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# A NOTE ON LOEWNER ENERGY, CONFORMAL RESTRICTION AND WERNER'S MEASURE ON SELF-AVOIDING LOOPS

by Yilin WANG (\*)

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**ABSTRACT.** — We establish an expression of the Loewner energy of a Jordan curve in terms of Werner's measure on simple loops of  $\text{SLE}_{8/3}$  type. The proof is based on a formula for the change of the Loewner energy under a conformal map that is reminiscent of SLE processes' conformal restriction property.

**RÉSUMÉ.** — Nous établissons une expression de l'énergie de Loewner d'une courbe de Jordan en termes de la mesure de Werner sur les boucles auto-évitantes du type  $\text{SLE}_{8/3}$ . La preuve est basée sur la variation de l'énergie de Loewner sous l'effet des transformations conformes. Cette formule rappelle la propriété de la restriction conforme de SLE.

## 1. Introduction

Loewner's idea [9] of encoding a simple curve into a real-valued driving function provides a powerful tool in the analysis of univalent functions. This idea led to the solution of the Bieberbach conjecture by De Branges [1] and has also more recently received a lot of attention since 1999 with the construction of random fractal simple curves, the SLEs, by Oded Schramm [14].

The study of the Loewner energy lies at the interface between classical Loewner theory and the SLE theory. The Loewner energy is a deterministic quantity associated to regular deterministic curves, but simultaneously reflects the large deviation structure of SLEs [16] driven by a vanishing multiple of Brownian motion. The Loewner energy is studied in [2, 4, 16, 13, 17] in various similar settings. Let us review briefly its definition for chords and then for loops as introduced by Rohde and the author [13].

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Let  $\gamma$  be a simple curve (chord) from 0 to  $\infty$  in the upper half-plane  $\mathbb{H}$ , one chooses to parametrize  $\gamma$  by the half-plane capacity of  $\gamma[0, t]$  seen from infinity, which means the conformal map  $g_t$  from  $\mathbb{H} \setminus \gamma[0, t]$  to  $\mathbb{H}$ , that is normalized near infinity by  $g_t(z) = z + o(1)$  does satisfy  $g_t(z) = z + 2t/z + o(1/z)$ . The function  $g_t$  can be extended continuously to the tip  $\gamma_t$  of the slit  $\gamma[0, t]$  which enables to define  $W(t) := g_t(\gamma_t)$ . The function  $W : \mathbb{R}_+ \rightarrow \mathbb{R}$  is called the driving function of  $\gamma$ . The Loewner energy in  $(\mathbb{H}, 0, \infty)$  of the chord  $\gamma$  is defined to be

$$I_{\mathbb{H}, 0, \infty}(\gamma) := I(W) := \frac{1}{2} \int_0^\infty W'(t)^2 dt$$

when  $W$  is absolutely continuous and is  $\infty$  otherwise.

The definition extends to a chord  $\gamma$  connecting two prime ends  $a$  and  $b$  in a simply connected domain  $D \subset \mathbb{C}$ , via a uniformizing conformal map  $\varphi : D \rightarrow \mathbb{H}$  such that  $\varphi(a) = 0$  and  $\varphi(b) = \infty$ , that is

$$I_{D, a, b}(\gamma) := I_{\mathbb{H}, 0, \infty}(\varphi(\gamma)).$$

Following [13], we define *Loewner loop energy* via a limiting procedure. Let  $\Gamma : [0, 1] \rightarrow \widehat{\mathbb{C}}$  be an oriented Jordan curve with a marked point  $\Gamma(0) = \Gamma(1) \in \Gamma$ , where  $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \simeq S^2$  is the Riemann sphere. For every  $\varepsilon > 0$ ,  $\Gamma[\varepsilon, 1]$  is a chord connecting  $\Gamma(\varepsilon)$  to  $\Gamma(1)$  in the simply connected domain  $\widehat{\mathbb{C}} \setminus \Gamma[0, \varepsilon]$ . The rooted Loewner loop energy is then defined as

$$I^L(\Gamma, \Gamma(0)) := \lim_{\varepsilon \rightarrow 0} I_{\widehat{\mathbb{C}} \setminus \Gamma[0, \varepsilon], \Gamma(\varepsilon), \Gamma(0)}(\Gamma[\varepsilon, 1]).$$

It was observed in [13] that the Loewner energy depends only on the trace of the curve (in particular, not on its root) which is a priori not obvious from the definition. To understand the presence of these symmetries, three identities of the Loewner energy are established in [17]. For the purpose of the present work, let us review the link to the determinants of Laplacians.

Let  $g$  be a Riemannian metric on  $S^2$  and  $\Gamma$  a smooth Jordan curve on  $S^2$ . We define

$$\begin{aligned} \mathcal{H}(\Gamma, g) := & \log \det'_\zeta(-\Delta_{S^2, g}) - \log \text{vol}_g(S^2) \\ & - \log \det_\zeta(-\Delta_{D_1, g}) - \log \det_\zeta(-\Delta_{D_2, g}), \end{aligned}$$

where  $D_1, D_2 \subset S^2$  are the connected components of  $S^2 \setminus \Gamma$ ,  $\Delta_{D_i, g}$  the Laplace–Beltrami operator on  $D_i$  with Dirichlet boundary condition, and  $\det'_\zeta$  (resp.  $\det_\zeta$ ) the zeta-regularized determinant for operators with non-trivial (resp. trivial) kernel. The zeta-regularization of determinants is introduced by Ray and Singer [12]. It uses an analytically continued zeta

function defined from the spectrum of the operator. Since the precise regularization procedure is not instrumental in our proofs nor in providing the intuition in the present work, we refer the interested readers to [11] for the definition and [17, Sec. 7] for more details on the functional  $\mathcal{H}$ . The relation between  $\mathcal{H}$  and  $I^L$  is the following.

