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LOCAL-TO-GLOBAL-RIGIDITY OF LATTICES IN $SL_n(\mathbb{K})$

by Amandine ESCALIER

Abstract. — A vertex-transitive graph $\mathcal{G}$ is called *Local-to-Global rigid* if there exists $R > 0$ such that every other graph whose balls of radius R are isometric to the balls of radius R in $\mathcal{G}$ is covered by $\mathcal{G}$. An example of such a graph is given by the Bruhat–Tits building of $PSL_n(\mathbb{K})$ with $n \geqslant 4$ and $\mathbb{K}$ a non-Archimedean local field of characteristic zero. In this paper we extend this rigidity property to a class of graphs quasi-isometric to the building including torsion-free lattices of $SL_n(\mathbb{K})$.

The proof is the opportunity to prove a result on the local structure of the building. We show that if we fix a $PSL_n(\mathbb{K})$-orbit in it, then a vertex is uniquely determined by the neighbouring vertices in this orbit.

Résumé. — Un graphe transitif $\mathcal{G}$ est dit *Local-Global rigide* s'il existe $R > 0$ tel que tout autre graphe dont les boules de rayon R sont isométriques aux boules de rayon R de $\mathcal{G}$ est revêtu par $\mathcal{G}$. Un exemple de tel graphe est donné par l'immeuble de Bruhat–Tits de $PSL_n(\mathbb{K})$ lorsque $n \geqslant 4$ et $\mathbb{K}$ est un corps local non-Archimédien de caractéristique nulle. Dans cet article nous étendons cette propriété de rigidité à une classe de graphes quasi-isométriques à l'immeuble, incluant les réseaux sans torsion de $SL_n(\mathbb{K})$.

La démonstration est l'occasion de prouver un résultat sur la structure locale des immeubles. Nous montrons que si l'on fixe une $PSL_n(\mathbb{K})$-orbite dans l'immeuble, alors un sommet est uniquement déterminé par les sommets voisins contenus dans cette orbite.

1. Introduction

A recurring theme in geometric group theory is that *local* properties of an object can have *global* implication for its geometry. A classical example is given by Lie groups and their locally defined Lie algebras. Another striking illustration is provided by the work of Tits [12] who gave a local characterization of a particular family of graphs called "buildings of type

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$\widetilde{A}_{d-1}$" (see Section 2.1 for a definition). Precisely, graphs and their local-to-global properties are the objects we focus on in this article. All graphs will be equipped with the usual metric, fixing the length of an edge to one.

A natural local condition to impose on a graph is to be d-regular for some $d \in \mathbb{N}$, which means that all the vertices must have degree d. A well-known result about such a graph is that the d-regular tree is its universal convering. This is a first example of a global information deduced only by a local knowledge of the graph.

One can now ask what happens if we impose a local condition which is stronger than d-regularity. We formalize this in the next definition.

Definition 1.1. — *Let $R > 0$ and let X and Y be two graphs.*

We say that Y is R-locally X *if every ball of radius R in Y is isometric to a ball of radius R in X.*

If Y is R-locally X and X is R-locally Y then we say that they are R-locally the same.

Example 1.2. — In Figure 1.1, $B_X(x_0, 2)$ is isometric to $B_Y(y_0, 2)$.

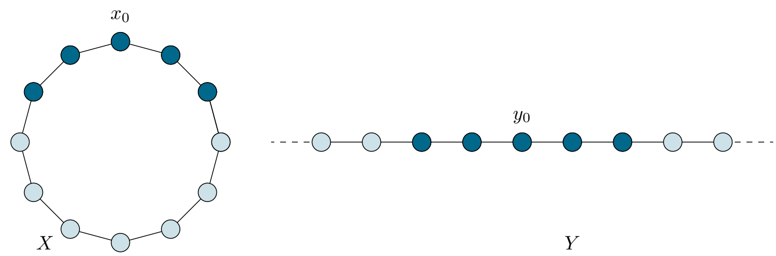

Figure 1.1. Two graphs 2-locally the same.

The previous covering result on the d-regular tree is a first example of a more general notion called the *Local-to-Gobal rigidity*, also named *LG-rigidity*.

Definition 1.3. — *Let $R > 0$. We say that X is* Local-to-Global-rigid at scale R *(or R-*LG-rigid *for short) if every graph Y which is R-locally X is covered by X.*

We say that a graph X is LG-rigid *if there exists $R > 0$ such that X is R-LG-rigid.*

Example 1.4. — Benjamini and Ellis [5] showed that for any $d \geqslant 2$ the Cayley graph of $\mathbb{Z}^d$ endowed with its usual generating set is 3-LG-rigid.

They also proved that 3 was optimal showing that $\mathbb{Z}^3$ is not LG-rigid at scale 2.

Example 1.5. — De la Salle et Tessera [11, Theorem C] proved that every cocompact graph quasi-isometric to a tree is LG-rigid.

Benjamini [4] and Georgakopoulos [8] conjectured that any Cayley graph of a finitely presented group is LG-rigid at some scale $R > 0$. That conjecture was proven to be false in [11, Theorem B], where the authors built counter-examples using groups with *torsion* elements.

COUNTER-EXAMPLE 1.6. — *The groups $F_2 \times F_2 \times \mathbb{Z}/2\mathbb{Z}$ and $SL_4(\mathbb{Z})$ admit Cayley graphs that are not LG-rigid.*

Remark here that we do not state that every Cayley graph of these groups is non-LG-rigid, but that each group *admits* a non-LG-rigid Cayley graph. Indeed, in [11, Theorem J] the authors also showed that every finitely presented group with an element of infinite order has a Cayley graph which is LG-rigid. Hence, LG-rigidity for a Cayley graph depends on the generating set. In particular LG-rigidity is not invariant under quasi-isometries.

With a little bit more of material, we will be able to give a topological interpretation of Local-to-Global rigidity (see page 1745).

That rigidity notion can be refined in what is called the *Strong Local-to-Global rigidity*, also named *SLG-rigidity*.

DEFINITION 1.7. — *Let $r, R > 0$. We say that X is* SLG-rigid at scale (r, R) *if for all Y which is R-locally X and for all isometry f from $B_X(x, R)$ to $B_Y(y, R)$, the restriction of f to $B_X(x, r)$ extends to a covering of Y by X.*

We say that X is SLG-rigid *if there exist two radius r and R such that X is SLG-rigid at scale (r, R).*

Such a refinement is far more than just a subtlety: it actually proves necessary to obtain our main result (see page 1762 for more details).

The following proposition gives us many examples of SLG-rigid graphs.

PROPOSITION 1.8 (de la Salle, Tessera [11, Proposition 3.8]). — *A graph with cocompact isometry group is LG-rigid if and only if it is SLG-rigid.*

For example, any LG-rigid Cayley graph is actually SLG-rigid. In the same article, de la Salle and Tessera proved a powerful condition relating to the isometry group of a Cayley graph. We will refer to the isometry group of a Cayley graph (Γ, S) as $\mathrm{Isom}(\Gamma, S)$.

THEOREM 1.9 (de la Salle, Tessera [11, Theorem E]). — *Let* Γ *be a finitely presented group and* S *be a symmetric generating set and denote by* (Γ, S) *the corresponding Cayley graph. If* $\mathrm{Isom}(\Gamma, S)$ *is discrete, then* (Γ, S) *is SLG-rigid.*

As stated in [11, Corollary F], we can deduce two new classes of examples from the above theorem. But before, let us introduce what we call *LG-rigid groups*.

DEFINITION 1.10. — *We say that a finitely presented group is* LG-rigid *(resp.* SLG-rigid*) if all its Cayley graphs are LG-rigid (resp.* SLG-rigid*).*

Example 1.11. — Torsion-free groups of polynomial growth are SLG-rigid.

Example 1.12. — Torsion-free, non-virtually free lattices in connected simple real Lie groups are SLG-rigid.

So far, the graphs chosen as examples are mostly Cayley graphs, but these are not the only LG-rigid ones. Indeed, besides the case of quasi-trees seen above, another interesting example is given by *Bruhat-Tits buildings* (see Section 2.1 for a definition).

THEOREM 1.13 (de la Salle, Tessera [10, Theorem 0.1]). — *Let* $\mathbb{K}$ *be a non-Archimedean local skew field.*

If $\mathbb{K}$ *has positive characteristic and* $n \geqslant 3$*, then the Bruhat-Tits building of* $PSL_n(\mathbb{K})$ *is not LG-rigid.*

If $\mathbb{K}$ *has characteristic zero and* $n \geqslant 4$*, then the Bruhat-Tits building of* $PSL_n(\mathbb{K})$ *is SLG-rigid.*

Keeping in mind the above theorem, consider the following question asked in [11].

QUESTION 1.14. — Among lattices in semi-simple Lie groups, which ones are LG-rigid?

This question concerns *real* Lie groups but one can also wonder what happens for the *p-adic* case. Indeed, by a well known result of Svarc and Milnor, any lattice of $SL_n(\mathbb{K})$ is quasi-isometric to the associated building (see Section 5.2 for more details). The fact that such a lattice is "almost" a building encouraged us to study the p-adic version of Question 1.14.

QUESTION 1.15. — Among lattices in p-adic Lie groups, which ones are LG-rigid?

De la Salle and Tessera showed [10] that if $\mathbb{K}$ has positive characteristic, then there exist p-adic lattices that are torsion-free, cocompact but not LG-rigid.

Example 1.16. — Let $n \geqslant 3$. There exists a torsion-free cocompact lattice in $PGL_n(\mathbb{F}_p)$ that is not LG-rigid.

When $\mathbb{K}$ is a non-Archimedean local skew field of characteristic zero, an element of response to Question 1.15 is provided by our first result hereunder.

THEOREM 1.17. — *Let $n \neq 3$ and $\mathbb{K}$ be a non-Archimedean local skew field of characteristic zero. The torsion-free lattices of $SL_n(\mathbb{K})$ are SLG-rigid.*

This result is actually a corollary of our main theorem below which goes beyond the lattices framework and gives a rigidity result in a more general case.

THEOREM 1.18. — *Let $n \neq 3$ and $\mathbb{K}$ be a non-Archimedean local skew field of characteristic zero. Let $\mathcal{X}$ be the Bruhat–Tits building of $PSL_n(\mathbb{K})$ and X be a transitive graph. If X verifies that*

- *There is an injective homomorphism ρ from $\mathrm{Isom}(X)$ to $\mathrm{Isom}(\mathcal{X})$ such that $\rho(\mathrm{Isom}(X))$ is of finite index in $\mathrm{Isom}(\mathcal{X})$;*
- *There is a $\mathrm{Isom}(X)$-equivariant injective quasi-isometry q from X to $\mathcal{X}$;*

then X is SLG-rigid.

Let us discuss the hypothesis, starting with the assumption made on n. If $n = 2$ then $\mathcal{X}$ is the $(p+1)$-regular tree, thus by Example 1.5 any graph quasi-isometric to $\mathcal{X}$ is LG-rigid which proves the theorem. Now, as we will see in the sketch of the proof, the main tool of our demonstration is the LG-rigidity of the building. But if $n = 3$ the question of the rigidity of $\mathcal{X}$ is still open. Indeed in that case a lot of flexibility seems to be allowed (see [3]). Thus our demonstration deals mainly with the case where $n \geqslant 4$.

Secondly, let us look at the hypothesis made on the characteristic of $\mathbb{K}$. According to [10, Theorem 0.4] and more precisely according to its *proof*, we get Counter-Example 1.19 below. It implies in particular that if we omit the characteristic zero hypothesis, then Theorems 1.18 and 1.17 are not true.

COUNTER-EXAMPLE 1.19. — *There exists a non-LG-rigid torsion-free cocompact lattice in $PGL_n\big(\mathbb{F}_{\mathrm{p}}((T))\big)$.*

Finally, before moving to the sketch of the proof let us discuss the hypothesis made on the torsion in Theorem 1.17. First, introducing torsion in a group is in some case a useful way to build non-LG-rigid graphs. Indeed the Counter-Example 1.6 is built this way. Second, in order to link (Γ, S) to $\mathcal{X}$ we will need an injection of $\mathrm{Isom}(\Gamma, S)$ into $\mathrm{Isom}(\mathcal{X})$. Using a famous result of Kleiner and Leeb we will show that $\mathrm{Isom}(\Gamma, S)$ acts on the buildings by isometries. The injection into $\mathrm{Isom}(\mathcal{X})$ will then be allowed by the following proposition.

Proposition 1.20 (de la Salle, Tessera [11, Proposition 6.2]). *Let Γ be an infinite, torsion-free, finitely generated group and let S be a finite symmetric generating subset of Γ. Then the isometry group of (Γ, S) has no non-trivial compact normal subgroup.*

For more details on how we use this proposition see the proof of Lemma 5.3.

Sketch of the proof of Theorem 1.18

As stated in the discussion below Theorem 1.18, the proof deals mainly with the case where $n \geqslant 4$. So, Let $n \geqslant 4$ and $\mathbb{K}$ be non-Archimedean local skew field of characteristic zero and and denote by $\mathcal{X}$ the Bruhat–Tits building of $PSL_n(\mathbb{K})$. Let X be the studied graph and Y be a graph R-locally the same as X. The main idea of the proof is to use the rigidity of $\mathcal{X}$ to build the wanted covering from X to Y (see Figure 1.2), thus we need to build a graph locally the same as $\mathcal{X}$. We will denote such a graph $\mathcal{Y}$.

