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ARITHMETICITY OF THE MONODROMY OF THE WIMAN–EDGE PENCIL

by Benson FARB & Eduard LOOIJENGA (*)

Abstract. — The *Wiman–Edge pencil* is the universal family of projective, genus 6, complex-algebraic curves endowed with a faithful action of the icosahedral group. The goal of this paper is to prove that its monodromy group is commensurable with a Hilbert modular group; in particular is arithmetic. We then give a modular interpretation of this, as well as a uniformization of its base.

Résumé. — Le *pinceau de Wiman–Edge* est une famille universelle de courbes projectives non singulières de genre 6 et munie d'une action fidèle du groupe icosahédral. Le but principal de ce travail est la détermination de son groupe de monodromie. Nous montrons que ce groupe est arithmétique et commensurable avec un groupe modulaire de Hilbert. Nous donnons une interprétation modulaire de ce fait et décrivons en plus une uniformisation de la base.

1. Introduction

The *Wiman–Edge pencil* is the universal family $\mathscr{C}/\mathcal{B}$ of projective, genus 6, complex-algebraic curves admitting a faithful action of the icosahedral group $\mathfrak{A}_5$. It has 5 singular members; including a reducible curve of 10 lines with intersection pattern the Petersen graph, and a union of 5 conics with intersection pattern the complete graph on 5 vertices. Discovered by Wiman [9] and Edge [6], the Wiman–Edge pencil appears in a variety of contexts, including:

(1) $\mathscr{C}/\mathcal{B}$ is a natural pencil of curves on the quintic del Pezzo surface S. It is invariant by the full automorphism group of S, i.e., the symmetric group of degree five, $\mathfrak{S}_5$, with each $C_t \in \mathcal{B}$ being $\mathfrak{A}_5$-invariant,

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and with a unique smooth member C_0 that is $\mathfrak{S}_5$-invariant, called the *Wiman curve*.

(2) $\mathcal{B}$ is the moduli space of K3-surfaces with (a certain) faithful $\mu_2 \times \mathfrak{A}_5$ action; see Section 5.3.

(3) $\mathscr{C}/\mathcal{B}$ is the quotient of one of the two 1-parameter families of lines on a nonsingular member of the Dwork pencil of Calabi–Yau quintic threefolds by its group of automorphisms.

For a number of recent papers on the Wiman–Edge pencil, see [3, 4, 5, 10].

Given a family of varieties, it is a basic problem to compute its monodromy, to relate this to geometric properties of the family, and to use this information to uniformize (if possible) the base in terms of a period mapping, via Hodge structures. While general theory has been developed around these questions, explicit computations can be quite difficult, and accordingly there are fewer of these. The purpose of this paper is to solve these problems for the Wiman–Edge pencil $\mathscr{C}/\mathcal{B}$. We prove that the monodromy of $\mathscr{C}/\mathcal{B}$ is commensurable with a Hilbert modular group; in particular that it is arithmetic. We then give a modular interpretation of this, and use it to uniformize $\mathcal{B}$.

Restricting to the smooth locus $\mathscr{C}/\mathcal{B}^\circ$, we obtain a family of smooth, genus 6 curves, and so (choosing, say, the Wiman curve C_0 as representing the base point) a *monodromy* representation

$$\rho : \pi_1(\mathcal{B}^\circ) \to \mathrm{Aut}(H_1(C_0;\mathbb{Z})) \cong \mathrm{Sp}_{12}(\mathbb{Z}) \tag{1.1}$$

that records how the fibers C_t twist along loops in $\mathcal{B}^\circ$. The isomorphism in (1.1) comes from the fact that diffeomorphisms of C_0 preserve the algebraic intersection number on C_0, which is a symplectic pairing on $H_1(C_0;\mathbb{Z})$. But the monodromy preserves more structure, for example it commutes with the $\mathfrak{A}_5$ action on C_0. The main result of this paper is to determine (up to finite index) the *monodromy group* $\rho(\pi_1(\mathcal{B}^\circ))$. To state our main result, first note that $\mathbb{Z}[\sqrt{5}]$ is a subring of the ring of integers of $\mathbb{Q}(\sqrt{5})$ of index 2. We will see that the monodromy representation ρ factors through $\mathrm{SL}_2(\mathbb{Z}[\sqrt{5}])$. In fact we will prove the following.

Theorem 1.1 (Arithmeticity of the monodromy). — *The monodromy group of the Wiman–Edge pencil is isomorphic to a finite index subgroup of* $\mathrm{SL}_2(\mathbb{Z}[\sqrt{5}])$*; in particular it is arithmetic.*

In Section 5 we apply Theorem 1.1 to various period mappings associated to the Wiman–Edge pencil. For example, let $\mathfrak{H}$ denote the hyperbolic upper half-plane. The group $\mathrm{SL}_2(\mathbb{Z}[\sqrt{5}])$ acts properly discontinuously on $\mathfrak{H} \times \mathfrak{H}$.

The quotient of this action is a quasi-projective, complex-algebraic surface, called a *Hilbert modular surface*.

The above monodromy representation ρ is induced by an algebraic map

$$\mathscr{B}^\circ = \Gamma\backslash\mathfrak{H} \to \mathrm{SL}_2(\mathbb{Z}[\sqrt{5}])\backslash\mathfrak{H}^2,$$

called the *period map*, which assigns to a curve with faithful $\mathfrak{A}_5$-action its Jacobian with the induced $\mathfrak{A}_5$-action. In Section 5.1 we use Theorem 1.1 and its proof to study this period map.

Finally, in Section 5.3, we show that one can attach to the Wiman–Edge pencil a family of K3 surfaces with base $\mathscr{B}$. We then show how this description can be used to uniformize $\mathscr{B}$. We find:

Theorem 1.2 (Uniformization of $\mathscr{B}$). — *The smooth, projective curve $\mathscr{B}$ (which we recall, is a copy of $\mathbb{P}^1$) supports in a natural manner a family of polarized K3 surfaces endowed with a particular faithful action of $\mu_2 \times \mathfrak{A}_5$ (described explicitly in Section 5.3), and the associated period map gives $\mathscr{B}$ the structure of a Shimura curve.*

Method of proof of Theorem 1.1. — As is usual with computations of monodromies, the proof of Theorem 1.1 consists of two main steps. First, in Section 3, we find constraints on the monodromy in order to narrow its target to a copy of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{5}])$; such restrictions come not only from the necessary commutation with the $\mathfrak{A}_5$-actions on the members of the family, but also from torsion in the Picard group of C_0, as well as an involutive structure coming from the extra symmetry of the Wiman curve. The final result is to prove that in fact ρ takes its values in $\mathrm{SL}_2(\mathbb{Z}[\sqrt{5}])$.

The second step in the proof of Theorem 1.1, which we accomplish in Section 4, is to prove that the image of ρ has finite index. To do this, we first use Picard–Lefschetz theory to find the conjugacy classes of the local monodromies about each of the 5 cusps of $\mathscr{B}^\circ$. These cusps correspond to the singular members of $\mathscr{C}$: two irreducible curves, 6-noded rational curves C_{ir} and C'_{ir}; two curves C_c and C'_c, each consisting of 5 conics whose intersection graph is the complete graph on 5 vertices; and a union C_∞ of 10 lines whose intersection graph is the Petersen graph. The group $\mathfrak{S}_5$ acts on $\mathscr{C}$ with $\mathfrak{A}_5$ leaving each member of $\mathscr{C}$ invariant. This action has two $\mathfrak{S}_5$-invariant members: the singular curve C_∞ and the Wiman curve C_0. The main effort of Section 4 is to understand these degenerations and the structures they preserve. After improving "up to conjugacy" to actual elements, we are able to apply an arithmeticity criterion due to Benoist–Oh [1] to deduce Theorem 1.1.

Added in proof

Although we prove the monodromy group to be arithmetic and give a presentation of it, we were not able to characterize it by a complete set of congruence conditions. Recently Matthew Stover informed us that with the help of a computer he has been able to extract such a description from our presentation: he finds a complete set of congruence conditions which involve the primes 5 (the one observed in this paper) and 2.

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2. Some algebra of $\mathbb{Z}\mathfrak{A}_5$-modules

We found in an earlier paper ([5], Cor. 3.6) that the first homology group $H_1(C_o;\mathbb{C})$ of the Wiman curve is, as a $\mathbb{C}\mathfrak{S}_5$-module, twice an irreducible representation $E_{\mathbb{C}}$ of degree six. Since it is known that the characters of the irreducible $\mathbb{Q}\mathfrak{S}_5$-modules are those of the irreducible $\mathbb{C}\mathfrak{S}_5$-modules, it follows that $H_1(C_o;\mathbb{Q})$ is as a $\mathbb{Q}\mathfrak{S}_5$-module also twice an irreducible representation of degree six (denoted here by $E_{\mathbb{Q}}$). This implies that if we replace C_o by an arbitrary smooth member C, then it is still true that $H_1(C;\mathbb{Q})\cong E_{\mathbb{Q}}^2$ as $\mathbb{Q}\mathfrak{A}_5$-modules.

The main goal of this section and the subsequent one is to lift this to the integral level, while also taking into account the intersection pairing. In other words, we want to identify $H_1(C_o)$ as a symplectic $\mathbb{Z}\mathfrak{S}_5$-module. This will be used in Section 4 to determine the monodromy of the Wiman–Edge pencil. The present section is only concerned with the algebraic aspects of the symplectic $\mathbb{Z}\mathfrak{S}_5$-modules that appear here.

Convention. — In this section we identify $\mathfrak{A}_5$ with a triangle group defined by the group of motions of a regular icosahedron. By this we mean that we make use of the following presentation of $\mathfrak{A}_5$: a set of generators is

$$\sigma_5 = (01234),$$
$$\sigma_2 = (04)(23),$$
$$\sigma_3 = (142)$$

and a complete set of relations is given by prescribing their order (indicated by the subscript) and the identity $\sigma_2\sigma_3\sigma_5 = 1$. We make this more concrete in Remark 2.7.

2.1. Irreducible $\mathbb{Z}\mathfrak{S}_5$-modules of degree six

Recall that the reflection representation of $\mathfrak{S}_5$ is the quotient of its "natural" representation on $\mathbb{C}^5$ (given by permutation of its basis vectors) modulo the main diagonal $\mathbb{C} \hookrightarrow \mathbb{C}^5$ (which is a trivial representation). It is irreducible and so is the degree 6 representation $E_{\mathbb{C}} := \wedge^2(\mathbb{C}^5/\mathbb{C})$. It is clear that this construction is defined over $\mathbb{Q}$ (even over $\mathbb{Z}$) and so let us write $E_{\mathbb{Q}}$ for the irreducible $\mathbb{Q}\mathfrak{S}_5$-module $\wedge^2(\mathbb{Q}^5/\mathbb{Q})$. If we consider this a $\mathbb{Q}\mathfrak{A}_5$-module, it is still irreducible, but if we extend the scalars to $\mathbb{Q}(\sqrt{5})$, it will split into two absolutely irreducible representations of dimension 3. To be precise (we will recall and explain this below), the endomorphism ring $K := \operatorname{End}_{\mathbb{Q}\mathfrak{A}_5} E_{\mathbb{Q}}$ is isomorphic to $\mathbb{Q}(\sqrt{5})$ and if we tensor $E_{\mathbb{Q}}$ over K with $\mathbb{R}$ via one of the two field embeddings $\sigma, \sigma' : K \hookrightarrow \mathbb{R}$, we obtain real forms of the two complex $\mathfrak{A}_5$-representations of degree 3 that differ from each other by an outer automorphism of $\mathfrak{A}_5$ (these were denoted in [5] by I and I').

An obvious integral form of $E_{\mathbb{Q}}$ is the $\mathbb{Z}\mathfrak{S}_5$-module $\wedge^2(\mathbb{Z}^5/\mathbb{Z})$. If $\{f_i\}_{i\in\mathbb{Z}/5}$ is the standard basis of $\mathbb{Z}^5$ and f_{ij} denotes the image of $f_i \wedge f_j$ $(i \neq j)$ in $\wedge^2(\mathbb{Z}^5/\mathbb{Z})$, then the set $\{f_{ij}\}_{i\neq j}$ generates $\wedge^2(\mathbb{Z}^5/\mathbb{Z})$ and a complete set of linear relations among them is $f_{ij} = -f_{ji}$ and $\sum_j f_{ij} = 0$. Note that $\{f_{ij}\}_{i\neq j}$ is an $\mathfrak{A}_5$-orbit and consists of 10 antipodal pairs. We take as our integral form the $\mathbb{Z}\mathfrak{S}_5$-submodule E_o of $\wedge^2(\mathbb{Z}^5/\mathbb{Z})$ defined as follows. Let $\phi : \mathbb{Z}^5 \to \mathbb{Z}$ be the coordinate sum (this is a generator of $\operatorname{Hom}(\mathbb{Z}^5, \mathbb{Z})^{\mathfrak{S}_5}$) and denote by E_o the image of the $\mathbb{Z}\mathfrak{S}_5$-homomorphism

$$\delta : \wedge^3(\mathbb{Z}^5) \xrightarrow{\iota_\phi} \wedge^2(\mathbb{Z}^5) \to \wedge^2(\mathbb{Z}^5/\mathbb{Z}), \qquad \text{(Definition } E_o\text{)}$$

where ι_ϕ is the inner product with ϕ and the second map is the obvious one. In other words, E_o is generated by the vectors $\delta(f_i\wedge f_j\wedge f_k) = f_{ij}+f_{jk}+f_{kj}$. The lattice E_o comes with an $\mathfrak{A}_5$-invariant basis, given up to signs:

Lemma 2.1. — *Let $e := \sum_i f_{i,i+1} \in \wedge^2(\mathbb{Z}^5/\mathbb{Z})$. Then the $\mathfrak{A}_5$-orbit of e is the union of a basis of E_o and its antipode. In particular, the inner product*

$$s : E_o \times E_o \to \mathbb{Z}$$

for which this basis is orthonormal is $\mathfrak{A}_5$-invariant.

Proof. — We first note that e is fixed by the 5-cycle (01234) and that (14)(23) takes e to $-e$. So the $\mathfrak{A}_5$-orbit of e consists of antipodal pairs, at most $60/(2.5) = 6$ in number. Since $E_{\mathbb{Q}}$ is irreducible, it must be spanned by this orbit and so we have equality: we have 6 antipodal pairs and the

$\mathfrak{A}_5$-stabilizer of e is generated by (01234). It remains to show that this orbit spans E_o.

The identity $(f_{01}+f_{12}+f_{20})+(f_{02}+f_{23}+f_{30})+(f_{03}+f_{34}+f_{40})=e$ shows that E_o contains e and hence the $\mathbb{Z}\mathfrak{A}_5.e$-submodule generated by e. On the other hand, it is straightforward to check that e and its translates under (04)(23) and (124) sum up to $\delta(f_1\wedge f_2\wedge f_4)=f_{12}+f_{24}+f_{41}$ and since $\wedge^3(\mathbb{Z}^5)$ is generated by the $\mathfrak{A}_5$-orbit of $f_1\wedge f_2\wedge f_4$, it follows that $\mathbb{Z}\mathfrak{A}_5.e$ contains E_o. □

Remark 2.2. — The $\mathfrak{A}_5$-orbit of e and the inner product s determine each other, but this $\mathfrak{A}_5$-orbit is not a $\mathfrak{S}_5$-orbit, and so s is not $\mathfrak{S}_5$-invariant. Indeed, the $\mathfrak{S}_5$-stabilizer of e is its $\mathfrak{A}_5$-stabilizer (namely the cyclic group of order 5 generated by (01234)) and so the $\mathfrak{S}_5$-orbit of e has size 24. On the other hand, it is clear that the vectors in E_o that have unit length for s make up the $\mathfrak{A}_5$-orbit of e, and so s cannot be $\mathfrak{S}_5$-invariant.

