Shuhei Maruyama

The translation number and quasi-morphisms on groups of symplectomorphisms of the disk

Article à paraître, mis en ligne le 29 juillet 2022, 12 p.
THE TRANSLATION NUMBER AND QUASI-MORPHISMS ON GROUPS OF SYMPLECTOMORPHISMS OF THE DISK

by Shuhei MARUYAMA

Abstract. — On groups of symplectomorphisms of the disk, we construct two homogeneous quasi-morphisms which relate to the Calabi invariant and the flux homomorphism respectively. We also show the relation between the quasi-morphisms and the translation number introduced by Poincaré.


1. Introduction

A quasi-morphism on a group $\Gamma$ is a function $\phi : \Gamma \to \mathbb{R}$ such that the value
\[
\sup_{\gamma_1, \gamma_2 \in \Gamma} |\phi(\gamma_1 \gamma_2) - \phi(\gamma_1) - \phi(\gamma_2)|
\]
is bounded. A quasi-morphism $\phi$ is called homogeneous if the condition $\phi(\gamma^n) = n\phi(\gamma)$ holds for any $\gamma \in \Gamma$ and $n \in \mathbb{Z}$. Let $Q(\Gamma)$ denote the $\mathbb{R}$-vector space of homogeneous quasi-morphisms on the group $\Gamma$. Given a quasi-morphism $\phi$, we obtain the homogeneous quasi-morphism $\overline{\phi}$ associated to $\phi$ by
\[
\overline{\phi}(g) = \lim_{n \to \infty} \frac{\phi(g^n)}{n}.
\]
This map $\overline{\phi}$ is called the homogenization of $\phi$.

Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be the unit disk in $\mathbb{R}^2$ and $\omega = dx \wedge dy$ be the standard symplectic form on $D$. Let $G = \text{Symp}(D)$ be the group of...
symplectomorphisms of $D$ (which may not be the identity on the boundary $\partial D$). In the present paper, we construct a homogeneous quasi-morphism on $G$. Let $\eta$ be a 1-form on $D$ satisfying $d\eta = \omega$. The map $\tau_\eta : G \to \mathbb{R}$ is defined by

$$
\tau_\eta(g) = \int_D g^* \eta \wedge \eta.
$$

Let $G_{\text{rel}}$ denote the kernel of the homomorphism $G \to \text{Diff}_+ (S^1)$, where $\text{Diff}_+ (S^1)$ denotes the group of orientation preserving diffeomorphisms of the circle. Then the map $\tau$ coincides with the Calabi invariant on $G_{\text{rel}}$. Although the Calabi invariant $\text{Cal} : G_{\text{rel}} \to \mathbb{R}$ is a homomorphism, the map $\tau_\eta : G \to \mathbb{R}$ is not a homomorphism. However, this map $\tau_\eta$ gives rise to a quasi-morphism. Thus, by the homogenization, we have the homogeneous quasi-morphism $\overline{\tau_\eta}$. Since $\overline{\tau_\eta}$ is independent of the choice of $\eta$, we simply denote it by $\overline{\tau}$. This $\overline{\tau}$ is the main object of the present paper.

It is known that the Calabi invariant $\text{Cal} : G_{\text{rel}} \to \mathbb{R}$ cannot be extended to a homomorphism $G \to \mathbb{R}$ (see Tsuboi [9]). However, the Calabi invariant can be extended to a homogeneous quasi-morphism on $G$. Indeed, we will show in Proposition 2.1 that the homogeneous quasi-morphism $\tau : G \to \mathbb{R}$ gives rise to an extension of the Calabi invariant. There is another extension $R$ of the Calabi invariant, which is introduced by Tsuboi [9] (see also Banyaga [1]). This extension $R$ is defined as a homomorphism to $\mathbb{R}$ from the universal covering group $\widetilde{G}$ of $G$ by

$$
R([g_t]) = \int_0^1 \left( \int_D f_{X_t} \omega \right) dt.
$$

Here $g_t$ is a path in $G$ and $f_{X_t}$ is the Hamiltonian function associated to $g_t$ (see Section 4). Then, it is natural to ask what the relation between two extensions $\tau$ and $R$ of the Calabi invariant is. The following theorem answers this.

**THEOREM 1.1** (Theorem 2.5). — Let $p : \widetilde{G} \to G$ be the projection. Then, we have

$$
p^* \tau + 2R = \pi^2 \text{rot} : \widetilde{G} \to \mathbb{R}.
$$

Here the map $\text{rot} : \widetilde{G} \to \mathbb{R}$ is the pullback of Poincaré’s translation number by the surjection $\widetilde{G} \to \text{Diff}_+ (S^1)$.