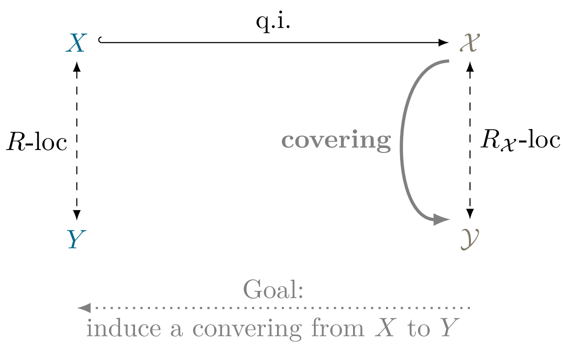

Figure 1.2. Sketch of the proof

Moreover, for the rigidity of the building to induce a covering between X and Y, we want $\mathcal{Y}$ to contain a copy of the vertices of Y. Hence the goal is to define the vertices of $\mathcal{Y}$ to be composed of the vertices of Y and a copy of each vertex in $\mathcal{X}\backslash q(X)$ and define the edges to correspond to edges in X.

With such a description $\mathcal{Y}$ is a "hybrid" graph and to define its edges we might need to understand how to link a vertex coming from Y to a vertex coming from $\mathcal{X}$. Hence, to avoid such a hybridation we chose to define the vertices only with informations encoded in Y. That is why we introduce the notion of *print* in the building (see Section 3.1). It allows us to characterize a vertex in $\mathcal{X}$ by a set of neighbouring vertices in $im(q)$ and, using a well chosen set of isometries from Y to X, to transfer this print notion to Y. Each print in Y corresponds to a vertex in $\mathcal{X}\backslash q(X)$. The vertices of the wanted graph $\mathcal{Y}$ will be composed of the vertices of Y and of prints in Y. It will now be easier to build edges between these vertices; the key argument to construct such edges is presented in Section 2.3.

Using the rigidity of the building we will obtain an isometry between $\mathcal{X}$ and $\mathcal{Y}$. To conclude the proof we will show that this isometry induces the wanted covering between Y and X.

Organization of the paper

The first section is devoted to the definition of our framework. We recall some material about Bruhat–Tits buildings and large scale simple connectedness and present a fundamental result on isometries' extension. The second and third sections are devoted to the proof of Theorem 1.18.

In the second section we develop the necessary engineering to build a graph locally the same as the building —this is where we define and study prints—while in the third one we use the rigidity of the building to prove the rigidity of the studied graph. We prove Theorem 1.17 in the fourth section where we check that the lattice verifies the hypothesis of our main theorem.

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2. Framework

Let us start by setting up the framework of the next sections. We first recall some material about Bruhat–Tits buildings, and large scale simple connectedness. Then we present a useful tool concerning the extension of isometries. We conclude by a result one step further in to the proof of our main theorem, linking the $PSL_n(\mathbb{K})$-orbits in the building and the image $q(X)$ of the graph studied.

2.1. Bruhat–Tits building

Let $n \geqslant 2$. Since it is the object at the center of our proof, let us recall the description of the Bruhat–Tits building associated to $PSL_n(\mathbb{K})$ where $n \geqslant 2$, see [1] for more details.

2.1.1. Non-Archimedean local skew fields

Let $\mathbb{K}$ be a field (not necessarily commutative). A *discrete valuation* on $\mathbb{K}$ is a surjective homomorphism $v : \mathbb{K}^* \to \mathbb{Z}$ satisfying $v(x+y) \geqslant \min\{v(x), v(y)\}$ for all $x, y \in \mathbb{K}^*$ such that $x + y \neq 0$. If $\mathbb{K}$ is endowed with such a valuation, we can extend v on all $\mathbb{K}$ by setting $v(0) = +\infty$. We say that $\mathbb{K}$ is a *non-Archimedean local skew field* if it is locally compact for the topology associated to a discrete valuation.

Example 2.1. — If $\mathbb{K} = \mathbb{Q}$ and p is a prime, then every $x \in \mathbb{K}$ can be written as $x = p^n a/b$ where a and b are integers non-divisible by p. The map defined by $v(p^n a/b) := n$ is a discrete valuation over $\mathbb{K}$. The field $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the *p-adic absolute value* defined by $|x|_p = p^{-v(x)}$.

Example 2.2. — Let $\mathbb{K} = \mathbb{F}_p((T))$, the field of *formal Laurent series* over $\mathbb{F}_p$. Denote by $f = \sum_{k \in \mathbb{Z}} a_k T^k$ an element in $\mathbb{F}_p((T))$ then the map defined by $v(f) := \min\{k : a_k \neq 0\}$ is a valuation over $\mathbb{K}$.

Let $\mathcal{O}$ denote the *ring of integers* of $\mathbb{K}$ with respect to v, that is to say $\mathcal{O} := \{x \in \mathbb{K} : v(x) \geqslant 0\}$. This ring has a unique *prime ideal* $\mathfrak{m} := \{x \in K : v(x) > 0\}$. Finally, let π be a generator of $\mathfrak{m}$ as a $\mathcal{O}$-module.

Example 2.3. — If $\mathbb{K} = \mathbb{Q}_p$ then its ring of integers is $\mathcal{O} = \mathbb{Z}_p$. Moreover $\mathfrak{m} = p\mathbb{Z}_p$ and $\pi = p$.

Example 2.4. — If $K = \mathbb{F}_p((T))$ then $\mathcal{O} = \mathbb{F}_p[[T]]$. Moreover $\mathfrak{m} = X\mathbb{F}_p[[T]]$ and $\pi = X$.

2.1.2. Buildings

Let $\mathbb{K}$ be a non-Archimedean local skew field endowed with a valuation v. A $\mathcal{O}$-lattice of $\mathbb{K}^n$ is a $\mathcal{O}$-submodule which generates $\mathbb{K}^n$ as a $\mathbb{K}$ vector space. Such a lattice can be written as $\mathcal{O}e_1 + \cdots + \mathcal{O}e_n$ for a basis $(e_1, \ldots, e_n)$ of $\mathbb{K}^n$. Since for any $a \in \mathbb{K}^*$ and any lattice L, the module aL is also a lattice, we can define the equivalence relation of *lattices modulo homothety*. We denote by $[L]$ the class of a lattice L.

The Bruhat–Tits building of $PSL_n(\mathbb{K})$ is a simplicial complex of dimension $n-1$ denoted by $\widehat{\mathcal{X}}$ whose 1-skeleton (denoted by $\mathcal{X}$) is described as follows. The vertices are the classes of $\mathcal{O}$-lattices modulo homothety. Two vertices x_1 and x_2 are linked by an edge if there exists representatives L_1 of x_1 and L_2 of x_2 such that:

$$pL_1 \subset L_2 \subset L_1.$$

Example 2.5. — One can show that the building of $PSL_2(\mathbb{Q}_p)$ is a $(p+1)$-regular tree. Figure 2.1a gives a representation of the building when $p = 2$.

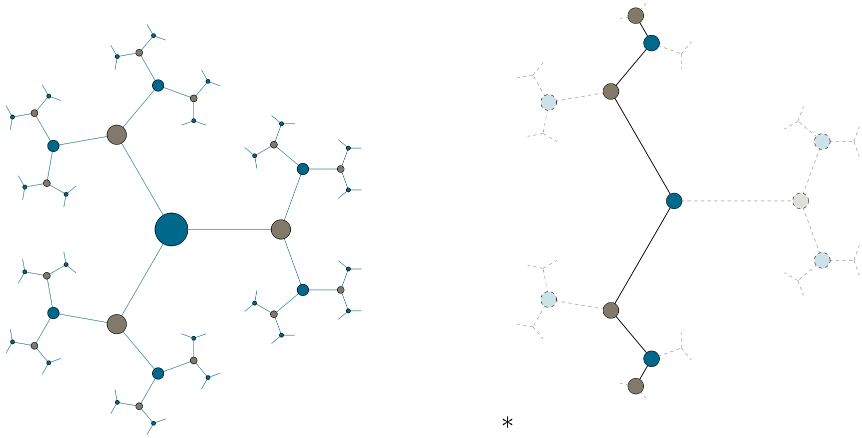
2.1.3. Orbits and types

The usual action of $GL_n(\mathbb{K})$ on $\mathbb{K}^n$ induces an action of $PGL_n(\mathbb{K})$ on $\mathcal{X}$ by isometry. Since $GL_n(\mathbb{K})$ acts transitively on the bases, the action of $PGL_n(\mathbb{K})$ on the vertices of $\mathcal{X}$ is also transitive.

If $L = \oplus_i \mathcal{O}e_i$ is a lattice we define its *type* to be $v\left(\det(e_1, \ldots, e_n)\right)$. Since:

$$\forall\, a \in \mathbb{K}^* \quad v\left(\det(ae_1, \ldots, ae_n)\right) = v\left(\det(e_1, \ldots, e_n)\right) \mod n,$$

one can define the *type of a vertex* x in $\mathcal{X}$ to be the value modulo n of the type of one of its representatives. We denote by $\tau(x)$ the type of x.



*

(a) The building has two $SL_2(\mathbb{Q}_2)$-orbits

(b) Representation of one apartment

Figure 2.1. The building of $PSL_2(\mathbb{Q}_2)$

If L' is a second lattice, we can choose our basis $e_1, \ldots, e_n$ for L in such a way that L' admits a basis of the form $a_1 e_1, \ldots, a_n e_n$ for some $a_i \in \mathbb{K}^*$. The scalars a_i can be taken to be powers of π. The incidence relation defined above implies that if the classes of L and L' are linked by an edge in $\mathcal{X}$, then they have different types.

Remark 2.6. — Remark that if $L = \oplus_i \mathcal{O} e_i$ and

$$L' = \mathcal{O}\pi e_1 \oplus \cdots \oplus \mathcal{O}\pi e_j \oplus e_{j+1} \oplus \cdots \oplus e_n,$$

then $\tau\big([L']\big) = \tau\big([L]\big) + j \mod n$.

The action of $SL_n(\mathbb{K})$ on $\mathcal{X}$ preserves the determinant and is transitive on the pairs of vertices of the same type. So there are exactly n orbits under the action of $SL_n(\mathbb{K})$ (see Figure 2.1a and Figure 2.2 for examples).

2.1.4. Apartments

If $\mathbf{e}$ is a basis of $\mathbb{K}^n$ then the sub-complex $\mathcal{A}$ induced by the set of vertices $\left\{\oplus_{i=1}^n \mathcal{O}\pi^{k_i} e_i \mid k_i \in \mathbb{Z}\right\}$ is isometric to a $(n-1)$-dimensional Euclidean space tiled by regular $(n-1)$-simplices. We call such sub-complexes *apartments*. For example an apartment in the building of $PSL_2(\mathbb{Q}_2)$ is isometric to $\mathbb{R}^1$ tiled with segments of length 1 (see Figure 2.1b), whereas for $PSL_3(\mathbb{Q}_2)$ the apartment are isometric to $\mathbb{R}^2$ and tiled with triangles (see Figure 2.2).

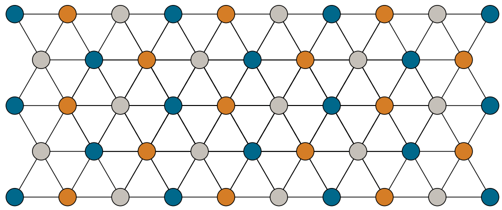


Figure 2.2. Apartment in the building of $PSL_3(\mathbb{Q}_2)$. The colors correspond to $SL_3(\mathbb{Q}_2)$-orbits.

For any two points in $\widehat{\mathcal{X}}$ there exists an apartment containing them. If $x,y\in\widehat{\mathcal{X}}$ let $\mathcal{A}$ be an apartment containing x and y and define $d_{\hat{\mathcal{X}}}(x,y)$ to be equal to the euclidean distance $d_{\mathcal{A}}(x,y)$. This definition does not depend on the choice of apartment $\mathcal{A}$ and thus endows $\widehat{\mathcal{X}}$ with a well defined distance. Moreover, this distance verifies the *negative curvature inequality:* for all $x,y,z\in\widehat{\mathcal{X}}$ and $t\in[0,1]$

$$d^2_{\hat{\mathcal{X}}}(z,tx+(1-t)y)\leqslant td^2_{\hat{\mathcal{X}}}(z,x)+(1-t)d^2_{\hat{\mathcal{X}}}(z,y)-t(1-t)d^2_{\hat{\mathcal{X}}}(x,y). \tag{2.1}$$

Denote by $d_{\mathcal{X}}$ the distance on the 1-skeleton $\mathcal{X}$ assigning length 1 to an edge. Then $d_{\mathcal{X}}(x,y)$ is greater than $d_{\hat{\mathcal{X}}}(x,y)$ for all vertices x and y in $\mathcal{X}$.

2.1.5. Contractibility

Using the above inequality one can show that the building is contractible (see [1] for more details). We can actually show that *convex sets* in $\widehat{\mathcal{X}}$ are themselves contractible.