For later use (in Subsection 3.1), we note that there is an equivariant map from E_o to a $\mathbb{F}_5\mathfrak{S}_5$-module N_5 that can be defined as follows. In terms of our basis, N_5 is the set of $\mathbb{Z}$-linear combinations of $f_0,\dots,f_4$ with coordinate sum zero, modulo the sublattice generated by the elements $(-5f_i+\sum_{j\in\mathbb{Z}/5}f_j)_{i\in\mathbb{Z}/5}$. Since $\{f_i\wedge f_j\wedge f_k\}_{0\leqslant i<j<k\leqslant 4}$ is a basis of $\wedge^3\mathbb{Z}^5$, we can define a homomorphism $\widetilde{\psi}:\wedge^3\mathbb{Z}^5\to N_5$ by assigning to $f_i\wedge f_j\wedge f_k$ the image of f_l-f_m in N_5 which is characterized by the property that (i,j,k,l,m) is an even permutation of $(0,1,2,3,4)$. This map is clearly onto and it is easy to see that it is also $\mathfrak{S}_5$-equivariant.

Lemma 2.3. — *The homomorphism $\widetilde{\psi}$ factors through a surjection $\psi: E_o\to N_5$ of $\mathbb{Z}\mathfrak{S}_5$-modules.*

Proof. — We must show that the kernel of the map $\wedge^3(\mathbb{Z}^5)\xrightarrow{\iota_\phi}\wedge^2(\mathbb{Z}^5)\to\wedge^2(\mathbb{Z}^5/\mathbb{Z})$ is contained in the kernel of $\widetilde{\psi}$. The kernel of the former is generated by the $\mathfrak{S}_5$-orbit of $f_0\wedge f_1\wedge(f_2+f_3+f_4)$ and $\widetilde{\psi}(f_0\wedge f_1\wedge(f_2+f_3+f_4))=(f_3-f_4)+(f_4-f_2)+(f_2-f_3)=0$. □

Some special orbits in E_o

We now select an element from each antipodal pair in the $\mathfrak{A}_5$-orbit of e:

$$e_0:=\sigma_2(e)=f_{41}+f_{13}+f_{32}+f_{20}+f_{04}$$
$$e_i=\sigma_5^i e_0,\ (i\in\mathbb{Z}/5).$$

The icosahedral generators act on this basis as follows:

$$\sigma_5 : e_0 \mapsto e_1 \mapsto e_2 \mapsto e_3 \mapsto e_4 \mapsto e_0 \text{ (fixes } e\text{)},$$
$$\sigma_2 : e \leftrightarrow e_0\,;\, e_1 \leftrightarrow e_4\,;\, e_2 \leftrightarrow -e_2\,;\, e_3 \leftrightarrow -e_3 \text{ (fixes } e+e_0\text{)},$$
$$\sigma_3 : e \mapsto e_0 \mapsto e_1 \mapsto e\,;\, e_2 \mapsto e_4 \mapsto -e_3 \mapsto e_2 \text{ (fixes } e+e_0+e_1\text{)}.$$

This is the matrix representation of $\mathfrak{A}_5$ that we will use. We first note that the sublattice $E \subset E_0$ consisting of integral linear combinations of our basis with even coefficient sum is $\mathfrak{A}_5$-invariant and of index 2 in E_o.

The next lemma reproduces some of the preceding in terms of this basis:

Lemma 2.4. — *We have the following special orbits and its stabilizers:*

(i) *The $\mathfrak{A}_5$-stabilizer of e is generated by σ_5. Its $\mathfrak{A}_5$-orbit generates E_o over $\mathbb{Z}$ and consists of the 6 antipodal pairs $\Delta_{ir} := \{\pm e, \pm e_0, \ldots, \pm e_4\}$.*

(ii) *The $\mathfrak{A}_5$-stabilizer of $e+e_0$ is generated by σ_2. Its $\mathfrak{A}_5$-orbit generates E over $\mathbb{Z}$ and consists of the 15 antipodal pairs $\Delta_\infty := \{\pm(e+e_i), \pm(e_i+e_{i+1}), \pm(e_{i+1}-e_{i+1})\}_i$.*

(iii) *The $\mathfrak{A}_5$-stabilizer of $e+e_0+e_1$ is generated by σ_3. Its $\mathfrak{S}_5$-orbit equals its $\mathfrak{A}_5$-orbit, generates E_o over $\mathbb{Z}$ and consists of the 10 antipodal pairs $\Delta_c := \{\pm(e+e_i+e_{i+1}), \pm(e_i-e_{i-2}-e_{i+2})\}_i$.*

The lattice E is $\mathfrak{S}_5$-invariant.

Proof. — We already established the first assertion.

Since σ_2 stabilizes $e+e_0$, its orbit has at most 30 elements. That it contains the 15 pairs listed is straightforward to verify (for example, $\sigma_5^i(e+e_0) = e+e_i$ and then note that for $i = 0,1,2,3,4$, the vector $\sigma_2(e+e_i)$ equals resp. e_0+e, e_0+e_4, e_0-e_2, e_0-e_3, e_0+e_1). This orbit is contained in E and the subset $(e_1-e_2, e_2-e_3, e_3-e_4, e_4-e_0, e_0-e, e_0+e)$ of this orbit is a basis of E.

We next consider the orbit of $e+e_0+e_1$. We compute

$$e+e_0+e_1 = f_{12}+f_{24}+f_{41}$$

and this shows that $e+e_0+e_1$ is not only stabilized by $\sigma_3 = (142)$, but also by the transposition (03). This implies that its $\mathfrak{S}_5$-orbit of $e+e_0+e_1$ equals its $\mathfrak{A}_5$-orbit. This orbit has at most 20 elements and we show that this orbit contains the 10 pairs listed. We have $\sigma_2\sigma_5(e+e_0+e_1) = \sigma_2(e+e_1+e_2) = e_0+e_4-e_2$ and the σ_5-orbits of $e+e_0+e_1$ and $e_0+e_4-e_2$ yield all the listed pairs up to sign. Since $\sigma_5^2\sigma_2\sigma_5^{-2}(e_0+e_4-e_2) = \sigma_5^2\sigma_2(e_3+e_2-e_0) = \sigma_5^2(-e_3-e_2-e) = -e_1-e_0-e$, it is also invariant under taking the opposite. This orbit generates E_o: it contains $\sigma_2(e+e_0+e_1)-(e+e_0+e_1) = e_4-e_1$ and with it then the span of the $\mathfrak{A}_5$-orbit of e_4-e_1, that is E. Since $e+e_0+e_1 \notin E$ and E has index 2 in E_o, we get all of E_o.

As to the last statement, we have seen in the proof of Lemma 2.1 that $\delta(f_1\wedge f_2\wedge f_4)=e+e_0+e_1$. Since the $\mathfrak{A}_5$-orbit of the latter generates E_o, it follows that E can also be characterized as the set of $\mathbb{Z}$-linear combinations of the $\delta(f_i\wedge f_j\wedge f_k)$ with even coefficient sum. This lattice is clearly $\mathfrak{S}_5$-invariant. □

The s-dual of E, denoted $E^\vee$, consists by definition of the $e\in E_{\mathbb{Q}}$ with $s(e,e')\in\mathbb{Z}$ for all $e'\in E_o$. It contains E_o as a sublattice of index 2 and a representative of the nontrivial coset is $\varepsilon:=\frac{1}{2}(e+\sum_{i\in\mathbb{Z}/5}e_i)$. We have $E^\vee/E\cong\mathbb{Z}/2\oplus\mathbb{Z}/2$ with the nonzero elements being represented by ε, e and $\varepsilon+e$. The action of $\mathfrak{A}_5$ on $E^\vee/E$ is trivial; this can be verified by computation, but this also follows from the fact that $\mathfrak{A}_5$ is simple so that any action of $\mathfrak{A}_5$ on a 3-element set must be trivial.

Remark 2.5. — This situation is familiar in the theory of root systems: the $\alpha\in E$ with $s(\alpha,\alpha)=2$ make up a root system of type D_6 that generates E (so E is the root lattice) and $E^\vee$ the weight lattice. The fact that $\mathfrak{A}_5$ acts trivially on $E^\vee/E$ implies that $\mathfrak{A}_5$ embeds in the Weyl group of this root system.)

2.2. Commutants of $\mathbb{Z}\mathfrak{A}_5$-modules

We shall see that the $\mathbb{Z}\mathfrak{A}_5$-modules above admit endomorphisms that are nontrivial in the sense that they are not multiples of the identity. One such element is $X\in\operatorname{End}(E^\vee)$, defined by

$$\begin{aligned}X(\varepsilon) &:=\varepsilon+e,\\ X(e_i) &:=\varepsilon-(e_{i+2}+e_{i-2}).\end{aligned}$$

Lemma 2.6. — *The endomorphism X is selfadjoint with respect to s and satisfies $X^2=X+1$. In particular, X preserves E and $E^\vee$ so that X also acts on $E^\vee/E$. We have $X(e)=\varepsilon$ and (so) X acts transitively on the set of the (3) nonzero elements of $E^\vee/E$. Moreover, X commutes with the $\mathfrak{A}_5$-action.*

Proof. — We only verify the last assertion, as checking the others is straightforward. Since σ_5 and σ_2 generate $\mathfrak{A}_5$, it suffices to check that these elements commute with X. This is obvious for σ_5. In the case of σ_2, we must verify that

$$\sigma_2: X(e)\leftrightarrow X(e_0)\,;\ X(e_1)\leftrightarrow X(e_4)\,;\ X(e_2)\leftrightarrow -X(e_2)\,;\ X(e_3)\leftrightarrow -X(e_3).$$

This is also straightforward. □

Thus E becomes a module over the ring $\mathcal{O} := \mathbb{Z}[X]/(X^2 - X - 1)$. Notice that the map $X \mapsto \frac{1}{2} + \frac{1}{2}\sqrt{5}$ identifies $K = \mathbb{Q}[X]/(X^2 - X - 1)$ with the number field $\mathbb{Q}(\sqrt{5})$ and $\mathcal{O}$ with its ring of integers. Any unit of $\mathcal{O}$ is an integral power of X up to sign. It is clear that E is a torsion free $\mathcal{O}$-module of rank 3 and $E_{\mathbb{Q}}$ a K-vector space of dimension 3. Since K has class number 1, E is in fact a free $\mathcal{O}$-module. For the same reason this is true for $E^\vee$. Since X acts transitively on the nonzero elements of $E^\vee/E$, there are no intermediate $\mathcal{O}$-submodules $E \subsetneq L \subsetneq E^\vee$. We note that the K-stabilizer of E_o in $E_{\mathbb{Q}}$ is the subring

$$\mathcal{O}_o := \mathbb{Z} + \mathbb{Z}.2X \cong \mathbb{Z} + \mathbb{Z}\sqrt{5}$$

of $\mathcal{O}$ of index 2. The group of units of $\mathcal{O}_o$ is generated by -1 and $X^3 = 2X+1$; it contains the subgroup of totally positive units of $\mathcal{O}_o$ as a subgroup of index 2, and the latter is generated by $X^6 = 8X + 5$.

Remark 2.7 (The icosahedral realizations). — When we regard $E_{\mathbb{Q}}$ as a $K\mathfrak{A}_5$-module of degree 3, it is absolutely irreducible. For example, we have two field embeddings $\sigma, \sigma' : K \hookrightarrow \mathbb{R}$ characterized by $\sigma(X) = \frac{1}{2}(1 + \sqrt{5})$ resp. $\sigma'(X) = \frac{1}{2}(1 - \sqrt{5})$ which are exchanged by the nontrivial Galois involution of K and the associated $\mathbb{R}\mathfrak{A}_5$-modules $\mathbb{R} \otimes_{K,\sigma} E_{\mathbb{Q}}$ and $\mathbb{R} \otimes_{K,\sigma'} E_{\mathbb{Q}}$ are irreducible. Since the K-action on $E_{\mathbb{Q}}$ is self-adjoint with respect to s, the inner product extends to a symmetric K-bilinear form $s_K : E_{\mathbb{Q}} \times E_{\mathbb{Q}} \to K$ and the two field embeddings define $\mathfrak{A}_5$-invariant inner products on $I_{\mathbb{R}}$ and $I'_{\mathbb{R}}$ preserved by the $\mathfrak{A}_5$-action. The convex hull of the $\mathfrak{A}_5$-orbit of the image of e in each of these is a regular icosahedron relative to this inner product, thus making explicit the realization of $\mathfrak{A}_5$ as the icosahedral group.

LEMMA 2.8. — *The commutant of the $\mathbb{Z}\mathfrak{A}_5$-module E resp. E_o is $\mathcal{O}$ resp. $\mathcal{O}_o$. Conjugation with an element of $\mathfrak{S}_5 \smallsetminus \mathfrak{A}_5$ induces in these rings the Galois involution (which sends X to $1 - X$).*

Proof. — Since E_K is absolutely irreducible as $\mathfrak{A}_5$-representation, $\mathrm{End}_{K\mathfrak{A}_5}(E_K) = K$ by Schur's lemma. So $\mathrm{End}_{\mathbb{Z}\mathfrak{A}_5}(E)$ is a subring of K. The integrality implies that this subring must be contained in $\mathcal{O}$. On the other hand, the previous lemma shows that it contains $\mathcal{O}$ so that we have equality. The proof that $\mathrm{End}_{\mathbb{Z}\mathfrak{A}_5}(E_o) = \mathcal{O}_o$ is similar.

We also know that $E_{\mathbb{C}}$ is irreducible as an $\mathbb{C}\mathfrak{S}_5$-module, and so Schur's lemma implies that the commutant of the $\mathbb{Z}\mathfrak{S}_5$-module E is just $\mathbb{Z}$. Hence conjugation with an element of $\mathfrak{S}_5 \smallsetminus \mathfrak{A}_5$ induces a nontrivial involution of the ring $\mathcal{O}$ with fixed point ring $\mathbb{Z}$. There is only such involution, namely the Galois involution of $\mathcal{O}$. $\square$

It is clear that $E^\vee$ is also $\mathcal{O}$-invariant. The definition of X shows that the action of $\mathcal{O}$ on $E^\vee/E$ factors through a faithful action of $\mathcal{O}/2\mathcal{O}$. But $2\mathcal{O}$ is a prime ideal of $\mathcal{O}$ so that the finite ring $\mathcal{O}/2\mathcal{O}$ is a field with 4 elements (hence denoted $\mathbb{F}_4$). It has the order 2 subring $\mathcal{O}_o/2\mathcal{O}$ as its prime field $\mathbb{F}_2 \subset \mathbb{F}_4$. Thus $E^\vee/E$ acquires the structure of a 1-dimensional vector space over $\mathcal{O}/2\mathcal{O} = \mathbb{F}_4$. The subgroup $E_o/E \subset E^\vee/E$ is a module over $\mathcal{O}_o/2\mathcal{O} = \mathbb{F}_2$, and so defines an $\mathbb{F}_2$-form of the $\mathbb{F}_4$-line $E^\vee/E$.

Remark 2.9. — One may check that the $\mathfrak{S}_5$-orbit of e is the union of two $\mathfrak{A}_5$-orbits, namely of e and of $X^3e = (2X+1)e$. One computes that $s(X^3e, X^3e) = 9$. Since $s(e,e) = 1$, this makes it evident that s is *not* preserved by $\mathfrak{S}_5$. Nevertheless, since $2X$ takes $E^\vee$ to E, $\mathfrak{S}_5$ will preserve each coset of E in $E^\vee$ (in other words, it will act as the identity in $E^\vee/E$).

2.3. The functors V and V_o

Let H be a finitely generated $\mathbb{Z}\mathfrak{A}_5$-module. Then the isogeny module

$$V_o(H) := \mathrm{Hom}_{\mathbb{Z}\mathfrak{A}_5}(E_o, H) \quad \text{resp.} \quad V(H) := \mathrm{Hom}_{\mathbb{Z}\mathfrak{A}_5}(E, H)$$

is in a natural manner an $\mathcal{O}_o$-module resp. $\mathcal{O}$-module (acting by precomposition). So V_o resp. V is a functor from the category of finitely generated $\mathbb{Z}\mathfrak{A}_5$-modules to the category of finitely generated $\mathcal{O}_o$-modules resp. $\mathcal{O}$-modules. Restriction defines a natural transformation $V_o \to V$. The evaluation map $V_o(H) \times E_o \to H$ factors through a homomorphism $V_o(H) \otimes_{\mathcal{O}_o} E \to H$ of $\mathbb{Z}\mathfrak{A}_5$-modules. There will be two cases of special interest to us.

