that if $f: \mathfrak{X} \to \mathcal{Y}$ is a morphism of fine log S_i -schemes.

3.1. From log schemes to formal log schemes

Point 3.1 (Charts for log formal \mathcal{V} -schemes). — Let P be a fine monoid and $\mathcal{V}\{P\}$ be the *p*-adic completion of $\mathcal{V}[P]$. Since \mathcal{V} is fixed, we denote by \mathfrak{A}_P the fine log formal \mathcal{V} -scheme whose underlying formal \mathcal{V} -scheme is $\operatorname{Spf}(\mathcal{V}\{P\})$ and whose log structure is the log structure associated with the pre-log structure induced canonically by $P \to \mathcal{V}\{P\}$.

Let \mathfrak{X} be a fine S-log formal scheme. We denote by $P_{\mathfrak{X}}$ the sheaf associated to the constant presheaf of P over \mathfrak{X} . Following Shiho's definition of [30, 2.1.7], a chart of \mathfrak{X} is a morphism of monoids $\alpha \colon P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ whose associated log structure is isomorphic to $M_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$. A chart of \mathfrak{X} is equivalent to the data of a strict morphism of the form $\mathfrak{X} \to \mathfrak{A}_P$.

LEMMA 3.2. — Let \mathfrak{X} be a fine S-log formal scheme. Let $i \ge 0$ be an integer. Then, the morphisms $\mathcal{O}_{\mathfrak{X}}^* \to \mathcal{O}_{X_i}^*$ and $M_{\mathfrak{X}} \to M_{X_i}$ are surjective.

Proof. — The fact that $\mathcal{O}_{\mathfrak{X}}^* \to \mathcal{O}_{X_i}^*$ is surjective comes from the fact that $\mathcal{O}_{\mathfrak{X}}$ is complete for the *p*-adic topology. The fact that $M_{\mathfrak{X}} \to M_{X_i}$ is surjective is étale local on \mathfrak{X} . Hence, we can suppose there exists a fine monoid *P* and a morphism of sheaves of monoids $\alpha: P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ (here $P_{\mathfrak{X}}$ means the sheaf associated to the constant presheaf of *P* over \mathfrak{X}) which induces the isomorphism of sheaves of monoids $P_{\mathfrak{X}} \oplus_{\alpha^{-1}(\mathcal{O}_{\mathfrak{X}}^*)} \mathcal{O}_{\mathfrak{X}}^* \xrightarrow{\sim} M_{\mathfrak{X}}$ and the isomorphism $P_{X_i} \oplus_{\alpha_i^{-1}(\mathcal{O}_{X_i}^*)} \mathcal{O}_{X_i}^* \xrightarrow{\sim} M_{X_i}$. Since $\mathcal{O}_{\mathfrak{X}}^* \to \mathcal{O}_{X_i}^*$ is surjective, we conclude. PROPOSITION 3.3. — Let \mathfrak{X} be a fine S-log formal scheme. Then, in the category of fine S-log formal schemes, \mathfrak{X} is the inductive limit of the system $(X_i)_i$.

Proof. — From [13, I.10.6.1], \mathfrak{X} is the inductive limit of the system $(X_i)_i$. It remains to check that the canonical morphism of sheaves of monoids $M_{\mathfrak{X}} \to \lim_{i} M_{X_{i}}$ is an isomorphism. Since this is étale local on \mathfrak{X} and since \mathfrak{X} is fine then we can suppose there exists a fine monoid P and a morphism of sheaves of monoids $\alpha: P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ which induces the isomorphism of sheaves of monoids $P_{\mathfrak{X}} \oplus_{\alpha^{-1}(\mathcal{O}_{\mathfrak{X}}^*)} \mathcal{O}_{\mathfrak{X}}^* \xrightarrow{\sim} M_{\mathfrak{X}}$. Let $i \ge 0$ be an integer. We get the morphism of sheaves of monoids $\alpha_i \colon P_{X_i} \to \mathcal{O}_{X_i}$ which induces the isomorphism $P_{X_i} \oplus_{\alpha_i^{-1}(\mathcal{O}_{X_i}^*)} \mathcal{O}_{X_i}^* \xrightarrow{\sim} M_{X_i}$. Hence, we reduce to prove that the canonical map $P_{\mathfrak{X}} \oplus_{\alpha^{-1}(\mathcal{O}_{\mathfrak{X}}^*)} \mathcal{O}_{\mathfrak{X}}^* \to \varprojlim_i P_{X_i} \oplus_{\alpha_i^{-1}(\mathcal{O}_{X_i}^*)} \mathcal{O}_{X_i}^*$ is an isomorphism. We put $\mathcal{F}_i := P_{X_i} \oplus "_{\alpha_i^{-1}(\mathcal{O}_{X_i}^*)} \mathcal{O}_{X_i}^*$ where " \oplus " means that the amalgamated sum is computed in the category of presheaves. We put $\mathcal{E}_i := P_{X_i} \oplus \mathcal{O}^*_{X_i}, \theta_i \colon \mathcal{E}_i \to \mathcal{F}_i$ the canonical surjective morphism, $\mathcal{G}_i :=$ $P_{X_i} \oplus_{\alpha_i^{-1}(\mathcal{O}_{X_i}^*)} \mathcal{O}_{X_i}^*$ and $\epsilon_i \colon \mathcal{F}_i \to \mathcal{G}_i$ the canonical morphism from a presheaf to its associated sheaf. We put $\phi_i := \epsilon_i \circ \theta_i$. We denote by $\pi_i : \mathcal{O}_{X_{i+1}}^* \to \mathcal{O}_{X_i}^*$, $\pi_i \colon \mathcal{E}_{i+1} \to \mathcal{E}_i \pi_i \colon \mathcal{F}_{i+1} \to \mathcal{F}_i, \pi_i \colon \mathcal{G}_{i+1} \to \mathcal{G}_i$ the canonical projections. Let $\mathfrak{U} \to \mathfrak{X}$ be an étale map such that \mathfrak{U} is connected.

Step 1. — Let $s_{i+1} \in \mathcal{F}_{i+1}(U_{i+1})$ and $s_i := \pi_i(s_{i+1}) \in \mathcal{F}_i(U_i)$. Then the canonical map $\pi_i : \theta_{i+1}^{-1}(s_{i+1}) \to \theta_i^{-1}(s_i)$ induced by $\pi_i : \mathcal{E}_{i+1}(U_{i+1}) \to \mathcal{E}_i(U_i)$ is a bijection.

(a). — Let us check the injectivity. Let $(x, a), (x', a') \in \theta_{i+1}^{-1}(s_{i+1})$ such that $\pi_i(x, a) = \pi_i(x', a')$ (where $x, x' \in P$ and $a, a' \in \mathcal{O}^*_{X_{i+1}}(U_{i+1})$). The latter equality yields x = x'. Since P is integral, $\theta_{i+1}(x, a) = \theta_{i+1}(x, a')$ implies a = a' (for the computation, use the remark of [21, 1.3]).

(b). — Let us check the surjectivity. Let $(y, b) \in \theta_i^{-1}(s_i)$. We remark that $\alpha^{-1}(\mathcal{O}^*_{\mathfrak{X}})(\mathfrak{U}) = \alpha_i^{-1}(\mathcal{O}^*_{X_i})(U_i)$ and we denote it by Q. Since θ_{i+1} is an epimorphism (in the category of presheaves) then there exists $(x, a) \in$ $\theta_{i+1}^{-1}(s_{i+1})$. Since $\pi_i(x, a) = (x, \pi_i(a)) \in \theta_i^{-1}(s_i)$, there exists $q, q' \in Q(U_i)$ such that $\pi_i(a)\alpha_i(q) = b\alpha_i(q')$ and xq' = yq (see the remark of [21, 1.3]). Set $a' := a\alpha_{i+1}(q)\alpha_{i+1}(q')^{-1}$. Then $\pi_i(a') = b$ and $\theta_{i+1}(x, a) = \theta_{i+1}(y, a')$, i.e. $\pi_i(y, a') = (y, b)$ and $(y, a') \in \theta_{i+1}^{-1}(s_{i+1})$.

Step 2. — Let $t_{i+1} \in \mathcal{G}_{i+1}(U_{i+1})$ and $t_i := \pi_i(t_{i+1}) \in \mathcal{G}_i(U_i)$. Then the canonical map $\pi_i : \phi_{i+1}^{-1}(t_{i+1}) \to \phi_i^{-1}(t_i)$ is a bijection.

(a). — Let us check the injectivity. Let $r, r' \in \phi_{i+1}^{-1}(t_{i+1})$ such that $\pi_i(r) = \pi_i(r')$. There exists an étale covering $(\mathfrak{U}_{\lambda} \to \mathfrak{U})_{\lambda}$ of \mathfrak{U} such that $\theta_{i+1}(r)|U_{\lambda} = \theta(r')_{i+1}|U_{\lambda}$. From 1) (applied for \mathfrak{U}_{λ} instead of \mathfrak{U}), this yields $r|U_{\lambda} = r'|U_{\lambda}$. Hence, r = r'.

(b). — Let us check the surjectivity. Let $r \in \phi_i^{-1}(t_i)$. Put $s := \theta_i(r)$. There exist an étale covering $(\mathfrak{U}_{\lambda} \to \mathfrak{U})_{\lambda}$ of \mathfrak{U} and sections $s_{\lambda} \in \mathcal{F}_{i+1}(U_{\lambda})$ such that $\epsilon_{i+1}(s_{\lambda}) = t_{i+1}|U_{\lambda}$ and $\pi_i(s_{\lambda}) = s|U_{\lambda}$. From 1.b), there exists $r_{\lambda} \in \mathcal{E}_{i+1}(U_{\lambda})$ such that $\pi_i(r_{\lambda}) = r|U_{\lambda}$ and $\theta_{i+1}(r_{\lambda}) = s_{\lambda}$. Hence, $\pi_i(r_{\lambda}) = r|U_{\lambda}$ and $\phi_{i+1}(r_{\lambda}) = t_{i+1}|U_{\lambda}$. From 2 (a), this yields that $(r_{\lambda})_{\lambda}$ come from a section of $\mathcal{E}_{i+1}(U_{i+1})$.

