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THE TRANSLATION NUMBER AND QUASI-MORPHISMS ON GROUPS OF SYMPLECTOMORPHISMS OF THE DISK

by Shuhei MARUYAMA

Abstract. — On groups of symplectomorphisms of the disk, we construct two homogeneous quasi-morphisms which relate to the Calabi invariant and the flux homomorphism respectively. We also show the relation between the quasi-morphisms and the translation number introduced by Poincaré.

Résumé. — Sur des groupes de symplectomorphismes du disque, nous construisons deux quasi-morphismes homogènes reliés à l'invariant de Calabi et l'homomorphisme du flux respectivement. Nous montrons également la relation entre les quasi-morphismes et le nombre de translation introduit par Poincaré.

1. Introduction

A quasi-morphism on a group Γ is a function $\varphi : \Gamma \to \mathbb{R}$ such that the value

$$\sup_{\gamma_1,\gamma_2 \in \Gamma} |\varphi(\gamma_1\gamma_2) - \varphi(\gamma_1) - \varphi(\gamma_2)|$$

is bounded. A quasi-morphism φ is called homogeneous if the condition $\varphi(\gamma^n) = n\varphi(\gamma)$ holds for any $\gamma \in \Gamma$ and $n \in \mathbb{Z}$. Let $Q(\Gamma)$ denote the $\mathbb{R}$-vector space of homogeneous quasi-morphisms on the group Γ. Given a quasi-morphism φ, we obtain the homogeneous quasi-morphism $\overline{\varphi}$ associated to φ by

$$\overline{\varphi}(g) = \lim_{n \to \infty} \frac{\varphi(g^n)}{n}.$$

This map $\overline{\varphi}$ is called the homogenization of φ.

Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\}$ be the unit disk in $\mathbb{R}^2$ and $\omega = dx \wedge dy$ be the standard symplectic form on D. Let $G = \mathrm{Symp}(D)$ be the group of

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symplectomorphisms of D (which may not be the identity on the boundary ∂D). In the present paper, we construct a homogeneous quasi-morphism on G. Let η be a 1-form on D satisfying $d\eta = \omega$. The map $\tau_\eta : G \to \mathbb{R}$ is defined by

$$\tau_\eta(g) = \int_D g^*\eta \wedge \eta.$$

Let G_{rel} denote the kernel of the homomorphism $G \to \mathrm{Diff}_+(S^1)$, where $\mathrm{Diff}_+(S^1)$ denotes the group of orientation preserving diffeomorphisms of the circle. Then the map τ coincides with the Calabi invariant on G_{rel}. Although the Calabi invariant $\mathrm{Cal} : G_{\mathrm{rel}} \to \mathbb{R}$ is a homomorphism, the map $\tau_\eta : G \to \mathbb{R}$ is not a homomorphism. However, this map τ_η gives rise to a quasi-morphism. Thus, by the homogenization, we have the homogeneous quasi-morphism $\overline{\tau_\eta}$. Since $\overline{\tau_\eta}$ is independent of the choice of η, we simply denote it by $\overline{\tau}$. This $\overline{\tau}$ is the main object of the present paper.

It is known that the Calabi invariant $\mathrm{Cal} : G_{\mathrm{rel}} \to \mathbb{R}$ cannot be extended to a homomorphism $G \to \mathbb{R}$ (see Tsuboi [9]). However, the Calabi invariant *can be* extended to a homogeneous quasi-morphism on G. Indeed, we will show in Proposition 2.1 that the homogeneous quasi-morphism $\overline{\tau} : G \to \mathbb{R}$ gives rise to an extension of the Calabi invariant. There is another extension R of the Calabi invariant, which is introduced by Tsuboi [9] (see also Banyaga [1]). This extension R is defined as a homomorphism to $\mathbb{R}$ from the universal covering group $\widetilde{G}$ of G by

$$R([g_t]) = \int_0^1 \left(\int_D f_{X_t} \omega \right) dt.$$

Here g_t is a path in G and f_{X_t} is the Hamiltonian function associated to g_t (see Section 4). Then, it is natural to ask what the relation between two extensions τ and R of the Calabi invariant is. The following theorem answers this.