Let $G_o = \{ g \in G \mid g(o) = o \}$ be the subgroup of $G$ consisting of symplectomorphisms which preserve the origin $o = (0,0) \in D$. On the group $G_o$, we also construct a homogeneous quasi-morphism $\sigma = \sigma_{\eta,\gamma} : G_o \to \mathbb{R}$, where $\sigma_{\eta,\gamma} : G_o \to \mathbb{R}$ is defined by
\[ \sigma_{\eta, \gamma}(g) = \int_{\gamma} g^* \eta - \eta. \]

Here the symbol \( \gamma \) is a path from the origin to a point on the boundary.

Let \( \tilde{G}_o \) be the universal covering group of \( G_o \). By using the homomorphism \( S : \tilde{G}_o \to \mathbb{R} \) introduced in Section 3, we describe the relation between \( \sigma \) and the translation number, which is similar to Theorem 1.1.

**Theorem 1.2** (Theorem 3.4). — Let \( p : \tilde{G}_o \to G_o \) be the projection. Then, we have
\[ p^* \sigma - S = \pi \tilde{\text{rot}} : \tilde{G}_o \to \mathbb{R}. \]

Here the map \( \tilde{\text{rot}} : \tilde{G}_o \to \mathbb{R} \) is the pullback of the translation number by the surjection \( \tilde{G}_o \to \text{Diff}_+^+(S^1) \).

The coboundary of the translation number \( \tilde{\text{rot}} \) gives the canonical Euler cocycle (Matsumoto [5]). Similarly, the coboundary of homogeneous quasi-morphisms \( \tau \) and \( \sigma \) also give cocycles which represents the bounded Euler class of \( \text{Diff}_+^+(S^1) \) (Propositions 2.2, 3.1).

By comparing the two homogeneous quasi-morphisms \( \tau \) and \( \sigma \), we obtain the following theorem.

**Theorem 1.3** (Theorem 4.1). — The difference \( \tau - \pi \sigma : G_o \to \mathbb{R} \) is a continuous surjective homomorphism.

Note that, in this paper, we assume the notation of group cohomology and bounded cohomology in [2].

**Acknowledgements**

The author would like to thank Professor Hitoshi Moriyoshi for his helpful advice. He also thanks Morimichi Kawasaki, who told him that there is another extension \( R \) of the Calabi invariant and suggested to investigate a connection between \( R \) and the quasi-morphism \( \tau \) constructed in this paper. He also thanks Professor Masayuki Asaoka for his comments.

**2. The Calabi invariant case**
2.1. Calabi invariant and the quasi-morphism $\tau$

Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be the unit disk with the standard symplectic form $\omega = dx \wedge dy$. Let $G = \text{Symp}(D)$ denote the symplectomorphism group of $D$ and $\text{Diff}^+(S^1)$ the orientation preserving diffeomorphism group of the unit circle $S^1 = \partial D$. Then the homomorphism $\rho : G \to \text{Diff}^+(S^1)$ is surjective (see Tsuboi [9]). Thus we have an exact sequence

$$1 \longrightarrow G_{\text{rel}} \longrightarrow G \overset{\rho}{\longrightarrow} \text{Diff}^+(S^1) \longrightarrow 1,$$

where the group $G_{\text{rel}}$ is the kernel of the map $\rho : G \to \text{Diff}^+(S^1)$.

The Calabi invariant $\text{Cal} : G_{\text{rel}} \to \mathbb{R}$ is defined by

$$\text{Cal}(h) = \int_D h^*\eta \wedge \eta$$

where $\eta$ is a 1-form satisfying $d\eta = \omega$. The Calabi invariant $\text{Cal}$ is a surjective homomorphism and is independent of the choice of $\eta$ (see Banyaga [1]). On the group $G$, the map $\tau_\eta : G \to \mathbb{R}$ is defined in the same way as in (2.1), that is, we put

$$\tau_\eta(g) = \int_D g^*\eta \wedge \eta.$$

Note that the map $\tau_\eta$ is not a homomorphism and does depend on the choice of $\eta$. In [6], for $\lambda = (x dy - y dx)/2$, Moriyoshi proved the transgression formula

$$\text{Cal}(h) = \tau_\lambda(h) \quad (h \in G_{\text{rel}})$$

$$-\delta \tau_\lambda(g, h) = \pi^2 \chi(\rho(g), \rho(h)) + \pi^2/2 \quad (g, h \in G).$$

Here $\delta$ is the coboundary operator of group cohomology and the symbol $\chi$ is a bounded 2-cocycle defined in Moriyoshi [6], which represents the bounded Euler class $e_b \in H^2_b(\text{Diff}^+(S^1); \mathbb{R})$. Since the cocycle $\chi$ is bounded, the map $\tau_\lambda : G \to \mathbb{R}$ is a quasi-morphism. Moreover, since the function $\tau_\eta - \tau_\lambda$ is bounded for any 1-form $\eta$ satisfying $d\eta = \omega$, the map $\tau_\eta$ is a quasi-morphism for any $\eta$ and the homogenizations of $\tau_\eta$ and $\tau_\lambda$ coincide. Thus we simply denote by $\tau$ the homogenization of $\tau_\eta$.

