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SINGULAR SUBALGEBROIDS

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With an appendix by Iakovos ANDROULIDAKIS

ABSTRACT. — We introduce singular subalgebroids of an integrable Lie algebroid, extending the notion of Lie subalgebroid by dropping the constant rank requirement. We lay the bases of a Lie theory for singular subalgebroids: we construct the associated holonomy groupoids, adapting the procedure of Androulidakis—Skandalis for singular foliations, in a way that keeps track of the choice of Lie groupoid integrating the ambient Lie algebroid. In the regular case, this recovers the integration of Lie subalgebroids by Moerdijk—Mrčun. The holonomy groupoids are topological groupoids, and are suitable for noncommutative geometry as they allow for the construction of the associated convolution algebras. Further we carry out the construction for morphisms in a functorial way.

RÉSUMÉ. — Nous introduisons les sous-algébroïdes singuliers d'un algébroïde de Lie intégrable, qui étendent la notion de sous-algébroïde de Lie par la suppression de la condition de rang constant. Nous posons les bases d'une théorie de Lie pour les sous-algébroïdes singuliers : nous construisons les groupoïdes d'holonomie associés, en adaptant la procédure d'Androulidakis—Skandalis pour les feuilletages singuliers, de manière à tenir compte du choix du groupoïde de Lie intégrant l'algébroïde de Lie. Dans le cas régulier, on retrouve bien l'intégration des sous-algébroïdes de Lie par Moerdijk—Mrčun. Les groupoïdes d'holonomie sont des groupoïdes topologiques, et ils sont adaptés à la géométrie non commutative car ils permettent la construction des algèbres de convolution associées. Enfin, nous réalisons la construction des morphismes de manière fonctorielle.

Introduction

Lie algebroids arise in differential geometry, mathematical physics and control theory. The standard viewpoint is to declare their sub-objects to be

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(wide) Lie subalgebroids, i.e. involutive constant-rank subbundles. However there is a multitude of interesting "singular" examples that violate the constant-rank requirement. This leads us to introduce here a new class of subalgebroids, quite more singular than the usual Lie subalgebroids: we call them singular subalgebroids.

Our aim is to build a Lie theory for singular subalgebroids. In this paper we construct a topological groupoid canonically associated to them, called *holonomy groupoid*, which depends on a choice of integration G of the ambient Lie algebroid. The construction parallels the one of [1], and here too the holonomy groupoid is a topological groupoid. This construction encompasses the integration of wide Lie subalgebroids by Moerdijk–Mrčun [18] and the holonomy groupoids of singular foliations of Androulidakis–Skandalis [1]. A novel feature is the presence of many interesting morphisms. We prove a version Lie's second theorem in this context (integration of morphisms), showing that our holonomy groupoid construction is functorial.

Building on the present work, in a follow-up paper with Androulidakis [7] we provide a version of Lie's third theorem, making precise how one can view the holonomy groupoid as an "integration" of the singular subalgebroid. This requires us to work in the realm of diffeological groupoids. In that paper we also show that although the holonomy groupoid is not smooth, it is still possible to do differential geometry on it.

Finally here, using the holonomy groupoid we attach a C^* -algebra to a singular subalgebroid, paving the way to the development of pseudo-differential calculus, index theory, and other noncommutative geometry constructions for such structures.

Recently Laurent-Gengoux–Lavau–Strobl [15, 16] showed that singular foliations are tightly connected with higher algebraic structures: under reasonable assumptions, a singular foliation admits a canonical L_{∞} -algebroid which "resolves" it and which provides fine invariants. We expect their construction to extend to singular subalgebroids.

Singular subalgebroids

Fix a Lie algebroid A over a manifold M. A singular subalgebroid is a $C^{\infty}(M)$ -submodule \mathcal{B} of $\Gamma_c(A)$ (the module of compactly supported sections of A), which is locally finitely generated and closed w.r.t. the Lie bracket.

Let us display two obvious classes of singular subalgebroids, whose intersection consists exactly of the regular foliations. Example (Wide Lie subalgebroids). — Let B be a wide Lie subalgebroid of A, i.e. a Lie subalgebroid supported on the whole of M. Then $\Gamma_c(B)$ is a singular subalgebroid.

Example (Singular foliations). — The singular subalgebroids of A = TM are exactly the singular foliations on M. Here singular foliation is meant in the sense of [1], a notion inspired by the work of Stefan and Sussman in the 1970's.

There is an interesting class that strictly contains the first one: the singular subalgebroids \mathcal{B} which are *projective*, i.e., so that there exists a vector bundle over M whose module of compactly supported sections is \mathcal{B} . Notice that such a vector bundle is then a Lie algebroid (but not necessarily a Lie subalgebroid of A).

In turn, projective subalgebroids are contained in a larger class, that of singular subalgebroids which are images of Lie groupoid morphisms covering the identity. Other examples of singular subalgebroids will be given in Section 1.3.

Singular subalgebroids can also be viewed as a nice class of Lie–Rinehart algebras [20], more general than Lie algebroids.

Main results

For singular foliations \mathcal{F} on M, which as we saw are exactly the singular subalgebroids of TM, the holonomy groupoid was constructed by Androulidakis–Skandalis [1]. There the crucial idea was that of a bisubmersion. Bisubmersions are manifolds U endowed with two submersive maps to M, and are defined locally from the data provided by the singular foliation. Their dimension is variable, and the holonomy groupoid $H(\mathcal{F})$ is a quotient of a disjoint union of bisubmersions.

For singular subalgebroids \mathcal{B} of an integrable Lie algebroid A, after choosing an integrating Lie groupoid G, taking a new point of view we extend the notion of [1] by defining bisubmersions to be smooth maps $U \to G$ satisfying certain conditions. With this notion we can construct the holonomy groupoid $H^G(\mathcal{B})$ in a way analogous to [1].

A feature of the construction we give here is that it keeps track of the choice of Lie groupoid G integrating A. More precisely, the holonomy groupoid $H^G(\mathcal{B})$ comes together with a canonical morphism to G. For instance the groupoid $H(\mathcal{F})$ given in [1], in the current context, is the holonomy groupoid associated to \mathcal{F} (viewed as a singular subalgebroid) when we choose G to be the pair groupoid $M \times M$. The canonical morphism to G is just the target-source map.

The main result of the paper is Theorem 3.8, which can be paraphrased in a simplified way as follows:

Theorem A. — Let \mathcal{B} be a singular subalgebroid of an integrable Lie algebroid A, and G a Lie groupoid integrating A. There exists a canonical map

$$\Phi \colon H^G(\mathcal{B}) \to G$$

where

- (1) $H^G(\mathcal{B})$ is a topological groupoid which is "nice" and "integrates \mathcal{B} ",
- (2) Φ is a topological groupoid morphism "integrating" the inclusion $\iota \colon \mathcal{B} \hookrightarrow \Gamma_c(A)$.

In joint work with Androulidakis [7]

- we show that $H^G(\mathcal{B})$ has some smoothness properties: first, it is leafwise smooth in the sense that its restriction to the leaves of \mathcal{B} are Lie groupoids [7, §2], and second, it has a structure of diffeological groupoid that allows to recover \mathcal{B} [7, §5.3, §6.4]. This is what we mean by "nice" and "integrates \mathcal{B} " in (1) above.
- we consider certain diffeological groupoids endowed with maps to Lie groupoids, and show that a morphism of such objects always induces a morphism of singular subalgebroids [7, §5.4]. Together with Example 4.12, this explains "integrating" in (2) above. Further, for every leaf L of \mathcal{B} , the map $H^G(\mathcal{B})|_L \to G$ obtained restricting Φ is a Lie groupoid morphism integrating the Lie algebroid morphism $\mathcal{B}_L \to A$ induced by the inclusion ι , see [7, §2.5]. Here \mathcal{B}_L is a transitive Lie algebroid over L; when L is an embedded leaf, its sections are $\mathcal{B}/I_L\mathcal{B}$ for I_L the ideal of functions on M vanishing on L.

Further, for singular subalgebroids which are images of Lie groupoid morphisms (this includes singular foliations), we give a minimality property for $H^G(\mathcal{B})$ in Proposition 3.17. This minimality property was postulated first by Moerdijk–Mrčun for wide Lie subalgebroids and is satisfied by the holonomy groupoid of a singular foliation.

The construction of the holonomy groupoid $H^G(\mathcal{B})$ allows to transfer almost verbatim the construction of the convolution *-algebra of [1], see Appendix A by Iakovos Androulidakis. Recall [1, 2, 3] that in the case of singular foliations the K-theory of the corresponding C^* -algebra is the recipient of the analytic index for longitudinal elliptic pseudodifferential operators.

A feature of singular subalgebroids compared to singular foliations is that morphisms abound. In Theorem 4.6 we show that the holonomy groupoid construction extends to morphisms covering the identity. (For more general morphisms we refer to Appendix D.) This provides new statements even for singular foliations.

THEOREM B. — Let $F: G_1 \to G_2$ be a morphism of Lie groupoids covering Id_M . Let \mathcal{B}_i be a singular subalgebroid of $\mathrm{Lie}(G_i)$ for i=1,2, such that $F_*(\mathcal{B}_1) \subset \mathcal{B}_2$. Then there is a canonical morphism of topological groupoids

$$\Xi \colon H^{G_1}(\mathcal{B}_1) \to H^{G_2}(\mathcal{B}_2)$$

covering Id_M and making the following diagram commute:

$$H^{G_1}(\mathcal{B}_1) - \stackrel{\Xi}{\longrightarrow} H^{G_2}(\mathcal{B}_2)$$

$$\downarrow^{\Phi_1} \qquad \qquad \downarrow^{\Phi_2}$$

$$G_1 \xrightarrow{F} G_2$$

The Moerdijk-Mrčun integration of wide Lie subalgebroids

Theorem A extends and unifies previous results by Moerdijk–Mrčun [18] and Androulidakis–Skandalis [1] for the two obvious classes of singular subalgebroids displayed above – wide Lie subalgebroids and singular foliations. We now elaborate on the first class, as

it is instructive to compare Theorem A with the results of Moerdijk–Mrčun in [18].

Let $A \to M$ be a Lie algebroid, and fix a Lie groupoid G integrating A. Let $B \to M$ be a wide Lie subalgebroid of A. Moerdijk–Mrčun show:

THEOREM ([18, Thm. 2.3]). — There exists a unique map

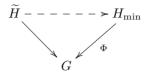
$$\Phi \colon H_{\min} \to G$$

where

- (1) H_{\min} is a Lie groupoid integrating B,
- (2) Φ is a Lie groupoid morphism integrating the inclusion $\iota \colon B \hookrightarrow A$,
- (3) minimality property: for any Lie groupoid morphism $\widetilde{H} \to G$ integrating⁽¹⁾ ι , there exists a surjective Lie groupoid morphism $\widetilde{H} \to G$

 $^{^{(1)}}$ So \widetilde{H} is necessarily a Lie groupoid integrating B, and the morphism is an immersion.

 H_{\min} integrating Id_B and making this diagram commute:



Moerdijk-Mrčun refer to H_{\min} as the minimal integral of B over G. By 3) above, H_{\min} is unique up to isomorphism. The construction of Theorem A, applied to $\mathcal{B} := \Gamma_c(B)$, yields exactly the minimal integral H_{\min} together with the above Lie groupoid morphism Φ , see Proposition 3.20.

To put this result into context, recall that the wide Lie algebroid B is integrable (because A is), and that the inclusion ι integrates to a morphism $H_{\text{max}} \to G$, where H_{max} is the source simply connected Lie groupoid integrating B. All other Lie groupoids integrating B are quotients of H_{max} . In general they do not admit a morphism to G integrating ι , and H_{min} is the "smallest" integration admitting such a morphism. We want to stress that the result of Moerdijk–Mrčun, just as our Theorem A, does not contain as a special case the integration of Lie algebroids (indeed, an integration G of the Lie algebroid A is part of the hypotheses).

As an example of the above theorem, take the case A = TM. Then a wide Lie subalgebroid is just an involutive distribution, which by the Frobenius theorem corresponds of a (regular) foliation on M. The Lie groupoids $H_{\rm max}$ and $H_{\rm min}$ are nothing else than the monodromy and holonomy groupoids of this foliation.

Notice that the above theorem of Moerdijk–Mrčun starts with a Lie subalgebroid B of A (rather than with an abstract Lie algebroid B), and that it produces a Lie groupoid morphism to G integrating the inclusion $\iota \colon B \hookrightarrow A$ (rather than only a Lie groupoid integrating B). In other words, in the above theorem H_{\min} is naturally endowed with a morphism $\Phi \colon H_{\min} \to G$. Notice that it would not be wise to disregard this morphism and consider only its image $\Phi(H_{\text{proj}})$. First, the latter is a set-theoretic subgroupoid of G, which usually fails to be smooth. (When B is an involutive distribution on M, $\Phi(H_{\min})$ is the graph of the equivalence relation given by the associated regular foliation, and its failure to be smooth was one of the reasons to introduce the holonomy groupoid in the first place, see the remarks in [19]). Second, the morphism of Lie groupoids Φ is usually not injective and hence contains more information than its image.

Androulidakis–Skandalis' holonomy groupoids of singular foliations

We now highlight the aspects of this work that represent the main novelties in comparison with the work of Androulidakis–Skandalis [1]. (For singular foliations, the construction of Theorem A yields the holonomy groupoid of Androulidakis–Skandalis.) Let G be a Lie groupoid and \mathcal{B} be a singular subalgebroid of $A := \operatorname{Lie}(G)$.

- Our definition of bisubmersion for \mathcal{B} (Definition 2.2) is not a straight-forward generalization of the one of [1]. Ours is given by a smooth map to G, which typically fails to be a submersion. In the case that \mathcal{B} is a singular foliation, our definition does not recover on the nose the notion of bisubmersion from [1], but it corresponds bijectively to it if one assembles two submersions into M to one map to the pair groupoid $M \times M$ (Proposition 2.10). Just as in [1], bisubmersions for \mathcal{B} have the following features: their bisections induce automorphisms of the singular subalgebroid \mathcal{B} (see Remark 2.33), and they allow for the construction of the holonomy groupoid by providing its "building blocks" (Theorem 3.8).
- The holonomy groupoid depends of the choice of Lie groupoid G integrating A. In Section 3.4 we display how the holonomy groupoid changes if we replace G by another Lie groupoid of which G is a quotient. For singular foliations, i.e. when A = TM, there is a canonical choice for G, namely the pair groupoid $M \times M$. With this choice we recover the holonomy groupoid of a singular foliation of [1].
- Morphisms between Lie algebroids abound, even in the special case of morphisms covering the identity on the base (take for instance the anchor map). We show that when such a morphism maps a singular subalgebroid into another, there is a canonically induced morphisms between the corresponding holonomy groupoids (Theorem B), and that this assignment is functorial. When one restricts to singular foliations, there are not as many morphisms, and the natural ones are given by smooth maps between manifolds with singular foliations. Their effect at the level of holonomy groupoids is not considered in [1] and plays an important role in [11].

Conventions. — All Lie groupoids are assumed to be source connected, not necessarily Hausdorff, but with Hausdorff source-fibers. Given a Lie groupoid $G \rightrightarrows M$, we denote by \mathbf{t} and \mathbf{s} its target and source maps, and

by $i : G \to G$ the inversion map. We denote by $1_x \in G$ the identity element corresponding to a point $x \in M$, and by $1_M \subset G$ the submanifold of identity elements. Two elements $g, h \in G$ are composable if $\mathbf{s}(g) = \mathbf{t}(h)$. We identify the Lie algebroid of G, which we sometimes denote by $\mathrm{Lie}(G)$, with $\mathrm{ker}(\mathrm{d}\mathbf{s})|_M$.

We use the term "generators" only in the context of modules over $C^{\infty}(U)$, for U a manifold, and to mean that they generate as a $C^{\infty}(U)$ -module.

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1. Singular subalgebroids

In this section we introduce the notion of singular subalgebroid, give several examples, and in Section 1.4 make some observations for later use. Throughout this section we are going to consider a Lie algebroid $A \to M$ with anchor $\rho: A \to TM$ (see for instance [8, 9, 17]).

1.1. Definition of singular subalgebroid

We define the main object of interest of this paper:

DEFINITION 1.1. — A singular subalgebroid of A is an involutive, locally finitely generated $C^{\infty}(M)$ -submodule \mathcal{B} of $\Gamma_c(A)$.

The notion of singular subalgebroid is obtained from the notion of wide Lie subalgebroid, by dropping the requirement of being a (constant rank) subbundle of A. This is achieved by focusing on the $C^{\infty}(M)$ -module $\Gamma_c(A)$ of compactly supported sections of A, rather than on the Lie algebroid A itself.

For any $C^{\infty}(M)$ -submodule \mathcal{B} of $\Gamma_c(A)$, we define its global hull [1, §1.1][7] to be

$$\widehat{\mathcal{B}} := \{ Z \in \Gamma(A) : fZ \in \mathcal{B} \text{ for all } f \in C_c^{\infty}(M) \}.$$

It is a $C^{\infty}(M)$ -submodule of $\Gamma(A)$ containing \mathcal{B} .

A subset \mathcal{G} of $\widehat{\mathcal{B}}$ is said to be a generating set for \mathcal{B} if $\mathcal{B} = \operatorname{Span}_{C_c^{\infty}(M)}(\mathcal{G})$, where the latter is the set of finite $C_c^{\infty}(M)$ -linear combinations of elements of \mathcal{G} .

Now we can explain the meaning of \mathcal{B} being "locally finitely generated" in Definition 1.1: it means that for every point of M there is an open neighbourhood $i \colon U \hookrightarrow M$ such that the submodule

$$i^*\mathcal{B} \coloneqq \{Z|_U : Z \in \mathcal{B} \text{ has support in } U\}$$

of $\Gamma_c(A|_U)$ admits a finite generating set. In other words, there are finitely many $Y_1, \ldots, Y_n \in \widehat{i^*\mathcal{B}}$ such that (2) every element of $i^*\mathcal{B}$ is a $C_c^{\infty}(U)$ -linear combination of the Y_j 's.

1.2. Motivating examples

This notion of singular subalgebroid is motivated by the following two special cases (whose intersection are exactly the regular foliations).

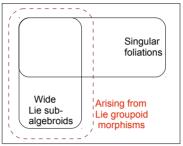
Example 1.2 (Singular foliations). — Recall from [1] that a singular foliation on a manifold M is an involutive, locally finitely generated submodule of the $C^{\infty}(M)$ -module of vector fields with compact support $\mathfrak{X}_c(M)$. The singular subalgebroids of A = TM are exactly the singular foliations on M.

Example 1.3 (Wide Lie subalgebroids). — Recall from [17, Def. 3.3.21] that a wide Lie subalgebroid of A is a vector subbundle $B \to M$, whose sections are closed with respect to the Lie bracket. In this case, $\Gamma_c(B)$ is a singular subalgebroid.

Example 1.3 belongs to the larger class of singular subalgebroids arising from Lie groupoid morphism, which we introduce in Section 1.3.1. In this paper we will focus mainly on these 3 classes, which at present are the only ones for which we are able to describe in an explicit way the holonomy

⁽²⁾ Explicitly, $\widehat{i^*\mathcal{B}} = \{Y \in \Gamma(A|_U) : fY \in i^*\mathcal{B} \text{ for all } f \in C_c^\infty(U)\}.$

groupoid.



Singular subalgebroids

1.3. Further examples

Let us now display four geometric contexts in which singular subalgebroids arise. Both Section 1.3.1 and Section 1.3.3 contain as a special case wide Lie subalgebroids (Example 1.3 above).

1.3.1. Arising from Lie algebroid morphisms

Let $\psi \colon E \to A$ be a morphism of Lie algebroids covering the identity on the base manifolds. Then the image of the induced map of compactly supported sections,

$$\mathcal{B} \coloneqq \psi(\Gamma_c(E)),$$

is a singular subalgebroid of A. We say that \mathcal{B} arises from the Lie algebroid morphism ψ .

(The above can be vastly generalized, replacing $\Gamma_c(E)$ by any singular subalgebroid of E, and by allowing ψ to cover a diffeomorphism of the base or even a surjective submersion (see Lemma D.1).)

Remark 1.4. — Given two Lie algebroids $A_1 \to M_1$ and $A_2 \to M_2$, there is a notion of $comorphism^{(3)}$ from A_1 to A_2 (see [17, Def. 4.3.16]). It induces a linear map $\Gamma(A_2) \to \Gamma(A_1)$. The $C_c^{\infty}(M_1)$ -module generated by its image is a singular subalgebroid of A_1 . Example 1.9 below is of this kind, since a Poisson map between Poisson manifolds $M_1 \to M_2$ induces a comorphism from T^*M_1 to T^*M_2 .

⁽³⁾ That is, a pair (Φ, f) where $f \colon M_1 \to M_2$ is any differentiable map and $\Phi \colon f^!A_2 \to A_1$ is a vector bundle map over Id_{M_1} , where $f^!A_2$ denotes the pullback of the vector bundle A_2 via f, such that the induced map of sections $\Gamma(A_2) \to \Gamma(A_1)$ preserves the Lie bracket and the anchor maps satisfy $f_* \circ \rho_{A_1} \circ \Phi = \rho_{A_2}$.

Examples 1.5.

(1) A singular subalgebroid \mathcal{B} of A is called *projective* if there exists a vector bundle B over M such that $\Gamma_c(B) \cong \mathcal{B}$ as $C^\infty(M)$ -modules. In that case, there is [7] a Lie algebroid structure on B and almost injective Lie algebroid morphism $\tau \colon B \to A$ inducing the isomorphism $\Gamma_c(B) \cong \mathcal{B}$, and these data are unique. In particular, \mathcal{B} arises from the Lie algebroid morphism τ .

A special case occurs when \mathcal{B} is the space of compactly supported sections of a wide Lie subalgebroid B of A. In that case $\tau \colon B \to A$ is the inclusion.

- (2) Given any Lie algebroid A, the anchor map $\rho \colon A \to TM$ is a Lie algebroid morphism. In this case $\mathcal{B} := \rho(\Gamma_c(A))$ is the singular foliation underlying A. Further, any Lie algebroid morphism (covering the identity) giving rise to \mathcal{B} must be the anchor map of a Lie algebroid.
- (3) Let A be a Lie algebroid. A Nijenhuis operator [14] is an endomorphism of vector bundles $N \colon A \to A$ over Id_M , whose Nijenhuis torsion $T_N(X,Y) := [NX,NY] N[X,Y]_N$ vanishes. Here $[X,Y]_N := [NX,Y] + [X,NY] N[X,Y]$. In this case N is a Lie algebroid morphism from $(A,[\cdot,\cdot]_N)$ to $(A,[\cdot,\cdot])$, so $\mathcal{B} := N(\Gamma_c(A))$ is a singular subalgebroid of the original Lie algebroid $(A,[\cdot,\cdot])$.

For later use we make the following definition.

DEFINITION 1.6. — Let \mathcal{B} be a singular subalgebroid of an integrable Lie algebroid A over M. We say that \mathcal{B} arises from a Lie groupoid morphism (covering the identity) if there is a Lie groupoid morphism $\Psi \colon K \to G$ over Id_M , where G is any Lie groupoid integrating A, such that

$$\mathcal{B} = \Psi_*(\Gamma_c(\mathrm{Lie}(K))).$$

Examples include compactly supported sections of wide Lie subalgebroids of A, for the latter are integrable. Clearly Definition 1.6 implies that $\mathcal B$ arises from a Lie algebroid morphism, namely $\Psi_*\colon \operatorname{Lie}(K)\to A$. Conversely, if singular subalgebroid arises from a Lie algebroid morphism $\psi\colon E\to A$ with E an integrable Lie algebroid, then this singular subalgebroid arises from a Lie groupoid morphism (namely, the Lie groupoid morphism $\Psi\colon K\to G$ integrating ψ , where K is the source simply connected Lie groupoid integrating E).

1.3.2. Globally finitely generated singular subalgebroids

Let $\alpha_1, \ldots, \alpha_n \in \Gamma(A)$ satisfying the following involutivity condition: For every $1 \leq i, j \leq n$ there exist smooth functions $f_{ij}^1, \ldots, f_{ij}^n \in C^{\infty}(M)$ such that $[\alpha_i, \alpha_j] = \sum_{k=1}^n f_{ij}^k \alpha_k$. Then the $C^{\infty}(M)$ -submodule of $\Gamma(A)$

$$\mathcal{B} := C_c^{\infty}(M)\alpha_1 + \dots + C_c^{\infty}(M)\alpha_n$$

is a singular subalgebroid.

Examples 1.7.

- (1) Given a single section $\alpha \in \Gamma(A)$, the $C^{\infty}(M)$ -module $\mathcal{B} = C_c^{\infty}(M)\alpha$ is a singular subalgebroid.
- (2) Let \mathfrak{g} be a Lie algebra and $\varphi \colon \mathfrak{g} \to \Gamma(A)$ a Lie algebra morphism. Defining \mathcal{B} as the $C_c^{\infty}(M)$ -span of $\{\varphi(v):v\in\mathfrak{g}\}$ we obtain a singular subalgebroid of the kind above. As α_1,\ldots,α_n we can take the image of a basis of \mathfrak{g} ; notice that in this case the functions f_{ij}^k are constant. This example also falls⁽⁴⁾ into the class considered in Section 1.3.1.

A concrete example is the following. Let (M, ω) be a symplectic manifold, \mathfrak{g} a Lie algebra, and $J \colon M \to \mathfrak{g}^*$ the moment map for some hamiltonian action on M. Then the comoment map (pullback of functions) $J^* \colon \mathfrak{g} \to C^{\infty}(M)$ delivers a Lie algebra morphism into the central extension of TM by the trivial vector bundle $M \times \mathbb{R}$ twisted by $\omega^{(5)}$:

$$\mathfrak{g} \to \Gamma(TM \oplus_{\omega} (M \times \mathbb{R})), v \mapsto (X_{J^*v}, J^*v)$$

where X_{J^*v} is the Hamiltonian vector field of $J^*v \in C^{\infty}(M)$. When ω is an integral 2-form, $TM \oplus_{\omega} (M \times \mathbb{R})$ is the Atiyah algebroid of a circle bundle prequantizing (M, ω) .

1.3.3. From Lie subalgebroids supported on submanifolds

Recall that a *Lie subalgebroid* of A over a closed embedded submanifold⁽⁶⁾ N of M (see [17, Def. 4.3.14]) is a subbundle $B \to N$, such that:

⁽⁴⁾ Indeed \mathcal{B} is the image of the Lie algebroid morphism $\mathfrak{g} \times M \to A, (v, x) \mapsto (\varphi(v))|_x$ over Id_M , where $\mathfrak{g} \times M$ is the transformation Lie algebroid of the infinitesimal action $\mathfrak{g} \to \mathfrak{X}(M), v \mapsto \rho(\varphi(v))$ induced by φ and the anchor of A.

⁽⁵⁾ Recall that $TM \oplus_{\omega} (M \times \mathbb{R})$ is a Lie algebroid with the bracket $[X \oplus V, Y \oplus W] = [X, Y] \oplus \{X(W) - Y(V) - \omega(X, Y)\}.$

⁽⁶⁾ A Lie subalgebroid of A over an immersion $\iota: N \to M$ can be defined as well: It is a vector bundle $B \to N$ together with a vector bundle morphism $j: B \to A$ over $\iota: N \to M$ such that: i) $\rho(j(B)) \subset \iota_*(TN)$, ii) $\widetilde{\Gamma}(B) \coloneqq \{\alpha \in \Gamma(A) \colon \alpha|_{\iota(N)} \subset j(B)\}$ is involutive, iii) $[\widetilde{\Gamma}(B), \widetilde{\Gamma}(0_N)] \subset \widetilde{\Gamma}(0_N)$.

- (1) $\rho(B) \subset TN$,
- (2) $\widetilde{\Gamma}(B) := \{ \alpha \in \Gamma(A) : \alpha|_N \subset B \}$ is involutive,
- (3) $[\widetilde{\Gamma}(B), \widetilde{\Gamma}(0_N)] \subset \widetilde{\Gamma}(0_N)$, where $\widetilde{\Gamma}(0_N) := \{ \alpha \in \Gamma(A) : \alpha|_N = 0 \}$.

If B is a Lie subalgebroid of A over a closed embedded submanifold N, then

$$\mathcal{B} := \{ \boldsymbol{\alpha} \in \Gamma_c(A) : \boldsymbol{\alpha}|_N \subset B \}$$

is a singular subalgebroid of A.

Let us describe \mathcal{B} near a point p of N. Choose coordinates $\{x_i\}$ around p adapted to N, i.e. $\{x_i\}_{i>n}$ vanish on N and $\{x_i\}_{i\leqslant n}$, once restricted to N, provide coordinates there (here $n=\dim(N)$). Let $\{\alpha_j\}$ be a frame of compactly supported sections of A adapted to B, i.e. $\{\alpha_j|_N\}_{j\leqslant b}\subset B$ where $b=\operatorname{rank}(B)$. Then \mathcal{B} , locally near p, is generated by

$$\{\boldsymbol{\alpha}_j\}_{j\leqslant b}\cup\{x_i\cdot\boldsymbol{\alpha}_j\}_{i>n,j>b},$$

while on open sets disjoint from N, \mathcal{B} is just given by restrictions of compactly supported sections of A. When N has codimension one in M, \mathcal{B} is projective (see Definition 1.5). If $\operatorname{codim}(N) \geq 2$ and $B \neq A|_N$, then \mathcal{B} is not projective, because the number of generators above is strictly larger than $\operatorname{rank}(A)$.

Examples 1.8.

- (1) Let A = TM. Let B be the zero vector subbundle over N. Then \mathcal{B} consists of the vector fields on M which vanish at points of N.
- (2) When N has codimension one in M, as mentioned above, B is a projective singular subalgebroid, i.e. it consists of compactly supported sections of an honest Lie algebroid over M, which Gualtieri and Li [12, Def. 2.11] call elementary modification of A along B and denote by [A:B]. We remark that they construct an integration of [A:B] applying a blow-up procedure to a Lie groupoid integrating A (assuming that A is integrable) [12, Thm. 2.9, Cor. 2.10].

In particular, when A = TM and B = TN, the Lie algebroid [TM:TN] is called the *log tangent bundle* associated to N, and \mathcal{B} is the projective singular foliation consisting of vector fields on M tangent to N.

