Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Igor Belegradek & David González-Álvaro

Diffeomorphic souls and disconnected moduli spaces of nonnegatively curved metrics

Tome 72, nº 1 (2022), p. 109-122.

https://doi.org/10.5802/aif.3471

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
72, 1 (2022) 109-122

DIFFEOMORPHIC SOULS AND DISCONNECTED MODULI SPACES OF NONNEGATIVELY CURVED METRICS

by Igor BELEGRADEK & David GONZÁLEZ-ÁLVARO (*)

Abstract. — We give examples of open manifolds that carry infinitely many complete metrics of nonnegative sectional curvature such that they all have the same soul, and their isometry classes lie in different connected components of the moduli space. All previously known examples of this kind have souls of codimension one. In our examples the souls have codimensions three and two.

Résumé. — Nous donnons des exemples de variétés ouvertes qui admettent une infinité de métriques complètes à courbure sectionnelle non négative telles que leurs âmes soient identiques et que leurs classes d'équivalence se trouvent dans des composantes connexes différentes de l'espace de modules. Tous les exemples de ce genre, connus auparavant, ont une âme de codimension un. Dans les exemples que nous présentons, les âmes sont de codimension trois et deux.

1. Motivation and results

There has been considerable recent interest in studying spaces of metrics with various curvature restrictions, such as nonnegative sectional curvature, to be denoted $K > 0$, see [36] and references therein. For a manifold V let $R_{K>0}(V)$ denote the space of complete Riemannian metrics on V of $K > 0$ with the topology of smooth ($= C^\infty$) uniform convergence on compact sets, and $M_{K>0}(V)$ be the corresponding moduli space, the quotient space of $R_{K>0}(V)$ by the $\mathrm{Diff}(V)$-action via pullback.

The soul construction [10] takes as the input a complete metric of $K > 0$ on an open connected manifold V, and a basepoint of V, and produces a

Keywords: Nonnegative curvature, soul, moduli space, positive scalar curvature.
2020 *Mathematics Subject Classification:* 53C20.
(*) This work was partially supported by the Simons Foundation grant 524838 (Belegradek) and by MINECO grant MTM2017-85934-C3-2-P (González-Álvaro).

totally convex compact boundaryless submanifold S of V, called the *soul*, such that V is is diffeomorphic to a tubular neighborhood of S. If we fix a metric and vary the basepoint, the resulting souls are ambiently isotopic [39] and isometric [34]. Consider the map soul that sends an isometry class of a complete metric of $K > 0$ on V to the isometry class of its soul:

$$\mathrm{soul}\colon\ \mathcal{M}_{K>0}(V) \to \coprod_{S \in V} \mathcal{M}_{K>0}(S)$$

where the co-domain is given the topology of disjoint union, and V is a set of pairwise non-diffeomorphic manifolds such that $S \in V$ if and only if S is diffeomorphic to a soul of a complete metric of $K > 0$ on V.

A tantalizing open problem is to decide if the map soul is continuous; the difficulty is that the soul is constructed via asymptotic geometry which is not captured by the compact-open topology on the space of metrics. The following is immediate from [5, Theorem 2.1].

Theorem 1.1. — *If V is indecomposable, then the map* soul *is continuous.*

An open manifold is *indecomposable* if it admits a complete metric of $K > 0$ such that the normal sphere bundle to a soul has no section. It follows from [39] that for indecomposable V the soul is uniquely determined by the metric (and not the basepoint). Moreover, [5] implies that the souls of nearby metrics are ambiently isotopic by a small compactly supported isotopy. In particular, metrics with non-diffeomorphic souls in an indecomposable manifold lie in different connected components of $\mathcal{M}_{K>0}(V)$.

There are many examples where the diffeomorphism (or even homeomorphism) type of the soul depends on the metric, see [4, 6, 7, 19, 25, 32], and if the ambient open manifold V is indecomposable, this gives examples where $\mathcal{M}_{K>0}(V)$ is not connected, or even has infinitely many connected components.

If V has a complete metric with $K > 0$ with soul of codimension one, then soul is a homeomorphism, see [6]. Thus if for some soul S the space $\mathcal{M}_{K>0}(S)$ has infinitely many connected components, then so does $\mathcal{M}_{K>0}(V)$; for example, this applies to $V = S \times \mathbb{R}$.

Examples of closed manifolds S for which $\mathcal{M}_{K>0}(S)$ has infinitely many connected components can be found in [12, 13, 14, 15, 21, 25]. These metrics on S have $K > 0$ and scal > 0, and the connected components are distinguished by index-theoretic invariants that are constant on paths of scal > 0.

The papers mentioned in the previous paragraph only prove existence of infinitely many path-components. We take this opportunity to note that they actually get infinitely many connected components.

Theorem 1.2. — *Let M be a closed manifold. If two points in the same connected component of $\mathcal{M}_{K>0}(M)$ have* scal > 0*, then they can be joined by a path of isometry classes of* scal > 0.