**Proposition 2.1.** — The homogenization $\tau : G \to \mathbb{R}$ is an extension of the Calabi invariant, that is, $\tau|_{G_{\text{rel}}} = \text{Cal}$. In particular, the map $\tau$ is a surjective homogeneous quasi-morphism.

**Proof.** — For $h \in G_{\text{rel}}$, we have

$$\tau(h) = \lim_{n \to \infty} \frac{\tau_\eta(h^n)}{n} = \lim_{n \to \infty} \frac{\text{Cal}(h^n)}{n} = \lim_{n \to \infty} \frac{n\text{Cal}(h)}{n} = \text{Cal}(h).$$
Since the Calabi invariant is surjective, the homogenization $\tau$ is also surjective. \qed

The homogeneous quasi-morphism $\tau$ relates to the bounded Euler class as follows.

**Proposition 2.2.** — The bounded cohomology class $[\delta \tau] \in H^2_b(G; \mathbb{R})$ is equal to $-\pi^2$ times the pullback $\rho^* e_b$ of the bounded Euler class $e_b$.

**Proof.** — Recall that the difference between a quasi-morphism and its homogenization is a bounded function. Thus we have $\delta \tau - \delta \bar{\tau} = \delta b$ where $b = \tau - \tau$ is a bounded function. This implies that the bounded cohomology class $[\delta \tau]$ coincides with $[\delta \bar{\tau}]$. Moreover, the class $[\delta \tau]$ is equal to the pullback $\rho^* e_b$ up to non-zero constant multiple because of the transgression formula (2.2). \qed

### 2.2. Two extensions $\tau$ and $R$ of the Calabi invariant

By Proposition 2.1, the homogeneous quasi-morphism $\tau : G \to \mathbb{R}$ is considered as an extension of the Calabi invariant. There is another extension $R$ of the Calabi invariant, which is introduced by Tsuboi [9] (see also Banyaga [1]). This extension is defined as a homomorphism $R : \tilde{G} \to \mathbb{R}$, where the group $\tilde{G}$ is the universal covering group of $G$ with respect to the $C^\infty$-topology. In this section, we investigate the relation between these two extensions $\tau$ and $R$.

We recall the definition of the homomorphism $R$. Let $\mathcal{L}_\omega(D)$ be the set of divergence free vector fields which are tangent to the boundary. For any vector field $X$ in $\mathcal{L}_\omega(D)$, there is a unique function $f_X : D \to \mathbb{R}$ such that $i_X \omega = df_X$ and $f_X|_{\partial D} = 0$. For any path $g_t$ in $G$, we define the time-dependent vector field $X_t$ by $X_t = (\partial g_t / \partial t) \circ g_t^{-1}$. Since $g_t$ is a symplectomorphism for any $t \in [0, 1]$, the vector field $X_t$ is in $\mathcal{L}_\omega(D)$. Then the map $R : \tilde{G} \to \mathbb{R}$ is defined by

$$R([g_t]) = \int_0^1 \left( \int_D f_{X_t} \omega \right) dt.$$ 

This map $R$ is a well-defined homomorphism (see Banyaga [1]).

We reproduce the following lemma, which is essentially proved in Tsuboi [9, Lemme 1.5].

**Lemma 2.3.** — Let $g_t$ be a path in $G$ such that $g_0 = \text{id}$ and $X_t$ the time-dependent vector field defined by $X_t = (\partial g_t / \partial t) \circ g_t^{-1}$, then

$$\tau_\eta(g_1) + 2R([g_t]) = \int_{\partial D} \left( \int_0^1 g_t^* (i_{X_t}\eta) dt \right) \eta.$$ 

2.2. Two extensions $\tau$ and $R$ of the Calabi invariant

By Proposition 2.1, the homogeneous quasi-morphism $\tau : G \to \mathbb{R}$ is considered as an extension of the Calabi invariant. There is another extension $R$ of the Calabi invariant, which is introduced by Tsuboi [9] (see also Banyaga [1]). This extension is defined as a homomorphism $R : \tilde{G} \to \mathbb{R}$, where the group $\tilde{G}$ is the universal covering group of $G$ with respect to the $C^\infty$-topology. In this section, we investigate the relation between these two extensions $\tau$ and $R$.

We recall the definition of the homomorphism $R$. Let $\mathcal{L}_\omega(D)$ be the set of divergence free vector fields which are tangent to the boundary. For any vector field $X$ in $\mathcal{L}_\omega(D)$, there is a unique function $f_X : D \to \mathbb{R}$ such that $i_X \omega = df_X$ and $f_X|_{\partial D} = 0$. For any path $g_t$ in $G$, we define the time-dependent vector field $X_t$ by $X_t = (\partial g_t / \partial t) \circ g_t^{-1}$. Since $g_t$ is a symplectomorphism for any $t \in [0, 1]$, the vector field $X_t$ is in $\mathcal{L}_\omega(D)$. Then the map $R : \tilde{G} \to \mathbb{R}$ is defined by

$$R([g_t]) = \int_0^1 \left( \int_D f_{X_t} \omega \right) dt.$$ 

This map $R$ is a well-defined homomorphism (see Banyaga [1]).