First assume that H is free as a $\mathbb{Z}$-module. Then both isogeny modules are torsion free, but in the case of $V_o(H)$, it need not be free. In fact, V_o applied to the chain $E \subset E_o \subset E^\vee$ yields the chain of $\mathcal{O}_o$-modules $\mathcal{O} \xrightarrow{\times 2} \mathcal{O}_o \subset \mathcal{O}$, and $\mathcal{O}$ is not free as a $\mathcal{O}_o$-module. On the other hand, $V(H)$ is a free $\mathcal{O}$-module, as $\mathcal{O}$ has class number 1. (Indeed, if we apply V to the above chain we find that $E \subset E_o$ induces an isomorphism $V(E) = V(E_o) = \mathcal{O}$ and that $E_o \subset E^\vee$ induces $V(E_o) = \mathcal{O} \xrightarrow{\times 2} \mathcal{O} = V(E^\vee)$.)

Suppose now H is also endowed with a $\mathfrak{A}_5$-invariant symplectic form $(x,y) \in H \times H \mapsto x \cdot y \in \mathbb{Z}$. Then for every pair $v_1, v_2 \in V_o(C)$, the form

$$(x,y) \in E_o \times E_o \mapsto v_1(x) \cdot v_2(y) \in \mathbb{Z}$$

is also $\mathfrak{A}_5$-invariant. This means that there exists a unique $\mathfrak{A}_5$-equivariant endomorphism $A(v_1, v_2)$ of E_o such that

$$v_1(x) \cdot v_2(y) = s(A(v_1, v_2)(x), y)$$

for all $x, y \in E$. But any such endomorphism is in $\mathcal{O}_o$. Using the fact that X is self-adjoint with respect to s, one checks that the resulting map $A : V_o(C) \times V_o(C) \to \mathcal{O}_o$ is symplectic: it is $\mathcal{O}_o$-bilinear, antisymmetric and becomes nondegenerate over K.

The other case is of interest is when H is the $\mathbb{F}_5\mathfrak{A}_5$-module N_5 defined in Subsection 3.1. Recall that we constructed in Lemma 2.3 a surjection $\psi : E_o \to N_5$. So this is a nontrivial element of $V_o(N_5)$. As N_5 is an $\mathbb{F}_5$-vector space, so will be $V_o(N_5)$. At the same time it is an $\mathcal{O}_o$-module. Indeed, the prime 5 ramifies in $\mathcal{O}_o$, for $\mathcal{O}_o$ is additively generated by 1 and $2X-1$ and $2X-1$ is a square root of 5. So $2X-1$ generates a prime ideal $\mathcal{O}_o$ with residue field $\mathbb{F}_5$ and the $\mathcal{O}_o$-module structure on $V_o(N_5)$ factors through this residue field.

Lemma 2.10. — *The $\mathcal{O}_o$-module $V_o(N_5)$ is a vector space over $\mathbb{F}_5$ of dimension one, generated by ψ.*

Proof. — Let $\psi' \in V_o(N_5)$. Since $f_{01} + f_{12} + f_{20}$ generates E_o as a $\mathfrak{A}_5$-module, it suffices to prove that $\psi'(f_{01} + f_{12} + f_{20})$ is unique up to a scalar in $\mathbb{F}_5$. Let us represent $\psi'(f_{01} + f_{12} + f_{20})$ by $\sum_i a_i f_i$ with $a_i \in \mathbb{Z}$ such that $\sum_i a_i = 0$. If we sum over the orbit of (012), we find that $3\psi'(f_{01}+f_{12}+f_{20})$ is represented by an element of the form $a(f_0 + f_1 + f_2) + bf_3 + cf_4$ with $3a + b + c = 0$. This element is also represented by

$$\begin{aligned}a(f_0 + f_1 + f_2) + bf_3 + cf_4 - a(f_0 + f_1 + f_2 + f_3 - 4f_4)&\\ &= (b-a)f_3 + (c+4a)f_4\\ &= (b-a)(f_3 - f_4).\end{aligned}$$

This proves that $\psi'(f_{01} + f_{12} + f_{20})$ is unique up to scalar. □

3. Structures preserved by the monodromy

The Wiman–Edge pencil $\mathcal{B}^\circ$ is a family of genus 6 smooth algebraic curves. Since the action of $\pi_1(\mathcal{B}^\circ)$ on the integral first homology of the fiber preserving algebraic intersection number, we obtain a *monodromy representation* $\pi_1(\mathcal{B}^\circ) \to \mathrm{Sp}_{12}(\mathbb{Z})$. The monodromy action preserves a lot more structure, for example it intertwines the $\mathfrak{A}_5$ automorphism group of each fiber. Our goal in this section is to find and describe other structure preserved by the monodromy, thus giving strong restrictions on its image.

3.1. Torsion in the Picard group of the Wiman–Edge pencil

Recall (see for example [4, 5]) that the Wiman–Edge pencil (when regarded as lying on the del Pezzo surface S) has as its base locus the unique $\mathfrak{S}_5$-orbit Σ in S of size 20. We follow [5] and denote by $H^2_0(S) \subset \mathrm{Pic}(C)$ the orthogonal complement of the anticanonical class. This is a negative definite lattice spanned by its elements of self-intersection -2. Each such (-2)-vector can be represented by the difference of two disjoint lines and together they make up a root system of type A_4.

For the discussion in this subsection, we shall regard S as obtained from $\mathbb{P}^2$ by blowing up in 4 points in general position. Then a basis of $H^2_0(S)$ is $(\ell, \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$, where the ε_i's are the classes of the exceptional curves and ℓ is the image of a class of a line in $\mathbb{P}^2$ (see [5, §3], where the notation slightly differs from the one used there). The anti-canonical class of S is $-K_S = 3\ell - \varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$, a root basis of its orthogonal complement $H^2_0(S)$ is $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\varepsilon_0 - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ell - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)$ and the 10 line classes of S are $\{\varepsilon_i\}_{i=0}^3$ and $\{\ell - \varepsilon_i - \varepsilon_j\}_{0 \leqslant i < j \leqslant 3}$. The $\mathfrak{S}_5$-action is realized as the Weyl group of this root system and we choose an identification which makes $\mathfrak{S}_4$ correspond with the stabilizer of ℓ, i.e., the full symmetric group of $\{\varepsilon_i\}_{i=0}^3$.

A finite group associated with the root system in $H_2(S)$

The intersection pairing identifies the dual lattice $H^2_0(S)^\vee = \mathrm{Hom}(H^2_0(S), \mathbb{Z})$ with a sublattice of $H^2_0(S; \mathbb{Q})$ of vectors that have integral intersection product with vectors in $H^2_0(S)$. This is the weight lattice of the above root system; it contains $H^2_0(S)$. By definition, $H^2_0(S)$ is the orthogonal complement of K_S in the unimodular lattice $H^2(S)$. Since $K_S \cdot K_S = 5$, it follows from the basic theory of lattices that $H^2_0(S)^\vee / H^2_0(S)$ is cyclic of order 5. We have $\ell \cdot K_S = -1$ and so a generator of $H^2_0(S)^\vee / H^2_0(S)$ is representable by the orthogonal projection of $-\ell$ in $H^2_0(S); \mathbb{Q})$. This is just the fundamental weight $\varpi_4 \in H^2_0(S)^\vee$ defined by $\varpi_4 \cdot \alpha_i = -\delta_{i4}$. The orbit of ϖ_4 under the Weyl group generates $H^2_0(S)^\vee$. Since $5\varpi_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4$ is indivisible in $H^2_0(S)$,

$$N_5 := H^2_0(S) / 5 H^2_0(S)^\vee$$

is an $\mathbb{F}_5$-vector space of dimension $4 - 1 = 3$. Essentially by construction, the intersection pairing induces a nonsingular quadratic form $N_5 \times N_5 \to \mathbb{F}_5$. Note that N_5 is an $\mathbb{F}_5 \mathfrak{S}_5$-module on which $\mathfrak{S}_5$ acts orthogonally (it is an incarnation of the standard representation $\mathrm{SL}_2(\mathbb{F}_5)$ of degree 3, which indeed leaves invariant a quadratic form). In terms of the usual description of the root system A_4 (see for example [2]), $H^2_0(S)$ is identified with the corank one sublattice of $\mathbb{Z}^5$ consisting of vectors with coordinate sum zero and $5H^2_0(S)^\vee$ with the sublattice generated by the vectors with four coordinates equal to -1 and the remaining coordinate equal to 4. The $\mathfrak{S}_5$-action is the obvious one and from this we easily see that every $\mathfrak{S}_5$-invariant element of N_5 and every $\mathfrak{S}_5$-invariant $\mathbb{F}_5$-valued linear form on N_5 is zero. Since $\mathbb{F}_5\mathfrak{A}_5$ has only the trivial representation in dimension one, this implies that N_5 is irreducible as an $\mathbb{F}_5\mathfrak{A}_5$-module.

A map to the Jacobian

Let C be a member of the Wiman–Edge pencil. Every point of Σ lies in the smooth part of C so that it defines a Cartier divisor of degree one on C. We thus obtain a homomorphism $H_0(\Sigma) \to \mathrm{Pic}(C)$. This map restricts to a homomorphism $\widetilde{H}_0(\Sigma) \to \mathrm{Pic}_0(C)$, where the source is reduced homology and the target is the degree zero part of $\mathrm{Pic}(C)$ (which is also the Jacobian of C, when C is smooth). A line in S meets C in Σ in an opposite pair in Σ whose sum represents the image of this line under the restriction map $H^2(S) = \mathrm{Pic}(S) \to \mathrm{Pic}(C)$. So if $\widetilde{H}_0(\Sigma)^+ \subset \widetilde{H}_0(\Sigma)$ stands for the sublattice for which opposite pairs have the same coefficient (so that $\widetilde{H}_0(\Sigma)^+$ is the generated by differences of opposite pairs), then the homomorphism $\widetilde{H}_0(\Sigma)^+ \to \mathrm{Pic}_0(C)$ factors through $\widetilde{H}_0(\Sigma)^+ \to H^2_0(S)$. The last map will be onto because $H^2_0(S)$ is generated by differences of lines. Observe that all these maps are $\mathfrak{A}_5$-equivariant.

Proposition 3.1. — *When C is smooth, the kernel of the restriction map $H^2_0(S) \to \mathrm{Pic}_0(C)$ factors through an $\mathfrak{A}_5$-equivariant embedding of N_5 in the 5-torsion of $\mathrm{Pic}_0(C)$.*

The proof rests on the fact that if we restrict the $\mathfrak{A}_5$-action to (a copy of) Klein's Vierergruppe $\mathfrak{V}_4 \subset \mathfrak{S}_4$ (the abelian subgroup of $\mathfrak{A}_4$ whose nontrivial elements are the three elements of order 2), then the orbit space is still of genus zero:

Lemma 3.2. — *Let $C \to \mathfrak{V}_4\backslash C := \overline{C}$ form the orbit space. Then $\overline{C}$ is rational and the degree 4 cover $C \to \overline{C}$ has nine singular fibers in which we have simple ramification.*

Proof. — In order to apply Riemann–Hurwitz, we need to determine the ramification data. Since $H_1(C;\mathbb{C})$ is as a $\mathfrak{A}_5$-representation of type $2I+2I'$, the trace of every order 2 element of $\mathfrak{S}_5$ on $H_1(C;\mathbb{C})$ is -4 and so the Lefschetz number of such an element is $1-(-4)+1=6$. Hence it has as many fixed points. The stabilizer of a point of C is cyclic and so the fixed point sets of distinct order 2 elements are disjoint. We therefore find that $C\to\overline{C}$ has $6.3=18$ points of simple ramification. Since $C\to\overline{C}$ is a Galois cover of degree 4, it follows that we have $18/2=9$ singular fibres. The identity of Euler numbers $4e(\overline{C})-18=e(C)=-10$ shows that $e(\overline{C})=2$. This proves that $\overline{C}$ is rational. □

Since Pic_0 of a rational curve is trivial, the fibers of $C\to\overline{C}$ all define the same class in $\mathrm{Pic}(C)$. We will use the preceding lemma via this implication.

Proof of Proposition 3.1. — Every element of Σ has an $\mathfrak{A}_5$-stabilizer of order 3 and so is not a ramification point of the projection $C\to\overline{C}$. It follows that Σ is the union of $60/(4\cdot 3)=5$ regular fibers of $C\to\overline{C}$. We want to understand how the antipodal involution of Σ acts in these fibers.

Since we agreed that the subgroup $\mathfrak{S}_4$ of $\mathfrak{S}_5$ is realized as the permutation group of $\{\varepsilon_i\}_i$, the Vierergruppe $\mathfrak{V}_4$ becomes a subgroup of this permutation group. We see that the $\mathfrak{V}_4$-orbits in the set of line classes are the 4-element set $\{\varepsilon_i\}_i$ and the three 2-element sets of the form $\{\ell-\varepsilon_i-\varepsilon_j,$ $\ell-\varepsilon_k-\varepsilon_l\}$, where i,j,k,l are mutually distinct. The image of $\sum_i\varepsilon_i$ in $\mathrm{Pic}(C)$ is the represented by the sum of 4 antipodal pairs and since $\mathfrak{V}_4$ permutes the ε_i's transitively, it follows that they are also the sum of two regular fibers of $C\to\overline{C}$. By the same reasoning, the image of the $\mathfrak{V}_4$-invariant element $(\ell-\varepsilon_0-\varepsilon_1)+(\ell-\varepsilon_2-\varepsilon_3)$ is represented 2 antipodal pairs in a single fiber. These fibers are linearly equivalent and so the kernel of $\mathrm{Pic}(S)\to\mathrm{Pic}(C)$ contains $n:=-\sum_i\varepsilon_i+2(2\ell-\varepsilon_0-\varepsilon_1-\varepsilon_2-\varepsilon_3)=4\ell-3\sum_i\varepsilon_i$. This is a sum of roots: $n=\sum_{0\leqslant i<j<k\leqslant 3}(\ell-\varepsilon_i-\varepsilon_j-\varepsilon_k)=\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4$. It is in fact equal to $5\varpi_4$. Since n is fixed by a reflection, its $\mathfrak{A}_5$-orbit is the full $\mathfrak{S}_5$-orbit. The lattice generated by this orbit is $5H^2_0(S)^\vee$ and as the kernel must be invariant under $\mathfrak{A}_5$, it will contain this sublattice. This proves the factorization.

So restriction defines a homomorphism from $N_5\to\mathrm{Pic}(C)[5]$. This map is $\mathfrak{A}_5$-equivariant and hence its kernel is a $\mathfrak{A}_5$-invariant subspace of N_5. Since N_5 is irreducible, this kernel is either trivial or all of $H^2_0(S)/5H^2_0(S)^\vee$. Suppose the latter. This then means that for every antipodal pair in C its sum is a degree 2 divisor whose class is independent of that pair. The linear system of this class then defines a pencil of degree 2 on C for which the antipodal pairs are fibers. A degree 2 pencil on a curve of positive genus must define a hyperelliptic involution and hence is intrinsic to the curve. In our case, this hyperelliptic involution must be normalized by the A_5-action. But according to Corollary 3.6 of [5], the automorphism group of C is contained in $\mathfrak{S}_5$ (with equality for the Wiman curve) and no order two element of $\mathfrak{S}_5$ commutes with $\mathfrak{A}_5$. As this yields a contradiction, this proves that $N_5 \to \mathrm{Pic}(C)[5]$ has trivial kernel. □

Remark 3.3. — Since $\mathrm{Pic}_0(C)[5]$ can be identified with $\mathrm{Hom}(H_1(C), \mu_5)$, we have an embedding of N_5 in $\mathrm{Hom}(H_1(C), \mu_5)$. Dually, this yields an surjection $H_1(C) \twoheadrightarrow \mathrm{Hom}(N_5, \mu_5)$. If we choose a primitive 5th root of unity (which identifies μ_5 with $\mathbb{F}_5$) and use the quadratic form to identify N_5 with its dual, then we obtain an isomorphism $\mathrm{Hom}(N_5, \mu_5) \cong N_5$ and there results a surjection $H_1(C) \twoheadrightarrow N_5$. Proposition 3.1 has therefore a topological consequence: the surjection $H_1(C) \twoheadrightarrow N_5$ is locally constant as C varies in the smooth fibers of the Wiman–Edge pencil, and so this imposes restriction (a 'level structure') on the monodromy of this family. We shall see this illustrated when we compute the monodromy group in Section 3 and Section 4.

