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# THE PERVERSE EIGENSPACE OF ONE FOR THE MILNOR MONODROMY

by David MASSEY

Abstract. — In this short paper, we describe the relationship between intersection cohomology with integral coefficients and the perverse eigenspace of one for the Milnor mondromy on the vanishing cycles.

Résumé. — Dans ce court article, nous décrivons la relation entre la cohomologie d'intersection avec coefficients intégraux et l'espace propre pervers de un pour la mondromie de Milnor sur les cycles de fuite.

## 1. Introduction

Suppose that $U$ is a non-empty open neighborhood of the origin in $\mathbb{C}^{n+1}$, where $n > 1$, and let $f : (U, \mathbf{0}) \to (\mathbb{C}, 0)$ be a reduced, nowhere locally constant, complex analytic function. Then $X := V(f) = f^{-1}(0)$ is a hypersurface in $U$ of pure dimension $n$, where for convenience we have assumed that $\mathbf{0} \quad V(f)$.

Since $f$ is reduced, the singular set $\Sigma$ of $X$ is equal to the intersection of the hypersurface with the critical locus, $\Sigma f$, of $f$. In fact, near $\mathbf{0}$, $\Sigma f \quad V(f)$, so that the intersection with $V(f)$ is unnecessary if we rechoose $U$ small enough.

In order to motivate the main theorem of this paper, we wish to consider the case where $\dim_{\mathbf{0}} \Sigma f = 0$, i.e., when $X$ has an isolated singularity at the origin, and see what classical results tell us about the relationship between the Milnor monodromy of $f$ and the intersection cohomology of $X$ at $\mathbf{0}$.

As explained in [5] (but translated from homology to cohomology), the stalk cohomology of intersection cohomology can be obtained from the

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ordinary cohomology of the *real link* of $X$ at $\mathbf{0}$; this real link is $K_{X,\mathbf{0}} := S \cap X$, where $S$ is a small sphere of radius centered at the origin. Letting $\mathbf{I}^\bullet$ denote the intersection cohomology complex, the relationship is:

$$H^k(\mathbf{I}^\bullet_X)_\mathbf{0} = \begin{cases} H^{n+k}(K_{X,\mathbf{0}};\,\mathbb{Z}), & \text{if } k \leqslant -1;\\ 0, & \text{if } k \geqslant 0.\end{cases}$$

Now let $F_{f,\mathbf{0}}$ be the Milnor fiber of $f$ at $\mathbf{0}$ and let $T$ be the Milnor monodromy automorphism on $H^n(F_{f,\mathbf{0}};\mathbb{Z})$. Then, the cohomological version of the Wang sequence given by Milnor in Lemma 8.4 of [14], combined with Alexander duality, tells us that, for $k \leqslant -1$,

$$H^{n+k}(K_{X,\mathbf{0}};\,\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = -n;\\ H^{n-1}(K_{X,\mathbf{0}};\,\mathbb{Z}), & \text{if } k = -1;\\ 0, & \text{otherwise.}\end{cases}$$

and that

$$H^{n-1}(K_{X,\mathbf{0}};\,\mathbb{Z}) = \ker\{\mathrm{id} - T\}.$$

Thus, we quickly conclude from classical results:

Proposition 1.1. — *Suppose that* $\dim_\mathbf{0} \Sigma f = 0$*. Then the stalk cohomology of the integral intersection cohomology sheaf* $\mathbf{I}^\bullet_X$ *on* $X := V(f)$ *is given by*

$$H^k(\mathbf{I}^\bullet_X)_\mathbf{0} = \begin{cases} \mathbb{Z}, & \textit{if } k = -n;\\ \ker\{\mathrm{id} - T\}, & \textit{if } k = -1;\\ 0, & \textit{otherwise,}\end{cases}$$

*where $T$ is the Milnor monodromy automorphism on* $H^n(F_{f,\mathbf{0}};\mathbb{Z})$ *and* $F_{f,\mathbf{0}}$ *is the Milnor fiber of $f$ at* $\mathbf{0}$*.*

The question, of course, is: *Does this generalize to the case where the dimension of the critical locus is greater than zero?*

The answer is "yes", though the generalization that we give is not about stalk cohomology. It is rather what we consider to be a beautiful, elegant generalization inside the Abelian category $\mathrm{Perv}(X)$ of perverse sheaves on $X$.

Our main theorem is:

Theorem 1.2 (Theorem 2.5). — *Suppose that* $\dim_\mathbf{0} \Sigma f$ *is arbitrary. Let* $X := V(f)$ *and let* $\widetilde{T}_f$ *denote the monodromy automorphism on the perverse sheaf of shifted vanishing cycles* $\varphi_f[-1]\mathbb{Z}^\bullet_U[n+1]$.