We reproduce the following lemma, which is essentially proved in Tsuboi [9, Lemme 1.5].

**Lemma 2.3.** — Let $g_t$ be a path in $G$ such that $g_0 = \text{id}$ and $X_t$ the time-dependent vector field defined by $X_t = (\partial g_t / \partial t) \circ g_t^{-1}$, then

$$\tau_\eta(g_1) + 2R([g_t]) = \int_{\partial D} \left( \int_0^1 g_t^* (i_{X_t}\eta) dt \right) \eta.$$ 

2.2. Two extensions $\tau$ and $R$ of the Calabi invariant

By Proposition 2.1, the homogeneous quasi-morphism $\tau : G \to \mathbb{R}$ is considered as an extension of the Calabi invariant. There is another extension $R$ of the Calabi invariant, which is introduced by Tsuboi [9] (see also Banyaga [1]). This extension is defined as a homomorphism $R : \tilde{G} \to \mathbb{R}$, where the group $\tilde{G}$ is the universal covering group of $G$ with respect to the $C^\infty$-topology. In this section, we investigate the relation between these two extensions $\tau$ and $R$.

We recall the definition of the homomorphism $R$. Let $\mathcal{L}_\omega(D)$ be the set of divergence free vector fields which are tangent to the boundary. For any vector field $X$ in $\mathcal{L}_\omega(D)$, there is a unique function $f_X : D \to \mathbb{R}$ such that $i_X \omega = df_X$ and $f_X|_{\partial D} = 0$. For any path $g_t$ in $G$, we define the time-dependent vector field $X_t$ by $X_t = (\partial g_t / \partial t) \circ g_t^{-1}$. Since $g_t$ is a symplectomorphism for any $t \in [0, 1]$, the vector field $X_t$ is in $\mathcal{L}_\omega(D)$. Then the map $R : \tilde{G} \to \mathbb{R}$ is defined by

$$R([g_t]) = \int_0^1 \left( \int_D f_{X_t} \omega \right) dt.$$ 

This map $R$ is a well-defined homomorphism (see Banyaga [1]).

We reproduce the following lemma, which is essentially proved in Tsuboi [9, Lemme 1.5].

**Lemma 2.3.** — Let $g_t$ be a path in $G$ such that $g_0 = \text{id}$ and $X_t$ the time-dependent vector field defined by $X_t = (\partial g_t / \partial t) \circ g_t^{-1}$, then

$$\tau_\eta(g_1) + 2R([g_t]) = \int_{\partial D} \left( \int_0^1 g_t^* (i_{X_t}\eta) dt \right) \eta.$$ 

2.2. Two extensions $\tau$ and $R$ of the Calabi invariant

By Proposition 2.1, the homogeneous quasi-morphism $\tau : G \to \mathbb{R}$ is considered as an extension of the Calabi invariant. There is another extension $R$ of the Calabi invariant, which is introduced by Tsuboi [9] (see also Banyaga [1]). This extension is defined as a homomorphism $R : \tilde{G} \to \mathbb{R}$, where the group $\tilde{G}$ is the universal covering group of $G$ with respect to the $C^\infty$-topology. In this section, we investigate the relation between these two extensions $\tau$ and $R$.

We recall the definition of the homomorphism $R$. Let $\mathcal{L}_\omega(D)$ be the set of divergence free vector fields which are tangent to the boundary. For any vector field $X$ in $\mathcal{L}_\omega(D)$, there is a unique function $f_X : D \to \mathbb{R}$ such that $i_X \omega = df_X$ and $f_X|_{\partial D} = 0$. For any path $g_t$ in $G$, we define the time-dependent vector field $X_t$ by $X_t = (\partial g_t / \partial t) \circ g_t^{-1}$. Since $g_t$ is a symplectomorphism for any $t \in [0, 1]$, the vector field $X_t$ is in $\mathcal{L}_\omega(D)$. Then the map $R : \tilde{G} \to \mathbb{R}$ is defined by

$$R([g_t]) = \int_0^1 \left( \int_D f_{X_t} \omega \right) dt.$$ 

This map $R$ is a well-defined homomorphism (see Banyaga [1]).

We reproduce the following lemma, which is essentially proved in Tsuboi [9, Lemme 1.5].