Step 3. — Now, let us check that the canonical map $P_{\mathfrak{X}} \oplus_{\alpha^{-1}(\mathcal{O}_{\mathfrak{X}}^*)} \mathcal{O}_{\mathfrak{X}}^* \to \lim_{i \to i} P_{X_i} \oplus_{\alpha_i^{-1}(\mathcal{O}_{\mathfrak{X}_i}^*)} \mathcal{O}_{\mathfrak{X}_i}^*$ is an isomorphism. First, we start with the injectivity. As above, put $\mathcal{E} := P_{\mathfrak{X}} \oplus \mathcal{O}_{\mathfrak{X}}^*, \mathcal{F} := P_{\mathfrak{X}}^* \oplus_{\alpha^{-1}(\mathcal{O}_{\mathfrak{X}}^*)} \mathcal{O}_{\mathfrak{X}}^*, \mathcal{G} := P_{\mathfrak{X}} \oplus_{\alpha^{-1}(\mathcal{O}_{\mathfrak{X}}^*)} \mathcal{O}_{\mathfrak{X}}^*, \theta : \mathcal{E} \to \mathcal{F}, \epsilon : \mathcal{F} \to \mathcal{G}, \phi := \epsilon \circ \theta$. Let $(x, a), (y, b) \in \mathcal{E}(\mathfrak{U})$ such that the image of $\phi(x, a)$ and $\phi(y, b)$ in $\lim_{i \to i} P_{X_i} \oplus_{\alpha_i^{-1}(\mathcal{O}_{X_i}^*)} \mathcal{O}_{X_i}^*(\mathfrak{U})$ are equal (where $x, y \in P$, and $a, b \in \mathcal{O}_{\mathfrak{X}}^*(\mathfrak{U})$). Since the injectivity is locally etale, we reduce to check that $\phi(x, a) = \phi(y, b)$. Denote by $(x, a_i), (y, b_i) \in \mathcal{E}_i$ the image of (x, a), (y, b). Shrinking \mathfrak{U} if necessary, we can suppose that $\theta_0(x, a_0) = \theta_0(y, b_0)$. Doing the same computation as in 1(b), we can check there exists $c \in \mathcal{O}_{\mathfrak{X}}^*$ such that $\theta(x, a) = \theta(y, c)$. Moreover, since P is integral, we can check that $\phi_i(y, c_i) = \phi_i(y, b_i)$ if and only if $c_i = b_i$ (since this is etale local, we reduce to check $\theta_i(y, c_i) = \theta_i(y, b_i)$ if and only if $c_i = b_i$). Hence, b = c, which implies $\phi(x, a) = \phi(y, b)$. Hence, we have checked the injectivity. The surjectivity is an easy consequence of Step 2.

DEFINITION 3.4. — We define the category of strict inductive systems of noetherian fine log schemes over $(S_i)_{i\in\mathbb{N}}$ as follows. A strict inductive system of noetherian fine log schemes over $(S_i)_{i\in\mathbb{N}}$ is the data, for any integer $i \in \mathbb{N}$, of a noetherian fine S_i -log scheme X_i , of an exact closed S_i immersion $X_i \hookrightarrow X_{i+1}$ such that the induced morphism $X_i \to X_{i+1} \times_{S_{i+1}} S_i$ is an isomorphism. A morphism $(X_i)_{i\in\mathbb{N}} \to (Y_i)_{i\in\mathbb{N}}$ of strict inductive systems of noetherian fine log schemes over $(S_i)_{i\in\mathbb{N}}$ is a family of S_i -morphism $X_i \to Y_i$ making commutative the diagram



DEFINITION 3.5. — Let X be a scheme. Let M be a sheaf (for the étale topology) of monoids over X

- (1) Following [28, II.1.1.3], we say that M is integral if for every $x \in X$, M_x is integral.
- (2) We say that M is coherent if there exists an open covering \$\mu\$ such that the restriction of M to each U in \$\mu\$ admits a chart subordinate to a finitely generated monoid (see definition [28, II.2.1.5]).
- (3) We say that M is fine if M is integral and coherent.
- (4) Let f: M → N be a morphism of sheaves (for the étale topology) of monoids over X. We say that f is local if the induced morphism M* → M×_N N* is an isomorphism (see Definitions [28, I.4.1.1] for monoids and we have taken a similar to [28, II.1.1.4] definition for sheaves of monoids).

LEMMA 3.6. — Let X be a scheme. Let $M \to N$ be a local morphism of sheaves (for the étale topology) of monoids over X. Suppose M integral, N fine. Then M is fine.

Proof. — Let us fix some notation. Let \bar{x} be a geometric point of X. Since N is fine, using [28, I.1.3.3 and II.1.8.1], we can check that $\overline{N}_{\bar{x}}$ is fine. Since $\overline{M} = \overline{N}$, we get that $\overline{M}_{\bar{x}}$ is fine. Hence, there exist a free \mathbb{Z} -module of finite type L endowed with a morphism $\alpha \colon L \to M_{\bar{x}}^{\mathrm{gr}}$ such that the composition of α with the projection $M_{\bar{x}}^{\mathrm{gr}} \to \overline{M}_{\bar{x}}^{\mathrm{gr}}$ is surjective (following Ogus's terminology appearing in [28, II.3.3], this means $\alpha \colon L \to M_{\bar{x}}^{\mathrm{gr}}$ is a markup of $M_{\bar{x}}$). We put $P := L \times_{M_{\bar{x}}^{\mathrm{gr}}} M_{\bar{x}}$. Since $M_{\bar{x}}$ is integral, then following [28, I.4.2.1] the homomorphism of monoids $M_{\bar{x}} \to \overline{M}_{\bar{x}}$ is exact. Hence, we get the equality $P := L \times_{M_{\bar{x}}^{\mathrm{gr}}} M_{\bar{x}} = L \times_{\overline{M}_{\bar{x}}^{\mathrm{gr}}} \overline{M}_{\bar{x}}$. Following [28, I.2.1.17.6], since L and $\overline{M}_{\bar{x}}$ are fine and since $\overline{M}_{\bar{x}}^{\mathrm{gr}}$ is integral, then $P = L \times_{\overline{M}_{\bar{x}}^{\mathrm{gr}}} \overline{M}_{\bar{x}}$ is fine.

Let $P \to M_{\overline{x}}$ be the projection. Using [28, II.2.2.4], there exist an étale neighborhood $u: U \to X$ of \overline{x} and a morphism of monoids $P \to M(U)$ inducing $P \to M_{\overline{x}}$. Let $\beta: P_U \to u^*M$ be the corresponding morphism. We get the factorization of β of the form $P_U \to P_U^{\beta} \xrightarrow{\beta^a} u^*M$, where β^a is the log structure associated to β i.e. the homomorphism of monoids β^a is logarithmic and is universal for such a factorization (see [28, II.1.1.5]).

We prove that, shrinking U is necessary, the morphism $\beta^{\mathbf{a}}$ is an isomorphism (and then M is coherent). Following [28, I.4.1.2], for any geometric point \overline{y} of U, since $\beta^{\mathbf{a}}_{\overline{y}}$ is sharp and since $M_{\overline{y}}$ is quasi-integral, we can check that the morphism $\beta^{\mathbf{a}}_{\overline{y}} : (P_U^{\beta})_{\overline{y}} \to M_{\overline{y}}$ is an isomorphism if and only if $\overline{\beta^{\mathbf{a}}}_{\overline{y}} : : (\overline{P_U^{\beta}})_{\overline{y}} \to \overline{M}_{\overline{y}}$ is an isomorphism (recall $\overline{M}_{\overline{y}} = \overline{M_{\overline{y}}}$). Using [28,

II.1.8.1], we can check that the canonical morphism $P/\beta_{\overline{y}}^{-1}(M_{\overline{y}}^*) \to (P_U^\beta)_{\overline{y}}$ is an isomorphism. Hence, $\beta_{\overline{y}}^{\mathbf{a}}$ is an isomorphism if and only if the canonical morphism $P/\beta_{\overline{u}}^{-1}(M_{\overline{y}}^*) \to \overline{M}_{\overline{y}}$ is an isomorphism.

Let β_0 be the composition of β with $u^*M \to u^*N$. We get the factorization of β_0 of the form $P_U \to P_U^{\beta_0} \xrightarrow{\beta_0^a} u^*N$, where β_0^a is the sharp localisation of β_0 . Since $P := L \times_{M_{\overline{x}}^{\mathrm{gr}}} M_{\overline{x}} = L \times_{\overline{M}_{\overline{x}}^{\mathrm{gr}}} \overline{M}_{\overline{x}} = L \times_{\overline{N}_{\overline{x}}^{\mathrm{gr}}} \overline{N}_{\overline{x}} = L \times_{N_{\overline{x}}^{\mathrm{gr}}} N_{\overline{x}}$, since N is fine, following [28, II.3.4] (which is checked similarly than [21, 2.10]), replacing U if necessary, we can suppose that β_0^a is an isomorphism. For any geometric point \overline{y} of U, this yields that the morphism $\beta_{0,\overline{y}}^a$: $(P_U^{\beta_0})_{\overline{y}} \to N_{\overline{y}}$ is an isomorphism. Hence so is $\overline{\beta_0^a}_{\overline{y}}: (\overline{P_U^{\beta_0}})_{\overline{y}} \to \overline{N}_{\overline{y}}$, i.e., $P/\beta_{0,\overline{y}}^{-1}(N_{\overline{y}}^*) \to \overline{N}_{\overline{y}}$ is an isomorphism.

Since the morphism $M \to N$ is local, the induced morphism $M^* \to M \times_N N^*$ is an isomorphism. Hence, we get $M_{\overline{y}}^* \to M_{\overline{y}} \times_{N_{\overline{y}}} N_{\overline{y}}^*$, i.e. the morphism $M_{\overline{y}} \to N_{\overline{y}}$ is local. Hence, we get $\beta_{0,\overline{y}}^{-1}(N_{\overline{y}}^*) = \beta_{\overline{y}}^{-1}(M_{\overline{y}}^*)$. Recalling that $\overline{M}_{\overline{y}} = \overline{N}_{\overline{y}}$, this implies that $P/\beta_{\overline{y}}^{-1}(M_{\overline{y}}^*) \to \overline{M}_{\overline{y}}$ is an isomorphism. Hence, we are done.

PROPOSITION 3.7. — Let $(X_i)_{i \in \mathbb{N}}$ be a strict inductive systems of noetherian fine log schemes over $(S_i)_{i \in \mathbb{N}}$. Then $\varinjlim_i X_i$ is a fine S-log formal scheme. Moreover, the canonical morphism $X_i \to (\varinjlim_i X_i) \times_S S_i$ is an isomorphism of fine log schemes.

Proof. — We already know that $\underline{\mathfrak{X}} := \varinjlim_i \underline{X}_i$ is a formal \mathcal{V} -scheme such that $\underline{X}_i \xrightarrow{\sim} \underline{\mathfrak{X}} \times_{\underline{S}} \underline{S}_i$. We have $\varinjlim_i X_i = (\varinjlim_i \underline{X}_i, \varprojlim_i M_{X_i})$. Put $M := \varinjlim_i M_{X_i}, \mathfrak{X} := \varinjlim_i X_i$. It remains to check that M is fine log structure of $\underline{\mathfrak{X}}$. This is checked in Step I.