Claim 2.7. — *Let $r>0$. Any convex set in $\widehat{\mathcal{X}}$ is contractible.*

Proof. — Let $r>0$ and $\mathcal{C}$ a convex set in $\widehat{\mathcal{X}}$ and endow it with the distance induced by $d_{\hat{\mathcal{X}}}$. Take $x_0\in\mathcal{C}$ and define,

$$\mathcal{H}:\begin{cases}[0,1]\times\mathcal{C} & \to\mathcal{C},\\ (t,x) & \mapsto tx+(1-t)x_0.\end{cases}$$

Since $\mathcal{C}$ is convex, the map $\mathcal{H}$ is well-defined. Moreover $\mathcal{H}(0,\cdot)=id_{\mathcal{C}}$ and $\mathcal{H}(1,x)=x_0$ for all x in $\mathcal{C}$. Let us show that $\mathcal{H}$ is continuous. Take $x,x'\in\mathcal{C}$ and $t,t'\in[0,1]$ and let $z=t'x'+(1-t')x_0$. By eq. (2.1)

$$d^2_{\hat{\mathcal{X}}}\big(z,tx+(1-t)x_0\big)\leqslant td^2_{\hat{\mathcal{X}}}(z,x)+(1-t)d^2_{\hat{\mathcal{X}}}(z,x_0)-t(1-t)d^2_{\hat{\mathcal{X}}}(x,x_0). \tag{2.2}$$

But if $\mathcal{A}$ is a an apartment containing z and x_0, then by property of the Euclidean distance $d_{\mathcal{A}}$

$$d_{\hat{\mathcal{X}}}(z,x_0) = d_{\mathcal{A}}\left(t'x' + (1-t')x_0, x_0\right) = t' d_{\mathcal{A}}(x',x_0) = t' d_{\hat{\mathcal{X}}}(x',x_0),$$

which tends to $t d_{\hat{\mathcal{X}}}(x,x_0)$ as (t',x') tends to (t,x). Similarly

$$\begin{aligned} d_{\hat{\mathcal{X}}}(z,x) &= d_{\mathcal{A}}\left(t'x' + (1-t')x_0, x\right) \leqslant d_{\hat{\mathcal{X}}}\left(z,x'\right) + d_{\hat{\mathcal{X}}}\left(x',x\right) \\ &= d_{\hat{\mathcal{X}}}\left(t'x' + (1-t')\,x_0, x'\right) + d_{\hat{\mathcal{X}}}\left(x',x\right), \\ &= (1-t')\,d_{\hat{\mathcal{X}}}\left(x',x_0\right) + d_{\hat{\mathcal{X}}}(x',x), \end{aligned}$$

which converges to $(1-t)d_{\hat{\mathcal{X}}}(x,x_0) + d_{\hat{\mathcal{X}}}(x',x)$ as (t',x') tends to (t,x). Thus the right term of eq. (2.2) converges to 0 as (t',x') tends to (t,x). Hence the continuity of $\mathcal{H}$ and the contractibility of $\mathcal{C}$. □

2.2. Large scale simple connectedness

For a graph $\mathcal{G}$ and $k \in \mathbb{N}$, we define a 2-complex, noted $P_k(\mathcal{G})$, such that:

- Its 1-skeleton is given by $\mathcal{G}$;
- Its 2-skeleton is composed of m-gons (for $m \in [0,k]$) defined by the simple loops of length m in $\mathcal{G}$ (up to cyclic permutations).

Definition 2.8. — *We say that $\mathcal{G}$ is* k-simply connected *or simply connected at scale k if $P_k(\mathcal{G})$ is simply connected.*

Example 2.9. — Let G be a finitely generated group and T a finite symmetric generating set. The Cayley graph (G,T) is simply connected at scale k if and only if G has a presentation $\langle T,\mathcal{R}\rangle$ with relations of length at most k.

Example 2.10. — Let $n \geqslant 2$. The Bruhat–Tits building of $PSL_n(\mathbb{K})$ is simply connected at scale 3.

Remark 2.11. — If $k \leqslant k'$, then every k-simply connected graph is k'-simply connected.

The following proposition allows us to restrict the study of the LG-rigidity of a graph $\mathcal{G}$ to some smaller class of graphs.

Proposition 2.12 (de la Salle, Tessera, [10, Proposition 1.5]).
Let $k \in \mathbb{N}$ and $\mathcal{G}$ be a k-simply connected graph, with cocompact isometry group. Then $\mathcal{G}$ is LG-rigid if and only if there exists R such that every k-simply connected graph which is R-locally $\mathcal{G}$ is isometric to $\mathcal{G}$.

To apply this result to our proof we will need to show that the studied graph X is simply connected. The following proposition shows that being simply connected is invariant under quasi-isometry.

Proposition 2.13 (de la Salle, Tessera, [10, Theorem 2.2]). — *Let $k \in \mathbb{N}^*$ and let $\mathcal{G}$ be a k-simply connected graph. If $\mathcal{H}$ is quasi-isometric to $\mathcal{G}$, then there exists $k' \in \mathbb{N}^*$ such that $\mathcal{H}$ is simply connected at scale k'.*

Before moving to the next section, let us mention a consequence of Proposition 2.12. Indeed, this result allows us to look at the LG-rigidty notion with a topological point of view. Let's denote $\mathfrak{G}_k$ the set of isometry classes of locally finite k-simply connected graphs. We can define a distance on this set by:

$$d_{\mathfrak{G}_k}(X, Y) := \inf \left\{ 2^{-R} \,:\, X \text{ and } Y \text{ are } R\text{-close} \right\},$$

which endows $\mathfrak{G}_k$ with a topology. Proposition 2.12 implies that a graph is LG-rigid if and only if its isometry class in $\mathfrak{G}_k$ is isolated for this topology.

2.3. Extension of isometries

In order to build the "hybrid" graph mentionned above, we will need some result to extend globally our local definition of edges. We recall here the result of de la Salle and Tessera [11, Lemma 4.1] that will serve our purpose.

Proposition 2.14 (de la Salle, Tessera). — *Let $\mathcal{G}$ be a graph with co-compact isometry group. Given some $r_1 \geqslant 0$, there exists $r_2 > 0$ such that: for every $g \in \mathcal{G}$, the restriction to $B_{\mathcal{G}}(g, r_1)$ of an isometry $f : B_{\mathcal{G}}(g, r_2) \to \mathcal{G}$ coincides with the restriction of an element of* $\mathrm{Isom}(\mathcal{G})$.

It is however not necessarily true that f coincides on the whole $B(g, r_2)$ with an isometry of $\mathcal{G}$. Indeed, truncating the entire graph to some ball might allow some kind of flexibility near the boundary of the ball (see Example 2.15 and Figure 2.3). Hence, in order to coincide with a global isometry we need to restrict the local isometry f to a smaller ball which does not contain the flexible area.

Example 2.15. — Let $\mathcal{G}$ be the Cayley graph of $\mathbb{Z}^2$ endowed with its generating part. We consider in Figure 2.3 an isometry f defined on $B\big((0,0),1\big)$ such that f fixes $(0,0)$, $(-1,0)$ and $(0,-1)$ (represented by the blue vertices) and exchange $(1,0)$ with $(0,1)$ (the orange and brown vertices). Then f is an isometry from $B\big((0,0),1\big)$ to $B\big((0,0),1\big)$, but can not coincide with a global isometry of $\mathcal{G}$ on that ball. Indeed, if such a global isometry existed, then it should send the vertex $(-1,1)$ (represented by the light brown vertex on the left part of the figure) at distance 1 from both $f(-1,0)=(-1,0)$ and $f(0,1)=(1,0)$. Which is impossible since the only point at distance 1 from $(1,0)$ and $(-1,0)$ is $(0,0)$ and it is already the image of $(0,0)$.

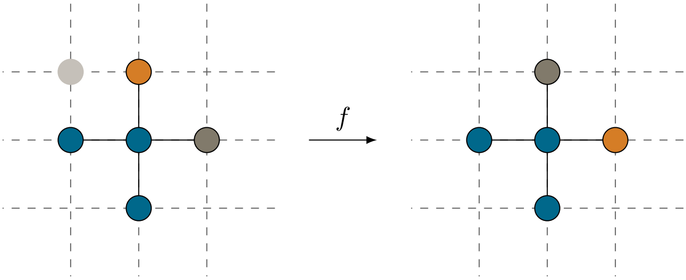


Figure 2.3. Local isometry that can not coincide with a global isometry on its entire domain of definition

2.4. Preliminary results on X

Lemma 2.16. — *If X verifies the hypothesis of Theorem 1.18, then $PSL_n(\mathbb{K})$ is included in $\rho(\mathrm{Isom}(X))$. Moreover, if $q(X)$ contains a vertex of a certain type i, then $q(X)$ contains all the vertices of type i.*

Proof. — Since $\rho(\mathrm{Isom}(X))$ is of finite index in the isometry group of the building $\mathcal{X}$, the same goes for its normal core $\cap_{g\,\in\,\mathrm{Isom}(\mathcal{X})} g\rho(\mathrm{Isom}(X))g^{-1}$. Then, by simplicity of $PSL_n(\mathbb{K})$, the normal core of $\rho(\mathrm{Isom}(X))$ contains $PSL_n(\mathbb{K})$. Hence the result.

Then, the second part of the lemma follows from the equivariance of q and the transitivity of $PSL_n(\mathbb{K})$ on vertices of the same type. □

Without loss of generality, we can assume that $im(q)$ contains type 0 vertices, that is to say $\tau^{-1}(0) \subset im(q)$. Moreover, using Proposition 2.13 we obtain that X is simply connected at some scale $k>0$.

*

The aim of the next two sections is to prove Theorem 1.18 for $n \geqslant 4$. For the sake of clarity we recapitulate here the needed assumptions for the proof.

Hypothesis (H). —

(1) *Let X be a k-simply-connected transitive graph;*
(2) *Let Y be a graph R-locally X and k-simply connected;*
(3) *Let $n \geqslant 4$ and $\mathbb{K}$ a non-Archimedean local skew field of characteristic zero. Denote by $\mathcal{X}$ be the Bruhat–Tits building of $PSL_n(\mathbb{K})$;*
(4) *Let $\rho : \operatorname{Isom}(X) \to \operatorname{Isom}(\mathcal{X})$ be an injective homomorphism and q: $X \to \mathcal{X}$ an $\operatorname{Isom}(X)$-equivariant injective quasi-isometry;*
(5) *Assume that $\rho(\operatorname{Isom}(X))$ is of finite index in $\operatorname{Isom}(\mathcal{X})$ and that $q(X)$ contains $\tau^{-1}(0)$.*

3. Tracking vertices through their prints

This section is dedicated to the definition of a graph locally the same as $\mathcal{X}$ which we will call $\mathcal{Y}$. Before moving to the detailed definition let us explain the idea of the construction. Recall that the vertices of $\mathcal{X}$ are partitioned into different types (see Section 2.1) denoted by integers in $\{0, \dots, n-1\}$. By Lemma 2.16, if $q(X)$ contains a vertex of a certain type then it contains all the vertices of that type. Denote by T the set of types that are not contained in $q(X)$, namely $T = \{0, \dots, n-1\} \backslash \tau(q(X))$. We have the following partition

$$\mathcal{X} = q(X) \sqcup \left(\sqcup_{i \in T} \tau^{-1}(i)\right). \tag{3.1}$$

Example 3.1. — Take $\mathbb{K} = \mathbb{Q}_2$ and assume that $im(q)$ is composed only of type zero vertices.

When $n = 2$ we have $T = \{1\}$ and the building is represented in Figure 2.1a. The partition in eq. (3.1) corresponds to the partition of vertices in two different colors.

When $n = 3$, we get $T = \{1, 2\}$. An apartment of $\mathcal{X}$ is represented in Figure 2.2 and the partition of this part of $\mathcal{X}$ corresponds to the partition in three different colors.

Example 3.2. — Let $n = 4$ and $\mathbb{K} = \mathbb{Q}_2$ and assume that $im(q)$ contains type zero and type 2 vertices. Then $T = \{1, 3\}$. We will not try to represent $\mathcal{X}$ or an apartment but recall that it is tiled by tetrahedrons. The partition is illustrated on a tetrahedron in Figure 3.1, where $im(q)$ corresponds to the two blue vertices.

The idea of the construction of $\mathcal{Y}$ is to take the vertices of Y and add to them vertices of the missing types, ie. vertices with type in T (see Figure 3.5 for an example). But we want to build this vertices only with informations encoded in $V(Y)$. That is why we introduce the local characterization of a vertex in the building (see Section 3.1). Then, using a well chosen set of isometries from Y to X, we transfer this print notion to Y, each print in Y corresponding to a vertex of a missing type.

3.1. Prints in a building

In this section we show that a vertex in $\mathcal{X}$ can be determined by a part of its 1-neighbourhood. More precisely, we prove that a vertex in the building is entirely determined by the vertices in its 1-neighbourhood having type zero.

Definition 3.3. — *Let x be a vertex of $\mathcal{X}$. We define the* print *of x, denoted by $\mathcal{P}(x)$, to be the intersection of the 1-neighbourhood of x with the vertices of type zero,* viz. $\mathcal{P}(x) := B_{\mathcal{X}}(x, 1) \cap \tau^{-1}(0)$.

Remark 3.4. — We choose to define a print as a set of vertices of type zero because (in order to simplify notations and proofs) we assumed from the beginning that $\tau^{-1}(0)$ was contained in $im(q)$. But we could have taken any other type.

Example 3.5. — Figure 3.2 represents a ball of radius 1 in two different cases. The case when $n = 2$ and $|\mathcal{O}/\pi\mathcal{O}| = 2$ (for example when $\mathbb{K} = \mathbb{Q}_2$) is represented on the left figure. The case when $\mathbb{K} = \mathbb{Q}_2$ and $n = 3$ is represented on the right figure. In each case, the print of x corresponds to the set of blue vertices.

The following result proves that a vertex in $\mathcal{X}$ is uniquely determined by its print.

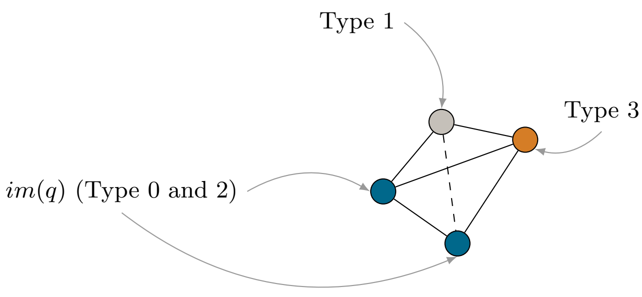


Figure 3.1. Partition of a simplex

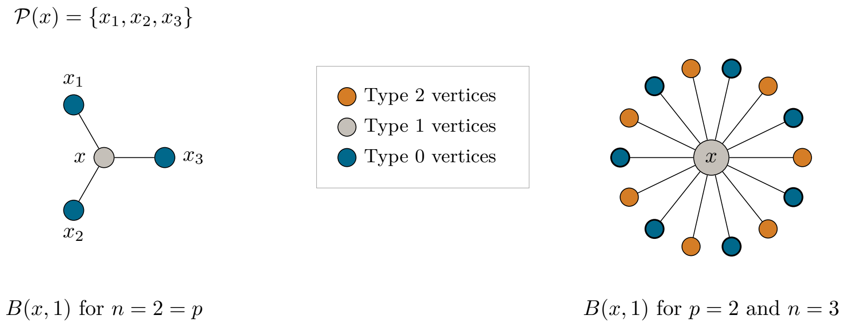


Figure 3.2. Prints and 1-neighbourhood of a vertex in $\mathcal{X}$

Proposition 3.6. — *Let $x_1, x_2 \in \mathcal{X}$. If $\mathcal{P}(x_1) = \mathcal{P}(x_2)$, then $x_1 = x_2$.*

Before showing the above property, let us recall (and prove) a useful fact concerning the choice of representative of a vertex.