*Then, in* Perv$(X)$*, the kernel of the canonical perverse surjection* $\mathbb{Z}^\bullet_X[n] \xrightarrow{\tau_X} \mathbf{I}^\bullet_X$ *is isomorphic to* $\ker\{\mathrm{id} - \widetilde{T}_f\}$*, that is, we have a short exact sequence in* Perv$(X)$*:*

$$0 \longrightarrow \ker\{\mathrm{id} - \widetilde{T}_f\} \longrightarrow \mathbb{Z}^\bullet_X[n] \xrightarrow{\tau_X} \mathbf{I}^\bullet_X \longrightarrow 0.$$

Therefore, in a sense, $\ker\{\mathrm{id} - \widetilde{T}_f\}$ should be thought of as the "reduced" intersection cohomology complex on $X$. In the case where $f$ has an isolated critical point, one can recover Proposition 1.1 from Theorem 2.5 by taking the induced long exact sequence on stalk cohomology. This does not work more generally because, in the non-isolated case, the stalk cohomology of $\ker\{\mathrm{id} - \widetilde{T}_f\}$ is not isomorphic to the kernel of the induced map on stalk cohomology. We will discuss this further in Section 3.

Also in Section 3, we will discuss two applications of Theorem 2.5 to describing the cohomology of Milnor fibers. We shall also discuss the results for eigenvalues other than 1 for $\widetilde{T}_f$. Finally, we will mention some open questions which naturally arise from Theorem 2.5.

Before leaving the introduction, we should mention that in the algebraic setting with field coefficients, Theorem 2.5 can be obtained from the work of Beilinson in [1] (see also [15]) and the work of M. Saito in [16]. However, we do care about $\mathbb{Z}$ coefficients and the analytic case, and the proofs by these other techniques are not simpler.

## 2. The Main Theroem

We continue with $U$, $f$, and $X$ as in the introduction. We denote the various inclusions as follows: $j : X \to U$, $i : U \setminus X \to U$, $m : \Sigma \to X$, and $l : X \setminus \Sigma \to X$. Furthermore, we let $\hat{m} := j \circ m$ be the inclusion of $\Sigma$ into $U$. Our base ring is always $\mathbb{Z}$ (though all statements hold with base ring $\mathbb{C}$). In addition, we let $s := \dim \Sigma$ (or, when we focus on the germ at $x \in \Sigma$, we will let $s := \dim_x \Sigma$).

In this setting, there (at least) two canonical perverse sheaves on $X$: the shifted constant sheaf $\mathbb{Z}^\bullet_X[n] = j^*[-1]\mathbb{Z}^\bullet_U[n+1]$ and the intersection cohomology complex $\mathbf{I}^\bullet_X$ (with constant $\mathbb{Z}$ coefficients), which is the intermediate extension to all of $X$ of $\mathbb{Z}^\bullet_{X\setminus\Sigma}[n]$. In addition, as described by Goresky and MacPherson in [6], there is a canonical morphism $\tau_X$ from $\mathbb{Z}^\bullet_X[n]$ to $\mathbf{I}^\bullet_X$; this morphism is a surjection in the Abelian category Perv$(X)$ of perverse sheaves (with middle perversity) on $X$ (the surjectivity follows from the fact that $\tau_X$ is an isomorphism when restricted to $X \setminus \Sigma$ and that $\mathbf{I}^\bullet_X$ is the intermediate extension of the shifted constant sheaf on $X \setminus \Sigma$).

Definition 2.1. — *The* comparison complex, $\mathbf{N}^\bullet_X$, *on* $X$ *is the kernel (in* $\operatorname{Perv}(X)$*) of* $\tau_x$*. Hence, by definition, the support of* $\mathbf{N}^\bullet_X$ *is contained in* $\Sigma$ *and there is a short exact sequence*

$$0 \longrightarrow \mathbf{N}^\bullet_X \longrightarrow \mathbb{Z}^\bullet_X[n] \xrightarrow{\tau_X} \mathbf{I}^\bullet_X \longrightarrow 0.$$

*Remark 2.2.* — While $\mathbf{N}^\bullet_X$ is interesting as a perverse sheaf, on the level of stalks, $\mathbf{N}^\bullet_X$ merely gives the shifted, reduced intersection cohomology of $X$. To be precise, the long exact sequence on stalk cohomology at $x \quad X$ yields a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H^{-n}(\mathbf{I}^\bullet_X)_x \longrightarrow H^{-n+1}(\mathbf{N}^\bullet_X)_x \longrightarrow 0$$

and, for all $k = -n+1$, isomorphisms

$$H^{k-1}(\mathbf{I}^\bullet_X)_x = H^k(\mathbf{N}^\bullet_X)_x.$$

If we let $I\widetilde{H}$ denote reduced intersection cohomology, with topological indexing (as is used for intersection homology in [5]), then we have

$$H^k(\mathbf{N}^\bullet_X)_x = I\widetilde{H}^{n+k-1}(B^\circ(x) \cap X; \mathbb{Z}),$$

where $B^\circ(x)$ denotes a small open ball, of radius , centered at $x$.