Theorem 1.1 (Theorem 2.5). — *Let $p : \widetilde{G} \to G$ be the projection. Then, we have*

$$p^*\overline{\tau} + 2R = \pi^2 \widetilde{\mathrm{rot}} : \widetilde{G} \to \mathbb{R}.$$

Here the map $\widetilde{\mathrm{rot}} : \widetilde{G} \to \mathbb{R}$ is the pullback of Poincaré's translation number by the surjection $\widetilde{G} \to \widehat{\mathrm{Diff}}_+(S^1)$.

Let $G_o = \{g \in G \mid g(o) = o\}$ be the subgroup of G consisting of symplectomorphisms which preserve the origin $o = (0, 0) \in D$. On the group G_o, we also construct a homogeneous quasi-morphism $\overline{\sigma} = \overline{\sigma}_{\eta,\gamma} : G_o \to \mathbb{R}$, where $\sigma_{\eta,\gamma} : G_o \to \mathbb{R}$ is defined by

$$\sigma_{\eta,\gamma}(g) = \int_\gamma g^*\eta - \eta.$$

Here the symbol γ is a path from the origin to a point on the boundary. Let $\widetilde{G}_o$ be the universal covering group of G_o. By using the homomorphism $S : \widetilde{G}_o \to \mathbb{R}$ introduced in Section 3, we describe the relation between $\overline{\sigma}$ and the translation number, which is similar to Theorem 1.1.

Theorem 1.2 (Theorem 3.4). — *Let $p : \widetilde{G}_o \to G_o$ be the projection. Then, we have*

$$p^*\overline{\sigma} - S = \pi\widetilde{\mathrm{rot}} : \widetilde{G}_o \to \mathbb{R}.$$

Here the map $\widetilde{\mathrm{rot}} : \widetilde{G}_o \to \mathbb{R}$ *is the pullback of the translation number by the surjection* $\widetilde{G}_o \to \widetilde{\mathrm{Diff}}_+(S^1)$.

The coboundary of the translation number $\widetilde{\mathrm{rot}}$ gives the canonical Euler cocycle (Matsumoto [5]). Similarly, the coboundary of homogeneous quasi-morphisms $\overline{\tau}$ and $\overline{\sigma}$ also give cocycles which represents the bounded Euler class of $\mathrm{Diff}_+(S^1)$ (Propositions 2.2, 3.1).

By comparing the two homogeneous quasi-morphisms $\overline{\tau}$ and $\overline{\sigma}$, we obtain the following theorem.

Theorem 1.3 (Theorem 4.1). — *The difference $\overline{\tau} - \pi\overline{\sigma} : G_o \to \mathbb{R}$ is a continuous surjective homomorphism.*

Note that, in this paper, we assume the notation of group cohomology and bounded cohomology in [2].

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2. The Calabi invariant case

2.1. Calabi invariant and the quasi-morphism $\overline{\tau}$

Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\}$ be the unit disk with the standard symplectic form $\omega = dx \wedge dy$. Let $G = \mathrm{Symp}(D)$ denote the symplectomorphism group of D and $\mathrm{Diff}_+(S^1)$ the orientation preserving diffeomorphism group of the unit circle $S^1 = \partial D$. Then the homomorphism $\rho : G \to \mathrm{Diff}_+(S^1)$ is surjective(see Tsuboi [9]). Thus we have an exact sequence

$$1 \longrightarrow G_{\mathrm{rel}} \longrightarrow G \xrightarrow{\rho} \mathrm{Diff}_+(S^1) \longrightarrow 1,$$

where the group G_{rel} is the kernel of the map $\rho : G \to \mathrm{Diff}_+(S^1)$.