Claim 3.7. — *For any vertex in $\mathcal{X}$, we can always find a representative $\oplus_i \mathcal{O}\pi^{k_i} e_i$ of the vertex such that*

$$\begin{cases} \forall\, i \in \{1, \ldots, n\} & k_i \geqslant 0, \\ \exists\, i_0 \in \{1, \ldots, n\} & k_{i_0} = 0. \end{cases} \tag{3.2}$$

Proof of the claim. — Indeed, let $x \in \mathcal{X}$ and let $(l_1, \ldots, l_n)$ be a representative of x and let i_0 be such that $l_{i_0} = \min_i l_i$, then

$$\left[\oplus_{i=1}^{n} \mathcal{O}\pi^{l_i} e_i\right] = \pi^{-l_{i_0}} \left[\oplus_{i=1}^{n} \mathcal{O}\pi^{l_i - l_{i_0}} e_i\right] = \left[\oplus_{i=1}^{n} \mathcal{O}\pi^{l_i - l_{i_0}} e_i\right].$$

Thus $(l_1 - l_{i_0}, \ldots, l_1 - l_{i_0})$ is a representative of x and verifies eq. (3.2). ☐

Now, let us prove that the print determines the vertex.

Proof of Proposition 3.6. — Let $x_1, x_2 \in \mathcal{X}$ such that $\mathcal{P}(x_1) = \mathcal{P}(x_2)$.

First remark that if $\tau(x_1) = 0$ then $\mathcal{P}(x_1) = \{x_1\}$ which implies that $\mathcal{P}(x_2) = \mathcal{P}(x_1) = \{x_1\}$. But then x_2 has only one neighbour of type 0, which is only possible if $\tau(x_2) = 0$. Thus $\{x_2\} = \mathcal{P}(x_2) = \{x_1\}$ and so $x_1 = x_2$.

Now assume that $\tau(x_1) \neq 0$ and take $\mathcal{A}$ to be an apartment containing x_1 and x_2. Define $P := \mathcal{P}(x) \cap \mathcal{A}$ and let $\mathbf{e}$ be a basis such that

$$\mathcal{A} = \left\{\oplus_{i=1}^{n} \mathcal{O}\pi^{k_i} e_i \,\middle|\, k_i \in \mathbb{Z}\right\} \quad \text{and} \quad x_1 = (0, \ldots, 0).$$

By Claim 3.7, we can choose a representative $(k_1, \ldots, k_n)$ of x_2 such that $k_i \geqslant 0$ for all i and there exists $j \in \{1, \ldots, n\}$ such that $k_j = 0$. Now define the sequence $i_1, \ldots, i_n$ of indices such that $k_{i_1} \geqslant \cdots \geqslant k_{i_n} = 0$ and let

$$l_{i_1} = \cdots = l_{i_{\tau(x)}} = 0 \qquad l_{i_{n-\tau(x)+1}} = \cdots = l_{i_n} = 1.$$

Then by Remark 2.6 the vertex $z = (l_1, \ldots, l_n)$ has type 0. Moreover it is at distance 1 from x_1, so z belongs to P. But if $k_{i_1} > 0$, then $d(z, x_2) > 1$ thus z can not belong to $\mathcal{P}(x_2)$. Hence $k_{i_1} \leqslant 0$, that is to say $k_i = 0$ for all i and thus $x_2 = x_1$. □

This proves that a vertex in $\mathcal{X}$ is uniquely determined by its print. Thus, we can introduce the following definition without ambiguity.

Definition 3.8. — *Let x to be a vertex in $\mathcal{X}$. We say that x is the* source *of $\mathcal{P}(x)$.*

In order to prove Theorem 1.18, we will need to know how prints behave under the action of $PSL_n(\mathbb{K})$. So, let $x \in \mathcal{X}$ and let $\alpha \in PSL_n(\mathbb{K})$. Since α is an isometry, we get

$$\begin{aligned} \alpha\Big(\mathcal{P}(x)\Big) &= \alpha\Big(B(x,1) \cap \tau^{-1}(0)\Big) \\ &= \alpha\big(B(x,1)\big) \cap \alpha\tau^{-1}(0) \\ &= B\left(\alpha(x), 1\right) \cap \tau^{-1}(0). \end{aligned}$$

We deduce the following lemma.

Lemma 3.9. — *Let $x \in \mathcal{X}$. If α belongs to $PSL_n(\mathbb{K})$, then $\alpha\left(\mathcal{P}(x)\right) = \mathcal{P}\left(\alpha(x)\right)$.*

3.2. Atlas of local isometries

To build our graph locally the same as $\mathcal{X}$, we need to restrict ourselves to a particular set of local isometries from Y to X. More precisely, if y_1 and y_2 are close in Y and f_1 (resp. f_2) is an isometry from $B_Y(y_1, R)$ (resp. $B_Y(y_1, R)$) to X, we want the transition map $f_2 f_1^{-1}$ to coincide with an element in $\rho^{-1} PSL_n(\mathbb{K})$ on a small ball. This is what we formalize here and schematize in Figure 3.3.

In order to avoid any ambiguity regarding the notion of center of a ball, let us precise our definition of ball in a graph. What we mean when we talk of "a ball of radius R" is actually a *pointed ball of radius R* that is to say, a couple $(\mathcal{B}, y)$ such that y is a vertex in Y and $\mathcal{B} = B_Y(y, R)$. We will abuse notation by denoting such a pointed ball $B_Y(y, R)$ (instead of $(B_Y(y, R), y)$). This way, the center of a ball is always well defined.

Definition 3.10. — *Let $\mathfrak{A}$ be a set of isometries from balls of radius R in Y to X. We say that $\mathfrak{A}$ is an* atlas *of local isometries from Y to X if the map that associates to each isometry in $\mathfrak{A}$ the center of its ball of definition is a bijection from $\mathfrak{A}$ to Y. That is to say, we can write*

$$\mathfrak{A} := \{f_y \,:\, B_Y(y,R) \to X \mid y \in Y\},$$

where the map that associates f_y to y is bijective.

We say that f_y is the isometry associated to y *in $\mathfrak{A}$.*

Let $H_0 := \rho^{-1}PSL_n(\mathbb{K})$. Now, we show that we can construct an atlas of local isometries from Y to X such that the transition maps between two isometries defined on balls with neighbouring centers coincide with elements of H_0. We will note a path between two vertices v_1 and v_2 as a sequence $(v_1, \dots, v_l)$ of adjacent vertices.

Lemma 3.11. — *Let $r_A > 0$ and let $H_0 := \rho^{-1}PSL_n(\mathbb{K})$. For R large enough, if Y is R-locally X, then there exists an atlas $\mathfrak{A}$ such that for any two neighbours y and z in Y*

$$\exists\, a \in H_0 \quad f_y \cdot {f_z^{-1}}_{|B(f_z(z),r_A)} = a_{|B(f_z(z),r_A)}. \tag{3.3}$$

Before proving it, let us schematize the framework of this lemma. In Figure 3.3 we represent two isometries f_y and f_z with z neighbour to y. The larger discs correspond to balls of radius R and the smaller ones to balls of radius r_A. The map $f_y f_z^{-1}$ restricted to $B(f_z(z), r_A)$ takes $f_z(z)$ to $f_y(z)$ which is a neighbour of $f_y(y)$ and coincide on this ball with an element in H_0.

Let us discuss the idea of the proof. First, for two neighbours y and z we use Proposition 2.14 to prove that $f_y f_z^{-1}$ coincides on a small ball with an element a in $\mathrm{Isom}(X)$.

This isometry corresponds to the "default" of belonging to H_0 we want to correct.

Hence, we consider in our atlas the new isometry defined on $B(z,R)$ by af_z. Finally, we extend this construction along paths in Y and prove that the wanted property for $\mathfrak{A}$ does not depend on the choice of path.

Proof. — Let $r_A > 0$ and let $H_0 := \rho^{-1}PSL_n(\mathbb{K})$. Now, let $y \in Y$ and f_y be an isometry from $B(y,R)$ to X. Let z be a neighbour of y in Y and $\widetilde{f}_z$ be an isometry from $B(z,R)$ to X. Then the map

$$f_y \cdot \widetilde{f}_z^{-1} \,:\, B_X\left(\widetilde{f}_z(z), R-1\right) \to B_X\left(f_y(z), R-1\right)$$

is a well defined local-isometry of X. By Proposition 2.14 if R is large enough, there exists a in $\mathrm{Isom}(X)$ such that $f_y \cdot \widetilde{f}_z^{-1}$ coincides with a on

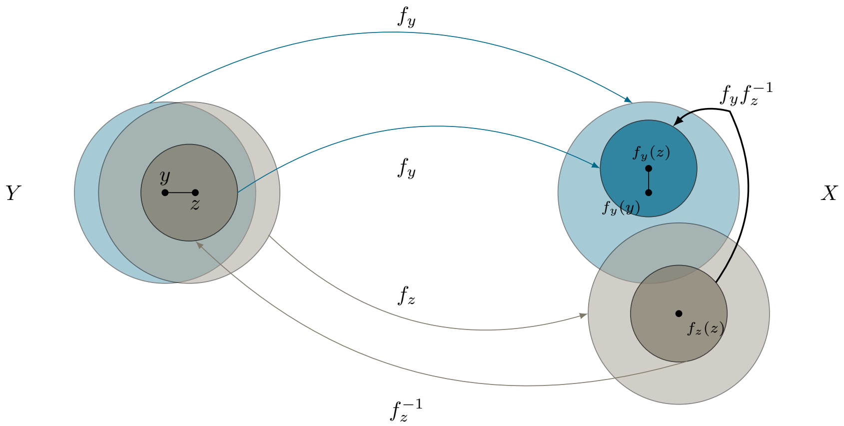


Figure 3.3. Composition of isometries with neighbouring centers

$B_X(\widetilde{f}_z(z), r_A+k)$, where we recall that k refers to the scale at which Y is simply connected. We will see below why we need to consider such a radius.

Now let $f_z := a\widetilde{f}_z$. By definition we have

$$f_z \ : \ \begin{cases} B_Y(z,R) & \to B_X\left(f_y(z),R\right), \\ z & \mapsto a\widetilde{f}_z(z) = f_y(z), \end{cases}$$

thus the transition map $f_y {f^{-1}}_z$ is well defined on $B_X(f_z(z), R-1)$. Moreover, by choice of f_z we get that $f_y {f^{-1}}_z$ restricted to $B(f_y(z), r_A+k)$ coincides with the identity and thus belongs to H_0.

Extending this construction along paths in Y we get an atlas $\mathfrak{A}$ of local isometries from Y to X.

Now if $y \in Y$ and f_y is the associated isometry in $\mathfrak{A}$, we want to show that (up to a multiplication by an element in $PSL_n(\mathbb{K})$) this isometry does not depend on the choice of path. So let $y \in Y$ and $(y_0 = y, y_1, \ldots, y_l = y)$ be a loop of length l. Take f_0 to be an isometry from $B_Y(y_0, R)$ to X and using the process detailed above, build a sequence of isometries $f_1, \ldots, f_l$ such that f_i is defined on $B_Y(y_i, R)$ and

$$\forall\, i \in \{1, \ldots, l\} \ \ \exists\, a_i \in H_0 \,|\, \left(f_{i-1} f_i^{-1}\right)_{|B(f_i(y_i), r_A+k)} = a_{i|B(f_i(y_i), r_A+k)}.$$

We have to prove that the restrictions to $B(y_0, r_A)$ of f_0 and f_l are equal up to a multiplication by an element in H_0. Since Y is simply connected at scale k, we only have to prove this for loop of length smaller than k. Hence, we assume that $l \leqslant k$.

First, remark that for all $i \in \{0, \ldots, l-1\}$

$$\begin{cases} f_{i-1}f_i^{-1} : B_X\left(f_i(y_i), r_A + k\right) & \to B_X\left(f_{i-1}(y_i), r_A + k\right), \\ f_i f_{i+1}^{-1} : B_X\left(f_{i+1}(y_{i+1}), r_A + k\right) & \to B_X\left(f_i(y_{i+1}), r_A + k\right). \end{cases}$$

Now since y_i and y_{i+1} are at distance 1, the ball $B_X(f_i(y_{i+1}), r_A + k - 1)$ is included in $B_X(f_i(y_i), r_A + k)$. Hence the map $(f_{i-1}f_i^{-1})(f_i f_{i+1}^{-1})$ is well defined and coincides with $a_i a_{i+1}$ on $B_X(f_{i+1}(y_{i+1}), r_A + k - 1)$. By induction we get that for all x in $B_X(f_{i+1}(y_{i+1}), r_A + k - l + 1)$

$$f_0 f_l^{-1}(x) = \left(f_0 f_1^{-1}\right) \cdots \left(f_{l-1} f_l^{-1}\right)(x) = a_1 \cdots a_l(x).$$

Since $\prod_{i=1}^{l} a_i$ belongs to H_0 and l is smaller than k, it implies that f_0 is equal to f_l on $B_Y(y_0, r_A)$ up to multiplication by an element in H_0. □

The atlas is defined such that a transition map between two isometries defined on balls with neighbouring centers belongs to H_0. But in fact, this property is also true when the centers are at a slightly bigger distance.