*Remark 2.3.* — Our interest in the comparison complex arose from considering parameterized hypersurfaces, i.e., hypersurfaces which have smooth normalizations. Our results in this case appear in our joint work with Brian Hepler in [8] (see also Hepler's paper [7]), where it was more natural to refer to $\mathbf{N}^\bullet_X$ as the *multiple-point complex*.

Now we denote the monodromy automorphisms on the nearby and vanishing cycles along $f$, respectively, by

$$T_f : \psi_f[-1]\mathbb{Z}^\bullet_U[n+1] \xrightarrow{\cong} \psi_f[-1]\mathbb{Z}^\bullet_U[n+1]$$

and

$$\widetilde{T}_f : \varphi_f[-1]\mathbb{Z}^\bullet_U[n+1] \xrightarrow{\cong} \varphi_f[-1]\mathbb{Z}^\bullet_U[n+1].$$

Recall that there are two canonical *nearby-vanishing short exact sequences* in $\operatorname{Perv}(X)$:

(2.1)

$$0 \longrightarrow j\ [-1]\mathbb{Z}^\bullet_U[n+1] \longrightarrow \psi_f[-1]\mathbb{Z}^\bullet_U[n+1] \xrightarrow{\text{can}} \varphi_f[-1]\mathbb{Z}^\bullet_U[n+1] \longrightarrow 0$$

and

$$0 \longrightarrow \varphi_f[-1]\mathbb{Z}^\bullet_U[n+1] \xrightarrow{\text{var}} \psi_f[-1]\mathbb{Z}^\bullet_U[n+1] \longrightarrow j^![1]\mathbb{Z}^\bullet_U[n+1] \longrightarrow 0,$$

where $\mathrm{var} \circ \mathrm{can} = \mathrm{id} - T_f$ and $\mathrm{can} \circ \mathrm{var} = \mathrm{id} - \widetilde{T}_f$. See, for instance, [9, 8.6.7 and 8.6.8] (but be aware that the $\varphi_f$ of Kashiwara and Schapira is our $\varphi_f[-1]$), [17, 6.0.4], or [4, Definition 4.2.4 and Remark 4.2.12].

Before we can prove the main theorem, we first need a lemma. We use ${}^{\mu}H^k(\mathbf{A}^\bullet)$ to denote the degree $k$ perverse cohomology of a complex (with middle perversity). See [9, Section 10.3] or [4, Sections 5.1 and 5.2].

Lemma 2.4. — *There is an isomorphism of perverse sheaves*

$$\mathbf{N}^\bullet_X = {}^{\mu}H^0(m_! m^! \mathrm{Z}^\bullet_X[n]).$$

*Proof.* — The intersection cohomology complex $\mathbf{I}^\bullet_X$ on $X$ is the intermediate extension of the shifted constant on $X \setminus \Sigma$, i.e., the image (in $\mathrm{Perv}(X)$) of the canonical morphism

$${}^{\mu}H^0(l_! \mathrm{Z}^\bullet_{X \setminus \Sigma}[n]) \xrightarrow{\alpha} {}^{\mu}H^0(l \;\; \mathrm{Z}^\bullet_{X \setminus \Sigma}[n]).$$

The morphism $\alpha$ factors through the perverse sheaf $\mathrm{Z}^\bullet_X[n]$; $\alpha$ is the composition of the canonical maps

$$\begin{aligned}{}^{\mu}H^0(l_! \mathrm{Z}^\bullet_{X \setminus \Sigma}[n]) = {}^{\mu}H^0(l_! l^! \mathrm{Z}^\bullet_X[n]) \xrightarrow{\beta} \mathrm{Z}^\bullet_X[n] \xrightarrow{\gamma} {}^{\mu}H^0(l \;\; l \;\; \mathrm{Z}^\bullet_X[n]) \\ = {}^{\mu}H^0(l \;\; \mathrm{Z}^\bullet_{X \setminus \Sigma}[n]).\end{aligned}$$