Step I. —

(1). — The canonical map $\theta: M \to \mathcal{O}_{\mathfrak{X}}$, canonically induced by the structural morphisms $\theta_i: M_{X_i} \to \mathcal{O}_{X_i}$, is a log structure. Indeed, we compute $M^*(\mathfrak{U}) = (\varprojlim_i M_{X_i}(U_i))^* = \varprojlim_i (M_{X_i}(U_i))^* = \varprojlim_i M_{X_i}^*(U_i) = \lim_{i \to i} \mathcal{O}_{\mathfrak{X}_i}^*(U_i) = \mathcal{O}_{\mathfrak{X}}^*(\mathfrak{U})$, for any etale morphism $\mathfrak{U} \to \mathfrak{X}$. Hence, $M^* = \mathcal{O}_{\mathfrak{X}}^*$. It remains to check that the morphism $M^* \to M \times_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}^*$ is an isomorphism, i.e. $M^*(\mathfrak{U}) \to M(\mathfrak{U}) \times_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})} \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})^*$ is an isomorphism. Since $M(\mathfrak{U}) \times_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})} \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})^* \subset M(\mathfrak{U})$, the injectivity is obvious. Let us check the surjectivity. Let $(a_i)_{i\in\mathbb{N}} \in \varprojlim_i M_{X_i}(U_i)$ such that $(\theta_i(a_i))_{i\in\mathbb{N}} \in \varprojlim_i \mathcal{O}_{X_i}^*(U_i)$. Since M_{X_i} is a log structure of X_i , we get $a_i \in \mathcal{O}_{X_i}^*(U_i)$, hence $(a_i)_{i\in\mathbb{N}} \in M^*(\mathfrak{U})$.

(2). — Let $\mathfrak{U} \to \mathfrak{X}$ be an etale morphism. Suppose \mathfrak{U} affine. We prove in this step that the canonical morphisms $M_{X_{i+1}}^{\mathrm{gr}}(U_{i+1})/\mathcal{O}_{X_{i+1}}^*(U_{i+1}) \to M_{X_i}^{\mathrm{gr}}(U_i)/\mathcal{O}_{X_i}^*(U_i)$ and $M_{X_{i+1}}(U_{i+1})/\mathcal{O}_{X_{i+1}}^*(U_{i+1}) \to M_{X_i}(U_i)/\mathcal{O}_{X_i}^*(U_i)$ are isomorphisms. First, we remark that since $(\pi^i \mathcal{O}_{X_{i+1}})^2 = 0$, then we have the canonical isomorphism of groups $(1 + \pi^i \mathcal{O}_{X_{i+1}}, \times) \xrightarrow{\sim} (\pi^i \mathcal{O}_{X_{i+1}}, +)$. Since U_{i+1} is affine and $\pi^i \mathcal{O}_{X_{i+1}}$ is quasi-coherent, this yields $H^1(U_{i+1}, 1 + \pi^i \mathcal{O}_{X_{i+1}}) = 0$. Hence, we get the commutative diagram (we use in the proof multiplicative notation)

$$1 \longrightarrow \mathcal{O}_{X_{i}}^{*}(U_{i}) \longrightarrow M_{X_{i}}^{\mathrm{gr}}(U_{i}) \longrightarrow M_{X_{i}}^{\mathrm{gr}}(U_{i}) / \mathcal{O}_{X_{i}}^{*}(U_{i}) \longrightarrow 1$$

$$(3.1) \quad 1 \longrightarrow \mathcal{O}_{X_{i+1}}^{*}(U_{i+1}) \longrightarrow M_{X_{i+1}}^{\mathrm{gr}}(U_{i+1}) \longrightarrow M_{X_{i+1}}^{\mathrm{gr}}(U_{i+1}) / \mathcal{O}_{X_{i+1}}^{*}(U_{i+1}) \to 1$$

$$1 + \pi^{i}\mathcal{O}_{X_{i+1}}(U_{i+1}) = 1 + \pi^{i}\mathcal{O}_{X_{i+1}}(U_{i+1})$$

$$1 + \pi^{i}\mathcal{O}_{X_{i+1}}(U_{i+1}) = 1 + \pi^{i}\mathcal{O}_{X_{i+1}}(U_{i+1})$$

whose two rows and two columns are exact. Hence, the morphism $M_{X_{i+1}}^{\mathrm{gr}}(U_{i+1})/\mathcal{O}_{X_{i+1}}^*(U_{i+1}) \to M_{X_i}^{\mathrm{gr}}(U_i)/\mathcal{O}_{X_i}^*(U_i)$ is an isomorphism. Since $M_{X_{i+1}} \to M_{X_i}$ is are exact closed immersion, then following this is a log thickening of finite order (see [28, Definition IV.2.1.1]). Following [28, IV.2.1.2.4], this yields that we have the surjective projection $M_{X_{i+1}}(U_{i+1}) = M_{X_{i+1}}^{\mathrm{gr}}(U_{i+1}) \times_{M_{X_i}}^{\mathrm{gr}}(U_i) M_{X_i}(U_i) \to M_{X_i}(U_i)$. Hence, $M_{X_{i+1}}(U_{i+1})/\mathcal{O}_{X_{i+1}}^*(U_{i+1}) \to M_{X_i}(U_i)/\mathcal{O}_{X_i}^*(U_i)$ is surjective. Since $M_{X_i}(U_i)$ and $M_{X_{i+1}}(U_{i+1})$ is integral, from [28, I.1.3.3], the horizontal morphisms of the commutative diagram

$$\begin{split} M_{X_{i+1}}(U_{i+1})/\mathcal{O}^*_{X_{i+1}}(U_{i+1}) & \longrightarrow M^{\mathrm{gr}}_{X_{i+1}}(U_{i+1})/\mathcal{O}^*_{X_{i+1}}(U_{i+1}) \\ & \downarrow & & \downarrow \sim \\ M_{X_i}(U_i)/\mathcal{O}^*_{X_i}(U_i) & \longleftarrow M^{\mathrm{gr}}_{X_i}(U_i)/\mathcal{O}^*_{X_i}(U_i) \end{split}$$

are injective. Hence, $M_{X_{i+1}}(U_{i+1})/\mathcal{O}^*_{X_{i+1}}(U_{i+1}) \to M_{X_i}(U_i)/\mathcal{O}^*_{X_i}(U_i)$ is an isomorphism.

(3). — Using Mittag-Leffler condition, we get the exact sequence

$$1 \to \varprojlim_{i} \mathcal{O}_{X_{i}}^{*}(U_{i}) \to \varprojlim_{i} M_{X_{i}}^{\mathrm{gr}}(U_{i}) \to \varprojlim_{i} M_{X_{i}}^{\mathrm{gr}}(U_{i})/\mathcal{O}_{X_{i}}^{*}(U_{i}) \to 1.$$

Since $\mathcal{O}^*_{\mathfrak{X}}(\mathfrak{U}) = \varprojlim_i \mathcal{O}^*_{X_i}(U_i)$, using Step 2 we obtain

$$(\varprojlim_{i} M_{X_{i}}^{\mathrm{gr}}(U_{i}))/\mathcal{O}_{\mathfrak{X}}^{*}(\mathfrak{U}) \xrightarrow{\sim} M_{X_{0}}^{\mathrm{gr}}(U_{0})/\mathcal{O}_{X_{0}}^{*}(U_{0}).$$

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By considering the commutative diagram

we get the injectivity of the map $M(\mathfrak{U})/\mathcal{O}_{\mathfrak{X}}^*(\mathfrak{U}) \to M_{X_0}(U_0)/\mathcal{O}_{X_0}^*(U_0)$. Since the maps $M_{X_{i+1}}(U_{i+1}) \to M_{X_i}(U_i)$ are surjective (this is checked in Step 2), we get that $M(\mathfrak{U}) \to M_{X_0}(U_0)$ is surjective and then so is $M(\mathfrak{U})/\mathcal{O}_{\mathfrak{X}}^*(\mathfrak{U}) \to M_{X_0}(U_0)/\mathcal{O}_{X_0}^*(U_0)$. Hence, the canonical morphism $M(\mathfrak{U})/\mathcal{O}_{\mathfrak{X}}^*(\mathfrak{U}) \to M_{X_0}(U_0)/\mathcal{O}_{X_0}^*(U_0)$ is an isomorphism. This yields $\overline{M} = M/M^* = M_{X_0}/M_{X_0}^* = \overline{M}_{X_0}$.

(4). — We compute that the induced morphism $M^* \to M \times_{M_{X_0}} M^*_{X_0}$ is an isomorphism, i.e. that the morphism $M \to M_{X_0}$ is local. Hence, since $\overline{M} = \overline{M_{X_0}}$ (see Step 3), since M_{X_0} is fine and M is integral, using Lemma 3.6, we get that M is fine.

Step II. — In this last step, we establish that $X_i \to \mathfrak{X} \times_S S_i$ is an isomorphism of fine log schemes. Let $u_i: X_i \to \mathfrak{X}$ be the canonical morphism. We already know that $\underline{X}_i \xrightarrow{\sim} \mathfrak{X} \times_S \underline{S}_i$. It remains to check that the morphism $u_i^*M \to M_{X_i}$ is an isomorphism. Since this is a morphism of fine log structures, then using [28, I.4.1.2], this is equivalent to check the isomorphism $\overline{u_i^*M} \xrightarrow{\sim} \overline{M_{X_i}}$. Following [21, 1.4.1], $\overline{u_i^*M} = \overline{M}$. From Step I(3), we have $\overline{M} \xrightarrow{\sim} \overline{M_{X_i}}$ and we are done.

THEOREM 3.8. — The functors $\mathfrak{X} \mapsto (X_i)_{i \in \mathbb{N}}$ and $(X_i)_{i \in \mathbb{N}} \mapsto \varinjlim_i X_i$ are quasi-inverse equivalences of categories between the category of fine S-log formal schemes to that of strict inductive systems of noetherian fine log schemes over $(S_i)_{i \in \mathbb{N}}$.

Proof. — This is a consequence of Propositions 3.3 and 3.7. \Box

LEMMA 3.9. — Let $f: \mathfrak{X} \to \mathcal{Y}$ be a morphism of fine S-log formal schemes. Then f is strict if and only if, for any $i \in \mathbb{N}$, f_i is strict.

Proof. — If f is strict then f_i , the base change of f by $S_i \hookrightarrow S$ is strict. Conversely, suppose that for any $i \in \mathbb{N}$, f_i is strict. Let Z be the fine S-log formal scheme whose underlying fine formal S-scheme is \mathfrak{X} and whose log structure is $f^*(M_{\mathcal{Y}})$. Then $Z_i \to Y_i$ is strict and $\underline{Z_i} = \underline{X_i}$. Hence, $Z_i = X_i$. Using Proposition 3.3, this yields that $\mathfrak{X} = Z$, i.e. f is strict.

3.2. Around log etaleness

DEFINITION 3.10. — Let $f: \mathfrak{X} \to \mathcal{Y}$ be a morphism of fine S-log formal schemes. We say that f is "log étale" (resp. "log smooth, resp. "formally log étale", resp. "log p-étale", resp. "formally log étale of level m", resp. "log p-étale of level m") if for any integer $i \in \mathbb{N}$ the morphism f_i is log étale (resp. log smooth, resp. fine formally log étale, resp. log p-étale, resp. formally log étale of level m, resp. log p-étale of level m).