**THEOREM 1.1** ([17, Prop. 7.1, Thm. 7.3]). — *If  $g = e^{2\sigma} g_0$  is conformally equivalent to the spherical metric  $g_0$  on  $S^2$ , then*

- (i)  $\mathcal{H}(\cdot, g) = \mathcal{H}(\cdot, g_0)$ ;
- (ii) any circle (denoted as  $S^1$ ) minimizes  $\mathcal{H}(\cdot, g)$  among all smooth Jordan curves;
- (iii) we have the identity

$$(1.1) \quad I^L(\Gamma, \Gamma(0)) = 12\mathcal{H}(\Gamma, g) - 12\mathcal{H}(S^1, g).$$

The right-hand side of (1.1) clearly does not depend on the parametrization of  $\Gamma$ .

It was pointed out in [8, 3] that  $-\log \det_\zeta(-\Delta_M)$  can be thought as a renormalization of the total mass of Brownian loops contained in  $M$  under the Brownian loop measure  $\mu_M^{\text{loop}}$  introduced by Lawler and Werner [7]. Roughly speaking,

$$\text{“} -\log \det_\zeta(-\Delta_M) = \int d\mu_M^{\text{loop}}. \text{”}$$

Hence the identity (1.1) can be interpreted as

$$(1.2) \quad \text{“} I^L(\Gamma) = 12\mu_{\mathbb{C}}^{\text{loop}}(\delta, \delta \cap S^1 \neq \emptyset) - 12\mu_{\mathbb{C}}^{\text{loop}}(\delta, \delta \cap \Gamma \neq \emptyset). \text{”}$$

We can also speculate this identity from the large deviations of SLE that we will comment on later. However, both terms on the right-hand side of (1.2) diverge. In this work, we make sense of the above identity for all Jordan curves using a renormalization of Werner's measure (see Section 2) on simple loops of  $\text{SLE}_{8/3}$  type which is closely related to the Brownian loop measure. The precise statement is the following (see Theorem 5.1 for the proof): Let  $f$  be a conformal map from the unit disk  $\mathbb{D}$  to a connected component of  $\widehat{\mathbb{C}} \setminus \Gamma$ . For  $0 < \varepsilon < 1$ , let  $S^{(1-\varepsilon)}$  denote the circle of radius  $1 - \varepsilon$  and center 0 and  $\Gamma^{(1-\varepsilon)} := f(S^{(1-\varepsilon)})$  the equi-potential.

**THEOREM 1.2.** — We have

$$I^L(\Gamma) = \lim_{\varepsilon \rightarrow 0} 12\mathcal{W}(S^1, S^{(1-\varepsilon)}; \widehat{\mathbb{C}}) - 12\mathcal{W}(\Gamma, \Gamma^{(1-\varepsilon)}; \widehat{\mathbb{C}}),$$

where  $\mathcal{W}(K_1, K_2; \widehat{\mathbb{C}})$  denotes the total mass of loops that intersect both  $K_1$  and  $K_2$  under Werner's measure.

As an intermediate step, we derive a variation formula for the Loewner energy under a conformal mapping (see Theorem 4.1).

**THEOREM 1.3.** — *If  $\eta$  is a Jordan curve with finite energy and  $\Gamma = f(\eta)$ , where  $f : A \rightarrow \tilde{A}$  is conformal on an open neighborhood  $A$  of  $\eta$ , then*

$$I^L(\Gamma) - I^L(\eta) = 12\mathcal{W}(\eta, A^c; \hat{\mathbb{C}}) - 12\mathcal{W}(\Gamma, \tilde{A}^c; \hat{\mathbb{C}}).$$

Let us make some loose comments in relation to SLE at the end of this introduction. Notice that the Dirichlet energy is the action functional of the standard Brownian motion  $B$ , therefore it is also the large deviation rate function of  $\sqrt{\kappa}B$  under appropriate norm as  $\kappa \rightarrow 0$ . Hence, the Loewner energy  $I(\cdot)$  is naturally the large deviation rate function of  $\text{SLE}_\kappa$  in the small  $\kappa$  limit [16], namely

$$\mathbb{P}\{\text{SLE}_\kappa \text{ loop stays close to } \Gamma\} \approx \exp(-I^L(\Gamma)/\kappa), \quad \text{as } \kappa \rightarrow 0+.$$

This relation to SLE indeed led us to Theorem 1.3 where we use an argument similar to the conformal restrictions for SLE.

On the other hand, the partition function for an SLE loop is formally given by [3]:

$$\mathbb{P}\{\text{SLE}_\kappa \text{ loop stays close to } \Gamma\} \propto \exp\left(-\frac{c}{2}\mu_{\mathbb{C}}^{\text{loop}}(\delta, \delta \cap \Gamma \neq \emptyset)\right),$$

where  $c = (3\kappa - 8)(6 - \kappa)/2\kappa \sim -24/\kappa$  is the central charge associated to  $\text{SLE}_\kappa$ . Therefore, by comparing the above two expressions, we identify  $I^L(\Gamma)$  with  $-12\mu_{\mathbb{C}}^{\text{loop}}(\delta, \delta \cap \Gamma \neq \emptyset)$  up to an additive constant as suggested by (1.2). Hence, the Loewner energy does not only quantify the decay of a certain volume of the infinitesimal neighborhood of a given curve measured by the total mass of  $\text{SLE}_{0+}$ , but also the mass of Brownian loops (whose boundary are of  $\text{SLE}_{8/3}$  type) attached to the curve as confirmed by this work's results.

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## 2. Werner's measure on self-avoiding loops

In this section, we briefly recall the definition and main features of Werner's measure on self-avoiding loops.