1.3.4. From Poisson geometry

For the cotangent Lie algebroid of a Poisson manifold, certain singular subalgebroids can be constructed out of functions. Let (M, π) be a Poisson

manifold, and consider the Poisson algebra $(C^{\infty}(M), \cdot, \{\cdot, \cdot\})$. Let \mathcal{S} be a *Poisson* subalgebra of $C^{\infty}(M)$, which is locally finitely generated as a multiplicative algebra. Then

$$\mathcal{B} := \operatorname{Span}_{C_c^{\infty}(M)} \{ \mathrm{d}f : f \in \mathcal{S} \}$$

is a singular subalgebroid of the cotangent Lie algebroid T^*M . To see that \mathcal{B} is locally finitely generated as a $C^{\infty}(M)$ -module one just uses the product rule, and to see that \mathcal{B} is involutive, use the Leibniz rule for the Poisson bracket and the fact that $[\mathrm{d}f,\mathrm{d}g]=\mathrm{d}\{f,g\}$.

Examples 1.9.

- (1) Let $\phi: M \to P$ a Poisson map between Poisson manifolds, and \mathcal{S}_P a Poisson subalgebra of $C^{\infty}(P)$ which is locally finitely generated as a multiplicative algebra. Then the same holds for $\mathcal{S} := \phi^*(\mathcal{S}_P) \subset C^{\infty}(M)$.
- (2) A special case of the above is given by Poisson maps $M \to \mathfrak{g}^*$ to the dual of a Lie algebra, and choosing $\mathcal{S}_{\mathfrak{g}^*}$ to consist of polynomial functions on \mathfrak{g}^* . Notice that in this case \mathcal{B} is of the kind⁽⁷⁾ described in Example 1.7(2).
- (3) Let $f \in C^{\infty}(M)$. Then $\operatorname{Span}_{C^{\infty}_{c}(M)}\{\mathrm{d}f\}$ is a singular subalgebroid of $T^{*}M$. (This can be seen as a special case of Section 1.3.2, or as a special case of the above taking \mathcal{S} to be the Poisson subalgebra of $C^{\infty}(M)$ generated by f.) For instance, take $M = \mathbb{R}^{2}$ with the standard "symplectic" structure $\pi = \partial_{x} \wedge \partial_{y}$, and let f(x,y) = xy. Then \mathcal{B} is given by all $C^{\infty}_{c}(\mathbb{R}^{2})$ -multiples of $\mathrm{d}(xy)$. The anchor map $\Pi \colon T^{*}\mathbb{R}^{2} \to T\mathbb{R}^{2}$ of the cotangent Lie algebroid is just contraction with π . The singular foliation $\Pi(\mathcal{B})$ of \mathbb{R}^{2} induced by \mathcal{B} is interesting: its leaves agree with the connected components of the f-fibers, except on the preimage of 0: $f^{-1}(0)$ is the union of the axes, and it consists of 5 leaves, namely the 4 open half-axes and the origin.

1.4. Singular subalgebroids and right-invariant vector fields

Let \mathcal{B} be a singular subalgebroid of a Lie algebroid A over M with anchor ρ . \mathcal{B} induces a singular foliation on M, whose leaves are contained in the orbits of the Lie algebroid A, namely

(1.1)
$$\mathcal{F}_{\mathcal{B}} := \{ \rho(\alpha) : \alpha \in \mathcal{B} \}.$$

⁽⁷⁾ Indeed, $\mathfrak{g} \to \Gamma(T^*M), v \mapsto d(\phi^*(v))$ is a Lie algebra morphism.

In this subsection we are concerned with another singular foliation associated to \mathcal{B} , which will be very important to carry out our constructions. Assume A is integrable and fix a Lie groupoid $G \rightrightarrows M$ integrating A. We denote the source and target maps of G by $\mathbf{s}, \mathbf{t} \colon G \to M$ and identify the Lie algebroids $\ker(\mathbf{ds}) \mid_M$ and A.

Given a section $\alpha \in \mathcal{B} \subset \Gamma_c(M, \ker(\mathrm{d}\mathbf{s})|_M)$, put $\overrightarrow{\alpha}$ the right-invariant vector field on G which extends α . Recall that $\overrightarrow{\alpha}$ is an element of $\Gamma(G, \ker(\mathrm{d}\mathbf{s})) \subset \mathfrak{X}(G)$, and it is given by the formula $\overrightarrow{\alpha}_g = (R_g)_* \alpha_{\mathbf{t}(g)}$, for all $g \in G$. We will consider the singular foliation

(1.2)
$$\overrightarrow{\mathcal{B}} := \operatorname{Span}_{C_c^{\infty}(G)} \left\{ \overrightarrow{\alpha} \mid \alpha \in \mathcal{B} \right\}.$$

Likewise, we denote by $\overleftarrow{\mathcal{B}}$ the $C_c^{\infty}(G)$ -module generated by the left-invariant vector fields $\overleftarrow{\alpha}$ for all $\alpha \in \mathcal{B}$. All the statements made in this subsection for $\overrightarrow{\mathcal{B}}$ hold in a similar way for $\overleftarrow{\mathcal{B}}$ too.

Notice that $\operatorname{support}(\overrightarrow{\alpha}) = \mathbf{t}^{-1}(\operatorname{support}(\alpha))$, hence $\overrightarrow{\alpha}$ is not necessarily compactly supported. However, we have:

LEMMA 1.10. — For every section $\alpha \in \mathcal{B}$ the vector field $\overrightarrow{\alpha} \in \overrightarrow{\mathcal{B}}$ is complete.

Proof. — Identifying the anchor map $\rho: A \to TM$ with

$$d\mathbf{t}|_{M} \colon \ker(d\mathbf{s})|_{M} \to TM,$$

we get that $\overrightarrow{\alpha}$ is **t**-related with the vector field $\rho(\alpha)$, which has the same support as α , whence it is complete. It follows from [17, Thm. 3.6.4] that $\overrightarrow{\alpha}$ is complete as well.

We now relate local generators of \mathcal{B} with local generators of $\overrightarrow{\mathcal{B}}$. Given $x \in M$, let I_x^M denote the ideal of functions on M vanishing at x, and I_x^G the ideal of functions on G vanishing at x.

Remark 1.11. — Let $\alpha_1, \ldots, \alpha_n \in \mathcal{B}$. These elements are generators of \mathcal{B} in a neighborhood of x iff their images $[\alpha_1], \ldots, [\alpha_n]$ in $\mathcal{B}/I_x^M \mathcal{B}$ are a spanning set of this vector space. This is proved exactly as in the case of singular foliations [1, Prop. 1.5 a)]. If the latter form a basis of $\mathcal{B}/I_x^M \mathcal{B}$, we say that $\alpha_1, \ldots, \alpha_n$ is a minimal set of local generators.

LEMMA 1.12. — Let $\alpha_1, \ldots, \alpha_n$ be a finite subset of \mathcal{B} . Then $[\alpha_1], \ldots, [\alpha_n]$ is a basis of $\mathcal{B}/I_x^M \mathcal{B}$ iff $[\overrightarrow{\alpha}_1], \ldots, [\overrightarrow{\alpha}_n]$ is a basis (8) of $\overrightarrow{\mathcal{B}}/I_x^G \overrightarrow{\mathcal{B}}$.

"\(\infty\)". — We first show that the $[\overrightarrow{\alpha}_i]$ are linearly independent. Let $c_1, \ldots, c_n \in \mathbb{R}$ with $\sum c_i \overrightarrow{\alpha}_i \in I_x^G \overrightarrow{\mathcal{B}}$. Restricting from G to M we obtain $\sum c_i \alpha_i \in I_x^M \mathcal{B}$, therefore all coefficients c_i are zero. We now show that the $[\overrightarrow{\alpha}_i]$ are a spanning set of $\overrightarrow{\mathcal{B}}/I_x^G \overrightarrow{\mathcal{B}}$. The α_i generate the $C^{\infty}(M)$ -module \mathcal{B} in a neighborhood of x (see Remark 1.11), and hence the $\overrightarrow{\alpha}_i$ generate the $C^{\infty}(G)$ -module $\overrightarrow{\mathcal{B}}$ near x. Given any $X \in \overrightarrow{\mathcal{B}}$, there are $f_i \in C_c^{\infty}(G)$ such that $X = \sum f_i \overrightarrow{\alpha}_i = \sum f_i(x) \overrightarrow{\alpha}_i + \sum (f_i - f_i(x)) \overrightarrow{\alpha}_i$, and since $f_i - f_i(x) \in I_x^G$ we obtain $[X] = \sum f_i(x) [\overrightarrow{\alpha}_i]$.

" \Leftarrow ". — The $[\alpha_i]$ are linearly independent: if $\sum c_i \alpha_i \in I_x^M \mathcal{B}$ then $\sum c_i \overrightarrow{\alpha}_i \in \mathbf{t}^*(I_x^M) \overrightarrow{\mathcal{B}} \subset I_x^G \overrightarrow{\mathcal{B}}$, showing that the c_i all vanish. To show that the $[\alpha_i]$ are a spanning set of $\mathcal{B}/I_x^M \mathcal{B}$, notice that by assumption any element of $\overrightarrow{\mathcal{B}}$ can be written as $\sum c_i \overrightarrow{\alpha}_i$ (for suitable $c_i \in \mathbb{R}$) plus an element of $I_x^G \overrightarrow{\mathcal{B}}$. We pick $\alpha \in \mathcal{B}$ and write $\overrightarrow{\alpha}$ in the above form. Restricting to M we see that α equals $\sum c_i \alpha_i$ plus an element of $I_x^M \mathcal{B}$, i.e. $[\alpha] = \sum c_i [\alpha_i]$. \square

2. Bisubmersions for singular subalgebroids

In this whole section we fix an integrable Lie algebroid $A \to M$ and a singular subalgebroid \mathcal{B} . Further, we fix a Lie groupoid G integrating A.

Recall from [1] that the key ingredient for the construction of the holonomy groupoid of a singular foliation is the notion of bisubmersion. Here, in order to carry out the construction in the case of singular subalgebroids, we reformulate the notion of bisubmersion in Section 2.2. We then present examples, including path holonomy bisubmersions. The latter, upon applying the operations we outline in Section 2.5, will be used to construct the holonomy groupoid in the next Section. Our Section 2.4 and Section 2.5 follow closely [1].

2.1. Pullbacks of singular foliations

We collect background material on pullbacks and generating sets for singular foliations (see Section 1.2).

⁽⁸⁾ Actually the $\overrightarrow{\alpha_i}$ do not lie in $\overrightarrow{\mathcal{B}}$ but rather in the global hull $\widehat{\overrightarrow{\mathcal{B}}}$, see Section 1.1. This does not pose any problems since the inclusion of $\overrightarrow{\mathcal{B}}$ in $\widehat{\overrightarrow{\mathcal{B}}}$ induces an isomorphism $\overrightarrow{\mathcal{B}}/I_x^G\overrightarrow{\mathcal{B}}\cong\widehat{\overrightarrow{\mathcal{B}}}/I_x^G\widehat{\overrightarrow{\mathcal{B}}}$.

Definition 2.1. — Let $\varphi \colon U \to V$ a smooth map between smooth manifolds.

- (1) Let $X \in \mathfrak{X}(U)$ and $Y \in \mathfrak{X}(V)$. We say that X is φ -related to Y iff $\varphi_*(X(p)) = Y(\varphi(p))$ for all $p \in U$.
- (2) Let \mathcal{F} be a $C^{\infty}(V)$ -submodule of $\mathfrak{X}_c(V)$. Define, as in [1] (see also [5, §1.1])

$$\varphi^{-1}(\mathcal{F}) := \{ X \in \mathfrak{X}_c(U) : \mathrm{d}\varphi(X) = \sum f_i(Y_i \circ \varphi)$$
 for finitely many $f_i \in C_c^{\infty}(U)$ and $Y_i \in \mathcal{F}\}.$

Here $d\varphi \colon TU \to \varphi^*(TV)$ is a vector bundle map covering Id_U , where $\varphi^*(TV)$ denotes the pullback vector bundle. Notice that $\varphi^{-1}(\mathcal{F})$ is a $C^{\infty}(U)$ -submodule of $\mathfrak{X}_c(U)$. It is a foliation, called pullback foliation, whenever \mathcal{F} is a foliation and φ is transverse to \mathcal{F} [1].

Now let $\varphi \colon U \to V$ a smooth map and \mathcal{F} be a $C^{\infty}(V)$ -submodule of $\mathfrak{X}_c(V)$. Fix a generating set \mathcal{G} of \mathcal{F} , as defined in Section 1.1. We display two technical lemmas, which are not completely obvious due to the fact that neither $\mathcal{G} \subset \mathcal{F}$ nor $\mathcal{G} \supset \mathcal{F}$ in general.

Lemma 2.2. — The following conditions are equivalent:

- for every $Y \in \mathcal{F}$ there is a $Z \in \mathfrak{X}(U)$ which is φ -related to Y.
- for every $Y \in \mathcal{G}$ there is a $Z \in \mathfrak{X}(U)$ which is φ -related to Y.

Proof. — We only show that the first condition implies the second (the converse is similar). Take a partition of unity $\{\psi_a\}$ on V by functions with compact support. Given $Y \in \mathcal{G}$, by assumption there is $Z_a \in \mathfrak{X}(U)$ that is φ -related to $\psi_a Y \in \mathcal{F}$. Since the partition of unity is locally finite, one can arrange that the sum $\sum_a Z_a$ is locally finite, and the resulting vector field is φ -related to $\sum_a \psi_a Y = Y$.

Under certain conditions, $\varphi^{-1}(\mathcal{F})$ has a distinguished generating set.

LEMMA 2.3. — Assume that any of the equivalent conditions in Lemma 2.2 is satisfied. (This happens for instance when φ is a submersion). Then

$$\begin{split} \varphi^{-1}(\mathcal{F}) &= \operatorname{Span}_{C_c^\infty(U)} \{ Z \in \mathfrak{X}(U) \colon Z \text{ is } \varphi\text{-related to an element of } \mathcal{F} \} \\ &= \operatorname{Span}_{C_c^\infty(U)} \{ Z \in \mathfrak{X}(U) \colon Z \text{ is } \varphi\text{-related to an element of } \mathcal{G} \text{ or to } 0 \}. \end{split}$$

Proof. — In the first equality, the inclusion " \supset " is easily checked to hold even when the assumption is not satisfied. For " \subset ", take $X \in \mathfrak{X}_c(U)$ such that $d\varphi(X) = \sum f_i(Y_i \circ \varphi)$ where the sum is finite, $f_i \in C_c^{\infty}(U)$ and

 $Y_i \in \mathcal{F}$. By the assumption, there exist $Z_i \in \mathfrak{X}(U)$ that is φ -related to Y_i , i.e. $d\varphi(Z_i) = Y_i \circ \varphi$. Hence we can write

$$d\varphi(X) = \sum f_i d\varphi(Z_i) = d\varphi\left(\sum f_i Z_i\right).$$

This means that $X = \sum f_i Z_i + Z$ where $Z \in \mathfrak{X}_c(U)$ is φ -related to the zero vector field on V, which is an element of \mathcal{F} .

For the second equality, to prove "\(\supset\)", take $Z \in \mathfrak{X}(U)$ which is φ -related to $Y \in \mathcal{G}$ and $f \in C_c^{\infty}(U)$. We can choose a function $\psi \in C_c^{\infty}(V)$ with is one on $\varphi(\operatorname{Supp}(f))$. We then have $fZ = f\varphi^*(\psi)Z$, and clearly $\varphi^*(\psi)Z$ is φ -related to $\psi Y \in \mathcal{F}$. To show " \subset ", take $X \in \mathfrak{X}(U)$ which is φ -related to an element of \mathcal{F} , i.e. to some $\sum g_i Y_i$ where $g_i \in C_c^{\infty}(U)$ and $Y_i \in \mathcal{G}$. By assumption there is $X_i \in \mathfrak{X}(U)$ that is φ -related to Y_i , hence $\sum \varphi^*(g_i)X_i$ is also related to $\sum g_i Y_i$. Fix $f \in C_c^{\infty}(U)$. Then $fX = f \sum \varphi^*(g_i) X_i + Z$ where $Z \in \mathfrak{X}_c(U)$ is φ -related to the zero vector field on V.

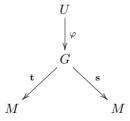
2.2. Definition of bisubmersion

Let A be an integrable Lie algebroid and G a Lie groupoid integrating A. Let \mathcal{B} be a singular subalgebroid of A.

DEFINITION 2.4. — A bisubmersion for \mathcal{B} is a smooth map $\varphi \colon U \to G$, where U is a manifold, such that

- (1) $\mathbf{s}_U := \mathbf{s} \circ \varphi$ and $\mathbf{t}_U := \mathbf{t} \circ \varphi \colon U \to M$ are submersions,
- (2) for every $\alpha \in \mathcal{B}$, there is $Z \in \mathfrak{X}(U)$ which is φ -related to $\overrightarrow{\alpha}$ and $W \in \mathfrak{X}(U)$ which is φ -related to $\overleftarrow{\boldsymbol{\alpha}}$, (3) $\varphi^{-1}(\overrightarrow{\mathcal{B}}) = \Gamma_c(U, \ker d\mathbf{s}_U)$ and $\varphi^{-1}(\overrightarrow{\mathcal{B}}) = \Gamma_c(U, \ker d\mathbf{t}_U)$.

Notation 2.5. — We denote a bisubmersion of \mathcal{B} by (U, φ, G) . One can bear in mind the following diagram:



Remark 2.6. — In Definition 2.4 the map φ is not required to be transverse to $\overrightarrow{\mathcal{B}}$, and the conditions in Definition 2.4 do not imply transversality in general (see the examples in Section 2.3.2 with K there being the trivial groupoid).

The rest of this subsection is devoted to explanations about conditions (2) and (3).

Remark 2.7. — The first part condition (2) in Definition 2.4 is expressed in terms of the generating set $\{\overrightarrow{\alpha} \mid \alpha \in \mathcal{B}\}\$ of the singular foliation $\overrightarrow{\mathcal{B}}$. Using Lemma 2.2 it can be rephrased saying that any element of $\overrightarrow{\mathcal{B}}$ can be lifted to U. Notice that any lift will lie in ker ds_U, since right-invariant vector fields on G lie in ker ds. An analogue statement holds for the second part of condition (2).

We now phrase the first part of condition (3) in Definition 2.4 more explicitly. We have

(2.1)

$$\varphi^{-1}(\overrightarrow{\mathcal{B}}) = \operatorname{Span}_{C_c^{\infty}(U)} \{ Z \in \mathfrak{X}(U) : Z \text{ is } \varphi\text{-related to } \overrightarrow{\alpha} \text{ for some } \alpha \in \mathcal{B} \}$$
$$= \operatorname{Span}_{C_c^{\infty}(U)} \Gamma(U, \ker \operatorname{ds}_U)^{\operatorname{proj}, \mathcal{B}},$$

where the first equation holds by Lemma 2.3 (which can be applied due to condition (2)), and in the second equation we used that $\overrightarrow{\mathcal{B}}$ lies in the kernel of ds. Here,

$$\Gamma(U, \ker d\mathbf{s}_U)^{\operatorname{proj}, \mathcal{B}}$$
:= $\{Z \in \Gamma(U, \ker d\mathbf{s}_U) : Z \text{ is } \varphi\text{-related to } \overrightarrow{\alpha} \text{ for some } \alpha \in \mathcal{B}\}.$

Lemma 2.8. — Given any map $\varphi \colon U \to G$, the following statements are equivalent:

- (1) $\varphi^{-1}(\overrightarrow{\mathcal{B}}) = \Gamma_c(U, \ker d\mathbf{s}_U)$
- (2) $\Gamma(U, \ker d\mathbf{s}_U)^{\operatorname{proj}, \mathcal{B}}$ generates $\Gamma_c(U, \ker d\mathbf{s}_U)$ as a $C_c^{\infty}(U)$ -module
- (3) $\Gamma(U, \ker d\mathbf{s}_U)^{\operatorname{proj}, \mathcal{B}}$ spans $\ker(d\mathbf{s}_U)_u$ at every $u \in U$.

Proof. — Using equation (2.1) it is clear that (1) and (2) are equivalent. Clearly (2) implies (3). For the converse, we first make an observation. By (3), for any $u \in U$, we can take a basis of $\ker(\mathrm{d}\mathbf{s}_U)_u$ and extend it to $\alpha^1, \ldots, \alpha^n \in \Gamma(U, \ker \mathrm{d}\mathbf{s}_U)^{\mathrm{proj},\mathcal{B}}$. These vector fields are linearly independent on a small open neighborhood U' of u. Therefore can write any section of $\ker \mathrm{d}\mathbf{s}_U$ with support in U' as a $C_c^{\infty}(U')$ -linear combination of the α^i .

Now fix $X \in \Gamma_c(U, \ker \operatorname{ds}_U)$. The support of X, being compact, can be covered by finitely many open neighborhoods as above. Extend this cover to a open cover $\{U_i'\}$ of U, and choose a partition of unity $\{g_i\} \subset C_c^{\infty}(U)$ subordinate to it. Then $X = \sum g_i(X|_{U_i'})$ is a finite sum. The summands are sections of $\ker \operatorname{ds}_U$ with support in U_i' . Applying the above

observation to each summand, we see that X is written as a finite $C_c^{\infty}(U)$ -linear combination of elements of $\Gamma(U, \ker d\mathbf{s}_U)^{\operatorname{proj},\mathcal{B}}$.

We mention another characterization of Definition 2.4:

Remark 2.9. — The first parts of conditions (2) and (3) in Definition 2.4 are equivalent to the following: the map of $C^{\infty}(U)$ -modules

$$d\varphi \colon \Gamma_c(U; \ker d\mathbf{s}_U) \to \varphi^*(\overrightarrow{\mathcal{B}})$$

is well-defined and surjective. Here $\varphi^*(\overrightarrow{\mathcal{B}})$ is the $C^\infty(U)$ -submodule of $\mathfrak{X}_c(U)$ generated by $f(\xi \circ \varphi)$ with $f \in C_c^\infty(U)$ and $\xi \in \overrightarrow{\mathcal{B}}$.

Indeed, the first part of condition (3) is equivalent to

$$\varphi^{-1}(\overrightarrow{\mathcal{B}}) \supset \Gamma_c(U, \ker \mathrm{d}\mathbf{s}_U)$$

(the other inclusion is obvious since $\overrightarrow{\mathcal{B}}$ lies in the kernel of ds), and therefore is equivalent to the fact that the above map is well-defined. The first part of condition (2), by Remark 2.7, says that this map is onto.

2.3. Examples

We exhibit examples of bisubmersions for singular foliations and wide Lie subalgebroids (the two motivating examples displayed in Section 1.2), and generalizing the latter, for the examples treated in Section 1.3.1.

2.3.1. Bisubmersions of singular foliations

In [1, Definition 2.1] a bisubmersion for a singular foliation (M, \mathcal{F}) is defined as a triple $(U, \mathbf{t}_U, \mathbf{s}_U)$ consisting of a manifold U with two submersions \mathbf{t}_U and \mathbf{s}_U to M, such that

(2.2)
$$\mathbf{t}_U^{-1}(\mathcal{F}) = \Gamma_c(U, \ker d\mathbf{t}_U) + \Gamma_c(U, \ker d\mathbf{s}_U) = \mathbf{s}_U^{-1}(\mathcal{F}).$$

On the other hand, singular foliations are special cases of singular subalgebroids; namely, they are the singular subalgebroids of TM. We show that the two notions of bisubmersion for singular foliations essentially agree, since there is a canonical bijective correspondence between them.

PROPOSITION 2.10. — Let (M, \mathcal{F}) be a singular foliation, U a manifold, and $\mathbf{t}_U \colon U \to M$ and $\mathbf{s}_U \colon U \to M$ submersions. The following are equivalent:

(1) $(U, \mathbf{t}_U, \mathbf{s}_U)$ is a bisubmersion for the singular foliation \mathcal{F} (in the sense of [1, Definition 2.1]);

(2) the map $(\mathbf{t}_U, \mathbf{s}_U) \colon U \to M \times M$ is a bisubmersion (in the sense of Definition 2.4), where $M \times M$ is endowed with the pair groupoid structure.

We give the following lemma without proof.

LEMMA 2.11. — Let $\mathbf{t}_U \colon U \to M$ and $\mathbf{s}_U \colon U \to M$ be smooth maps. If $Z \in \mathfrak{X}(U)$ and $X,Y \in \mathfrak{X}(M)$, then Z is $(\mathbf{t}_U,\mathbf{s}_U)$ -related to X and X-related to X. Here $X \mapsto Y$ denotes the vector field (X,Y) on $X \mapsto M$.

Proof of Proposition 2.10.

 $(1) \Rightarrow (2)$. — Property (1) in Definition 2.4 is obviously satisfied. For property (2) we argue as follows. Every $X \in \mathcal{F}$ can be \mathbf{t}_U -lifted to $Z \in \Gamma(\ker \operatorname{ds}_U)$, by the proof of [1, Prop. 2.10 b)]. Such a Z is $(\mathbf{t}_U, \mathbf{s}_U)$ -related to \overrightarrow{X} by Lemma 2.11. Further, X can be \mathbf{s}_U -lifted to $W \in \Gamma(\ker \operatorname{dt}_U)$, by the same argument in the proof of [1, Prop. 2.10 b)](interchanging the roles of source and target), and such a W is $(\mathbf{t}_U, \mathbf{s}_U)$ -related to X.

We show property (3), that is, $(\mathbf{t}_U, \mathbf{s}_U)^{-1}(\overrightarrow{\mathcal{F}}) = \Gamma_c(U, \ker d\mathbf{s}_U)$. We do so using Lemma 2.8(2). Let $Z \in \Gamma_c(U, \ker d\mathbf{s}_U)$. By the first equality in equation (2.2), we have $Z = \sum g_i Z_i$ where $g_i \in C_c^{\infty}(U)$ and $Z_i \in \mathfrak{X}(U)$ is \mathbf{t}_U -projectable to an element of \mathcal{F} . We can write each Z_i as the sum of a vector field in $\Gamma(\ker d\mathbf{t}_U)$ and one in $\Gamma(\ker d\mathbf{s}_U)$ – which we denote by the respective subscrips. To see this, we have to apply with some care the first equality in equation (2.2): choose a partition of unity $\{\psi_a\}$ with compact support on U. Then $\psi_a Z_i$ equals an element of $\Gamma_c(\ker d\mathbf{t}_U)$ plus an element of $\Gamma_c(\ker d\mathbf{s}_U)$, and we may assume⁽⁹⁾ that their support is contained in a small enough neighborhood of $\operatorname{Supp}(\psi_a)$. Hence $Z_i = \sum_a \psi_a Z_i$ equals a locally finite sum of elements of $\Gamma_c(\ker d\mathbf{t}_U)$. Altogether we obtain

$$Z = Z' + \sum g_i(Z_i)_{\ker d\mathbf{s}_U}, \text{ where } Z' \coloneqq \sum g_i(Z_i)_{\ker d\mathbf{t}_U}.$$

Notice that Z' lies in $\Gamma_c(U, \ker d\mathbf{s}_U)$ (being the difference of two vector fields with this property), hence it is $(\mathbf{t}_U, \mathbf{s}_U)$ -related to the zero vector field. Similarly, each $(Z_i)_{\ker d\mathbf{s}_U}$ is \mathbf{t}_U -projectable to an element of \mathcal{F} (being the difference $Z_i - (Z_i)_{\ker d\mathbf{t}_U}$ of two vector fields with this property), hence by Lemma 2.11 it is $(\mathbf{t}_U, \mathbf{s}_U)$ -related to an element of $\{\overrightarrow{X} : X \in \mathcal{F}\}$. In conclusion, Z is a $C_c^{\infty}(U)$ -linear combination of vector fields which are $(\mathbf{t}_U, \mathbf{s}_U)$ -related to elements of $\{\overrightarrow{X} : X \in \mathcal{F}\}$, proving that the condition

⁽⁹⁾ By multiplying them with a suitable function with value 1 on Supp(ψ_a).

in Lemma 2.8(2) holds. The second equality in item (3) of Definition 2.4 follows in an analogous way.

 $(2) \Rightarrow (1)$. — \mathbf{t}_U and \mathbf{s}_U are submersions by item (1) of Definition 2.4. Hence we just need to show equation (2.2). By Lemma 2.11 and item (3) of Definition 2.4 we have

$$\mathbf{t}_{U}^{-1}(\mathcal{F}) \cap \mathbf{s}_{U}^{-1}(\mathcal{F}) \supset (\mathbf{t}_{U}, \mathbf{s}_{U})^{-1}(\overrightarrow{\mathcal{F}}) + (\mathbf{t}_{U}, \mathbf{s}_{U})^{-1}(\overleftarrow{\mathcal{F}})$$

$$= \Gamma_{c}(U, \ker d\mathbf{s}_{U}) + \Gamma_{c}(U, \ker d\mathbf{t}_{U}).$$

We show $\mathbf{t}_U^{-1}(\mathcal{F}) \subset \Gamma_c(U, \ker d\mathbf{s}_U) + \Gamma_c(U, \ker d\mathbf{t}_U)$ (the argument for $\mathbf{s}_U^{-1}(\mathcal{F})$ is analogous). Let $Z \in \mathfrak{X}(U)$ be \mathbf{t}_U -related to some $X \in \mathcal{F}$. By condition (2), there is a vector field W on U which is $(\mathbf{t}_U, \mathbf{s}_U)$ -related to \overrightarrow{X} , i.e. W lies in $\Gamma(\ker d\mathbf{s}_U)$ and is \mathbf{t}_U -related to X. We conclude by writing Z = (Z - W) + W, with $Z - W \in \Gamma(\ker d\mathbf{t}_U)$.

Further, a bisubmersion for any singular subalgebroid \mathcal{B} gives rise to a bisubmersion for $\mathcal{F}_{\mathcal{B}}$, the induced singular foliation defined in equation (1.1). We have:

LEMMA 2.12. — Let (U, φ, G) a bisubmersion of \mathcal{B} . Then $(U, \mathbf{t}_U, \mathbf{s}_U)$ is a bisubmersion of the singular foliation $\mathcal{F}_{\mathcal{B}}$ (in the sense of [1]).

Proof. — Since \mathbf{t}_U and \mathbf{s}_U are submersions, we just have to show that

$$\mathbf{t}_U^{-1}(\mathcal{F}_{\mathcal{B}}) = \Gamma_c(U, \ker d\mathbf{t}_U) + \Gamma_c(U, \ker d\mathbf{s}_U)$$

and similarly for $\mathbf{s}_U^{-1}(\mathcal{F}_{\mathcal{B}})$. We start proving the above equality.