When the morphism is strict, we remove the word "log". For instance, f is p-étale means that f is strict and f is log p-étale.

Remark 3.11.

- We remark that our definition of log étaleness was named by Shiho formal log étaleness (see [30, 2.2.2]). We hope there will be no confusion.
- It is also possible define some "fine saturated" definition similar to Definition 3.10 but we leave it to the interested reader. Since we only consider the "fine" case, we have removed the word "fine" in the terminology of Definition 3.10. We might also consider the notion of "fine log relatively perfect" morphism of fine *S*-log formal schemes, but it seems useless for us.

Point 3.12. — The following diagram summarizes the relations between our definitions:

formally log étale.

PROPOSITION 3.13. — Let \mathcal{Y} be a fine \mathcal{S} -log formal schemes. Let f_0 : $X_0 \to Y_0$ be a log smooth morphism of fine log S_0 -schemes such that $\underline{X_0}$ is affine. Then there exists a log smooth morphism of fine \mathcal{S} -log formal schemes of the form $f: \mathfrak{X} \to \mathcal{Y}$ whose reduction modulo π is f_0 . We say that such morphism f is a log smooth lifting of f_0 .

Proof. — From [21, 3.14.(1)], there exists a unique up to isomorphism log smooth morphism of fine log S_i -schemes $f_i: X_i \to Y_i$ endowed with an isomorphism $X_0 \xrightarrow{\sim} X_i \times_{Y_i} Y_0$. Put $\mathcal{Y} := \varinjlim_i Y_i$. Let $f: \mathcal{Y} \to \mathfrak{X}$ be the induced morphism. Following Theorem 3.8, \mathcal{Y} is a fine S-log formal schemes. By construction, f is log smooth since f_i is log smooth for any $i \in \mathbb{N}$.

PROPOSITION 3.14. — Let $f: \mathfrak{X} \to \mathcal{Y}$ be a morphism of fine S-log formal schemes. The morphism f is log étale (resp. log p-étale, resp. formally log étale of level m, resp. log p-étale of level m) if and only if f is formally log étale and f_0 is log étale (resp. log p-étale, resp. formally log étale of level m, resp. log p-étale of level m).

Proof. — If f_0 is log étale then $\underline{f_0}$ is of finite type. This yields that $\underline{f_i}$ is of finite type, which proves the non respective case. The respective cases are consequences of Lemma 1.20 or Lemma 1.56.

Example 3.15. — For instance, when $\mathfrak{X} := \operatorname{Spf} \mathcal{V}[\![t]\!]$ and $\mathcal{Y} := \operatorname{Spf} \mathcal{V}[\![t]\!]$, the canonical morphism $f : \mathfrak{X} \to \mathcal{Y}$ is *p*-étale but not étale.

Let us check this fact. It follows from it from the example Example 1.26 that f_0 is relatively perfect and then *p*-étale. By using Proposition 3.14, it remains to check that f is formally (log) étale. Since $W(k) \to \mathcal{V}$ is finite (where W(k) is the ring of Witt vectors of k), since formal étaleness is stable under base change, then we reduce to the case where $W(k) = \mathcal{V}$. Since $W_i(k)[t] \to W_i(k)[t]$ is flat, then it follows from [22, Lemma 1.6], that Spec $W_i(k)[t] \to \text{Spec } W_i(k)[t]$ is relatively perfect. Hence, Spec $W_i(k)[t] \to$ Spec $W_i(k)[t]$ is in particular formally étale, and we are done.

Point 3.16. — Let $f: \mathfrak{X} \to \mathcal{Y}$ be a morphism of fine S-log formal schemes. We set $\Omega^1_{\mathfrak{X}/\mathcal{Y}} := \varprojlim_i \Omega^1_{X_i/Y_i}$. When f is log smooth, then from [21, 3.10] the $\mathcal{O}_{\mathfrak{X}}$ -module $\Omega^1_{\mathfrak{X}/S}$ is locally free of finite type. When f log smooth and $\mathcal{Y} = S$, then f is flat. Indeed, in that case, $f_i: X_i \to S_i$ is integral (use [21, 4.3]) and smooth and then flat (see [21, 4.4]). Since f is of finite type, then f is flat.

LEMMA 3.17. — Let $f: \mathfrak{X} \to \mathcal{Y}, g: \mathcal{Y} \to \mathcal{Z}$ be two morphisms of fine \mathcal{S} log formal schemes such that $\mathcal{O}_{\mathfrak{X}}$ has no p-torsion, the structural morphism $g \circ f$ is log smooth and $f_0: X_0 \to Y_0$ is log étale. Then f is log étale.

Proof. — We construct by p-adic completion the morphism $\phi: f^*\Omega^1_{\mathcal{Y}/\mathcal{Z}} \to \Omega^1_{\mathfrak{X}/\mathcal{Z}}$, where we put $f^*\Omega^1_{\mathcal{Y}/\mathcal{Z}} := \varprojlim_i f^*_i\Omega^1_{Y_i/Z_i}$. Since $g \circ f: \mathfrak{X} \to \mathcal{Z}$ is log smooth, the $\mathcal{O}_{\mathfrak{X}}$ -module $\Omega^1_{\mathfrak{X}/\mathcal{Z}}$ is locally free of finite type (see Point 3.16). In particular, $\Omega^1_{\mathfrak{X}/\mathcal{Z}}$ has no p-torsion. The reduction of ϕ modulo π is canonically isomorphic to $f^*_0\Omega^1_{Y_0/Z_0} \to \Omega^1_{X_0/Z_0}$. Since f_0 is log étale, this latter homomorphism is an isomorphism. Since $\Omega^1_{\mathfrak{X}/\mathcal{Z}}$ has no p-torsion, this yields

that ϕ is an isomorphism (e.g. use Lemma [7, 2.2.15]). This implies that the canonical morphism $f_i^*\Omega^1_{Y_i/Z_i} \to \Omega^1_{X_i/Z_i}$ is an isomorphism. Since $X_i \to Z_i$ is log smooth, from [21, 3.12], we conclude that f_i is log-étale.

PROPOSITION 3.18. — Let $f: \mathfrak{X} \to \mathcal{Y}$ be a morphism of fine S-log formal schemes such that $\mathcal{O}_{\mathfrak{X}}$ has no p-torsion. The morphism f is log smooth if and only if, étale locally on \mathfrak{X} there exists a log étale \mathcal{Y} -morphism of the form $\mathfrak{X} \to \mathcal{Y} \times_{\mathcal{Y}} \mathfrak{A}_{\mathbb{N}^r}$.

Proof. — Suppose f is log smooth. Since f_0 is log smooth, when can suppose there exists a morphism $X_0 \to A_{\mathbb{N}^r}$ such that the induced Y_0 morphism $X_0 \to Y_0 \times A_{\mathbb{N}^r}$ is log-étale. Using Lemma 3.2, we can suppose that $X_0 \to A_{\mathbb{N}^r}$ has the lifting of the form $\mathfrak{X} \to \mathfrak{A}_{\mathbb{N}^r}$. We get the \mathcal{Y} morphism $\mathfrak{X} \to \mathcal{Y} \times_{\mathcal{V}} \mathfrak{A}_{\mathbb{N}^r}$. We conclude by applying Lemma 3.17 that this latter morphism is log-étale.

3.3. Sheaf of differential operators over weakly log smooth S-log formal scheme

DEFINITION 3.19. — As in Definition 1.4 we define the category \mathfrak{C} of S-immersions of fine S-log formal schemes. For any integer n, we denote by \mathscr{C}_n the full subcategory of \mathscr{C} whose objects are exact closed immersions of order n.

LEMMA 3.20. — The inclusion functor $\operatorname{For}_n : \mathfrak{C}_n \to \mathfrak{C}$ has a right adjoint functor which we will denote by $P^n : \mathfrak{C} \to \mathfrak{C}_n$. Let $u : \mathbb{Z} \to \mathfrak{X}$ be an object of \mathfrak{C} . Then \mathbb{Z} is also the source of $P^n(u)$.

Proof. — Let $u: \mathcal{Z} \hookrightarrow \mathfrak{X}$ be an object of \mathfrak{C} . Since $u_i: Z_i \hookrightarrow X_i$ is an object of \mathscr{C} , from Proposition 1.11, we get the object $P^n(u_i): Z_i \hookrightarrow P^n(u_i)$ of \mathscr{C}^n such that $P^n(u_i) \to X_i$ is affine and $P^n(u_i)$ is noetherian. Hence, using Theorem 3.8, we get that $\varinjlim_i P^n(u_i)$ satisfies the universal property of $P^n(u)$.

DEFINITION 3.21. — Let $f: \mathfrak{X} \to \mathcal{Y}$ be a morphism of fine S-log formal schemes.

- (1) We say that a finite set $(b_{\lambda})_{\lambda=1,...,r}$ of elements of $\Gamma(\mathfrak{X}, M_{\mathfrak{X}})$ is a " formal log basis of f" if the induced \mathcal{Y} -morphism $\mathfrak{X} \to \mathcal{Y} \times_{\mathcal{V}} \mathfrak{A}_{\mathbb{N}^r}$ is formally log étale (concerning $\mathfrak{A}_{\mathbb{N}^r}$, see the notation of Point 3.1).
- (2) We say that f is "weakly log smooth" if, étale locally on \mathfrak{X} , f has formal log bases. Notice that this notion of weak log smoothness

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is étale local on \mathcal{Y} . When $\mathcal{Y} = \mathcal{S}$, we say that \mathfrak{X} is a "weakly log smooth \mathcal{S} -log formal scheme" (following the terminology, the log structure of such \mathfrak{X} is understood to be fine).

Remark 3.22. — Following Proposition 3.18, a log smooth morphism is weakly log smooth which justifies the terminology.