We claim that $\beta$ is a surjection and, hence, $\operatorname{im} \gamma = \mathbf{I}^\bullet_X$. To see this, take the canonical distinguished triangle

$$\longrightarrow l_! l^! \mathrm{Z}^\bullet_X[n] \longrightarrow \mathrm{Z}^\bullet_X[n] \longrightarrow m \;\; m \;\; \mathrm{Z}^\bullet_X[n] \xrightarrow{[1]}$$

and consider a portion of the long exact sequence in $\mathrm{Perv}(X)$ obtained by applying perverse cohomology:

$$\longrightarrow {}^{\mu}H^0(l_! l^! \mathrm{Z}^\bullet_X[n]) \xrightarrow{\beta} \mathrm{Z}^\bullet_X[n] \longrightarrow {}^{\mu}H^0(m \;\; m \;\; \mathrm{Z}^\bullet_X[n]) \longrightarrow .$$

We want to show that ${}^{\mu}H^0(m \;\; m \;\; \mathrm{Z}^\bullet_X[n]) = 0$.

We have

$${}^{\mu}H^0(m \;\; m \;\; \mathrm{Z}^\bullet_X[n]) = {}^{\mu}H^0(m \;\; \mathrm{Z}^\bullet_\Sigma[n]).$$

Then it is trivial that the complex $\mathrm{Z}^\bullet_\Sigma[s]$ satisfies the support condition, and so ${}^{\mu}H^k(\mathrm{Z}^\bullet_\Sigma[s]) = 0$ for $k > 1$. But

$${}^{\mu}H^0(m \;\; \mathrm{Z}^\bullet_\Sigma[n]) = m \;\; {}^{\mu}H^{n-s}(\mathrm{Z}^\bullet_\Sigma[s]),$$

which equals 0 since $n - s > 1$. Therefore, $\beta$ is a surjection and $\operatorname{im} \gamma = \mathbf{I}^\bullet_X$.

Now take the canonical distinguished triangle

$$\longrightarrow m_! m^! \mathrm{Z}^\bullet_X[n] \longrightarrow \mathrm{Z}^\bullet_X[n] \longrightarrow l \;\; l \;\; \mathrm{Z}^\bullet_X[n] \xrightarrow{[1]}$$

and consider a portion of the long exact sequence in $\mathrm{Perv}(X)$ obtained by applying perverse cohomology:

$$\longrightarrow {}^{\mu}H^{-1}(I\ I\ \mathbb{Z}^\bullet_X[n]) \longrightarrow {}^{\mu}H^{0}(m_!m^!\mathbb{Z}^\bullet_X[n])$$
$$\longrightarrow \mathbb{Z}^\bullet_X[n] \xrightarrow{\gamma} {}^{\mu}H^{0}(I\ I\ \mathbb{Z}^\bullet_X[n]) \longrightarrow .$$

We claim that ${}^{\mu}H^{-1}(I\ I\ \mathbb{Z}^\bullet_X[n]) = 0$. This is easy; as $\mathbb{Z}^\bullet_X[n]$ is perverse, $I\ \mathbb{Z}^\bullet_X[n]$ is perverse (since it is the restriction to an open subset) and, in particular, satisfies the cosupport condition. By 10.3.3.iv of [9], $I\ I\ \mathbb{Z}^\bullet_X[n]$ also satisfies the cosupport condition. Thus, ${}^{\mu}H^{-1}(I\ I\ \mathbb{Z}^\bullet_X[n]) = 0$.

Therefore, ${}^{\mu}H^{0}(m_!m^!\mathbb{Z}^\bullet_X[n])$ is the kernel of the map $\gamma$, whose image is $\mathbf{I}^\bullet_X$, i.e., this kernel is how we defined $\mathbf{N}^\bullet_X$, and we are finished.

Now we can prove our main theorem, which we stated in the introduction.

Theorem 2.5. — *In* $\mathrm{Perv}(X)$*, there is an isomorphism*

$$\mathbf{N}^\bullet_X = \ker\{\mathrm{id} - T_f\},$$

*that is, we have a short exact sequence in* $\mathrm{Perv}(X)$*:*

$$0 \longrightarrow \ker\{\mathrm{id} - T_f\} \longrightarrow \mathbb{Z}^\bullet_X[n] \xrightarrow{\tau_X} \mathbf{I}^\bullet_X \longrightarrow 0.$$