The Calabi invariant $\mathrm{Cal} : G_{\mathrm{rel}} \to \mathbb{R}$ is defined by

$$\mathrm{Cal}(h) = \int_D h^*\eta \wedge \eta \tag{2.1}$$

where η is a 1-form satisfying $d\eta = \omega$. The Calabi invariant Cal is a surjective homomorphism and is independent of the choice of η (see Banyaga [1]). On the group G, the map $\tau_\eta : G \to \mathbb{R}$ is defined in the same way as in (2.1), that is, we put

$$\tau_\eta(g) = \int_D g^*\eta \wedge \eta.$$

Note that the map τ_η is *not* a homomorphism and *does* depend on the choice of η. In [6], for $\lambda = (xdy - ydx)/2$, Moriyoshi proved the transgression formula

$$\begin{aligned} \mathrm{Cal}(h) &= \tau_\lambda(h) \quad (h \in G_{\mathrm{rel}}) \\ -\delta\tau_\lambda(g, h) &= \pi^2\chi\left(\rho(g), \rho(h)\right) + \pi^2/2 \quad (g, h \in G). \end{aligned} \tag{2.2}$$

Here δ is the coboundary operator of group cohomology and the symbol χ is a bounded 2-cocycle defined in Moriyoshi [6], which represents the bounded Euler class $e_b \in H_b^2(\mathrm{Diff}_+(S^1); \mathbb{R})$. Since the cocycle χ is bounded, the map $\tau_\lambda : G \to \mathbb{R}$ is a quasi-morphism. Moreover, since the function $\tau_\eta - \tau_\lambda$ is bounded for any 1-form η satisfying $d\eta = \omega$, the map τ_η is a quasi-morphism for any η and the homogenizations of τ_η and τ_λ coincide. Thus we simply denote by $\overline{\tau}$ the homogenization of τ_η.

Proposition 2.1. — *The homogenization $\overline{\tau} : G \to \mathbb{R}$ is an extension of the Calabi invariant, that is, $\overline{\tau}|_{G_{\mathrm{rel}}} = \mathrm{Cal}$. In particular, the map $\overline{\tau}$ is a surjective homogeneous quasi-morphism.*

Proof. — For $h \in G_{\mathrm{rel}}$, we have

$$\overline{\tau}(h) = \lim_{n\to\infty} \frac{\tau_\eta(h^n)}{n} = \lim_{n\to\infty} \frac{\mathrm{Cal}(h^n)}{n} = \lim_{n\to\infty} \frac{n\mathrm{Cal}(h)}{n} = \mathrm{Cal}(h).$$

Since the Calabi invariant is surjective, the homogenization $\overline{\tau}$ is also surjective.

The homogeneous quasi-morphism $\overline{\tau}$ relates to the bounded Euler class as follows.

Proposition 2.2. — *The bounded cohomology class* $[\delta\overline{\tau}] \in H_b^2(G;\mathbb{R})$ *is equal to* $-\pi^2$ *times the pullback* $\rho^* e_b$ *of the bounded Euler class* e_b.

Proof. — Recall that the difference between a quasi-morphism and its homogenization is a bounded function. Thus we have $\delta\tau_\lambda - \delta\overline{\tau} = \delta b$ where $b = \tau_\lambda - \overline{\tau}$ is a bounded function. This implies that the bounded cohomology class $[\delta\tau_\eta]$ coincides with $[\delta\overline{\tau}]$. Moreover, the class $[\delta\tau_\lambda]$ is equal to the pullback $\rho^* e_b$ up to non-zero constant multiple because of the transgression formula (2.2).

2.2. Two extensions $\overline{\tau}$ and R of the Calabi invariant

By Proposition 2.1, the homogeneous quasi-morphism $\overline{\tau} : G \to \mathbb{R}$ is considered as an extension of the Calabi invariant. There is another extension R of the Calabi invariant, which is introduced by Tsuboi [9] (see also Banyaga [1]). This extension is defined as a homomorphism $R : \widetilde{G} \to \mathbb{R}$, where the group $\widetilde{G}$ is the universal covering group of G with respect to the C^∞-topology. In this section, we investigate the relation between these two extensions $\overline{\tau}$ and R.