Point 3.23 (nth infinitesimal neighborhood). — Let \mathfrak{X} be a weakly log smooth S-log formal scheme. Let $\Delta_{\mathfrak{X}/S} \colon \mathfrak{X} \hookrightarrow \mathfrak{X} \times_S \mathfrak{X}$ be the diagonal immersion. Since $\Delta_{\mathfrak{X}/S}$ is not necessarily an object of \mathfrak{C} (because $\mathfrak{X} \times_S \mathfrak{X}$ is not noetherian in general, e.g. when $\mathfrak{X} = \operatorname{Spf}(\mathcal{V}[t])$ since its special fiber is Spec $(k[t] \otimes_k k[t])$ which is not noetherian), we can not use Lemma 3.20 and we can not put $\Delta_{\mathfrak{X}/S}^n := P^n(\Delta_{\mathfrak{X}/S})$. But this is possible to define $\Delta_{\mathfrak{X}/S}^n$ by taking inductive limits as follows. From Lemma 1.9, we have $\Delta_{X_i/S_i}^n =$ $\Delta_{X_{i+1}/S_{i+1}}^n \times_{S_{i+1}} S_i$. From Proposition 1.11, since $\underline{\Delta}_{X_i/S_i}^n$ are noetherian schemes, using Theorem 3.8, we get the fine S-log formal schemes $\Delta_{\mathfrak{X}/S}^n$ by putting $\Delta_{\mathfrak{X}/S}^n := \lim_{i \to i} \Delta_{X_i/S_i}^n$. Taking the inductive limits to the strict morphisms of fine log schemes $p_0^n \colon \Delta_{X_i/S_i}^n \to X_i$ (resp. $p_1^n \colon \Delta_{X_i/S_i}^n \to$ X_i), using Lemma 3.9 we get the strict morphism of fine log formal \mathcal{V} schemes $p_0^n \colon \Delta^n_{\mathfrak{X}/S} \to \mathfrak{X}$ (resp. $p_1^n \colon \Delta^n_{\mathfrak{X}/S} \to \mathfrak{X}$). Using Remark 2.4, we can check that the underlying morphism of formal \mathcal{V} -schemes of $p_0^n \colon \Delta^n_{\mathfrak{X}/\mathcal{S}} \to$ \mathfrak{X} and $p_1^n: \Delta_{\mathfrak{X}/S}^n \to \mathfrak{X}$ are finite (more precisely, we can check the local description (3.3)). Hence, we denote by $\mathcal{P}^n_{\mathfrak{X}/S}$ the coherent $\mathcal{O}_{\mathfrak{X}}$ -algebra such that $\operatorname{Spf} \mathcal{P}^n_{\mathfrak{X}/\mathcal{S}} = \underline{\Delta}^n_{\mathfrak{X}/\mathcal{S}}.$

If $a \in M_{\mathfrak{X}}$, we denote by $\mu_{(m)}(a)$ the unique section of $\ker(\mathcal{O}_{\Delta_{\mathfrak{X}/\mathcal{S}}^n}^* \to \mathcal{O}_{\mathfrak{X}}^*)$ such that we get in $M_{\mathfrak{X}/\mathcal{S}}^n$ the equality $p_1^{n*}(a) = p_0^{n*}(a)\mu^n(a)$ (see Lemma 1.81). We get $\mu^n \colon M_{\mathfrak{X}} \to \ker(\mathcal{O}_{\Delta_{\mathfrak{X}/\mathcal{S}}^n}^* \to \mathcal{O}_{\mathfrak{X}}^*)$ given by $a \mapsto \mu^n(a)$.

PROPOSITION 3.24 (Local description of $\mathcal{P}^n_{\mathfrak{X}/S}$). — Let $(a_{\lambda})_{\lambda=1,\ldots,r}$ be a formal log basis of f. Put $\eta_{\lambda,n} := \mu^n(a_{\lambda}) - 1$. We have the following isomorphism of $\mathcal{O}_{\mathfrak{X}}$ -algebras:

(3.3)
$$\mathcal{O}_{\mathfrak{X}}[T_1,\ldots,T_r]_n \xrightarrow{\sim} \mathcal{P}^n_{\mathfrak{X}/S}$$
$$T_{\lambda} \longmapsto \eta_{\lambda,n}.$$

Proof. — This is a consequence of Proposition 2.3.

DEFINITION 3.25. — The sheaf of differential operators of order $\leq n$ of f is defined by putting $\mathcal{D}_{\mathfrak{X}/\mathcal{S},n} := \mathscr{H} \mathrm{om}_{\mathcal{O}_{\mathfrak{X}}}(p_{0*}^n \mathcal{P}_{\mathfrak{X}/\mathcal{S}}^n, \mathcal{O}_{\mathfrak{X}})$. The sheaf of differential operators of f is defined by putting $\mathcal{D}_{\mathfrak{X}/\mathcal{S}} := \bigcup_{n \in \mathbb{N}} \mathcal{D}_{\mathfrak{X}/\mathcal{S},n}$.

Let $P \in \mathcal{D}_{\mathfrak{X}/S,n}$, $P' \in \mathcal{D}_{\mathfrak{X}/S,n'}$. We define the product $PP' \in \mathcal{D}_{\mathfrak{X}/S,n+n'}$ to be the composition

$$(3.4) \qquad PP' \colon \mathcal{P}^{n+n'}_{\mathfrak{X}/S} \xrightarrow{\delta^{n,n'}} \mathcal{P}^{n}_{\mathfrak{X}/S} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{P}^{n'}_{\mathfrak{X}/S} \xrightarrow{\mathrm{Id} \otimes P'} \mathcal{P}^{n}_{\mathfrak{X}/S} \xrightarrow{P} \mathcal{O}_{\mathfrak{X}}$$

Similarly to Proposition 2.10, we can check that the sheaf $\mathcal{D}_{\mathfrak{X}/S}$ is a sheaf of rings with the product as defined in (3.4)

3.4. Sheaf of differential operators of level m over weakly log smooth of level m fine S-log formal schemes

Let $m \ge 0$ be an integer. The principal ideal (p) of \mathcal{V} is endowed with a canonical *m*-PD-structure, which we will denote by γ_{\emptyset} .

DEFINITION 3.26. — As in Definition 1.48, we define the categories $\mathfrak{C}_n^{(m)}$ whose objects are pairs (u, δ) where u is an exact closed S-immersion of fine log S-schemes and δ is an m-PD-structure on the ideal \mathcal{I} defined by u(which is compatible with γ_{\emptyset}) and such that $\mathcal{I}^{\{n+1\}_{(m)}} = 0$ and whose morphisms $(u', \delta') \to (u, \delta)$ are morphisms $u' \to u$ of \mathfrak{C} which are compatible with the m-PD-structures δ and δ' .

PROPOSITION 3.27.

- (1) The canonical functor $\mathfrak{C}_n^{(m)} \to \mathfrak{C}$ has a right adjoint, which we will denote by $P_{(m)}^n \colon \mathfrak{C} \to \mathfrak{C}_n^{(m)}$.
- (2) Let u be an object of \mathfrak{C} . The source of $P_{(m)}^n(u)$ is the source of u.

Proof. — The first assertion is a consequence of Theorem 3.8 and Proposition 1.63 (we need in particular Proposition 1.63 (4). Since γ_{\emptyset} extends to any *S*-log formal schemes (because the ideal of the *m*-PD-structure γ_{\emptyset} is locally principal: see [4, 1.3.2.c)]), we get the second assertion.

Point 3.28. — Let u be an object of \mathfrak{C} . We call $P_{(m)}^n(u)$ the m-PDenvelope compatible of order n of u. We sometimes denote abusively by $P_{(m)}^n(u)$ the target of the arrow $P_{(m)}^n(u)$.

DEFINITION 3.29. — Let $f: \mathfrak{X} \to \mathcal{Y}$ be a morphism of fine S-log formal schemes.

 We say that a finite set (b_λ)_{λ=1,...,r} of elements of Γ(𝔅, M_𝔅) is a "log p-basis of f" (resp. "formal log basis of level m of f", resp. "log p-basis of level m of f") if the induced 𝔅-morphism 𝔅 → 𝔅 ×_𝔅 𝔅_{ℕ^r} is log p-étale (resp. formally log étale of level m, resp. log p-étale of level m). (2) We say that f is "log p-smooth" (resp. "weakly log smooth of level m", resp. "log p-smooth of level m") if, étale locally on X, f has log p-bases (resp. formal log bases of level m, resp. log p-bases of level m). When Y = S, we say that X is a log p-smooth log-formal Sscheme (resp. a weakly log smooth of level m log-formal S-scheme, resp. a log p-smooth of level m log-formal S-scheme). Remark that the log structure of such X is always fine following our terminology.

Point 3.30. — Using (3.2) and Proposition 3.18, we get the following diagram summarizing the relations between our definitions:

 $\begin{array}{rcl} \log \ \mathrm{smooth} & \Longrightarrow \ \log \ p \text{-smooth} \\ & \Longrightarrow \ \log \ p \text{-smooth of level } m \\ & \Longrightarrow & \mathrm{weakly} \ \log \ \mathrm{smooth} \ \mathrm{of} \ \mathrm{level} \ m \\ & \Longrightarrow & \mathrm{weakly} \ \log \ \mathrm{smooth} \end{array}$

Point 3.31. — Let \mathfrak{X} be a weakly log smooth of level m log-formal S-scheme. Using Remark 2.15, we can check the underlying scheme of $\Delta_{X_i/S_i,(m)}^n \xrightarrow{\sim} \Delta_{X_{i+1}/S_{i+1},(m)}^n \times_{S_{i+1}} S_i$ (recall also that $p_0^n \colon \Delta_{X_i/S_i,(m)}^n \to X_i$ is strict). Using Theorem 3.8, we get the fine S-log formal schemes $\Delta_{\mathfrak{X}/S,(m)}^n$ by putting $\Delta_{\mathfrak{X}/S,(m)}^n \coloneqq \lim_{\mathfrak{X}/S,(m)} \Delta_{\mathfrak{X}/S,(m)}^n$. Let $p_1^n, p_0^n \colon \Delta_{\mathfrak{X}/S,(m)}^n \to \mathfrak{X}$ be the morphisms induced respectively by $p_1^n, p_0^n \colon \Delta_{X_i/S_i,(m)}^n \to X_i$. From Notation 2.12, Lemma 3.9 and Remark 2.15, the morphisms $p_1^n, p_0^n \colon \Delta_{\mathfrak{X}/S,(m)}^n \to \mathfrak{X}$ are strict and finite (more precisely concerning the finiteness, we have the local description (3.5)).

We denote by $M^n_{\mathfrak{X}/\mathcal{S},(m)}$ the log structure of $\Delta^n_{\mathfrak{X}/\mathcal{S},(m)}$. We denote by $\mathcal{P}^n_{\mathfrak{X}/\mathcal{S},(m)}$ the coherent $\mathcal{O}_{\mathfrak{X}}$ -algebra corresponding to the underlying formal \mathcal{V} -scheme of $\Delta^n_{\mathfrak{X}/\mathcal{S},(m)}$. Hence, $\Delta^n_{\mathfrak{X}/\mathcal{S},(m)}$ is an exact closed immersion of the form $\Delta^n_{\mathfrak{X}/\mathcal{S},(m)}: \mathfrak{X} \hookrightarrow (\operatorname{Spf} \mathcal{P}^n_{\mathfrak{X}/\mathcal{S},(m)}, M^n_{\mathfrak{X}/\mathcal{S},(m)})$. We sometimes denote abusively by $\Delta^n_{\mathfrak{X}/\mathcal{S},(m)}$ the target of the arrow $\Delta^n_{\mathfrak{X}/\mathcal{S},(m)}$.

As in paragraph Definition 2.26, we can define $\mathcal{D}_{\mathfrak{X}/S}^{(m)}$, the sheaf of differential operator on \mathfrak{X} of level m.