The *Brownian loop measure* in the complex plane  $\mathbb{C}$  and its sub-domains has been introduced by Lawler and Werner in [7] and its definition can be immediately extended to a general Riemannian surface  $M$  (with or without boundary) in the following way:

Let  $x \in M$ ,  $t > 0$ , consider the sub-probability measure  $\mathbb{W}_x^t$  on the path of the Brownian motion (diffusion of infinitesimal generator the Laplace–Beltrami operator  $\Delta_M$ ) on  $M$  started from  $x$  on the time interval  $[0, t]$ , which is killed if it hits the boundary of  $M$ . The measures  $\mathbb{W}_{x \rightarrow y}^t$  on paths from  $x$  to  $y$  are obtained from the disintegration of  $\mathbb{W}_x^t$  according to its endpoint  $y$ :

$$\mathbb{W}_x^t = \int_M \mathbb{W}_{x \rightarrow y}^t \, d\text{vol}(y).$$

Note that  $\mathbb{W}_{x \rightarrow y}^t$  has total mass  $p_t(x, y)$ , where  $p_t$  is the heat kernel with Dirichlet boundary condition. Define the *Brownian loop measure* on  $M$ :

$$\mu_M^{\text{loop}} := \int_0^\infty \frac{dt}{t} \int_M \mathbb{W}_{x \rightarrow x}^t \, d\text{vol}(x).$$

Since the starting point coincide with the endpoint, it is a measure on the set of unrooted loops by forgetting the starting point and the time-parametrization (so that we distinguish loops only by their trace).

The Brownian loop measure satisfies the following two remarkable properties

- (*Restriction property*) If  $M' \subset M$ , then  $d\mu_{M'}^{\text{loop}}(\delta) = 1_{\delta \subset M'} d\mu_M^{\text{loop}}(\delta)$ .
- (*Conformal invariance*) Let  $M_1 = (M, g)$  and  $M_2 = (M, e^{2\sigma}g)$  be two conformally equivalent Riemann surfaces, where  $\sigma \in C^\infty(M, \mathbb{R})$ .

We have

$$\mu_{M_1}^{\text{loop}} = \mu_{M_2}^{\text{loop}}.$$

Notice that the total mass (under the Brownian loop measure) of loops contained in  $\mathbb{C}$  is infinite (in fact, that for all positive  $R$ , both the mass of loops of diameter greater than  $R$  and the mass of loops of diameter smaller than  $R$  are both infinite), which can be viewed as a consequence of its scale-invariance (or from the fact that the integral of  $1/t$  diverges both at infinity and at 0). However, when  $D \subset \mathbb{C}$  is a proper subset of  $\mathbb{C}$  with non-polar boundary, and  $K_1, K_2$  are two disjoint compact subsets of  $D$ , the total mass (under the Brownian loop measure) of the set of loops that do stay in  $D$  and intersect both  $K_1$  and  $K_2$  is finite (staying in  $D$  in some sense removes most large loops, and intersecting both  $K_1$  and  $K_2$  prevents the loops for being too small). We will denote this finite mass by

$$\mathcal{B}(K_1, K_2; D) := \mu_D^{\text{loop}}(\{\delta; \delta \cap K_1 \neq \emptyset, \delta \cap K_2 \neq \emptyset\}).$$

Werner's measure on simple (self-avoiding) loops in  $\widehat{\mathbb{C}}$  defined in [18] is simply the image of  $\mu_{\mathbb{C}}^{\text{loop}}$  under that map that associates to a (Brownian) loop its outer boundary (i.e., the boundary of the unbounded connected component of its complement). As shown in [18], this measure turns out to be invariant under the map  $z \mapsto 1/z$  (and more generally under any conformal automorphism of the Riemann sphere), which in turn makes it possible to define this measure  $\mu_{W,M}^{\text{loop}}$  in any Riemann surface  $M$ , in such a way that the above restriction and conformal invariance properties still hold for this family of measures on self-avoiding loops. In fact, shown in [18] that this is the unique (up to a multiplicative constant) such family of  $\sigma$ -finite measures on self-avoiding loops satisfying both the restriction property and the conformal invariance properties. For other characterizations (via a restriction-type formula, or as a measure on SLE loops) of Werner's measure and its properties (it is supported on  $\text{SLE}_{8/3}$ -type loops which have fractal dimension  $4/3$ ), see [18]. Since we will be discussing conformal restriction properties of the Loewner energy here, it is worth stressing here that the proofs in [18] are building on the work of Lawler, Schramm, and Werner [6] on chordal conformal restriction properties.

One feature that makes Werner's measure convenient to work with on Riemann surfaces is that if we consider two disjoint compact sets  $K_1, K_2 \subset \widehat{\mathbb{C}}$ , then the total mass of loops that intersect both  $K_1$  and  $K_2$  is finite (see [10, Lem. 4]):

$$\mathcal{W}(K_1, K_2; \widehat{\mathbb{C}}) := \mu_{W, \widehat{\mathbb{C}}}^{\text{loop}}(\{\delta; \delta \cap K_1 \neq \emptyset, \delta \cap K_2 \neq \emptyset\}) < \infty.$$

This contrasts with the fact that the total mass (for the Brownian loop measure) of loops that intersect both  $K_1$  and  $K_2$  is infinite, due to the many very large Brownian loops that intersect both  $K_1$  and  $K_2$  (but the outer boundary of these large loops tends not to intersect  $K_1$  or  $K_2$ , which explains why  $\mathcal{W}(K_1, K_2; \widehat{\mathbb{C}})$  is finite). This feature was also instrumental in the proof of the conformal invariance of simple Conformal Loop Ensembles on the Riemann sphere by Kemppainen and Werner in [5].

### 3. Conformal restriction for simple chord

We first recall the variation formula of the chordal Loewner energy under conformal restriction, first appeared in [2] and [16]: Let  $K$  be a compact hull in  $\mathbb{H}$  at positive distance to 0. The simply connected domain  $H_K := \mathbb{H} \setminus K$  coincides with  $\mathbb{H}$  in the neighborhoods of 0 and  $\infty$ . Let  $\Gamma$  be a simple chord contained in  $H_K$  connecting 0 to  $\infty$  with finite Loewner energy in  $(\mathbb{H}, 0, \infty)$ .