In this paper we show that some of these S as above can be realized as souls of codimensions 2 or 3 in indecomposable manifolds. The codimension 2 case is a fairly straightforward consequence of results in [15, 27, 37].

Theorem 1.3. — *For every positive integer n there are infinitely many homotopy types that contain a manifold M such that*

(a) *M is a simply-connected manifold that is the total space of a principal circle bundle over $S^2 \times CP^{2n}$, and*
(b) *if V is the total space of a non-trivial complex line bundle over M, then V has infinitely many complete metrics of $K > 0$ whose souls equal the zero section, and whose isometry classes lie in different connected components of $\mathcal{M}_{K>0}(V)$.*

The codimension 3 case requires a bit more work. Recall that if M is the total space of a linear S^3-bundle over S^4, then M admits a metric of $K > 0$ [23], and moreover, if the bundle has nonzero Euler number, then $\mathcal{M}_{K>0}(M)$ has infinitely many connected components [12, 21]. We prove:

Theorem 1.4. — *Let M be the total space of a linear S^3-bundle ξ over S^4 with Pontryagin number $p_1(\xi)$ and nonzero Euler number $e(\xi)$. If $\frac{p_1(\xi)}{2e(\xi)}$ is not an odd integer, then M is diffeomorphic to a codimension three submanifold S of an indecomposable manifold V that admits infinitely many complete metrics of $K > 0$ with soul S whose isometry classes lie in different connected components of $\mathcal{M}_{K>0}(V)$.*

Milnor famously showed that some S^3-bundles over S^4 are exotic spheres [29]. In fact, M is a homotopy sphere if and only if $e(\xi) = \pm 1$. Unfortunately, if $e(\xi) = \pm 1$, then $\frac{p_1(\xi)}{2}$ is an odd integer, so no M in Theorem 1.4 is a homotopy sphere. On the other hand, for every integer n with $n > 2$ there is M as in the conclusion of Theorem 1.4 with $H^4(M) = \mathbb{Z}_n$, see Section 7.

To prove Theorem 1.4 we use results of Grove-Ziller [23] and some topological considerations to find an indecomposable V with a codimension three soul, and then we observe that the metric on the soul can be moved by Cheeger deformation to metrics in [12, 21] that represent infinitely many connected components.

Let us conclude by mentioning that other results on connected components of moduli spaces corresponding to various nonnegative or positive curvature conditions can be found in [8, 20, 27, 35, 38].

Structure of the paper

Theorems 1.1 and 1.2 are proved in Section 2. Theorem 1.3 is established in Section 3. Theorem 1.4 is proved in Section 7, and the needed background is reviewed in Sections 4, 5, 6.

Acknowledgements

The authors are grateful to Luis Guijarro for hospitality during Belegradek's visit to Madrid where this project was started.

2. Continuity of souls, connectedness and path-connectedness

Proof of Theorem 1.1. — Theorem 2.1 in [5] says that the map that sends a complete metric of $K > 0$ on V to its soul, considered as a point in the space of smooth compact submanifolds of V with smooth topology, is continuous. Two nearby submanifolds are ambiently isotopic by a small isotopy with compact support. Hence, the isometry classes of the induced metrics on these submanifolds are close in the moduli space. Thus we get a continuous map

$$\mathcal{R}_{K>0}(V) \to \coprod_{S \subset V} \mathcal{M}_{K>0}(S)$$

that takes a metric to the isometry class of its soul, where the co-domain is given the disjoint union topology, i.e., the set in the co-domain is open if and only if its intersection with each $\mathcal{M}_{K>0}(S)$ is open. Finally, by the definition of quotient topology the above continuous map descends to a continuous map defined on $\mathcal{M}_{K>0}(V)$.

Let $\mathfrak{X}$ denote the the space of isometry classes of Riemannian metrics on a closed manifold M with smooth ($= C^\infty$) topology, and let $\mathfrak{X}_{\mathrm{scal}\geq 0}$, $\mathfrak{X}_{\mathrm{scal}>0}$ be the subspaces of $\mathfrak{X}$ of isometry classes of metrics of nonnegative and positive scalar curvature, respectively.

Lemma 2.1. — *X is metrizable.*

Proof. — This is well-known, but we cannot find a proof in the literature, and hence present it here for completeness. The smooth topology on the space of all Riemannian metrics on M is induced by a metric whose isometry group contains $\mathrm{Diff}(M)$ [16, Proposition 148], and every $\mathrm{Diff}(M)$-orbit is closed [16, Proposition 142]. The corresponding pseudometric on the set of orbits induces the quotient topology, and the pseudodistance is simply the infimum of distances between the orbits [24, Theorem 4]. Since the orbits are closed, the quotient space is T_1, so that the pseudometric is actually a metric.