Remark 3.4. — In the above argument we divided out by the Klein Vierergruppe. If we divide out by the bigger group $\mathfrak{A}_4$ instead, then the orbit space *a fortiori* of genus zero. This intermediate orbit space defines a (non-Galois) cover of the $\mathfrak{A}_5$-orbit space $\widetilde{P} \to P$ of degree 5. Such a covering can be given by a rational function f on the smooth rational curve $\widetilde{P}$ of degree 5. The monodromy group of f is $\mathfrak{A}_5$ so that the Galois closure of the associated degree 5 extension of rational function fields $C(\widetilde{P})/\mathbb{C}(f)$ defines an $\mathfrak{A}_5$-covering. This covering will be a copy of $C \to P$.

If C is a smooth member C of the Wiman–Edge pencil then $H_1(C)$ is a symplectic $\mathbb{Z}\mathfrak{A}_5$-module, and we then abbreviate $V_o(H_1(C))$ by $V_o(C)$. Recall that in Remark 3.3 we found (after identifying μ_5 with $\mathbb{F}_5$) a natural surjection $H_1(C) \to N_5$ of $\mathbb{Z}\mathfrak{A}_5$-modules. The following is then immediate from Lemma 2.10:

Corollary 3.5. — *There is a natural surjection $V_o(C) \to \mathbb{F}_5$ that is locally constant when C varies in the smooth members of the Wiman–Edge pencil.*

3.2. The homology of the Wiman curve as a symplectic $\mathbb{Z}\mathfrak{S}_5$-module

The goal of this subsection is to determine $H_1(C_o)$ as a symplectic $\mathbb{Z}\mathfrak{S}_5$-module. This information will help us to determine the global monodromy group of the Wiman–Edge pencil.

Recall that the $\mathfrak{S}_5$-action on the Wiman curve C_o makes it a $\mathfrak{S}_5$-orbifold cover of an orbifold P_o of type $(0; 6, 4, 2)$ and that the restriction of the action to $\mathfrak{A}_5$ defines an intermediate orbifold P of type $(0; 3, 2, 2, 2)$ such that $P \to P_o$ is of degree 2 and ramifies over the orbifold points of orders 6 and 4. In order to identify $H_1(C_o)$ as a symplectic $\mathbb{Z}\mathfrak{S}_5$-module, we take a closer look at this situation.

Let w be the affine coordinate on P_o such that the orbifold points of order 6, 4 and 2 are given by $w = \infty$, $w = 0$ and $w = 1$ respectively (this makes a P_o as a projective line defined over $\mathbb{R}$ (even over $\mathbb{Q}$). Then P is also defined over $\mathbb{R}$: it admits an affine coordinate z for which $w = z^2$, so that the orbifold point of order 3 is given by $z = \infty$, and the orbifold points of order 2 are $z = 0, 1, -1$. Note that z is unique up to sign.

The preimage of the real projective line (that is, where w is real) in C_o defines a $\mathfrak{S}_5$-invariant triangulation of C_o; if we endow C_o with its hyperbolic structure (i.e., the unique metric of constant curvature -1 inducing the given conformal structure), then this is in fact a hyperbolic triangulation. Let $\mathbb{K} \subset C_o$ be a 2-simplex of this triangulation which maps onto the upper half plane of P_o. We denote the vertices of $\mathbb{K}$ by p_6, p_4, p_2 according to the order of their stabilizer. The stabilizer of such a point is cyclic and the orientation of C_o singles out a natural generator τ_j (counter clockwise rotation over $2\pi/j$ around p_j). It is elementary to see that the cycle type of these generators is $(3, 2)$ for τ_6, (4) for τ_4 (so both are odd) and $(2, 2)$ for τ_2 (so τ_2 is even) and that $\tau_6\tau_4\tau_2 = 1$; see [7, §2.3]. In fact, Theorem 2.1 of [7] implies that any ordered triple (τ_6, τ_4, τ_2) of generators $\mathfrak{S}_5$ whose orders are as their subscript and satisfy $\tau_6\tau_4\tau_2 = 1$, differ from the triple above by an inner automorphism. So any such triple comes from some choice of $\mathbb{K}$. We shall exploit this below.

Let $\mathbb{K}^* \subset C_o$ be the geodesic 2-simplex adjacent to $\mathbb{K}$ that has in common with $\mathbb{K}$ the edge p_4p_6. Then $\mathbb{K} \cup \mathbb{K}^*$ is a fundamental domain of the $\mathfrak{S}_5$-action on C_o. We denote the vertex of $\mathbb{K}^*$ distinct from p_4 and p_6 by p_2'. So $p_2' = \tau_4^{-1}p_2$ and its $\mathfrak{S}_5$-stabilizer is generated by $\tau_2' := \tau_4^{-1}\tau_2\tau_4$. Every edge of $\mathbb{K}$ or $\mathbb{K}^*$ lies on a closed geodesic that lies over an interval in $P_o(\mathbb{R})$.

We write α resp. α' for the complete geodesic in C_o that contains the geodesic segment $[p_4, p_2]$ resp. $[p_4, p_2']$. Both map to the segment $[0, 1]$ in

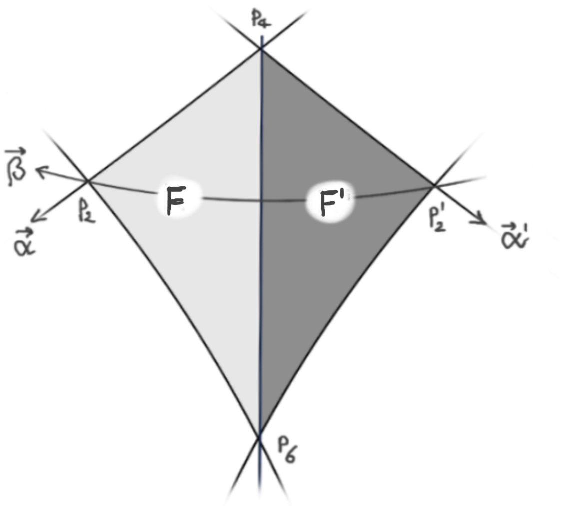

Figure 3.1. The closed oriented geodesics $\vec{\alpha}$, $\vec{\alpha}'$ and $\vec{\beta}$ on the Wiman curve.

$P_o(\mathbb{R})$, but their images in P are distinct and consist of $[0,1]$ resp. $[0,-1]$. It is clear that both τ_4^2 and τ_2 leave α invariant and act in α as a reflection. They generate in α a reflection group with $[p_4,p_2]$ as fundamental domain. Since $[p_4,p_2]$ maps injectively to the $\mathfrak{S}_5$-orbit space, it follows that this subgroup of $\mathfrak{S}_5$ is in fact the $\mathfrak{S}_5$-stabilizer of α. Both generators are even permutations, and so this stabilizer is in fact a subgroup of $\mathfrak{A}_5$. The product $\tau_4^2\tau_2$ is easily shown to be of order 3 (otherwise, see our specific choice for the τ_i's below), and so this stabilizer is a dihedral reflection group of order 6. It follows that α has $120/6=20$ $\mathfrak{S}_5$-translates and $60/6=10$ $\mathfrak{A}_5$-translates. Since α' is a translate of α under an odd permutation (namely τ_4), it cannot be a $\mathfrak{A}_5$-translate. Note that if $\vec{\alpha}$ stands for α with the orientation defined by $[p_4,p_2]$, then the stabilizer of $\vec{\alpha}$ is the cyclic group generated by $\tau_4^2\tau_2$ and hence its $\mathfrak{A}_5$-obit consists of 10 oppositely oriented pairs.

We will also be interested in the geodesic β on C_o that contains the geodesic segment $[p_2',p_2]$. This geodesic is also closed; it maps in P_o to the unit circle $|w|=1$ and hence (upon perhaps replacing z by $-z$) its image in P will be the semicircle $|z|=1$, $\Re(z)\geqslant 0$. A similar argument shows that the stabilizer of β is the reflection group generated by the even permutations τ_2 and τ_2'. The product $\tau_2\tau_2'$ has order 5 and hence the stabilizer of β

is a dihedral subgroup $\mathfrak{A}_5$ of order 10, whereas the stabilizer of $\vec{\beta}$ (the orientation being given by $[p_2', p_2]$) is generated by $\tau_2\tau_2'$. It follows that β has $120/10 = 12$ $\mathfrak{S}_5$-translates and $60/10 = 6$ $\mathfrak{A}_5$-translates. A $\mathfrak{S}_5$-translate which is not an $\mathfrak{A}_5$-translate is for instance $\beta' := \tau_4(\beta)$. Its image in P will be the semicircle $|z| = 1$, $\Re(z) \leqslant 0$.

Since the $\mathfrak{A}_5$-orbit of α resp. β is the preimage of an arc in P that connects two orbifold points of order 2, it must consist of resp. 10, 6 closed geodesics *that are pairwise disjoint*. As shown in [7, §2], these $\mathfrak{A}_5$-orbits make up a configuration of K_5-type resp. dodecahedral type. The same is true for the $\mathfrak{A}_5$-orbits of α' and β'. Recall that at the beginning of Section 2 we specified the generators $\sigma_5 = (01234)$, $\sigma_3 = (142)$ and $\sigma_2 = (04)(23)$ for $\mathfrak{A}_5$.

Lemma 3.6. — *We can choose $\mathbb{K}$ such that the associated triple (τ_6, τ_4, τ_2) in $\mathfrak{S}_5$ has the property that σ_3 resp. σ_5 generates the stabilizer of $\vec{\alpha}$ resp. $\vec{\beta}$ and $\sigma_5^{-1}\sigma_3\sigma_5$ stabilizes $\vec{\alpha}' := \tau_4(\vec{\alpha})$.*

Proof. — We take

$$\tau_6 = (012)(34), \quad \tau_4 = (0432), \quad \tau_2 = (03)(12).$$

Then $\tau_6\tau_4\tau_2 = 1$. Since $\tau_4^2 = (03)(24)$, α is stabilized by $\tau_4^2\tau_2 = (03)(24)(03)(12) = (24)(12) = (142) = \sigma_3$.

Furthermore, $\tau_2' = \tau_4\tau_2\tau_4^{-1} = (0432)(03)(12)(0234) = (01)(24)$ and β is stabilized by $\tau_2\tau_2' = (03)(12)(01)(24) = (02413) = \sigma_5^2$ and hence also by σ_5.

Finally, α' is stabilized by $\tau_2'\tau_4^2 = (01)(24)(03)(24) = (01)(03) = (031)$. But we also have $\sigma_5^{-1}\sigma_3\sigma_5 = (04321)(142)(01234) = (031)$. ☐

From now on we assume that $\mathbb{K}$ and (τ_6, τ_4, τ_2) are as in Lemma 3.6.

Lemma 3.7. — *Every $\mathfrak{A}_5$-translate of α' meets α transversally in at most one point. Similarly, exactly three $\mathfrak{A}_5$-translates of β meet α resp. α', and they do so simply in at most one point.*

Proof. — The $\mathfrak{A}_5$-translates of α' meet the fundamental segment $[p_4, p_2]$ of α in p_4 only and through that point passes just one member, namely $\alpha' = \tau_4\alpha$. It then follows that the $\mathfrak{S}_5$-translates of α distinct from α meeting α is the collection $\{\sigma_3^i\alpha'\}_i$. These are pairwise distinct, proving the first assertion.

The property regarding β is proved in a similar fashion. The cyclic group generated by σ_5 resp. σ_3 is the $\mathfrak{A}_5$-stabilizer of $\vec{\alpha}$ resp. $\vec{\beta}$. Any $\mathfrak{A}_5$-translate of β which intersects α is of the form $\sigma_3^i\beta$ for some $i \in \mathbb{Z}/3$. The $\mathfrak{A}_5$-stabilizer of $\sigma_3^i\vec{\beta}$ is generated by $\sigma_3^i\sigma_5\sigma_3^{-i}$. We have $\sigma_3\sigma_5\sigma_3^{-1} = (142)(01234)(124) = (04132)$ and $\sigma_3^{-1}\sigma_5\sigma_3 = (124)(01234)(142) = (02431)$ and neither is a power of $\sigma_5 = (01234)$. So these stabilizers are pairwise distinct. Hence so are the $\{\sigma_3^i\vec{\beta}\}_{i\in\mathbb{Z}/3}$, so that β meets α in at most one point.

Changing the orientation of C_o has the effect of replacing τ_4 by its inverse, and this exchanges $\mathbb{K}$ and $\mathbb{K}^*$, α and α', but preserves β. So β meets α' in at most one point. □

Note that $\vec{\alpha}\cdot\vec{\alpha}' = 1$, $\vec{\beta}\cdot\vec{\alpha} = 1$ and $\vec{\beta}\cdot\vec{\alpha}' = 1$. They define classes in $H_1(C_o)$ which we continue to denote by the same symbol. We write $\Delta(\alpha)$ for the $\mathfrak{A}_5$-orbit of $\vec{\alpha}$ in $H_1(C_o)$. This is a set of 10 antipodal pairs. We define $\Delta(\alpha')$ and $\Delta(\beta)$ likewise: these are sets of 10 resp. 6 antipodal pairs.

The following proposition gives us the structure of $H_1(C_o)$ as a symplectic $\mathfrak{S}_5$-module that we need in order to determine the monodromy group. The group $\mathfrak{S}_5$ acts in both E_o and $H_1(C_o)$, but elements of $V_o(C_o) = \mathrm{Hom}_{\mathbb{Z}A_5}(E_o, H_1(C_o))$ will rarely be $\mathfrak{S}_5$-equivariant. This gives therefore rise to an anti-involution ι in $V_o(C_o)$: for $\tau\in\mathfrak{S}_5$, the element $\iota(v) := \tau v\tau^{-1}$ is also in $V_o(C_o)$ and only depends on the image of τ in $\mathfrak{S}_5/\mathfrak{A}_5 \cong \mathbb{Z}/2$. The resulting involution is anti-linear with respect to the Galois involution in $\mathcal{O}_o$: for $\lambda\in\mathcal{O}_o$ and $v\in V_o(C_o)$, we have $\iota(\lambda v) = \lambda'\iota(v)$.

Proposition 3.8. — *There exists a basis (v, v') of $V_o(C_o)$ with the following properties:*

1. *The map $E_o^2 \to H_1(C_o)$ given by $(a_1, a_2)\mapsto v(a_1) + X^3v'(a_2)$ is an isomorphism of $\mathbb{Z}\mathfrak{S}_5$-modules which maps each summand onto a Lagrangian submodule of $H_1(C_o)$, and is symplectic in the sense that $A(v, X^3v') = 1$.*
2. *The anti-involution ι in $V_o(C_o)$ takes (v, v') to $\pm(v', v)$ (we leave the sign as an unknown here).*
3. *We have $v(\Delta_c) = \Delta(\alpha)$, $v'(\Delta_c) = \Delta(\alpha')$ and $(v - v')(X^3\Delta_{ir}) = \Delta(\beta)$.*

The proof of Proposition 3.8 will use the following Lemma.

Lemma 3.9. — *Suppose that $\Delta\subset E_{\mathbb{Q}}$ consists of* 10 *(resp. 15) antipodal pairs and has the property that s takes on $\Delta\times\Delta_c$ values in* $\{-1, 0, 1\}$. *In the second case, assume also that at most three antipodal pairs of Δ are not perpendicular to a member of Δ_c. Then Δ equals $X^{-3}\Delta_c$ (resp. Δ_{ir} or $X^{-3}\Delta_{ir}$).*

Proof. — Since Δ_c generates E_o and s is unimodular on E_o, our assumption implies that $\Delta\subset E_o$. The fact that Δ_c resp. Δ_{ir} generates E_o also implies that Δ is the image of Δ_c resp. Δ_{ir} under a $\mathfrak{A}_5$-equivariant homomorphism $E_o\to E_o$, so is given by a scalar $\lambda\in\mathcal{O}_o$. This scalar is unique up to sign, for any element of $\mathcal{O}_o^\times$ which leaves Δ invariant will be of finite order and hence equal to ±1.

A straightforward computation shows that $X^{-3}=2X-3$ sends the vector $e+e_0+e_1$ to $e_2-e_3+e_4$. Noting that $\{1,2X-3\}$ is a $\mathbb{Z}$-basis of $\mathcal{O}_o$, we write λ out on this basis: $\lambda=p+qX^{-3}=p+q(2X-3)$ with $p,q\in\mathbb{Z}$.