*Proof.* — Consider the two nearby-vanishing short exact sequences (2.1):

$$0 \longrightarrow j\ [-1]\mathbb{Z}^\bullet_U[n+1] \longrightarrow \psi_f[-1]\mathbb{Z}^\bullet_U[n+1] \xrightarrow{\mathrm{can}} \varphi_f[-1]\mathbb{Z}^\bullet_U[n+1] \longrightarrow 0$$

and

$$0 \longrightarrow \varphi_f[-1]\mathbb{Z}^\bullet_U[n+1] \xrightarrow{\mathrm{var}} \psi_f[-1]\mathbb{Z}^\bullet_U[n+1] \longrightarrow j^![1]\mathbb{Z}^\bullet_U[n+1] \longrightarrow 0$$

and apply $m_!m^!$ to obtain two distinguished triangles:

$$\longrightarrow m_!m^!j\ [-1]\mathbb{Z}^\bullet_U[n+1] \longrightarrow m_!m^!\psi_f[-1]\mathbb{Z}^\bullet_U[n+1]$$
$$\xrightarrow{m_!m^!\,\mathrm{can}} m_!m^!\varphi_f[-1]\mathbb{Z}^\bullet_U[n+1] \xrightarrow{[1]}$$

and

$$\longrightarrow m_!m^!\varphi_f[-1]\mathbb{Z}^\bullet_U[n+1] \xrightarrow{m_!m^!\,\mathrm{var}} m_!m^!\psi_f[-1]\mathbb{Z}^\bullet_U[n+1]$$
$$\longrightarrow m_!m^!j^![1]\mathbb{Z}^\bullet_U[n+1] \xrightarrow{[1]} .$$

As $j\ [-1]\mathbb{Z}^\bullet_U[n+1] = \mathbb{Z}^\bullet_X[n]$ and the support of $\varphi_f[-1]\mathbb{Z}^\bullet_U[n+1]$ is $\Sigma$, these distinguished triangles become

$$(2.2)\qquad \longrightarrow m_!m^!\mathbb{Z}^\bullet_X[n] \longrightarrow m_!m^!\psi_f[-1]\mathbb{Z}^\bullet_U[n+1]$$
$$\xrightarrow{m_!m^!\,\mathrm{can}} \varphi_f[-1]\mathbb{Z}^\bullet_U[n+1] \xrightarrow{[1]}$$

and

$$(2.3)\qquad \longrightarrow \varphi_f[-1]\mathbb{Z}^\bullet_U[n+1] \xrightarrow{m_! m^! \operatorname{var}} m_! m^! \psi_f[-1]\mathbb{Z}^\bullet_U[n+1] \longrightarrow m_! \hat{m}^![1]\mathbb{Z}^\bullet_U[n+1] \xrightarrow{[1]} .$$

By applying perverse cohomology to (2.2), using that $\varphi_f[-1]\mathbb{Z}^\bullet_U[n+1]$ is perverse, and using the lemma, we immediately conclude that

$$\mathbf{N}^\bullet_X = {}^{\mu}H^0(m_! m^! \mathbb{Z}^\bullet_X[n]) = \ker\left\{{}^{\mu}H^0 m_! m^! \operatorname{can}\right\}.$$

Now, it is an easy exercise to verify that $\hat{m}^! \mathbb{Z}^\bullet_U[2n+2-s]$ satisfies the cosupport condition. This implies that ${}^{\mu}H^k(\hat{m}^! \mathbb{Z}^\bullet_U[2n+2-s]) = 0$ for all $k \leqslant -1$, i.e.,

$${}^{\mu}H^k(m_! \hat{m}^![1]\mathbb{Z}^\bullet_U[n+1]) = m_! {}^{\mu}H^k(\hat{m}^![1]\mathbb{Z}^\bullet_U[n+1]) = 0, \text{ for } k \leqslant -1 + (n-s).$$

As $n - s \geqslant 1$, ${}^{\mu}H^k(m_! \hat{m}^![1]\mathbb{Z}^\bullet_U[n+1])$ is zero for $k \leqslant 0$. Therefore, applying perverse cohomology to (2.3), yields that

$$\varphi_f[-1]\mathbb{Z}^\bullet_U[n+1] \xrightarrow{{}^{\mu}H^0 m_! m^! \operatorname{var}} {}^{\mu}H^0(m_! m^! \psi_f[-1]\mathbb{Z}^\bullet_U[n+1])$$

is an isomorphism.