Also X is locally path-connected (because this property is inherited by quotients, and X is the quotient of the space of metrics, which is an open subset in the Fréchet space of 2-tensors on M). In fact, every point of X has a contractible neighborhood (as follows from the smooth version of Corollary 7.3 in [17] which can be deduced from the discussion after the corollary) but we do not need it here.

Theorem 2.2. — *If C is a connected subset of $X_{\mathrm{scal}>0}$ that contains no Ricci-flat metrics, then any two points $y, z \in C$ can be joined by a path in $\{y, z\} \cup X_{\mathrm{scal}>0}$.*

Proof. — By continuous dependence of Ricci flow on initial metric, see e.g. [2, Theorem A], for every point $x \in X$ there is a neighborhood U_x and a positive constant τ_x such that the Ricci flow that starts at any point of U_x exists in $[0, \tau_x]$.

Being a metrizable space X is paracompact Hausdorff, and hence has a locally finite open cover $\{R_{x_i}\}_{i\in I}$ such that $R_{x_i} \subset U_{x_i}$ for all i, and there is a continuous function $\tau \colon X \to (0, \infty)$ with $\tau(x) \leqslant \tau_{x_i}$ for all $x \in R_{x_i}$, see [31, Theorem 41.8]. Since C contains no Ricci-flat metrics, for every $x \in C$ the Ricci flow of x has scal > 0 for all times in $(0, \tau(x)]$, see [9, Proposition 2.18]. By continuous dependence of the Ricci flow on initial metric the map $T \colon X \to X$ that sends x to the Ricci flow of x at time $\tau(x)$ is continuous. Hence, if C is a connected subset of $X_{\mathrm{scal}>0}$ that contains y, z, then $T(C)$ is a connected subset of $X_{\mathrm{scal}>0}$.

Since $X_{\mathrm{scal}>0}$ is an open subset in the locally path-connected space X, every connected component of $X_{\mathrm{scal}>0}$ is path-connected. Hence the connected component of $X_{\mathrm{scal}>0}$ that contains $T(C)$ also contains a path from $T(y)$ to $T(z)$. Concatenating the path with Ricci flows from y, z to $T(y), T(z)$, respectively, we get a path from y to z with desired properties.

Proof of Theorem 1.2. — No flat manifold admits a metric of scal > 0 [22, Corollary A]. Hence M admits no flat metric. Since Ricci-flat metrics of $K > 0$ are flat, $\mathcal{M}_{K>0}(M)$ contains no Ricci-flat metrics. Applying Theorem 2.2 to the connected component of $\mathcal{M}_{K>0}(M)$ that contains y, z finishes the proof.

3. Codimension two

Proof of Theorem 1.3. — If $S^{2t+1} \to \mathbb{C}P^t$ is the circle bundle obtained by restricting the diagonal circle action on $\mathbb{C}^{t+1}$, where t is a positive integer, then its Euler class generates $H^2(\mathbb{C}P^t)$ as follows from the Gysin sequence and 2-connectedness of S^{2t+1}. Consider the product of two such circle bundles with $t = 1$ and $t = 2n$. Then the argument [37, p. 227] implies that any M as in (a) is the quotient of the Riemannian product of two unit spheres $S^3 \times S^{4n+1}$ by the free isometric circle action $e^{i\varphi}(x, y) = (e^{il\varphi} x, e^{-ik\varphi} y)$ for some coprime integers k, l. This gives a Riemannian submersion metric on M with $K > 0$ and Ric > 0.

Sometimes it happens that the quotients corresponding to different pairs (k, l) are diffeomorphic. In fact, $H^4(M)$ is a cyclic group of order l^2, so up to sign l is determined by the homotopy type of M, but for a given l the quotients fall into finitely many diffeomorphism types [15, Proposition 2.2]. Their diffeomorphism classification was studied in [27, 37] and finally in [15] where it was shown that for each n there are infinitely many homotopy types that contain M as in (a) and such that the Riemannian submersion metrics as above represent infinitely many connected components of $\mathcal{M}_{K>0}(M)$.

Similarly, since $S^3 \times S^{4n+1}$ is 2-connected, any complex line bundle over M is the quotient of $S^3 \times S^{4n+1} \times \mathbb{C}$ by the circle action $e^{i\varphi}(x, y, z) = (e^{il\varphi} x, e^{-ik\varphi} y, e^{im} z)$, cf. [7, Lemma 12.3]. In particular, V carries a complete Riemannian submersion metric of $K > 0$ with soul equal to the zero section, which is the quotient of $S^3 \times S^{4n+1} \times \{0\}$ by the above circle action, and hence is diffeomorphic to M.

If we fix l and the Euler class of the line bundle in $H^2(M) = \mathbb{Z}$, there are only finitely many possibilities for the diffeomorphism type of the pair $(V,$ soul) for the above metrics. By varying k appropriately, then we get a sequence of complete metrics of $K > 0$ on each V as above such that the metrics on the soul represent infinitely many connected components of $\mathcal{M}_{K>0}(M)$.