**PROPOSITION 3.1** ([16, Prop. 4.1]). — *The energy of  $\Gamma$  in  $(\mathbb{H}, 0, \infty)$  and in  $(H_K, 0, \infty)$  differ by*

$$\begin{aligned} I_{H_K, 0, \infty}(\Gamma) - I_{\mathbb{H}, 0, \infty}(\Gamma) &= 3 \log |\psi'(0)\psi'(\infty)| + 12\mathcal{B}(\Gamma, K; \mathbb{H}) \\ &= 3 \ln |\psi'(0)\psi'(\infty)| + 12\mathcal{W}(\Gamma, K; \mathbb{H}), \end{aligned}$$

where  $\psi$  is a conformal map  $H_K \rightarrow \mathbb{H}$  fixing  $0, \infty$ .

Notice that the derivatives of  $\psi$  at boundary points  $0$  and  $\infty$  are well-defined by Schwarz reflection principle since  $H_K$  coincides with  $\mathbb{H}$  in their neighborhood. The first equality is the analogy of the conformal restriction property of SLE derived in [6]. The second equality is due to the fact that  $\mathbb{H}$  is simply connected domain with non-polar boundary and both  $K$  and  $\Gamma$  are attached to the boundary, so that the Brownian loop hits both  $K$  and  $\Gamma$  if and only if the outer-boundary hits them. For readers' convenience, we include the derivation of the first equality below.

Without loss of generality, we choose the conformal map  $\psi : H_K \rightarrow \mathbb{H}$  as in Proposition 3.1 such that  $\psi'(\infty) = 1$ . Let  $K_t$  be the image of  $K$  under the flow  $g_t$  associated to  $\Gamma$ ,  $\tilde{g}_t$  the mapping-out function of  $\psi(\Gamma[0, t])$ , and  $\psi_t = \tilde{g}_t \circ \psi \circ g_t^{-1} : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$  making the diagram commute (see figure 3.1). It suffices to show that for  $T < \infty$ ,

$$(3.1) \quad \begin{aligned} I_{\mathbb{H}, 0, \infty}(\psi(\Gamma[0, T])) - I_{\mathbb{H}, 0, \infty}(\Gamma[0, T]) \\ = 3 \ln |\psi'_T(0)| + 12\mathcal{B}(\Gamma[0, T], K; \mathbb{H}) - 3 \ln |\psi'_T(0)| \end{aligned}$$

which implies Proposition 3.1 since the last term  $3 \ln |\psi'_T(0)| \rightarrow 0$  when  $T \rightarrow \infty$  and  $I_{\mathbb{H}, 0, \infty}(\psi(\Gamma[0, T])) = I_{H_K, 0, \infty}(\Gamma[0, T])$ .

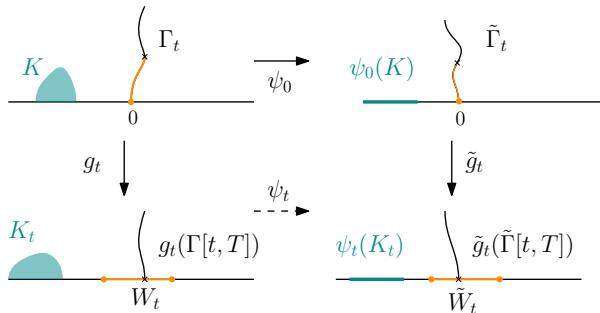


Figure 3.1. Maps in the proof of Proposition 3.1,  $\tilde{W}_t = \psi_t(W_t)$ .

*Proof.* — We write  $W_t$  for  $W(t)$  to shorten the notation. We show first

$$(3.2) \quad I_{\mathbb{H},0,\infty}(\psi(\Gamma[0,T])) = \frac{1}{2} \int_0^T \left[ \partial_t W_t - \frac{3\psi_t''(0)}{\psi_t'(0)} \right]^2 dt.$$

Notice that  $(\tilde{\Gamma}_t := \psi(\Gamma_t))_{t \geq 0}$  is not capacity-parametrized. We denote  $a(t)$  the capacity of  $\tilde{\Gamma}[0,t]$ , such that

$$\tilde{g}_t(z) = z + 2a(t)/z + o(1/z), \text{ as } z \rightarrow \infty.$$

By a scaling consideration, we have  $\partial_t a(t) = [\psi_t'(W_t)]^2$ . The family of conformal maps  $\tilde{g}_t$  satisfies the Loewner differential equation: For  $z \in \mathbb{H}$ ,

$$\partial_t \tilde{g}_t(z) = \partial_a \tilde{g}_t(z) \partial_t a(t) = \frac{2[\psi_t'(W_t)]^2}{\tilde{g}_t(z) - \tilde{W}_t}.$$

Now we compute the variation of  $\tilde{W}$ . Since  $\psi_t$  is defined by  $\tilde{g}_t \circ \psi \circ g_t^{-1}$ , we have

$$(3.3) \quad \begin{aligned} \partial_t \psi_t(z) &= \partial_t \tilde{g}_t(\psi \circ g_t^{-1}(z)) + (\tilde{g}_t \circ \psi)'(g_t^{-1}(z)) \partial_t(g_t^{-1}(z)) \\ &= \frac{2[\psi_t'(W_t)]^2}{\tilde{g}_t \circ \psi \circ g_t^{-1}(z) - \tilde{W}_t} + (\tilde{g}_t \circ \psi)'(g_t^{-1}(z)) \frac{-2(g_t^{-1})'(z)}{z - W_t} \\ &= \frac{2[\psi_t'(W_t)]^2}{\psi_t(z) - \tilde{W}_t} - \frac{2\psi_t'(z)}{z - W_t}. \end{aligned}$$