We recall the definition of the homomorphism R. Let $L_\omega(D)$ be the set of divergence free vector fields which are tangent to the boundary. For any vector field X in $L_\omega(D)$, there is a unique function $f_X : D \to \mathbb{R}$ such that $i_X\omega = df_X$ and $f_X|_{\partial D} = 0$. For any path g_t in G, we define the time-dependent vector field X_t by $X_t = (\partial g_t/\partial t) \circ g_t^{-1}$. Since g_t is a symplectomorphism for any $t \in [0,1]$, the vector field X_t is in $L_\omega(D)$. Then the map $R : \widetilde{G} \to \mathbb{R}$ is defined by

$$R([g_t]) = \int_0^1 \left(\int_D f_{X_t}\omega \right) dt.$$

This map R is a well-defined homomorphism (see Banyaga [1]).

We reproduce the following lemma, which is essentially proved in Tsuboi [9, Lemme 1.5].

Lemma 2.3. — *Let* g_t *be a path in* G *such that* $g_0 = \mathrm{id}$ *and* X_t *the time-dependent vector field defined by* $X_t = (\partial g_t/\partial t) \circ g_t^{-1}$, *then*

$$\tau_\eta(g_1) + 2R([g_t]) = \int_{\partial D} \left(\int_0^1 g_t^* (i_{X_t}\eta)\, dt \right) \eta. \tag{2.3}$$

In particular, for a path h_t in G_{rel} such that $h_0 = \mathrm{id}$, we have $\mathrm{Cal}(h_1) = -2R([h_t])$.

Let $\widetilde{\mathrm{Diff}}_+(S^1)$ denote the universal covering of $\mathrm{Diff}_+(S^1)$. Note that, in this paper, we identify the circle S^1 with the quotient $\mathbb{R}/2\pi\mathbb{Z}$. We consider an element $\tilde\gamma \in \widetilde{\mathrm{Diff}}_+(S^1)$ as an orientation preserving diffeomorphism of $\mathbb{R}$ satisfying $\tilde\gamma(\theta + 2\pi) = \tilde\gamma(\theta) + 2\pi$ for any $\theta \in \mathbb{R}$. Let ϕ_t be the path in $\mathrm{Diff}_+(S^1)$ defined by $\phi_t = g_t|_{\partial D}$. Let ξ_t be the time-dependent vector field defined by $\xi_t = (\partial\phi_t/\partial t) \circ \phi_t^{-1}$. Let $\tilde\phi_t \in \widetilde{\mathrm{Diff}}_+(S^1)$ be the lift of ϕ_t such that $\tilde\phi_0 = \mathrm{id}$. Note that $\lambda = (xdy - ydx)/2 = (r^2 d\theta)/2$ where $(r, \theta) \in D$ is the polar coordinates. Then the right-hand side of the equality (2.3) can be written as

$$\int_{\partial D}\left(\int_0^1 g_t^*(i_{X_t}\lambda)dt\right)\lambda = \frac{1}{4}\int_{S^1}\left(\int_0^1 \phi_t^*(i_{\xi_t}d\theta)dt\right)d\theta$$
$$= \frac{1}{4}\int_0^{2\pi}\left(\int_0^1 \frac{\partial\tilde\phi_t}{\partial t}dt\right)d\theta \tag{2.4}$$
$$= \frac{1}{4}\int_0^{2\pi}\left(\tilde\phi_1(\theta) - \theta\right)d\theta.$$

Let us define a map $f : \widetilde{\mathrm{Diff}}_+(S^1) \to \mathbb{R}$ by $f(\tilde\phi) = \frac{1}{4\pi^2}\int_0^{2\pi}(\tilde\phi(\theta) - \theta)d\theta$. Then we have

$$\tau_\lambda(g_1) + 2R([g_t]) = \pi^2 f(\tilde\phi_1). \tag{2.5}$$

Note that, for any $\tilde\phi, \tilde\psi$ in $\widetilde{\mathrm{Diff}}_+(S^1)$, the inequality $|\tilde\phi\tilde\psi(\theta) - \tilde\phi(\theta) - \tilde\psi(\theta) + \theta| < 4\pi$ holds. This implies that the map f is a quasi-morphism. Let $\overline{f}$ be the homogenization of f. By taking the homogenizations of the both sides of the equality (2.5), we have

$$\overline{\tau}(g_1) + 2R([g_t]) = \pi^2\overline{f}(\tilde\phi_1). \tag{2.6}$$