If the line bundle is non-trivial, then V is indecomposable, and the map **soul** is continuous by Theorem 1.1. Thus $\mathfrak{M}_{K>0}(V)$ has infinitely many connected components.

4. Equivariant Cheeger deformation

The purpose of this section is to review the Cheeger deformation, and note that it passes to quotients by free isometric actions.

Let G be a compact Lie group with a bi-invariant metric Q that acts isometrically on a Riemannian manifold (M, q_0). Consider the diagonal G-action on $M \times G$ given by

$$a \cdot (p, g) = (ap, ag), \quad p \in M, \quad a, g \in G.$$

Its orbit space is commonly denoted by $M \times_G G$. The map $\pi\colon M \times G \to M$ given by $\pi(p, g) = g^{-1}p$ descends to a diffeomorphism $\varphi\colon M \times_G G \to M$.

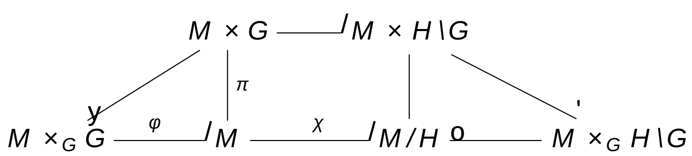
For any positive scalar t the G-action is isometric in the product metric $q_0 + \frac{Q}{t}$, which induces a metric q_t on M that makes π into a Riemannian submersion. Similarly, φ becomes an isometry between q_t, $t > 0$, and the Riemannian submersion metric on $M \times_G G$ induced by $q_0 + \frac{Q}{t}$.

The map $t \to g_t$ is continuous for $t > 0$; this is the *Cheeger deformation* of q_0, see e.g. [1, p. 140] or [40]. The key property is that if q_0 has $K > 0$, then so does q_t for all t.

Fix a closed subgroup H of G such that the H-action on M is free. For $t > 0$ let q_t be the metric on M/H that makes the H-orbit map into a Riemannian submersion $\chi\colon (M, q_t) \to (M/H, \mathsf{q}_t)$. The map $t \to \mathsf{q}_t$ is continuous for $t > 0$.

The H-action on $M \times G$ given by $h \cdot (p, g) = (p, gh^{-1})$ commutes with the diagonal G-action, and hence descends to a free H-action on $M \times_G G$. For this action the maps π and φ are H-equivariant, and descend to a Riemannian submersion $M \times (H\backslash G) \to M/H$ and an isometry $M \times_G (H\backslash G) \to M/H$, respectively, where $t > 0$ and $H\backslash G$ is given the Riemannian submersion metric induced by $\frac{Q}{t}$.

Thus in the following diagram all maps are Riemannian submersions for $t > 0$



and χ is also a Riemannian submersion for $t = 0$. In this diagram M and M/H are the only spaces where the metric corresponding to $t = 0$ is defined.

5. Some algebra and geometry of the 3-sphere

In this section we specialize the discussion of Section 4 to the case when $G = \mathbb{S}^3 \times \mathbb{S}^3$, where $\mathbb{S}^3$ is thought of as unit quaternions, and H is the diagonal subgroup of G, i.e., $H = \{(g, g) : g \in G\}$.

Consider the diffeomorphism $\psi \colon \mathbb{S}^3 \to H \backslash G$ given by $\psi(c) = (c, 1)H$; thus ψ^{-1} sends the coset $(a, b)H$ to ab^{-1}. With this identification the (left) G-action on $H \backslash G$ becomes $(a, b) \cdot c = acb^{-1}$, where $a, b, c \in \mathbb{S}^3$; indeed

$$(a, b)(c, 1)H = (ac, b)H = (acb^{-1}, 1)H.$$

Since $(-1, -1)$ acts trivially, the G-action on $H \backslash G$ descends to an SO(4) action with isotropy subgroups isomorphic to SO(3).

It follows that any G-invariant Riemannian metric on $H \backslash G$ is isometric to a round 3-sphere (i.e., a metric sphere in $\mathbb{R}^4$). Indeed, SO(3) acts transitively on every tangent 2-sphere, so G acts transitively on the unit tangent bundle, and hence the metric has constant Ricci curvature, which on the 3-sphere makes the metric round.

The discussion in Section 4 immediately gives the following.

Proposition 5.1. — *Let H be the diagonal subgroup of $G = \mathbb{S}^3 \times \mathbb{S}^3$. Given an isometric G-action on a Riemannian manifold (M, q_0) of $K > 0$ that restricts to a free H-action there is path of Riemannian metrics (M, q_t) of $K > 0$, defined for $t > 0$, such that*

- *for every $t > 0$ the G-action is q_t-isometric, and the Riemannian submersion metric $(M/H, \mathbf{q}_t)$ induced by q_t has $K > 0$, and $t \to \mathbf{q}_t$ is a continuous path of metrics on M/H,*
- *if $t > 0$ and $H \backslash G$ is given the Riemannian submersion metric induced by a bi-invariant metric on G, then $H \backslash G$ is isometric to a round sphere, and $(M/H, \mathbf{q}_t)$ is isometric to the Riemannian submersion metric on $(M, q_t) \times_G H \backslash G$.*

6. Bundle theoretic facts

This section reviews several well-known bundle theoretic facts.