Expanding  $\psi_t$  in the neighborhood of  $W_t$  (this is possible since  $\psi_t$  is analytic by Schwarz reflection principle), we obtain

$$\partial_t \psi_t(z) = -3\psi_t''(W_t) + O(z - W_t).$$

Therefore

$$\begin{aligned} \partial_t \tilde{W}_t &= \partial_t(\psi_t(W_t)) = (\partial_t \psi_t)(W_t) + \psi_t(W_t) \partial_t W_t \\ &= \left( -3\frac{\psi_t''(W_t)}{\psi_t'(W_t)} + \partial_t W_t \right) \psi_t'(W_t). \end{aligned}$$

Notice that since we assumed that  $\Gamma$  has finite Loewner energy in  $\mathbb{H}$ , it implies that  $W$  is in  $W^{1,2}$ . In particular,  $W$  is absolutely continuous. It is not hard to see that it implies that  $\tilde{W}$  is also absolutely continuous and the above computation of  $\partial_t \tilde{W}_t$  makes sense for almost every  $t$ . The Loewner energy of  $\psi(\Gamma[0,T])$  is given by

$$\begin{aligned} \frac{1}{2} \int_0^{a(T)} \left| \partial_a \tilde{W}(t(a)) \right|^2 da &= \frac{1}{2} \int_0^T \left| \partial_t \tilde{W}(t) \right|^2 (a'(t))^{-1} dt \\ &= \frac{1}{2} \int_0^T \left[ -3\frac{\psi_t''(W_t)}{\psi_t'(W_t)} + \partial_t W_t \right]^2 dt \end{aligned}$$

as we claimed.

Now we relate the right-hand side of (3.1) with the mass of Brownian loop measure attached to both  $\Gamma$  and  $K$ . Differentiating (3.3) in  $z$  and taking  $z \rightarrow W_t$ , we obtain

$$(\partial_t \psi'_t)(W_t) = \frac{\psi''_t(W_t)^2}{2\psi'_t(W_t)} - \frac{4\psi'''_t(W_t)}{3}.$$

We have also

$$\partial_t [\ln \psi'_t(W_t)] = \frac{1}{2} \left( \frac{\psi''_t(W_t)}{\psi'_t(W_t)} \right)^2 - \frac{4}{3} \frac{\psi'''_t(W_t)}{\psi'_t(W_t)} + \frac{\psi''_t(W_t)}{\psi'_t(W_t)} \partial_t W_t.$$

Therefore

$$(3.4) \quad \begin{aligned} \frac{1}{2} \left[ \partial_t W_t - 3 \frac{\psi''_t(W_t)}{\psi'_t(W_t)} \right]^2 - \frac{1}{2} (\partial_t W_t)^2 &= \frac{9}{2} \left( \frac{\psi''_t(W_t)}{\psi'_t(W_t)} \right)^2 - 3 \frac{\psi''_t(W_t)}{\psi'_t(W_t)} \partial_t W_t \\ &= -3\partial_t [\ln \psi'_t(W_t)] - 4S\psi_t(W_t), \end{aligned}$$

where

$$S\psi_t = \frac{\psi'''_t}{\psi'_t} - \frac{3}{2} \left( \frac{\psi''_t}{\psi'_t} \right)^2$$

is the Schwarzian derivative of  $\psi_t$ . Integrating (3.4) over  $[0, T]$ , we obtain the identity (3.1) by identifying the Schwarzian derivative term using the path decomposition of the Brownian loop measure (see [6, 7])

$$-4 \int_0^T S\psi_t(W_t) dt = 12\mathcal{B}(\Gamma[0, T], K; \mathbb{H}). \quad \square$$

From Proposition 3.1, we deduce a more general relation between the Loewner energy of the same chord in two domains  $D$  and  $D'$ , whose boundary coincides in a neighborhood of the two marked boundary points. It suffices to compare  $D$  and  $D'$  to the Riemann surface  $D \sqcup D'$  identified along the connected component of  $D \cap D'$  containing  $\Gamma$ .

**COROLLARY 3.2.** — *Let  $(D, a, b)$  and  $(D', a, b)$  be two simply connected domains in  $\widehat{\mathbb{C}}$  coinciding in a neighborhood of  $a$  and  $b$ , and  $\Gamma$  a simple curve in both  $(D, a, b)$  and  $(D', a, b)$ . Then we have*

$$\begin{aligned} I_{D',a,b}(\Gamma) - I_{D,a,b}(\Gamma) &= 3 \log |\psi'(a)\psi'(b)| \\ &\quad + 12\mathcal{W}(\Gamma, D \setminus D'; D) - 12\mathcal{W}(\Gamma, D' \setminus D; D'), \end{aligned}$$

where  $\psi : D' \rightarrow D$  is a conformal map fixing  $a$  and  $b$ .

#### 4. Conformal restriction for simple loop

We prove in this section the following conformal restriction formula for the loop energy. The loop version has the advantage compared to the chordal case of no longer having the boundary terms.

**THEOREM 4.1.** — *Let  $\eta$  be a Jordan curve with finite energy and  $\Gamma = f(\eta)$ , where  $f : A \rightarrow \tilde{A}$  is conformal on an open neighborhood  $A$  of  $\eta$ . Then*

$$I^L(\Gamma) - I^L(\eta) = 12\mathcal{W}(\eta, A^c; \hat{\mathbb{C}}) - 12\mathcal{W}(\Gamma, \tilde{A}^c; \hat{\mathbb{C}}).$$

The loop terms are finite since  $A^c$  and  $\eta$  are compact in  $\hat{\mathbb{C}}$  and are disjoint.