Lemma 3.12. — *Let $r > 0$ and $\mathfrak{A}$ be an atlas verifiying the conditions of Lemma 3.11 with $r_A > 3r$. Let y and z in Y be at distance less than $2r$ and f_y, f_z the associated isometries in $\mathfrak{A}$. Then*

$$\exists\, a \in H_0 \qquad \left(f_y f_z^{-1}\right)_{|B_Y(z,r)} = a_{|B_Y(z,r)}. \tag{3.4}$$

Proof. — Let $r > 0$ and assume $r_A > 3r$. Let $y, z \in Y$ be at distance $l \leqslant 2r$ and let f_y, f_z be two elements of $\mathfrak{A}$ such that

$$f_y : B_Y(y, R) \to X \quad f_z : B_Y(z, R) \to X.$$

Take $(y_0 = y, y_1, \ldots, y_l = z)$ to be a geodesic between y and z, and for all $i \in \{0, \ldots, l\}$, let $f_i \in \mathfrak{A}$ be the isometry associated to y_i. Remark that by definition of an atlas, it implies $f_0 = f_y$ and $f_l = f_z$ and

$$\forall\, i \in \{0, \ldots, l-1\} \quad \exists\, a_i \in H_0 \quad \left(f_i f_{i+1}^{-1}\right)_{|B(f_{i+1}(y_{i+1}), r_A)} = a_{i|B(f_{i+1}(y_{i+1}), r_A)}.$$

Now, if $r_A > 3r$ and $l \leqslant 2r$, then $B_Y(z, r)$ is contained in $B_Y(y, r_A)$. Hence the composition of transition maps $(f_0 f_1^{-1}) \cdots (f_{-1} f_r^{-1})$ is well defined on $B_Y(f_l(y_l), r_A - l)$ and verifies on that ball

$$f_0 f_l^{-1} = \left(f_0 f_1^{-1}\right) \cdots \left(f_{l-1} f_l^{-1}\right) = a_0 \cdots a_{l-1}. \tag{3.5}$$

Hence the result. □

3.3. Prints in Y

Using the atlas built above, we can now transfer this print notion to the graph Y. Let $r_{\mathcal{P}} > 0$ and assume that Y is endowed with an atlas of isometries $\mathfrak{A}$ as given by Lemma 3.11 with $r_A > 3r_{\mathcal{P}}$. Hence, we have

$$R > r_A > 3r_{\mathcal{P}} > r_{\mathcal{P}}.$$

Definition 3.13. — *Let P be a set of vertices in Y. We say that P is a* print *if there exists y in Y and $\in \mathfrak{A}$ an isometry from $B_Y(y, R)$ to X such that*

- *The set P is contained in $B_Y(y, r_{\mathcal{P}})$;*
- *There exists $x \in \mathcal{X} \backslash im(q)$ such that $\mathcal{P}(x) = qf(P)$.*

Remark 3.14. — Note that in the definition above we ask that x does *not* belong to $im(q)$. The definition would also make sense if x belonged to $im(q)$ but the purpose of these prints is to reconstruct the "missing" vertices, namely vertices that are not in the image of q. Thus to simplify formalism in the next pages, we chose to restrict now the definition to prints of vertices in $\mathcal{X} \backslash im(q)$.

Example 3.15. — If $n = 3$ and $p = 2$ there are exactly 3 types of vertices, each represented in Figure 3.4 by a different color. The 1-neighbourhood of a vertex x in $\mathcal{X}$ is then composed of fourteen vertices, represented on the right side of the aforementioned figure (where x is the brown vertex at the center). If $x \in \mathcal{X} \backslash im(q)$ then seven of these fourteen vertices are in $im(q)$ (the blue vertices). On the left side of the figure is represented P (the black dots) inside $B(y, r_{\mathcal{P}})$ (the darker disc). The set $qf(P)$ is exactly the set of blue vertices. Hence P is a print.

For now, let's say that P verifying the definition above is a print *associated to y and f*. We are going to show that this definition depends neither on y nor f.

Lemma 3.16. — *Let $y_1, y_2 \in Y$ and f_1, f_2 be the associated isometries in $\mathfrak{A}$. Let P be a print associated to y_1 and f_1. If $P \subset B(y_2, r_{\mathcal{P}})$ then P is a print associated to y_2 and f_2.*

Proof. — First, remark that since $P \subset B(y_2, r_{\mathcal{P}}) \cap B(y_1, r_{\mathcal{P}})$, then taking any y in P we get

$$d_Y(y_1, y_2) \leqslant d_Y(y_1, y) + d_Y(y, y_2) \leqslant 2r_{\mathcal{P}}.$$

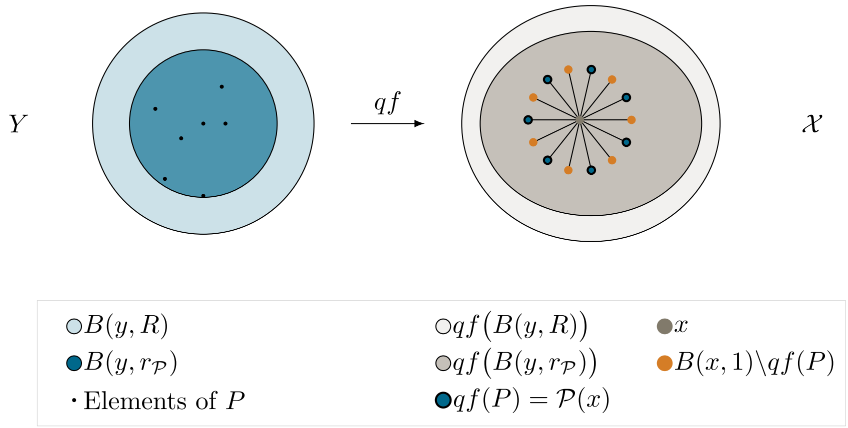


Figure 3.4. Definition of a print in $\mathcal{Y}$

Applying Lemma 3.12 with $r = r_{\mathcal{P}}$, we get that there exists $a \in H_0$ such that

$$\left(f_1 f_2^{-1}\right)_{|B_X(f_2(y_2),r_{\mathcal{P}})} = a_{|B_X(f_1(y_2),r_{\mathcal{P}})}.$$

Now let $x \in \mathcal{X}$ be such that $\mathcal{P}(x) = qf_1(P)$. Using the equivariance of q and Lemma 3.9, we get

$$qf_2(P) = \rho(a)^{-1} qf_1(P) = \rho(a)^{-1}\mathcal{P}(x) = \mathcal{P}\left(\rho(a)^{-1}(x)\right).$$

Hence P is a print associated to y_2 and f_2. ☐

This last lemma proves that being a print does not depend on the choice of local isometry.

Remark 3.17. — In the above proof $\rho(a)^{-1}(x)$ has same type as x since $\rho(a)$ is type preserving. Thus, once we have taken our atlas in $PSL_n(\mathbb{K})$, the type of the source of $qf(P)$ does not depend on the choice of local isometry f.

3.4. Definition of $\mathcal{Y}$: a building's replica

The following property defines the graph $\mathcal{Y}$ we will demonstrate to be locally the same as $\mathcal{X}$.

Proposition 3.18. — *Let $r_{\mathcal{P}} > 0$ and $\mathfrak{A}$ be the atlas given by Lemma 3.11 for $r_A > 3r_{\mathcal{P}}$. If R is large enough, then the following graph is well defined.*

Let $\mathcal{Y}$ be the graph whose vertices are given by

$$V(\mathcal{Y}) := V(Y) \sqcup \{P : \exists\, x \in \mathcal{X} \backslash im(q),\, \mathcal{P}(x) = P\},$$

and edges are given by:

- *If $y_1, y_2 \in V(\mathcal{Y})$, then (y_1, y_2) is an edge if there exists z in Y and $f \in \mathfrak{A}$ defined on $B_Y(z,R)$ such that $y_1, y_2 \in B(z, r_{\mathcal{P}})$ and $d_{\mathcal{X}}(qf(y_1), qf(y_2)) = 1$.*
- *If $y \in V(\mathcal{Y})$ and P is a print, then $(y,(i,P))$ is an edge if there exists z in Y and $f \in \mathfrak{A}$ defined on $B_Y(z,R)$ cointaining y and P and such that $qf(y)$ is at distance 1 from the source of $qf(P)$.*
- *If P_1 and P_2 are two prints, then (P_1, P_2) is an edge if there exists z in Y and $f \in \mathfrak{A}$ defined on $B_Y(z,R)$ such that $P_1, P_2 \subset B_Y(z, r_{\mathcal{P}})$ and such that the source of $qf(P_1)$ is at distance 1 from the source of $qf(P_2)$.*

Before looking to the proof of this property, let us sketch some part of this graph.

Example 3.19. — If $n = 4$ then $\mathcal{X}$ is composed of vertices of type 0, 1, 2 and 3. Assume that $q(X)$ is composed of vertices of type 0 and 2, then $T = \{1, 3\}$ and we saw the corresponding partition of $\mathcal{X}$ in Example 3.2 and Figure 3.1. The appearance of the corresponding $V(\mathcal{Y})$ is represented in Figure 3.5.

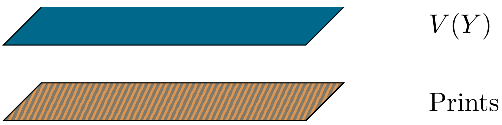


Figure 3.5. Schematic of $V(\mathcal{Y})$ in the case of Example 3.19

Proof. — Let $\mathcal{Y}$ be as in Proposition 3.18 and let us show that the definition of the edges does not depend on the choice of f in the atlas.

First, let $y_1, y_2 \in Y$ and $y, z \in Y$ such that y_1 and y_2 belongs to $B(y, r_{\mathcal{P}}) \cap B(z, r_{\mathcal{P}})$. Then, take two local maps f_y, f_z in $\mathfrak{A}$ associated to y and z respectively. Then $d(y,z) \leqslant 2r_{\mathcal{P}}$ and by Lemma 3.12 there exists $a \in$

$\mathrm{Isom}(X)$ verifying eq. (3.4). Hence, by $\mathrm{Isom}(X)$-equivariance of q we get

$$\begin{aligned} d_{\mathcal{X}}\Big(qf_z(y_1), qf_z(y_2)\Big) &= d_{\mathcal{X}}\Big(\rho(a)qf_z(y_1), \rho(a)qf_z(y_2)\Big) \\ &= d_{\mathcal{X}}\Big(q\left(af_z(y_1)\right), q\left(af_z(y_2)\right)\Big) \\ &= d_{\mathcal{X}}\Big(qf_y(y_1), qf_y(y_2)\Big). \end{aligned}$$

Thus $d_{\mathcal{X}}(qf_z(y_1), qf_z(y_2)) = 1$ if and only if $d_{\mathcal{X}}(qf_y(y_1), qf_y(y_2)) = 1$ and the definition of edges between two vertices of Y does not depend on the choice of local isometry.

Now take $y \in Y$ and let $P \subset Y$ be a print. Let z and z' such that y and P are contained in $B(z, r_{\mathcal{P}}) \cap B(z', r_{\mathcal{P}})$ and take f (resp. f') in $\mathfrak{A}$ defined on $B(z, R)$ (resp. $B(z', R)$). Then $d(z, z') \leqslant 2r_{\mathcal{P}}$ and by Lemma 3.12 there exists $a \in \mathrm{Isom}(X)$ verifying eq. (3.4). Hence,

$$\begin{aligned} d_{\mathcal{X}}\big(qf(y), x\big) &= d_{\mathcal{X}}\big(\rho(a)qf(y), \rho(a)(x)\big) \\ &= d_{\mathcal{X}}\big(q\left(af(y)\right), \rho(a)(x)\big) = d_{\mathcal{X}}\big(qf'(y), \rho(a)(x)\big). \end{aligned}$$

If x is the source of $qf(P)$ then, by Lemma 3.9 we get

$$\mathcal{P}(\rho(a)(x)) = \rho(a)\left(\mathcal{P}(x)\right) = \rho(a)qf(P) = qf'(P).$$

Thus, the existence of en edge between y and P in $\mathcal{Y}$ does not depend of the choice of map in $\mathfrak{A}$.

Finally, take $P_1, P_2 \subset Y$ two prints and let z, z' in Y and $f \in \mathfrak{A}$ (resp. f') defined on $B_Y(z, R)$ (resp. $B(z\prime, R)$) such that $P_1, P_2 \subset B_Y(z, r_{\mathcal{P}}) \cap B_Y(z', r_{\mathcal{P}})$. Again $d(z, z') \leqslant 2r_{\mathcal{P}}$ and by Lemma 3.12 there exists $a \in \mathrm{Isom}(X)$ verifying eq. (3.4). Hence if x_1 is the source of $qf(P_1)$ and x_2 the source of $qf(P_2)$, then $d(x_1, x_2) = 1$ if and only if $d(\rho(a)(x_1), \rho(a)(x_2)) = 1$. Moreover, by Lemma 3.9

$$\forall\, i = 1, 2 \quad \mathcal{P}(\rho(a)(x_i)) = \rho(a)\left(\mathcal{P}(x_1)\right) = \rho(a)qf(P_i) = qf'(P_i).$$

Hence the existence of en edge between P_1 and P_2 in $\mathcal{Y}$ does not depend of the choice of map in the atlas $\mathfrak{A}$. $\square$

4. From one graph to the other

In this section we prove the isometry between the graph $\mathcal{Y}$ built and the Bruhat–Tits building and show that it induces an isometry between X and Y.

4.1. Isometry with the building

We can now prove that $\mathcal{Y}$ is isometric the Bruhat–Tits building. Recall that r_A is the radius used to define our atlas $\mathfrak{A}$ (see Lemma 3.11) and $r_{\mathcal{P}}$ is the radius used to define prints in $\mathcal{Y}$ (see Definition 3.13). These constants verify $R > r_A > 3r_{\mathcal{P}} > r_{\mathcal{P}}$.