Lemma 6.1. — *Let $C \leqslant G$ be an order two normal subgroup of a topological group G. If $P \to X$ is a non-trivial principal G-bundle over a finite cell complex with $H^1(X;\mathbb{Z}_2) = 0$, then the associated principal G/C-bundle $P/C \to X$ is non-trivial.*

Proof. — The surjection $G \to G/C = H$ induces a fibration of classifying spaces $BC \to BG \to BH$ where BC is a homotopy fiber of $BG \to BH$, see [28]. As explained in [30, p. 139], for any finite complex X we get an exact sequence of pointed sets

$$[X, BC] \to [X, BG] \to [X, BH]$$

with constant maps as basepoints. Since $[X, BC] = H^1(X;\mathbb{Z}_2) = 0$, the rightmost arrow is injective.

A *k-plane bundle* is a vector bundle with fiber $\mathbb{R}^k$.

Lemma 6.2. — *Let X be a paracompact space with $H^1(X;\mathbb{Z}_2) = 0 = H^2(X)$. If a 3-plane bundle over X has a nowhere zero section, then it is trivial.*

Proof. — A nowhere zero section gives rise to a splitting of the bundle into the Whitney sum of a line and a 2-plane subbundles, which are orientable since $H^1(X;\mathbb{Z}_2) = 0$, and in fact, trivial because a line bundle is determined by its first Stiefel-Whitney class in $H^1(X;\mathbb{Z}_2)$, and an orientable 2-plane bundle is determined by its Euler class in $H^2(X)$.

Lemma 6.3. — *If X is a finite cell complex with $H^1(X;\mathbb{Z}_2) = 0 = H^4(X;\mathbb{Q})$, then the number of isomorphism classes of 3-plane bundles over X is finite.*

Proof. — Since $H^1(X;\mathbb{Z}_2) = 0$, any vector bundle over X is orientable. There are only finitely many isomorphism classes of orientable 3-plane bundles with a given first rational Pontryagin class [3, Theorem A.0.1], which lies in $H^4(X;\mathbb{Q}) = 0$.

7. Codimension three

This section ends with a proof of Theorem 1.4. First, we recall some results and notations from [23].

Following [23, p. 349] let $P_{k,l}$ denote the principal $S^3 \times S^3$-bundle over S^4 classified by the map $q \to (q^k, q^{-l})$ in $\pi_3(S^3 \times S^3) = \mathbb{Z} \times \mathbb{Z}$, where $q \in S^3$.

Let $M_{k,l}$ be the the associated bundle $P_{k,l} \times_{S^3 \times S^3} S^3$ where the action on S^3 is as in Section 5, see [23, p. 352]. Equivalently [33, Proposition 8.27], the action is given by the universal covering $S^3 \times S^3 \to SO(4)$ where the $SO(4)$-action on S^3 is standard. Hence, $M_{k,l}$ is a linear S^3-bundle over S^4.

The Euler number and the Pontryagin number of the S^3-bundle $M_{k,l} \to S^4$ are $\pm(k + l)$ and $\pm 2(k - l)$, see [26, p. 159, 169]. The Gysin sequence shows that $H^4(M_{k,l}) = \mathbb{Z}_{k+l}$ if $k+l = 0$, and then $H^4(M_{k,l}) = \mathbb{Z}$ if $k+l = 0$.

Remark 7.1. — Somewhat confusingly, the notation $M_{m,n}$ is also used in the literature to denote the total space of another S^3-bundle over S^4 based on a different choice of generators in $\pi_3(S^3 \times S^3)$. This usage goes back to James and Whitehead, and more to the point, appears in works quoted below. Thus $M_{k,l}$ of [23] equals $M_{m,n}$ of [11, 21] when $m = -l$, $n = k + l$. In what follows all results are rephrased in notations of [23].

According to Section 4, $M_{k,l}$ can be described as $P_{k,l}/H$ where H is the diagonal subgroup in $S^3 \times S^3$, cf. also Key Observation in [18]. Thus $M_{k,l}$ is the base of a principal S^3-bundle with total space $P_{k,l}$. Our strategy hinges on the following:

Problem. — *Find all k, l such that the principal H-bundle $P_{k,l} \to P_{k,l}/H = M_{k,l}$ is non-trivial.*

Some partial solutions are presented below. An especially interesting case (which we could not resolve in this paper) is when $|k+l| = 1$, or equivalently, $M_{k,l}$ is a homotopy sphere.