"\rightarrow". — By item (3) of Definition 2.4 and by Lemma 2.8(2), $\Gamma_c(U, \ker d\mathbf{t}_U) + \Gamma_c(U, \ker d\mathbf{s}_U)$ is generated by elements which are φ -related to elements of $\{\overrightarrow{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{B}\} + \{\overleftarrow{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{B}\}$. The latter are \mathbf{t} -related to elements of $\mathcal{F}_{\mathcal{B}}$. As $\mathbf{t} \circ \varphi = \mathbf{t}_U$, the above generators are \mathbf{t}_U -related to elements of $\mathcal{F}_{\mathcal{B}}$.

"C". — Let $Z \in \mathfrak{X}(U)$ be \mathbf{t}_U -related to some $X \in \mathcal{F}_{\mathcal{B}}$. There exists $\alpha \in \mathcal{B}$ with $\rho(\alpha) = X$. Since under the identification $A \cong \ker(\mathrm{d}\mathbf{s})|_M$ the anchor ρ is identified with $\mathrm{d}\mathbf{t}|_M$, we see that $\overrightarrow{\alpha}$ is \mathbf{t} -related to X. By item (2) of Definition 2.4, there is $W \in \mathfrak{X}(U)$ which is φ -related to $\overrightarrow{\alpha}$. Hence W is also related to X under the map $\mathbf{t}_U = \mathbf{t} \circ \varphi$. Notice that $W \in \Gamma(U, \ker \mathrm{d}\mathbf{s}_U)$, and further $Z - W \in \Gamma(U, \ker \mathrm{d}\mathbf{t}_U)$. Writing Z = W + (Z - W) we conclude the proof of the inclusion.

To show the above equality for \mathbf{s}_U^{-1} in place of \mathbf{t}_U^{-1} we proceed as follows: consider the above equality for the inverse bisubmersion $(U, \bar{\varphi}, G)$ (see Definition 2.24), and use $\bar{\mathbf{t}}_U = \mathbf{s}_U$, $\mathbf{s}_U = \bar{\mathbf{t}}_U$.

2.3.2. Lie groupoid morphisms as bisubmersions

If a singular subalgebroid arises from a Lie groupoid morphism (see Definition 1.6, this includes wide Lie subalgebroids), then that morphism is automatically a bisubmersion:

PROPOSITION 2.13. — Let $\varphi \colon K \to G$ be a morphism of Lie groupoids covering the identity on M. Denote by $\mathcal{B} := \varphi_*(\Gamma_c(\text{Lie}(K)))$ the singular subalgebroid of Lie(G) it gives rise to. Then $\varphi \colon K \to G$ is a bisubmersion for \mathcal{B} .

Proof. — We check the properties of Definition 2.4:

- (1) is satisfied since K is a Lie groupoid over M.
- (2) is an immediate consequence of the Claim below, since any element of \mathcal{B} is of the form φ_*X for some $X \in \mathcal{B}$.
- (3) for all $X \in \Gamma_c(\text{Lie}(K))$, the claim below implies that \overrightarrow{X} lies in $\Gamma(K, \ker d\mathbf{s}_K)^{\text{proj},\mathcal{B}}$. Such \overrightarrow{X} span $\ker d\mathbf{s}_K$ at every point of K, so we can conclude using Lemma 2.8(3) . The same argument applies to \overline{X} .

Claim. — Denote E := Lie(K) and A := Lie(G). Let $X \in \Gamma_c(E)$, and denote $\alpha_X := \varphi_* X \in \Gamma(A)$. Then \overrightarrow{X} is φ -related to $\overrightarrow{\alpha_X}$. Similarly, \overleftarrow{X} is φ -related to $\overleftarrow{\alpha_X}$.

Consider \overrightarrow{X} , the right-invariant vector field on K which restricts to X along M. Let $k \in K$. We have

$$\begin{split} \varphi_*(\overrightarrow{X}_k) &= \varphi_*(R_k^K)_* \left(X_{\mathbf{t}(k)}\right) = (R_{\varphi(k)}^G)_* \varphi_* \left(X_{\mathbf{t}(k)}\right) \\ &= (R_{\varphi(k)}^G)_* \left((\alpha_X)_{\mathbf{t}(k)}\right) = \overrightarrow{(\alpha_X)}_{\varphi(k)}, \end{split}$$

showing that \overrightarrow{X} is φ -related to $\overrightarrow{\alpha_X}$. Here we denote R^G the right-translation by in G, and likewise R^K the right-translation in K. In the second equality we used that φ is a groupoid morphism.

Now consider X. We have $X = -i_*^K X$, where i^K is the inversion map of the Lie groupoid K. Hence

$$\varphi_*(\overrightarrow{X}) = -\varphi_*i_*^K(\overrightarrow{X}) = -i_*^G\varphi_*(\overrightarrow{X}) = -i_*^G(\overrightarrow{\alpha_X}) = \overleftarrow{\alpha_X},$$

where in the second equality we used that φ is a groupoid morphism, and in the third that \overrightarrow{X} is φ -related to $\overrightarrow{\alpha_X}$.

We spell out Proposition 2.13 in the case of a wide Lie subalgebroid:

COROLLARY 2.14. — Let A be an integrable Lie algebroid and B a wide Lie subalgebroid of A. Let $\varphi \colon K \to G$ be a morphism⁽¹⁰⁾ of Lie groupoids which integrates the inclusion $\iota \colon B \hookrightarrow A$.

Then $\varphi \colon K \to G$ is a bisubmersion for the singular subalgebroid $\mathcal{B} = \Gamma_c(B)$.

Example 2.15. — For $\mathcal{B} = \Gamma_c(A)$ we obtain that id: $G \to G$ is a bisubmersion.

2.4. Path-holonomy bisubmersions

We give an explicit construction of bisubmersions starting from local generators of \mathcal{B} . Recall from Lemma 1.10 that if $\alpha \in \mathcal{B}$, the vector field $\overrightarrow{\alpha}$ is complete. We denote by $\exp_y \overrightarrow{\alpha}$ its time-1 flow applied to the point $y \in M$.

Definition 2.16.

(1) Let $x \in M$ and $\alpha_1, \ldots, \alpha_n \in \mathcal{B}$ such that $[\alpha_1], \ldots, [\alpha_n]$ span $\mathcal{B}/I_x\mathcal{B}$. The associated path holonomy bisubmersion is the map

$$\varphi \colon U \to G, \ (\lambda, y) \mapsto \exp_y \sum \lambda_i \overrightarrow{\alpha_i},$$

where U is a neighborhood of (0, x) in $\mathbb{R}^n \times M$ such that $\mathbf{t} \circ \varphi$ is (11) a submersion, and we write $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$.

(2) We say that (U, φ, G) is minimal (at x) if $[\alpha_1], \ldots, [\alpha_n]$ is a basis of the vector space $\mathcal{B}/I_x\mathcal{B}$.

Example 2.17. — Consider the case when A = TM, so that $\mathcal{F} := \mathcal{B}$ is a singular foliation on M. Fix $x \in M$ and let $X_1, \ldots, X_n \in \mathcal{F}$ so that they induce a basis of $\mathcal{F}/I_x\mathcal{F}$. We make two choices of Lie groupoid G integrating the Lie algebroid TM.

(1) As G let us choose the pair groupoid $M \times M$. Then the path holonomy bisubmersion we obtain is

$$\varphi \colon U \to M \times M, \ (y, \lambda) \mapsto \left(\exp_y\left(\sum \lambda_i X_i\right), y\right).$$

Notice that under the bijection given by Proposition 2.10, it corresponds to the path holonomy bisubmersion $(U, \mathbf{t}, \mathbf{s})$ for the singular foliation \mathcal{F} (induced by X_1, \ldots, X_n) as in [1].

 $^{^{(10)}}$ The map φ is always a (not necessarily injective) immersion, and covers the identity on M.

⁽¹¹⁾ Such a neighborhood exists since φ is a submersion at (0, x).

(2) Now as G we take the fundamental groupoid $\Pi(M)$. Then the path holonomy bisubmersion we obtain is

$$\widetilde{\varphi} \colon U \to \Pi(M), \ (y,\lambda) \mapsto \text{homotopy class of } \gamma_{(y,\lambda)},$$

where $\gamma_{(y,\lambda)}$ is the path $[0,1] \ni t \mapsto \exp_y(t \sum \lambda_i X_i)$, i.e. the integral curve of $\sum \lambda_i X_i$ that starts at y.

To see this, use the canonical Lie groupoid isomorphism $\widetilde{M} \times_{\pi_1(M)} \widetilde{M} \cong \Pi(M)$, where \widetilde{M} is the universal covering space of M, and take the unique lift of $\sum \lambda_i X_i$ to a vector field on \widetilde{M} .

Notice that the above two path holonomy bisubmersions are related by $\varphi = \pi \circ \widetilde{\varphi}$, where $\pi \colon \Pi(M) \to M \times M$ is the morphism of Lie groupoids sending the homotopy class of a path γ in M to $(\gamma(0), \gamma(1))$. We will elaborate on this example in Section 3.4.

Let us show now that the object defined in Definition 2.16 is really a bisubmersion as in Definition 2.4. To this aim recall that $\mathbf{s}_U := \mathbf{s} \circ \varphi$ and $\mathbf{t}_U := \mathbf{t} \circ \varphi \colon U \to M$.

PROPOSITION 2.18. — Let $x \in M$, let $\alpha_1, \ldots, \alpha_n \in \mathcal{B}$ such that $[\alpha_1], \ldots, [\alpha_n]$ span $\mathcal{B}/I_x\mathcal{B}$, and let (U, φ, G) be as in Definition 2.16. Then (U, φ, G) is a bisubmersion.

Remark 2.19. — Recall from Section 1.4 that there are two foliations associated to \mathcal{B} . In the proof of Proposition 2.18 we will use the following relations between (U, φ, G) (as in Definition 2.16) and bisubmersions for these two foliations:

(1) For the singular foliation $\mathcal{F}_{\mathcal{B}}$ on M:

$$(U, \mathbf{t}_{U}, \mathbf{s}_{U})$$

is a bisubmersion for $\mathcal{F}_{\mathcal{B}}$ (in the sense of [1, Definition 2.1]). Indeed, it is the "path-holonomy bisubmersion" of $\mathcal{F}_{\mathcal{B}}$ constructed using the generators $X_i = \rho(\alpha_i)$ of the foliation $\mathcal{F}_{\mathcal{B}}$ on M. This is proven⁽¹²⁾ exactly as in [1, Prop. 2.10 a)], which holds since the X_i define a spanning set of $\mathcal{F}_x := \mathcal{F}/I_x\mathcal{F}$ (it does not matter that they do not define a basis in general).

(2) For the singular foliation $\overrightarrow{\mathcal{B}}$ on G: a set of local generators of $\overrightarrow{\mathcal{B}}$ is $\overrightarrow{\alpha_1}, \ldots, \overrightarrow{\alpha_n}$, by Lemma 1.12. Let

$$(V, \mathbf{t}_V, \mathbf{s}_V)$$

 $^{^{(12)}}$ Once we establish Proposition 2.18, an alternative proof is obtained applying Lemma 2.12.

be the path-holonomy bisubmersion (in the sense of [1]) associated to these generators near x. Notice that, as M embeds in G as the identity section, we can view $U \subset \mathbb{R}^n \times M$ as a subset of $V \subset \mathbb{R}^n \times G$. Restricting the source and target map of V we see that $\mathbf{s}_U = \mathbf{s}_V|_U \colon U \to M$ (the vector bundle projection) and $\varphi = \mathbf{t}_V|_U \colon U \to G$.

Proof of Proposition 2.18. — We show that the three conditions listed in Definition 2.4 hold.

Condition (1) holds since $(U, \mathbf{t}_U, \mathbf{s}_U)$ is a bisubmersion for the foliation $\mathcal{F}_{\mathcal{B}}$, by Remark 2.19(1).

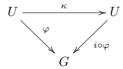
In the rest of the proof we use the bisubmersion $(V, \mathbf{t}_V, \mathbf{s}_V)$ for the foliation $\overrightarrow{\mathcal{B}}$ described in Remark 2.19(2). We show the first part of condition (2), i.e., that for any $\alpha \in \mathcal{B}$, there is a vector field on U which is φ -related to $\overrightarrow{\alpha}$. This holds because there exists $\widetilde{Z} \in \Gamma(V, \ker d\mathbf{s}_V)$ which \mathbf{t}_V -projects to $\overrightarrow{\alpha}$ (for example by [1, Prop. 2.10(b)], and therefore $\widetilde{Z}|_U$ is tangent to U and φ -related to $\overrightarrow{\alpha}$.

We show the first equation in condition (3). We do so using Lemma 2.8 (2). Let $Z \in \Gamma_c(U, \ker d\mathbf{s}_U)$, and take an extension $\widetilde{Z} \in \Gamma_c(V, \ker d\mathbf{s}_V)$ to V. By Proposition 2.10 $(\mathbf{t}_V, \mathbf{s}_V) : V \to G \times G$ is a bisubmersion for the singular subalgebroid $\overrightarrow{\mathcal{B}}$. Applying Lemma 2.8(2) to it we see that \widetilde{Z} is a $C_c^{\infty}(V)$ -linear combination of vector fields which are $(\mathbf{t}_V, \mathbf{s}_V)$ -related to elements of $\overrightarrow{\mathcal{B}}$. In other words, $\widetilde{Z} = \sum \widetilde{g_i} \widetilde{Y_i}$ where $\widetilde{g_i} \in C_c^{\infty}(V)$ and the vector fields Y_i lie in $\Gamma(V, \ker d\mathbf{s}_V)$ and are \mathbf{t}_V -related to some element of $\overrightarrow{\mathcal{B}}$. In particular, $(\widetilde{Y_i})|_U$ is tangent to U and, since $\varphi = \mathbf{t}_V|_U$, it is φ -related to some element of $\overrightarrow{\mathcal{B}}$. Hence $Z = \sum_i (\widetilde{g_i})|_U (\widetilde{Y_i})|_U$ lies in $\varphi^{-1}(\overrightarrow{\mathcal{B}})$.

The second parts of conditions (2) and (3) in Definition 2.4 are proven analogously to the above. \Box

The following Lemma allows to simplify some of the later constructions and proofs, see Remark 2.26.

LEMMA 2.20. — Let $\alpha_1, \ldots, \alpha_n \in \mathcal{B}$ and let (U, φ, G) be a path holonomy bisubmersion as in Definition 2.16. There exists a diffeomorphism κ making the following diagram commute:



In particular, $\mathbf{s}_U \circ \kappa = \mathbf{t}_U$ and $\mathbf{t}_U \circ \kappa = \mathbf{s}_U$.

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Proof. — Consider the map

$$\kappa \colon U \to U, \kappa(\lambda, x) = (-\lambda, \mathbf{t}_U(\lambda, x)).$$

One computes easily that $\kappa^2 = \operatorname{Id}_U$, hence κ is a diffeomorphism⁽¹³⁾. To check that the diagram commutes, we fix $(\lambda, x) \in U$ and compute

$$(i\circ\varphi\circ\kappa)(\lambda,x)=i\left(\exp_{\mathbf{t}_U(\lambda,x)}\left(-\sum\lambda_i\overrightarrow{\boldsymbol{\alpha}_i}\right)\right).$$

By definition we have $\varphi(\lambda, x) = \exp_x(\sum \lambda_i \overrightarrow{\alpha_i})$. We need to show that these two points of G agree, or equivalently that

(2.3)
$$\exp_{\mathbf{t}_{U}(\lambda,x)}\left(-\sum \lambda_{i}\overrightarrow{\alpha_{i}}\right) \cdot \exp_{x}\left(\sum \lambda_{i}\overrightarrow{\alpha_{i}}\right) = 1_{x}.$$

Use the short form notation $\alpha := \sum \lambda_i \alpha_i$. For all $\epsilon \in \mathbb{R}$ define section of **s** by $\psi_{\epsilon} : M \to G, \psi_{\epsilon}(x) = \exp_x(\epsilon \overrightarrow{\alpha})$. Its image defines a bisection, at least for ϵ is small enough. The right invariance of $\overrightarrow{\alpha}$ implies that the above family of bisections satisfies⁽¹⁴⁾ $\psi_{\epsilon} * \psi_{\sigma} = \psi_{\epsilon+\sigma}$. In particular we obtain

$$(\psi_{-1} * \psi_1)(x) = \psi_0(x) = x,$$

which is exactly equation (2.3).

2.5. Operations involving bisubmersions

We explain how to handle bisubmersions algebraically. This will be used the construction of the holonomy groupoid in Section 3. To this end, we fix a Lie groupoid G and a singular subalgebroid \mathcal{B} of its Lie algebroid.

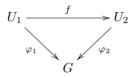
Recall that, given a singular foliation \mathcal{F} , in Proposition 2.18 we established a bijection between bisubmersions for \mathcal{F} in the sense of [1] and bisubmersions for \mathcal{F} regarded as a singular subalgebroid (Definition 2.4). All the operations we define in this subsection, in the special case of singular foliations, correspond under the above bijection with the operations introduced in [1].

⁽¹³⁾ The idea of considering κ comes from [1, Prop. 2.10 a)], where κ is shown to satisfy $\mathbf{s}_U \circ \kappa = \mathbf{t}_U$ and $\mathbf{t}_U \circ \kappa = \mathbf{s}_U$.

⁽¹⁴⁾ Recall from [17, Prop. 1.4.2, p. 22] that the product of bisections is defined as $(\psi_{\epsilon} * \psi_{\sigma})(y) = \psi_{\epsilon}(\mathbf{t}(\psi_{\sigma}(y))) \cdot \psi_{\sigma}(y)$ for all $y \in M$.

2.5.1. Morphisms

DEFINITION 2.21. — Let (U_i, φ_i, G) , i = 1, 2 be bisubmersions for \mathcal{B} . A morphism of bisubmersions is a map $f: U_1 \to U_2$ such that $\varphi_1 = \varphi_2 \circ f$.



There is a simple way to construct new bisubmersions out of old ones, which we will use in the sequel.

LEMMA 2.22. — Let (V, φ, G) be a bisubmersion for \mathcal{B} . Let U be a manifold and $p: U \to V$ be a submersion. Then $(U, \varphi \circ p, G)$ is a bisubmersion for \mathcal{B} . Further, p is a morphism of bisubmersions.



Remark 2.23. — If $q: V \to U$ is a section of p (i.e. , $p \circ q = \mathrm{Id}_V$), then q is also a morphism of bisubmersions. Notice that such a global section might not exist, but local sections of p exist around every point of the open subset p(U) of V.

Proof. — We check that $(U, \varphi \circ p, G)$ satisfies the conditions of Definition 2.4. Since p is a submersion, (1) clearly holds, and (2) holds too as p allows to lift vector fields on V to vector fields on U. For (3), by Lemma 2.8(3) that it suffices to show that

$$\mathcal{S}_U \coloneqq \{W \in \Gamma(U, \ker \mathrm{d}\mathbf{s}_U) : W \text{ is } (\varphi \circ p)\text{-related to an element } \overrightarrow{\alpha} \ \}$$

spans $d_u \mathbf{s}_U$ at every $u \in U$. By assumption,

$$S_V := \{ Z \in \Gamma(V, \ker d\mathbf{s}_V) : Z \text{ is } \varphi\text{-related to an element } \overrightarrow{\alpha} \}$$

spans $d_v \mathbf{s}_V$ at every $v \in V$. Since $\ker d\mathbf{s}_U = p_*^{-1}(\ker d\mathbf{s}_V)$, taking all lifts (via p) of all elements of \mathcal{S}_V we obtain a subset of \mathcal{S}_U which contains $\Gamma_c(\ker(p_*))$ and spans $d_u \mathbf{s}_U$ at every $u \in U$. Finally, p is a morphism of bisubmersions by construction.

2.5.2. Inverses

DEFINITION 2.24. — Let (U, φ, G) be a bisubmersion for \mathcal{B} . Its inverse is $\bar{\varphi} := i \circ \varphi \colon U \to G$, where $i \colon G \to G, i(g) = g^{-1}$. We denote it $(U, \bar{\varphi}, G)$.

PROPOSITION 2.25. — The inverse of a bisubmersion is a bisubmersion.

Proof. — Given a bisubmersion (U, φ, G) , first notice that $\bar{\mathbf{s}}_U := \mathbf{s} \circ \bar{\varphi} = \mathbf{t}_U$ and likewise $\bar{\mathbf{t}}_U := \mathbf{t} \circ \bar{\varphi} = s_U$ are submersions. For condition (2) of Definition 2.4, let $\alpha \in \mathcal{B}$. Since this condition holds for (U, φ, G) , there is $Z \in \mathfrak{X}(U)$ which is φ -related to $\overrightarrow{\alpha}$. Hence Z is $i \circ \varphi$ -related to $i_*\overrightarrow{\alpha} = -\overleftarrow{\alpha}$, and therefore -Z is $\bar{\varphi}$ -related to $\overleftarrow{\alpha}$. Similarly, there is $W \in \mathfrak{X}(U)$ which is φ -related to $\overleftarrow{\alpha}$, and -W is $\bar{\varphi}$ -related to $\overrightarrow{\alpha}$.

For condition (3) of Definition 2.4 notice that the inversion map $i: G \to G$ gives rise to an isomorphism $i_*: \overrightarrow{\mathcal{B}} \to \overleftarrow{\mathcal{B}}$, whence $\overline{\varphi}^{-1}(\overrightarrow{\mathcal{B}}) = \varphi^{-1}(\overleftarrow{\mathcal{B}})$ and $\overline{\varphi}^{-1}(\overleftarrow{\mathcal{B}}) = \varphi^{-1}(\overleftarrow{\mathcal{B}})$. We conclude because (U, φ, G) satisfies condition (3) of Definition 2.4 and $\overline{\mathbf{s}}_U = \mathbf{t}_U, \overline{\mathbf{t}}_U = \mathbf{s}_U$.

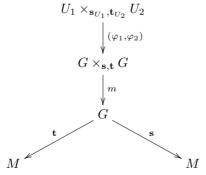
Remark 2.26. — If U is a path holonomy bisubmersion, then its inverse bisubmersion is isomorphic to U itself. This follows from Lemma 2.20.

2.5.3. Compositions

DEFINITION 2.27. — Let (U_j, φ_j, G) be bisubmersions for \mathcal{B} , j = 1, 2. Their composition is

$$m \circ (\varphi_1, \varphi_2) \colon U_1 \times_{\mathbf{s}_{U_1}, \mathbf{t}_{U_2}} U_2 \to G \times_{\mathbf{s}, \mathbf{t}} G \to G$$

where m is the groupoid multiplication on G and $\mathbf{s}_{U_1} = \mathbf{s} \circ \varphi_1$ and $\mathbf{t}_{U_2} = \mathbf{t} \circ \varphi_2$. We denote it $(U_1 \circ U_2, \varphi_1 \cdot \varphi_2, G)$.



Notice that if the open subsets $\mathbf{s}_1(U_1)$ and $\mathbf{t}_2(U_2)$ of M are disjoint, then the composition $U_1 \circ U_2$ is the empty set.

Proposition 2.28. — The composition of two bisubmersions is a bisubmersion.

Proof. — Consider two bisubmersions (U_i, φ_i, G) , i = 1, 2. For condition (1) of Definition 2.4 consider the two natural maps

$$\mathbf{t}_{12}, \mathbf{s}_{12} \colon U_1 \circ U_2 \to M$$

given by $\mathbf{t}_{12} := \mathbf{t} \circ (\varphi_1 \cdot \varphi_2) = \mathbf{t}_{U_1} \circ p_1$ and $\mathbf{s}_{12} := \mathbf{s} \circ (\varphi_1 \cdot \varphi_2) = \mathbf{s}_{U_2} \circ p_2$, where $p_i : U_1 \circ U_2 \to U_i$ are the projections. They are both submersions because the p_i are submersions and because \mathbf{t}_{U_1} and \mathbf{s}_{U_2} are submersions.

For condition (2), let $\alpha \in \mathcal{B}$ and take a vector field $Z_1 \in \mathfrak{X}(U_1)$ which is φ_1 -related to $\overrightarrow{\alpha}$ (it exists since condition (2) holds for the bisubmersions (U_1, φ_1, G)). Notice that $(\overrightarrow{\alpha}, 0)$ restricts to a vector field on $G \times_{\mathbf{s,t}} G$ which is m-related to $\overrightarrow{\alpha}$. Hence the vector field $(Z_1, 0)$ on $U_1 \circ U_2$ is related by $\varphi_1 \cdot \varphi_2 = m \circ (\varphi_1, \varphi_2)$ to $\overrightarrow{\alpha}$. Similarly, taking $W_2 \in \mathfrak{X}(U_2)$ which is φ_2 -related to $\overleftarrow{\alpha}$, we see that $(0, W_2)$ is $\varphi_1 \cdot \varphi_2$ -related to $\overleftarrow{\alpha}$.

Now we prove that $(U_1 \circ U_2, \varphi_1 \cdot \varphi_2, G)$ satisfies condition (3) of Definition 2.4. For simplicity we'll show the first of the equalities appearing there, namely

$$(\varphi_1 \cdot \varphi_2)^{-1}(\overrightarrow{\mathcal{B}}) = \Gamma_c(U_1 \circ U_2; \ker d\mathbf{s}_{12})$$

(the second is proven similarly). First notice that there are distinguished elements of $\Gamma_c(U_1 \circ U_2; \ker d\mathbf{s}_{12})$:

- (*) $(W_1, Z_2) \in \mathfrak{X}(U_1 \circ U_2)$ such that W_1 is φ_1 -related to $i_* \overrightarrow{\alpha}$, and Z_2 is φ_2 -related to $\overrightarrow{\alpha}$, for some $\alpha \in \mathcal{B}$,
- (**) $(Z_1,0)$ such that Z_1 is φ_1 -related to $\overrightarrow{\alpha}$ for some $\alpha \in \mathcal{B}$.

Notice that both families (*) and (**) consist of vector fields which are $\varphi_1 \cdot \varphi_2$ -related to $\overrightarrow{\alpha}$ for some $\alpha \in \mathcal{B}$. Indeed, for any $\alpha \in \mathcal{B}$, $(i_*\overrightarrow{\alpha}, \overrightarrow{\alpha})$ is m-related to the zero vector field on G, and $(\overrightarrow{\alpha}, 0)$ is m-related to $\overrightarrow{\alpha}$, where m is the multiplication. Hence Lemma 2.8(3), together with the following claim, finishes the proof.

CLAIM. — The union of the families of vector fields (*) and (**), evaluated at any point $(u_1, u_2) \in U_1 \circ U_2$, span the kernel of $d\mathbf{s}_{12}$ at (u_1, u_2) .

Fix a vector in the kernel of ds₁₂, that is, $(X_1, X_2) \in T_{u_1}U_1 \times T_{u_2}U_2$ such that

$$d\mathbf{s}_{U_1}(X_1) = d\mathbf{t}_{U_2}(X_2) \text{ and } d\mathbf{s}_{U_2}(X_2) = 0.$$

Since $(\varphi_2)^{-1}(\overrightarrow{\mathcal{B}}) = \Gamma_c(\ker \operatorname{ds}_{U_2})$, by Lemma 2.8(3) we can extend X_2 to a vector field $Z_2 \in \mathfrak{X}(U_2)$ which is φ_2 -related to $\overrightarrow{\alpha}$ for some $\alpha \in \mathcal{B}$. Since U_1 satisfies condition (2) in Definition 2.4, there is $W_1 \in \mathfrak{X}(U_1)$ which is

 φ_1 -related to $i_*\overrightarrow{\alpha}=-\overleftarrow{\alpha}$. Notice that (W_1,Z_2) belongs to the family of vector fields (*).

Further, the map $d_{u_1}\mathbf{s}_{U_1}$ sends both X_1 and $(W_1)|_{u_1}$ to the same vector $d_{u_2}\mathbf{t}_{U_2}(X_2)$. This means that $X_1 - (W_1)|_{u_1}$ lies in $\ker d_{u_1}\mathbf{s}_{U_1}$, and therefore can be extended to a vector field Z_1 lying in $\Gamma(U_1, \ker d\mathbf{s}_{U_1})$. Since $(\varphi_1)^{-1}(\overrightarrow{\mathcal{B}}) = \Gamma_c(U_1, \ker d\mathbf{s}_{U_1})$, by Lemma 2.8(3) we can choose Z_1 so that it is φ_1 -related to some $\overrightarrow{\alpha'}$, i.e. so that $(Z_1, 0)$ belongs to the family of vector fields (**). Now

$$(X_1,X_2) = ((W_1)|_{u_1},X_2) + (X_1 - (W_1)|_{u_1},0) = ((W_1,Z_2) + (Z_1,0))|_{(u_1,u_2)},$$
 proving the claim. \Box

Example 2.29. — Let $\mathcal{B} = \Gamma_c(A)$, where A = Lie(G). We saw in Example 2.15 that (G, Id_G, G) is a bisubmersion for \mathcal{B} . From Proposition 2.28 we deduce that $(G \times_{\mathbf{s,t}} G, m, G)$ is also a bisubmersion for \mathcal{B} . The multiplication $m \colon G \times_{\mathbf{s,t}} G \to G$ is a morphism of bisubmersions.

2.5.4. Bisections

DEFINITION 2.30. — Let (U, φ, G) be a bisubmersion for \mathcal{B} .

- (1) A bisection of (U, φ, G) is a locally closed submanifold **b** of U such that the restrictions of both \mathbf{s}_U and \mathbf{t}_U to **b** are diffeomorphisms from **b** onto open subsets of M.
- (2) Let $u \in U$ and \mathbf{c} a bisection of the Lie groupoid G with $\varphi(u) \in \mathbf{c}$. We say that \mathbf{c} is carried by (U, φ, G) at u if there exists a bisection \mathbf{b} of U such that $u \in \mathbf{b}$ and $\varphi(\mathbf{b})$ is an open subset of \mathbf{c} .

Notice that if **b** is a bisection of (U, φ, G) , then $\varphi(\mathbf{b})$ is a bisection of the Lie groupoid G, and further $\varphi|_{\mathbf{b}} \colon \mathbf{b} \to \varphi(\mathbf{b})$ is a diffeomorphism. This is best seen viewing bisections of (U, φ, G) as images of maps b which are sections of $\mathbf{s}_U \colon U \to M$ so that $\mathbf{t}_U \circ b$ is local diffeomorphism of M.