Assume now that $\Delta=\lambda\Delta_c$. We have $e+e_1+e_2\in\Delta_c$ and so by our assumption

$$\begin{aligned}
&s(\lambda(e+e_0+e_1),e+e_0+e_1)\\
&\qquad\qquad=s(p(e+e_0+e_1)+q(e_2-e_3+e_4),e+e_0+e_1)=3p,\\
&s(\lambda(e+e_0+e_1),e+e_1+e_2)\\
&\qquad\qquad=s(p(e+e_0+e_1)+q(e_2-e_3+e_4),e+e_1+e_2)=2p+q
\end{aligned}$$

both lie in $\{-1,0,1\}$. It follows that $p=0$ and $q=\pm1$, so that $\lambda=\pm X^{-3}$.

If $\Delta=\lambda\Delta_{ir}$, then $\lambda(e+e_0+e_1)=p(e+e_0+e_1)+q(e_2-e_3+e_4)$ has s-inner product of absolute value $\leqslant 1$ with the elements of Δ_c. This means that $|p|\leqslant 1$ and $|q|\leqslant 1$. It remains to show that (p,q) is either $(\pm1,0)$ or $(0,\pm1)$. If $(p,q)\neq(0,0)$, then for every $i\in\mathbb{Z}/5$,

$$\begin{aligned}
s(\lambda e_i,e+e_0+e_1)&=s(e_i,\lambda(e+e_0+e_1))\\
&=s(e_i,p(e_2-e_3+e_4)+q(e+e_0+e_1)\neq 0,
\end{aligned}$$

so that at least 5 antipodal pairs in Δ would be not orthogonal to $e+e_0+e_1$. This we excluded. $\square$

Proof of Proposition 3.8. — Recall that $e+e_0+e_1$ is stabilized by σ_3 and that its $\mathfrak{A}_5$-orbit generates E_o. Since $H_1(C_o;\mathbb{Q})$ is isotypical as a $\mathbb{Q}\mathfrak{A}_5$-module (it is isomorphic to two copies of $E_{\mathbb{Q}}$), it follows that there exists a $v\in V_o(C_o)_{\mathbb{Q}}=\mathrm{Hom}_{\mathbb{Z}\mathfrak{A}_5}(E_o,H_1(C_o;\mathbb{Q}))$ such that $v(e+e_0+e_1)=\vec{\alpha}$. Then v will take its values in $H_1(C_o)$ and so will lie in $V_o(C_o)$. By Lemma 3.6, the vector $\vec{\alpha}'\in H_1(C_o)$ is stabilized by $\sigma_5^{-1}\sigma_3\sigma_5$. Since $\sigma_5^{-1}\sigma_3\sigma_5$ stabilizes $\sigma_5^{-1}(e+e_0+e_1)$, it follows that there exists a $v'\in V_o(C_o)$ such that $v'\sigma_5^{-1}(e+e_0+e_1)=\vec{\alpha}'$.

Consider the pairing $(x,y)\in E_o\times E_o\mapsto v(x)\cdot v'(y)$. This pairing is $\mathfrak{A}_5$-invariant. We saw in Lemma 3.7 that it takes on $\Delta_c\times\Delta_c$ values only in $\{-1,0,1\}$. It then follows from Lemma 3.9 that $v(x)\cdot v'(y)=\pm s(x,X^{-3}y)$ when $x,y\in\Delta_c$. In order to determine the sign, we note that $v(e+e_0+e_1)\cdot v'\sigma_5^{-1}(e+e_0+e_1)=\vec{\alpha}\cdot\vec{\alpha}'=1$ and

$$\begin{aligned}
s(e+e_0+e_1,X^{-3}\sigma_5^{-1}(e+e_0+e_1))&=s(X^{-3}(e+e_0+e_1),\sigma_5^{-1}(e+e_0+e_1))\\
&=s(e_2-e_3+e_4,e+e_4+e_0)=1
\end{aligned}$$

This shows that $v(x)\cdot v'(y)=s(x,X^{-3}y)$. Equivalently: $A(v,X^3v')=1$. So (v,X^3v') is a symplectic basis of $V_o(C_o)$ in the sense that the $\mathfrak{A}_5$-equivariant map $E_o\oplus E_o\to H(C_o;\mathbb{Z})$ defined by (v,X^3v') pulls back the intersection pairing on $H(C_o;\mathbb{Z})$ to the symplectic pairing on $E_o \oplus E_o$ defined by s. Since the latter is unimodular (since s is), this also implies that the map $E_o \oplus E_o \to H(C_o;\mathbb{Z})$ is an isomorphism of symplectic modules.

According to Lemma 2.4, Δ_c is $\mathfrak{S}_5$-invariant. On the other hand, $\tau_4 \in \mathfrak{S}_5 \smallsetminus \mathfrak{A}_5$ takes α to α', and so it follows that $\iota(v) = \tau_4 v \tau_4^{-1}$ is an element of $V_o(C_o)$ that takes Δ_c to $\Delta(\alpha')$. But v' also has this property. It is the only element of $V_o(C_o)$ with this property up to sign, for any two elements of $V_o(C_o)$ for which the images of Δ_c coincide will differ by a factor in $\mathcal{O}_o^\times$ which is of finite order, in other words, will differ by a sign. As we are only interested in the effect of conjugation with ι, this sign is unimportant for us and we leave it as an unknown: we have $\iota(v) = \pm v'$. The map ι is an antilinear involution of $V_o(C_o)$ and so $v = \iota^2(v) = \iota(\pm v')$, which shows that $\iota(v') = \pm v$.

Finally, since the $\mathfrak{A}_5$-stabilizers of e and $\vec{\beta}$ are generated by σ_5, there exists a $u \in V_o(C_o)$ such that $u(e) = \vec{\beta}$. The set $\Delta(\beta)$ consists of 6 antipodal pairs in $H_1(C_o)$. Lemma 3.7 tells us that the hypotheses of Lemma 3.9 are fulfilled (second case): $(x, y) \in \Delta_{ir} \times \Delta_c \mapsto u(x) \cdot v(y)$ takes its values in $\{-1, 0, 1\}$ with for a given $y \in \Delta_c$, a nonzero value occurring for at most three antipodal pairs in Δ_{ir}. It then follows that either $u(x) \cdot v(y) = \pm s(x, y)$ or $u(x) \cdot v(y) = \pm s(x, X^{-3}y)$. We have $u(e) \cdot v(e + e_0 + e_1) = \vec{\beta} \cdot \vec{\alpha} = 1$, $s(e, e + e_0 + e_1) = 1$ and $s(e, X^{-3}(e + e_0 + e_1)) = s(e, e_2 - e_3 + e_4) = 0$. It follows that $u(x) \cdot v(y) = s(x, y)$, in other words, $A(u, v) = 1$.

The same argument works if we replace v by v': $u(x) \cdot v'(y) = \pm s(x, y)$ or $u(x) \cdot v'(y) = \pm s(x, X^{-3}y)$. Since we have $u(e) \cdot v'(e + e_4 + e_0) = \vec{\beta} \cdot \vec{\alpha}' = 1$, $s(e, \sigma_5^{-1}(e + e_0 + e_1)) = s(\sigma_5(e), e + e_0 + e_1) = 1$ and

$$\begin{aligned} s(e, X^{-3}\sigma_5^{-1}(e + e_0 + e_1)) &= s(e, \sigma_5^{-1}X^{-3}(e + e_0 + e_1)) \\ &= s(\sigma_5(e), e_2 - e_3 + e_4) = s(e, e_2 - e_3 + e_4) = 0, \end{aligned}$$

it follows that $u(x) \cdot v'(y) = s(x, y)$, so that $A(u, v') = 1$. Since $A(v, v') = X^{-3}$, this proves that $u = X^3(v - v')$ and so $\Delta(\beta) = (v - v')(X^3\Delta_{ir})$. $\square$

Remark 3.10. — The second property implies that if an endomorphism T of $V_o(C_o)$ has on the basis (v, v') the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $\iota T \iota$ has the matrix $\left(\begin{smallmatrix} d' & c' \\ b' & a' \end{smallmatrix}\right)$.

Remark 3.11. — We can realize $\Delta(\alpha)$, $\Delta(\beta)$ and their ι-transforms as sets of vanishing cycles as follows. Consider in P the union of the four arcs $\alpha, \alpha', \beta, \beta'$. So this is the union of the unit circle and its center line $[-1, 1]$. It contains all the order 2-orbifold points. By moving these points we deform C_o as a $\mathfrak{A}_5$-curve. In the present case, each of the four arcs determines a simple way to do this and gives a path in $\mathcal{B}$ from c_o to one of the points $c_c, c_c', c_{ir}, c_{ir}'$. For the arc α, we move the central point 0 along the ray $[0,1]$ to its end point 1 and for the arc β we move 1 along the semi-circle to its end point -1 (we fix the other orbifold points) and we do the obvious analogue for α' and β'. The arcs γ_α, γ_β, $\gamma_{\alpha'}$, $\gamma_{\beta'}$ in $\mathcal{B}$ thus obtained have the property that they do not meet away from c_o and avoid discriminant points except at the end point. We assume the labeling such that the end points are c_c, c_c', c_{ir} and c_{ir}' respectively. Notice that ι exchanges the items of $\gamma_\alpha, \gamma_{\alpha'}$ and $\gamma_\beta, \gamma_{\beta'}$

To the arc γ_α from c_o to c_c there is associated element $[\gamma_\alpha] \in \pi_i(\mathcal{B}^\circ, c_o)$ represented by a positive simple loop based at c_o around the end point c_c of γ_α and similarly for the other arcs. The four elements $[\gamma_\alpha], [\gamma_\beta], [\gamma_{\alpha'}], [\gamma_{\beta'}]$ generate $\pi_i(\mathcal{B}^\circ, c_o)$ freely and $[\gamma_\alpha][\gamma_\beta][\gamma_{\alpha'}][\gamma_{\beta'}]$ (read from right to left) represents a negative simple loop around c_∞.

3.3. The local system of isogeny modules of the Wiman–Edge pencil

The $V_o(C_t)$ define a local system $\mathbb{V}_o$ of symplectic $\mathcal{O}_o$-modules of rank 2 over $\mathcal{B}^\circ$. The involution ι of $\mathcal{B}^\circ$ (that is given by precomposition with a non-inner automorphism of $\mathfrak{A}_5$) lifts to an isomorphism between the pull-back $\iota^*\mathbb{V}_o$ and the twist of $\mathbb{V}_o$ as a $\mathcal{O}_o$-module by the Galois involution. In other words, the involution ι lifts in an anti-linear manner to $\mathbb{V}_o$. As proved in Theorem 1.1 (see also Remark 2.5) of [7], there is an identification $\mathcal{B}^\circ = \Gamma\backslash\mathfrak{H}$ with $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ being torsion free and so $\mathbb{V}_o$ pulls back to $\mathfrak{H}$ as a trivial symplectic local system with Γ-action. The basis (v, v') of $V_o(C_o)$ constructed in Proposition 3.8 extends to one of the pull-back of $\mathbb{V}_o$ to $\mathfrak{H}$ (so we use c_o as our base point). Now the Γ-action (and hence the monodromy of $\mathbb{V}_o$) is given by a homomorphism

$$\rho : \Gamma \to \mathrm{SL}_2(\mathcal{O}_o)$$

that is compatible with the involutions named ι (it acts in $\mathrm{SL}_2(\mathcal{O}_o)$ as prescribed by Remark 3.10. (So this yields a group homomorphism between the semi-direct products defined by these involutions; this can be understood as a monodromy representation of a local system on the Deligne–Mumford stack $\mathcal{B}^\circ/\iota$.)

Our goal is to describe this monodromy representation. We first do this locally.

The cusps of $\mathrm{SL}_2(\mathcal{O}_o)$

We observed that $\mathcal{O}/2\mathcal{O} \cong \mathbb{F}_4$ and that the Galois involution of $\mathcal{O}$ induces in $\mathcal{O}/2\mathcal{O}$ the Frobenius map (so its fixed point set is the prime field $\mathcal{O}_o/2\mathcal{O} = \mathbb{F}_2$). Reduction modulo 2 defines a homomorphism $\mathrm{SL}_2(\mathcal{O}) \to \mathrm{SL}_2(\mathbb{F}_4)$. It is surjective and the permutation representation of $\mathrm{SL}_2(\mathbb{F}_4)$ on $\mathbb{P}^1(\mathbb{F}_4)$ identifies $\mathrm{SL}_2(\mathbb{F}_4)$ with the alternating group $\mathfrak{A}_5$. (It is also known that the full permutation group of $\mathbb{P}^1(\mathbb{F}_4)$ is the semi-direct product of $\mathrm{SL}_2(\mathbb{F}_4)$ and the Frobenius.)

We shall write $\mathrm{SL}_2(\mathcal{O})[2]$ for the kernel of $\mathrm{SL}_2(\mathcal{O}) \to \mathrm{SL}_2(\mathbb{F}_4)$. Since $\mathbb{P}^1(\mathbb{F}_4)$ has 5 elements, $\mathrm{SL}_2(\mathcal{O})[2]$ has as many cusps (= $\mathrm{SL}_2(\mathcal{O})[2]$-orbits in $\mathbb{P}^1(K)$). These are represented by $[1:0]$, $[0:1]$, $[1:1]$, $[X:1]$, $[1:X]$. Note that the involution $I : (x_0, x_1) \mapsto (x_1, x_0)$ exchanges $[1:0]$ and $[0:1]$ and $[X:1]$ and $[1:X]$, whereas the Frobenius only exchanges $[X:1]$ and $[1:X]$ (for $[X^2:1] = [1:X^{-2}] = [1:-X+2]$ and $[1:X]$ define the same element of $\mathbb{P}^1(\mathbb{F}_4)$).

It is clear that $\mathrm{SL}_2(\mathcal{O}_o)$ is the preimage of $\mathrm{SL}_2(\mathbb{F}_2)$, when regarded as a subgroup of $\mathrm{SL}_2(\mathbb{F}_4)$. The subgroup $\mathrm{SL}_2(\mathbb{F}_2) \subset \mathrm{SL}_2(\mathbb{F}_4)$ has two orbits in $\mathbb{P}^1(\mathbb{F}_4)$, namely $\{[1:0],[1:1],[0:1]\}$ and $\{[X:1],[1:X]\}$, and so $\mathrm{SL}_2(\mathcal{O}_o)$ has only 2 cusps (which we shall denote ∞_0 resp. ∞_X), both of which are invariant under the involution I. This has the following implication.

Lemma 3.12. — *A rank one $\mathcal{O}_o$-submodule $L \subset \mathcal{O}_o^2$ which is primitive in the sense that $\mathcal{O}_o^2/L$ is torsion free, is a $\mathrm{SL}_2(\mathcal{O}_o)$-transform of either*

- (type ∞_0): *the first summand of $\mathcal{O}_o^2$ (the associated $\mathrm{SL}_2(\mathcal{O}_o)$-cusp is ∞_0), or*
- (type ∞_X): *the image of $a \in \mathcal{O} \mapsto (2a, 2Xa) \in \mathcal{O}_o^2$ (the associated $\mathrm{SL}_2(\mathcal{O}_o)$-cusp is ∞_X).*

Proof. — By regarding $\mathbb{Q} \otimes_{\mathbb{Z}} L = K \otimes_{\mathcal{O}_o} L$ as a K-linear subspace of K^2 of dimension one, we get an element of $\mathbb{P}^1(K)$. Since the $\mathrm{SL}_2(\mathcal{O}_o)$-orbits in $\mathbb{P}^1(K)$ are represented by $[1:0]$ and $[1:X]$, we can assume that either $\mathbb{Q} \otimes_{\mathbb{Z}} L$ is the first summand of K^2 or the graph of X. In the first case, it is clear that L is the first summand of $\mathcal{O}_o^2$. In the second case, we note that if $u \in \mathcal{O}_o$ is such that $Xu \in \mathcal{O}_o$, then by writing u as an integral linear combination of 1 and X, we find that $u \in 2\mathcal{O}$. Conversely, every element of $2\mathcal{O}$ has that property. □

The cusps of the principal level 2 subgroup of $\mathrm{SL}_2(\mathcal{O}_o)$

We next consider the mod 2 reduction of $\mathcal{O}_o$ and $\mathrm{SL}_2(\mathcal{O}_o)$. A $\mathbb{Z}$-basis of $\mathcal{O}_o$ consists of 1 and $Y := 2X$. Since $Y^2 = 4X+4 = 2Y+4 \in 2\mathcal{O}_o$, it follows that $\mathcal{O}_o/2\mathcal{O}_o \cong \mathbb{F}_2[Y]/(Y^2)$ as a ring. Its group of units is $\{1, 1+Y\}$. So a nonzero submodule of $(\mathcal{O}_o/2\mathcal{O}_o)^2$ generated by a single element are the ones generated by $(1,0),(1,Y),(1,1),(1+Y,1),(Y,Y),(Y,1),(0,1)$. These seven submodules are pairwise distinct. Note that the involution $(x_0,x_1) \mapsto (x_1,x_0)$ exchanges the items of $\{[1:0],[0:1]\}$ and $\{[Y:1],[1:Y]\}$ and fixes the other three $[1:1]$, $[1+Y:1]$, $[Y:Y]$. The reduction homomorphism $\mathrm{SL}_2(\mathcal{O}_o) \to \mathrm{SL}_2(\mathcal{O}_o/2\mathcal{O}_o)$ is onto.