**Remark 4.2.** — The right-hand side of the above identity remains the same if we replace  $A$  by a subset  $B$  such that  $\eta \subset B \subset A$ . In fact, since  $A \setminus B$  is at positive distance to  $\eta$ , we have  $\mathcal{W}(\eta, A \setminus B; A) < \infty$ . We then decompose the loop measure

$$\mathcal{W}(\eta, B^c; \hat{\mathbb{C}}) = \mathcal{W}(\eta, A^c; \hat{\mathbb{C}}) + \mathcal{W}(\eta, A \setminus B; A).$$

Similarly, we have

$$\mathcal{W}(\Gamma, f(B)^c; \hat{\mathbb{C}}) = \mathcal{W}(\Gamma, \tilde{A}^c; \hat{\mathbb{C}}) + \mathcal{W}(\Gamma, \tilde{A} \setminus f(B); \tilde{A}).$$

The conformal invariance of Werner's measure provides that

$$\mathcal{W}(\eta, A \setminus B; A) = \mathcal{W}(f(\eta), f(A \setminus B); f(A)) = \mathcal{W}(\Gamma, \tilde{A} \setminus f(B); \tilde{A}).$$

Hence

$$\mathcal{W}(\eta, A^c; \hat{\mathbb{C}}) - \mathcal{W}(\Gamma, \tilde{A}^c; \hat{\mathbb{C}}) = \mathcal{W}(\eta, B^c; \hat{\mathbb{C}}) - \mathcal{W}(\Gamma, f(B)^c; \hat{\mathbb{C}}).$$

Therefore, since  $A$  contains an annular neighborhood of  $\eta$ , it suffices to prove Theorem 4.1 when  $A$  is an annulus.

*Proof of Theorem 4.1.* — Since the loop energy is a limit of chordal Loewner energy, the idea of the proof is to apply the conformal restriction for chordal energies.

More precisely, let  $a, b$  be two points on the curve  $\eta$ ,  $\tilde{a} = f(a)$  and  $\tilde{b} = f(b)$ . We write  $D := \hat{\mathbb{C}} \setminus (ab)_\eta$  and  $\tilde{D} := \hat{\mathbb{C}} \setminus (\tilde{a}\tilde{b})_\Gamma$ . We take a “stick”  $T$  attached to the arc  $(ab)_\eta$  of the curve  $\eta$ , such that  $D \setminus K$  is simply connected, where  $K = A^c \cup T$  is the union of two “lollipops”. Define  $\tilde{T} := f(T) \subset \tilde{A}$  the conformal image of  $T$  and similarly  $\tilde{K} = \tilde{A}^c \cup \tilde{T}$  (see Figure 4.1).

Now we compare the chordal Loewner energy of  $(ba)_\eta$  (the complement of  $(ab)_\eta$  in the curve  $\eta$ ) in  $D$  and  $(\tilde{b}\tilde{a})_\Gamma$  in  $\tilde{D}$ . Notice that  $D$  and  $D \setminus K$  coincide in a neighborhood of both  $a$  and  $b$ . Let  $\psi$  and  $\tilde{\psi}$  be a choice of

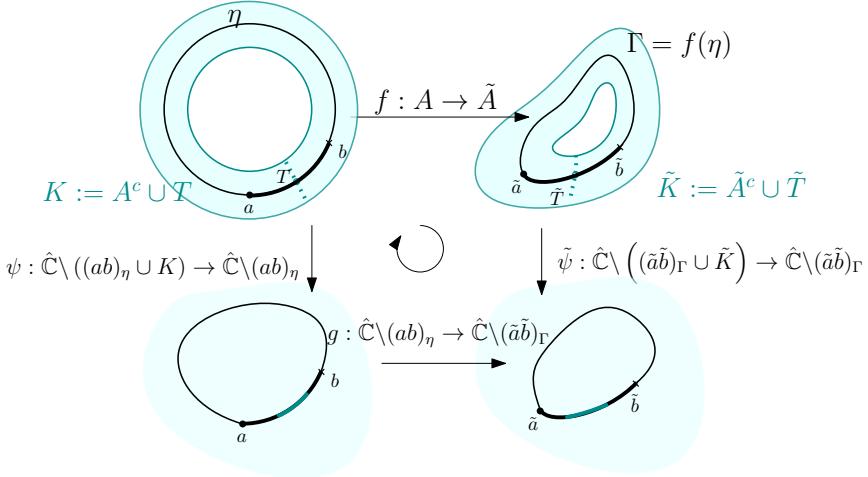


Figure 4.1. Maps in the proof of Theorem 4.1.

conformal maps as in Figure 4.1, and  $g$  factorizes the diagram. Applying Corollary 3.2 to  $(ba)_\eta$  in  $D$ , we have

$$(4.1) \quad I_{D \setminus K}((ba)_\eta) - I_D((ba)_\eta) = 3 \log |\psi'(a)\psi'(b)| + 12\mathcal{W}((ba)_\eta, K; D),$$

and similarly,

$$(4.2) \quad I_{\tilde{D} \setminus \tilde{K}}((\tilde{b}\tilde{a})_\Gamma) - I_{\tilde{D}}((\tilde{b}\tilde{a})_\Gamma) = 3 \log |\tilde{\psi}'(\tilde{a})\tilde{\psi}'(\tilde{b})| + 12\mathcal{W}((\tilde{b}\tilde{a})_\Gamma, \tilde{K}; \tilde{D}).$$

From the construction,

$$\begin{aligned} I_{D \setminus K}((ba)_\eta) &= I_D(\psi[(ba)_\eta]) = I_{\tilde{D}}(g \circ \psi[(ba)_\eta]) \\ &= I_{\tilde{D}}(\tilde{\psi}[(\tilde{b}\tilde{a})_\Gamma]) = I_{\tilde{D} \setminus \tilde{K}}((\tilde{b}\tilde{a})_\Gamma), \end{aligned}$$

where the second equality follows from the conformal invariance of the chordal Loewner energy.