The existence of bisections at every $u \in U$ of a bisubmersion (U, φ, G) is proven exactly like in [1, Prop. 2.7]. Notice that the proof uses only the fact that $\mathbf{t}_U, \mathbf{s}_U \colon U \to M$ are submersions. The following Proposition is given without proof.

PROPOSITION 2.31. — Let (U, φ, G) and (U_i, φ_i, G) , i = 1, 2 be bisubmersions.

(1) Let $u \in U$ and \mathbf{c} a local bisection of G carried by (U, φ, G) at u. Then \mathbf{c}^{-1} is carried by the inverse bisubmersion $(U, \bar{\varphi}, G)$ at u. (2) Let $u_i \in U_i$, i = 1, 2 be such that $\mathbf{s}_{U_1}(u_1) = \mathbf{t}_{U_2}(u_2)$ and let \mathbf{c}_i be local bisections of G carried by (U_i, φ_i, G) at u_i respectively, i = 1, 2. Then $\mathbf{c}_1 \cdot \mathbf{c}_2$ is carried by the composition $(U_1 \circ U_2, \varphi_1 \cdot \varphi_2, G)$ at (u_1, u_2) .

2.5.5. Modifying a bisubmersion by a bisection

Let $(U, \mathbf{t}_U, \mathbf{s}_U)$ be a bisubmersion for a singular foliation (M, \mathcal{F}) (in the sense of [1], see Section 2.3.1), and **b** a bisection of U. An easy but important fact is that the diffeomorphism carried by the bisection, namely $\phi \colon M \to M, \mathbf{s}_U(u) \mapsto \mathbf{t}_U(u)$ where $u \in \mathbf{b}$, is an automorphism⁽¹⁵⁾ of the singular foliation \mathcal{F} , i.e. $\phi_*\mathcal{F} = \mathcal{F}$. This fact was used in [1, §2.3] to obtain a new bisubmersion for \mathcal{F} out of $(U, \mathbf{t}_U, \mathbf{s}_U)$ and **b**.

We now derive an analog statement for bisubmersions of singular subalgebroids. This will be used in the proof of Corollary 3.3 and in Lemma A.2.

PROPOSITION 2.32. — Let (U, φ, G) be a bisubmersion for the singular subalgebroid \mathcal{B} , and let \mathbf{b} be a bisection of U. Denote by $\mathbf{c} := \varphi(\mathbf{b})$ the induced bisection of the Lie groupoid G, and by $L_{\mathbf{c}} : G \to G$ the left multiplication by \mathbf{c} . Denote by \mathbf{c}^{-1} the image of \mathbf{c} under the inversion map. Then

- $(1) (L_{\mathbf{c}})_* \overrightarrow{\mathcal{B}} = \overrightarrow{\mathcal{B}}.$
- (2) $(U, L_{\mathbf{c}^{-1}} \circ \varphi, G)$ is also a bisubmersion for \mathcal{B} .

Remark 2.33. — Item (1) above can be rephrased saying that the singular subalgebroid \mathcal{B} is preserved by the Lie algebroid automorphism of $A = \ker(\mathbf{s}_*|_M)$ induced by \mathbf{c} . Recall that the conjugation by the bisection \mathbf{c} is a Lie groupoid automorphism of G, which differentiates to the Lie algebroid automorphism $A \to A$, $a_x \mapsto (R_{\mathbf{c}(x)^{-1}})_*(L_{\mathbf{c}})_*(a_x)$, where $\mathbf{c}(x)$ is the unique point of \mathbf{c} with source x. The fact that this Lie algebroid automorphism preserves \mathcal{B} is an immediate consequence of Proposition 2.32(1), upon using the facts that vector fields of the form $(L_{\mathbf{c}})_*(\overrightarrow{\alpha})$ are right-invariant and that the right invariant vector fields in the global hull of $\overrightarrow{\mathcal{B}}$ are exactly the right-translates of elements of \mathcal{B} .

⁽¹⁵⁾ To see this, notice that $(\mathbf{t}_U)|_{\mathbf{b}}$ is an isomorphism of foliated manifolds between the submanifold \mathbf{b} – endowed with the restriction of the pullback foliation $\mathbf{t}_U^{-1}(\mathcal{F})$ – and (M, \mathcal{F}) . The same holds for \mathbf{s}_U in place of \mathbf{t}_U , and the results follows since $\mathbf{t}_U^{-1}(\mathcal{F}) = \mathbf{s}_U^{-1}(\mathcal{F})$.

Proof.

(1). — We view the bisection **b** as a map **b**: $M \to U$ which is a right-inverse to \mathbf{s}_U . In Proposition B.1 we established that $\widehat{U} := U \times_{\mathbf{s}_U, \mathbf{t}} G$, with the target and source maps indicated there, is a bisubmersion (in the sense of [1]) for the singular foliation $\overrightarrow{\mathcal{B}}$ on G. From **b** we obtain a bisection of \widehat{U} , namely

$$\widehat{\mathbf{b}} \colon G \to U \times_{\mathbf{s}_U, \mathbf{t}} G, \ g \mapsto (\mathbf{b}(\mathbf{t}(g)), g).$$

By the text just before Proposition 2.32, the diffeomorphism $\mathbf{t}_{\widehat{U}} \circ \widehat{\mathbf{b}} \colon G \to G$ carried by this bisection is an automorphism of the singular foliation $\overrightarrow{\mathcal{B}}$. This diffeomorphism reads $g \mapsto \mathbf{c}(\mathbf{t}(g)) \cdot g$, i.e. it is exactly $L_{\mathbf{c}}$.

(2). — We check that the fact that (U, φ, G) is a bisubmersion for \mathcal{B} implies that $(U, L_{\mathbf{c}^{-1}} \circ \varphi, G)$ satisfies the three conditions in Definition 2.4. Condition (1) holds because $\mathbf{s} \circ L_{\mathbf{c}^{-1}} = \mathbf{s}$, and $\mathbf{t} \circ L_{\mathbf{c}^{-1}} = \phi \circ \mathbf{t}$ for some diffeomorphism ϕ of M.

For (2), let $\alpha \in \mathcal{B}$. By item (1) (or more precisely Remark 2.33) we have $(L_{\mathbf{c}})_*\overrightarrow{\alpha} = \overrightarrow{\alpha'}$ for some $\alpha' \in \mathcal{B}$. Hence any $Z \in \mathfrak{X}(U)$ which is φ -related to $\overrightarrow{\alpha'}$ is also $(L_{\mathbf{c}^{-1}} \circ \varphi)$ -related to $\overrightarrow{\alpha}$. The second part of condition (2) holds trivially since $(L_{\mathbf{c}})_*\overleftarrow{\alpha} = \overleftarrow{\alpha}$.

Finally, the first part of condition (3) holds because $(L_{\mathbf{c}^{-1}} \circ \varphi)^{-1}(\overrightarrow{\mathcal{B}}) = \varphi^{-1}(L_{\mathbf{c}})_*(\overrightarrow{\mathcal{B}}) = \varphi^{-1}(\overrightarrow{\mathcal{B}})$, where the last equation holds by item (1). The second part of condition (3) holds similarly.

Remark 2.34. — A variation of Proposition 2.32(2) is the following: under the same hypotheses of the proposition, $(U, L_{\mathbf{c}} \circ \varphi, G)$ is also a bisubmersion for \mathcal{B} . Indeed, since $(L_{\mathbf{c}})^{-1} = L_{\mathbf{c}^{-1}}$, Proposition 2.32(1) implies that $(L_{\mathbf{c}^{-1}})_* \overrightarrow{\mathcal{B}} = \overrightarrow{\mathcal{B}}$, and the proof of Proposition 2.32(2) gives the claimed result.

3. The Holonomy Groupoid

In this whole section we fix an integrable Lie algebroid $A \to M$ and a singular subalgebroid \mathcal{B} . Further, we fix a Lie groupoid G integrating A.

We give the construction of the holonomy groupoid associated with a singular subalgebroid \mathcal{B} , relying on the methods developed in [1]. In particular, our Section 3.1 and Section 3.2 follow closely [1]. A new feature is that the holonomy groupoid depends on the choice of G; in Section 3.4 we describe this dependence.

3.1. Comparison of bisubmersions

We start with a technical result, needed in the proof of Proposition 3.2.

LEMMA 3.1. — Let (U, φ, G) be a bisubmersion for the singular subalgebroid $\mathcal{B}, u \in U$, and $\alpha_1, \ldots, \alpha_n \in \mathcal{B}$ which induce a linearly independent set of vectors in $\mathcal{B}/I_{\mathbf{t}_U(u)}\mathcal{B}$. Let $Y_1, \ldots, Y_n \in \Gamma(U, \ker d\mathbf{s}_U)$ such that Y_i is φ -related to $\overrightarrow{\alpha}_i$ for every $i = 1, \ldots, n$. Then $Y_1(u), \ldots, Y_n(u)$ are linearly independent.

Proof. — The existence of the Y_i follows from Definition 2.4(2). We show that they are linearly independent at u.

First, recall from Remark 2.9 that there is a well-defined map of $C^{\infty}(U)$ modules $d\varphi \colon \Gamma_c(U; \ker d\mathbf{s}_U) \to \varphi^*(\overrightarrow{\mathcal{B}})$. Denote by $\mathbf{t} \colon G \to M$ the target
map of the Lie groupoid. Upon the identification between the pullback
vector bundle $\mathbf{t}^*(\ker d\mathbf{s}|_M)$ and $\ker d\mathbf{s}$ given by right-translation, we have $\mathbf{t}^*\mathcal{B} = \overrightarrow{\mathcal{B}}$. Hence, since $\mathbf{t}_U = \mathbf{t} \circ \varphi$, the above map can be written as $d\varphi \colon \Gamma_c(U; \ker d\mathbf{s}_U) \to \mathbf{t}_U^*\mathcal{B}$.

Take constants λ_i such that $\sum \lambda_i Y_i$ vanishes at u. We need to show that $\lambda_i = 0$ for all i. By Definition 2.4(3) and Lemma 2.8(3) there are elements $W_1, \ldots, W_k \in \Gamma(U, \ker \operatorname{ds}_U)$ which form a frame for $\ker \operatorname{ds}_U$ over a neighborhood U_0 of u, and which are φ -related to elements $\overrightarrow{\beta}_j$, for some $\beta_i \in \mathcal{B}$.

On U_0 we have $\sum \lambda_i Y_i = \sum f_j W_j$ for some $f_j \in I_u \subset C^{\infty}(U_0)$. Applying $d\varphi$ to this equation we obtain

(3.1)
$$\sum \lambda_i \, \mathbf{t}_U^*(\boldsymbol{\alpha}_i) = \sum f_j \, \mathbf{t}_U^*(\boldsymbol{\beta}_j)$$

Choose a (locally defined) section τ of the submersion $\mathbf{t}_U : U \to M$. Notice that the l.h.s. of equation (3.1) is the pullback by \mathbf{t}_U of an element of \mathcal{B} , namely $\sum \lambda_i \alpha_i$, hence the above expression is determined by its value on the image $\operatorname{Im}(\tau)$. Therefore the value of the r.h.s. of equation (3.1) is unchanged if we replace the coefficients f_j with $\mathbf{t}_U^* F_j$, where $F_j = \tau^* f_j \in C^{\infty}(M)$. Hence we have the following equality of elements of \mathcal{B} :

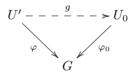
$$\sum \lambda_i \alpha_i = \sum F_j \beta_j.$$

Since $F_j \in I_{\mathbf{t}_U(u)}$, the image of this element in $\mathcal{B}/I_{\mathbf{t}_U(u)}\mathcal{B}$ vanishes. But the image is $\sum \lambda_i[\alpha_i]$, and from the linear independence of the $[\alpha_i]$, we conclude that $\lambda_i = 0$ for all i.

For the following fundamental result, recall that the minimality of a set of generators is defined in Remark 1.11.

PROPOSITION 3.2. — Let $x \in M$ and $\alpha_1, \ldots, \alpha_n \in \mathcal{B}$ which form a minimal set of generators of \mathcal{B} around x. Let (U_0, φ_0, G) be the path holonomy bisubmersion they define (see Definition 2.16). Let (U, φ, G) be a bisubmersion of \mathcal{B} and suppose that $u \in U$ with $\varphi(u) = 1_x$ carries the identity bisection 1_M of G.

Then there exists an open neighborhood U' of u in U and a submersion $g: U' \to U_0$ which is a morphism of bisubmersions and g(u) = (0, x).



Proof. — Replacing U by an open subset containing u, we may assume $\mathbf{s}_U(U) \subset \mathbf{s}_{U_0}(U_0)$. By Lemma 3.1 there are $Y_1, \ldots, Y_n \in \Gamma(U, \ker \mathrm{d}\mathbf{s}_U)$ such that Y_i is φ -related to $\overrightarrow{\alpha}_i$ for every $i=1,\ldots,n$, and the $Y_1(u),\ldots,Y_n(u)$ are linearly independent. Let $Z'_{n+1},\ldots,Z'_k \in \Gamma(U,\ker \mathrm{d}\mathbf{s}_U)$ such that $(Y_1,\ldots,Y_n,Z'_{n+1},\ldots,Z'_k)$ is a frame of $\ker \mathrm{d}\mathbf{s}_U$ nearby u. Consider as in Remark 2.9 the map $\mathrm{d}\varphi\colon\Gamma_c(U;\ker \mathrm{d}\mathbf{s}_U)\to\varphi^*(\overrightarrow{\mathcal{B}})$. For all $i=n+1,\ldots,k$ consider also $\mathrm{d}\varphi(Z'_i)\in\varphi^*(\overrightarrow{\mathcal{B}})$. Since $\varphi^*(\overrightarrow{\mathcal{B}})$ is generated by $\{\varphi^*\overrightarrow{\alpha}_j\}_{j=1,\ldots,n}$ nearby u, there exist functions f_i^j nearby u such that $\mathrm{d}\varphi(Z'_i)=\mathrm{d}\varphi(\xi_i)$, where $\xi_i:=\sum_{j=1}^n f_i^j Y_j$. Put $Z_i=Z'_i-\xi_i$. Then $(Y_1,\ldots,Y_n,Z_{n+1},\ldots,Z_k)$ is also a frame nearby u, since the ξ 's are linear combinations of the Y_i 's, and further

(3.2)
$$\varphi_*(Y_i) = \overrightarrow{\alpha}_i \text{ for } i \leq n \qquad \varphi_*(Z_i) = 0 \text{ for } i > n.$$

To unify the notation, denote $Y_i := Z_i$ for i > n.

For $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ small enough, we denote by ψ_{λ} the partially defined diffeomorphism $\exp(\sum_{i=1}^k \lambda_i Y_i)$ of U. Denote by \mathbf{b} a bisection of U through u carrying the identity bisection of G. There is an open neighborhood $\mathbf{b}' \subset \mathbf{b}$ of u and an open ball B^k in \mathbb{R}^k such that

$$h \colon \mathbf{b}' \times B^k \to U', \quad (y, \lambda) \mapsto \psi_{\lambda}(y)$$

is a diffeomorphism of $\mathbf{b}' \times B^k$ into an open neighborhood U' of u in U. Notice that for all $y \in \mathbf{b}'$ we have

(3.3)
$$\varphi(\psi_{\lambda}(y)) = \exp_{\varphi(y)} \left(\sum_{i=1}^{n} \lambda_{i} \overrightarrow{\boldsymbol{\alpha}}_{i} \right),$$

where the sum runs only until n as a consequence of equation (3.2). Let $p: \mathbb{R}^k \to \mathbb{R}^n$ be the projection to the first n coordinates. Use the diffeomorphism $\varphi|_{\mathbf{b}'}$ to identify \mathbf{b}' with an open subset of M (thereby changing

the domain of h). We define

$$g := p \circ h^{-1} \colon U' \to U_0.$$

The map g is a morphism of bisubmersions by equation (3.3), and it is a submersion.

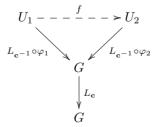
Corollary 3.3 below allows to define the equivalence relation giving rise to the holonomy groupoid.

COROLLARY 3.3. — Let (U_i, φ_i, G) , i = 1, 2 be bisubmersions of \mathcal{B} and $u_i \in U_i$ such that $\varphi_1(u_1) = \varphi_2(u_2)$.

- (1) If the identity bisection 1_M of G is carried by U_i at u_i , for i = 1, 2, there exists an open neighborhood U'_1 of U_1 and a morphism of bisubmersions $f: U'_1 \to U_2$ such that $f(u_1) = u_2$.
- (2) If there is a bisection of G carried by both U_1 at u_1 and by U_2 at u_2 , there exists an open neighborhood U'_1 of u_1 in U_1 and a morphism of bisubmersions $f: U'_1 \to U_2$ such that $f(u_1) = u_2$.
- (3) If there is a morphism of bisubmersions $g: U_1 \to U_2$ such that $g(u_1) = u_2$, then there exists an open neighborhood U'_2 of u_2 in U_2 and a morphism of bisubmersions $f: U'_2 \to U_1$ such that $f(u_2) = u_1$.

Proof. — Given Proposition 3.2, (1) is proven exactly as in [1, Cor. 2.11 a)].

(2). — By assumption there are bisections \mathbf{b}_i of U_i through u_i (i=1,2) and a bisection \mathbf{c} of G through $\varphi_1(u_1) = \varphi_2(u_2)$ such that $\varphi_1(\mathbf{b}_1)$ and $\varphi_2(\mathbf{b}_2)$ are open subsets of \mathbf{c} . By Proposition 2.32, $(U_i, L_{\mathbf{c}^{-1}} \circ \varphi_i, G)$ are bisubmersions for \mathcal{B} , where $L_{\mathbf{c}^{-1}}$ denotes the automorphism of G given by left multiplication with the bisection $\mathbf{c}^{-1} \coloneqq \{g^{-1} : g \in \mathbf{c}\}$. The points u_i of these bisubmersions carry 1_M , for the \mathbf{b}_i are bisections of these bisubmersions whose images in G are contained in 1_M . By (1) we therefore obtain a map f making the following diagram commute:



Therefore f is the desired morphism.

(3). — Let **b** be a bisection of U_1 through u_1 . Then $g(\mathbf{b}_1)$ is a bisection of U_2 through u_2 . Both **b** and $g(\mathbf{b}_1)$ carry the same bisection of G, that

is, $\varphi_2(g(\mathbf{b})) = \varphi_1(\mathbf{b})$. Hence we can apply item (2) to obtain the existence of f.

3.2. Construction of the holonomy groupoid

Recall that we fixed an integrable Lie algebroid $A \to M$, a singular subalgebroid \mathcal{B} , and a Lie groupoid G integrating A.

DEFINITION 3.4. — Consider a family $(U_i, \varphi_i, G)_{i \in I}$ of source-connected minimal path-holonomy bisubmersions defined as in Definition 2.16 such that $M = \bigcup_{i \in I} \mathbf{s}_{U_i}(U_i)$. Let \mathcal{U} be the collection of all such bisubmersions, together with their inverses and finite compositions. (We can omit the inverses, by Remark 2.26). We call \mathcal{U} a path holonomy atlas of \mathcal{B} .

Corollary 3.3(3) shows that for $u_1 \in (U_1, \varphi_1, G), u_2 \in (U_2, \varphi_2, G)$ the relation

 $u_1 \sim u_2 \iff$ there is an open neighborhood U_1' of u_1 , there is a morphism of bisubmersions $f: U_1' \to U_2$

such that
$$f(u_1) = u_2$$

is an equivalence relation. This allows us to give the following definition:

DEFINITION 3.5. — Let G be a Lie groupoid and \mathcal{B} a singular subalgebroid of Lie(G). The holonomy groupoid of \mathcal{B} over G is

$$H^G(\mathcal{B}) \coloneqq \coprod_{U \in \mathcal{U}} U / \sim$$

We write $H(\mathcal{B})$ instead of $H^G(\mathcal{B})$ when the choice of G is understood.

Remark 3.6. — The equivalence relation \sim can be made more explicit as follows, as a consequence of Corollary 3.3(2):

$$u_1 \sim u_2 \Leftrightarrow \varphi_1(u_1) = \varphi_2(u_2),$$

 \exists bisections \mathbf{b}_1 through u_1 , \mathbf{b}_2 through u_2 , s.t. $\varphi_1(\mathbf{b}_1) = \varphi_2(\mathbf{b}_2)$.

Denote by $\sharp: \coprod_{U \in \mathcal{U}} U \to H^G(\mathcal{B})$ the quotient map.

Remark 3.7. — In the following, we endow $H^G(\mathcal{B})$ with the quotient topology induced by \natural . For any bisubmersion $U \in \mathcal{U}$, the image $\natural U$ is open in $H^G(\mathcal{B})$, by the very same argument used at the beginning of $[4, \S 3.4]$.

The next proposition justifies the use of the term "groupoid" for $H^G(\mathcal{B})$.

THEOREM 3.8. — Denote $q_U := \natural|_U$ for all $U \in \mathcal{U}$.

- (1) There are maps $\mathbf{s}_H, \mathbf{t}_H \colon H^G(\mathcal{B}) \to M$ such that $\mathbf{s}_H \circ q_U = \mathbf{s}_U$ and $\mathbf{t}_H \circ q_U = \mathbf{t}_U$ for all $U \in \mathcal{U}$.
- (2) There is a topological groupoid structure on $H^G(\mathcal{B})$, with objects M, source and target maps $\mathbf{s}_H, \mathbf{t}_H$ defined above, and with multiplication $q_U(u)q_V(v) := q_{U \circ V}(u,v)$.
- (3) The canonical map

$$\Phi \colon H^G(\mathcal{B}) \to G,$$

determined by $\Phi \circ q_U = \varphi_U$ for all $U \in \mathcal{U}$, is a morphism of topological groupoids covering Id_M .

Proof.

- (1). First notice that the map Φ introduced in (3) is well-defined, by the definition of morphism of bisubmersions (Definition 2.21). Hence $\mathbf{s}_H := \mathbf{s} \circ \Phi$ and $\mathbf{t}_H := \mathbf{t} \circ \Phi$ are well-defined maps $H^G(\mathcal{B}) \to M$. They clearly satify $\mathbf{s}_H \circ q_U = \mathbf{s}_U$ and $\mathbf{t}_H \circ q_U = \mathbf{t}_U$ for all $U \in \mathcal{U}$.
- (2). We prove that the multiplication is well-defined. Let $U, V, U', V' \in \mathcal{U}$ and consider elements satisfying $q_U(u) = q_{U'}(u')$ and $q_V(v) = q_{V'}(v')$. Then there is exist local morphisms of bisubmersions $f: U \to U'$ with $u \mapsto u'$, and $h: V \to V'$ with $v \mapsto v'$. Assume that $\mathbf{s}_U(u) = \mathbf{t}_V(v)$, which implies $\mathbf{s}_{U'}(u') = \mathbf{t}_{V'}(v')$. Since morphisms of bisubmersions preserve the source and target maps, the product map restricts to a well-defined map $(f,h): U \circ V \to U' \circ V'$ with $(u,v) \mapsto (u',v')$ and which is a morphism of bisubmersions, showing that $(u,v) \sim (u',v')$ and therefore $q_{U \circ V'}(u,v) = q_{U' \circ V'}(u',v')$.

For any $x \in M$, the identity element $1_x \in H^G(\mathcal{B})$ is represented by any point u in a bisubmersion $U \in \mathcal{U}$ with $\mathbf{s}_U(u) = \mathbf{t}_U(u) = x$ and so that u carries (locally) the identity bisection of G. (For instance, one can take $U \subset \mathbb{R}^n \times M$ to be a minimal path holonomy bisubmersion and u = (0, x).). The inverse of $q_V(v) \in H^G(\mathcal{B})$ is $q_{\bar{V}}(v)$, where \bar{V} denotes the inverse bisubmersion to V as in Definition 2.24. It is clear that, with these operations, $H^G(\mathcal{B})$ forms a topological groupoid.

(3). — The map Φ is a morphism of topological groupoids, by the definition of inverse and composition of bisubmersions (Definitions 2.24 and 2.27).

Remark 3.9. — In Appendix C we define the notion of atlas for singular subalgebroids, of which \mathcal{U} appearing in Definition 3.4 is an example, and from each atlas we construct a topological groupoid.

Let us work out by hand an elementary example. More classes of examples will be given in Section 3.3.

Example 3.10 (Lie algebras). — Let $\mathcal{B} = A = \mathfrak{g}$ a Lie algebra and G any connected Lie group integrating \mathfrak{g} . There is a neighborhood U of the origin in \mathfrak{g} such that the exponential map $U \to G$ is a bisubmersion. Indeed, for any basis of \mathfrak{g} , consider the induced path-holonomy bisubmersion $\mathbb{R}^n \to G$; upon the identification $\mathfrak{g} \cong \mathbb{R}^n$ given by the basis, it is the exponential map, as the latter is obtained taking integral curves starting at the identity of right-invariant vector fields. The n-fold composition of this bisubmersion is

$$U^{\times n} := U \times \cdots \times U \to G, \quad (v_1, \dots, v_n) \mapsto \exp(v_1) \dots \exp(v_n).$$

The map $\cup_n U^{\times n} \to G$ is surjective and, by the definition of holonomy groupoid, it descends to an injective map $H^G(\mathfrak{g}) \to G$. We conclude that $H^G(\mathfrak{g})$ is isomorphic (as a topological groupoid) to G.

3.3. Examples

We give some examples of the holonomy groupoid defined in Section 3.2. We do so for the two basic examples of singular subalgebroid displayed in Section 1.2: singular foliations and wide Lie subalgebroids. For wide Lie subalgebroids we show that $H^G(\mathcal{B})$ agrees with H_{\min} , the minimal integral of B over G of Moerdijk and Mrčun recalled in the Introduction. Wide Lie subalgebroids are treated as a special case of images of Lie algebroid morphisms covering diffeomorphisms of the base (Section 1.3.1). Unfortunately at the moment we have no way to describe explicitly the holonomy groupoids for the other classes of singular subalgebroids displayed in Section 1.3.

3.3.1. For singular foliations

Example 3.11 (Singular foliations). — When A = TM (so \mathcal{B} is a singular foliation on M), and $G = M \times M$, then $H^{M \times M}(\mathcal{B})$ is the holonomy groupoid of the singular foliation as defined in [1]. This follows from Example 2.17 and comparing the construction in Section 3.2 to the one of [1].

In Example 3.29 we will take $G = \Pi(M)$ (the fundamental groupoid of M, which is source simply connected) and construct the topological groupoid $H^{\Pi(M)}(\mathcal{B})$. We will see that $H^{\Pi(M)}(\mathcal{B})$ is not source simply connected in general, and has $H^{M \times M}(\mathcal{B})$ as a quotient.

3.3.2. For singular subalgebroids arising from Lie groupoid morphisms

The next proposition will allow us to construct holonomy groupoids for several classes of singular subalgebroids. It is based on the ideas explained in [1, Ex. 3.4(4)].

PROPOSITION 3.12. — Let G be a Lie groupoid over M and \mathcal{B} a singular subalgebroid of $\mathrm{Lie}(G)$. Let K be a Lie groupoid over M. Let $\varphi \colon K \to G$ be a morphism of Lie groupoids covering Id_M which is also a bisubmersion for \mathcal{B} . Then:

- (1) $H^G(\mathcal{B}) = K/\mathcal{I}$ as topological groupoids, where
- $\mathcal{I} := \{k \in K : \exists \text{ a local bisection } \mathbf{b} \text{ through } k \text{ such that } \varphi(\mathbf{b}) \subset 1_M \}$
- (2) the canonical map $\Phi \colon H^G(\mathcal{B}) \to G$ coincides with the map $K/\mathcal{I} \to G$ induced by φ .

Remark 3.13. — We explain the notation and terminology in the definition of \mathcal{I} in the above proposition. The symbol 1_M denotes the set of identity elements of the Lie groupoid G. The term "bisection" refers to bisection for the Lie groupoid K. This coincides with the notion of bisection for the bisubmersion (K, φ, G) (in the sense of Definition 2.30) since the morphism of Lie groupoids φ covers Id_M .

The set \mathcal{I} is a normal subgroupoid of K. This means that \mathcal{I} is a wide subalgebroid and is contained in the union of the isotropy subgroups of K (a consequence of $\mathcal{I} \subset \ker(\varphi)$). Further, it means that \mathcal{I} is invariant under conjugation: for every $k \in K$ we have $k\mathcal{I}_y k^{-1} \subset \mathcal{I}_x$ where $y = \mathbf{s}_K(k)$ and $x = \mathbf{t}_K(k)$. The quotient K/\mathcal{I} has a unique topological groupoid structure such that the quotient map $K \to K/\mathcal{I}$ is a groupoid morphism (see [17, Prop. 2.2.3]).

Remark 3.14. — Every normal subgroupoid \mathcal{I} of K corresponds to the equivalence relation

$$k_1 \sim_K k_2 \Leftrightarrow k_1(k_2)^{-1} \in \mathcal{I}$$

For \mathcal{I} as in Proposition 3.12 this relation can be written down⁽¹⁶⁾ explicitly:

(3.4)
$$k_1 \sim_K k_2 \Leftrightarrow \varphi(k_1) = \varphi(k_2)$$
 and

 \exists local bisections \mathbf{b}_1 through k_1 , \mathbf{b}_2 through k_2 , s.t. $\varphi(\mathbf{b}_1) = \varphi(\mathbf{b}_2)$.

⁽¹⁶⁾ To see that $k_1(k_2)^{-1} \in \mathcal{I}$ implies that k_1 and k_2 satisfy the conditions on the r.h.s. of (3.4), notice the following: if **b** is a bisection through $k_1(k_2)^{-1}$ such that $\varphi(\mathbf{b}) \subset 1_M$ and \mathbf{b}_2 is any bisection through k_2 , then $\mathbf{b} \cdot \mathbf{b}_2$ is a bisection through k_1 such that $\varphi(\mathbf{b} \cdot \mathbf{b}_2) = \varphi(\mathbf{b}_2)$.

Analogously to Remark 3.6, the equivalence (3.4) can be rephrased as follows:

$$k_1 \sim_K k_2 \iff \exists \text{ neighborhood } K' \text{ of } k_1$$

and $f \colon K' \to K \text{ satisfying } \varphi \circ f = \varphi \text{ and } f(k_1) = k_2.$

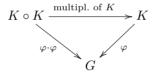
Proof of Proposition 3.12. — The proof relies on Appendix C.

(1). — We first show the following claim:

Claim. —
$$H^G(\mathcal{B}) = K/\sim_K$$
 as topological spaces.