Lemma 7.2. — *If $kl = 0$, the principal H-bundle $P_{k,l} \to M_{k,l}$ is trivial.*

Proof. — The principal $S^3 \times S^3$-bundle $P_{k,0}$ is isomorphic to $P \times S^3$ for some principal S^3-bundle P over S^4. The inclusion $i\colon P \to P \times S^3$ given by $i(p) = (p, 1)$ is transverse to the H-orbits, hence it descends to an immersion $i\colon P \to (P \times S^3)/H$, which is a diffeomorphism because both domain and co-domain are closed manifolds of the same dimension. Then $i \circ i^{-1}$ is a section of $P \times S^3 \to (P \times S^3)/H$, and any principal bundle with a section is trivial.

Lemma 7.4 below sheds some light on why the assumption "$\frac{p_1(\xi)}{2e(\xi)}$ is not odd" is relevant. Let us first restate the assumption:

Lemma 7.3. — *$\frac{k-l}{k+l}$ is an odd integer if and only if $\frac{k}{k+l}$ Z.*

Proof. — $\frac{k-l}{k+l}$ is odd if and only if $\frac{k-l}{k+l} + 1 = \frac{2k}{k+l}$ is even if and only if $\frac{k}{k+l}$ Z.

Lemma 7.4. — *If $kl = 0$ and the principal H-bundle $P_{k,l} \to P_{k,l}/H = M_{k,l}$ is trivial, then $k + l = 0$ and $\frac{k}{k+l}$ Z.*

Proof. — If the bundle is trivial, $P_{k,l}$ is diffeomorphic to $S^3 \times M_{k,l}$. By the Künneth formula $H^4(P_{k,l}) = H^4(M_{k,l})$ which is $\mathbb{Z}_{k+l}$ if $k + l \neq 0$, and $\mathbb{Z}$ if $k + l = 0$. As was mentioned on [23, p. 349], the quotient of the principal $S^3 \times S^3$-bundle $P_{k,l}$ by the subgroup $1 \times S^3$ can be identified with P_k, the principal S^3-bundle over S^4 with Euler number k. Since $k \neq 0$, we get $H^4(P_k) = \mathbb{Z}_k$ [23, p. 346]. The Gysin sequence for the S^3-bundle $P_{k,l} \to P_k$ reads

$$\mathbb{Z}_k = H^4(P_k) \longrightarrow H^4(P_{k,l}) \longrightarrow H^1(P_k) = 0,$$

which shows that $k + l$ is a nonzero integer that divides k.

Remark 7.5. — The asymmetry in the conclusion of Lemma 7.4 is an illusion: $\frac{k}{k+l} \in \mathbb{Z}$ if and only if $\frac{l}{k+l} \in \mathbb{Z}$ because $\frac{k}{k+l} + \frac{l}{k+l} = 1$.

Proof of Theorem 1.4. — By Proposition 3.11 of [23] each $P_{k,l}$ admits a cohomogeneity one action by $S^3 \times S^3 \times S^3$ with codimension two singular orbits, and such that the action of the subgroup $G := S^3 \times S^3 \times \{1\}$ coincides with the principal bundle action. Hence by [23, Theorem E] the space $P_{k,l}$ carries a G-invariant metric $\gamma_{k,l}$ of $K \geq 0$.

Let $S^3(r)$ be the round 3-sphere of radius r on which $S^3 \times S^3$ acts as in Section 5. Let $h_{k,l,r}$ be the Riemannian submersion metrics on $M_{k,l} = P_{k,l} \times_{S^3 \times S^3} S^3$ induced by the product of $\gamma_{k,l}$ and $S^3(r)$. Then $h_{k,l,r}$ has $K \geq 0$ and scal > 0 by [21, Theorem 2.1].

An essential point is that there are infinitely many ways to represent M as $M_{k,l}$. Indeed, by assumption $M = M_{k,l}$ for some $k, l \in \mathbb{Z}$ with $k + l \neq 0$ and such that $\frac{k-l}{k+l}$ is not an odd integer. The latter is equivalent to $\frac{k}{k+l} \notin \mathbb{Z}$ by Lemma 7.3. For $i \in \mathbb{Z}$ let

$$l_i = l - 56(k + l)i \quad \text{and} \quad k_i = k + l - l_i = -l + (k + l)(56i + 1).$$

Then M_{k_i,l_i} are orientation-preserving diffeomorphic to M [11, Corollary 1.6]. By [21, Section 3.1] there is r and infinitely many values of i for which the metrics $h_{k_i,l_i,r}$ lie in different connected components of $\mathcal{M}_{K\geq 0}(M)$.