Lemma 4.1. — *Let $R_{\mathcal{X}} > 0$. If $r_{\mathcal{P}}$ (and hence R) is large enough, then $\mathcal{Y}$ is $R_{\mathcal{X}}$-locally $\mathcal{X}$.*

To prove this lemma, we define explicitely the local isometries on balls of radius $R_{\mathcal{X}}$ and prove that these maps are well defined injections. Then, we compute the minimal value of $r_{\mathcal{P}}$ necessary for these applications to be surjective on balls of radius $R_{\mathcal{X}}$. We conclude by showing that these maps preserve the distance.

Proof. — Let $v \in V(\mathcal{Y})$. If $v \in V(Y)$ let $f \in \mathfrak{A}$ be the isometry defined on $B_Y(v, R)$. If v is a print P let y and $f \in \mathfrak{A}$ be such that P is a print associated to y and f. Our goal is to show that the map

$$\phi_f \ : \ \begin{cases} B_{\mathcal{Y}}(v, R_{\mathcal{X}}) & \to \mathcal{X}, \\ z \in Y & \mapsto qf(y), \\ Q & \mapsto x \quad \text{where } \mathcal{P}(x) = qf(Q), \end{cases}$$

is an isometry.

By Proposition 3.6, it is a well defined map. Moreover, using the injectivity of q and Proposition 3.6 and eq. (3.1) we get that ϕ_f is an injective map.

Now, recall that since q is a quasi-isometry, two elements $q(x_1)$ and $q(x_2)$ joined by an edge in $\mathcal{X}$ might be at distance greater than 1 in X. If we want to prove that ϕ_f is surjective on$B_{\mathcal{X}}(\phi_f(v), R_{\mathcal{X}})$ and preserves the distance, we have to show that there exists a radius $r_{\mathcal{P}}$ allowing us to “reconstruct” all the edges of $B_{\mathcal{X}}(\phi_f(v), R_{\mathcal{X}})$ in $B_{\mathcal{Y}}(v, R_{\mathcal{X}})$. Let $L, \varepsilon > 0$ be such that q is a (L, ε)-quasi-isometry. We distinguish three cases, represented in Figure 4.1.

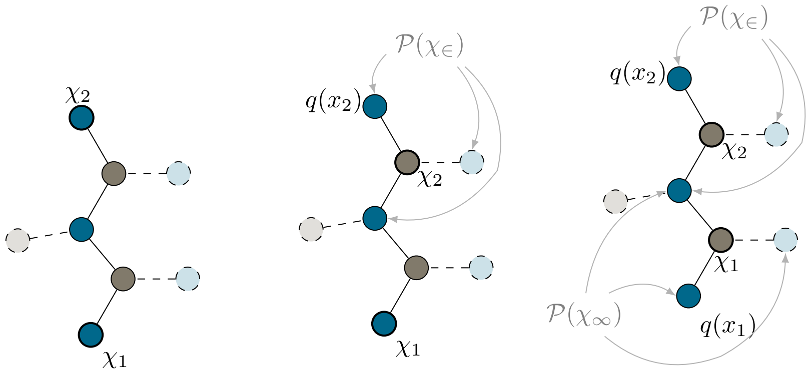
If $\chi_1, \chi_2 \in im(q)$, then let $x_1, x_2 \in X$ such that $q(x_i) = \chi_i$. They verify $d_X(x_1, x_2) \leqslant L d_{\mathcal{X}}(\chi_1, \chi_2) + \varepsilon$. This case is represented in Figure 4.1a.

If $\chi_1 \in im(q)$ and $\chi_2 \notin im(q)$, let $x_1 = q^{-1}(\chi_1)$. For all $x_2 \in X$ such that $q(x_2) \in \mathcal{P}(\chi_2)$, we have (see Figure 4.1b)

$$d_{\mathcal{X}}\left(q(x_1), q(x_2)\right) \leqslant 1 + d_{\mathcal{X}}\left(\chi_1, \chi_2\right) \Rightarrow d_X\left(x_1, x_2\right) \leqslant L d_{\mathcal{X}}\left(\chi_1, \chi_2\right) + L + \varepsilon.$$

If $\chi_1, \chi_2 \notin im(q)$, let $x_i \in X$ such that $q(x_i) \in \mathcal{P}(\chi_i)$ for $i = 1, 2$. Then (see Figure 4.1b)

$$\begin{aligned} d\left(\chi_1, \chi_2\right) = 1 &\Rightarrow d_{\mathcal{X}}\left(q(x_1), q(x_2)\right) \leqslant 2 + d_{\mathcal{X}}\left(\chi_1, \chi_2\right) \\ &\Rightarrow d_X\left(x_1, x_2\right) \leqslant L d_{\mathcal{X}}\left(\chi_1, \chi_2\right) + 2L + \varepsilon. \end{aligned}$$



(a) First case *(b)* Second case *(c)* Third case

Figure 4.1. The three cases ($im(q)$ is represented by the blue vertices)

Hence, assume $r_{\mathcal{P}} > LR_{\mathcal{X}} + 2L + \varepsilon$ and let us show that ϕ_f is an isometry.

Let $\chi \in B_{\mathcal{X}}(\phi_f(v), R_{\mathcal{X}})$, by choice of $r_{\mathcal{P}}$ either $\chi \in im(q)$ and then there exists $z \in B_Y(y, r_{\mathcal{P}})$ such that $qf(z) = \chi$ or $\chi \notin im(q)$ and then there exists $P \subset B_Y(y, r_{\mathcal{P}})$ such that $qf(P) = \mathcal{P}(\chi)$. Hence, in both cases $\chi \in im(\phi_f)$ and thus, ϕ_f is a bijection from $B_{\mathcal{Y}}(v, R_{\mathcal{X}})$ to $B_{\mathcal{X}}(\phi_f(v), R_{\mathcal{X}})$. Now take v_1, v_2 in $B_{\mathcal{Y}}(v, R_{\mathcal{X}})$ at distance l in $\mathcal{Y}$ and let $(w_0 = v_1, w_1, \ldots, w_l = v_2)$ be a geodesic in $\mathcal{Y}$. By definition of $\mathcal{Y}$ and choice of $r_{\mathcal{P}}$, for all $i \in \{0, \ldots, l-1\}$ if there is an edge between w_i and w_{i+1}, then $d(\phi_f(w_i), \phi_f(w_{i+1})) = 1$. Hence $d_{\mathcal{X}}(\phi_f(v_1), \phi_f(v_2)) \leqslant l$.

To get the reversed inequality, take χ_1, χ_2 in $B_{\mathcal{X}}(\phi_f(v), R_{\mathcal{X}})$. Since ϕ_f is bijective there exists $v_0, \ldots, v_l$ in $\mathcal{Y}$ such that $(\phi_f(v_0), \ldots, \phi_f(v_l))$ is a geodesic between χ_1 and χ_2. Again, by definition of $\mathcal{Y}$ and choice of $r_{\mathcal{P}}$, an edge between $\phi_f(v_i)$ and $\phi_f(v_{i+1})$ gives an edge between v_i and v_{i+1} in $\mathcal{Y}$ and thus $d_{\mathcal{Y}}(v_1, v_2) \leqslant l$.

Hence, if $r_{\mathcal{P}} > LR_{\mathcal{X}} + 2L + \varepsilon$ then ϕ_f is an isometry. □

The LG-rigidity of the building will give us a covering from $\mathcal{X}$ to $\mathcal{Y}$. In order to obtain an *isometry* we need to prove (by Proposition 2.12) that $\mathcal{Y}$ is simply connected at the same scale than $\mathcal{X}$.

Lemma 4.2. — *If $R_{\mathcal{X}}$ (and hence R) is large enough, then $\mathcal{Y}$ is simply connected at scale* 3.

We first prove that $\mathcal{Y}$ is quasi-isometric to Y and use it to show that $\mathcal{Y}$ is simply connected at some scale k'. We conclude using the contractibility of the building and the fact that $\mathcal{Y}$ is locally the same as the building. But before looking at the detail of the proof, let us make a remark.

Remark 4.3. — Let P be a print associated to some $z \in Y$ and $f \in \mathfrak{A}$ and let $y \in P$. If x is the source of $qf(P)$, then $d_{\mathcal{Y}}(P, y) = d_{\mathcal{X}}(x, qf(y)) = 1$.

Proof of Lemma 4.2. — Let us show that $\mathcal{Y}$ is quasi-isometric to Y. Define $\pi : \mathcal{Y} \to Y$ such that if $y \in V(Y)$ then $\pi(y) = y$ and if P is a print then $\pi(P) = y$ for some $y \in P$ arbitrarily chosen. Let $(v_0, \dots, v_m)$ be a geodesic in $\mathcal{Y}$ and for all $i \in \{0, \dots, m\}$ define $y_i := \pi(v_i)$ and f_i to be the isometry of $\mathfrak{A}$ associated to y_i. Using that q is a (L, ε)-quasi-isometry, we get

$$\begin{aligned} d_Y\big(\pi(v_0), \pi(v_m)\big) = d_Y(y_0, y_m) &\leqslant \sum_{i=0}^{m} d_Y\left(y_i, y_{i+1}\right), \\ &\leqslant \sum_{i=0}^{m} \big[L d_{\mathcal{X}}\left(qf_i(y_i), qf_i(y_{i+1})\right) + \varepsilon\big]. \end{aligned}$$

Now let $i \in \{0, \dots, m\}$. If v_i is a print, denote by x_i the source of $qf(v_i)$ and if v_i belongs to the copy of $V(Y)$ contained in $\mathcal{Y}$ let $x_i := qf_i\pi(v_i)$. Then $d_{\mathcal{Y}}(v_i, v_{i+1}) = d_{\mathcal{X}}(x_i, x_{i+1})$ for all i. Thus, using Remark 4.3, we get

$$\begin{aligned} &d_{\mathcal{X}}\big(qf_i(y_i), qf_i(y_{i+1})\big) \\ &\qquad \leqslant d_{\mathcal{X}}\big(qf_i(y_i), x_i\big) + d_{\mathcal{X}}\left(x_i, x_{i+1}\right) + d_{\mathcal{X}}\big(qf_i(y_{i+1}), x_{i+1}\big), \\ &\qquad \leqslant 2 + d_{\mathcal{X}}\left(x_i, x_{i+1}\right) = 2 + d_{\mathcal{Y}}\left(v_i, v_{i+1}\right). \end{aligned}$$

Since $d_{\mathcal{Y}}(v_i, v_{i+1}) = 1$, we obtain

$$\begin{aligned} d_Y(\pi(v_0), \pi(v_m)) = d_Y(y_0, y_m) &\leqslant \sum_{i=0}^{m} \big[L2 + L d_{\mathcal{Y}}\left(v_i, v_{i+1}\right) + \varepsilon\big], \\ &= (3L + \varepsilon)m = (3L + \varepsilon) d_{\mathcal{Y}}\left(v_0, v_m\right). \end{aligned}$$

Now let $v, v' \in \mathcal{Y}$ and let $(\pi(v) = z_0, \dots, \pi(v') = z_l)$ be a geodesic in Y. For all $i \in \{0, \dots, l\}$ take $f'_i \in \mathfrak{A}$ the isometry associated to z_i. Then

$$d_{\mathcal{Y}}\left(v, v'\right) \leqslant d_{\mathcal{Y}}\left(v, z_0\right) + \sum_{i=0}^{l-1} d_{\mathcal{Y}}\left(z_i, z_{i+1}\right) + d_{\mathcal{Y}}\left(z_l, v'\right).$$

But by Remark 4.3 if v (resp. v') is a print then $d_{\mathcal{Y}}(v, z_0) = 1$ (resp. $d_{\mathcal{Y}}(v', z_l) = 1$). And if v (resp. v') belongs to $V(Y)$ then $v = z_0$ (resp. $v' = z_l$). Thus both $d_{\mathcal{Y}}(v, z_0)$ and $d_{\mathcal{Y}}(v', z_l)$ are always smaller than 1. Hence,

$$\begin{aligned} d_{\mathcal{Y}}(v, v') \leqslant 2 + \sum_{i=0}^{l-1} d_{\mathcal{Y}}(z_i, z_{i+1}) \\ &= 2 + \sum_{i=0}^{l-1} d_{\mathcal{X}}\left(qf_i'(z_i), qf'\left(z_{i+1}\right)\right), \\ &\leqslant 2 + \sum_{i=0}^{l-1} \left[L d_Y(z_i, z_{i+1}) + \varepsilon\right], \\ &= 2 + (L+\varepsilon)l = 2 + (L+\varepsilon) d_Y(\pi(v), \pi(v')). \end{aligned}$$

Thus π is a quasi-isometry between $\mathcal{Y}$ and Y. Hence Proposition proposition 2.13 implies that there exists $k' \in \mathbb{N}^*$ such that $\mathcal{Y}$ is simply-connected at scale k'.

Finally, let ℓ be loop in $\mathcal{Y}$ of length less than k'. If $R_{\mathcal{X}}$ is large enough then ℓ is contained in some ball B in $\mathcal{Y}$. By Lemma 4.1 there exists a local isometry ϕ from B to some ball $\mathcal{B}$ in $\mathcal{X}$. But $\phi(\ell)$ is contractible inside its convex hull, by Claim 2.7. In particular it is simply-connected. Since $\mathcal{X}$ is 3-simply-connected and if $R_{\mathcal{X}}$ is large enough, the convex hull of $\phi(\ell)$ is contained in the complex obtained by gluing triangles on all the loops of length 3 in $\mathcal{B}$. Which, by local isometry with B, proves the wanted assertion. □

Thanks to the previous Lemma 4.2, we can now use the rigidity of the Bruhat–Tits building.

PROPOSITION 4.4. — *If $R_{\mathcal{X}}$ (and hence R) is large enough, then $\mathcal{Y}$ is isometric to $\mathcal{X}$.*

Proof. — Recall that we have $R > r_A > 3r_{\mathcal{P}} > r_{\mathcal{P}} > 3R_{\mathcal{X}} + 2L + \varepsilon > R_{\mathcal{X}}$.