**Lemma 2.3.** — Let $g_t$ be a path in $G$ such that $g_0 = \text{id}$ and $X_t$ the time-dependent vector field defined by $X_t = (\partial g_t / \partial t) \circ g_t^{-1}$, then

$$\tau_\eta(g_1) + 2R([g_t]) = \int_{\partial D} \left( \int_0^1 g_t^* (i_{X_t}\eta) dt \right) \eta.$$
In particular, for a path $h_t$ in $G_{\text{rel}}$ such that $h_0 = \text{id}$, we have $\text{Cal}(h_1) = -2R([h_t])$.

Let $\widetilde{\text{Diff}}_+(S^1)$ denote the universal covering of $\text{Diff}_+(S^1)$. Note that, in this paper, we identify the circle $S^1$ with the quotient $\mathbb{R}/2\pi\mathbb{Z}$. We consider an element $\tilde{\gamma} \in \widetilde{\text{Diff}}_+(S^1)$ as an orientation preserving diffeomorphism of $\mathbb{R}$ satisfying $\tilde{\gamma}(\theta + 2\pi) = \tilde{\gamma}(\theta) + 2\pi$ for any $\theta \in \mathbb{R}$. Let $\varphi_t$ be the path in $\text{Diff}_+(S^1)$ defined by $\varphi_t = g_t|_{\partial D}$. Let $\xi_t$ be the time-dependent vector field defined by $\xi_t = \left(\partial \varphi_t / \partial t\right) \circ \varphi_t^{-1}$. Let $\tilde{\varphi}_t \in \widetilde{\text{Diff}}_+(S^1)$ be the lift of $\varphi_t$ such that $\tilde{\varphi}_0 = \text{id}$. Note that $\lambda = \left(x dy - y dx\right) / 2 = (r^2 d\theta) / 2$ where $(r, \theta) \in D$ is the polar coordinates. Then the right-hand side of the equality (2.3) can be written as

$$\int_{\partial D} \left(\int_0^1 g_t^1 (i_X, \lambda) dt\right) = \frac{1}{4} \int_{S^1} \left(\int_0^1 \varphi_t^1 (i_{\xi_t} \lambda) dt\right) d\theta$$

$$= \frac{1}{4} \int_0^{2\pi} \left(\int_0^1 \partial \tilde{\varphi}_t / \partial t dt\right) d\theta$$

$$= \frac{1}{4} \int_0^{2\pi} (\tilde{\varphi}_1(\theta) - \theta) d\theta. \quad (2.4)$$

Let us define a map $f : \widetilde{\text{Diff}}_+(S^1) \to \mathbb{R}$ by $f(\tilde{\varphi}) = \frac{1}{4\pi^2} \int_0^{2\pi} (\tilde{\varphi}(\theta) - \theta) d\theta$. Then we have

$$\tau_\lambda(g_1) + 2R([g_t]) = \frac{\pi^2}{2} f(\tilde{\varphi}_1). \quad (2.5)$$

Note that, for any $\tilde{\varphi}, \tilde{\psi}$ in $\widetilde{\text{Diff}}_+(S^1)$, the inequality $|\tilde{\varphi} \tilde{\psi}(\theta) - \tilde{\varphi}(\theta) + \tilde{\psi}(\theta) + \theta| < 4\pi$ holds. This implies that the map $f$ is a quasi-morphism. Let $\overline{f}$ be the homogenization of $f$. By taking the homogenizations of the both sides of the equality (2.5), we have

$$\tau(g_1) + 2R([g_t]) = \pi^2 \overline{f}(\tilde{\varphi}_1). \quad (2.6)$$

To explain the map $\overline{f} : \widetilde{\text{Diff}}_+(S^1) \to \mathbb{R}$, we recall the translation number introduced by Poincaré [7]. The translation number is a homogeneous quasi-morphism $\text{rot} : \widetilde{\text{Diff}}_+(S^1) \to \mathbb{R}$ defined by

$$\text{rot}(\tilde{\varphi}) = \lim_{n \to \infty} \frac{\tilde{\varphi}^n(0)}{2\pi n}. \quad (2.7)$$

Note that, in this paper, we identify the circle $S^1$ with the quotient $\mathbb{R}/2\pi\mathbb{Z}$.

**Proposition 2.4.** — The homogeneous quasi-morphism $\overline{f} : \widetilde{\text{Diff}}_+(S^1) \to \mathbb{R}$ coincides with the translation number.
Proof. — Since the sequence $\{ \frac{\tilde{\varphi}^n(x) - x}{n} \}$ converges uniformly to the constant function $\lim_{n \to \infty} \frac{\tilde{\varphi}^n(0)}{n}$ on the interval $[0, 2\pi]$, we have

$$J(\tilde{\varphi}) = \frac{1}{4\pi^2} \lim_{n \to \infty} \int_0^{2\pi} \frac{\tilde{\varphi}^n(x) - x}{n} dx = \frac{1}{4\pi^2} \int_0^{2\pi} \lim_{n \to \infty} \frac{\tilde{\varphi}^n(0)}{n} dx = \tilde{\text{rot}}(\tilde{\varphi}).$$

By Proposition 2.4 and equality (2.6), we obtain the following theorem.