Point 3.32 (Local description). — Suppose in this paragraph that $\mathfrak{X} \to \mathcal{S}$ is endowed with a formal log basis of level m $(b_{\lambda})_{\lambda=1,...,r}$ of f. Put $\eta_{\lambda(m)} := \mu_{(m)}^n(b_{\lambda}) - 1$ (or simply η_{λ}), where $\mu_{(m)}^n(a)$ is the unique section of $\ker(\mathcal{O}_{\Delta^n_{\mathfrak{X}/\mathcal{S},(m)}}^n \to \mathcal{O}_{\mathfrak{X}}^*)$ such that we get in $M^n_{\mathfrak{X}/\mathcal{S},(m)}$ the equality $p_1^{n*}(a) =$

 $p_0^{n*}(a)\mu_{(m)}^n(a)$. Taking the limits to Proposition 2.13, we get the isomorphism of m-PD- $\mathcal{O}_{\mathfrak{X}}$ -algebras

(3.5)
$$\mathcal{O}_{\mathfrak{X}}\langle T_1, \dots, T_r \rangle_{(m),n} \xrightarrow{\sim} \mathcal{P}^n_{\mathfrak{X}/\mathcal{S},(m)} \\ T_\lambda \longmapsto \eta_{\lambda,(m)},$$

where the first term is defined as in Notation 1.78. In particular, the elements $\{\underline{\eta}^{\{\underline{k}\}_{(m)}}\}_{|\underline{k}| \leq n}$ form an $\mathcal{O}_{\mathfrak{X}}$ -basis of $\mathcal{P}^n_{\mathfrak{X}/\mathcal{S},(m)}$. The corresponding dual basis of $\mathcal{D}^{(m)}_{\mathfrak{X}/\mathcal{S},n}$ will be denoted by $\{\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}\}_{|\underline{k}| \leq n}$. Let $\epsilon_1, \ldots, \epsilon_r$ be the canonical basis of \mathbb{N}^r , i.e. the coordinates of ϵ_i are 0 except for the *i*th term which is 1. We put $\partial_i := \underline{\partial}^{\langle \epsilon_i \rangle_{(m)}}$. We can define the logarithmic transposition as in Notation 2.38 and we can check that the properties analogous to the Section 2.5 are still satisfied in the formal context.

We finish the subsection by the formal version of the definition appearing in Definition 1.85.

DEFINITION 3.33. — Let $f: \mathfrak{X} \to \mathcal{Y}$ be a morphism of fine S-log formal schemes.

- We say that a finite set (t_λ)_{λ=1,...,r} of elements of Γ(𝔅, O_𝔅) are "log p-étale coordinates" (resp. "formal log étale coordinates", resp. "formal log étale coordinates of level m", resp. "log p-étale coordinates of level m"), if the corresponding 𝒱-morphism 𝔅 → 𝒱×_𝔅Â^𝑘_𝔅, where Â^𝑘_𝔅 is the p-adic completion of the rth affine space over 𝒱 endowed with the trivial logarithmic structure, is log p-étale (resp. formally log étale, resp. formally log étale of level m, log p-étale of level m). When f is strict we remove "log" in the terminology, e.g. we get the notion of "p-étale coordinates".
- (2) We say that f is "p-smooth" (resp. "weakly smooth", resp. "weakly smooth of level m", resp. "p-smooth of level m"), if f is strict and if, étale locally on X, f has p-étale coordinates" (resp. "formal étale coordinates", resp. "formal étale coordinates of level m", resp. "p-étale coordinates of level m"). Notice that these notions are étale local on Y.

3.5. Sheaf of differential operators of finite level over log p-smooth S-log formal schemes

Point 3.34. — Let \mathfrak{X} be a log *p*-smooth S-log formal schemes. We denote by $\widehat{\mathcal{D}}_{\mathfrak{X}/S}^{(m)}$ the *p*-adic completion of $\mathcal{D}_{\mathfrak{X}/S}^{(m)}$. As in Point 2.32, we can check that $\mathcal{D}_{X_i/S_i}^{(m)} \xrightarrow{\sim} \mathcal{D}_{\mathfrak{X}/S}^{(m)} \otimes_{\mathcal{V}} \mathcal{V}/\pi^{i+1}$. Hence $\widehat{\mathcal{D}}_{\mathfrak{X}/S}^{(m)} \xrightarrow{\sim} \lim_{i \to \infty} \mathcal{D}_{X_i/S_i}^{(m)}$. Point 3.35. — Let \mathfrak{X} be a log *p*-smooth S-log-formal schemes. We put $\mathcal{D}^{\dagger}_{\mathfrak{X}/S} := \varinjlim_{\mathfrak{X}/S} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/S}$. This is the "sheaf of differential operators of finite level of \mathfrak{X}/S ". When $\mathfrak{X} \to S$ is endowed with a log *p*-basis $(b_{\lambda})_{\lambda=1,\ldots,n}$, we get the usual description ([4, 2.4.4]): an operator P of $\Gamma(\mathfrak{X}, \mathcal{D}^{\dagger}_{\mathfrak{X}/S})$ is of the form

$$P = \sum_{\underline{k} \in \mathbb{N}^n} a_{\underline{k}} \underline{\partial}^{[\underline{k}]}$$

where $a_{\underline{k}} \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ satisfy the condition: there exist some constants $c, \eta \in \mathbb{R}$, with $\eta < 1$, such that for any $\underline{k} \in \mathbb{N}^n$ we have

$$||a_k|| \leqslant c \, \eta^{|\underline{k}|},$$

where $\|\cdot\|$ is the *p*-adic norm i.e. whose basis of open neighbourhoods of 0 is given by $(p^n \mathcal{O}_{\mathfrak{X}})_{n \in \mathbb{N}}$.

Point 3.36. — Let \mathfrak{X} be a log *p*-smooth *S*-log-formal scheme. As in [4, 2.2.5], we can check that if \mathfrak{U} is a Zariski open set of \mathfrak{X} having a log *p*-basis then $\Gamma(U_i, \mathcal{D}_{X_i/S_i}^{(m)})$ is right and left noetherian. As in [4, 3.1.2], we can check that the sheaf $\mathcal{D}_{X_i/S_i}^{(m)}$ is coherent on the right and on the left. As in [4, 3.3.4], this yields that $\widehat{\mathcal{D}}_{\mathfrak{X}/S}^{(m)}$ is coherent on the right and on the left. As in [4, 3.4.2], this implies that $\widehat{\mathcal{D}}_{\mathfrak{X}/S,\mathbb{Q}}^{(m)}$ is coherent on the right and on the left. As in [4, 3.4.2], this implies that $\widehat{\mathcal{D}}_{\mathfrak{X}/S,\mathbb{Q}}^{(m)}$ is coherent on the right and on the left. By copying the proof of [8, 4.2] (indeed, we have the same local description and \mathfrak{X} is noetherian), we can prove that the extension $\widehat{\mathcal{D}}_{\mathfrak{X}/S,\mathbb{Q}}^{(m)} \to \widehat{\mathcal{D}}_{\mathfrak{X}/S,\mathbb{Q}}^{(m+1)}$ is flat on the right and on the left. Hence, taking the inductive limits, we obtain the coherence on the right and on the left of $\mathcal{D}_{\mathfrak{X}/S}^{\dagger}$. Similarly, we have theorem of type *B* as in [4, 3]: if \mathfrak{X} is affine, for any integer $q \ge 1$ we have the vanishing $H^q(\mathfrak{X}, \mathcal{D}_{X_i/S_i}^{(m)}) = 0$, $H^q(\mathfrak{X}, \widehat{\mathcal{D}}_{\mathfrak{X}/S,\mathbb{Q}}^{(m)}) = 0$.

Point 3.37 (Overconvergent isocrystals). — Let \mathfrak{X} be a log *p*-smooth Slog-formal schemes and Z be a Cartier divisor of \underline{X}_0 . As in [4, 4.2.3] and with its notation, the commutative \mathcal{O}_{X_i} -algebra $\mathcal{B}_{X_i}^{(m)}(Z)$ can be endowed with a (canonical) compatible structure of left $\mathcal{D}_{X_i/S_i}^{(m)}$ -module (see Definition 2.36) such that $\mathcal{B}_{X_i}^{(m)}(Z) \to \mathcal{B}_{X_i}^{(m+1)}(Z)$ is $\mathcal{D}_{X_i/S_i}^{(m)}$ -linear. We get a structure of $\widehat{\mathcal{D}}_{\mathfrak{X}/S}^{(m)}$ -module on $\widehat{\mathcal{B}}_{\mathfrak{X}}^{(m)}(Z) = \varprojlim_i \mathcal{B}_{X_i}^{(m)}(Z)$. From Proposition 2.37, we get the $\mathcal{O}_{\mathfrak{X}}$ -algebra $\widehat{\mathcal{B}}_{\mathfrak{X}}^{(m)}(Z) \widehat{\otimes} \widehat{\mathcal{D}}_{\mathfrak{X}/S}^{(m)}$ such that the canonical map $\widehat{\mathcal{D}}_{\mathfrak{X}/S}^{(m)} \to \widehat{\mathcal{B}}_{\mathfrak{X}}^{(m)}(Z) \widehat{\otimes} \widehat{\mathcal{D}}_{\mathfrak{X}/S}^{(m)}$ is a morphism of $\mathcal{O}_{\mathfrak{X}}$ -algebras. We set $\mathcal{O}_{\mathfrak{X}}(^{\dagger}Z) := \varinjlim_{m} \widehat{\mathcal{B}}_{\mathfrak{X}}^{(m)}(Z) \text{ and } \mathcal{D}_{\mathfrak{X}/\mathcal{S}}^{\dagger}(^{\dagger}Z) := \varinjlim_{m} \widehat{\mathcal{B}}_{\mathfrak{X}}^{(m)}(Z) \widehat{\otimes} \widehat{\mathcal{D}}_{\mathfrak{X}/\mathcal{S}}^{(m)}. \text{ We define an isocrystal on } \mathfrak{X} \text{ overconvergent along } Z \text{ to be a coherent } \mathcal{D}_{\mathfrak{X}/\mathcal{S}}^{\dagger}(^{\dagger}Z)_{\mathbb{Q}}\text{-module which is also } \mathcal{O}_{\mathfrak{X}}(^{\dagger}Z)_{\mathbb{Q}}\text{-coherent (for the structure induced by the canonical morphism } \mathcal{O}_{\mathfrak{X}}(^{\dagger}Z)_{\mathbb{Q}} \to \mathcal{D}_{\mathfrak{X}/\mathcal{S}}^{\dagger}(^{\dagger}Z)_{\mathbb{Q}}).$

3.6. Structure of right $\mathcal{D}_{\mathfrak{X}/S}^{\dagger}$ -module on $\omega_{\mathfrak{X}/S}$

PROPOSITION 3.38 (Structure of right $\mathcal{D}_{X/S}^{(0)}$ -module on $\omega_{X/S}$). — Let S be a fine log scheme over $\mathbb{Z}/p^{i+1}\mathbb{Z}$ and let (I_S, J_S, γ) be a quasi-coherent *m*-PD-ideal of \mathcal{O}_S . Let $f: X \to S$ be a weakly log smooth of level *m* compatible with γ morphism of fine log-schemes such that γ extends to X.