By Theorem 1.13, the building $\mathcal{X}$ is LG-rigid. Moreover, since its isometry group is transitive Proposition 2.12 gives us the existence of some radius $R_{sc} > 0$ such that every graph which is 3-simply connected and R_{sc}-locally $\mathcal{X}$ is isometric to $\mathcal{X}$.

By definition of the edges on $\mathcal{Y}$, this graph is simply connected at scale 3. Taking $r_{\mathcal{P}}$ (and hence R) large enough so that $R_{\mathcal{X}} \geqslant R_{sc}$ the preceding paragraph combined with Lemma 4.1 give us the existence of an isometry between $\mathcal{X}$ and $\mathcal{Y}$. □

4.2. Change of local map, change of global isometry

Let $y \in Y$ and $f_y \in \mathfrak{A}$ be the isometry defined on $B(y,R)$.

Let

$$\phi_y : \begin{cases} B_{\mathcal{Y}}(y, R_{\mathcal{X}}) & \to \mathcal{X} \\ z \in Y & \mapsto q f_y(z) \\ Q & \mapsto x \qquad \text{where}\, \mathcal{P}(x) = q f_y(Q). \end{cases} \tag{4.1}$$

Lemma 4.5. — *Let y and z be neighbours in Y and $a \in H_0$ such that $f_y f_z^{-1}$ coincide with a on $B_X(f(z), r_A)$. If $R_{\mathcal{X}}$ is large enough, then $\phi_y \phi_z^{-1}$ coincide with $\rho(a)$ on $B_{\mathcal{X}}(\phi_z(z), 2)$.*

Proof. — Let y and z be neighbours in Y and $a \in H_0$ such that $f_y f_z^{-1}$ coincide with a on $B_X(f(z), r_A)$. If $R_{\mathcal{X}}$ (and hence R) is large enough, then $B_{\mathcal{Y}}(z,2)$ is contained in $B_{\mathcal{Y}}(y, R_{\mathcal{X}})$. Thus, $\phi_y \phi_z^{-1}$ is well defined on $B_{\mathcal{X}}(\phi_z(z), 2)$.

Let $v \in B_{\mathcal{Y}}(z,2)$. If $v \in V(Y)$, then

$$\phi_y(v) = q f_y(v) = q a f_z(v) = \rho(a) q f_z(v) = \rho(a)\phi_z(v).$$

If $v = P$ with $P \subset Y$ a print, then

$$\mathcal{P}\big(\phi_y(v)\big) = q f_y(P) = q a f_z(P) = \rho(a) q f_z(P) = \mathcal{P}\big(\rho(a)\phi_z(v)\big),$$

Thus $\phi_y(v) = \rho(a)\phi_z(v)$, since the print determines the vertex. Hence the result. □

Now let $r_{\mathcal{X}} > 0$. If $R_{\mathcal{X}}$ is large enough then, by SLG-rigidity of $\mathcal{X}$ there exists an isometry ι_y from $\mathcal{Y}$ to $\mathcal{X}$ that coincides with ϕ_y on $B(y, r_{\mathcal{X}})$. Thus, the lemma above allows us to work with a set of isometries from $\mathcal{Y}$ to $\mathcal{X}$ that differs only by a multiplication by an element of $PSL_n(\mathbb{K})$.

Lemma 4.6. — *If y and z belong to Y and $R_{\mathcal{X}}$ is large enough, then $\iota_y \iota_z^{-1} \in PSL_n(\mathbb{K})$. Hence for all $y \in Y$, the isometry ι_y sends the copy of $V(Y)$ contained in $\mathcal{Y}$ to $im(q)$ and sends prints contained in $\mathcal{Y}$ to vertices in $\mathcal{X} \backslash im(q)$.*

Proof. — Let y and z be neighbours in Y. Since $\iota_y \iota_z^{-1}$ is an isometry of $\mathcal{X}$ it permutes the $PSL_n(\mathbb{K})$-orbits. Recall that ι_y coincides with ϕ_y on $B(y, r_{\mathcal{X}})$.

Hence, if $r_{\mathcal{X}}$ (and hence R) is large enough, then $B_{\mathcal{Y}}(z,2)$ is contained in $B_{\mathcal{Y}}(y, r_{\mathcal{X}})$, thus

$$\big(\iota_y \iota_z^{-1}\big)_{|B_{\mathcal{X}}(\iota_z(z),\,2)} = \phi_y \phi_z^{-1}.$$

But $\phi_y\phi_z^{-1}$ coincides with an element of $PSL_n(\mathbb{K})$ on $B_{\mathcal{X}}(\phi_z(z), 2)$, by Lemma 4.5. Hence $\iota_y\iota_z^{-1}$ restricted to a ball of radius 2 preserves the $PSL_n(\mathbb{K})$-orbits. Since such a ball contains a vertex of each type, it implies that $\iota_y\iota_z^{-1}$ preserves the $PSL_n(\mathbb{K})$-orbits and thus belongs to $PSL_n(\mathbb{K})$.

Now take y and z in Y (not necessarily neighbours), denote by $(y_0 = y, y_1 \ldots, y_l = z)$ a geodesic in Y. By the preceding paragraph, there exists a sequence $\alpha_1, \ldots, \alpha_l$ of elements in $PSL_n(\mathbb{K})$ such that

$$\forall\, i \in \{1, \ldots, l\} \quad \iota_{y_i}\iota_{y_{i-1}}^{-1} = \alpha_i.$$

Thus, recalling that $z = y_l$ and $y = y_0$, we get $\iota_z = \alpha_l \cdots \alpha_1 \iota_y$. Which proves the first assertion of the Lemma 4.6.

Let us now prove the second part of the lemma. Let $y \in Y$ and $v \in \mathcal{Y}$. There exists $z \in Y$ such that $v \in B_{\mathcal{Y}}(z, 2)$, and using the paragraph above, there exists $\alpha \in PSL_n(\mathbb{K})$ such that $\iota_y = \alpha\iota_z$. In particular, since v belongs to $B_{\mathcal{Y}}(z, R_{\mathcal{X}})$,

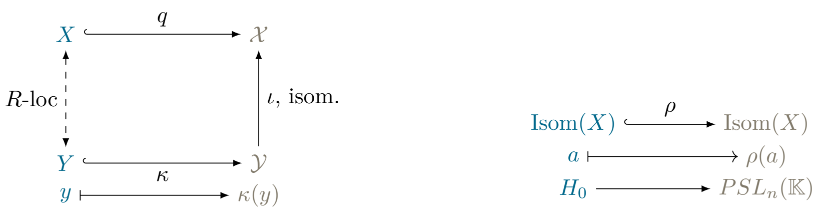
$$\iota_y(v) = \alpha\iota_z(v) = \alpha\phi_z(v).$$

By definition of ϕ_z, if $v \in V(Y)$ then $\phi_z(v)$ belongs to $im(q)$ and if $v = P$ with $P \subset Y$ a print, then $\phi_z(v)$ belongs to $\mathcal{X}\backslash im(q)$. This finishes the proof of the Lemma 4.6. ☐

Now we have all the tools we need to prove the isometry between Y and X.

4.3. Isometry from Y to X

Let κ be the natural injection of Y in $\mathcal{Y}_Z$ and ι an isometry given by Proposition 4.4. With the objects constructed so far we get the diagram in Figure 4.2.



(a) Maps between graphs *(b)* Relations between groups

Figure 4.2. Relations between the different graphs and groups

The aim of this section is to prove the following result.

Proposition 4.7. — *For $R_{\mathcal{X}}$ large enough, the graphs Y and X are isometric.*

Let us discuss the strategy of the proof. Using the preceding section, we chose an isometry ι from $\mathcal{Y}$ to $\mathcal{X}$ that coincides with a ϕ_y on a small ball. Then, we show that $\kappa\iota q^{-1}$ is locally an isometry, *viz.* there exists a radius r_Y such that $q^{-1}\iota\kappa$ restricted to any ball of radius r_Y preserves the distance. We conclude by showing that it forces $\kappa\iota q^{-1}$ to be an isometry.

Proof of Proposition 4.7. — By Lemma 4.6, for any $y \in Y$ the map $q^{-1}\iota_y\kappa$ is well defined. Now fix $y_0 \in Y$ and consider $\iota := \iota_{y_0}$. We want to prove that $q^{-1}\iota\kappa$ restricted to small balls preserves the distance. Then we will show that it is an isometry from Y to X.

Claim 4.8. — *Let $y \in Y$ and $r_Y \geqslant 1$. If R is large enough, then $q^{-1}\iota\kappa$ restricted to $B_Y(y, r_Y)$ preserves the distance.*

Proof of the claim. — Let $r_Y \geqslant 1$ and recall that we have $R > r_A > 3r_{\mathcal{P}} > r_{\mathcal{P}} > 3R_{\mathcal{X}} + 2L + \varepsilon > R_{\mathcal{X}} > r_{\mathcal{X}}$. Let $y \in Y$ and recall that L and ε are constants such that q is a (L, ε)-quasi-isometry. If $r_{\mathcal{X}} \geqslant Lr_Y + \varepsilon$ (and hence if R is large enough) then $\kappa(B_Y(y, r_Y))$ is included in $B_{\mathcal{Y}}(y, r_{\mathcal{X}})$. Indeed if $z \in B_Y(y, r_Y)$ then

$$\begin{aligned} d_{\mathcal{X}}\Big(qf_y(y), qf_y(z)\Big) &\leqslant Ld_X\Big(f_y(y), f_y(z)\Big) + \varepsilon \\ &= Ld_Y(y, z) + \varepsilon \leqslant Lr_Y + \varepsilon \leqslant r_{\mathcal{X}}. \end{aligned}$$

Thus $\phi_y(\kappa(z)) = qf_y(z)$ and

$$d_{\mathcal{Y}}\big(\kappa(y), \kappa(z)\big) = d_{\mathcal{X}}\left(\phi_y\big(\kappa(y)\big), \phi_y\big(\kappa(z)\big)\right) = d_{\mathcal{X}}\Big(qf_y(y), qf_y(z)\Big) \leqslant R_{\mathcal{X}}.$$

Now, recall that $H_0 = \rho^{-1}PSL_n(\mathbb{K})$. Then, by Lemma 4.6 there exists $a_y \in H_0$ such that $\iota_y\iota^{-1} = \rho(a_y)$. Hence, using the equivariance of q we get that for all z_1 and z_2 in $B_Y(y, r_Y)$

$$\begin{aligned} d_X\Big(q^{-1}\iota\kappa(z_1), q^{-1}\iota\kappa(z_1)\Big) &= d_X\Big(a_y q^{-1}\iota\kappa(z_1), a_y q^{-1}\iota\kappa(z_1)\Big) \\ &= d_X\Big(q^{-1}\rho(a_y)\iota\kappa(z_1), q^{-1}\rho(a_y)\iota\kappa(z_1)\Big) \\ &= d_X\Big(q^{-1}\iota_y\kappa(z_1), q^{-1}\iota_y\kappa(z_1)\Big). \end{aligned}$$

But z_1 and z_2 belong to $B_Y(y, r_Y)$, hence for $i = 1, 2$ we have $\iota_y\kappa(z_i) = qf_y(z_i)$. Thus,

$$\begin{aligned} d_X\Big(q^{-1}\iota\kappa(z_1), q^{-1}\iota\kappa(z_1)\Big) &= d_X\Big(q^{-1}qf_y(z_1), q^{-1}qf_y(z_2)\Big) \\ &= d_X\Big(f_y(z_1), f_y(z_2)\Big) = d_Y(z_1, z_2). \end{aligned}$$

Thus $q^{-1}\iota\kappa$ restricted to $B_Y(y, r_Y)$ preserves the distance. □

Let's show that the claim forces $q^{-1}\iota\kappa$ to be an isometry from Y to X. Take $r_Y \geqslant 2$ and let $y, y' \in Y$ and $(y_0 = y, y_1, \ldots, y_l = y')$ be a geodesic in Y. Since for all i the vertices y_i and y_{i+1} are adjacent, then Claim 4.8 implies that $d_X(q^{-1}\iota\kappa(y_i), q^{-1}\iota\kappa(y_{i+1})) = 1$. Hence

$$d_X\left(q^{-1}\iota\kappa(y), q^{-1}\iota\kappa(y')\right) \leqslant \sum_{i=0}^{l-1} d_X\left(q^{-1}\iota\kappa(y_i), q^{-1}\iota\kappa(y_i)\right) = l.$$

Moreover, if $(x_0 = q^{-1}\iota\kappa(y), x_1, \ldots, x_m = q^{-1}\iota\kappa(y'))$ is a geodesic in X, then by bijectivity of $q^{-1}\iota\kappa$ there exists $z_i \in Y$ such that $q^{-1}\iota\kappa(z_i) = x_i$ for all i in $\{1, \ldots, m-1\}$. Denote $z_0 = y$ and $z_m = y'$. Since for all i the vertices x_i and x_{i+1} are adjacent, then Claim 4.8 implies that $d_X(z_i, z_{i+1}) = d_{\mathcal{X}}(q^{-1}\iota\kappa(z_i), q^{-1}\iota\kappa(z_{i+1}))$. Thus

$$\begin{aligned} d_Y(y, y') &\leqslant \sum_{i=0}^{m-1} d_Y(z_i, z_{i+1}) \\ &= \sum_{i=0}^{m-1} d_X\left(q^{-1}\iota\kappa(z_i), q^{-1}\iota\kappa(z_{i+1})\right) \\ &= \sum_{i=0}^{m-1} d_X\left(x_i, x_{i+1}\right) = m. \end{aligned}$$ □

We conclude by the proof of Theorem 1.18.