We write  $H(a, b; D)$  for the Poisson excursion kernel between two boundary points  $a, b$  of the domain  $D$  (relatively to the local analytic coordinates). Choosing the same analytic coordinates near  $a, b$  in the above four pictures, then we have

$$\begin{aligned} \frac{\psi'(a)\psi'(b)}{\tilde{\psi}'(\tilde{a})\tilde{\psi}'(\tilde{b})} &= \frac{H(a, b; A \setminus (ab)_\eta)}{H(a, b; D)} \frac{H(\tilde{a}, \tilde{b}; \tilde{D})}{H(\tilde{a}, \tilde{b}; \tilde{A} \setminus (\tilde{a}\tilde{b})_\Gamma)} \\ &= \frac{H(a, b; A \setminus (ab)_\eta)}{H(\tilde{a}, \tilde{b}; \tilde{A} \setminus (\tilde{a}\tilde{b})_\Gamma)} \frac{H(\tilde{a}, \tilde{b}; \tilde{D})}{H(a, b; D)} = \frac{f'(a)f'(b)}{g'(a)g'(b)} \end{aligned}$$

which no longer depends on the stick  $T$  chosen.

Also notice that we can decompose the loop measure term as in the remark:

$$\mathcal{W}((ba)_\eta, K; D) = \mathcal{W}((ba)_\eta, A^c; D) + \mathcal{W}((ba)_\eta, T; A \setminus (ba)_\eta).$$

Since Werner's measure is conformally invariant, we have in particular

$$\mathcal{W}((ba)_\eta, T; A \setminus (ba)_\eta) = \mathcal{W}((\tilde{b}\tilde{a})_\Gamma, \tilde{T}; \tilde{A} \setminus (\tilde{b}\tilde{a})_\Gamma).$$

Taking the difference (4.1)–(4.2) combining the above observations, we get

$$(4.3) \quad I_{\tilde{D}}((\tilde{b}\tilde{a})_\Gamma) - I_D((ba)_\eta) \\ = 3 \log \left| \frac{f'(a)f'(b)}{g'(a)g'(b)} \right| + 12\mathcal{W}((ba)_\eta, A^c; D) - 12\mathcal{W}((\tilde{b}\tilde{a})_\Gamma, \tilde{A}^c; \tilde{D}).$$

We conclude the proof by taking  $b \rightarrow a$  on  $\eta$ , using the definition of loop energy

$$I_D((ba)_\eta) \xrightarrow{b \rightarrow a} I^L(\eta, a) = I^L(\eta)$$

and the fact that

$$\mathcal{W}((ba)_\eta, A^c; D) \xrightarrow{b \rightarrow a} \mathcal{W}(\eta, A^c; \widehat{\mathbb{C}}).$$

The log-derivative terms goes 0 thanks to the following lemma and concludes the proof of the theorem.  $\square$

**LEMMA 4.3.** — *With the same notations as in the proof of Theorem 4.1 (see Figure 4.1),*

$$\lim_{b \rightarrow a} \left| \frac{f'(a)f'(b)}{g'(a)g'(b)} \right| = 1.$$

*Proof.* — Without loss of generality, we may assume that  $A = \mathbb{D}$  the unit disk,  $a = 0$ , and  $\tilde{a} = f(a) = 0$ .

Let  $\psi$  be the conformal map  $\widehat{\mathbb{C}} \setminus (ab)_\eta \rightarrow \mathbb{D}$ , such that  $\psi(\infty) = 0$  and  $\psi(0) = 1$ . Similarly, let  $\varphi$  be the conformal map  $\widehat{\mathbb{C}} \setminus (\tilde{a}\tilde{b})_\Gamma \rightarrow \mathbb{D}$ , such that  $\varphi(\infty) = 0$  and  $\varphi(0) = 1$ . Define the conformal maps  $F = \varphi \circ f \circ \psi^{-1}$  and  $G = \varphi \circ f \circ \psi^{-1}$  between the blue-shaded area in Figure 4.2.

It is not hard to see that the diameter of  $\psi(\mathbb{D}^c)$  and  $\varphi(\mathbb{D}^c)$  shrinks to 0 as  $b \rightarrow a$ . Therefore

$$\frac{F'(1)F'(\psi(b))}{G'(1)G'(\psi(b))} \xrightarrow{b \rightarrow a} 1.$$

On the other hand,

$$\frac{F'(1)F'(\psi(b))}{G'(1)G'(\psi(b))} = \frac{H(a, b; A \setminus (ab)_\eta)}{H(\tilde{a}, \tilde{b}; \tilde{A} \setminus (\tilde{a}\tilde{b})_\Gamma)} \frac{H(\tilde{a}, \tilde{b}; \tilde{D})}{H(a, b; D)} = \frac{f'(a)f'(b)}{g'(a)g'(b)}$$

which concludes the proof.  $\square$

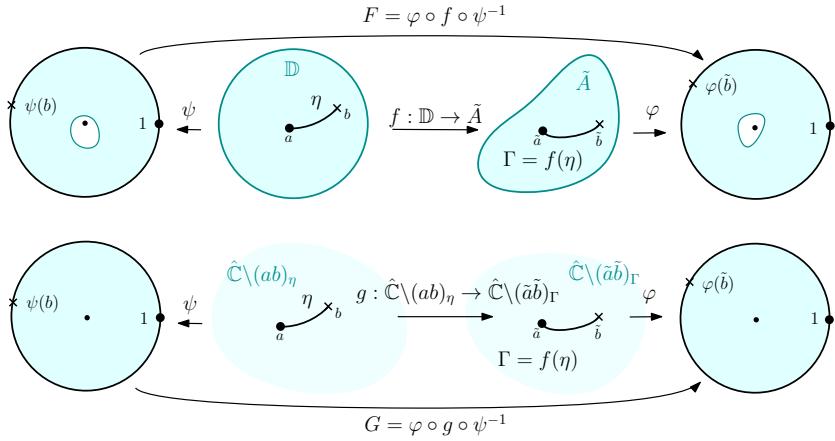


Figure 4.2. Maps in the proof of Lemma 4.3.