To explain the map $\overline{f} : \widetilde{\mathrm{Diff}}_+(S^1) \to \mathbb{R}$, we recall the translation number introduced by Poincaré [7]. The translation number is a homogeneous quasi-morphism $\widetilde{\mathrm{rot}} : \widetilde{\mathrm{Diff}}_+(S^1) \to \mathbb{R}$ defined by

$$\widetilde{\mathrm{rot}}(\tilde\phi) = \lim_{n\to\infty}\frac{\tilde\phi^n(0)}{2\pi n}.$$

Note that, in this paper, we identify the circle S^1 with the quotient $\mathbb{R}/2\pi\mathbb{Z}$.

Proposition 2.4. — *The homogeneous quasi-morphism $\overline{f} : \widetilde{\mathrm{Diff}}_+(S^1) \to \mathbb{R}$ coincides with the translation number.*

Proof. — Since the sequence $\{\frac{\tilde{\phi}^n(x)-x}{n}\}_n$ converges uniformly to the constant function $\lim_{n\to\infty}\tilde{\phi}^n(0)/n$ on the interval $[0,2\pi]$, we have

$$\overline{f}(\tilde{\phi}) = \frac{1}{4\pi^2}\lim_{n\to\infty}\int_0^{2\pi}\frac{\tilde{\phi}^n(x)-x}{n}dx = \frac{1}{4\pi^2}\int_0^{2\pi}\lim_{n\to\infty}\frac{\tilde{\phi}^n(0)}{n}dx = \widetilde{\mathrm{rot}}(\tilde{\phi}).$$

By Proposition 2.4 and equality (2.6), we obtain the following theorem.

Theorem 2.5. — *Let $p:\tilde{G}\to G$ be the projection. Then we have*

$$p^*\overline{\tau} + 2R = \pi^2\widetilde{\mathrm{rot}} : \tilde{G}\to \mathbb{R}.$$

Here the map $\widetilde{\mathrm{rot}}:\tilde{G}\to\mathbb{R}$ is the pullback of the translation number by the surjection $\tilde{G}\to \widehat{\mathrm{Diff}}_+(S^1)$.

Poincaré's translation number descends to the map $\mathrm{rot}:\mathrm{Diff}_+(S^1)\to\mathbb{R}/\mathbb{Z}$ and this is called Poincaré's rotation number. The homomorphism $2R/\pi^2:\tilde{G}\to\mathbb{R}$ also decends to the homomorphism $\underline{R}:G\to\mathbb{R}/\mathbb{Z}$ (see Tsuboi [9, Corollary 2.9]).

Theorem 2.6. — *Let $\underline{\tau}:G\to\mathbb{R}/\mathbb{Z}$ be the composition of the homogeneous quasi-morphism $\overline{\tau}/\pi^2:G\to\mathbb{R}$ and the projection $\mathbb{R}\to\mathbb{R}/\mathbb{Z}$, then*

$$\underline{\tau} + \underline{R} = \mathrm{rot}.$$

Here the rot $: G\to\mathbb{R}/\mathbb{Z}$ *is the pullback of the rotation number by the projection $G\to\mathrm{Diff}_+(S^1)$.*

3. The flux homomorphism case

3.1. The flux homomorphism and the quasi-morphism $\overline{\sigma}$

Let us consider the subgroup

$$G_o = \{g\in G \mid g(o) = o\in D\}$$

of G. Put $G_{o,\mathrm{rel}} = G_{\mathrm{rel}}\cap G_o$. Then the following sequence of groups

$$1 \longrightarrow G_{o,\mathrm{rel}} \longrightarrow G_o \xrightarrow{\rho} \mathrm{Diff}_+\left(S^1\right) \longrightarrow 1$$

is an exact sequence. On the group $G_{o,\mathrm{rel}}$, the Calabi invariant is defined as the restriction $\mathrm{Cal}|_{G_{o,\mathrm{rel}}}:G_{o,\mathrm{rel}}\to\mathbb{R}$. In [4] the author studied a version of flux homomorphism defined on $G_{o,\,\mathrm{rel}}$ which is denoted by $\mathrm{Flux}_{\mathbb{R}}$. This flux homomorphism $\mathrm{Flux}_{\mathbb{R}}$ is defined by

$$\mathrm{Flux}_{\mathbb{R}}(h) = \int_{\gamma} h^{*}\eta - \eta$$

where γ is a path from the origin o to a point on the boundary ∂D. Note that the flux homomorphism is a surjective homomorphism and is independent of the choice of η and γ.