4. Arithmeticity of the monodromy

In Section 3 we proved that the monodromy representation of the Wiman–Edge pencil has target $\mathrm{SL}_2(\mathcal{O}_o)$, giving a representation $\rho : \Gamma \to \mathrm{SL}_2(\mathcal{O}_o)$. The goal of this section is to prove Theorem 1.1, that the image of ρ has finite index in $\mathrm{SL}_2(\mathcal{O}_o)$. The first step in doing this is to compute the image under ρ of the generators of Γ. As explained above, Γ is generated by loops around the cusps, i.e. the degenerations of the pencil. We start by applying classical Picard–Lefschetz theory to compute the conjugacy classes of these local monodromies. Computing them on the nose requires more work, which we do later in the section.

4.1. The monodromy around a cusp

In order to gain a better understanding of what ρ is like, let us recall that Γ is a free group and is generated by simple loops around the punctures of the 5-punctured sphere $\mathcal{B}^\circ$. We therefore concentrate on the monodromy around each puncture.

Assume that C_s is singular, and choose a disk-like neighborhood U of s in $\{s\}\cup\mathcal{B}^\circ$ (so that $C_s \subset \mathscr{C}_U$ is a homotopy equivalence). Choose also $\eta \in U \smallsetminus \{s\}$ and write C for C_η. Then the natural map $H_1(C) \to H_1(\mathscr{C}_U) \cong H_1(C_s)$ is onto, and the kernel is a $\mathfrak{A}_5$-invariant isotropic sublattice. If G_s denotes the dual intersection graph of C_s then there is a natural homotopy class of maps $C_s \to G_s$. Since the irreducible components of the normalization of C_s are all of genus zero, this homotopy class induces an isomorphism on the first (co)homology. We note that in each case $H_1(G_s)$ is free of rank 6, and so the kernel of $H_1(C) \to H_1(C_s) \cong H_1(G_s)$ is in fact a primitive Lagrangian sublattice. The intersection pairing identifies this kernel with the dual $H^1(G_s)$ of $H_1(C_s)$, so that we have an exact sequence of $\mathbb{Z}\mathfrak{A}_5$-modules

(vanishing sequence) $$0 \to H^1(G_s) \to H_1(C) \to H_1(G_s) \to 0,$$

where $H^1(G_s) \to H_1(C)$ is the composition of $H^1(G_s) \to H^1(C)$ with the isomorphism $H^1(C) \cong H_1(C)$ defined by the intersection pairing. The sequence is preserved by the monodromy operator of the family $\mathscr{C}_{U \smallsetminus \{s\}}$, with the monodromy acting nontrivially only on the middle term. Its difference with the identity (which is also called the *variation of the monodromy*) is therefore given by a homomorphism

$$\text{(variation of the local monodromy)} \qquad \nu_s : H_1(G_s) \to H^1(G_s)$$

of $\mathbb{Z}\mathfrak{A}_5$-modules. This homomorphism can be read off from G_s. To see this, we recall that the monodromy is given by the classical Picard–Lefschetz formula. Each node of C_s determines a vanishing circle on C up to isotopy, and hence, after orienting it, an element of $H_1(C)$ up to sign (a *vanishing cycle*). We denote the collection of vanishing cycles by $\Delta_C \subset H_1(C)$. It is clear from the preceding that Δ_C lies in the image of $H^1(G_s) \to H^1(C) \cong H_1(C)$. In fact, Δ_C generates that image. The monodromy around C_s is a multi Dehn twist which acts on $H_1(C)$ as:

$$T_s : x \in H_1(C) \mapsto x + \sum_{\delta \in \Delta_C/\{\pm 1\}} (\delta \cdot x)\delta \in H_1(C)$$

(the sum makes sense because replacing δ by $-\delta$ does not alter $(\delta \cdot x)\delta$). We now see that if we identify $H_1(G_s)$ with the dual of $H^1(G_s)$, then $\nu_s \in \operatorname{Hom}(H_1(G_s), H^1(G_s)) \cong H^1(G_s) \otimes H^1(G_s)$ is represented by the symmetric tensor

$$t_s := \sum_{\delta \in \Delta_C/\{\pm 1\}} \delta \otimes \delta \in H^1(G_s) \otimes H^1(G_s).$$

We shall refer to t_s as the *variation tensor*. It is the sum over the squares of the edges of G_s and so canonically associated with G_s. It is of course also $\mathfrak{A}_5$-invariant. By construction $T_s(x) - x$ is obtained by contracting t_s on the right with the image $[x]$ of x in $H_1(G_s)$ and regard the resulting element of $H^1(G_s)$ as sitting in $H_1(C)$ via Poincaré duality. In order to determine t_s in each case, we first note that the inverse form of s (i.e., the quadratic form on $E_o^\vee$) is the tensor

$$\check{s} = e \otimes e + \sum_{i \in \mathbb{Z}/(5)} e_i \otimes e_i \in E_o \otimes E_o.$$

Proposition 4.1 (Variation tensors of the singular fibers). — *For a singular fiber C_s of the Wiman–Edge pencil, its homology as a $\mathfrak{A}_5$-module and its variation tensor are as follows:*

(1) *When C_s is irreducible, there exists an isomorphism $v_s : E_o \cong H^1(G_s)$ of $\mathbb{Z}\mathfrak{A}_5$-modules (so the associated cusp is ∞_0) such that $t_s = v_s \otimes v_s(\check{s})$.*
(2) *When C_s consists of five conics, there exists an isomorphism $v_s : E_o \cong H^1(G_s)$ (so the associated cusp is ∞_0) such that $t_s = (3+4X)v_s \otimes v_s(\check{s})$.*
(3) *When C_s consists of ten lines, there exists an isomorphism $v_s : E \cong H^1(G_s)$ (so the associated cusp is ∞_X) such that $t_s = (4+2X)v \otimes v(\check{s})$.*

Proof when C_s is irreducible. — In this case $G := G_s$ has one vertex with 6 loops attached. Choose an orientation of each loop of G. This selects a vanishing cycle from each of our six antipodal pairs and the associated classes in $H^1(G)$ make up a basis and the variation of the monodromy assigns to the oriented loop the associated vanishing cycle. An $\mathfrak{A}_5$-isomorphism $v : E_0 \cong H^1(G) \subset H_1(C)$ is defined by assigning to e one such vanishing cycle. It is clear that then t_s is as asserted. □

Proof when C_s is the union of 5 conics. — In this case G_s is the K_5 graph (which has $\mathfrak{S}_5$ as its automorphism group). It has 20 oriented edges and $\mathfrak{A}_5$ acts transitively on this set. We have in fact an $\mathfrak{A}_5$-equivariant bijection between the order 3-elements in $\mathfrak{A}_5$ (the conjugacy class of 3-cycles) and the oriented edges of K_5 by assigning to the 3-cycle $h = (\tau_1\tau_2\tau_3)$ the oriented edge $[\tau_4, \tau_5]$ which is characterized by the property that the permutation $i \mapsto \tau_i$ is even. Note that this assigns to the inverse element $(\tau_1\tau_3\tau_2)$ the oppositely oriented edge $[\tau_5, \tau_4]$. Thus the group $Z^1(K_5)$ of simplicial 1-cochains on K_5 is identified with the $\mathbb{Z}$-module of rank 10 generated by the order 3 elements $h \in \mathfrak{A}_5$, subject to the relations $h + h^{-1} = 0$. Its $\mathbb{Z}\mathfrak{A}_5$-module structure is defined by conjugation. The 1-coboundary submodule $B^1(K_5)$ has rank 4 and is spanned by the vertices subject to the relation that the sum of the vertices is zero. So $B^1(K_5)_{\mathbb{C}}$ is the reflection representation; it is irreducible as a $\mathfrak{A}_5$-module.

According to Lemma 2.4, the $\mathfrak{A}_5$-orbit Δ_c of $e + e_0 + e_1 \in E_o$ consists of the 10 opposite pairs $\{\pm(e + e_i + e_{i+1})\}_{i\in\mathbb{Z}/(5)}, \pm(e_i - e_{i-2} - e_{i+2})\}_{i\in\mathbb{Z}/(5)}$, spans E_o over $\mathbb{Z}$ and $\sigma_3 \in \mathfrak{A}_5$ generates the $\mathfrak{A}_5$-stabilizer of $e + e_0 + e_1$. A $\mathbb{Z}\mathfrak{A}_5$-module epimorphism $Z^1(K_5) \to E_o$ is then defined by demanding that it takes the oriented edge fixed by σ_3 to $e+e_0+e_1$. Since $E_{\mathbb{Q}}$ is irreducible as a $\mathbb{Q}\mathfrak{A}_5$-module, the image of $B^1(K_5)_{\mathbb{Q}}$ in $E_{\mathbb{Q}}$ is zero. So $B^1(K_5)$ is contained in the kernel of $Z^1(K_5) \to E_o$. But $B^1(K_5)$ is a primitive submodule of $Z^1(K_5)$ (for $H^1(K_5)$ is torsion free) and has the same rank as this kernel. So it is equal to the kernel and we have an induced isomorphism $\mathbb{Z}\mathfrak{A}_5$-module isomorphism $v_s : E_o \to H^1(K_5)$.

The associated quadratic tensor is the image under $v_s \otimes v_s$ of

$$\sum_{i\in\mathbb{Z}/(5)} (e + e_{i-1} + e_{i+1})^{\otimes 2} + \sum_{i\in\mathbb{Z}/(5)} (e_i - e_{i-2} - e_{i+2})^{\otimes 2}.$$

If we write this tensor as $u \otimes e + \sum_i u_i \otimes e_i$ we find that $u = 5e + \sum_i e_{i-1} + \sum_i e_{i+1} = 3e + 2(e + \sum_i e_i) = 3e + 4\varepsilon = (3 + 4X)(e)$. Likewise we find that $u_i = (3 + 4X)(e_i)$. □

Proof when C_s is the union of 10 lines. — In this case G_s is the Petersen graph P. Recall that the vertices of P are indexed by the 2-element subsets of $\mathbb{Z}/5$ (a set of size 10) and that two such 2-element subsets span an edge if and only they are disjoint (a set of size 15). This makes it plain that $\mathfrak{A}_5$ acts transitively on its set of oriented edges (a set of 15 antipodal pairs), so that the stabilizer of an oriented edge is of order 2. The elements of order 2 in $\mathfrak{A}_5$ make up a single conjugacy class, and a given order 2 element preserves just one edge in an orientation preserving manner. The centralizer of that element is a copy of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$; it preserves the edge, but may reverse orientation.

Similarly, $e + e_o \in E$ has as its $\mathfrak{A}_5$-stabilizer a subgroup of order 2 (namely σ_2). It is clear that if $g \in \mathfrak{A}_5$ maps $e + e_o$ to $-(e + e_o)$, then it must centralize σ_2. It follows that there exists a $\mathfrak{A}_5$-equivariant bijection of the set of oriented edges of P onto the $\mathfrak{A}_5$-orbit of $e + e_o$ with the property that orientation reversal corresponds to taking antipode. In view of Lemma 2.4 this homomorphism is onto.

Recall that $E_\mathbb{C}$ is an $\mathbb{C}\mathfrak{A}_5$-module isomorphic to $I_\mathbb{C} \oplus I'_\mathbb{C}$, where $I_\mathbb{C}$ and $I'_\mathbb{C}$ are irreducible of degree 3. So to prove that $Z^1(P) \to E$ factors through an isomorphism $H^1(P) \to E$, it suffices to show that the coboundary space $B^1(P;\mathbb{C})$, does not contain a copy of $I_\mathbb{C}$ or $I'_\mathbb{C}$. As we observed in the proof of Lemma 2.1 of part I [5], the vertex set of P spans a $\mathbb{C}\mathfrak{A}_5$-module which decomposes into a trivial representation and two irreducible representations of dimension 4 and 5. Hence the latter two will span $B^1(P;\mathbb{C})$. In particular, neither $I_\mathbb{C}$ nor $I'_\mathbb{C}$ appears in $B^1(P;\mathbb{C})$ and so we have an isomorphism $H^1(P) \to E$. We denote by $v_s : E \cong H^1(P)$ its inverse.

The associated quadratic tensor is then the image under $v_s \otimes v_s$ of

$$\sum_{i\in\mathbb{Z}/(5)} (e + e_i)^{\otimes 2} + \sum_{i\in\mathbb{Z}/(5)} (e_i + e_{i+1})^{\otimes 2} + \sum_{i\in\mathbb{Z}/(5)} (e_{i-1} - e_{i+1})^{\otimes 2}.$$

Proceeding as in the previous case, we write this as $u \otimes e + \sum_i u_i \otimes e_i$ and find that $u = 5e + \sum_i e_i = 4e + 2\varepsilon = (4+2X)(e)$ and likewise that $u_i = (4+2X)(e_i)$. $\square$

Remark 4.2. — The descriptions in Proposition 4.1 also tell us what the monodromy variations, or rather their $\mathrm{SL}_2(\mathcal{O}_o)$-conjugacy classes, are in terms of $\mathbb{V}_o(\eta)$: as this endomorphism of $\mathbb{V}_o(\eta)$ is $\mathcal{O}_o$-linear, they are in the three cases given by respectively the images of $v \otimes v$, $(3+4X)v \otimes v$, and $(4+2X)v \otimes v$ in $\mathbb{V}_o(\eta) \otimes_{\mathcal{O}_o} \mathbb{V}_o(\eta)$ (in the last case this element lies *a priori* only in $\mathbb{V}(\eta) \otimes_{\mathcal{O}} \mathbb{V}(\eta)$, but one checks that it actually lies in the image of $\mathbb{V}_o(\eta) \otimes_{\mathcal{O}_o} \mathbb{V}_o(\eta)$).

As we might expect, each of these three conjugacy classes is Galois invariant. In the first case this is obvious. To see this in the two other cases, recall that the group of units of $\mathcal{O}_o$ is generated by $X^3 = 2X + 1$. So we can change the representative of the conjugacy class by conjugation with the diagonal matrix in $\mathrm{SL}_2(\mathcal{O}_o)$ with diagonal entries X^3 and X^{-3}. Its effect on $\binom{1}{0} \otimes_K \binom{1}{0}$ is multiplication with X^6. Since $(3+4X)' = 7 - 4X = X^{-6}(3+4X)$, it follows that the conjugacy class defined by the five conics is indeed Galois invariant. This also the case for the conjugacy class defined by the ten lines: $A := \left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ takes $\binom{1}{X'} = \binom{1}{1-X}$ to $\binom{X}{1+X} = X\binom{1}{X}$ and hence takes $(4+2X')\binom{1}{X'} \otimes_K \binom{1}{X'}$ to

$$(6-2X)X^2 \begin{pmatrix} 1 \\ X \end{pmatrix} \otimes_K \begin{pmatrix} 1 \\ X \end{pmatrix} = (4+2X) \begin{pmatrix} 1 \\ X \end{pmatrix} \otimes_K \begin{pmatrix} 1 \\ X \end{pmatrix}.$$

We can be more precise in that we can obtain actual monodromies rather than just their conjugacy classes. Recall that the involution of the Wiman–Edge pencil determines an involution ι of $\mathcal{B}$ with fixed points c_o and c_∞ representing the Wiman curve resp. the union of ten lines. This involution is covered by an involution of $\mathbb{V}_o$ which is anti-linear: if $\mathbb{V}'_o$ denotes the same local system but for which the $\mathcal{O}_o$-module structure has been precomposed with nontrival Galois element $X \mapsto X' = 1 - X$, then we have an identification $\mathbb{V}_o \cong \iota_* \mathbb{V}'_o$ such that applying this twice gives the identity. The singular fibers $\neq C_\infty$ come in two (unordered) pairs $\{C_{ir}, C'_{ir}\}$, $\{C_c, C'_c\}$ and lie over points denoted $c_{ir}, c'_{ir} = \iota(c_{ir})$ resp. $c_c, c'_c = \iota(c_c)$. We focus on the affine line $\mathcal{B} \smallsetminus \{c_\infty\}$.