By taking  $\eta = S^1$ , we deduce immediately the interpretation of the Loewner energy of an analytic Jordan curve:

COROLLARY 4.4. — If  $\Gamma = f(S^1)$  is an analytic curve, then

$$I^L(\Gamma) = 12\mathcal{W}(S^1, A^c; \widehat{\mathbb{C}}) - 12\mathcal{W}(\Gamma, \tilde{A}^c; \widehat{\mathbb{C}}),$$

where  $f : A \rightarrow \tilde{A}$  maps conformally a neighborhood  $A$  of  $S^1$  to a neighborhood  $\tilde{A}$  of  $\Gamma$ .

*Proof.* — It follows immediately from Theorem 4.1 and  $I^L(S^1) = 0$ .  $\square$

## 5. Loewner energy as renormalized Werner's measure

Let  $\Gamma$  be a Jordan curve in  $\widehat{\mathbb{C}}$ ,  $D$  a connected component of  $\widehat{\mathbb{C}} \setminus \Gamma$  and  $f$  a conformal map from the unit disk  $\mathbb{D}$  to  $D$ . For  $0 < \varepsilon < 1$ , let  $S^{(1-\varepsilon)}$  denote the circle of radius  $1 - \varepsilon$  and center 0 and  $\Gamma^{(1-\varepsilon)} := f(S^{(1-\varepsilon)})$ .

THEOREM 5.1. — We have

$$I^L(\Gamma) = \lim_{\varepsilon \rightarrow 0} 12\mathcal{W}(S^1, S^{(1-\varepsilon)}; \widehat{\mathbb{C}}) - 12\mathcal{W}(\Gamma, \Gamma^{(1-\varepsilon)}; \widehat{\mathbb{C}}).$$

*Proof.* — For each  $\varepsilon$ , we apply Corollary 4.4 to the analytic curve  $\Gamma^{(1-\varepsilon)}$  with  $A := D$  which gives

$$I^L(\Gamma^{(1-\varepsilon)}) = 12\mathcal{W}(S^1, S^{(1-\varepsilon)}; \widehat{\mathbb{C}}) - 12\mathcal{W}(\Gamma, \Gamma^{(1-\varepsilon)}; \widehat{\mathbb{C}}).$$

Now it suffices to see that  $I^L(\Gamma^{(1-\varepsilon)})$  converges to  $I^L(\Gamma)$ .

If  $I^L(\Gamma) < \infty$ , it is shown in [17, Sec. 8] that  $\Gamma$  is a quasicircle of Weil–Petersson class, which is equivalent to

$$\int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 dz^2 < \infty,$$

where  $dz^2$  represents the Lebesgue measure. Let  $f_\varepsilon(z) := f((1-\varepsilon)z)$  denote the uniformizing conformal map from  $\mathbb{D}$  to the connected component of  $\widehat{\mathbb{C}} \setminus \Gamma^{(1-\varepsilon)}$ . We have for  $\varepsilon < \varepsilon_0 < 1/2$ ,

$$\begin{aligned} & \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} - \frac{f''_\varepsilon(z)}{f'_\varepsilon(z)} \right|^2 dz^2 \\ &= \int_{|z| < 1-\varepsilon_0} \left| \frac{f''(z)}{f'(z)} - \frac{f''_\varepsilon(z)}{f'_\varepsilon(z)} \right|^2 dz^2 + \int_{1-\varepsilon_0 \leq |z| < 1} \left| \frac{f''(z)}{f'(z)} - \frac{f''_\varepsilon(z)}{f'_\varepsilon(z)} \right|^2 dz^2 \\ &\leq \int_{|z| < 1-\varepsilon_0} \left| \frac{f''(z)}{f'(z)} - \frac{f''_\varepsilon(z)}{f'_\varepsilon(z)} \right|^2 dz^2 + 4 \int_{1-2\varepsilon_0 \leq |z| < 1} \left| \frac{f''(z)}{f'(z)} \right|^2 dz^2 \\ &\xrightarrow{\varepsilon \rightarrow 0} 4 \int_{1-2\varepsilon_0 \leq |z| < 1} \left| \frac{f''(z)}{f'(z)} \right|^2 dz^2, \end{aligned}$$

the convergence is due to the fact that  $f''_\varepsilon/f'_\varepsilon$  converges uniformly on compacts to  $f''/f'$ . As  $\varepsilon_0 \rightarrow 0$ , the above integral converges to 0, and we conclude that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} - \frac{f''_\varepsilon(z)}{f'_\varepsilon(z)} \right|^2 dz^2 = 0.$$

It yields that  $\Gamma^{(1-\varepsilon)}$  converges in the Weil–Petersson metric to  $\Gamma$  (see [15, Cor. A.4] or [17, Lem. H]) and therefore  $I^L(\Gamma^{(1-\varepsilon)})$  converges as well to  $I^L(\Gamma)$ .

If  $I^L(\Gamma) = \infty$ , from the lower-semicontinuity of the Loewner loop energy [13, Lem. 2.9] and the fact that  $\Gamma^{(1-\varepsilon)}$  converges uniformly (parametrized by  $S^1$  via  $f_\varepsilon$ ) to  $\Gamma$ , we have

$$\liminf_{\varepsilon \rightarrow 0} I^L(\Gamma^{(1-\varepsilon)}) \geq I^L(\Gamma) = \infty.$$

Hence  $I^L(\Gamma^{(1-\varepsilon)})$  converges to  $\infty$  as  $\varepsilon \rightarrow 0$ . □

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