The set $\mathcal{U} := \{(K, \varphi, G)\}$ is an atlas for \mathcal{B} (see Definition C.1) consisting of just one bisubmersion. This holds by the following two arguments which are consequences of the fact that φ is a groupoid morphism:

- The inverse bisubmersion $(K, i_G \circ \varphi, G)$ is adapted to \mathcal{U} because the inversion map i_K is a morphism of bisubmersions from $(K, i_G \circ \varphi, G)$ to (K, φ, G) .
- The composition of bisubmersions $K \circ K$ is also adapted to \mathcal{U} . Indeed it agrees with the space of composable arrows of K and the groupoid multiplication of K is a morphism of bisubmersions, i.e. this diagram commutes:



The atlas \mathcal{U} is equivalent to a path holonomy atlas (Definition 3.4). To prove this, by Proposition C.3, we just need to show that the bisubmersion (K, φ, G) is adapted to a path holonomy atlas. Proposition 3.2 implies that for any $x \in M$, there exists a morphism of bisubmersions from an open neighborhood of 1_x in K to some path holonomy bisubmersion. Then simply use that the Lie groupoid K is generated by such neighborhoods.

As $\mathcal{U} = \{(K, \varphi, G)\}$ is equivalent to a path holonomy atlas, by Proposition C.2 we have $H(\mathcal{B})^{\mathcal{U}} = H^{G}(\mathcal{B})$. The former, as a topological space, is defined as the quotient of K by the equivalence relation \sim_{K} (see Equation (C.1)). This proves the claim.

Now we can show that $H^G(\mathcal{B}) = K/\sim_K$ as groupoids. Let $k, k' \in K$ such that $\mathbf{s}_K(k) = \mathbf{t}_K(k')$. First, the product of [k] and [k'] in $H(\mathcal{B})^{\mathcal{U}} = H^G(\mathcal{B})$ is the class of $k \circ k'$ in the composition of bisubmersions $K \circ K$. Second, as seen above, the groupoid multiplication is a morphism of bisubmersions

from $K \circ K$ to K, hence $k \circ k' \sim_K kk'$. Combining the last two statements we obtain $[k] \cdot [k'] = [kk']$.

(2). — The canonical map $H(\mathcal{B})^{\mathcal{U}} \to G$ is the map $K/\mathcal{I} \to G$ induced by φ . Now apply Proposition C.2(2).

We immediately obtain a construction for the holonomy groupoid of singular subalgebroids arising from Lie groupoid morphisms (covering the identity), see Definition 1.6:

PROPOSITION 3.15. — Assume that the singular subalgebroid \mathcal{B} arises from a Lie groupoid morphism $\Psi \colon K \to G$ covering the identity. Then $H^G(\mathcal{B}) = K/\mathcal{I}$, where \mathcal{I} is an in Proposition 3.12, and the canonical map $\Phi \colon H^G(\mathcal{B}) \to G$ is the map $K/\mathcal{I} \to G$ induced by Ψ .

Proof. — By Proposition 2.13, $\Psi \colon K \to G$ is a bisubmersion for \mathcal{B} . Hence we can apply Proposition 3.12.

Examples 3.16 (Singular foliations arising from Lie algebroids).

(1) The following example is a rephrasing of [1, Ex. 3.4(4)]. Let A be a Lie algebroid over M, and let $\mathcal{F} := \rho(\Gamma_c(A))$ be the singular foliation on M associated to A (here ρ is the anchor map). Let $K \rightrightarrows M$ be a Lie groupoid integrating A. The Lie groupoid morphism given by the target-source map

$$\Psi := (\mathbf{t}, \mathbf{s}) \colon K \to M \times M$$

integrates the anchor map, so it gives rise to \mathcal{F} , i.e. $\mathcal{F} = \{\Psi_*(\Gamma_c(A))\}$. Therefore Proposition 3.15 implies that the holonomy groupoid $H(\mathcal{F})$ of the foliation is

$$H(\mathcal{F}) = K/\mathcal{I},$$

where \mathcal{I} consists of the elements $k \in K$ through which passes a local bisection inducing the identity (local) diffeomorphism on M.

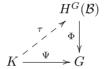
(2) We now spell out a special case of the above (for linear actions compare with [1, Ex. 3.7]). Consider an action of a Lie group G on M. It gives rise to a singular foliation \mathcal{F} on M, which is generated by the image of the associated infinitesimal action $\psi \colon \mathfrak{g} \to \mathfrak{X}(M), v \to v_M$. Let $A := \mathfrak{g} \times M$ be the transformation algebroid of the infinitesimal action. Its anchor map $\rho \colon \mathfrak{g} \times M \to TM, (v, p) \mapsto v_M(p)$ satisfies $\mathcal{F} = \rho(\Gamma_c(A))$. By (1), the holonomy groupoid of \mathcal{F} is obtained from the transformation groupoid $G \times M \rightrightarrows M$ as

$$H(\mathcal{F}) = (G \times M)/\mathcal{I},$$

where \mathcal{I} is very explicit: it consists of $(g,x) \in G \times M$ (necessarily with $g \cdot x = x$) for which there is a neighborhood U of x in M and a smooth map $\widetilde{g} \colon U \to G$ such that $\widetilde{g}(x) = g$ and $\widetilde{g}(y) \cdot y = y$ for all $y \in U$. In other words, it consists of elements of $G \times M$ through which there is a local section of the second projection $G \times M \to M$ which lies in the isotropy groups of the action of G on M.

For singular subalgebroids arising from Lie groupoid morphisms, the holonomy groupoid satisfies a minimality property. In particular, it is a quotient of any Lie groupoid giving rise to the given singular subalgebroid.

PROPOSITION 3.17. — Any Lie groupoid morphism covering the identity $\Psi \colon K \to G$ giving rise to a singular subalgebroid \mathcal{B} factors as



where $\tau \colon G \to H^G(\mathcal{B})$ is a surjective morphism of topological groupoids.

Proof. — Thanks to Proposition 3.15, we can take τ to be the quotient map $K \to K/\mathcal{I} = H^G(\mathcal{B})$.

3.3.3. For wide Lie subalgebroids

The proof of the following statement is similar to the one of [6, Prop. 1.9] but has the advantage of relying only on elementary facts.

PROPOSITION 3.18. — Let B be a wide Lie subalgebroid of A, let G be a Lie groupoid integrating A, and denote $\mathcal{B} := \Gamma_c(B)$. Then

- (1) $H^G(\mathcal{B})$ is a Lie groupoid integrating B
- (2) the canonical Lie groupoid morphism $\Phi \colon H^G(\mathcal{B}) \to G$ integrates the inclusion $\iota \colon B \hookrightarrow A$.

Proof. — The Lie algebroid B is integrable since A is [18]. Let K be the source simply connected Lie groupoid integrating B. Let $\Psi \colon K \to G$ the morphism of Lie groupoids which integrates the inclusion $\iota \colon B \hookrightarrow A$.

On one hand, clearly the Lie groupoid morphism Ψ gives rise to \mathcal{B} . Therefore we can apply Proposition 3.15, which states that $H^G(\mathcal{B}) = K/\mathcal{I}$ and the canonical map $\Phi \colon H^G(\mathcal{B}) \to G$ is the map $K/\mathcal{I} \to G$ induced by Ψ .

On the other hand we have the following

Claim. — The subgroupoid \mathcal{I} of K, defined as in Proposition 3.12, satisfies:

- (1) Set-theoretically, \mathcal{I} is a normal subgroupoid of K lying in the union of the isotropy groups of K.
- (2) Topologically, \mathcal{I} is an embedded Lie subgroupoid of K and it is s-discrete (i.e. the intersection of \mathcal{I} with any s-fiber is discrete).

The claim implies (see for example [12, Thm. 1.20]) that K/\mathcal{I} is also a Lie groupoid integrating B. Clearly the Lie groupoid morphism $K/\mathcal{I} \to G$ induced by Ψ integrates the inclusion $\iota \colon B \hookrightarrow A$. This concludes the proof of the proposition, modulo the claim which we prove right now.

That (1) in the claim holds was explained in Remark 3.13. We argue that (2) holds. There is a neighborhood $V \subset K$ of the set of identities 1_M on which the morphism Ψ is injective (this follows from Ψ being a Lie groupoid morphism covering the identity and whose Lie algebroid map is injective). Since $\mathcal{I} \subset \ker(\Psi)$, we have $\mathcal{I} \cap V = 1_M$. Let $k \in \mathcal{I}$, and take a bisection \mathbf{b} of K through k as in the definition of \mathcal{I} . Denote $U := \mathbf{s}(\mathbf{b})$, an open subset of M. Denote $r_{\mathbf{b}} \colon \mathbf{s}^{-1}(U) \to \mathbf{s}^{-1}(U)$ the diffeomorphism given by right-multiplication by the bisection \mathbf{b} . It maps $1_{\mathbf{s}(g)}$ to k and it preserves \mathcal{I} , since \mathbf{b} lies in the subgroupoid \mathcal{I} . Applying $r_{\mathbf{b}}$ to

$$\mathcal{I} \cap (V \cap \mathbf{s}^{-1}(U)) = 1_U$$

we obtain that the intersection of \mathcal{I} with $r_{\mathbf{b}}(V) \cap \mathbf{s}^{-1}(U)$ (an open neighbourhood of k) is exactly \mathbf{b} . This shows both that \mathcal{I} is an embedded submanifold (hence, a *Lie subgroupoid*) of K and that \mathcal{I} is \mathbf{s} -discrete.

Remark 3.19. — Recall from Section 1.3.1 that \mathcal{B} is a projective singular subalgebroid if $\mathcal{B} \cong \Gamma_c(E)$ for some vector bundle $E \to M$, which automatically comes with a Lie algebroid structure and a morphism $\tau \colon E \to A$. In [7], generalizing Proposition 3.18 and its proof, we show that $H^G(\mathcal{B})$ is a Lie groupoid iff \mathcal{B} is projective, that the Lie groupoid $H^G(\mathcal{B})$ integrates E, and that the canonical morphism $\Phi \colon H^G(\mathcal{B}) \to G$ integrates τ .

We now refine Proposition 3.18, showing that in the case of wide Lie subalgebroids $H^G(\mathcal{B})$ is exactly the the minimal integral of B over G defined in the work of Moerdjik and Mrčun [18, Thm. 2.3] recalled in the Introduction.

PROPOSITION 3.20. — Let B be a wide Lie subalgebroid of an integrable Lie algebroid A, and fix a Lie groupoid G integrating A. Let $\mathcal{B} = \Gamma_c(B)$. Then

- (1) $H^G(\mathcal{B})$ agrees with H_{\min} , the minimal integral of B over G recalled in the introduction,
- (2) the canonical map $\Phi \colon H^G(\mathcal{B}) \to G$ agrees with the map recalled in the introduction.

Proof. — By Proposition 3.18 the canonical map $\Phi \colon H^G(\mathcal{B}) \to G$ satisfies properties (1) and (2) from [18, Thm. 2.3] recalled in the Introduction. Any Lie groupoid morphism $\widetilde{H} \to G$ integrating the inclusion $\iota \colon B \hookrightarrow A$ is a Lie groupoid morphism giving rise to \mathcal{B} . Hence, by Proposition 3.17, property (3) from that theorem holds too. The uniqueness statement in that theorem finishes the proof.

The following two examples generalize the elementary Example 3.10.

Example 3.21 (Lie algebroids). — For any integrable Lie algebroid A take $\mathcal{B} = \Gamma_c(A)$, and let G be a Lie groupoid integrating A. Then $H^G(\mathcal{B}) = G$ and $\Phi = \mathrm{Id}_G$. This follows taking $\Psi = \mathrm{Id}_G$ in Corollary 3.15. (It also follows directly from Proposition 3.20).

Example 3.22 (Lie subalgebras). — Let \mathfrak{g} a Lie algebra, \mathfrak{k} a Lie subalgebra, and fix a connnected Lie group G integrating \mathfrak{g} . Let $\Psi \colon K \to G$ be any morphism of Lie groups integrating the inclusion $\iota \colon \mathfrak{k} \hookrightarrow \mathfrak{g}$, where K is assumed to be connected. (For instance, take K to be the simply connected integration of \mathfrak{k} .) Then

$$H^G(\mathfrak{k}) = K/\ker(\Psi).$$

This follows from Corollary 3.15, noticing that since the space of objects of K is just a point, we have $\mathcal{I} = \ker(\Psi)$. Using Proposition 3.18 we see that $H^G(\mathfrak{k})$ is a Lie group integrating \mathfrak{k} , and the map $\Phi \colon H^G(\mathfrak{k}) \to G$ induced by φ is an injective immersion and group homomorphism. In other words, $(H^G(\mathfrak{k}), \Phi)$ is the Lie subgroup of G integrating \mathfrak{k} .

3.4. Dependence of $H^G(\mathcal{B})$ on G

Let $A \to M$ be a Lie algebroid and \mathcal{B} a singular subalgebroid. Fix a Lie groupoid G integrating A. In Section 3.2 we constructed the holonomy groupoid $H(\mathcal{B}) := H^G(\mathcal{B})$ over M, as the quotient of a path-holonomy atlas of G-bisubmersions (17) by the equivalence relation given by morphisms of G-bisubmersions. Clearly the construction depends on G.

 $^{^{(17)}}$ In this section we use the terminology "G-bisubmersion" instead "bisubmersion", to emphatize the dependence on the Lie groupoid G.

Now take another Lie groupoid \widetilde{G} integrating A, and so that G is a quotient of it, i.e. there is a surjective Lie groupoid morphism

$$\pi\colon \widetilde{G}\to G$$

with discrete fibers. Denote by $\widetilde{H}(\mathcal{B}) := H^{\widetilde{G}}(\mathcal{B})$ the holonomy groupoid constructed using \widetilde{G} . In this subsection we describe $\widetilde{H}(\mathcal{B})$ in terms of $H(\mathcal{B})$.

3.4.1. A theorem describing $\widetilde{H}(\mathcal{B})$ in terms of $H(\mathcal{B})$

Consider the fiber product of the canonical groupoid morphism $\Phi \colon H(\mathcal{B}) \to G$ and $\pi \colon \widetilde{G} \to G$, i.e.

$$H(\mathcal{B}) \times_{\Phi,\pi} \widetilde{G} \rightrightarrows M$$

with the component-wise groupoid structure. Upon the identification between M and the diagonal $\Delta M \subset M \times M$, it is the subgroupoid of the product groupoid $H(\mathcal{B}) \times \widetilde{G} \rightrightarrows M \times M$ given by the preimage of ΔG under the groupoid morphism $(\Phi, \pi) \colon H(\mathcal{B}) \times \widetilde{G} \to G \times G$. Notice that the latter morphism does not have connected fibers in general, so that $H(\mathcal{B}) \times_{\Phi,\pi} \widetilde{G}$ will not be source-connected in general. Hence we consider the source-connected component of the identities:

$$(3.5)$$
 $H(\mathcal{B}) \times_{\Phi,\pi} \widetilde{G})_0$

$$=\{(h,\widetilde{g})\in H(\mathcal{B})\times\widetilde{G}:\exists \text{ a continuous path } (h(t),\widetilde{g}(t))\subset (\mathbf{s}_H,\widetilde{\mathbf{s}})^{-1}(x,x)\}$$

from
$$(1_x^{H(\mathcal{B})}, 1_x^{\widetilde{G}})$$
 to (h, \widetilde{g}) with $\Phi(h(t)) = \pi(\widetilde{g}(t))$,

where $x := \mathbf{s}_H(h) = \widetilde{\mathbf{s}}(\widetilde{g}) \in M$.

Example 3.23. — Take the simple case $\mathcal{B} = \{0\}$. For any choice of G, we have that $H(\mathcal{B})$ is the trivial groupoid $M \rightrightarrows M$. We obtain

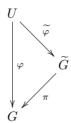
$$H(\mathcal{B}) \times_{\Phi,\pi} \widetilde{G} = M \times_{(\iota_M,\pi)} \widetilde{G} \cong \pi^{-1}(1_M^G) = \ker(\pi)$$

where ι_M is the inclusion of the identity elements 1_M^G into G. Hence $(H(\mathcal{B}) \times_{\Phi,\pi} \widetilde{G})_0$ consists of the identity elements of $\widetilde{G} \rightrightarrows M$, and therefore is isomorphic to the trivial groupoid $M \rightrightarrows M$.

Remark 3.24. — As $\pi \colon \widetilde{G} \to G$ has discrete fibers, for any path γ in the s-fiber of G starting at 1_x^G there exists a unique lift starting at $1_x^{\widetilde{G}}$, i.e. a unique path τ in \widetilde{G} with $\gamma = \pi \circ \tau$ and $\tau(0) = 1_x^{\widetilde{G}}$.

This shows that, in the characterization (3.5) of $(H(\mathcal{B}) \times_{\Phi,\pi} \widetilde{G})_0$, the path $\widetilde{g}(t)$ (hence in particular \widetilde{g}) is determined by the path h(t): indeed, $\widetilde{g}(t)$ is the unique π -lift of $\Phi(h(t))$ starting at $1_x^{\widetilde{G}}$.

Notice that, if we fix a local generating set $\alpha_1, \ldots, \alpha_n$ for \mathcal{B} , it gives rise to two path-holonomy bisubmersions: a G-bisubmersion and a \widetilde{G} -bisubmersion. The domains of both bisubmersions coincide (they are an open subset in $\mathbb{R}^n \times M$). The domains of the compositions of path-holonomy bisubmersions also coincide, hence if \mathcal{U} is a G-path-holonomy atlas, then $\coprod_{U \in \mathcal{U}} U$ will be the domain of a \widetilde{G} -path-holonomy atlas too. Given corresponding bisubmersions (U, φ, G) and $(U, \widetilde{\varphi}, \widetilde{G})$, from Definition 2.16 we see that $\varphi = \pi \circ \widetilde{\varphi} \colon U \to G$:



We denote the quotient maps to the holonomy groupoids by $\sharp|_U: U \to H(\mathcal{B})$ and $\widetilde{\sharp}|_U: U \to \widetilde{H}(\mathcal{B})$ respectively, as in Section 3.2.

The central result of this subsection is the following.

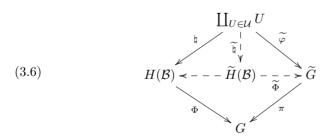
Theorem 3.25. — There is a canonical isomorphism over Id_M of topological groupoids

$$T : \widetilde{H}(\mathcal{B}) \to (H(\mathcal{B}) \times_{\Phi} \pi \widetilde{G})_0 \quad \widetilde{h} \mapsto (\natural u, \widetilde{\Phi}(\widetilde{h}))$$

where u is any point in a path-holonomy atlas with $\widetilde{\mathfrak{h}}u = \widetilde{h}$.

Clearly, under the above isomorphism, the canonical map $\widetilde{\Phi} \colon \widetilde{H}(\mathcal{B}) \to \widetilde{G}$ corresponds to the second projection $(h, \widetilde{g}) \mapsto \widetilde{g}$.

Remark 3.26. — The relevant diagram is



Proof of Theorem 3.25.

Claim. — T is well-defined.

We show that the map

$$\widetilde{H}(\mathcal{B}) \to H(\mathcal{B}), \ \widetilde{\natural} u \mapsto \natural u$$

is well-defined. Let U, V be \widetilde{G} -bisubmersions, and u, v points with $\widetilde{\natural}u = \widetilde{\natural}v$. This means that there is a morphism of \widetilde{G} -bisubmersions $f: U \to V$ with f(u) = v (possibly shrinking U). Composing $\widetilde{\varphi}_V \circ f = \widetilde{\varphi}_U \colon U \to \widetilde{G}$ with $\pi \colon \widetilde{G} \to G$ we find $\varphi_V \circ f = \varphi_U \colon U \to G$ (using $\pi \circ \widetilde{\varphi}_U = \varphi_U$). This shows that f is also a morphism of G-bisubmersions, therefore $\natural u = \natural v$.

Further, the image of T is really contained in $(H(\mathcal{B}) \times_{\Phi,\pi} \widetilde{G})_0$. Indeed, it is contained in the fiber product $H(\mathcal{B}) \times_{\Phi,\pi} \widetilde{G}$ because we have $\Phi(\natural u) = \varphi(u) = \pi(\widetilde{\varphi}(u)) = \pi(\widetilde{\Phi}(\widetilde{h}))$ for all u. It is contained in the source-connected component of the identities because $\widetilde{H}(\mathcal{B})$ is source-connected and, as we shall see immediately, T is a continuous groupoid morphism.

Claim. — T is a groupoid morphism.

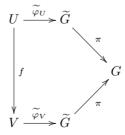
One checks directly that $\widetilde{H}(\mathcal{B}) \to H(\mathcal{B})$, $\widetilde{\natural} u \mapsto \natural u$ is morphism of groupoids. Also, $\widetilde{\Phi} \colon \widetilde{H}(\mathcal{B}) \to \widetilde{G}$ is a morphism of groupoids by Theorem 3.8.

Claim. - T is continuous.

This holds since $\widetilde{H}(\mathcal{B}) \to H(\mathcal{B})$ is continuous (being the map induced by $\xi \colon \coprod_{U \in \mathcal{U}} U \to H(\mathcal{B})$ on the quotient $\widetilde{H}(\mathcal{B})$) and since $\widetilde{\Phi}$ is continuous.

Claim. — T is injective.

Since T is a morphism of groupoids, it suffices to check that if $\natural v = 1_x^{H(\mathcal{B})}$ and $\widetilde{\varphi}_V(v) = 1_x^{\widetilde{G}}$ then $\widetilde{\natural}v = 1_x^{\widetilde{H}(\mathcal{B})}$, for any element v in a bisubmersion V. Let $(U,\widetilde{\varphi}_U,\widetilde{G})$ be a path-holonomy \widetilde{G} -bisubmersion containing (0,x), so $U \subset \mathbb{R}^n \times M$. There exists a morphism of G-bisubmersions $f \colon U \to V$ with f(0,x) = v, by the first assumption above. Notice that the diagram



commutes, since $\varphi_U = \pi \circ \widetilde{\varphi}_U$.

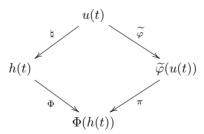
Now consider $S := (\{0\} \times M) \cap U$. We have $\widetilde{\varphi}_U(S) = 1_S^{\widetilde{G}}$, an open subset of the identity bisection of \widetilde{G} , and $(\widetilde{\varphi}_V \circ f)(S)$ is a bisection of \widetilde{G} which, by the second hypothesis above, contains $1_x^{\widetilde{G}}$. Both bisections map under π to 1_S^G , an open subset of the identity bisection of G, by the commutativity of the above diagram. Since π has discrete fibers, it follows that the two bisections of \widetilde{G} agree, i.e. $\widetilde{\varphi}_U(S) = \widetilde{\varphi}_V(f(S))$. Hence Corollary 3.3(2) implies that there is morphism of \widetilde{G} -bisubmersions $U \to V$ with $(0,x) \mapsto v$, hence $\widetilde{\mathfrak{f}}_V = \widetilde{\mathfrak{f}}(0,x) = 1_x^{\widetilde{H}(\mathcal{B})}$.

Claim. — T is surjective.

Let $(h, \widetilde{g}) \in H(\mathcal{B}) \times \widetilde{G}$ so that there is a continuous path $(h(t), \widetilde{g}(t))$ from $(1_x^{H(\mathcal{B})}, 1_x^{\widetilde{G}})$ to (h, \widetilde{g}) with $\mathbf{s}_H(h(t)) = \widetilde{\mathbf{s}}(\widetilde{g}(t)) = x$ and $\Phi(h(t)) = \pi(\widetilde{g}(t))$. We have to show that there is a bisubmersion U in the path-holonomy atlas and $u \in U$ with $\natural u = h, \widetilde{\varphi}(u) = \widetilde{g}$. Since T is a groupoid morphism and every source-connected topological groupoid is generated by any symmetric neighbourhood of the identities, we can assume that (h, \widetilde{g}) is arbitrarily close to the set of identities. Hence, by Remark 3.7, we can assume that there is a path holonomy bisubmersion U_0 such that $h \in \natural U_0$.

Denote by $L \subset M$ the leaf of the foliation $\rho(\mathcal{B})$ through x. As we show in [7], $H(\mathcal{B})|_L := \mathbf{s}_H^{-1}(L)$ has a smooth structure such that for any G-bisubmersion U in the path-holonomy atlas of \mathcal{B} , the quotient map $\natural \colon U|_L := \mathbf{s}_U^{-1}(L) \to H(\mathcal{B})|_L$ is a submersion. Hence there exists a continuous curve u(t) in U_0 with $\natural u(t) = h(t)$ and u(0) = (0, x), where (0, x) lies in a minimal path-holonomy G-bisubmersion. We claim that u := u(1) satisfies the above properties.

By definition we have abla u(1) = h(1) = h. Now consider the part of diagram (3.6) with solid arrows and the paths:



The path $\widetilde{\varphi}(u(t))$ is a π -lift of $\Phi(h(t))$, and its starting point is $\widetilde{\varphi}((0,x)) = 1_x^{\widetilde{G}}$. The same holds for $\widetilde{g}(t)$. Hence by the uniqueness of the π -lift starting at $1_x^{\widetilde{G}}$ (see Remark 3.24) we obtain $\widetilde{\varphi}(u(t)) = \widetilde{g}(t)$, and evaluating at t = 1 we get $\widetilde{\varphi}(u) = \widetilde{g}$.

Claim. — T is A homeomorphism.

It suffices to show that T is an open map. Let (U, φ, G) be a G-bisubmersion in the path-holonomy atlas. Then $\widetilde{\natural}U$ is open in $\widetilde{H}(\mathcal{B})$, by Remark 3.7. We will show that its image $T(\widetilde{\natural}U)$ is open. The Lie groupoid morphism $\pi \colon \widetilde{G} \to G$ is a local homeomorphism, hence the first projection $pr_1 \colon H(\mathcal{B}) \times_{\Phi,\pi} \widetilde{G} \to H(\mathcal{B})$ is also a local homeomorphism. Shrinking U if necessary, we can assume that $T(\widetilde{\natural}U) = \{(\natural u, \widetilde{\varphi}u) : u \in U\}$ is contained in an open subset N of $(H(\mathcal{B}) \times_{\Phi,\pi} \widetilde{G})_0$ such that $pr_1|_N \colon N \to pr_1(N)$ is a homeomorphism onto an open subset of $H(\mathcal{B})$. The subset $pr_1(T(\widetilde{\natural}U))$ equals $\natural U$, which is open in $H(\mathcal{B})$ by Remark 3.7. Hence $T(\widetilde{\natural}U)$ is open in N and therefore in $(H(\mathcal{B}) \times_{\Phi,\pi} \widetilde{G})_0$.

Summarizing: for small enough bisubmersions U in the path-holonomy atlas, T maps the open subsets $\widetilde{\natural}U$ of $\widetilde{H}(\mathcal{B})$ to open subsets. Since any open subset of $\widetilde{H}(\mathcal{B})$ is a union of such $\widetilde{\natural}U$, we are done.

Remark 3.27. — There is a groupoid morphism

$$\widetilde{H}(\mathcal{B}) \to H(\mathcal{B}), \ \widetilde{\natural} u \mapsto \natural u$$

(see the first two claims in the proof of Theorem 3.25), which is clearly surjective. Under the canonical isomorphism $\widetilde{H}(\mathcal{B}) \cong (H(\mathcal{B}) \times_{\Phi,\pi} \widetilde{G})_0$ it is given by the projection onto the first component. Its kernel is

$$(3.7) \qquad \qquad (1_M^{H(\mathcal{B})} \times \widetilde{G}) \cap (H(\mathcal{B}) \times_{\Phi, \pi} \widetilde{G})_0,$$

which is contained in $\{(1_x^{H(\mathcal{B})}, \widetilde{g}) : x \in M, \pi(\widetilde{g}) = 1_x^G\} \cong \ker \pi$. A more explicit description of the kernel can be obtained using equation (3.5). (Here $\pi \colon \widetilde{G} \to G$ is the original covering map.) An explicit example of the groupoid morphism $\widetilde{H}(\mathcal{B}) \to H(\mathcal{B})$ is given in Example 3.31.

Let L be a leaf of \mathcal{B} . Then $\widetilde{H}(\mathcal{B})|_L$ and $\widetilde{H}(\mathcal{B})|_L$ are transitive Lie groupoids over L [7, §2]. The kernel of the surjective morphism $\widetilde{H}(\mathcal{B})|_L \to \widetilde{H}(\mathcal{B})|_L$ is the restriction of (3.7) to L. This kernel is contained in the product of $1_L^{H(\mathcal{B})|_L}$ with $(\mathbf{s}_{\widetilde{G}}^{-1}(L) \cap \mathbf{t}_{\widetilde{G}}^{-1}(L))_0$, the Lie groupoid given by the source connected component of the restriction of \widetilde{G} to L. This follows from the fact that L is contained in a leaf of the Lie groupoid \widetilde{G} .

3.4.2. Examples for Theorem 3.25.

We present a few examples for Theorem 3.25. Example 3.31 in particular shows that even when we use a source simply connected Lie groupoid G to construct the holonomy groupoid $H^G(\mathcal{B})$, the latter might not be source simply connected.

Example 3.28. — When $\mathcal{B} = \Gamma_c(A)$, Theorem 3.25 and Example 3.21 recover the obvious isomorphism $\widetilde{G} \cong G \times_{\mathrm{Id},\pi} \widetilde{G}$.

Example 3.29. — Consider the case of a singular foliation \mathcal{F} . First we integrate the Lie algebroid TM to the pair groupoid $G \coloneqq M \times M$, giving rise to $H(\mathcal{F}) \coloneqq H^G(\mathcal{F})$, the holonomy groupoid of the singular foliation as in [1] (see Example 3.11). We have $\Phi = (\mathbf{t}_H, \mathbf{s}_H) \colon H(\mathcal{F}) \to M \times M$. We can also integrate TM to the fundamental groupoid $\widetilde{G} \coloneqq \Pi(M)$. The construction of Section 3.2 gives rise to another topological groupoid $\widetilde{H}(\mathcal{F}) \coloneqq H^{\widetilde{G}}(\mathcal{F})$, which has a canonical groupoid morphism $\widetilde{\Phi}$ to $\Pi(M)$. By Theorem 3.25 we have

$$\widetilde{H}(\mathcal{F}) \cong (H(\mathcal{F}) \times_{(\mathbf{t}_H, \mathbf{s}_H), \pi} \Pi(M))_0,$$

where $\pi \colon \Pi(M) \to M \times M$ is the target-source map of $\Pi(M)$ (sending the homotopy class of a path in M to its endpoints).