We observed in Remark 3.11 that the elements of $\pi_1(\mathcal{B}^\circ, c_o) \cong \Gamma$ defined by $\gamma_\alpha, \gamma_{\alpha'}$ and $\gamma_\beta, \gamma_{\beta'}$ freely generate $\pi_1(\mathcal{B}^\circ, c_o) \cong \Gamma$; that ι exchanges each pair; and that a simple negative loop around c_∞ is represented by the product $[\gamma_\alpha][\gamma_\beta][\gamma_{\alpha'}][\gamma_{\beta'}]$. We shall abuse notation a bit by writing $\rho(\gamma_\alpha)$ for $\rho([\gamma_\alpha])$ and likewise for the other generators.

Corollary 4.3. — *The monodromies satisfy the following identities:*

$$\rho(\gamma_\alpha) = \begin{pmatrix} 1 & -1+2X \\ 0 & 1 \end{pmatrix}, \quad \rho(\gamma_{\alpha'}) = \begin{pmatrix} 1 & 0 \\ 1-2X & 1 \end{pmatrix}$$

and

$$\rho(\gamma_\beta) = \begin{pmatrix} 1+X^3 & X^3 \\ -X^3 & 1-X^3 \end{pmatrix}, \quad \rho(\gamma_{\beta'}) = \begin{pmatrix} 1+X^{-3} & X^{-3} \\ -X^{-3} & 1-X^{-3} \end{pmatrix}.$$

These elements determine the monodromy representation of $\mathbb{V}_o$ and generate its monodromy group. This monodromy group fixes the map $\mathcal{O}_o^2 \to \mathbb{F}_5$ defined by taking the symplectic product with $\binom{1}{-1}(= v - v')$ followed by the reduction $\mathcal{O}_o \to \mathbb{F}_5$. Up to a scalar this is the map $V_o(C_o) \to \mathbb{F}_5$ found in Corollary 3.5.

Proof. — The set of vanishing cycles for the degeneration along γ_α is $v(\Delta_c)$. According to Proposition 4.1, the variation tensor of $\rho(\gamma_\alpha)$ is then $(3+4X)\check{s}(v \otimes v)$. Now note that $(3+4X)A(v,v') = (3+4X)(-3+2X) = -1+2X$. Hence $\rho(\gamma_\alpha)(v) = v$ and

$$\rho(\gamma_\alpha)(v') = v' + (3+4X)A(v,v')v = v' + (-1+2X)v$$

so that $\rho(\gamma_\alpha)$ is as asserted. Proposition 3.8 also shows that the set of vanishing cycles for the degeneration along γ_β is the image of Δ_{ir} under $u = X^3v - X^3v'$. We have $A(u,v') = A(u,v) = +1$ and so

$$\rho(\gamma_\beta)(v) = v + u = (1+X^3)v - X^3v',$$
$$\rho(\gamma_\beta)(v') = v' + u = X^3v + (1-X^3)v',$$

which yields the matrix for $\rho(\gamma_\beta)$. The matrices for $\rho(\gamma_{\alpha'})$ and $\rho(\gamma_{\beta'})$ are then obtained using Remark 3.10 and the fact that $(-1+2X)' = 1-2X$ and $(X^3)' = -X^{-3}$.

We note that the images of $\rho(\gamma_\alpha)$ and $\rho(\gamma_{\alpha'})$ are trivial in $\mathrm{SL}_2(\mathbb{F}_5)$. The images of $\rho(\gamma_\beta)$ and $\rho(\gamma_{\beta'})$ must generate the same one-parameter subgroup $U \subset \mathrm{SL}_2(\mathbb{F}_5)$, namely the additive copy of $\mathbb{F}_5$ defined by the quadratic tensor $\binom{1}{-1} \otimes \binom{1}{-1}$. So the monodromy fixes pointwise the line in $\mathbb{F}_5 \otimes V_o(C_o)$ spanned by $v - v'$ and it is the only line spanned by that property. So it is the one determined by Corollary 3.5. ☐

Question 4.4. — *It is well-known that the reduction homomorphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{F}_5)$ is onto and hence so is $\mathrm{SL}_2(\mathcal{O}_o) \to \mathrm{SL}_2(\mathbb{F}_5)$. Are there no other restrictions on the monodromy group other than the one given in Corollary 4.3, in the sense that it contains the kernel of the reduction map $\mathrm{SL}_2(\mathcal{O}_o) \to \mathrm{SL}_2(\mathbb{F}_5)$?*

We note that $\gamma_\beta\gamma_\alpha\gamma_{\beta'}\gamma_{\alpha'}$ represents a negative loop around c_∞ and so its image under ρ is in the conjugacy class of the unipotent element defined by the variation $-2(2+X)v_s\otimes v_s(\check{s})$ for some vector v_s of type ∞_X (as defined in Lemma 3.12). Hence the following proposition gives an additional check on our computations.

Proposition 4.5. — *Let $v_\infty := v + Xv'$. Then $\rho(\gamma_\alpha\gamma_\beta\gamma'_\alpha\gamma'_\beta)$ is given by the variation tensor $-2(2+X)v_\infty\otimes v_\infty(\check{s})$.*

Proof. — We put $B := \rho(\gamma_\alpha\gamma_\beta)$, so that $\rho(\gamma_\alpha\gamma_\beta\gamma'_\alpha\gamma'_\beta) = B\iota B\iota$. We compute

$$\begin{aligned}B = \rho(\gamma_\alpha)\rho(\gamma_\beta) &= \begin{pmatrix}1 & -1+2X\\ 0 & 1\end{pmatrix}\begin{pmatrix}1+X^3 & X^3\\ -X^3 & 1-X^3\end{pmatrix}\\ &= \begin{pmatrix}-1-2X & -3\\ -1-2X & -2X\end{pmatrix}.\end{aligned}$$

Then in view of Remark 3.10,

$$\iota B\iota = \rho(\gamma_{\beta'})\rho(\gamma_{\alpha'}) = \begin{pmatrix}-2X' & -1-2X'\\ -3 & -1-2X'\end{pmatrix} = \begin{pmatrix}-2+2X & -3+2X\\ -3 & -3+2X\end{pmatrix}$$

and so

$$\begin{aligned}B\iota B\iota &= \begin{pmatrix}-1-2X & -3\\ -1-2X & -2X\end{pmatrix}\cdot\begin{pmatrix}-2+2X & -3+2X\\ -3 & -3+2X\end{pmatrix}\\ &= \begin{pmatrix}7-2X & 8-6X\\ -2+4X & -5+2X\end{pmatrix}\end{aligned}$$

It follows that

$$B\iota B\iota - \mathbb{1} = 2\begin{pmatrix}3-X & 4-3X\\ -1+2X & -3+X\end{pmatrix} = -2(2+X)\begin{pmatrix}-X^{-2} & X^{-3}\\ -X^{-1} & X^{-2}\end{pmatrix}.$$

Now note that $A(v_\infty, v)v_\infty, = -X^{-2}(v+Xv') = -X^{-2}v - X^{-1}v'$ and $A(v'_\infty, v')v_\infty = X^{-3}(v+Xv') = X^{-3}v + X^{-2}v'$ and so the last matrix is indeed the matrix of the endomorphism $x\in V_o(C_o)\mapsto -2(2+X)A(v_\infty,x)v_\infty\in V_o(C_o)$. □

4.2. Arithmeticity

Now that we know the image of the generators of Γ under the monodromy representation, we can prove arithmeticity of the monodromy group. We use a criterion due to Benoist–Oh, namely Theorem 1.1 of [1]. That theorem gives the following as a special case.

Theorem 4.6 (Benoist–Oh). — *Let K be a real quadratic number field, $\mathcal{O}_K$ its ring of integers, and $\Omega < K$ a lattice. Let $\Lambda < \mathrm{SL}_2(\mathcal{O}_K)$ be the subgroup generated by a matrix of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \neq 0$, together with the set of matrices*

$$\left\{ \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} : \omega \in \Omega \right\}.$$

If $\sigma, \sigma' : K \to \mathbb{R}$ are the two real embeddings of K, then the associated embedding $\mathrm{SL}_2(\mathcal{O}_K) \to \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ maps Λ onto a lattice in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$; in particular Λ has finite index in $\mathrm{SL}_2(\mathcal{O}_K)$.

Proof of Theorem 1.1. — It suffices to check that $\rho(\Gamma)$ satisfies the criteria of Theorem 4.6. Of course it suffices to do this after a single conjugation by an element of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$; that is, after a single change of basis.

For parabolic property, we note that the elements $\rho(\gamma_\beta)$ and $\rho(\gamma_{\beta'})$ both stabilize $v + v'$. The variation construction shows that under the above homomorphism, $\rho(\gamma_\beta)$ resp. $\rho(\gamma_{\beta'})$ is the image of $X^3 = 2X + 1$ resp. $X^{-3} = 3 - 2X$. The additive span Ω of these elements is of finite index in $\mathcal{O}_o$ and so Ω is a lattice in K. We conclude that the parabolic condition of Theorem 4.6 is satisfied.

We now claim that, after conjugating $\rho(\Gamma)$ so that the parabolic subgroup $P := \langle \rho(\gamma_\beta), \rho(\gamma_{\beta'} \rangle$ has upper triangular form with 1 on the diagonal, $\rho(\Gamma)$ contains a matrix of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \neq 0$. If this were not the case then the Zariski closure G of the image of monodromy group in $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ would lie in the group of upper triangular matrices. We claim that G is in fact all of $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$, a contradiction, finishing the proof of the theorem.

To prove the claim, note that the images $\rho(\gamma_\alpha)$ and $\rho(\gamma_{\alpha'})$ in $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ are

$$\left(\left(\begin{smallmatrix} 1 & -\sqrt{5} \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \sqrt{5} \\ 0 & 1 \end{smallmatrix}\right)\right) \text{ resp. } \left(\left(\begin{smallmatrix} 1 & 0 \\ \sqrt{5} & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ -\sqrt{5} & 1 \end{smallmatrix}\right)\right).$$

It is clear that G contains the subgroup generated by the one-parameter subgroups obtained by replacing $\sqrt{5}$ by a complex variable. These two groups generate the subgroup of $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ that is in fact the graph of an automorphism u of $\mathrm{SL}_2(\mathbb{C})$, namely that assigns to $g \in \mathrm{SL}_2(\mathbb{C})$ the transpose inverse of g followed by conjugation with $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $U \subset \mathrm{SL}_2(\mathbb{C})$ be the image of the one parameter group $t \in \mathbb{C} \mapsto \left(\begin{smallmatrix} 0 & t \\ 1 & 0 \end{smallmatrix}\right)$. The parabolic property shows that G also contains the $U \times U$. Since $\mathrm{SL}_2(\mathbb{C})$ is generated by the conjugacy class of U, the $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$-conjugates of $U \times \{1\}$ generate $\mathrm{SL}_2(\mathbb{C}) \times \{1\}$. Similarly, the $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$-conjugates of $\{1\} \times U$ generate $\{1\} \times \mathrm{SL}_2(\mathbb{C})$ and so $G = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ as asserted. $\square$

5. The period map

In this section we use Theorem 1.1 to study various period maps associated to the Wiman–Edge pencil.

5.1. The period map of the Wiman–Edge pencil

We shall see that the monodromy representation $\rho : \pi_1(\mathcal{B}^\circ) \to \mathrm{SL}_2(\mathcal{O}_o)$ is induced by an algebraic map from $\mathcal{B}^\circ$ to a quotient of a period domain $\mathfrak{D}$ isomorphic to $\mathfrak{H}^2$ by an action of $\mathrm{SL}_2(\mathcal{O}_o)$. This is the *period map*, which assigns to a curve with faithful $\mathfrak{A}_5$-action its Jacobian with the induced $\mathfrak{A}_5$-action. (Beware however, that there is *a priori* no obvious relation between Γ as a subgroup in $\mathrm{PSL}_2(\mathbb{Z})$ and its image in $\mathrm{SL}_2(\mathcal{O}_o)$.) The quotient $Y^\circ := \mathrm{SL}_2(\mathcal{O}_o)\backslash\mathfrak{D}$ is a quasi-projective, complex-algebraic surface, called a *Hilbert modular surface*. In this subsection we prove some properties of the period mapping; in particular we prove that it comes equipped with some extra structure.

The two ring embeddings $\sigma, \sigma' : \mathcal{O}_o \hookrightarrow \mathbb{R}$ define an algebra-isomorphism $(\sigma, \sigma') : \mathbb{R}\otimes_\mathbb{Z}\mathcal{O}_o = \mathbb{R}\otimes_\mathbb{Q} K \cong \mathbb{R}\oplus\mathbb{R}$. So for a member C of the Wiman–Edge pencil we have the decomposition

$$H_1(C;\mathbb{R}) = \mathbb{R}\otimes_\mathbb{Z} H_1(C) \cong \mathbb{R}\otimes_\mathbb{Z} V_o(C)\otimes_{\mathcal{O}_o} E_o$$

Note that for $C = C_o$, the basis (v, v') of $V_o(C_o)$ introduced in Proposition 3.8 yields the $\mathbb{R}$-basis $\{\sigma v, \sigma v', \sigma' v, \sigma' v'\}$ for $\mathbb{R}\otimes_\mathbb{Z} V(C_o)$. The anti-involution ι exchanges v and v' up to a common sign, but also exchanges the two real embeddings of K. In other words, it exchanges the basis elements $\sigma v, \sigma' v'$ resp. $\sigma' v, \sigma v'$ up to a common sign. The $\mathcal{O}_o$-module $V_o(C_o)$ does not have a $\mathbb{Z}$-basis consisting of elements invariant under ι, but the K-vector space $V_o(C_o)_\mathbb{Q}$ does, namely $(v+v', X^3(v-v'))$ or $(v-v', X^3(v+v'))$, depending on whether ι exchanges v and v' of v and $-v'$.

Let (w, w') be such a basis. Assuming that C is smooth, then $H_1(C)$ acquires a Hodge structure of weight -1 polarized by the intersection pairing. It is completely given by the complex subspace $F^0H_1(C)\subset H_1(C;\mathbb{C})$. This is an $\mathfrak{A}_5$-invariant subspace that can be written as the graph of a $\mathfrak{A}_5$-equivariant map from the image of $w'_\mathbb{C}$ to the image of $w_\mathbb{C}$. The positivity property of the associated Hermitian form implies that there exist $\tau, \tau' \in \mathfrak{H}$ such that $F^0H_1(C)$ is spanned by images of $\tau\sigma w + \sigma w'$ and $\tau'\sigma' w + \sigma' w'$. The action of $\mathrm{SL}_2(K)$ on $\mathfrak{H}^2$ is then the standard one:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau, \tau') = \left(\frac{\sigma(a)\tau + \sigma(b)}{\sigma(c)\tau + \sigma(d)}, \frac{\sigma'(a)\tau' + \sigma'(b)}{\sigma'(c)\tau' + \sigma'(d)}\right)$$

We prefer however to work with (v, v'). Let us simply write $\mathfrak{D}$ for the space of Hodge structures of weight -1 on $V_o(C_o)$ of the above type so that (by the above discussion) $\mathfrak{D}$ is a domain isomorphic to $\mathfrak{H}^2$. Then $Y^\circ := \mathrm{SL}\,2(\mathcal{O}_o)\backslash\mathfrak{D}$ is an algebraic surface equipped with an involution ι. We note that the fixed-point set $(Y^\circ)^\iota$ of ι in Y° contains the image $D^\circ \subset Y^\circ$ of what corresponds to the diagonal of $\mathfrak{H}^2$. Since the stabilizer of the diagonal in $\mathrm{SL}_2(\mathcal{O}_o)$ is $\mathrm{SL}_2(\mathbb{Z})$, the latter is just a copy of the j-line $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$. The closure D of D° in Y adds the cusp ∞_0, and is of course contained in Y^ι. But Y^ι has other curves as irreducible components. It also contains the other cusp.