As in the case of Calabi invariant, the flux homomorphism can be extended to the group G_o, that is, we define the map $\sigma_{\eta,\gamma} : G_o \to \mathbb{R}$ by

$$\sigma_{\eta,\gamma}(g) = \int_{\gamma} g^{*}\eta - \eta.$$

The following transgression formula

$$(3.1)\qquad \begin{cases} \mathrm{Flux}_{\mathbb{R}}(h) = \sigma_{\eta,\gamma}(h) & (h \in G_{o,\,\mathrm{rel}}) \\ -\delta\sigma_{\eta,\gamma}(g,h) = \pi\xi\left(\rho(g),\rho(h)\right) & (g,h \in G_o)\,, \end{cases}$$

holds, where $\xi \in C^2(\mathrm{Diff}_+(S^1);\mathbb{R})$ is an Euler cocycle (see [4], where, in [4], the map $\sigma_{\eta,\gamma}$ is denoted by τ and the Euler cocycle ξ is denoted by χ). Since ξ is bounded, the map $\sigma_{\eta,\gamma}$ is a quasi-morphism. Let $\overline{\sigma}$ denote the homogenization of $\sigma_{\eta,\gamma}$. By arguments similar to those in Section 2, we obtain the following proposition.

Proposition 3.1.

(1) *The homogenization* $\overline{\sigma} : G_o \to \mathbb{R}$ *is independent of the choice of* η *and* γ.

(2) *The homogenization* $\overline{\sigma} : G_o \to \mathbb{R}$ *is an extension of the flux homomorphism. In particular,* $\overline{\sigma}$ *is a surjective homogeneous quasi-morphism.*

(3) *The bounded cohomology class* $[\delta\overline{\sigma}]$ *is equal to* $-\pi$ *times the class* $\rho^{*}e_b$*, where* e_b *is the bounded Euler class.*

Remark 3.2. — For an inner point $a \in D$, put $G^a = \{g \in G \mid g(a) = a\}$. We can define the homogeneous quasi-morphism $\overline{\sigma}_a : G^a \to \mathbb{R}$ in the same way. We can also show that $[\delta\overline{\sigma}_a] = -\pi\rho^{*}e_b$. Thus, for inner points $a, b \in D$, we have a homomorphism

$$\overline{\sigma}_a - \overline{\sigma}_b : G^a \cap G^b \to \mathbb{R}$$

and this is equal to the action difference defined in Polterovich [8](see also [3]).

3.2. Two extensions $\overline{\sigma}$ and S of the flux homomorphism

Let $\widetilde{G}_o$ be the universal covering group of G_o with respect to the C^∞-topology. In this section, we introduce a homomorphism $S : \widetilde{G}_o \to \mathbb{R}$ and show that the difference of $\overline{\sigma}$ and S is equal to the translation number.

For a path g_t in G_o such that $g_0 = \mathrm{id}$, the time-dependent vector field X_t is defined as in Section 2. Then we put

$$S(g_t) = \int_0^1 \left(\int_\gamma i_{X_t}\omega \right) dt, \tag{3.2}$$

where $\gamma : [0,1] \to D$ is a path from the origin $o \in D$ to a point on the boundary ∂D. Take the time-dependent C^∞-function $f_t : D \to \mathbb{R}$ satisfying $i_{X_t}\omega = df_t$ and $f_t(o) = 0$. Then we have

$$S(g_t) = \int_0^1 \left(\int_\gamma i_{X_t}\omega \right) dt = \int_0^1 \left(\int_\gamma df_t \right) dt = \int_0^1 f_t(\gamma(1))dt.$$

Note that, for any $t \in [0,1]$, the restriction $f_t|_{\partial D} : \partial D \to \mathbb{R}$ is a constant function. This implies that the function S is independent of the choice of γ.