Example 3.29 can be made more explicit when \mathcal{F} is a regular foliation.

Proposition 3.30. — When \mathcal{F} is a regular foliation one obtains

(3.8)
$$(H(\mathcal{F}) \times_{(\mathbf{t}_H, \mathbf{s}_H), \pi} \Pi(M))_0$$

= $\{([\gamma], \langle \gamma \rangle_M) : \gamma \text{ is a path in a leaf of the foliation}\}$

where $[\gamma] \in H(\mathcal{F})$ denotes the holonomy class of γ , and $\langle \gamma \rangle_M \in \Pi(M)$ its homotopy class (fixing endpoints) as a path in M.

Proof. — Use that, by equation (3.5), $(H(\mathcal{F}) \times_{(\mathbf{t}_H, \mathbf{s}_H), \pi} \Pi(M))_0$ equals

$$\{([\delta], \langle \sigma \rangle_M) \in H(\mathcal{F}) \times \Pi(M) :$$

 \exists homotopy $\{\delta^t\}$ in L with $\delta^0 = (\text{loop with trivial holonomy}), <math>\delta^1 = \delta$

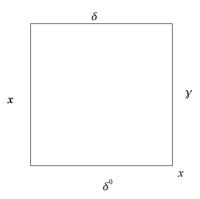
 \exists homotopy $\{\sigma^t\}$ in M with σ^0 =(contractible loop), $\sigma^1=\sigma$

such that
$$\delta^t(0) = \sigma^t(0) = x$$
 and $\delta^t(1) = \sigma^t(1)$ for all t ,

where L denotes the leaf through $x := \delta(0) = \sigma(0)$. Fix a path δ in L and a path σ in M as above (in particular, both start at x and have the same endpoint). We first focus on δ . Consider the map

$$S \colon [0,1] \times [0,1] \to L, (s,t) \mapsto \delta^t(s).$$

As the square is contractible, the holonomy of the restriction of S to the boundary is zero. S maps the left edge of the square to the constant path at x, and the lower edge to the loop δ^0 which has trivial holonomy. Considering the remaining two edges we conclude that $t \mapsto \gamma(t) := \delta^t(1)$ is a path in the leaf L with the same starting/ending point and same holonomy as δ , i.e., $|\delta| = |\gamma|$.



Further $\bar{\gamma} \circ \sigma$, which is defined composing σ with $\bar{\gamma}(t) := \gamma(1-t)$, is a contractible loop in M based at x. Indeed, recalling that $\gamma(t) = \sigma^t(1)$, we see that the family of loops $\gamma|_{[0,t]} \circ \sigma^t$ parametrized by $t \in [0,1]$ provides a contraction, since at time t=0 it equals the contractible loop σ^0 . In other words, $\langle \sigma \rangle_M = \langle \gamma \rangle_M$. Altogether we get the inclusion " \subset " in Equation (3.8). For the opposite inclusion, given a path γ in a leaf, use the homotopy $\{\gamma^t\}$ defined by $\gamma^t(s) = \gamma(st)$ to deform it to the constant path at $\gamma(0)$.

Example 3.31. — As above, let \mathcal{F} be a regular foliation. Denote by D the associated involutive distribution, which in particular is a Lie algebroid. We display three Lie groupoids integrating the Lie algebroid D. The first one is $H(\mathcal{F})$, the holonomy groupoid of \mathcal{F} . The second is $Mon(\mathcal{F})$, the monodromy groupoid of \mathcal{F} . It is a source simply connected Lie groupoid, consisting of all homotopy classes (fixing endpoints) $\langle \gamma \rangle_{\text{leaf}}$ of paths γ in the leaves of the foliation. The third one is $\widetilde{H}(\mathcal{F})$ as in Example 3.29. (It integrates D since it is the minimal integral of D over $\Pi(M)$, by Proposition 3.20.)

As for any (source connected) Lie groupoid integrating D, the Lie groupoid $\widetilde{H}(\mathcal{F})$ is a quotient of $Mon(\mathcal{F})$ and maps surjectively onto $H(\mathcal{F})$. Due to Proposition 3.30 the quotient maps read

$$Mon(\mathcal{F}) \to \widetilde{H}(\mathcal{F}) \to H(\mathcal{F}), \ \langle \gamma \rangle_{\text{leaf}} \mapsto ([\gamma], \langle \gamma \rangle_M) \mapsto [\gamma].$$

In particular we see that, even though $\widetilde{H}(\mathcal{F})$ was constructed using a source simply connected Lie groupoid \widetilde{G} in Example 3.29, $\widetilde{H}(\mathcal{F})$ itself is not source simply connected in general.

4. Morphisms of holonomy groupoids covering the identity

In Section 3, starting from a Lie groupoid G and singular subalgebroid \mathcal{B} of the Lie algebroid $\operatorname{Lie}(G)$, we constructed the holonomy groupoid $H^G(\mathcal{B})$ endowed with a map of topological groupoids to G. In this section we extend this construction to morphisms covering the identity: given a suitable "morphism between singular subalgebroids", we construct a morphism between the associated holonomy groupoids. More precisely, given a morphism of Lie groupoids $F: G_1 \to G_2$ covering the identity on the base and singular subalgebroids \mathcal{B}_i of $\operatorname{Lie}(G_i)$ (i=1,2) with $F_*(\mathcal{B}_1) \subset \mathcal{B}_2$, there is a canonical morphism of topological groupoids

$$H^{G_1}(\mathcal{B}_1) \to H^{G_2}(\mathcal{B}_2)$$

commuting with the canonical maps (see Theorem 4.6).

We consider the simple case of "surjective morphisms between singular subalgebroids" in Section 4.1. The integration of arbitrary morphisms (covering the identity) then follows easily in Section 4.2. All examples are collected in Section 4.3, where we recover in a unified fashion several of the constructions we already gave. We describe the resulting functor in Section 4.4.

It is only for the sake of presentation that in this section we restrict ourselves to morphisms covering the identity on the base manifold. Analogous results for morphisms covering surjective submersions hold, and are collected in Appendix D (see Theorem D.6).

4.1. Surjective morphisms covering the identity

PROPOSITION 4.1. — Let $F: G_1 \to G_2$ be a morphism of Lie groupoids over M, covering Id_M . Let \mathcal{B}_1 be a singular subalgebroid of $A_1 := \mathrm{Lie}(G_1)$, and

$$\mathcal{B}_2 \coloneqq F_*(\mathcal{B}_1) \coloneqq \{F_*(\boldsymbol{\alpha}) : \boldsymbol{\alpha} \in \mathcal{B}_1\},\$$

which clearly is a singular subalgebroid of $Lie(G_2)$.

Then there is a canonical, surjective morphism of topological groupoids

$$\Xi \colon H^{G_1}(\mathcal{B}_1) \to H^{G_2}(\mathcal{B}_2)$$

covering Id_M and making the following diagram commute:

$$H^{G_1}(\mathcal{B}_1) - \stackrel{\Xi}{\longrightarrow} H^{G_2}(\mathcal{B}_2)$$

$$\downarrow^{\Phi_1} \qquad \qquad \downarrow^{\Phi_2}$$

$$G_1 \xrightarrow{F} G_2$$

Proof. — Let $x \in M$ and $\alpha_1, \ldots, \alpha_n \in \mathcal{B}_1$ such that $[\alpha_1], \ldots, [\alpha_n]$ is a basis of $\mathcal{B}_1/I_x\mathcal{B}_1$. Denote by (U, φ, G_1) the associated path holonomy bisubmersion, where $U \subset \mathbb{R}^n \times M$.

Claim. — $(U, F \circ \varphi, G_2)$ is the path holonomy bisubmersion for \mathcal{B}_2 associated to the (not necessarily minimal) set of local generators $\{F_*(\alpha_1), \ldots, F_*(\alpha_n)\}$ of \mathcal{B}_2 .

$$\begin{array}{c}
U \\
\downarrow^{\varphi} \\
G_1 \xrightarrow{F} G_2
\end{array}$$

By Definition 2.16 we have $\varphi \colon U \to G_1, (\lambda, x) \mapsto \exp_x \sum \lambda_i \overrightarrow{\alpha}_i$. Composing with F we obtain

$$(F\circ\varphi)((\lambda,x))=F\left(\exp_x\sum\lambda_i\overrightarrow{\alpha_i}\right)=\exp_{f(x)}\sum\lambda_i\overline{F_*(\alpha_i)}.$$

Here the second equality holds because F being a Lie groupoid morphism implies that $F_*(\vec{\alpha_i}) = \overrightarrow{F_*(\alpha_i)}$.

Consider a family $(U_i, \varphi_i, G_1)_{i \in I}$ of path holonomy bisubmersions for \mathcal{B}_1 such that $M = \cup_{i \in I} \mathbf{s}_i(U_i)$. Let \mathcal{U} be the path holonomy atlas it generates (Definition 3.4), i.e. the collection of the U_i 's together with their inverses and finite compositions. Since F is a Lie groupoid morphism over Id_M , the family $\{(U, F \circ \varphi, G_2) : U \in \mathcal{U}\}$ defines an atlas of bisubmersions for \mathcal{B}_2 . Denote by \sim_i the equivalence relation defined on $\coprod_{U \in \mathcal{U}} U$ viewing the U's as G_i -bisubmersions for \mathcal{B}_i , for i = 1, 2. The equivalence classes of \sim_1 are contained in those of \sim_2 , inducing a surjective morphism of topological groupoids

$$H^{G_1}(\mathcal{B}_1) = \coprod_{U \in \mathcal{U}} U / \sim_1 \to \coprod_{U \in \mathcal{U}} U / \sim_2.$$

The latter groupoid is the holonomy groupoid $H^{G_2}(\mathcal{B}_2)$, by Proposition C.5 and Remark C.6. The morphism is independent of the chosen path-holonomy bisubmersions and hence canonical.

We consider the special case in which the Lie groupoid morphism $F: G_1 \to G_2$ is injective (then $F_*: A_1 \to A_2$ is injective, and therefore induces an isomorphism of $C^{\infty}(M)$ -modules $\mathcal{B}_1 \cong \mathcal{B}_2$). We show that in that case it makes no difference whether we regard \mathcal{B}_1 a singular subalgebroid of $\text{Lie}(G_1)$ or as a singular subalgebroid of $\text{Lie}(G_2)$, for the corresponding holonomy groupoids are isomorphic.

COROLLARY 4.2. — If the Lie groupoid morphism $F: G_1 \to G_2$ is injective, then the canonical morphism of topological groupoids $\Xi: H^{G_1}(\mathcal{B}_1) \to H^{G_2}(\mathcal{B}_2)$ of Proposition 4.1 is an isomorphism.

Proof. — We just have to show that Ξ is injective. To this aim, we show that the equivalence classes of the equivalence relation \sim_2 appearing in the proof of Proposition 4.1 are contained in those of \sim_1 . Let $u \in U$ and $v \in V$ be points of bisubmersions in \mathcal{U} . Assume that $u \sim_2 v$. By definition, this means that there is a locally defined map $\tau \colon U \to V$ taking u to v and which is a morphism of G_2 -bisubmersions, that is,

$$(F \circ \varphi_U) = (F \circ \varphi_V) \circ \tau.$$

Since F is injective, it follows that $\varphi_U = \varphi_V \circ \tau$, that is, τ is a morphism of G_1 -bisubmersions. In particular, $u \sim_1 v$.

Remark 4.3. — If we only assume that Lie algebroid map F_* is injective, but $F: G_1 \to G_2$ itself not, then Ξ is not injective in general. This can be seen already in the case that G_1 and G_2 are non-isomorphic Lie groupoids integrating the same Lie algebroid A, and $F: G_1 \to G_2$ is a surjective but not injective map differentiating the identity at the level of Lie algebroids (use Example 3.21).

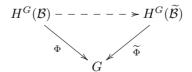
4.2. Arbitrary morphisms covering the identity

We now extend Proposition 4.1 by removing the assumption that the map $F_*|_{\mathcal{B}_1} : \mathcal{B}_1 \to \mathcal{B}_2$ be surjective.

We first look at two singular subalgebroids of the same Lie algebroid, one containing the other.

LEMMA 4.4. — Let G be a Lie groupoid over M, and denote its Lie algebroid by A. Let $\mathcal{B}, \widetilde{\mathcal{B}}$ be a singular subalgebroids of A, with $\mathcal{B} \subset \widetilde{\mathcal{B}}$.

Then there is a canonical morphism of topological groupoids $H^G(\mathcal{B}) \to H^G(\widetilde{\mathcal{B}})$ making the following diagram commute:



Proof. — Let $x \in M$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a minimal set of local generators of \mathcal{B} near x, that is, $[\alpha_1], \ldots, [\alpha_n]$ form a basis of the vector

space $\mathcal{B}/I_x\mathcal{B}$. Let (U_0, φ, G) be the corresponding (minimal) path holonomy bisubmersion for \mathcal{B} , hence $U_0 \subset \mathbb{R}^n \times M$.

The inclusion $\mathcal{B} \subset \widetilde{\mathcal{B}}$ induces a linear map $J \colon \mathcal{B}/I_x\mathcal{B} \to \widetilde{\mathcal{B}}/I_x\widetilde{\mathcal{B}}$, which is generally not injective. Completing the image of the above basis to a spanning set of $\widetilde{\mathcal{B}}/I_x\widetilde{\mathcal{B}}$ we obtain a generating set $\{\alpha_1,\ldots,\alpha_n,\gamma_1,\ldots,\gamma_k\}$ of $\widetilde{\mathcal{B}}$ near x, see Remark 1.11. Let $(\widetilde{U}_0,\widetilde{\varphi},G)$ be the corresponding path holonomy bisubmersion for $\widetilde{\mathcal{B}}$, so $\widetilde{U}_0 \subset \mathbb{R}^n \times \mathbb{R}^k \times M$. The inclusion

$$\iota \colon U_0 \to \widetilde{U}_0, (\lambda, y) \mapsto (\lambda, 0, y)$$

commutes with the maps to $\varphi \colon U_0 \to G$ and $\widetilde{\varphi} \colon \widetilde{U}_0 \to G$ associated to the bisubmersions as in Definition 2.16.

Let \mathcal{U} denote the atlas for \mathcal{B} generated by bisubmersions U_0 as above, and $\widetilde{\mathcal{U}}$ the atlas for $\widetilde{\mathcal{B}}$ generated by the corresponding \widetilde{U}_0 as above. The inclusion ι extends in a straightforward way to compositions of bisubmersions, and hence to a map

(4.1)
$$\iota \colon \coprod_{U \in \mathcal{U}} U \to \coprod_{\widetilde{U} \in \widetilde{\mathcal{U}}} \widetilde{U}$$

commuting with the canonical maps to G.

The latter property assures that if $u \in U$ for some element U of the atlas \mathcal{U} , and \mathbf{b} is a bisection of U through u, then its image $\iota(\mathbf{b})$ is a bisection of \widetilde{U} through $\iota(u)$ and both carry the same bisection of G. By Remark 3.6 this implies that the map (4.1) descends to a morphism of topological groupoids

(4.2)
$$H^{G}(\mathcal{B}) = \coprod_{U \in \mathcal{U}} U / \sim \rightarrow \coprod_{\widetilde{U} \in \widetilde{\mathcal{U}}} \widetilde{U} / \sim = H^{G}(\widetilde{\mathcal{B}})$$

which commutes the canonical maps to G, and also that this morphism is independent of the chosen atlas \mathcal{U} and hence canonical. Notice that the topological groupoid on the r.h.s. is really $H^G(\widetilde{\mathcal{B}})$, by Proposition C.5 and Remark C.6.

Remark 4.5. — The morphism $H^G(\mathcal{B}) \to H^G(\widetilde{\mathcal{B}})$ in Lemma 4.4 is not injective in general. For instance, taking $G = M \times M$, \mathcal{B} a foliation and $\widetilde{\mathcal{B}} = \Gamma_c(TM)$, this map is the target-source map of the holonomy groupoid of the foliation.

The next theorem generalizes Proposition 4.1 and establishes the functoriality of the holonomy groupoid construction:

THEOREM 4.6. — Let $F: G_1 \to G_2$ be a morphism of Lie groupoids covering Id_M . Let \mathcal{B}_i be a singular subalgebroid of $\mathrm{Lie}(G_i)$ for i=1,2,

such that $F_*(\mathcal{B}_1) \subset \mathcal{B}_2$. Then there is a canonical morphism of topological groupoids

$$\Xi \colon H^{G_1}(\mathcal{B}_1) \to H^{G_2}(\mathcal{B}_2)$$

covering Id_M and making the following diagram commute:

$$H^{G_1}(\mathcal{B}_1) - \stackrel{\Xi}{\longrightarrow} H^{G_2}(\mathcal{B}_2)$$

$$\downarrow^{\Phi_1} \qquad \qquad \downarrow^{\Phi_2}$$

$$G_1 \xrightarrow{F} G_2$$

Proof. — Compose the canonical morphism $\Xi \colon H^{G_1}(\mathcal{B}_1) \to H^{G_2}(F_*(\mathcal{B}_1))$ given by Proposition 4.1 with the canonical morphism $H^{G_2}(F_*(\mathcal{B}_1)) \to H^{G_2}(\mathcal{B}_2)$ given by Lemma 4.4.

4.3. Examples

We display a few examples for Proposition 4.1, recovering in a unified manner several results obtained so far. Then we display examples for Theorem 4.6.

Example 4.7 (Images of Lie algebroid morphisms). — Let $F: K \to G$ be a Lie groupoid morphism over Id_M . Applying Proposition 4.1 with $\mathcal{B}_1 := \Gamma_c(\mathrm{Lie}(K))$ we obtain a canonical surjective morphism

$$H^K(\mathcal{B}_1) \to H^G(F_*(\mathcal{B}_1)).$$

In particular the latter groupoid is a quotient of $H^K(\mathcal{B}_1) = K$ (here we used Example 3.21), recovering part of Corollary 3.15.

Example 4.8 (The underlying singular foliation $\mathcal{F}_{\mathcal{B}}$). — Let $G \rightrightarrows M$ be a Lie groupoid, and \mathcal{B} a singular subalgebroid of A := Lie(G). Consider the Lie groupoid morphism

$$F = (\mathbf{t}, \mathbf{s}) \colon G \to M \times M$$

over the identity Id_M . The corresponding Lie algebroid morphism is the anchor map $\rho \colon A \to TM$, hence $F_*(\mathcal{B}) = \mathcal{F}_{\mathcal{B}}$, the singular foliation on M induced by \mathcal{B} . Proposition 4.1 implies the existence of a canonical surjective morphism of topological groupoids

$$(4.3) H^G(\mathcal{B}) \to H(\mathcal{F}_{\mathcal{B}})$$

to the holonomy groupoid [1] of the singular foliation $\mathcal{F}_{\mathcal{B}}$.

We describe explicitly the kernel of the morphism $H^G(\mathcal{B}) \to H(\mathcal{F}_{\mathcal{B}})$. In the special case that $H^G(\mathcal{B})$ is a Lie groupoid, the kernel is

$$\left\{ h \in H^G(\mathcal{B}) \middle| \begin{array}{l} \exists \text{ a local bisection through } h \text{ carrying} \\ \text{the identity diffeomorphism on } M \end{array} \right\}.$$

Indeed, when $H^G(\mathcal{B})$ is a Lie groupoid⁽¹⁸⁾ the canonical morphism $\Phi \colon H^G(\mathcal{B}) \to G$ is a Lie groupoid morphism giving rise to \mathcal{B} [7, §3.2]. Hence it is a bisubmersion for \mathcal{B} , by Proposition 2.13. Actually it is an atlas of bisubmersions, equivalent to the path-holonomy atlas (the argument is exactly as in [1, Ex. 3.4(4)]). The construction of the morphism in Proposition 4.1 then implies the claim.

In the general case, the kernel is still described as indicated above, but the smoothness of the bisection is to be understood w.r.t. the canonical diffeology on $H^G(\mathcal{B})$ [7, §4.2].

Remark 4.9. — We give an alternative description of the morphism (4.3) in the special case that \mathcal{B} arises from a Lie groupoid morphism (see Definition 1.6), i.e there is a Lie groupoid morphism $\Psi \colon K \to G$ over Id_M such that $\mathcal{B} = \Psi_*(\Gamma_c(\mathrm{Lie}(K)))$. The composition

$$K \stackrel{\Psi}{\to} G \stackrel{(\mathbf{t}_G, \mathbf{s}_G)}{\to} M \times M$$

is the target-source map $(\mathbf{t}_K, \mathbf{s}_K)$ of K. It gives rise to the singular foliation $\mathcal{F}_{\mathcal{B}}$, hence $H(\mathcal{F}_{\mathcal{B}}) = K/\sim_2$ where \sim_2 identifies two points of K iff there are bisections through them that carry the same diffeomorphism of M (see Example 3.16). Since by assumption \mathcal{B} arises from the Lie groupoid morphism Ψ , by Proposition 3.15 we have $H^G(\mathcal{B}) = K/\sim_1$, where \sim_1 identifies two points of K iff there are bisections through them that map under Ψ to the same bisection of G (see Remark 3.14). The equivalence classes of \sim_1 are contained in those of \sim_2 , giving rise to a morphism $H^G(\mathcal{B}) \to H(\mathcal{F}_{\mathcal{B}})$. The latter agrees with the morphism (4.3).

Example 4.10 (Covers of the Lie groupoid G). — Let

$$F\colon \widetilde{G}\to G$$

a morphism of Lie groupoids integrating the identity on the Lie algebroid $A := \text{Lie}(\widetilde{G}) = \text{Lie}(G)$. Let \mathcal{B} be a singular subalgebroid of A. Clearly, $F_*(\mathcal{B}) = \mathcal{B}$. Proposition 4.1 implies the existence of a canonical surjective morphism of topological groupoids

$$H^{\widetilde{G}}(\mathcal{B}) \to H^{G}(\mathcal{B})$$

 $^{^{(18)}}$ As an aside, this is equivalent to ${\cal B}$ being a projective singular subalgebroid [7, §3.2].

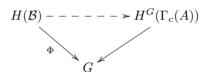
from the holonomy groupoid of \mathcal{B} constructed using \widetilde{G} to the holonomy groupoid of \mathcal{B} constructed using G. Under the identification of Theorem 3.25, this map is just the first projection, since both maps are induced by the identity map at the level of bisubmersions. Hence this map is exactly the one discussed in Remark 3.27.

We now present examples of Theorem 4.6. We start with a simple statement about singular foliations:

Example 4.11 (Singular foliations). — Let \mathcal{F}_1 , \mathcal{F}_2 be singular foliations on a manifold M, with $\mathcal{F}_1 \subset \mathcal{F}_2$. Then there is a canonical morphism of topological groupoids $H(\mathcal{F}_1) \to H(\mathcal{F}_2)$ covering the identity. This follows applying Theorem 4.6 with F the identity map on the pair groupoid $M \times M$.

The following example shows that the canonical map $\Phi: H^G(\mathcal{B}) \to G$ arises from the inclusion $\mathcal{B} \hookrightarrow \Gamma_c(A)$.

Example 4.12 (Recovering Φ). — Let G be a Lie groupoid, and \mathcal{B} be a singular subalgebroid of A := Lie(G). By Lemma 4.4, the inclusion $\mathcal{B} \subset \Gamma_c(A)$ induces a canonical morphism of topological groupoids $H^G(\mathcal{B}) \to H^G(\Gamma_c(A))$ making the following diagram commute:



But $H^G(\Gamma_c(A)) = G$ and the right map above is Id_G , by Example 3.21. Hence the canonical morphism dotted above is exactly Φ .

Specializing Theorem 4.6 to wide Lie subalgebroids and using Proposition 3.20 to relate holonomy groupoids with minimal integrals we obtain the following example.

Example 4.13. — Let $F: G_1 \to G_2$ be a morphism of Lie groupoids covering the identity on M. Let B_i be a wide Lie subalgebroid of $\text{Lie}(G_i)$ for i = 1, 2, such that $F_*(B_1) \subset B_2$. Denote by H^i_{\min} the minimal integral of B_i over G_i . Then there is a canonical morphism of Lie groupoids

$$\Xi \colon H^1_{\min} \to H^2_{\min}$$

covering Id_M and which, together with F, intertwines the immersions $H^i_{\min} \to G_i$ integrating the inclusions.

4.4. The integration functor

The purpose of this subsection is put in place our "integration" process, describing it as a functor. The term "integration" is in quotes, since we have not specified in which sense the holonomy groupoid $H^G(\mathcal{B})$ is an integration of the singular subalgebroid \mathcal{B} of Lie(G). Indeed this is the topic of a separate publication [7].

We fix a manifold M and consider two categories. The first one, denoted by $\mathsf{SingSub}_M^{Gpd}$, is:

• objects:

$$\left\{ (G,\mathcal{B}) \left| \begin{array}{c} G \text{ a Lie groupoid over } M, \\ \mathcal{B} \text{ a singular subalgebroid of } \operatorname{Lie}(G) \end{array} \right. \right\}$$

• arrows from (G_1, \mathcal{B}_1) to (G_2, \mathcal{B}_2) : $\begin{cases} F \colon G_1 \to G_2 \text{ a morphism of Lie groupoids covering } \mathrm{Id}_M, \\ \text{such that } F_*(\mathcal{B}_1) \subset \mathcal{B}_2 \end{cases}$

The second category, denoted by TopGrd_M , is:

• objects:

$$\left\{ \begin{split} \Phi \colon H \to G & \middle| \begin{array}{l} H \text{ a topological groupoid over } M, \\ G \text{ a Lie groupoid over } M, \\ \Phi \text{ a morphism of topological groupoids covering } \mathrm{Id}_M \\ \end{split} \right\}$$

• arrows from $(\Phi_1: H_1 \to G_1)$ to $(\Phi_2: H_2 \to G_2)$:

 $\left\{ (\Xi,F) \middle| \begin{array}{l} \Xi\colon H_1\to H_2 \text{ a morphism of topological groupoids over } \mathrm{Id}_M, \\ F\colon G_1\to G_2 \text{ a morphism of Lie groupoids over } \mathrm{Id}_M, \\ \mathrm{s.t. \ the \ diagram \ below \ commutes} \end{array} \right\}$

$$\begin{array}{c|c} H_1 & \xrightarrow{\Xi} & H_2 \\ & & & \downarrow \Phi_2 \\ & & & & \downarrow \Phi_2 \\ G_1 & \xrightarrow{F} & G_2 \end{array}$$

Our construction provides a functor

$$\begin{array}{cccc} \mathsf{SingSub}_M^{Gpd} & \to & \mathsf{TopGrd}_M \\ (G,\mathcal{B}) & \mapsto & H^G(\mathcal{B}) \\ F & \mapsto & (\Xi,F) \end{array}$$

where $H^G(\mathcal{B})$ is constructed as in Definition 3.5 and Ξ is constructed as in Theorem 4.6. This is really a functor, due to the canonicity of our constructions.

Appendix A. The convolution algebra of a singular subalgebroid (by Iakovos Androulidakis)

Here we explain how the construction of the C^* -algebra(s) of a singular foliation given in [1, §4] can be adapted to the context of singular subalgebroids. The only thing that needs to be explained here is the construction of the *-algebra associated with a singular subalgebroid \mathcal{B} ; its completion(s) is exactly as given in [1, §4.4, §4.5]. In fact, this appendix aims to exhibit that it is possible to do Noncommutative Geometry with singular subalgebroids; recall from [1, 2, 3] that this convolution algebra is the starting point in order to develop longitudinal pseudodifferential operators and the associated index theory.

The *-algebra of a singular subalgebroid \mathcal{B} (Definition A.6) is constructed using the holonomy groupoid $H^G(\mathcal{B})$, or more precisely a path holonomy atlas for \mathcal{B} .

Remark A.1. — It turns out that a singular subalgebroid \mathcal{B} corresponds to a singular foliation $\overrightarrow{\mathcal{B}}$ on G. Hence, following [18], we can consider the holonomy groupoid $H(\overrightarrow{\mathcal{B}})$ of the singular foliation. The latter has an induced action of G, and the quotient is the holonomy groupoid $H^G(\mathcal{B})$, as is shown in [7]. This construction is satisfactory from a geometric point of view, but is less suited for the purposes of *-algebras, for it is not clear that the *-algebra associated to $H(\overrightarrow{\mathcal{B}})$ and the G action on $H(\overrightarrow{\mathcal{B}})$ give rise to the *-algebras of $H^G(\mathcal{B})$.

Remarks on the topology of the holonomy groupoid

Just like the case of singular foliations, the topology of $H^G(\mathcal{B})$ is quite bad. Here we extend some statements from [1, §3.3]. We will need this material for the construction of the convolution algebra. In fact, our constructions can be carried out for the groupoid associated to any atlas of \mathcal{B} (cf. Appendix C), not only the path-holonomy one, so we give them in full generality.

Fix a singular subalgebroid \mathcal{B} . Fix an atlas $\mathcal{U} = (U_i, \varphi_i, G)_{i \in I}$ (see Appendix C) and let $H(\mathcal{B})^{\mathcal{U}}$ be the associated groupoid. For every bisubmersion (U, φ, G) adapted to \mathcal{U} , consider the set

$$\Gamma_U = \{ u \in U : \dim(U) = \dim(M) + \dim(\mathcal{B}_{\mathbf{s}(u)}) \} \subset U,$$

where $\mathcal{B}_{\mathbf{s}(u)} = \mathcal{B}/I_{\mathbf{s}(u)}\mathcal{B}$ for $I_{\mathbf{s}(u)}$ the smooth functions on M vanishing at $\mathbf{s}(u)$. Note that Γ_U is either empty or it consists of a union of fibers of $\mathbf{s}: U \to M$. It is an open subset of U when U is endowed with the smooth structure along the leaves of the regular foliation $\varphi^{-1}(\overrightarrow{\mathcal{B}}) = \Gamma_c(U; \ker d\mathbf{s}_U)$, i.e. the longitudinal smooth structure in [1, Prop. 1.14].

LEMMA A.2. — For every $h \in H(\mathcal{B})^{\mathcal{U}}$ there exists a bisubmersion (U, ψ, G) adapted to \mathcal{U} and $u \in \Gamma_U$ such that $q_U(u) = h$.