Finally, we have

$$\begin{aligned}\ker\left\{\mathrm{id} - \widetilde{T}_f\right\} &= \ker\left\{{}^{\mu}H^0 m_! m^!(\mathrm{id} - \widetilde{T}_f)\right\} \\ &= \ker\left\{{}^{\mu}H^0 m_! m^! \operatorname{can} \circ {}^{\mu}H^0 m_! m^! \operatorname{var}\right\} = \ker\left\{{}^{\mu}H^0 m_! m^! \operatorname{can}\right\} = \mathbf{N}^\bullet_X.\end{aligned}$$

By dualizing the above argument, one can easily show:

Theorem 2.6. — *In* $\operatorname{Perv}(X)$*, there exists a short exact sequence*

$$0 \longrightarrow \mathbf{I}^\bullet_X \longrightarrow j^![1]\mathbb{Z}^\bullet_U[n+1] \longrightarrow \operatorname{coker}\{\mathrm{id} - \widetilde{T}_f\} \longrightarrow 0.$$

## 3. Remarks, Applications, and Future Directions

In Theorem 2.5, it was absolutely crucial that we used the vanishing cycles and their monodromy $\widetilde{T}_f$ rather than the nearby cycles and their monodromy $T_f$. Why? Because the nearby-vanishing short exact sequences (2.1), combined with the fact that $\operatorname{var} \circ \operatorname{can} = \mathrm{id} - T_f$, immediately allow us to conclude:

Proposition 3.1. — *There is an isomorphism of perverse sheaves*

$$\mathbb{Z}^\bullet_X[n] = \ker\left\{\mathrm{id} - T_f\right\}.$$

This is very unsatisfying, since it means that $\ker(\mathrm{id} - T_f)$ does not detect the singular set of $X$ at all.

While we have proved Theorem 2.5 with integral coefficients, the proof of the analogous statement with coefficients in an arbitrary field $\mathfrak{K}$ is precisely the same. That is, if $\mathbf{N}^\bullet_X(\mathfrak{K})$ is defined as the kernel of the canonical surjection from the constant sheaf $\mathfrak{K}^\bullet_X[n]$ to the intersection cohomology with constant $\mathfrak{K}$ coefficients $\mathbf{I}^\bullet_X(\mathfrak{K})$, then $\mathbf{N}^\bullet_X(\mathfrak{K})$ is isomorphic to the kernel of

$$\mathrm{id} - \tilde{T}^{\mathfrak{K}}_f : \varphi_f[-1]\mathfrak{K}^\bullet_U[n+1] \longrightarrow \varphi_f[-1]\mathfrak{K}^\bullet_U[n+1].$$

With field coefficients, we proved a simple corollary of Theorem 2.5 in Theorem 3.2 of [13], where we showed that $\ker(\mathrm{id} - \tilde{T}^{\mathfrak{K}}_f) = 0$ if and only if $\mathfrak{K}^\bullet_X[n]$ is isomorphic to $\mathbf{I}^\bullet_X(\mathfrak{K})$.

In fact, if we use $\mathbb{C}$ as our base ring, it is easy to analyze what happens for the other eigenvalues of the monodromy.

We choose real and $\delta$, $0 < \delta$ $1$, such that $B^\circ(\mathbf{0})$ $U$ and

$$B^\circ(\mathbf{0}) \cap f^{-1}(\mathbb{D}^\circ_\delta \backslash \{\mathbf{0}\}) \xrightarrow{\hat{f}} \mathbb{D}^\circ_\delta \backslash \{\mathbf{0}\}$$

is a locally trivial fibration, whose fiber is the Milnor fiber of $f$ at $\mathbf{0}$, where $\hat{f}$ denotes the restriction of $f$. Replace the open set $U$ with the open set $B^\circ(\mathbf{0}) \cap f^{-1}(\mathbb{D}^\circ_\delta)$. On the open dense subset $B^\circ(p) \cap f^{-1}(\mathbb{D}^\circ_\delta \backslash \{\mathbf{0}\})$, we take the local system $L_\xi$, which is in degree $-(n+1)$, has stalk cohomology $\mathbb{C}$ in degree $-(n+1)$ and is given by the representation

$$\pi_1\left(B^\circ(p) \cap f^{-1}(\mathbb{D}^\circ_\delta \backslash \{\mathbf{0}\})\right) \xrightarrow{\hat{f}} \pi_1(\mathbb{D}^\circ_\delta \backslash \{\mathbf{0}\}) = \mathbb{Z} \xrightarrow{h} \mathrm{Aut}(\mathbb{C}),$$

where $\mathbb{D}^\circ_\delta \backslash \{\mathbf{0}\}$ is oriented counterclockwise and $h$ is the homomorphism which takes the generator $1$ to multiplication by $\xi$. Thus, $L_\xi$ is the rank 1 local system, in degree $-(n+1)$, which multiplies by $\xi$ as one goes once around a counterclockwise meridian around the hypersurface $X$.