Lemma 3.3. — *Let g_t be a path in G_o such that $g_0 = \mathrm{id}$ and X_t the time-dependent vector field defined by $X_t = (\partial g_t/\partial t) \circ g_t^{-1}$, then*

$$\sigma_{\eta,\gamma}(g_1) - S(g_t) = \int_0^1 \left(g_t^* (i_{X_t}\eta) \right)(\gamma(1))dt. \tag{3.3}$$

Proof. — Note that the identity

$$g_1^*\eta - \eta = d\left(\int_0^1 g_t^* f_t dt \right) + \int_0^1 g_t^* (i_{X_t}\eta)\, dt$$

holds. Thus we have

$$\begin{aligned} \sigma_{\eta,\gamma}(g_1) &= \int_\gamma g_1^*\eta - \eta \\ &= \int_\gamma \left(d\left(\int_0^1 g_t^* f_t dt \right) + \int_0^1 g_t^* (i_{X_t}\eta)\, dt \right) \\ &= \left(\int_0^1 (g_t^* f_t)(\gamma(1))\, dt + \int_0^1 (g_t^* (i_{X_t}\eta))(\gamma(1))\, dt \right) \\ &\quad - \left(\int_0^1 (g_t^* f_t)(\gamma(0))\, dt + \int_0^1 (g_t^* (i_{X_t}\eta))(\gamma(0))\, dt \right). \end{aligned} \tag{3.4}$$

Since $(g_t^* f_t)(\gamma(0)) = 0$ and $X_t(\gamma(0)) = 0$ for any $t \in [0,1]$, the second term in (3.4) is eqaul to 0. Moreover, since the function $f_t|_{\partial D}$ is constant for any $t \in [0,1]$, the first term in (3.4) is equal to $S(g_t) + \int_0^1 (g_t^* (i_{X_t}\eta))(\gamma(1))dt$ and the lemma follows.

Put $\eta = (r^2 d\theta)/2$ and $\phi_t = g_t|_{\partial D}$ in $\mathrm{Diff}_+(S^1)$. Take a path $\gamma : [0,1] \to D$ defined by $\gamma(t) = (t, 0)$. Let $\tilde{\phi}_t \in \widehat{\mathrm{Diff}}_+(S^1)$ be the lift of ϕ_t such that $\tilde{\phi}_0 = \mathrm{id}$. As in the equation (2.4), we have

$$\int_0^1 g_t^* (i_{X_t}\eta)(\gamma(1))dt = \frac{1}{2}\int_0^1 \frac{\partial \tilde{\phi}_t}{\partial t}(0)dt = \frac{1}{2}\tilde{\phi}_1(0),$$

where we identify $\gamma(1) \in \partial D$ with $0 \in \mathbb{R}/2\pi\mathbb{Z}$ by the identification $\partial D = S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Thus we have

$$\sigma_{\eta,\gamma}(g_1) - S(g_t) = \frac{1}{2}\tilde{\phi}_1(0). \tag{3.5}$$

Equality (3.5) implies that the value $S(g_t)$ depends only on the homotopy class relatively to fixed ends of the path g_t in G_o. Henceforth, the map $S : \widetilde{G}_o \to \mathbb{R} : [g_t] \mapsto S(g_t)$ is well-defined. Moreover, the map S gives rise to a homomorphism. In fact, let g_t, h_t be paths in G_o, then

$$\begin{aligned} &S(g_t h_t) - S(g_t) - S(h_t) \\ &\quad = \sigma_{\eta,\gamma}(g_1 h_1) - \sigma_{\eta,\gamma}(g_1) - \sigma_{\eta,\gamma}(h_1) - \frac{1}{2}\left(\tilde{\phi}_1\tilde{\psi}_1(0) - \tilde{\phi}_1(0) - \tilde{\psi}_1(0)\right) \end{aligned}$$

and this is equal to 0 (see Maruyama [4]). Thus we have

$$\begin{aligned} &\overline{\sigma}(g_1) - S([g_t]) \\ &\quad = \lim_{n\to\infty} \frac{\sigma_{\eta,\gamma}(g_1^n) - S([g_t]^n)}{n} = \pi \lim_{n\to\infty} \frac{\tilde{\phi}_1^{\,n}(0)}{2\pi n} = \pi\widetilde{\mathrm{rot}}(\tilde{\phi}_1). \end{aligned} \tag{3.6}$$

By the above equality (3.6), we obtain the following theorem.