**Theorem 2.5.** — Let $p : \tilde{G} \to G$ be the projection. Then we have

$$p^* \tau + 2R = \pi^2 \tilde{\text{rot}} : \tilde{G} \to \mathbb{R}.$$

Here the map $\tilde{\text{rot}} : \tilde{G} \to \mathbb{R}$ is the pullback of the translation number by the surjection $\tilde{G} \to \text{Diff}_+(S^1)$.

Poincaré’s translation number descends to the map $\text{rot} : \text{Diff}_+(S^1) \to \mathbb{R}/\mathbb{Z}$ and this is called Poincaré’s rotation number. The homomorphism $2R/\pi^2 : \tilde{G} \to \mathbb{R}$ also descends to the homomorphism $\tilde{R} : G \to \mathbb{R}/\mathbb{Z}$ (see Tsuboi [9, Corollary 2.9]).

**Theorem 2.6.** — Let $\tau : G \to \mathbb{R}/\mathbb{Z}$ be the composition of the homogeneous quasi-morphism $\tau/\pi^2 : G \to \mathbb{R}$ and the projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$, then

$$\tau + \tilde{R} = \text{rot}.$$

Here the $\text{rot} : G \to \mathbb{R}/\mathbb{Z}$ is the pullback of the rotation number by the projection $G \to \text{Diff}_+(S^1)$.

### 3. The flux homomorphism case

#### 3.1. The flux homomorphism and the quasi-morphism $\sigma$

Let us consider the subgroup

$$G_o = \{ g \in G | g(o) = o \in D \}$$

of $G$. Put $G_{o, \text{rel}} = G_{\text{rel}} \cap G_o$. Then the following sequence of groups

$$1 \longrightarrow G_{o, \text{rel}} \longrightarrow G_o \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 1$$

is an exact sequence. On the group $G_{o, \text{rel}}$, the Calabi invariant is defined as the restriction $\text{Cal}|_{G_{o, \text{rel}}} : G_{o, \text{rel}} \to \mathbb{R}$. In [4] the author studied a version
of flux homomorphism defined on $G_{o,\text{rel}}$ which is denoted by $\text{Flux}_{\mathbb{R}}$. This flux homomorphism $\text{Flux}_{\mathbb{R}}$ is defined by

$$\text{Flux}_{\mathbb{R}}(h) = \int_\gamma h^*\eta - \eta$$

where $\gamma$ is a path from the origin $o$ to a point on the boundary $\partial D$. Note that the flux homomorphism is a surjective homomorphism and is independent of the choice of $\eta$ and $\gamma$.

As in the case of Calabi invariant, the flux homomorphism can be extended to the group $G_{o}$, that is, we define the map $\sigma_{\eta,\gamma} : G_{o} \to \mathbb{R}$ by

$$\sigma_{\eta,\gamma}(g) = \int_\gamma g^*\eta - \eta.$$ 

The following transgression formula

$$\text{Flux}_{\mathbb{R}}(h) = \sigma_{\eta,\gamma}(h) \quad (h \in G_{o,\text{rel}})$$

$$-\delta\sigma_{\eta,\gamma}(g, h) = \pi\xi(\rho(g), \rho(h)) \quad (g, h \in G_{o}),$$

holds, where $\xi \in C^2(\text{Diff}_+(S^1); \mathbb{R})$ is an Euler cocycle (see [4], where, in [4], the map $\sigma_{\eta,\gamma}$ is denoted by $\tau$ and the Euler cocycle $\xi$ is denoted by $\chi$). Since $\xi$ is bounded, the map $\sigma_{\eta,\gamma}$ is a quasi-morphism. Let $\bar{\sigma}$ denote the homogenization of $\sigma_{\eta,\gamma}$. By arguments similar to those in Section 2, we obtain the following proposition.

**Proposition 3.1.**

(1) The homogenization $\bar{\sigma} : G_{o} \to \mathbb{R}$ is independent of the choice of $\eta$ and $\gamma$.

(2) The homogenization $\bar{\sigma} : G_{o} \to \mathbb{R}$ is an extension of the flux homomorphism. In particular, $\bar{\sigma}$ is a surjective homogeneous quasi-morphism.

(3) The bounded cohomology class $[\delta\bar{\sigma}]$ is equal to $-\pi$ times the class $\rho^*e_b$, where $e_b$ is the bounded Euler class.