We have a canonical structure of right $\mathcal{D}_{X/S}^{(0)}$ -module on $\omega_{X/S}$ (see Remark 2.21). Locally, this structure is characterized by the following description. Suppose that $X \to S$ is endowed with a formal log basis $(b_i)_{i=1,...,n}$ of level m compatible with γ . Let dlog b_i denotes the image of η_i in $\Gamma(X, \Omega_{X/S}^1)$. The action of $P \in \mathcal{D}_{X/S}^{(0)}$ on the section $a \operatorname{dlog} b_1 \wedge \cdots \wedge \operatorname{dlog} b_n$, where a is section of \mathcal{O}_X is given by the formula

$$(3.6) \qquad (a \operatorname{dlog} b_1 \wedge \cdots \wedge \operatorname{dlog} b_n) \cdot P = P(a) \operatorname{dlog} b_1 \wedge \cdots \wedge \operatorname{dlog} b_n.$$

Proof.

Step 0. — It is sufficient to check the independence of the formula (3.6) with respect to the chosen formal log basis of level m compatible with γ . Suppose that $X \to S$ is endowed with two formal log bases $(b_i)_{i=1,...,n}$ and $(b'_i)_{i=1,...,n}$ of level m compatible with γ .

Step 1. — Let $A = (a_{ij}) \in M_n(\mathcal{O}_X)$ and $A' = (a'_{ij}) \in M_n(\mathcal{O}_X)$ be the matrices such that

$$\begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} = A \begin{pmatrix} \partial'_1 \\ \vdots \\ \partial'_n \end{pmatrix} \text{ and } \begin{pmatrix} \partial'_1 \\ \vdots \\ \partial'_n \end{pmatrix} = A' \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}.$$

Hence, we get

$$A' = A^{-1}, \quad \begin{pmatrix} \operatorname{dlog} b'_1 \\ \vdots \\ \operatorname{dlog} b'_n \end{pmatrix} = {}^t A \begin{pmatrix} \operatorname{dlog} b_1 \\ \vdots \\ \operatorname{dlog} b_n \end{pmatrix}$$

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and then dlog $b_1 \wedge \cdots \wedge d\log b_n = |A'| d\log b'_1 \wedge \cdots \wedge d\log b'_n$. We compute

$$\partial_{i'}\partial_i = \sum_{j=1}^n \partial_{i'}a_{ij}\partial_j'$$

= $\sum_{j=1}^n a_{ij}\partial_{i'}\partial_j' + \sum_{j=1}^n \partial_{i'}(a_{ij})\partial_j'$
= $\sum_{j,j'=1}^n a_{ij}a_{i'j'}\partial_{j'}\partial_j' + \sum_{j=1}^n \partial_{i'}(a_{ij})\partial_j'.$

Since $\partial_{i'}\partial_i = \partial_i\partial_{i'}$ and $\partial'_{j'}\partial'_j = \partial'_j\partial'_{j'}$, exchanging *i* with *i'* yields

$$\sum_{j=1}^n \partial_{i'}(a_{ij})\partial'_j = \sum_{j=1}^n \partial_i(a_{i'j})\partial'_j.$$

Hence, for any i, i', j, we have $\partial_{i'}(a_{ij}) = \partial_i(a_{i'j})$ and then (by symmetry) $\partial'_{i'}(a'_{ij}) = \partial'_i(a'_{i'j})$.

Step 2. — By symmetry and \mathcal{O}_X -linearity, it is sufficient to check that both actions of ∂_1 on dlog $b_1 \wedge \cdots \wedge$ dlog b_n coincides. With the first choice, this is straightforward that we get 0. Now, we consider the action ∂_1 on dlog $b_1 \wedge \cdots \wedge$ dlog b_n for the second choice of log *p*-basis. Since $\partial_1 =$ $\sum_{j=1}^n a_{1j}\partial'_j$, we get dlog $b_1 \wedge \cdots \wedge$ dlog $b_n \cdot \partial_1 = (|A'| \operatorname{dlog} b'_1 \wedge \cdots \wedge \operatorname{dlog} b'_n) \cdot$ $\partial_1 = -\sum_{j=1}^n \partial'_j (a_{1j}|A'|) \operatorname{dlog} b'_1 \wedge \cdots \wedge \operatorname{dlog} b'_n$. Hence, we have to check $\sum_{j=1}^n \partial'_j (a_{1j}|A'|) = 0$.

(a). — We compute $a_{1j}|A'| = \sum_{\sigma \in S_n, \sigma(1)=j} (-1)^{\epsilon(\sigma)} \prod_{i=2}^n a'_{\sigma(i)i}$. Indeed, let L'_1, \ldots, L'_n be the rows of A'. We remark that $a_{1j}|A'|$ is equal to the determinant of the matrix A' whose row L'_j is replaced by $a_{1j}L'_j$ and then by $\sum_{l=1}^n a_{1l}L'_l$. Since $AA' = I_n$, we get $\sum_{l=1}^n a_{1l}L'_l = (1, 0, \ldots, 0)$. This yields the desired formula.

$$\sum_{j=1}^{n} \partial'_{j}(a_{1j}|A'|) = \sum_{\sigma \in S_{n}, l \in \{2, \dots, n\}} (-1)^{\epsilon(\sigma)} \partial'_{\sigma(1)}(a'_{\sigma(l)l}) \prod_{i=2, i \neq l}^{n} a'_{\sigma(i)i}.$$

Indeed, this is a consequence of the formula

$$\partial'_{\sigma(1)}(\prod_{i=2}^{n} a'_{\sigma(i)i}) = \sum_{l=2}^{n} \partial'_{\sigma(1)}(a'_{\sigma(l)l}) \prod_{i=2, i \neq l}^{n} a'_{\sigma(i)i},$$

and of that of part (a).

(c). — We define on $S_n \times \{2, \ldots, n\}$ the following equivalence relation. Two elements (σ, l) and (σ', l') of $S_n \times \{2, \ldots, n\}$ are equivalent if either $(\sigma', l') = (\sigma, l)$ or $(\sigma', l') = (\sigma \circ (1, l), l)$. Let (σ, l) and $(\sigma', l') = (\sigma \circ (1, l), l)$ be a class of $S_n \times \{2, \ldots, n\}$. Using the formula checked in Step 1 of the proof, we get

$$(-1)^{\epsilon(\sigma')}\partial'_{\sigma'(1)}(a'_{\sigma'(l)l})\prod_{i=2,i\neq l}^{n}a'_{\sigma'(i)i}+(-1)^{\epsilon(\sigma)}\partial'_{\sigma(1)}(a'_{\sigma(l)l})\prod_{i=2,i\neq l}^{n}a'_{\sigma(i)i}=0.$$

Since we have a partition of $S_n \times \{2, \ldots, n\}$ by its classes, this implies

$$\sum_{\substack{\epsilon \in S_n, l \in \{2, \dots, n\}}} (-1)^{\epsilon(\sigma)} \partial'_{\sigma(1)} (a'_{\sigma(l)l}) \prod_{i=2, i \neq l}^n a'_{\sigma(i)i} = 0.$$

 \square

 $\sigma \in S_n, l \in \{2, \dots \}$ We conclude using (b).

PROPOSITION 3.39. — Let \mathfrak{X} be a log *p*-smooth *S*-log-formal scheme. We suppose that $\underline{\mathfrak{X}}$ has no *p*-torsion. We put $\omega_{\mathfrak{X}/S} := \varprojlim_i \omega_{X_i/S_i}$. There exists a canonical structure of right $\mathcal{D}_{\mathfrak{X}/S}^{\dagger}$ -module on $\omega_{\mathfrak{X}/S}$. It is characterized by the following local formula: suppose that \mathfrak{X} is endowed with a log *p*-basis $(b_{\lambda})_{\lambda=1,\ldots,n}$. Let dlog b_{λ} be the image of η_{λ} in $\Omega_{\mathfrak{X}/S}^{1}$. Then, for any integer *m*, for any differential operator $P \in \mathcal{D}_{\mathfrak{X}/S}^{(m)}$ and $a \in \mathcal{O}_{\mathfrak{X}}$ we have

$$(3.7) \qquad (a \operatorname{dlog} b_1 \wedge \cdots \wedge \operatorname{dlog} b_n) \cdot P := P(a) \operatorname{dlog} b_1 \wedge \cdots \wedge \operatorname{dlog} b_n.$$

Proof. — Using Proposition 3.38, we get a canonical structure of right $\widehat{\mathcal{D}}_{\mathfrak{X}}^{(0)}$ -module on $\omega_{\mathfrak{X}/S} = \varprojlim_{i} \omega_{X_i/S_i}$. Hence, we get a structure of right $\widehat{\mathcal{D}}_{\mathfrak{X},\mathbb{Q}}^{(0)}$ -module on $\omega_{\mathfrak{X}/S,\mathbb{Q}}$. Since $\mathcal{D}_{\mathfrak{X}/S}^{(m)} \subset \widehat{\mathcal{D}}_{\mathfrak{X}/S,\mathbb{Q}}^{(0)}$, we get a structure of right $\mathcal{D}_{\mathfrak{X}/S}^{(m)}$ -module on $\omega_{\mathfrak{X}/S,\mathbb{Q}}$. Let us check that $\omega_{\mathfrak{X}/S}$ is a sub $\mathcal{D}_{\mathfrak{X}/S}^{(m)}$ -module of $\omega_{\mathfrak{X}/S,\mathbb{Q}}$. Let us check that $\omega_{\mathfrak{X}/S}$ is a sub $\mathcal{D}_{\mathfrak{X}/S}^{(m)}$ -module of $\omega_{\mathfrak{X}/S,\mathbb{Q}}$. Since this is local, we can suppose that \mathfrak{X} is endowed with a log p-basis $(b_{\lambda})_{\lambda=1,\dots,n}$. We compute that the right $\mathcal{D}_{\mathfrak{X}/S}^{(m)}$ -action on $\omega_{\mathfrak{X}/S,\mathbb{Q}}$ is given by the formula (3.7). This implies that $\omega_{\mathfrak{X}/S}$ is a sub $\mathcal{D}_{\mathfrak{X}/S}^{(m)}$ -module of $\omega_{\mathfrak{X}/S,\mathbb{Q}}$. Using (a right log version of) [3, 3.1.3], this yields that $\omega_{\mathfrak{X}/S}$ is endowed with a canonical structure of $\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)}$ -module. Since these structures are compatible with $\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)} \to \widehat{\mathcal{D}}_{\mathfrak{X}}^{(m+1)}$, we are done.