Theorem 3.4. — *Let $p : \widetilde{G}_o \to G_o$ be the projection. Then, we have*

$$p^*\overline{\sigma} - S = \pi\widetilde{\mathrm{rot}} : \widetilde{G}_o \to \mathbb{R}.$$

Here the map $\widetilde{\mathrm{rot}} : \widetilde{G}_o \to \mathbb{R}$ is the pullback of the translation number by the surjection $\widetilde{G}_o \to \widehat{\mathrm{Diff}}_+(S^1)$.

Remark 3.5. — By considering the map to $\mathbb{R}/\mathbb{Z}$, we obtain a theorem similar to Theorem 2.6 for $\overline{\sigma}$, S, and the rotation number.

Remark 3.6. — By (3.5), we obtain the formula similar to [9, Corollary (2.9)] and thus the formula similar to [9, Proposition (3.1)]. This implies that the homomorphism $\mathrm{Flux}_{\mathbb{R}}$ cannot be extended to a homomorphism on G_o.

4. Relation between $\overline{\tau}$ and $\overline{\sigma}$

The restriction $\mathrm{Cal}|_{G_{o,\mathrm{rel}}} : G_{o,\mathrm{rel}} \to \mathbb{R}$ of the Calabi invariant remains surjective. So the restriction $\overline{\tau} : G_o \to \mathbb{R}$ is also surjective homogeneous quasi-morphism. Therefore we have two non-trivial homogeneous quasi-morphisms $\overline{\tau}, \overline{\sigma} \in Q(G_o)$. By Proposition 2.2 and Proposition 3.1, the class $[\delta\overline{\tau}]$ coincides with $\pi[\delta\overline{\sigma}]$ in $H_b^2(G_o;\mathbb{R})$. Thus the difference $\overline{\tau} - \pi\overline{\sigma}$ is a homomorphism on G_o. This implies that, in contrast with Cal and $\mathrm{Flux}_{\mathbb{R}}$, the difference $\mathrm{Cal} - \pi\mathrm{Flux}_{\mathbb{R}}$ can be extended to a homomorphism $\overline{\tau} - \pi\overline{\sigma} : G_o \to \mathbb{R}$.

Theorem 4.1. — *The difference $\overline{\tau} - \pi\overline{\sigma} : G_o \to \mathbb{R}$ is a continuous surjective homomorphism.*

Proof. — On the group $G_{o,\mathrm{rel}}$, the homomorphism $\overline{\tau} - \pi\overline{\sigma}$ is equal to $\mathrm{Cal} - \pi\mathrm{Flux}_{\mathbb{R}}$. Put the non-increasing C^∞-function $f : [0,1] \to \mathbb{R}$ which is equal to 1 near $r = 0$ and $f(1) = 0$. Then, for $s \in \mathbb{R}$, we define a diffeomorphism g_s in $G_{o,\mathrm{rel}}$ by

$$g_s(r,\theta) = \left(r, \theta + sf(r)\right)$$

where $(r,\theta) \in D$ is the polar coordinates. For

$$\eta = \left(r^2 d\theta\right)/2, \quad \gamma(r) = (r, 0) \in D,$$

we have

$$\mathrm{Cal}(g_s) = \frac{s\pi}{2}\int_0^1 r^4 \frac{\partial f}{\partial r} dr, \quad \pi\mathrm{Flux}_{\mathbb{R}}(g_s) = \frac{s\pi}{2}\int_0^1 r^2 \frac{\partial f}{\partial r} dr.$$

This implies that the difference $\overline{\tau} - \pi\overline{\sigma}$ is surjective on $G_{o,\mathrm{rel}}$, and so is on G_o.

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