Let $g_{k,l}$ be the Riemannian submersion metric of $K \geq 0$ induced on $M_{k,l} = H \backslash P_{k,l}$ by $\gamma_{k,l}$. Proposition 5.1 implies that $g_{k,l}$ and $h_{k,l,r}$ lie in the same path-component of $\mathcal{M}_{K\geq 0}(M_{k,l})$. Thus for k_i, l_i as in the previous paragraph g_{k_i,l_i} lie in different connected components of $\mathcal{M}_{K\geq 0}(M)$.

Consider the associated vector bundle $P_{k,l} \times_H \mathbb{R}^3$ over $M_{k,l}$ where $H = S^3$ acts on $\mathbb{R}^3$ via the universal covering $S^3 \to SO(3)$. We give $P_{k,l} \times_H \mathbb{R}^3$ the Riemannian submersion metric induced by the product of $\gamma_{k,l}$ and the standard Euclidean metric. This is a complete metric of $K \geq 0$ with soul $P_{k,l} \times_H \{0\}$ which is isometric to $(M_{k,l}, g_{k,l})$.

Since $\frac{k_i}{k_i+l_i} \notin Z$, the principal S^3-bundle $P_{k_i,l_i} \to M_{k_i,l_i}$ is non-trivial by Lemma 7.4. Consider the associated 3-plane bundle $P_{k_i,l_i} \times_{S^3} R^3$ over M_{k_i,l_i} where S^3 acts on R^3 via the universal covering $S^3 \to SO(3)$. Any such vector bundle is non-trivial by Lemma 6.1, and hence by Lemma 6.2 its total space is indecomposable. Pull back the vector bundles via diffeomorphisms $M \to M_{k_i,l_i}$. The pullback bundles fall into finitely many isomorphism classes by Lemma 6.3, so after passing to a subsequence we can assume that the bundles are isomorphic, and hence share the same ten-dimensional total space, which we denote V.

In summary, V is an indecomposable manifold with infinitely many complete metrics of $K > 0$ whose souls are all equal to the zero section, and diffeomorphic to M, and such that the induced metrics on the souls lie in different connected components of $\mathcal{M}_{K>0}(M)$. Theorem 1.1 finishes the proof.

BIBLIOGRAPHY

[1] M. M. Alexandrino & R. G. Bettiol, *Lie groups and geometric aspects of isometric actions*, Springer, Cham, 2015, x+213 pages.

[2] E. Bahuaud, C. Guenther & J. Isenberg, "Convergence stability for Ricci flow", *J. Geom. Anal.* **30** (2020), no. 1, p. 310-336.

[3] I. Belegradek, "Pinching, Pontrjagin classes, and negatively curved vector bundles", *Invent. Math.* **144** (2001), no. 2, p. 353-379.

[4] ———, "Vector bundles with infinitely many souls", *Proc. Amer. Math. Soc.* **131** (2003), no. 7, p. 2217-2221.

[5] I. Belegradek, F. T. Farrell & V. Kapovitch, "Space of nonnegatively curved metrics and pseudoisotopies", *J. Differential Geom.* **105** (2017), no. 3, p. 345-374.

[6] I. Belegradek, S. Kwasik & R. Schultz, "Moduli spaces of nonnegative sectional curvature and non-unique souls", *J. Differential Geom.* **89** (2011), no. 1, p. 49-85.

[7] ———, "Codimension two souls and cancellation phenomena", *Adv. Math.* **275** (2015), p. 1-46.

[8] B. Botvinnik & P. B. Gilkey, "The eta invariant and metrics of positive scalar curvature", *Math. Ann.* **302** (1995), no. 3, p. 507-517.

[9] S. Brendle, *Ricci flow and the sphere theorem*, Graduate Studies in Mathematics, vol. 111, American Mathematical Society, Providence, RI, 2010, viii+176 pages.

[10] J. Cheeger & D. Gromoll, "On the structure of complete manifolds of nonnegative curvature", *Ann. of Math. (2)* **96** (1972), p. 413-443.

[11] D. Crowley & C. M. Escher, "A classification of S^3-bundles over S^4", *Differential Geom. Appl.* **18** (2003), no. 3, p. 363-380.

[12] A. Dessai, "On the moduli space of nonnegatively curved metrics on Milnor spheres", https://arxiv.org/abs/1712.08821, 2017.

[13] ———, "Moduli space of nonnegatively curved metrics on manifolds of dimension $4k+1$", https://arxiv.org/abs/2005.04741, 2020.

[14] A. Dessai & D. González-Álvaro, "Moduli space of metrics of nonnegative sectional or positive Ricci curvature on homotopy real projective spaces", *Trans. Amer. Math. Soc.* **374** (2021), no. 1, p. 1-33.

[15] A. Dessai, S. Klaus & W. Tuschmann, "Nonconnected moduli spaces of nonnegative sectional curvature metrics on simply connected manifolds", *Bull. Lond. Math. Soc.* **50** (2018), no. 1, p. 96-107.

[16] D. G. Ebin, *On the space of Riemannian metrics*, ProQuest LLC, Ann Arbor, MI, 1968, Thesis (Ph.D.)–Massachusetts Institute of Technology.