**Remark 3.2.** — For an inner point $a \in D$, put $G^a = \{g \in G \mid g(a) = a\}$. We can define the homogeneous quasi-morphism $\bar{\sigma}_a : G^a \to \mathbb{R}$ in the same way. We can also show that $[\delta\bar{\sigma}_a] = -\pi\rho^*e_b$. Thus, for inner points $a, b \in D$, we have a homomorphism

$$\bar{\sigma}_a - \bar{\sigma}_b : G^a \cap G^b \to \mathbb{R}$$

and this is equal to the action difference defined in Polterovich [8](see also [3]).
3.2. Two extensions $\bar{\sigma}$ and $S$ of the flux homomorphism

Let $\widetilde{G}_o$ be the universal covering group of $G_o$ with respect to the $C^\infty$-topology. In this section, we introduce a homomorphism $S: \widetilde{G}_o \to \mathbb{R}$ and show that the difference of $\bar{\sigma}$ and $S$ is equal to the translation number.

For a path $g_t$ in $G_o$ such that $g_0 = \text{id}$, the time-dependent vector field $X_t$ is defined as in Section 2. Then we put

$$S(g_t) = \int_0^1 \int_\gamma i_{X_t} \omega \ dt,$$

where $\gamma : [0, 1] \to D$ is a path from the origin $o \in D$ to a point on the boundary $\partial D$. Take the time-dependent $C^\infty$-function $f_t : D \to \mathbb{R}$ satisfying $i_{X_t} \omega = df_t$ and $f_t(o) = 0$. Then we have

$$S(g_t) = \int_0^1 \int_\gamma i_{X_t} \omega \ dt = \int_0^1 \int f_t \ dt = \int_0^1 f_t(\gamma(1)) \ dt.$$

Note that, for any $t \in [0, 1]$, the restriction $f_t|_{\partial D} : \partial D \to \mathbb{R}$ is a constant function. This implies that the function $S$ is independent of the choice of $\gamma$.

**Lemma 3.3.** — Let $g_t$ be a path in $G_o$ such that $g_0 = \text{id}$ and $X_t$ the time-dependent vector field defined by $X_t = (\partial g_t/\partial t) \circ g_t^{-1}$, then

$$\sigma_{\eta, \gamma}(g_1) - S(g_t) = \int_0^1 g_t^*(i_{X_t} \eta)(\gamma(1)) \ dt.$$

**Proof.** — Note that the identity

$$g_1^* \eta - \eta = d \left( \int_0^1 g_t^* f_t \ dt + \int_0^1 g_t^* (i_{X_t} \eta) \ dt \right)$$

holds. Thus we have

$$\sigma_{\eta, \gamma}(g_1) = \int_\gamma g_1^* \eta - \eta$$

$$= \int_\gamma d \left( \int_0^1 g_t^* f_t \ dt + \int_0^1 g_t^* (i_{X_t} \eta) \ dt \right)$$

$$= \left( \int_0^1 (g_t^* f_t)(\gamma(1)) \ dt + \int_0^1 (g_t^* (i_{X_t} \eta))(\gamma(1)) \ dt \right)$$

$$- \left( \int_0^1 (g_t^* f_t)(\gamma(0)) \ dt + \int_0^1 (g_t^* (i_{X_t} \eta))(\gamma(0)) \ dt \right).$$

Since $(g_t^* f_t)(\gamma(0)) = 0$ and $X_t(\gamma(0)) = 0$ for any $t \in [0, 1]$, the second term in (3.4) is equal to 0. Moreover, since the function $f_t|_{\partial D}$ is constant for any $t \in [0, 1]$, the first term in (3.4) is equal to $S(g_t) + \int_0^1 (g_t^* (i_{X_t} \eta))(\gamma(1)) dt$ and the lemma follows. □
Put \( \eta = (r^2 d\theta)/2 \) and \( \varphi_t = g_t \mid_{\partial D} \) in \( \text{Diff}_+(S^1) \). Take a path \( \gamma : [0, 1] \to D \) defined by \( \gamma(t) = (t, 0) \). Let \( \tilde{\varphi}_t \in \text{Diff}_+(S^1) \) be the lift of \( \varphi_t \) such that \( \tilde{\varphi}_0 = \text{id} \). As in the equation (2.4), we have

\[
\int_0^1 g_t^* (i_X \eta) (\gamma(1)) dt = \frac{1}{2} \int_0^1 \frac{\partial \tilde{\varphi}_t}{\partial t}(0) dt = \frac{1}{2} \tilde{\varphi}_1(0),
\]

where we identify \( \gamma(1) \in \partial D \) with \( 0 \in \mathbb{R}/2\pi \mathbb{Z} \) by the identification \( \partial D = \mathbb{S}^1 = \mathbb{R}/2\pi \mathbb{Z} \). Thus we have

(3.5) \[
\sigma_{\eta, \gamma}(g_1) - S(g_t) = \frac{1}{2} \tilde{\varphi}_1(0).
\]