Proof. — Let (W, φ, G) be a bisubmersion in \mathcal{U} , let $w \in W$ such that $q_W(w) = h$ and $\mathbf{b} \subset W$ a bisection through w. Consider the minimal path holonomy bisubmersion (U, ψ_0, G) constructed by a basis of $\mathcal{B}_{\mathbf{s}(h)}$ (see Definition 2.16 and Proposition 2.18). Then $u \coloneqq (\mathbf{s}(h), 0) \in U$ carries the identity bisection of G. Consider the map $\psi \coloneqq L_{\varphi(\mathbf{b})} \circ \psi_0$, where $L_{\varphi(\mathbf{b})}$ is the diffeomorphism of G defined by left multiplication by the bisection $\varphi(\mathbf{b})$. As shown in Remark 2.34 we obtain a new bisubmersion (U, ψ, G) . The points $u \in (U, \psi, G)$ and $w \in W$ both carry the bisection $\varphi(\mathbf{b})$ of G, hence under the quotient map to $H(\mathcal{B})^{\mathcal{U}}$, the point u also maps to h. Further $u \in \Gamma_U$, because the path holonomy bisubmersion (U, ψ_0, G) is minimal at $\mathbf{s}(h)$, and since the source maps of (U, ψ_0, G) and (U, ψ, G) agree.

LEMMA A.3. — Consider bisubmersions (U, φ, G) and (U', φ', G) and $f: U \to U'$ a morphism of bisubmersions. Let $u \in U$.

- (1) If $u \in \Gamma_U$, then $(df)_u$ is injective;
- (2) If $f(u) \in \Gamma_{U'}$ then $(df)_u$ is surjective

Proof. — We denote by **s** and **s**' the source maps of U and U'. Since **s** and **s**' are submersions and $\mathbf{s}' \circ f = \mathbf{s}$, $(\mathrm{d}f)_u \colon T_uU \to T_{f(u)}U'$ is injective or surjective if and only if the restriction $(\mathrm{d}f)_u|_{\ker(\mathrm{d}\mathbf{s})_u} \colon \ker(\mathrm{d}\mathbf{s})_u \to \ker(\mathrm{d}\mathbf{s}')_{f(u)}$ is injective or surjective. Consider the composition

$$\ker(\mathrm{d}\mathbf{s})_u \stackrel{(\mathrm{d}f)_u|_{\ker(\mathrm{d}\mathbf{s})_u}}{\longrightarrow} \ker(\mathrm{d}\mathbf{s}')_{f(u)} \stackrel{\varphi'_*}{\longrightarrow} \vec{\mathcal{B}}_{\varphi(u)}.$$

By the definition of bisubmersions, the maps φ'_* and $\varphi_* = \varphi'_* \circ (\mathrm{d}f)_u$ are onto.

If $u \in \Gamma_U$, then by dimension reasons φ_* : $\ker(\mathrm{d}\mathbf{s})_u \to \overrightarrow{\mathcal{B}}$ is an isomorphism, implying (1).

If $f(u) \in \Gamma_{U'}$ then φ'_* : $\ker(\mathrm{d}\mathbf{s}')_{f(u)} \to \overrightarrow{\mathcal{B}}_{\varphi(u)}$ is an isomorphism, implying (2).

Preliminaries on densities

Let (U, φ_U, G) be a bisubmersion. Just like in [1, §4], we are going to work with bundles of (complex) a-densities $(a \in \mathbb{R})$ associated with a vector bundle $E \to U$. When $E = (\ker \operatorname{ds}_U \times \ker \operatorname{dt}_U) \to U$, we write $\Omega^a(U)$. Let us also denote $\Gamma_c(\Omega^a(U))$ the space of compactly supported sections of $\Omega^a(U)$. Now put $a = \frac{1}{2}$ and recall from [1, §4.1, §4.2]:

(1) If (V, φ_V, G) is another bisubmersion, then

$$\Omega^{1/2}(U \circ V)_{(u,v)} = (\Omega^{1/2}(U))_u \otimes (\Omega^{1/2}(V))_v$$

at all $(u, v) \in U \circ V$. So if $f \in \Gamma_c(\Omega^{1/2}(U))$ and $g \in \Gamma_c(\Omega^{1/2}(V))$, we obtain an element $f \otimes g \in \Gamma_c(\Omega^{1/2}(U \circ V))$ defined by

$$f \otimes g \colon (u, v) \mapsto f(u) \otimes g(v).$$

(2) If $\kappa\colon U\to U^{-1}$ is the identity isomorphism (where U^{-1} denotes the inverse bisubmersion) we put

$$\Gamma_c(\Omega^{1/2}(U)) \to \Gamma_c(\Omega^{1/2}(U^{-1}))$$

 $f \mapsto f^* := \overline{f} \circ \kappa^{-1}$

(3) If $p: U \to V$ is a submersion and a morphism of bisubmersions we have $\Omega^{1/2}(U) = \Omega^1(\ker dp) \otimes p^*(\Omega^{1/2}(V))$. So integration along the fibers of p gives a map

$$p_! \colon \Gamma_c(\Omega^{1/2}(U)) \to \Gamma_c(\Omega^{1/2}(V))$$

If p is a surjective submersion then $p_!$ is onto.

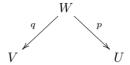
The *-algebra of an atlas of bisubmersions

Let us fix an atlas of bisubmersions $\mathcal{U} = (U_i, \varphi_i, G)_{i \in I}$ for the singular subalgebroid \mathcal{B} . Consider the disjoint union $U := \coprod_{i \in I} U_i$ and $\varphi \colon U \to G$ the map defined by $\varphi|_{U_i} = \varphi_i$. It is easy to see that (U, φ, G) is a bisubmersion and $\Gamma_c(\Omega^{1/2}(U)) = \bigoplus_{i \in I} \Gamma_c(\Omega^{1/2}(U_i))$.

For the construction of the convolution algebra, will have to identify smooth densities between two different bisubmersions, and this can be done by means of integration along the fibers of a submersion. Since there exist bisubmersions which are adapted to \mathcal{U} , but not necessarily through a submersive morphism, the next lemma is in order. Its proof is exactly the same as [1, Lem. 4.3], using our Lemma A.2, so we omit it.

LEMMA A.4. — Let (V, φ_V, G) be a bisubmersion adapted to \mathcal{U} . The following two statements hold:

(1) Let $v \in V$. Then there exists a bisubmersion (W, φ_W, G) and submersions $p \colon W \to U$, $q \colon W \to V$ which are morphisms of bisubmersions such that $v \in q(W)$.



(2) Let $f \in \Gamma_c(\Omega^{1/2}(V))$. Then there exists a bisubmersion (W, φ_W, G) and submersions $p \colon W \to U$, $q \colon W \to V$ which are morphisms of bisubmersions and $g \in \Gamma_c(\Omega^{1/2}(W))$ such that $q_!(g) = f$.

Analogously to $[1, \S 4.3]$, Lemma A.4 allows us to consider the quotient vector space

$$\mathcal{A}_{\mathcal{U}} = \bigotimes_{i \in I} \Gamma_c(\Omega^{1/2}(U_i)) / \mathcal{I}$$

Here \mathcal{I} is the linear subspace spanned by $p_!f$, where $p\colon W\to U$ is a submersion and morphism of bisubmersions and $f\in\Gamma_c(\Omega^{1/2}(W))$ is such that there exists a submersion and morphism of bisubmersions $q\colon W\to V$ with $q_!(f)=0$.

Remark A.5. — A bisubmersion (U, φ, G) for the singular subalgebroid \mathcal{B} can be viewed a bisubmersion $(U, \mathbf{t}_U, \mathbf{s}_U)$ for the underlying foliation $\mathcal{F}_{\mathcal{B}}$, see Remark 2.19. In spite of this, the space $\mathcal{A}_{\mathcal{U}}$ constructed here differs from the one constructed in [1, §4.3] out of the singular foliation $\mathcal{F}_{\mathcal{B}}$. That is because, while every morphism of bisubmersions $p: (U, \varphi, G) \to (U', \varphi', G)$ is a morphism of bisubmersions $p: (U, \mathbf{t}_U, \mathbf{s}_U) \to (U', \mathbf{t}_{U'}, \mathbf{s}_{U'})$, the converse is not true.

The proofs of [1, Prop. 4.4, 4.5] go through verbatim in the context of bisubmersions to endow $\mathcal{A}_{\mathcal{U}}$ with the following *-algebra structure:

- Given a bisubmersion (V, φ_V, G) adapted to \mathcal{U} there is a linear map $Q_V \colon \Gamma_c(\Omega^{1/2}(V)) \to \mathcal{A}_{\mathcal{U}}$ such that:
 - (1) If $(V, \varphi_V, G) = (U_i, \varphi_i, G)$ then Q_V is the quotient map

- (2) If $p: W \to V$ is a morphism of bisubmersions which is a submersion, then $Q_W = Q_V \circ p_!$.
- If (V, φ_V, G) , (W, φ_W, G) are bisubmersions adapted to \mathcal{U} then for sections $f \in \Gamma_c(\Omega^{1/2}(V))$, $g \in \Gamma_c(\Omega^{1/2}(W))$, the *-involution and product in $\mathcal{A}_{\mathcal{U}}$ are defined as follows:

$$(Q_V(f))^* = Q_{V^{-1}}(f^*), \qquad Q_V(f)Q_W(g) = Q_{V \circ W}(f \otimes g)$$

Definition A.6.

- (1) The *-algebra of the atlas \mathcal{U} is $\mathcal{A}_{\mathcal{U}}$ with the above operations.
- (2) The *-algebra $\mathcal{A}(\mathcal{B})$ of the singular subalgebroid \mathcal{B} is $\mathcal{A}_{\mathcal{U}}$ for \mathcal{U} the path-holonomy atlas of bisubmersions.

The completion of the *-algebra $\mathcal{A}_{\mathcal{U}}$ is verbatim as explained in [1, §4.4, §4.5] (but using the foliation $\mathcal{F}_{\mathcal{B}}$ for the definition of the L^1 -norm in [1, §4.4]). This way we obtain the full and reduced C^* -algebras $C^*(\mathcal{U})$ and $C^*_r(\mathcal{U})$ respectively. When \mathcal{U} is the path-holonomy atlas of bisubmersions associated with \mathcal{B} , we write $C^*(\mathcal{B})$ and $C^*_r(\mathcal{B})$.

Remark A.7. — The C^* -algebra $C^*(\mathcal{B})$ enjoys functorial properties. Consider for instance the anchor $\rho \colon A \to TM$, a Lie algebroid morphism which maps a given singular subalgebroid \mathcal{B} to the underlying singular foliation $\mathcal{F}_{\mathcal{B}}$. There is an induced canonical morphism $H^G(\mathcal{B}) \to H(\mathcal{F}_{\mathcal{B}})$, as we saw in Example 4.8. One can show that the anchor also induces a *-homomorphism $C^*(\rho) \colon C^*(\mathcal{B}) \to C^*(\mathcal{F}_{\mathcal{B}})$, given by integration along fibers of submersive morphisms between bisubmersions of \mathcal{B} and of $\mathcal{F}_{\mathcal{B}}$.

Appendix B. A technical lemma about bisubmersions

The proof of Proposition 2.32 relies on the following proposition, which relates bisubmersions for a singular subalgebroid on M with bisubmersions for the induced singular foliation on G.

PROPOSITION B.1. — Let (U, φ, G) be a bisubmersion for a singular subalgebroid \mathcal{B} . Then

$$\widehat{U} \coloneqq U \times_{\mathbf{s}_{U}, \mathbf{t}} G,$$

with target and source maps

$$\mathbf{t}_{\widehat{U}}(u,g) \coloneqq \varphi(u) \cdot g$$
$$\mathbf{s}_{\widehat{U}}(u,g) \coloneqq g$$

to G, is a bisubmersion for the singular foliation $\overrightarrow{\mathcal{B}}$ on G (in the sense of [1], see Section 2.3.1

The main step in the proof of Proposition B.1 is to check that equation (2.2) holds, i.e. that

(B.1)
$$\mathbf{t}_{\widehat{U}}^{-1}(\overrightarrow{\mathcal{B}}) = \Gamma_c(\ker d\mathbf{t}_{\widehat{U}}) + \Gamma_c(\ker d\mathbf{s}_{\widehat{U}}) = \mathbf{s}_{\widehat{U}}^{-1}(\overrightarrow{\mathcal{B}}).$$

We start making these submodules more explicit. By Lemma 2.3 we have (B.2)

$$\overrightarrow{\mathbf{t}}_{\widehat{U}}^{-1}(\overrightarrow{\mathcal{B}}) = \operatorname{Span}_{C_{c}^{\infty}(\widehat{U})} \bigcup_{\alpha \in \mathcal{B}} \left\{ (Y, Z) : Y \in \mathfrak{X}(U), Z \in \Gamma(\ker \operatorname{ds}_{U}) \right.$$
s.t. $(d_{u}\mathbf{s}_{U})Y = (d_{g}\mathbf{t})Z$ for all $(u, g) \in \widehat{U}$
and (Y, Z) is $\mathbf{t}_{\widehat{U}}$ -related to $\overrightarrow{\alpha}$

(B.3)
$$\mathbf{s}_{\widehat{U}}^{-1}(\overrightarrow{\mathcal{B}}) = \operatorname{Span}_{C_{c}^{\infty}(\widehat{U})} \bigcup_{\alpha \in \mathcal{B}} \left\{ (Y, \overrightarrow{\alpha}) : Y \in \mathfrak{X}(U) \text{ s.t. } (d_{u}\mathbf{s}_{U})Y = (d_{g}\mathbf{t})\overrightarrow{\alpha} \right.$$
for all $(u, g) \in \widehat{U}$.

Lemma B.2.

$$\Gamma_c(\ker \operatorname{dt}_{\widehat{U}}) = \operatorname{Span}_{C_c^{\infty}(\widehat{U})} \bigcup_{\alpha \in \mathcal{B}} \left\{ (Y, -\overrightarrow{\alpha}) : Y \in \Gamma(\ker \operatorname{dt}_U) \right.$$

$$s.t. \ Y \text{ is } \varphi\text{-related to } \overleftarrow{\alpha} \right\}.$$

Proof. — At every point $(u, g) \in \widehat{U}$, we have

$$\ker d_{(u,g)}\mathbf{t}_{\widehat{U}} = \{(Y,Z) : Y \in \ker(d_u\mathbf{t}_U), Z \in \ker(d_g\mathbf{s})$$
 s.t. $d_u\mathbf{s}_U(Y) = d_g\mathbf{t}(Z)$ and $(d_u\varphi(Y)) \cdot Z_g = 0\}.$

The inclusion "⊃" in the statement of the lemma holds because $\overleftarrow{\alpha} \cdot (-\overrightarrow{\alpha}) = \overleftarrow{\alpha} \cdot (i_* \overleftarrow{\alpha}) = 0$ for all $\alpha \in \mathcal{B}$, where i is the inversion on G. For the opposite inclusion, let $(Y,Z) \in \Gamma_c(\ker \operatorname{dt}_{\widehat{U}})$. In particular $Y \in \Gamma_c(U,\ker \operatorname{dt}_U)$, so by Lemma 2.8(2) (applied to t_U) we have $Y = \sum f_i Y_i$ where $f_i \in C_c^{\infty}(U)$ and the $Y_i \in \Gamma(U,\ker \operatorname{dt}_U)$ are φ -related to $\overleftarrow{\alpha}_i$ for suitable $\alpha_i \in \mathcal{B}$. As seen earlier, each $(Y_i, -\overrightarrow{\alpha}_i) \in \Gamma(\ker \operatorname{dt}_{\widehat{U}})$, so the difference $(Y,Z) - \sum (pr_U)^* f_i \cdot (Y_i, -\overrightarrow{\alpha}_i)$ lies in $\Gamma(\ker \operatorname{dt}_{\widehat{U}})$ too. Now this difference is of the form (0,*), and the condition $(\operatorname{d}\varphi(0)) \cdot * = 0$ implies that * = 0. Hence this difference is zero. Multiplying by an element of $C_c^{\infty}(\widehat{U})$ which is 1 on $\operatorname{supp}(Y,Z)$, we are done.

Proof of Proposition B.1. — The map $\mathbf{s}_{\widehat{U}}$ is a submersion because \mathbf{s}_U is. We now show that $\mathbf{t}_{\widehat{U}} \colon \widehat{U} \to G$ is a submersion, by showing that its

derivative at any point $(u,g) \in \widehat{U}$ is surjective. Let $\mathbf{b} \colon M \to U$ be a bisection of U through u. Then $\mathbf{c} \coloneqq \varphi \circ \mathbf{b} \colon M \to G$ is a bisection of G through $\varphi(u)$. Since the left multiplication $L_{\mathbf{c}}$ is a diffeomorphism, any vector in $T_{(\varphi(u)\cdot g)}G$ can be written as $(L_{\mathbf{c}})_*v$ for some $v \in T_gG$, i.e. as $(\mathbf{c}_*\mathbf{t}_*v) \cdot v$, which is the image under $(\mathbf{t}_{\widehat{U}})_*$ of $(\mathbf{b}_*\mathbf{t}_*v, v) \in T_{(u,g)}\widehat{U}$.

We now prove the first equality in equation (B.1). For " \supset ", it suffices to show that $\mathbf{t}_{\widehat{U}}^{-1}(\overrightarrow{B}) \supset \Gamma_c(\ker \mathrm{d}\mathbf{s}_{\widehat{U}})$. By Lemma 2.8(2), it suffices to consider (Y,0) where $Y \in \Gamma(U,\ker \mathrm{d}\mathbf{s}_U)$ is φ -related to $\overrightarrow{\alpha}$ for some $\alpha \in \mathcal{B}$. Such an element is as on the r.h.s. of equation (B.2), since $\overrightarrow{\alpha} \cdot 0 = \overrightarrow{\alpha}$. For " \subset ", let (Y,Z) be as in the r.h.s. of equation (B.2). Since (U,φ,G) is a bisubmersion for \mathcal{B} , there is $Y' \in \Gamma(\ker \mathrm{d}\mathbf{s}_U)$ which is φ -related to $\overrightarrow{\alpha}$. Then (Y',0) is $\mathbf{t}_{\widehat{U}}$ -related to $\overrightarrow{\alpha}$. Hence the difference satisfies $(Y,Z) - (Y',0) \in \Gamma(\ker \mathrm{d}\mathbf{t}_{\widehat{U}})$, and therefore $(Y,Z) \in \Gamma(\ker \mathrm{d}\mathbf{t}_{\widehat{U}}) + \Gamma(\ker \mathrm{d}\mathbf{s}_{\widehat{U}})$. Taking $C_c^{\infty}(\widehat{U})$ -linear combinations we are done.

We are left with proving the second equality in equation (B.1). For " \subset ", it suffices to show that $\Gamma_c(\ker d\mathbf{t}_{\widehat{U}}) \subset \mathbf{s}_{\widehat{U}}^{-1}(\overrightarrow{\mathcal{B}})$, which is easily seen to hold using Lemma B.2 and $i_*\overleftarrow{\alpha} = -\overrightarrow{\alpha}$. For the inclusion " \supset ", it suffices to consider elements $(Y, \overrightarrow{\alpha})$ as on the r.h.s. of equation (B.3). Since (U, φ, G) is a bisubmersion for \mathcal{B} , there is $Y' \in \Gamma(\ker d\mathbf{t}_U)$ which is φ -related to $-\overleftarrow{\alpha}$. By Lemma B.2 we have $(Y', \overrightarrow{\alpha}) \in \Gamma(\ker d\mathbf{t}_{\widehat{U}})$. Since $Y - Y' \in \Gamma(\ker d\mathbf{s}_U)$, the difference satisfies $(Y, \overrightarrow{\alpha}) - (Y', \overrightarrow{\alpha}) = (Y - Y', 0) \in \Gamma(\ker d\mathbf{s}_{\widehat{U}})$. Therefore $(Y, \overrightarrow{\alpha}) \in \Gamma(\ker d\mathbf{t}_{\widehat{U}}) + \Gamma(\ker d\mathbf{s}_{\widehat{U}})$.

Appendix C. On atlases and holonomy groupoids

We recall some material from [1, §3.1], spelling out part of it and rephrasing it in the context of singular subalgebroids (rather than singular foliations). This material is used in Proposition 3.12, and in the construction of morphisms between holonomy groupoids (covering the identity in Section 4 and covering submersions in Appendix D).

Fix a singular subalgebroid \mathcal{B} of a Lie algebroid A, and a Lie groupoid G integrating A.

DEFINITION C.1. — Let $\mathcal{U} := (U_i, \varphi_i, G)_{i \in I}$ be a family of bisubmersions for \mathcal{B} .

(1) Let (U, φ, G) be a bisubmersion of \mathcal{B} . We say that (U, φ, G) is adapted to \mathcal{U} if for every point $u \in U$ there is an open subset $U' \subset U$ containing u, an index $i \in I$ and a morphism of bisubmersions $U' \to U_i$.

- (2) \mathcal{U} is an atlas if
 - (a) $\bigcup_{i \in I} \mathbf{s}_i(U_i) = M$,
 - (b) the inverse of every element of \mathcal{U} is adapted to \mathcal{U} ,
 - (c) the composition of any two elements of \mathcal{U} is adapted to \mathcal{U} .
- (3) Let \(\mathcal{U}\) and \(\mathcal{V}\) be two atlases. \(\mathcal{U}\) is adapted to \(\mathcal{V}\) is every element of \(\mathcal{U}\) is adapted to \(\mathcal{V}\). The atlases \(\mathcal{U}\) and \(\mathcal{V}\) are equivalent if they are adapted to each other.

Let \mathcal{U} be an atlas for \mathcal{B} . By Corollary 3.3(3), the relation

 $u_1 \sim u_2 \iff$ there is an open neighborhood U_1' of u_1 , there is a morphism of bisubmersions $f\colon U_1' \to U_2$ such that $f(u_1) = u_2$,

is an equivalence relation on the disjoint union $\coprod_{U \in \mathcal{U}} U$. (It can be expressed in terms of bisections too, as in Remark 3.6.) The quotient

(C.1)
$$H(\mathcal{B})^{\mathcal{U}} := \coprod_{U \in \mathcal{U}} U / \sim$$

is a topological groupoid. It comes with a canonical morphism of topological groupoids $\Phi \colon H(\mathcal{B})^{\mathcal{U}} \to G$, induced by the bisubmersions $\varphi \colon U \to G$ in \mathcal{U} . This is proven in a way similar to [1, Prop. 3.2] and Theorem 3.8.

We now display some properties of this construction. The following is [1, Rem. 3.3], with details added.

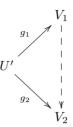
PROPOSITION C.2. — Let the atlas \mathcal{U} be adapted to the atlas \mathcal{V} . Then

- (1) there is a canonical, injective morphism of topological groupoids $\Theta \colon H(\mathcal{B})^{\mathcal{U}} \to H(\mathcal{B})^{\mathcal{V}}$
- (2) Θ commutes with the canonical maps from $H(\mathcal{B})^{\mathcal{U}}$ and $H(\mathcal{B})^{\mathcal{V}}$ to G.

Proof.

(1). — Take an element U of \mathcal{U} and $u \in U$. By definition, there is an open subset $U' \subset U$ containing u, an element V_1 of \mathcal{V} , and a morphism of bisubmersions $g_1 \colon U' \to V_1$. Assume that there is another element V_2 of \mathcal{V} , and a morphism of bisubmersions $g_2 \colon U' \to V_2$. Then there is a morphism of bisubmersions $V_1' \to V_2$ obtained applying Corollary 3.3(3) to g_1 —

defined on a neighborhood of $g_1(u)$ in V_1 , making this diagram commute:



This gives a well-defined map $\coprod_{U\in\mathcal{U}}U\to H(\mathcal{B})^{\mathcal{V}}$, mapping a point u_1 to $[g_1(u_1)]$, where g_1 is any morphism of bisubmersions into an element of \mathcal{V} . Now consider two morphisms of bisubmersions $g_1:U_1\to V_1$ and $g_2:U_2\to V_2$, where U_1,U_2 are open subsets of elements of \mathcal{U} , and $V_1,V_2\in\mathcal{V}$. Suppose f is a morphism of bisubmersions mapping a point $u_1\in U_1$ to $u_2\in U_2$. Then there is a morphism of bisubmersions mapping $g_1(u_1)$ to $g_2(u_2)$. This can be seen using Corollary 3.3(3) to "invert" the top horizontal map in

$$\begin{array}{c|c} U_1 & \xrightarrow{g_1} & V_1 \\ \downarrow^f & \downarrow \\ \downarrow^g & \forall \\ U_2 & \xrightarrow{g_2} & V_2 \end{array}$$

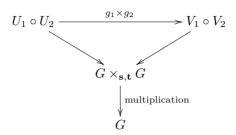
Hence, by $\Theta([u_1]) := [g_1(u_1)]$ we obtain a well-defined map $\Theta \colon H(\mathcal{B})^{\mathcal{U}} \to H(\mathcal{B})^{\mathcal{V}}$.

The injectivity of Θ can be seen applying the above reasoning to the bottom horizontal map in the diagram

$$\begin{array}{ccc} U_1 & \longrightarrow & V_1 \\ & & & \downarrow \\ & & & \downarrow \\ V_2 & \longrightarrow & V_2 \end{array}$$

Last, we show that Θ is a groupoid morphism. We indicate only how to check that the groupoid multiplications are preserved. Again, consider two morphisms of bisubmersions $g_1 \colon U_1 \to V_1$ and $g_2 \colon U_2 \to V_2$, where U_1, U_2 are open subsets of elements of \mathcal{U} , and $V_1, V_2 \in \mathcal{V}$. Let $u_1 \in U_1$ and $u_2 \in U_2$ so that their product $[u_1] \cdot [u_2]$ in $H(\mathcal{B})^{\mathcal{U}}$ is defined. Then $g_1 \times g_2$, which clearly maps $u_1 \circ u_2$ to $g_1(u_1) \circ g_2(u_2)$, is a morphism of bisubmersions, as

can be seen from the commutativity of the diagram below.



Hence

$$\Theta([u_1] \cdot [u_2]) = [(g_1 \times g_2)(u_1 \circ u_2)] = [g_1(u_1) \circ g_2(u_2)] = \Theta([u_1]) \cdot \Theta([u_2]).$$

(2). — The morphism $\Theta \colon H(\mathcal{B})^{\mathcal{U}} \to H(\mathcal{B})^{\mathcal{V}}$ commutes with the maps to G because Θ is constructed assembling morphisms of bisubmersions, which by definition commute with the respective maps to G (see the commutative diagram in Definition 2.21).

PROPOSITION C.3. — Let \mathcal{U} be a path holonomy atlas as in Definition 3.4. Then \mathcal{U} is adapted to any atlas \mathcal{V} .

Remark C.4.

- (1) In particular, any two path holonomy at lases are adapted to each other. Proposition C.2 implies that the corresponding topological groupoids agree. (We denote them $H(\mathcal{B})$ along the whole paper).
- (2) For any atlas \mathcal{V} , there is a canonical injective morphism of topological groupoids $H(\mathcal{B}) \to H(\mathcal{B})^{\mathcal{V}}$, as a consequence of Proposition C.2 and Proposition C.3.

In the converse, we can only say that there is an injective morphism of local topological groupoids from a neighborhood of the identity section of $H(\mathcal{B})^{\mathcal{V}}$ to $H(\mathcal{B})$. This can be seen using Proposition 3.2 and proceeding as in the proof of Proposition C.2.

Proof. — Let U be a minimal path holonomy bisubmersion lying in \mathcal{U} and $x \in M$ with $(0,x) \in U$. Take a preimage v of the identity element 1_x under the quotient map $\natural \colon \coprod_{V \in \mathcal{V}} V \to H(\mathcal{B})^{\mathcal{V}}$, and suppose that $v \in V$. Then by Proposition 3.2 and Corollary 3.3(3) there exists an open neighbourhood U' of (0,x) in U and a morphism of bisubmersions $U' \to V$ mapping (0,x) to v. Repeating for all x, we see that there is a neighbourhood N of $U \cap M$ in U, such that every point of N lies in the domain of some morphism of bisubmersions into some element of \mathcal{V} . In other words, N is adapted to \mathcal{V} .

This implies that the same holds for any arbitrary point of U: if $(\lambda, x) \in U$, then there is a positive integer k such that $(\lambda/k, x) \in N$, and there is a neighborhood U of (λ, x) and a morphism of bisubmersions $U \to N \circ \cdots \circ N$ mapping (λ, x) to $(\lambda/k, \ldots) \circ \cdots (\lambda/k, \mathbf{t}_U((\lambda/k, x))) \circ (\lambda/k, x)$. (It exists by Corollary 3.3(2) since the constant bisection of U with value λ and the constant bisection of $N \circ \cdots \circ N$ with value $(\lambda/k, \ldots, \lambda/k)$ map to the same bisection of G.) Using that N is is adapted to V, we obtain a morphism of bisubmersions $U' \to V_k \circ \cdots \circ V_1$ for some elements V_i of V. This shows that U is adapted to V.

For elements of \mathcal{U} which are compositions $U_n \circ \cdots \circ U_1$ of path holonomy bisubmersions, apply the above to each U_i . Notice that we do not need to consider inverses of path holonomy bisubmersions, due to Remark 2.26. \square

PROPOSITION C.5. — Let \mathcal{U} be an atlas generated by path-holonomy bisubmersions as in Definition 2.16 (not necessarily minimal ones). Then \mathcal{U} is adapted to a path-holonomy atlas.

Remark C.6. — Such an atlas \mathcal{U} is equivalent to a path-holonomy atlas, by Proposition C.3 and Proposition C.5, hence $H(\mathcal{B})^{\mathcal{U}} = H(\mathcal{B})$.

Proof. — Let $\{\alpha_1, \ldots, \alpha_n\}$ be a set of local generators of \mathcal{B} , giving rise to the bisubmersion (U, φ, G) . Let $x \in \mathbf{s}_U(U) \subset M$. We may assume that, for some $k \leq n$, $\{\alpha_1, \ldots, \alpha_k\}$ is a minimal set of local generators at x (see Remark 1.11). So on a neighborhood M' of x, for a > k, we can write $\alpha_a = \sum_{i=1}^k f_a^i \alpha_i$ for functions f_a^i on M'. It is straightforward to check that

$$\mathbb{R}^{n} \times M \to \mathbb{R}^{k} \times M, (\lambda_{1}, \dots, \lambda_{n}; x)$$

$$\mapsto \left(\lambda_{1} + \sum_{a=k+1}^{n} \lambda_{a} f_{a}^{1}, \dots, \lambda_{k} + \sum_{a=k+1}^{n} \lambda_{a} f_{a}^{k}; x\right)$$

restricts to a morphism of bisubmersions from $U|_{M'}$ to the minimal path holonomy bisubmersion constructed using $\{\alpha_1, \ldots, \alpha_k\}$.