We let $\mathbf{I}^\bullet_U(\xi)$ denote intersection cohomology using the local system $L_\xi$; this is the intermediate extension of $L_\xi$ to all of $U$. It is a simple object in the category of perverse sheaves; see [2]. Then, it is well-known that $j_*[-1]\mathbf{I}^\bullet_U(\xi)$ and $j^![1]\mathbf{I}^\bullet_U(\xi)$ are perverse, and it is not difficult to show:

Proposition 3.2. — *Using $\mathbb{C}$ as our base ring, and supposing the $\xi = 0$ or $1$, there is an isomorphism of perverse sheaves*

$$j_*[-1]\mathbf{I}^\bullet_U(\xi) = \ker\{\xi^{-1}\mathrm{id} - T_f\} = \ker\{\xi^{-1}\mathrm{id} - \tilde{T}_f\}.$$

*Dually,*

$$j^![1]\mathbf{I}^\bullet_U(\xi) = \mathrm{coker}\{\xi^{-1}\mathrm{id} - T_f\} = \mathrm{coker}\{\xi^{-1}\mathrm{id} - \tilde{T}_f\}.$$

We remark again, as in the introduction, that with field coefficients in the algebraic setting Theorem 2.5 and Proposition 3.2 can be obtained from the work of Beilinson in [1] (see, also, [15]) and the work of M. Saito in [16].

From a formal abstract point of view, Theorem 2.5 is very attractive. But are there applications of Theorem 2.5 to the cohomology of Milnor fibers? In fact, the situation is complicated. It is *not* true, in general, that cohomology of the stalk of the kernel is isomorphic to the kernel of the cohomology on the stalks, i.e., there may exist $x \in \Sigma$ and degrees $k$ such that

$$H^k\left(\ker\left(\mathrm{id} - \widetilde{T}_f\right)\right)_x \neq \ker\left(\mathrm{id} - (\widetilde{T}_f)^k_x\right),$$

where $(\widetilde{T}_f)^k_x$ is the induced map on $H^k(\varphi_f[-1]\mathbb{Z}^\bullet_U[n+1])_x$.

However, if $s = \dim_x \Sigma$, then it is easy to show that

$$H^{-s}\left(\ker\left(\mathrm{id} - \widetilde{T}_f\right)\right)_x \cong \ker\left(\mathrm{id} - (\widetilde{T}_f)^{-s}_x\right).$$

From this and Theorem 2.5, and using topological indexing for the reduced intersection cohomology $I\widetilde{H}$, one concludes:

**Proposition 3.3.** — *Let $x \in \Sigma$ and let $s := \dim_x \Sigma$. Let*

$$H^{n-s}(F_{f,x};\mathbb{Z}) \xrightarrow{(\widetilde{T}_f)^{-s}_x} H^{n-s}(F_{f,x};\mathbb{Z})$$

*be the standard Milnor monodromy on the degree $(n-s)$ cohomology of the Milnor fiber of $f$ at $x$.*

*Then, there are isomorphisms*

$$\ker\left(\mathrm{id} - (\widetilde{T}_f)^{-s}_x\right) \cong \mathbb{Z}^{\omega} \cong I\widetilde{H}^{\,n-s-1}(B^\circ(x)\cap X;\mathbb{Z}),$$

*where* $0 < \quad 1$ *and*

$$\omega := \begin{cases} \operatorname{rank} H^{n+s}(K_{X,x};\mathbb{Z}), & \text{if } s = n-1;\\ -1 + \operatorname{rank} H^{n+s}(K_{X,x};\mathbb{Z}), & \text{if } s = n-1.\end{cases}$$

*Remark 3.4.* — We should point out that one does not need Theorem 2.5 to prove the ordinary cohomology statement in Proposition 3.3. As in our discussion in the introduction which led to Proposition 1.1, Milnor's work in Section 8 of [14] tells us how the homology/cohomology of the complement of the real link $K_{X,x}$ inside $S^{2n+1}$ relates to the kernel of $\mathrm{id} - (\widetilde{T}_f)^{-s}_x$. Then, using Alexander Duality, one can recover the first isomorphism in Proposition 3.3.

Also, related to the case where $s = n-1$, it is well-known that the rank of $H^{2n-1}(K_{X,x};\mathbb{Z})$ is equal to the rank of $IH^0(B^\circ(x)\cap X;\mathbb{Z})$, which is equal to the number of irreducible components of $X$ at $x$.