Proposition 5.1. — *The period map*

$$\Pi^\circ : \mathcal{B}^\circ = \Gamma\backslash\mathfrak{H} \to \mathrm{SL}_2(\mathcal{O}_o)\backslash\mathfrak{D} = Y^\circ$$

is an ι-equivariant closed embedding. It extends to an ι-equivariant morphism $\Pi : \mathcal{B} \to Y$ such that $\Pi^{-1}Y^\iota$ is the union of the 5 points of $\mathcal{B} \smallsetminus \mathcal{B}^\circ$ and the point associated to the Wiman curve. The preimage of ∞_X is the C_∞-point of $\mathcal{B} \smallsetminus \mathcal{B}^\circ$ and the preimage of ∞_0 consists of the other four points over which there is a singular fiber.

Proof. — The ι-equivariance has already been established. The rest of the proposition follows essentially from the Torelli theorem, which asserts that a smooth projective curve can be reconstructed from its Jacobian as a principally polarized abelian variety. An automorphism of the latter is up to the composition with the involution -1 in $J(C)$ induced an automorphism of the curve. This automorphism is then unique unless the curve is hyperelliptic. A curve C in the Wiman–Edge pencil is nonhyperelliptic and hence $\mathrm{Aut}(C)$ is identified with the group of automorphisms of $J(C)$ *as a polarized abelian variety* modulo ± 1. Thus Π can be understood as the lift of the usual period map which also takes into account the identification of a group of automorphisms of the curve with $\mathfrak{A}_5$ up to inner automorphism. It is therefore an embedding. The cusps in $\mathcal{B}$ define stable degenerations. This implies that Π° is proper and extends to a morphism from $\mathcal{B}$ (which adds five points) to the Baily–Borel compactification Y of $\mathrm{SL}_2(\mathcal{O}_o)\backslash\mathfrak{D}$ (which adds two cusps). □

5.2. Relation with the Clebsch–Hirzebruch model

We briefly explain the relation between our Hilbert modular surface Y° and another one that was investigated by Hirzebruch [8] and described by him in terms of the Clebsch surface.

Recall that the natural map $\mathrm{SL}_2(\mathcal{O}) \to \mathrm{SL}_2(\mathcal{O}/2\mathcal{O}) \cong \mathrm{SL}_2(\mathbb{F}_4)$ is onto with kernel the principal level 2 congruence subgroup $\mathrm{SL}_2(\mathcal{O})[2]$ and that $\mathrm{SL}_2(\mathcal{O}_o)$ is the preimage of $\mathrm{SL}_2(\mathbb{F}_2) \subset \mathrm{SL}_2(\mathbb{F}_4)$. (Note that $\mathrm{SL}_2(\mathbb{F}_2)$ can be identified with the full permutation group of the three elements of $\mathbb{P}^1(\mathbb{F}_2)$.) This is replicated by applying the functor V_o to the chain $E \subset E_o \subset E^\vee$, for as we observed earlier, we then get $(2\mathcal{O})^2 \subset \mathcal{O}_o^2 \subset \mathcal{O}^2$. The group $\mathrm{SL}_2(\mathcal{O})[2]$ contains -1, hence acts on $\mathfrak{D}$ through $\mathrm{PSL}_2(\mathcal{O})[2] := \mathrm{SL}_2(\mathcal{O})[2]/\{\pm 1\}$. This action is faithful and even free so that $Y^\circ[2] := \mathrm{SL}_2(\mathcal{O})[2]\backslash\mathfrak{D}$ is a smooth surface. Its Baily–Borel compactification $Y^\circ[2] \subset Y[2]$ is a normal projective surface obtained by adding the five points of $\mathbb{P}^1(\mathbb{F}_4)$. All five points are cusp surface singularities of the same type; following Hirzebruch they are resolved by a toroidal resolution $\widehat{Y}[2] \to Y[2]$ for which the preimage of each cusp is a triangle of rational curves of self-intersection -3. We thus end up with a smooth surface $\widehat{Y}[2]$ endowed with an action of $\mathrm{SL}_2(\mathbb{F}_4)$, or rather, of $\mathrm{PSL}_2(\mathbb{F}_4)$. As we mentioned earlier, $\mathrm{PSL}_2(\mathbb{F}_4) \cong \mathfrak{A}_5$, but since we do not know whether that is a curious coincidence or that $\mathrm{PSL}_2(\mathbb{F}_4)$ is naturally identified with the automorphism group of a general member of the Wiman–Edge pencil, we prefer to make the notational distinction.

Recall that what we called in [5] the *Klein plane* and denoted by P, a projective plane with faithful $\mathrm{SL}_2(\mathbb{F}_4)$-action. It is obtained from complexification followed by projectivization of a real irreducible representation of degree 3 in which $\mathrm{PSL}_2(\mathbb{F}_4)$ is identified with the group of motions of a regular icosahedron. The 12 vertices of the icosahedron determine an $\mathrm{PSL}_2(\mathbb{F}_4)$-orbit in P of size 6. The blowup $\widetilde{P} \to P$ of this orbit is then a cubic surface with $\mathrm{PSL}_2(\mathbb{F}_4)$-action. It is the classical *Clebsch surface*: it is isomorphic to the cubic surface in the diagonal hyperplane $\sum_i z_i = 0$ in $\mathbb{P}^4$ (a copy of $\mathbb{P}^3$) defined by $\sum_i z_i^3 = 0$, where $\mathfrak{A}_5$ of course acts by permuting coordinates. Since the Clebsch surface actually comes with an $\mathfrak{S}_5$-action, so must $\widetilde{P}$. The barycenters of the 20 faces of the icosahedron determine $\mathrm{SL}_2(\mathbb{F}_4)$-orbit in P of size 10 and appear on $\widetilde{P}$ as its set of *Eckardt points*, that is, the set of points of $\widetilde{P}$ through which pass three distinct lines on $\widetilde{P}$. Hirzebruch proves in [8] that $\widehat{Y}[2]$ is equivariantly isomorphic to the blowup $\widehat{P} \to \widetilde{P}$ at this size 10 orbit. It is in particular a rational surface. It follows that our period map defines a morphism from the base of the Wiman–Edge pencil $\mathcal{B}$ to the $\mathrm{SL}_2(\mathbb{F}_2)$-orbit space of $\widehat{P}$. It would be worthwhile to determine its image in terms of the above construction.

5.3. The associated family of K3 surfaces

There is another period map for the Wiman–Edge pencil, which in the terminology of Kudla–Rapoport, is of occult type. Recall that the Wiman–Edge pencil is realized on a quintic del Pezzo surface S whose automorphism group (a copy of $\mathfrak{S}_5$) preserves the pencil and induces in each member $C_t \subset S$ the $\mathfrak{A}_5$-action. There exists a section α_t of ω_S^{-2} with divisor C_t. Then $\sqrt{\alpha_t}$ defines a surface $\widehat{S}_t$ in the total space of ω_S^{-1} (the determinant bundle of the tangent bundle) such that the projection $\widehat{S}_t \to S_t$ is a double cover ramified along C_t. Then $\widehat{\alpha}_t := \sqrt{\alpha_t}$ is unambiguously defined on $\widehat{S}_t$ as a 2-vector and is there nowhere vanishing.

Proposition 5.2. — *The surface $\widehat{S}_t$ is a $K3$-surface (with an ordinary double point over every node of C_t). The $\mathfrak{A}_5$-action on S lifts uniquely to one on $\widehat{S}_t$ (and hence commutes with the involution) and the orthogonal complement of the $\mathbb{Q}\mathfrak{A}_5$-embedding $H^2(S;\mathbb{Q}) \hookrightarrow H^2(\widehat{S}_t;\mathbb{Q})$ is as a $\mathbb{Q}\mathfrak{A}_5$-module isomorphic to $3.\mathbb{1} \oplus V \oplus 2W$. The 3-dimensional summand on which $\mathfrak{A}_5$ acts trivially has signature $(2,1)$, and its complexification contains $H^{2,0}(\widehat{S}_t)$.*

Proof. — The inverse of $\widehat{\alpha}_t$ is a nowhere zero 2-form. In order to conclude that $\widehat{S}_t$ is a $K3$ surface, it suffices to show that $H^1(\widehat{S}_t) = 0$. If $\pi : \widehat{S}_t \to S$ is the projection, then we have $H^1(\widehat{S}_t) = H^1(S;\pi_*\mathbb{Z})$. The cokernel of $\mathbb{Z}_S \to \pi_*\mathbb{Z}$ is a rank one local system $\mathcal{L}$ on $S \smallsetminus C_t$ and since $H^1(S) = 0$, it follows that $H^1(\widehat{S}_t)$ embeds in $H^1(S;\pi_*\mathbb{Z}/\mathbb{Z}_S) = H^1_c(S \smallsetminus C_t;\mathcal{L})$. But $S \smallsetminus C_t$ is affine, and hence $H^1_c(S \smallsetminus C_t;\mathcal{L}) = 0$.

A priori, there is a central extension of order 2 of $\mathfrak{A}_5$ that lifts the $\mathfrak{A}_5$-action on S, with the nontrivial center acting as involution. The definition makes it clear that the center takes $\widehat{\alpha}_t$ to $-\widehat{\alpha}_t$. This implies that the central extension must be split. In particular, the $\mathfrak{A}_5$-action on S lifts to $\widehat{S}_t$. It is unique, since any homomorphism from $\mathfrak{A}_5$ to a cyclic group is trivial.

The $\mathfrak{A}_5$-representation $H^2(\widehat{S}_t;\mathbb{Q})$ contains $H^2(S_t;\mathbb{Q})$ as a direct summand that is nondegenerate for the intersection pairing and so the orthogonal complement, denoted $H^2(\widehat{S}_t;\mathbb{Q})^-$, is a $\mathbb{Q}\mathfrak{A}_5$-module of dimension $22-5=17$. We determine its character by computing some Lefschetz numbers. We assume here that C_t is smooth, so that the $\mathfrak{A}_5$-character of $H^1(C_t;\mathbb{Q})$ is $2E_{\mathbb{Q}}$.

The element (01234) has 2 fixed points in S; this is best seen using the modular interpretation $(S, S\smallsetminus C_\infty) = (\overline{\mathcal{M}}_{0,5}, \mathcal{M}_{0,5})$. The fixed points are then represented by the stable 5-tuples on $\mathbb{P}^1$ given by $(1,\zeta_5,\zeta_5^2,\zeta_5^3,\zeta_5^4)$ and $(1,\zeta_5^2,\zeta_5^4,\zeta_5,\zeta_5^3)$, where ζ_5 is a 5th root of unity $\neq 1$. These do not lie on C_∞

and hence not on C_t for general t (in fact, each of the singular dodecahedral curves contains one of them). It follows that (01234) has 4 fixed points in $\widehat{S}_t$. So the trace of (01234) acting on $H^2(\widehat{S}_t;\mathbb{Q})^-$ is $4-2=2$. On the other hand, (012) has 4 fixed points and they are represented by taking as the first three points $(1,\zeta_3,\zeta_3^2)$ and letting the last two be arbitrary chosen in $\{0,\infty\}$. So exactly two lie outside C_∞ so that (012) has $2.2+2=6$ fixed points in $\widehat{S}_t$. It follows that its trace on $H^2(\widehat{S}_t;\mathbb{Q})^-$ is $6-4=2$.

Write

$$H^2(\widehat{S}_t;\mathbb{Q})^- = a\mathbb{1}\oplus bV\oplus cW\oplus dE$$

as $\mathbb{Q}\mathfrak{A}_5$-modules. The character table of $\mathfrak{A}_5$ shows that we have $a+4b+5c+6d=17$, $a-b+d=2$, $a+b-c=2$. We noted that $H^{2,0}(\widehat{S}_t)\oplus H^{0,2}(\widehat{S}_t)$ is a subspace of $H^2(\widehat{S}_t;\mathbb{C})^-$ on which $\mathfrak{A}_5$ acts trivially. This subspace cannot be constant in t, and so we must have $a\geqslant 3$. We then find that the only solution is $(a,b,c,d)=(3,1,2,0)$.

We observed that the complexification 3-dimensional subspace of $H^2(\widehat{S}_t;\mathbb{Q})^-$ defined by the trivial character contains $H^{2,0}(\widehat{S}_t)$. This implies that its signature is $(2,1)$. □

With the above in hand, we are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. — Since $(H^2(\widehat{S}_t;\mathbb{Q})^-)^{\mathfrak{A}_5}$ has signature $(2,1)$, a connected component of its associated symmetric domain, which we shall denote by $\mathfrak{C}$, is of dimension one: it is copy of $\mathfrak{H}$. This domain parametrizes the Hodge structures of the $K3$-surfaces with a faithful action of $\mu_2\times\mathfrak{A}_5$ of the type above. If M is the subgroup of the orthogonal transformations of $H^2(\widehat{S}_t)$ of spinor norm one and acting trivially on the vectors perpendicular to $(H^2(\widehat{S}_t;\mathbb{Q})^-)^{\mathfrak{A}_5}$, then our period map is defined on all of $\mathcal{B}$ and lands in the Shimura curve $M\backslash\mathfrak{C}$. The Torelli theorem for $K3$-surfaces implies that this morphism is injective. So it must be an isomorphism. In particular, $M\backslash\mathfrak{C}$ is compact. This means that the intersection form on $(H^2(\widehat{S}_t;\mathbb{Q})^-)^{\mathfrak{A}_5}$ does not represent zero, and that $M\backslash\mathfrak{C}$ is of quaternionic type. □

We remark that the structure of $\mathcal{B}$ as a Shimura curve of quaternionic type cannot be induced from the period map defined by the Hodge structure $H^1(C_t)$, for the latter has cusps (and goes to a Hilbert modular surface). Simply put, the monodromy along a simple loop around a puncture is of finite order for the former and of infinite order for the latter, so the monodromy representation $\Gamma\to M$ is not injective.

BIBLIOGRAPHY

[1] Y. Benoist & H. Oh, "Discreteness criterion for subgroups of products of SL(2)", *Transform. Groups* **15** (2010), no. 3, p. 503-515.

[2] N. Bourbaki, *Groupes et algèbres de Lie. Chapitres 4, 5 et 6*, Élements de Mathématique, Masson, 1981.

[3] I. Cheltsov, A. Kuznetsov & C. Shramov, "Coble fourfold, S_6-invariant quartic threefolds, and Wiman-Edge sextics", `https://arxiv.org/abs/1712.08906`, to appear in *Algebra Number Theory*, 2017.

[4] I. Cheltsov & C. Shramov, *Cremona groups and the icosahedron*, Monographs and Research Notes in Mathematics, CRC Press, 2016.

[5] I. Dolgachev, B. Farb & E. Looijenga, "Geometry of the Wiman Pencil. I: algebro-geometric aspects", *Eur. J. Math.* **4** (2018), no. 3, p. 879-930.

[6] W. L. Edge, "A pencil of four-nodal plane sextics", *Math. Proc. Camb. Philos. Soc.* **89** (1981), p. 413-421.

[7] B. Farb & E. Looijenga, "Geometry of the Wiman-Edge pencil and the Wiman curve", *Geom. Dedicata* **208** (2020), p. 197-220.

[8] F. Hirzebruch, "The Hilbert modular group for the field $\mathbb{Q}(\sqrt{5})$ and the cubic diagonal surface of Clebsch and Klein", *Usp. Mat. Nauk* **31** (1976), p. 153-166, English traslation in *Russ. Math. Surveys* **31** (2019), no. 5, p. 96-110.

[9] A. Wiman, "Ueber die algebraische Curven von den Geschlecht $p = 4, 5$ and 6 welche eindeutige Transformationen in sich besitzen", *Stockh. Akad. Bihang* **21** (1895), no. 3, p. 2-41.

[10] A. Zamora, "Some remarks on the Wiman–Edge pencil", *Proc. Edinb. Math. Soc.* **61** (2018), no. 2, p. 401-412.

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