As such bisubmersions (U, φ, G) generate the atlas \mathcal{U} , we are done. \square

Appendix D. Morphisms of holonomy groupoids covering submersions

In Section 3, starting from a Lie groupoid G and singular subalgebroid \mathcal{B} of the Lie algebroid Lie(G), we constructed a holonomy groupoid $H^G(\mathcal{B})$ endowed with a map of topological groupoids to G. In Section 4 we extended

this construction to morphisms covering the identity on the base, obtaining canonical morphisms of topological groupoids

$$H^{G_1}(\mathcal{B}_1) o H^{G_2}(\mathcal{B}_2)$$

commuting with the canonical maps. In this appendix we do the same for morphisms covering surjective submersions, see Theorem D.6.

Let $F_*: A_1 \to A_2$ be a morphism of Lie algebroids, covering a surjective submersion $f: M_1 \to M_2$. Let \mathcal{B}_1 be a singular subalgebroid of A_1 satisfying the following condition:

(D.1)
$$\widehat{\mathcal{B}}_1^{\operatorname{proj}} := \{ \boldsymbol{\alpha} \in \widehat{\mathcal{B}}_1 : \boldsymbol{\alpha} \text{ is } F_*\text{-projectable to a section of } A_2 \}$$
 generates \mathcal{B}_1 as a $C_c^{\infty}(M_1)$ -module.

Recall that the global hull $\widehat{\mathcal{B}}_1$ was defined in Section 1.1. Here by " α is F_* -projectable to a section of A_2 " we mean that there exists $\mathbf{b} \in \Gamma(A_2)$ such that $F_*(\alpha|_x) = \mathbf{b}|_{f(x)}$ for all $x \in M_1$. As f is surjective, such a section \mathbf{b} is unique, and will be denoted by $F_*\alpha$.

LEMMA D.1. — Condition (D.1) implies that

$$F_*(\mathcal{B}_1) := \operatorname{Span}_{C^{\infty}(M_2)} \{ F_* \alpha : \alpha \in \widehat{\mathcal{B}}_1^{\operatorname{proj}} \}$$

is a singular subalgebroid of A_2 .

Proof. — First notice that $F_*(\mathcal{B}_1)$ is a well-defined $C^{\infty}(M_2)$ -submodule of $\Gamma_c(A_2)$, since f is surjective.

 $F_*(\mathcal{B}_1)$ is involutive: the fact that F_* is a morphism of Lie algebroids implies that if $\boldsymbol{\alpha}, \mathbf{b} \in \Gamma(A_1)$ are F_* -projectable, then their bracket also is, and $F_*[\boldsymbol{\alpha}, \mathbf{b}] = [F_*\boldsymbol{\alpha}, F_*\mathbf{b}]$ [17, §3.4, equation (24)].

We now show that $F_*(\mathcal{B}_1)$ is locally finitely generated. For all $x \in M_1$, the restriction to $\widehat{\mathcal{B}_1}^{\mathrm{proj}}$ of the linear map $\mathcal{B}_1 \to \mathcal{B}_1/I_x\mathcal{B}_1$ is surjective, as a consequence of condition (D.1). Choosing $\{\alpha_1,\ldots,\alpha_n\}\subset\widehat{\mathcal{B}_1}^{\mathrm{proj}}$ so that its image forms a basis of $\mathcal{B}_1/I_x\mathcal{B}_1$, we obtain a set of generators of \mathcal{B}_1 near x (see Remark 1.11, which remains true for global hulls). We claim that $\{F_*\alpha_1,\ldots,F_*\alpha_n\}$ is a set of generators of $F_*(\mathcal{B}_1)$ near f(x). To this aim, it suffices to show that $F_*\alpha$ is a $C^\infty(M_2)$ -linear combination of the $F_*\alpha_i$'s, for all $\alpha \in \widehat{\mathcal{B}_1}^{\mathrm{proj}}$. We can write

(D.2)
$$\alpha = \sum g_i \alpha_i$$

nearby x, for some $g_i \in C^{\infty}(M_1)$. Take a small enough submanifold S of M_1 through x which is transverse to the f-fibers (hence $f|_S$ is a diffeomorphism onto an open subset of M_2). Restricting the equation (D.2) to S and applying F_* gives the desired conclusion.

We display two classes of singular subalgebroids that satisfy condition (D.1).

Lemma D.2.

- (1) If f is a diffeomorphism, any singular subalgebroid \mathcal{B}_1 satisfies condition (D.1).
- (2) Assume that $F_*: A_1 \to A_2$ has constant rank and \mathcal{B}_2 is a singular subalgebroid of A_2 , such that any element of \mathcal{B}_2 can be F_* -lifted to a section of A_1 . Then

 $\mathcal{B}_1 := \operatorname{Span}_{C_c^{\infty}(M_1)} \{ \boldsymbol{\alpha} \in \Gamma(A_1) : \boldsymbol{\alpha} \text{ is } F_*\text{-projectable to an element of } \mathcal{B}_2 \},$ is a singular subalgebroid of A_1 satisfying condition (D.1). Further $F_*(\mathcal{B}_1) = \mathcal{B}_2$.

Remark D.3. — If F_* is fiberwise surjective, then any singular subalgebroid \mathcal{B}_2 satisfies the assumptions of Lemma D.2(2). The singular subalgebroid \mathcal{B}_1 appearing there deserves to be called the *pullback* of \mathcal{B}_2 . Indeed, when $A_1 = TM_1$ and $A_2 = TM_2$ are tangent bundles, \mathcal{B}_1 is exactly the pullback of the singular foliation \mathcal{B}_2 by f, as defined in [1, Proposition 1.10]. See also Remark D.7 below.

Proof. — (1) is clear since in that case $\widehat{\mathcal{B}}_1^{\text{proj}} = \widehat{\mathcal{B}}_1$. For (2) we proceed as follows.

We first check that \mathcal{B}_1 is a singular subalgebroid of A_1 . The involutivity of \mathcal{B}_1 follows from that of \mathcal{B}_2 . To see that \mathcal{B}_1 is locally finitely generated nearby a given point $x \in M_1$, choose a finite set of local generators $\{\mathbf{b}^i\}$ of \mathcal{B}_2 in a neighborhood V of f(x). Shrinking V if necessary, we can assume that the \mathbf{b}^i 's are compactly supported, and hence elements of \mathcal{B}_2 . By assumption, there are sections $\{\boldsymbol{\alpha}^i\} \subset \Gamma(A_1)$ which lift the $\{\mathbf{b}^i\}$, i.e., each $\boldsymbol{\alpha}^i$ is F_* -projectable, and $F_*\boldsymbol{\alpha}^i = \mathbf{b}^i$. We have $\boldsymbol{\alpha}^i \in \widehat{\mathcal{B}_1}$ by the definition of the latter. Now let $\{\mathbf{c}^j\} \subset \Gamma(\ker(F_*))$ be a finite set which forms a frame for the vector bundle $\ker(F_*)$ near x. It exists since F_* has constant rank. We claim that

$$\{oldsymbol{lpha}^i\} \cup \{oldsymbol{\mathbf{c}}^j\}$$

is a generating set for \mathcal{B}_1 near x. Indeed, given $\alpha \in \Gamma(A_1)$ which is F_* -projectable to an element of \mathcal{B}_2 , there exist $\{g_i\} \subset C^{\infty}(M_2)$ such that $F_*\alpha = \sum g_i \mathbf{b}^i$ on V. On $f^{-1}(V)$, the section $\sum f^*(g_i)\alpha^i$ projects to $F_*\alpha$. Hence, in neighborhood of x, we have $\alpha = \sum f^*(g_i)\alpha^i + \sum h_j \mathbf{c}^j$ for certain $\{h_j\} \subset C^{\infty}(M_1)$. This shows that \mathcal{B}_1 is locally finitely generated nearby x.

By construction \mathcal{B}_1 satisfies condition (D.1). By definition $F_*\mathcal{B}_1 \subset \mathcal{B}_2$, and the reverse inclusion holds since by assumption elements of \mathcal{B}_2 can be lifted to sections of A_1 .

Surjective morphisms

The following proposition address surjective morphisms, and a special case (corresponding to the case f = Id) was already addressed in Proposition 4.1.

PROPOSITION D.4. — Let $F: G_1 \to G_2$ be a morphism of Lie groupoids, covering a surjective submersion $f: M_1 \to M_2$. Let \mathcal{B}_1 be a singular subalgebroid of $\text{Lie}(G_1)$, and assume that it satisfies condition (D.1) above. Let $\mathcal{B}_2 := F_*(\mathcal{B}_1)$, which is a singular subalgebroid of $\text{Lie}(G_2)$ by Lemma D.1.

Then there is a canonical, surjective morphism of topological groupoids

$$\Xi \colon H^{G_1}(\mathcal{B}_1) \to H^{G_2}(\mathcal{B}_2)$$

making the following diagram commute:

(D.3)
$$H^{G_1}(\mathcal{B}_1) - \stackrel{\Xi}{-} > H^{G_2}(\mathcal{B}_2)$$

$$\downarrow^{\Phi_1} \qquad \qquad \downarrow^{\Phi_2}$$

$$G_1 \xrightarrow{F} G_2$$

To construct the morphism Ξ we will relate path-holonomy bisubmersions for \mathcal{B}_1 with bisubmersions for \mathcal{B}_2 . We first need a lemma⁽¹⁹⁾

LEMMA D.5. — Assume the set-up of Proposition D.4. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a minimal set of local generators of \mathcal{B}_1 lying in $\widehat{\mathcal{B}_1}^{\text{proj}}$. Let (U, φ_1, G_1) the corresponding path-holonomy G_1 -bisubmersion for \mathcal{B}_1 . Then $(U, \varphi_2 := F \circ \varphi_1, G_2)$ is a G_2 -bisubmersion for \mathcal{B}_2 .

The situation is visualized in the following diagram, which is helpful to follow the proof of Proposition D.4 too.

⁽¹⁹⁾Lemma D.5 is concerned with path-holonomy bisubmersions. One can check that the statement of the lemma holds for all G_1 -bisubmersion for \mathcal{B}_1 , but we will not need this.

Proof. — Recall that U is an open subset of $\mathbb{R}^n \times M_1$, and by Definition 2.16 we have $\varphi_1 \colon U \to G_1, (\lambda, x) \mapsto \exp_x \sum \lambda_i \overrightarrow{\alpha}_i$. Hence

$$(\mathrm{D.4}) \qquad \qquad \varphi_2((\lambda,x)) = F(\exp_x \sum \lambda_i \overrightarrow{\alpha_i}) = \exp_{f(x)} \sum \lambda_i \overrightarrow{F_*(\alpha_i)}.$$

Here the second equality holds because, since F is a Lie groupoid morphism and α_i is F_* -projectable (to $F_*(\alpha_i) \in \mathcal{B}_2$), we have $F_*(\overrightarrow{\alpha_i}) = \overline{F_*(\alpha_i)}$.

Let (W, φ_W, G_2) be the path-holonomy G_2 -bisubmersion for \mathcal{B}_2 (not necessarily minimal) associated to the set of local generators $\{F_*(\boldsymbol{\alpha}_1), \dots, F_*(\boldsymbol{\alpha}_n)\}$ of \mathcal{B}_2 . In particular, W is an open subset of $\mathbb{R}^n \times M_2$. The submersion $(\mathrm{Id}, f) \colon \mathbb{R}^n \times M_1 \to \mathbb{R}^n \times M_2$ restricts to map $p \colon U \to W$ which by equation $(\mathrm{D}.4)$ makes this diagram commute:



By Lemma 2.22, we conclude that (U, φ_2, G_2) is a G_2 -bisubmersion for \mathcal{B}_2 .

Notice that if if (U, φ_1, G_1) and (U', φ'_1, G_1) are G_1 -bisubmersion for \mathcal{B}_1 as in Lemma D.5, and k is a (locally defined) morphism between them, then k is is also a morphism of bisubmersions between the corresponding G_2 -bisubmersion for \mathcal{B}_2 . Applying this fact in the proof of Proposition D.4 implies that the morphism Ξ is canonical in a neighborhood of the identity section of $H^{G_1}(\mathcal{B}_1)$, and by source-connectedness on the whole of $H^{G_1}(\mathcal{B}_1)$.

Proof of Proposition D.4. — Take a family S of path-holonomy G_1 -bisubmersions for \mathcal{B}_1 , constructed out of minimal sets of local, F-projectable generators of \mathcal{B}_1 , and so that $\bigcup_{U \in S} \mathbf{s}_U(U) = M_1$. It exists since \mathcal{B}_1 satisfies property (D.1). Denote by \mathcal{U}_1 the atlas (see Definitions 3.4 and C.1) generated by the elements of S, viewed as G_1 -bisubmersions for \mathcal{B}_1 . In other words, \mathcal{U}_1 is constructed from elements of S, taking their finite compositions as G_1 -bisubmersions for \mathcal{B}_1 (we do not need to take inverse bisubmersions by Remark 2.26.) Similarly, denote by \mathcal{U}_2 the atlas generated by the elements of S, viewed as G_2 -bisubmersions for \mathcal{B}_2 as in Lemma D.5 (again⁽²⁰⁾ we do not need to take inverses).

 $^{^{(20)}}$ Indeed, since F is a groupoid morphism, an isomorphism of bisubmersions between $(U, \varphi, G_1) \in \mathcal{S}$ and its inverse $(U, i_{G_1} \circ \varphi, G_1)$ provides an isomorphism of G_2 -bisubmersions between $(U, F \circ \varphi, G_2)$ and its inverse $(U, i_{G_2} \circ F \circ \varphi, G_2)$.

Claim. — There is a canonical injective map ι that makes the following diagram commute:

$$\coprod_{U_1 \in \mathcal{U}_1} U_1 \xrightarrow{\iota} \coprod_{U_2 \in \mathcal{U}_2} U_2$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$G_1 \xrightarrow{F} G_2$$

By Lemma D.5, for every $(U, \varphi_1, G_1) \in \mathcal{S}$, we have that $(U, \varphi_2 := F \circ \varphi_1, G_2)$ is a G_2 -bisubmersion for \mathcal{B}_2 . We take $\iota|_U$ to be simply the identity. Let $U, V \in \mathcal{S}$. Then $U \circ_1 V := U \times_{(\mathbf{s}_U)_1, (\mathbf{t}_V)_1} V$, their composition as G_1 -bisubmersions for \mathcal{B}_1 , is contained in $U \circ_2 V := U \times_{(\mathbf{s}_U)_2, (\mathbf{t}_V)_2} V$, their composition as G_2 -bisubmersions for \mathcal{B}_2 . (Notice that the former is a fiber product over M_1 , the latter a fiber product over M_2 .) We define $\iota|_{U \circ_1 V}$ to be this inclusion. The following diagram commutes, since F is a morphism of groupoids:

$$\begin{array}{c|c} U \circ_1 V & \longrightarrow U \circ_2 V \\ (\varphi_U)_1 \cdot (\varphi_V)_1 & & & & & & \\ G_1 & \xrightarrow{F} & G_2 \end{array}$$

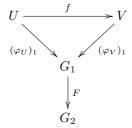
The same holds for the composition of any finite number of bisubmersions in S.

Claim. — The map ι descends to a map

(D.6)
$$H^{G_1}(\mathcal{B}_1) = \coprod_{U_1 \in \mathcal{U}_1} U_1 / \sim_1 \to \coprod_{U_2 \in \mathcal{U}_2} U_2 / \sim_2$$

where \sim_1 (respectively \sim_2) denotes the equivalence relation of G_1 -bisubmersions (respectively G_2 -bisubmersions) as in Appendix C. It is clearly a morphism of topological groupoids.

It is clear that if $U, V \in \mathcal{S}$ and $f \colon U \to V$ is a morphism for G_1 -bisubmersions, then it is also a morphism for G_2 -bisubmersions. Hence, for all $u \in U$ and $v \in V$, $u \sim_1 v$ implies that $u \sim_2 v$.



The equivalences between points in arbitrary elements of \mathcal{U}_1 are more delicate. We first describe how to construct certain bisections. Let $U_1 \in \mathcal{U}_1$, so that $U_1 = U^1 \circ_1 \cdots \circ_1 U^k$ for $U^1, \ldots, U^k \in \mathcal{S}$. We denote by U_2 the "corresponding" element of \mathcal{U}_2 , i.e., $U_2 := U^1 \circ_2 \cdots \circ_2 U^k$. As remarked in the previous claim, we have a (usually strict) inclusion $\iota : U_1 \hookrightarrow U_2$. Fix $u \in U_1$. We construct a bisection for U_2 passing through $\iota(u)$, as follows. Let \mathbf{b} be a $\dim(M_2)$ -dimensional submanifold of U_1 so that U_1 is transverse to both $(\mathbf{s}_U)_1^{-1}(\ker(f_*))$ and $(\mathbf{t}_U)_1^{-1}(\ker(f_*))$. Notice that $(\mathbf{s}_U)_1(\mathbf{b})$ and $(\mathbf{t}_U)_1(\mathbf{b})$ are both submanifolds of M_1 transverse to the fibers of $f: M_1 \to M_2$, and that the map

$$(\mathbf{s}_U)_1(\mathbf{b}) \to (\mathbf{t}_U)_1(\mathbf{b}),$$

obtained $(\mathbf{s}_U)_1$ -lifting to points of **b** and applying $(\mathbf{t}_U)_1$, is a diffeomorphism. Together with the commutativity of (D.5), this implies that $\iota(\mathbf{b})$ is a bisection for the G_2 -bisubmersion U_2 .

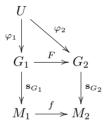
Now let U_1, V_1 be arbitrary elements of U_1 , and U_2, V_2 the "corresponding" elements of U_2 . Let $u \in U_1$ and $v \in V_1$ with $u \sim_1 v$. By definition, this means that there is a locally defined morphism $f \colon U_1 \to V_1$ of bisubmersions over G_1 , mapping u to v (see Section 3.2). Choosing a submanifold $\mathbf{b} \subset U_1$ through u as above, we obtain bisections $\iota(\mathbf{b})$ for the bisubmersion U_2 , and $\iota(f(\mathbf{b}))$ for the bisubmersion V_2 . Notice that these bisections pass through $\iota(u)$ and $\iota(v)$ respectively. Both bisections carry the same bisection of G_2 , by the commutativity of (D.5). By Corollary 3.3(2) we obtain $\iota(u) \sim_2 \iota(v)$.

Claim. — We have $\coprod_{U_2 \in \mathcal{U}_2} U_2 / \sim_2 = H^{G_2}(\mathcal{B}_2)$, hence the morphism in (D.6) reads $H^{G_1}(\mathcal{B}_1) \to H^{G_2}(\mathcal{B}_2)$.

Let $(U, \varphi_1, G_1) \in \mathcal{S}$, constructed out of a minimal set of local generators $\{\alpha_1, \ldots, \alpha_n\}$ of \mathcal{B}_1 . Consider $(U, \varphi_2 := F \circ \varphi_1, G_2)$, a G_2 -bisubmersion for \mathcal{B}_2 . Further consider (W, φ_W, G_2) , the G_2 -bisubmersion for \mathcal{B}_2 (not necessarily minimal) associated to the set of local generators $\{F_*(\alpha_1), \ldots, F_*(\alpha_n)\}$ of \mathcal{B}_2 . In the proof of Lemma D.5 we saw that there is a morphism of bisubmersions $p \colon U \to W$, so the bisubmersion (U, φ_2, G_2) is adapted (see Definition C.1) to the bisubmersion (W, φ_W, G_2) . Hence the atlas \mathcal{U}_2 , which is generated by all (U, φ_2, G_2) 's as above, is adapted to the atlas generated by the corresponding W's, which we denote by W. In turn, by Remark C.6, W is equivalent to a path holonomy atlas. The latter is adapted to \mathcal{U}_2 by Proposition C.3. In conclusion, \mathcal{U}_2 is equivalent to a path holonomy atlas, so by Proposition C.2 there is a canonical isomorphism of topological groupoids $\coprod_{U_2 \in \mathcal{U}_2} U_2 / \sim_2 \cong H^{G_2}(\mathcal{B}_2)$.

Claim. — The map $H^{G_1}(\mathcal{B}_1) \to H^{G_2}(\mathcal{B}_2)$ constructed above is surjective.

The inclusion ι in diagram (D.5) is not surjective. We have to show that any point in the codomain is equivalent, under \sim_2 , to a point in the image of ι . We do so only for points in the codomain which lie in the product of two bisubmersions, as the general case is similar. Consider path holonomy G_1 -bisubmersions U and V lying in S, i.e. constructed out of minimal sets of local F-projectable generators $\{a_i\}_{i\leqslant I}$ and $\{b_k\}_{k\leqslant K}$ of \mathcal{B}_1 . Consider points $(\lambda,y)\in U\subset\mathbb{R}^I\times M_1$ and $(\eta,x)\in V\subset\mathbb{R}^K\times M_1$ such that their images under (x,y) and (x,y) and (x,y) respectively coincide. We want to show that $(x,y)\circ_2(\eta,x)\in U\circ_2 V$ is equivalent under $(x,y)\circ_2(\eta,x)$ and $(x,y)\circ_1(\eta,x)$ might not be well-defined.



Define $y' := (\mathbf{t}_V)_1(\eta, x) \in M_1$. Notice that $f(y') = (\mathbf{t}_V)_2(\eta, x) = f(y)$. Choose a path holonomy bisubmersion U', constructed out of a minimal set of local, F-projectable generators $\{a'_j\}_{j \in J}$ of \mathcal{B}_1 defined nearby y'. There is $\lambda' \in \mathbb{R}^J$ such that

(D.7)
$$(\lambda, y) \sim_2 (\lambda', y').$$

To see this, first use the fact that the $\{F_*a_j'\}_{j\leqslant J}$ generate \mathcal{B}_2 nearby f(y') to write $F_*a_i = \sum_j c_i^j(F_*a_j')$ where $c_i^j \in C^{\infty}(M_2)$. Then check using (D.4) that

$$U \to U', (\gamma_1, \dots, \gamma_I; z) \mapsto \left(\sum c_i^1 \gamma_i, \dots, \sum c_i^J \gamma_i; \psi(z)\right)$$

is a morphism of G_2 -bisubmersions, which maps (λ, y) to a point of the form (λ', y') for some $\lambda' \in \mathbb{R}^J$. Here ψ is any local diffeomorphism of M_1 with $\psi(y) = y'$ preserving the f-fibers, which exists since f is a submersion.

⁽²¹⁾Recall that $(\mathbf{s}_U)_2$ denotes the composition of $\varphi_2 \colon U \to G_2$ and the source map of G_2 .

Notice that $(\lambda', y') \circ_1 (\eta, x) \in U' \circ_1 V$ is well-defined. Thanks to equation (D.7) we can hence write

$$(\lambda, y) \circ_2 (\eta, x) \sim_2 (\lambda', y') \circ_2 (\eta, x) = \iota((\lambda', y') \circ_1 (\eta, x)),$$

proving the claim.

To finish the proof, notice that diagram (D.3) commutes since diagram (D.5) does. \Box

Arbitrary morphisms

The next theorem, which specializes to Theorem 4.6 for morphisms covering the identity, extends Proposition D.4.

THEOREM D.6. — Let $F: G_1 \to G_2$ be a morphism of Lie groupoids, covering a surjective submersion $f: M_1 \to M_2$ between their spaces of identity elements. Let \mathcal{B}_1 be a singular subalgebroid of $A_1 := \operatorname{Lie}(G_1)$, and assume that it satisfies condition (D.1) above. Let \mathcal{B}_2 be a singular subalgebroid of $A_2 := \operatorname{Lie}(G_2)$, such that $F_*(\mathcal{B}_1) \subset \mathcal{B}_2$. Then there is a canonical morphism of topological groupoids

$$\Xi \colon H^{G_1}(\mathcal{B}_1) \to H^{G_2}(\mathcal{B}_2)$$

over f making the following diagram commute:

(D.8)
$$H^{G_1}(\mathcal{B}_1) - \stackrel{\Xi}{-} > H^{G_2}(\mathcal{B}_2)$$

$$\downarrow^{\Phi_1} \qquad \qquad \downarrow^{\Phi_2}$$

$$G_1 \xrightarrow{F} G_2$$

Proof. — Compose the canonical morphism $\Xi: H^{G_1}(\mathcal{B}_1) \to H^{G_2}(F_*(\mathcal{B}_1))$ given by Proposition D.4 with the canonical morphism $H^{G_2}(F_*(\mathcal{B}_1)) \to H^{G_2}(\mathcal{B}_2)$ given by Lemma 4.4.

Remark D.7. — One can wonder whether an analogue of Theorem D.6 holds if one replaces the Lie groupoid morphism $F: G_1 \to G_2$ by a generalized morphism. A useful characterization of generalized morphisms is the one [13, Remark 4.5.3] as a diagram of Lie groupoid morphisms

$$G_1 \stackrel{\phi}{\leftarrow} K \stackrel{\psi}{\rightarrow} G_2,$$

where ϕ is a strong equivalence (hence the base map $f: P \to M_1$ is a surjective submersion, $K \cong f^{-1}G_1 := P \times_M G_1 \times_M P$ is the pullback groupoid,

and ϕ is the second projection). We have $\text{Lie}(K) \cong f^{-1} \text{Lie}(G_1)$ (the pullback Lie algebroid). Pulling back the singular subalgebroid \mathcal{B}_1 one obtains a singular subalgebroid $f^{-1}\mathcal{B}_1$ of Lie(K). Indeed one can define the pullback of singular subalgebroids by fiber-wise surjective Lie algebroid morphisms just as for singular foliations [1, Def. 1.9] (notice that [1, Prop. 1.10 b)] extends straightforwardly to singular subalgebroids).

Under the assumption that $f: P \to M_1$ has connected fibers, we expect that $f^{-1}(H^{G_1}(\mathcal{B}_1)) \cong H^K(f^{-1}\mathcal{B}_1)$: this is true for singular foliations [10, Thm. 3.21], and we expect the proof given there to extend to singular subalgebroids. If so, one obtains a strong equivalence of topological groupoids $\Xi_1: H^K(f^{-1}\mathcal{B}_1) \to H^{G_1}(\mathcal{B}_1)$. If the assumptions of Theorem D.6 are satisfied for the Lie groupoid morphism $\psi: K \to G_2$ and the singular subalgebroids $f^{-1}\mathcal{B}_1$ and \mathcal{B}_2 , we can then apply that theorem to obtain a morphism of topological groupoids $\Xi_2: H^K(\phi^{-1}\mathcal{B}_1) \to H^{G_2}(\mathcal{B}_2)$. Altogether, this would yield a generalized morphism from $H^{G_1}(\mathcal{B}_1)$ to $H^{G_2}(\mathcal{B}_2)$.

Examples

The following are applications of Proposition D.4.

Example D.8 (The singular foliation $\overrightarrow{\mathcal{B}}$ on G). — Let $G \rightrightarrows M$ be a Lie groupoid, and \mathcal{B} a singular subalgebroid of $A := \operatorname{Lie}(G)$. Consider the Lie groupoid $G \times_{\mathbf{s},\mathbf{s}} G \rightrightarrows G$ associated to the submersion $\mathbf{s} : G \to M$. (It is a subgroupoid of the pair groupoid associated to G, where we regard G as a mere manifold). Consider the morphism of Lie groupoids

$$F \colon G \times_{\mathbf{s},\mathbf{s}} G \to G, \ (g,h) \mapsto gh^{-1}$$

over $\mathbf{t}: G \to M$. The corresponding Lie algebroid morphism $F_*: \ker(\mathbf{s}_*) \to \ker(\mathbf{s}_*)|_M = A$ sends $v \in \ker(\mathbf{s}_*)_h$ to $(R_{h^{-1}})_*v$. From this we see that the singular foliation $\overrightarrow{\mathcal{B}}$ satisfies condition (D.1) and $F_*(\overrightarrow{\mathcal{B}}) = \mathcal{B}$.

Notice that, since $G \times_{\mathbf{s},\mathbf{s}} G$ is a subgroupoid of the pair groupoid $G \times G$, by Proposition 4.2 we have $H^{G \times_{\mathbf{s},\mathbf{s}} G}(\overrightarrow{\mathcal{B}}) \cong H^{G \times G}(\overrightarrow{\mathcal{B}})$, that is: the holonomy groupoid of $\overrightarrow{\mathcal{B}}$ constructed using $G \times_{\mathbf{s},\mathbf{s}} G$ agrees with the one constructed using the pair groupoid $G \times G$. Proposition D.4 implies the existence of a canonical surjective morphism of topological groupoids

$$H^{G \times G}(\overrightarrow{\mathcal{B}}) \to H^G(\mathcal{B}).$$

Notice that the domain of the above morphism is simply the holonomy groupoid of the singular foliation $\overrightarrow{\mathcal{B}}$, in the sense of [1]. In [7] we use

this morphism to study the geometric properties of the holonomy groupoid $H^G(\mathcal{B})$.

Example D.9 (Singular foliations). — Let $f: M_1 \to M_2$ be a surjective submersion. Let \mathcal{F}_1 be a singular foliation on M_1 , such that $\widehat{\mathcal{F}_1}^{\operatorname{proj}} := \{X \in \widehat{\mathcal{F}_1}: X \text{ is } f\text{-projectable to a vector field on } M_2\}$ generates \mathcal{F}_1 as a $C_c^{\infty}(M_1)$ -module. Then $f_*\mathcal{F}_1$ is a singular foliation on M_2 , by Lemma D.1. Let

$$(f,f)\colon M_1\times M_1\to M_2\times M_2$$

be the induced map on pair groupoids. Proposition D.4 implies the existence of a canonical surjective morphism of topological groupoids

$$H(\mathcal{F}_1) \to H(f_*\mathcal{F}_1)$$

between the holonomy groupoids of the two singular foliations.

We remark that, under the additional assumptions that f has connected fibers and that \mathcal{F}_1 contains $\Gamma_c(\ker(f_*))$, the foliation \mathcal{F}_1 agrees with the pullback foliation $f^{-1}(f_*\mathcal{F}_1)$, by [4, Lemma 3.2]. It is shown in [10] that $H(\mathcal{F}_1)$ then agrees with the pullback of the groupoid $H(f_*\mathcal{F}_1)$ via f, and the above morphism is the canonical projection of the pullback groupoid.

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