Proof of Theorem 1.18. — Let $n \neq 3$ and X verifying the hypothesis of Theorem 1.18. If $n = 2$ then $\mathcal{X}$ is the $(p+1)$-regular tree, thus by Example 1.5 if X is quasi-isometric to $\mathcal{X}$ then X is LG-rigid. If $n \geqslant 4$, let $k \in \mathbb{N}$ such that X is simply connected at scale k. Then by Proposition 4.7 for R large enough, any k-simply-connected graph Y being R-locally the same as X is isometric to X. Thus X is LG-rigid. Finally for any $n \neq 3$, since X is assumed transitive it is actually SLG-rigid by Proposition 1.8. □

5. Application to p-adic lattices

In this section we prove Theorem 1.17 which we recall below.

Corollary 5.1. — *Let $n \neq 3$ and $\mathbb{K}$ be a non-Archimedean skew field of characteristic zero. The torsion-free lattices of $SL_n(\mathbb{K})$ are SLG-rigid.*

Let $n \neq 3$, let $\mathbb{K}$ be a non-Archimedean skew field of characteristic zero and $\Gamma \leqslant SL_n(\mathbb{K})$ be a lattice without torsion. Denote by (Γ, S) one of its Cayley graphs. Recall that any lattice in $SL_n(\mathbb{K})$ is uniform (i.e. cocompact).

5.1. Quasi-isometry between the lattice and the building

To show the corollary, we first check that the lattice is quasi-isometric to the building. Then, using a famous result of Kleiner and Leeb we show that the isometry group of the lattice acts on the building and that the quasi-isometry can be chosen to be equivariant under this action.

Lemma 5.2. — *Let Λ be a lattice of $SL_n(\mathbb{K})$. Then Λ is quasi-isometric to $\mathcal{X}$.*

Proof. — First, recall that any lattice in $SL_n(\mathbb{K})$ is uniform, *viz.* cocompact (see for example [6]).

Since Λ is a lattice of $SL_n(\mathbb{K})$, there is a natural action on the Bruhat–Tits building induced by the action of $PSL_n(\mathbb{K})$. Moreover, since Λ is cocompact and the $PSL_n(\mathbb{K})$ action has exactly n orbits, the Λ action is also cocompact. Hence by the Svarc–Milnor's lemma Λ is quasi-isometric to X. □

By a result of Kleiner and Leeb [9] and Cornulier [7, Theorem 3.B.1] applied to our lattice Γ, this quasi-isometry implies the existence of a homomorphism from $\operatorname{Isom}(\Gamma, S)$ to $\operatorname{Isom}(X)$ and a quasi-isometry from (Γ, S) to X which is $\operatorname{Isom}(\Gamma, S)$-equivariant. Since Γ is assumed to be torsion-free, we can refine the informations about these two applications.

Lemma 5.3. — *Let Λ be a lattice of $SL_n(\mathbb{K})$ and T a symmetric generating set. If Λ is torsion-free, then there exists an injective homomorphism*

$$\rho : \operatorname{Isom}(\Lambda, T) \to \operatorname{Isom}(X),$$

and an injective quasi-isometry which is $\operatorname{Isom}(\Lambda, T)$*-equivariant*

$$q : (\Lambda, T) \to X.$$

Proof. — Since we assumed that Λ has no torsion element, by Proposition 1.20 the isometry group of (Λ, T) contains no non-trivial compact normal subgroup. Hence the morphism ρ given by Kleiner–Leeb's theorem is injective.

Assume that there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda_1 \neq \lambda_2$ and $q(\lambda_1) = q(\lambda_2)$. Then, the equivariance of q implies that

$$q\left(\left\{\left(\lambda_1\lambda_2^{-1}\right)^n \ : \ n \in \mathbb{N}\right\}\right) = \{q(e)\},$$

which contradicts the fact that q is a quasi-isometry. □

5.2. Relation between the isometry groups

To apply Theorem 1.18, we still need to check that $\mathrm{Isom}(\Gamma, S)$ is of finite index in $\mathrm{Isom}(\mathcal{X})$. As stated in the lemma below, this is not always the case: the lattice's isometry group can also be discrete. But as we will see in Section 5.3 we will be able to prove the rigidity of the lattice in that case too.

Lemma 5.4. — *Using the previous notations,*

- *Either* $\mathrm{Isom}(\Gamma, S)$ *is discrete.*
- *Or* $\mathrm{Isom}(\Gamma, S)$ *is of finite index in* $\mathrm{Isom}(X)$ *and contains* $PSL_n(\mathbb{K})$*.*

Before proving this lemma, let us recall a useful consequence of a theorem of Benoist and Quint. The original and more general statement can be found in [6, Corollary 4.5].

Proposition 5.5 (Benoist, Quint [6]). — *Let G be p-adic Lie group and H be a finite covolume closed subgroup of G, with Lie algebra $\mathfrak{h}$. If G has no proper cocompact normal subgroup, then G normalizes $\mathfrak{h}$.*

Proof of Lemma 5.4. — Let $G = PSL_n(\mathbb{K})$ and $H = \mathrm{Isom}(\Gamma, S) \cap G$ and note $\mathfrak{h} =: Lie(H)$ and $\mathfrak{G} := Lie(G)$ their respective Lie algebras. Since Γ is a lattice in $SL_n(\mathbb{K})$, we get that $\rho(\Gamma) \cap PSL_n(\mathbb{K})$ is a lattice in $PSL_n(\mathbb{K})$. Hence H contains the uniform lattice $\rho(\Gamma) \cap G$ of G, thus H has finite covolume in $PSL_n(\mathbb{K})$. If $\mathbb{K}$ is a non-Archimedean local skew field of characteristic zero then it is an extension of $\mathbb{Q}_p$ for some prime p (see for example [10, Section 1]). In particular G is a p-adic Lie group. Thus the above property applied to G and H implies that G normalises $\mathfrak{h}$, in other words $\mathfrak{h}$ is an ideal of $\mathfrak{G}$. Since $\mathfrak{G}$ is simple, we get that $\mathfrak{h}$ is either trivial or the full Lie algebra $\mathfrak{G}$. If $\mathrm{Isom}(\Gamma, S)$ isn't discrete, then it is a closed subgroup of $\mathrm{Isom}(X)$. Hence H is a closed subgroup of G and its Lie algebra is non-trivial. By the previous point it can only be $\mathfrak{G}$. Hence, it implies that is an open subgroup of G. Since it is also cocompact, it is necessarily of finite index in G. Thus, we get that $\rho\left(\mathrm{Isom}(\Gamma, S)\right)$ is of finite index in $\mathrm{Isom}(X)$.

Let's show that $PSL_n(\mathbb{K})$ is a subgroup of $\rho(\mathrm{Isom}(\Gamma, S))$. First assume that $\rho(\mathrm{Isom}(\Gamma, S))$ is strictly contained in $PSL_n(\mathbb{K})$. Since these two groups are of finite index in $\mathrm{Isom}(X)$, we get that $\rho(\mathrm{Isom}(\Gamma, S))$ is of finite index in $PSL_n(\mathbb{K})$. But then the core:

$$\bigcap_{g \in PSL_n} g \cdot \rho\left(\mathrm{Isom}(\Gamma, S)\right) \cdot g^{-1}$$

of $\rho(\mathrm{Isom}(\Gamma, S))$ is itself of finite index in $PSL_n(\mathbb{K})$ (and different from $PSL_n(\mathbb{K})$), which contradicts the simplicity of $PSL_n(\mathbb{K})$.

Now, let's go back to the general case. Assume that $PSL_n(\mathbb{K})$ isn't included in $\rho(\mathrm{Isom}(\Gamma, S))$ and remark that:

$$\mathfrak{h} = Lie\left(\mathrm{Isom}(X)\right) = Lie\left(PSL_n(\mathbb{K})\right).$$

In particular $\rho(\mathrm{Isom}(\Gamma, S))$ is "locally" $PSL_n(\mathbb{K})$ so, up to apply what precedes to an open set centered on e_Γ sufficiently small of $\rho(\mathrm{Isom}(\Gamma, S))$, we obtain a contradiction. Hence $PSL_n(\mathbb{K})$ is contained in $\rho(\mathrm{Isom}(\Gamma, S))$. □

5.3. Rigidity of p-adic lattices

We conclude by the proof of Corollary 5.1.

Proof of Corollary 5.1. — Let $n \neq 3$ and p be a prime. Let Γ be a torsion-free lattice of $PSL_n(\mathbb{K})$ and S be a symmetric generating part.

If $n = 2$, then $\mathcal{X}$ is the $(p+1)$-regular tree. Since by Lemma 5.2, the graph (Γ, S) is quasi-isometric to $\mathcal{X}$, Example 1.5 implies that (Γ, S) is LG-rigid.

Assume now that $n > 3$. If $\mathrm{Isom}(\Gamma, S)$ is discrete the LG-rigidity of the lattice is given by Theorem 1.9.

If $\mathrm{Isom}(\Gamma, S)$ is non-discrete, then by Lemma 5.4 it has finite index in $\mathrm{Isom}(X)$ and in this case the hypothesis of Theorem 1.18 are satisfied, hence the rigidity of the lattice.

Finally, for all $n \neq 3$ the lattice Γ acts transitively on (Γ, S) thus, by Proposition 1.8, it is SLG-rigid. □

6. Conclusion and open problems

Our main result is proved for graphs quasi-isometric to the Bruhat-Tits building of $PSL_n(\mathbb{K})$ and the key idea of the proof is to use the rigidity of this building to "transfer it" to the graph quasi-isometric thereto. One can ask wether we can generalize this idea to other LG-rigid graphs.

Question 6.1. — *Let $\mathcal{G}$ be quasi-isometric to a LG-rigid graph $\mathcal{H}$, both having cocompact isometry group. If the quasi-isometry is* $\mathrm{Isom}(\mathcal{G})$*-equivariant, is $\mathcal{G}$ LG-rigid?*

Remark that if $\mathcal{H}$ and $\mathcal{G}$ are two Cayley graphs of the same group, we can chose $\mathcal{H}$ to be LG-rigid and $\mathcal{G}$ to be non-rigid (see the discussion below Counter-Example 1.6 for more details). In that case the hypothesis of the preceding question are satisfied without $\mathcal{G}$ being LG-rigid. Thus, more restrictive hypothesis will be needed to get the rigidity of $\mathcal{G}$.

Our result on lattices is proved for $n \neq 3$; when $n = 3$ we don't know (yet) the answer. Indeed, ou proof is based on the rigidity of the Bruhat–Tits building of $PSL_n(\mathbb{K})$, a result known to be true only for $n \neq 3$. In the $n = 3$ case, a lot of flexibility seems to be allowed (see for example [3]) obstructing any local recognizability result. Hence the following question:

Question 6.2. — *Are torsion-free lattices of $SL_3(\mathbb{K})$ LG-rigid?*

Lattices in p-adic Lie groups can be viewed as particular cases of S-arithmetic lattices.

Definition 6.3. — *Let S be a set of prime. We say that Γ an S-*arithmetic *lattice if it's a lattice in a product of the form $\prod_i G_i$ where G_i is either a real Lie group or a p-adic Lie group for $p \in S$.*

Hence, one we can ask what happens in that more general case.

Question 6.4. — *Are torsion-free S-arithmetic lattices LG-rigid?*

A result by Bader, Furman and Sauer [2, Theorem B] can be used to deal with irreducible torsion-free S-arithmetic lattices. Indeed, if the product $\prod_i G_i$ contains at least a non-compact real factor, then the aforementioned theorem implies that the isometry group of a Cayley graph of Γ is discrete. Thus, by Theorem 1.9 the lattice is LG-rigid. Now, if the product contains a compact real factor then the isometry group of the Cayley graph might not be discrete and in that case, the problem is still open.

When the lattice is reducible, we now know that the projection on the p-adic factors gives LG-rigid lattices. Moreover, if we suppose the real factors to be simple and connected, then a result by de la Salle and Tessera [11] shows that the projection on these factors are also LG-rigid. Hence it remains to understand how to combine these results on the *factors* in order to get a result on the *product*.

Notations Index

$\mathfrak{A}$: Atlas of isometries from Y to X.
$\mathcal{A}$: An apartment in $\mathcal{X}$.
(Γ, S): Cayley graph of Γ with respect to the generating part S.
H_0: The group $\rho^{-1}(PSL_n(\mathbb{K}))$.
$\mathrm{Isom}(\mathcal{G})$: Isometry group of $\mathcal{G}$.
ι_y: Isometry from $\mathcal{Y}$ to $\mathcal{X}$ based at y (see page 1762).
κ: Natural injection of Y in $\mathcal{Y}$ (see section 4.3).
$[L]$: Class modulo homothety of the lattice L.
$\mathcal{P}(x)$: The print of the vertex x (see Definition 3.3).
P: A print in Y (see Definition 3.13).
ϕ_y: Local isometry from $\mathcal{Y}$ to $\mathcal{X}$ based at y (see eq. (4.1)).
q: Quasi-isometry between X and $\mathcal{X}$.
R: Radius such that Y is R-locally the same as X.
ρ: Injective homomorphism from $\mathrm{Isom}(X)$ to $\mathrm{Isom}(\mathcal{X})$.
r_A: See Lemma 3.11.
$r_{\mathcal{P}}$: Radius considered to define prints (see Definition 3.13).
$R_{\mathcal{X}}$: Radius such that $\mathcal{Y}$ is $R_{\mathcal{X}}$-locally $\mathcal{X}$.
$r_{\mathcal{X}}$: Radius such that ι_y coincide with ϕ_y on $B_{\mathcal{Y}}(y, r_{\mathcal{X}})$ (see page 1762).
r_Y: See Claim 4.8.
$\tau(x)$: The type of the vertex x, where x belongs to the Bruhat–Tits building of $PSL_n(\mathbb{K})$.
$\mathcal{X}$: The Bruhat–Tits building of $PSL_n(\mathbb{K})$.
$\mathcal{Y}$: Hybrid graph built to be locally the same as the building (see Section 3.4).
$(y_1, \ldots, y_l)$: A path of adjacent vertices $y_1, y_2, \ldots, y_l$.

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