[17] ———, "The manifold of Riemannian metrics", in *Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968)* (Providence, R.I.), Amer. Math. Soc., 1970, p. 11-40.

[18] S. Goette, M. Kerin & K. Shankar, "Highly connected 7-manifolds and non-negative sectional curvature", *Ann. of Math. (2)* **191** (2020), no. 3, p. 829-892.

[19] D. González-Álvaro & M. Zibrowius, "Open manifolds with non-homeomorphic positively curved souls", *Math. Proc. Cambridge Philos. Soc.* **169** (2020), no. 2, p. 357-376.

[20] M. J. Goodman, "Moduli spaces of Ricci positive metrics in dimension five", https://arxiv.org/abs/2002.00333, 2020.

[21] ———, "On the moduli spaces of metrics with nonnegative sectional curvature", *Ann. Global Anal. Geom.* **57** (2020), no. 2, p. 305-320.

[22] M. Gromov & H. B. Lawson, Jr., "Positive scalar curvature and the Dirac operator on complete Riemannian manifolds", *Inst. Hautes Études Sci. Publ. Math.* (1983), no. 58, p. 83-196 (1984).

[23] K. Grove & W. Ziller, "Curvature and symmetry of Milnor spheres", *Ann. of Math. (2)* **152** (2000), no. 1, p. 331-367.

[24] C. J. Himmelberg, "Pseudo-metrizability of quotient spaces", *Fund. Math.* **63** (1968), p. 1-6.

[25] V. Kapovitch, A. Petrunin & W. Tuschmann, "Non-negative pinching, moduli spaces and bundles with infinitely many souls", *J. Di erential Geom.* **71** (2005), no. 3, p. 365-383.

[26] M. Kreck, *Di erential algebraic topology. From stratifolds to exotic spheres*, Graduate Studies in Mathematics, vol. 110, American Mathematical Society, Providence, RI, 2010, xii+218 pages.

[27] M. Kreck & S. Stolz, "Nonconnected moduli spaces of positive sectional curvature metrics", *J. Amer. Math. Soc.* **6** (1993), no. 4, p. 825-850.

[28] P. May & D. Nardin, "A fibration of classifying spaces", mathoverflow.net/questions/182618.

[29] J. Milnor, "On manifolds homeomorphic to the 7-sphere", *Ann. of Math. (2)* **64** (1956), p. 399-405.

[30] R. E. Mosher & M. C. Tangora, *Cohomology operations and applications in homotopy theory*, Harper & Row, Publishers, New York-London, 1968, x+214 pages.

[31] J. R. Munkres, *Topology*, Prentice Hall, Inc., Upper Saddle River, NJ, 2000, Second edition of [MR0464128], xvi+537 pages.

[32] S. Ottenburger, "A classification of 5-dimensional manifolds, souls of codimension two and non-di eomorphic pairs", https://arxiv.org/abs/1103.0099, 2011.

[33] I. R. Porteous, *Cli ord algebras and the classical groups*, Cambridge Studies in Advanced Mathematics, vol. 50, Cambridge University Press, Cambridge, 1995, x+295 pages.

[34] V. A. Šarafutdinov, "Convex sets in a manifold of nonnegative curvature", *Mat. Zametki* **26** (1979), no. 1, p. 129-136, 159.

[35] W. Tuschmann & M. Wiemeler, "On the topology of moduli spaces of non-negatively curved Riemannian metrics", https://arxiv.org/abs/1712.07052, 2017.

[36] W. Tuschmann & D. J. Wraith, *Moduli spaces of Riemannian metrics*, Oberwolfach Seminars, vol. 46, Birkhäuser Verlag, Basel, 2015, Second corrected printing, x+123 pages.

[37] M. Y. Wang & W. Ziller, "Einstein metrics on principal torus bundles", *J. Differential Geom.* **31** (1990), no. 1, p. 215-248.

[38] D. J. Wraith, "On the moduli space of positive Ricci curvature metrics on homotopy spheres", *Geom. Topol.* **15** (2011), no. 4, p. 1983-2015.

[39] J.-W. Yim, "Space of souls in a complete open manifold of nonnegative curvature", *J. Di erential Geom.* **32** (1990), no. 2, p. 429-455.

[40] W. Ziller, "On M. Mueter's Ph.D. Thesis on Cheeger deformations", https://arxiv.org/abs/0909.0161.

Manuscrit reçu le 26 mai 2020,
révisé le 20 novembre 2020,
accepté le 7 janvier 2021.

Igor BELEGRADEK
School of Mathematics
Georgia Institute of Technology
Atlanta, GA, USA 30332
ib@math.gatech.edu

David GONZÁLEZ-ÁLVARO
ETSI de Caminos, Canales y Puertos
Universidad Politécnica de Madrid
28040, Spain
david.gonzalez.alvaro@upm.es