Equality (3.5) implies that the value \( S(g_t) \) depends only on the homotopy class relatively to fixed ends of the path \( g_t \) in \( G_o \). Henceforth, the map \( S : \widetilde{G}_o \to \mathbb{R} : [g_t] \mapsto S(g_t) \) is well-defined. Moreover, the map \( S \) gives rise to a homomorphism. In fact, let \( g_t, h_t \) be paths in \( G_o \), then

\[
S(g_t h_t) - S(g_t) - S(h_t) = \sigma_{\eta, \gamma}(g_1 h_1) - \sigma_{\eta, \gamma}(g_1) - \sigma_{\eta, \gamma}(h_1) - \frac{1}{2} \left( \tilde{\varphi}_1 \psi_1(0) - \tilde{\varphi}_1(0) - \psi_1(0) \right)
\]

and this is equal to 0 (see Maruyama [4]). Thus we have

(3.6) \[
\varpi(g_1) - S([g_t]) = \lim_{n \to \infty} \frac{\sigma_{\eta, \gamma}(g_1^n) - S([g_t]^n)}{n} = \pi \lim_{n \to \infty} \frac{\tilde{\varphi}_1^n(0)}{2\pi n} = \pi \text{rot} (\tilde{\varphi}_1).
\]

By the above equality (3.6), we obtain the following theorem.

**Theorem 3.4.** — Let \( p : \widetilde{G}_o \to G_o \) be the projection. Then, we have

\[
p^* \varpi - S = \pi \text{rot} : \widetilde{G}_o \to \mathbb{R}.
\]

Here the map \( \pi \text{rot} : \widetilde{G}_o \to \mathbb{R} \) is the pullback of the translation number by the surjection \( \widetilde{G}_o \to \text{Diff}_+(S^1) \).

**Remark 3.5.** — By considering the map to \( \mathbb{R}/\mathbb{Z} \), we obtain a theorem similar to Theorem 2.6 for \( \varpi, S \), and the rotation number.

**Remark 3.6.** — By (3.5), we obtain the formula similar to [9, Corollary (2.9)] and thus the formula similar to [9, Proposition (3.1)]. This implies that the homomorphism \( \text{Flux}_{\mathbb{R}} \) cannot be extended to a homomorphism on \( G_o \).
4. Relation between $\tau$ and $\sigma$

The restriction $\text{Cal}|_{G_{o,\text{rel}}} : G_{o,\text{rel}} \to \mathbb{R}$ of the Calabi invariant remains surjective. So the restriction $\tau : G_{o} \to \mathbb{R}$ is also surjective homogeneous quasi-morphism. Therefore we have two non-trivial homogeneous quasi-morphisms $\tau, \sigma \in Q(G_{o})$. By Proposition 2.2 and Proposition 3.1, the class $[\delta \tau]$ coincides with $\pi[\delta \sigma]$ in $H_{b}^{2}(G_{o};\mathbb{R})$. Thus the difference $\tau - \pi \sigma$ is a homomorphism on $G_{o}$. This implies that, in contrast with Cal and Flux$_{\mathbb{R}}$, the difference $\text{Cal} - \pi \text{Flux}_{\mathbb{R}}$ can be extended to a homomorphism $\tau - \pi \sigma : G_{o} \to \mathbb{R}$.

**Theorem 4.1.** — The difference $\tau - \pi \sigma : G_{o} \to \mathbb{R}$ is a continuous surjective homomorphism.

**Proof.** — On the group $G_{o,\text{rel}}$, the homomorphism $\tau - \pi \sigma$ is equal to $\text{Cal} - \pi \text{Flux}_{\mathbb{R}}$. Put the non-increasing $C^{\infty}$-function $f : [0,1] \to \mathbb{R}$ which is equal to 1 near $r = 0$ and $f(1) = 0$. Then, for $s \in \mathbb{R}$, we define a diffeomorphism $g_{s}$ in $G_{o,\text{rel}}$ by

$$g_{s}(r, \theta) = (r, \theta + sf(r))$$

where $(r, \theta) \in D$ is the polar coordinates. For

$$\eta = (r^{2}d\theta)/2, \quad \gamma(r) = (r,0) \in D,$$

we have

$$\text{Cal}(g_{s}) = \frac{s\pi}{2} \int_{0}^{1} r^{4} \frac{\partial f}{\partial r} dr, \quad \pi \text{Flux}_{\mathbb{R}}(g_{s}) = \frac{s\pi}{2} \int_{0}^{1} r^{2} \frac{\partial f}{\partial r} dr.$$ 

This implies that the difference $\tau - \pi \sigma$ is surjective on $G_{o,\text{rel}}$, and so is on $G_{o}$. 

---

**BIBLIOGRAPHY**


Manuscrit reçu le 20 février 2020,
révisé le 6 octobre 2020,
accepté le 8 avril 2021.

Shuhei MARUYAMA
Nagoya University,
Graduate School of Mathematics,
Furocho, Chikusaku,
Nagoya (Japan)
m17037h@math.nagoya-u.ac.jp