Another application of the short exact sequence in Theorem 2.5 is that, for each $x \in X$, we can apply the vanishing cycle functor, $\varphi_L[-1]$, which is exact on $\mathrm{Perv}(X)$, where $L$ is the restriction to $X$ of a generic affine linear form such that $L(x) = 0$. We then obtain a short exact sequence of $\mathbb{Z}$-modules:

$$(3.1)\quad 0 \longrightarrow \ker\left\{\mathrm{id} - (\varphi_L[-1]T_f)^0_x\right\} \longrightarrow H^0\left(\varphi_L[-1]\mathbb{Z}^\bullet_X[n]\right)_x \longrightarrow H^0\left(\varphi_L[-1]\mathbf{I}^\bullet_X\right)_x \longrightarrow 0,$$

where $(\varphi_L[-1]T_f)^0_x$ is the automorphism induced by the $f$-monodromy on

$$H^0\left(\varphi_L[-1]\varphi_f[-1]\mathbb{Z}^\bullet_U[n+1]\right)_x = \mathbb{Z}^{\lambda^0_{f,L}(x)},$$

where $\lambda^0_{f,L}(x)$ is the 0-th Lê number of $f$ with respect to $L$ at $x$ (see [12]).

Now, rank $H^0\left(\varphi_L[-1]\mathbf{I}^\bullet_X\right)_x$ is the coefficient of $\{x\}$ in the characteristic cycle of intersection cohomology; this is a great importance in some settings (see, for instance, [3] for a discussion). (There are different shifting/sign conventions on the characteristic cycle; we are using a convention that, for a perverse sheaf, gives us that all of the coefficients are non-negative.) However, without the language of the derived category and using topological indexing, $H^0\left(\varphi_L[-1]\mathbf{I}^\bullet_X\right)_x$ is isomorphic to

$$\mathrm{coker}\left\{IH^{n-1}(B^\circ(x) \cap X;\, \mathbb{Z}) \xrightarrow{r_{x,x}} IH^{n-1}(B^\circ(x) \cap X \cap L^{-1}(y);\, \mathbb{Z})\right\},$$

where $r_{x,x}$ is induced by the restriction and, as before, $L$ is a generic affine linear form such that $L(x) = 0$ and $0 < |y| \ll 1$.

Furthermore, as the result of Lê in [10] tells us that the complex link $\mathbb{L}_{X,x}$ of $X$ at $x$ has the homotopy-type of a bouquet of $(n-1)$-spheres, the number of spheres in this homotopy-type is precisely rank $H^0\left(\varphi_L[-1]\mathbb{Z}^\bullet_X[n]\right)_x$, which is known to equal the intersection number $\left(\Gamma^1_{f,L} \cdot V(L)\right)_x$, where $\Gamma^1_{f,L}$ is the relative polar curve of $f$ with respect to $L$; see Corollary 2.6 of [11] (though this was known earlier by Hamm, Lê, Siersma, and Teissier).

Now, in the case where $\dim \Sigma_X = 1$, we can conclude that the short exact sequence (3.1) tells us:

**Corollary 3.5.** — *Suppose that* $\dim_x \Sigma = 1$ *and that* $\dim_x \Sigma(f_{|_{V(L)}}) = 0$. *Then, there is an equality*

$$\mathrm{rank}\,\mathrm{coker}\{r_{x,x}\} = \left(\Gamma^1_{f,L} \cdot V(L)\right)_x - \mathrm{rank}\ker\left\{\mathrm{id} - T_{f,L}\right\},$$

*where*

$$r_{x,x} : IH^{n-1}(B^\circ(x) \cap X;\, \mathbb{Z}) \longrightarrow IH^{n-1}(B^\circ(x) \cap X \cap L^{-1}(y);\, \mathbb{Z}),$$

*is induced by restriction, and where* $\widetilde{T}_{f,L}$ *is the automorphism induced by the Milnor monodromy of* $f$ *on the relative cohomology*

$$H^n\left(F_{f,x}, \bigcup F_{f,x_i}; \mathbb{Z}\right),$$

*where the union is over* $x_i \in B^\circ(x) \cap \Sigma \cap L^{-1}(y)$*, where* $0 < |y| \ll 1$*, and* $F_{f,x}$ *and* $F_{f,x_i}$ *denote the Milnor fibers of* $f$ *at the respective points.*

However, Corollary 3.5 is really a pseudo-application of Theorem 2.5; it is difficult to use this corollary to obtain information about either the Milnor fiber or the intersection cohomology.

Finally, without worrying about more down-to-Earth applications, one could look for generalizations of the abstract result in Theorem 2.5. One obvious generalization would be to functions on local complete intersections. Other generalizations could involve functions on arbitrarily singular spaces $U$, but replacing the initial shifted constant sheaf on $U$ with either the intersection cohomology sheaf or the perverse cohomology of the constant sheaf. Or one could try starting with an arbitrary perverse sheaf on $U$.

We have looked at all of these generalizations, but have yet to obtain any nice results.

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