COROLLARY 3.40. — Let \mathfrak{X} be a log *p*-smooth S-log-formal scheme such that $\underline{\mathfrak{X}}$ has no *p*-torsion. Let *i* be an integer. There exists a canonical structure of right $\mathcal{D}_{X_i/S_i}^{(m)}$ -module on ω_{X_i/S_i} . It is characterized by the following local formula: suppose that \mathfrak{X} is endowed with a log *p*-basis $(b_\lambda)_{\lambda=1,\dots,n}$. Let dlog b_λ be the image of η_λ in $\Omega^1_{X_i/S_i}$. Then, for any integer *m*, for any differential operator $P \in \mathcal{D}_{X_i/S_i}^{(m)}$ and $a \in \mathcal{O}_{X_i}$ we have

$$(3.8) \qquad (a \operatorname{dlog} b_1 \wedge \dots \wedge \operatorname{dlog} b_n) \cdot P := P(a) \operatorname{dlog} b_1 \wedge \dots \wedge \operatorname{dlog} b_n.$$

Proof. — This is a consequence of Proposition 3.39.

COROLLARY 3.41. — Let \mathfrak{X} be a log *p*-smooth S-log-formal scheme such that $\underline{\mathfrak{X}}$ has no *p*-torsion. The functor $- \otimes_{\mathcal{O}_{X_i}} \omega_{X_i}$ (resp. $- \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}}$) is an equivalence of categories between that of left $\mathcal{D}_{X_i/S_i}^{(m)}$ -modules (resp. left $\mathcal{D}_{\mathfrak{X}/S}^{\dagger}$ -modules) and that of right $\mathcal{D}_{X_i/S_i}^{(m)}$ -modules (resp. right $\mathcal{D}_{\mathfrak{X}/S}^{\dagger}$ modules). The functor $- \otimes_{\mathcal{O}_{X_i}} \omega_{X_i}^{-1}$ (resp. $- \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}}^{-1}$) is a quasi-inverse functor. Both functors preserve the coherence. These functors are the "twisted structures" of \mathcal{D} -module.

3.7. Pushforwards, extraordinary pull-backs and duality

Let $f: \mathfrak{X} \to \mathcal{Y}$ be a morphism of log *p*-smooth *S*-log-formal schemes. We suppose that the formal \mathcal{V} -schemes $\underline{\mathfrak{X}}$ and $\underline{\mathcal{Y}}$ have no *p*-torsion. We can follow Berthelot's construction of pushforwards, extraordinary pull-backs and duality as explained in [6] or [5]. For the reader, let's briefly sketch the construction.

Notation 3.42 (dimension and rank of a p-basis).

- (1) The $\mathcal{O}_{\mathfrak{X}}$ -module $\Omega_{\mathfrak{X}/S}$ is locally free of finite rank. We denote by $\delta_{\mathfrak{X}/S} \colon \mathfrak{X} \to \mathbb{N}$ the locally constant function given by $x \mapsto \operatorname{rank}_{\mathcal{O}_{\mathfrak{X},x}} \Omega_{\mathfrak{X}/S,x}$. Since X is noetherian and formally smooth over Spec k, then X is regular (see [15, 0_{IV} .19.3.8] and [15, 0_{IV} .19.6.4]). This yields that X is the sum of its irreducible components (see [13, 6.1.10]). If \mathfrak{U} is an irreducible component of \mathfrak{X} , then $\delta_{\mathfrak{X}}|\mathfrak{U}$ is a constant function. If moreover \mathfrak{U}/S has a finite p-basis, then $\delta_{\mathfrak{X}}|\mathfrak{U}$ is the constant function equal to the rank of $\Omega_{\mathfrak{U}/S}$, which is equal to the number of elements of the p-basis.
- (2) We have the locally constant function $d_{\mathfrak{X}} \colon \mathfrak{X} \to \mathbb{N}$, given by $x \mapsto \dim_x \mathfrak{X}$. In general $d_{\mathfrak{X}} \neq \delta_{\mathfrak{X}}$ For instance, when $\mathfrak{X} = \operatorname{Spf} \mathcal{V}((t))$, the Krull dimension of \mathfrak{X} is 0 but $\Omega_{\mathfrak{X}/S}$ is $\mathcal{O}_{\mathfrak{X}}$ -free of rank 1. The function $d_{\mathfrak{X}}$ is not the right one in our context. The function $\delta_{\mathfrak{X}/S}$ seems as fine as the dimension in the case of smooth formal \mathcal{V} -schemes. For instance, by copying the usual proof, with (3.11), we can check the isomorphism $\mathbb{D}_{\mathfrak{X}}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$.
- (3) We set $\delta_{\mathfrak{X}/\mathcal{Y}} := \delta_{\mathfrak{X}} \delta_{\mathcal{Y}} \circ f$.

Point 3.43. — By functoriality, the left $\mathcal{D}_{X_i/S_i}^{(m)}$ -module $f^*\mathcal{D}_{Y_i/S_i}^{(m)}$ is in fact endowed with a structure of $(\mathcal{D}_{X_i/S_i}^{(m)}, f^{-1}\mathcal{D}_{Y_i/S_i}^{(m)})$ -bimodule which we will

denote by $\mathcal{D}_{X_i \to Y_i/S_i}^{(m)}$. By twisting (see Corollary 3.41), we get a $(f^{-1}\mathcal{D}_{Y_i/S_i}^{(m)}, \mathcal{D}_{X_i/S_i}^{(m)})$ -bimodule by setting $\mathcal{D}_{Y_i \leftarrow X_i/S_i}^{(m)} \coloneqq \omega_{X_i/S_i} \otimes_{\mathcal{O}_{X_i}} f_{\mathfrak{g}}^*(\mathcal{D}_{Y_i/S_i}^{(m)} \otimes_{\mathcal{O}_{Y_i}} \omega_{Y_i/S_i}^{-1})$, where the index g means that we choose the left structure of the left $\mathcal{D}_{Y_i/S_i}^{(m)}$ -bimodule $\mathcal{D}_{Y_i/S_i}^{(m)} \otimes_{\mathcal{O}_{Y_i}} \omega_{Y_i/S_i}^{-1}$. Next, we put $\widehat{\mathcal{D}}_{\mathfrak{X} \to \mathcal{Y}/S}^{(m)} \coloneqq \lim_{\mathfrak{X} \to \mathcal{Y}/S_i} \inf \widehat{\mathcal{D}}_{\mathcal{Y} \leftarrow \mathfrak{X}/S}^{(m)} \cong \lim_{\mathfrak{X} \to \mathcal{Y}/S_i} \mathfrak{D}_{\mathfrak{X} \to \mathcal{Y}/S_i}^{(m)}$ and $\widehat{\mathcal{D}}_{\mathcal{Y} \leftarrow \mathfrak{X}/S}^{(m)} \coloneqq \lim_{\mathfrak{X} \to \mathcal{Y}/S_i} \widehat{\mathcal{D}}_{\mathfrak{X} \to \mathcal{Y}/S}^{(m)}$. Finally, $\mathcal{D}_{\mathfrak{X} \to \mathcal{Y}/S}^{\dagger} \coloneqq \lim_{\mathfrak{X} \to \mathcal{Y}/S} \widehat{\mathcal{D}}_{\mathfrak{X} \to \mathcal{Y}/S}^{(m)}$ is a d $\widehat{\mathcal{D}}_{\mathcal{Y} \leftarrow \mathfrak{X}/S}^{(m)} \cong \lim_{\mathfrak{X} \to \mathcal{Y}/S_i} \widehat{\mathcal{D}}_{\mathfrak{X} \to \mathcal{Y}/S}^{(m)}$. We denote by $D^{\mathrm{b}}(\mathcal{D}_{\mathfrak{X}/S,\mathbb{Q}}^{\dagger})$, (resp. $\mathcal{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathcal{D}_{\mathfrak{X}/S,\mathbb{Q}}^{\dagger})$, resp. $\mathcal{D}_{\mathrm{parf}}(\mathcal{D}_{\mathfrak{X}/S,\mathbb{Q}}^{\dagger})$) the derived category of bounded (resp. and bounded and coherent, resp. perfect) complexes of left $\mathcal{D}_{\mathfrak{X}/S,\mathbb{Q}}^{\dagger}$ -modules.

(1) As in [6, 4.3.2.2], we get the functor $f^!: D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{Y}/\mathcal{S},\mathbb{Q}}) \to D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}})$ by setting, for any object \mathcal{F} of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{Y}/\mathcal{S},\mathbb{Q}})$,

(3.9)
$$f^{!}(\mathcal{F}) := \mathcal{D}_{\mathfrak{X} \to \mathcal{Y}/\mathcal{S}, \mathbb{Q}}^{\dagger} \otimes_{f^{-1} \mathcal{D}_{\mathcal{Y}/\mathcal{S}, \mathbb{Q}}}^{\mathbb{L}} f^{-1} \mathcal{F}[\delta_{\mathfrak{X}/\mathcal{Y}}].$$

When f is log p-smooth, only by copying Berthelot's proof in the classical theory, we expect that the functor $f^!$ preserves the coherence, i.e. we have the factorization $f^!: D^{\rm b}_{\rm coh}(\mathcal{D}^{\dagger}_{\mathcal{X}/\mathcal{S},\mathbb{Q}}) \to D^{\rm b}_{\rm coh}(\mathcal{D}^{\dagger}_{\mathcal{X}/\mathcal{S},\mathbb{Q}}).$

(2) As in [6, 4.3.7.1], we get the functor $f_+: D^{\rm b}_{\rm coh}(\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}}) \to D^{\rm b}(\mathcal{D}^{\dagger}_{\mathcal{Y}/\mathcal{S},\mathbb{Q}})$ by setting, for any object \mathcal{E} of $D^{\rm b}_{\rm coh}(\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}})$,

(3.10)
$$f_{+}(\mathcal{E}) := \mathbb{R}f_{*}(\mathcal{D}_{\mathcal{Y}\leftarrow\mathfrak{X}/\mathcal{S},\mathbb{Q}}^{\dagger}\otimes_{\mathcal{D}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}}^{\dagger}}^{\mathbb{L}}\mathcal{E}).$$

When f is proper, only by copying Berthelot's proof in the classical theory, we expect that the functor f^{\dagger} preserves the coherence, i.e. we have the factorization $f_{+}: D^{\rm b}_{\rm coh}(\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathcal{S},\mathbb{O}}) \to D^{\rm b}_{\rm coh}(\mathcal{D}^{\dagger}_{\mathcal{Y}/\mathcal{S},\mathbb{O}}).$

(3) As in [6, 4.3.10], we get the functor $\mathbb{D}_{\mathfrak{X}} \colon D_{\mathrm{parf}}(\mathcal{D}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}}^{\dagger}) \to D_{\mathrm{parf}}(\mathcal{D}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}}^{\dagger})$ by posing, for any object \mathcal{E} of $D_{\mathrm{parf}}(\mathcal{D}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}}^{\dagger})$,

$$(3.11) \qquad \mathbb{D}_{\mathfrak{X}}(\mathcal{E}) := \mathbb{R} \,\mathscr{H}_{\mathrm{Om}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^{\dagger}_{\mathfrak{X}/\mathcal{S},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \omega_{\mathfrak{X}/\mathcal{S},\mathbb{Q}}^{-1})[\delta_{\mathfrak{X}}].$$

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