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# CLASSIFICATION OF STRING LINKS UP TO $2n$ -MOVES AND LINK-HOMOTOPY

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ABSTRACT. — Two string links are equivalent up to  $2n$ -moves and link-homotopy if and only if their all Milnor link-homotopy invariants are congruent modulo  $n$ . Moreover, the set of the equivalence classes forms a finite group generated by elements of order  $n$ . The classification induces that if two string links are equivalent up to  $2n$ -moves for every  $n > 0$ , then they are link-homotopic.

RÉSUMÉ. — Deux enlacements d'intervalles sont équivalents à  $2n$ -mouvements et homotopie près si et seulement si leurs invariants d'homotopie de Milnor sont congrus modulo  $n$ . De plus, l'ensemble des classes d'équivalence forme un groupe fini engendré par des éléments d'ordre  $n$ . Cette classification implique que si deux enlacements d'intervalles sont équivalents à  $2n$ -mouvements près pour tout  $n > 0$ , alors ils sont homotopes.

## 1. Introduction

In the 1950s, J. Milnor [22, 23] defined a family of link invariants, known as *Milnor  $\bar{\mu}$ -invariants*. For an ordered oriented  $m$ -component link  $L$  in the 3-sphere  $S^3$ , the *Milnor number*  $\mu_L(I)$  ( $\in \mathbb{Z}$ ) of  $L$  is specified by a finite sequence  $I$  of elements in  $\{1, \dots, m\}$ . This number is only well-defined up to a certain indeterminacy  $\Delta_L(I)$ , i.e. the residue class  $\bar{\mu}_L(I)$  of  $\mu_L(I)$  modulo  $\Delta_L(I)$  is a link invariant. The invariant  $\bar{\mu}_L(ij)$  for a sequence  $ij$  is just the linking number between the  $i^{\text{th}}$  and  $j^{\text{th}}$  components of  $L$ . This justifies regarding  $\bar{\mu}$ -invariants as “generalized linking numbers”.

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*Keywords:* Milnor invariant, link, string link,  $2n$ -move, link-homotopy, Fox's congruence class, clasper.

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In [13], N. Habegger and X.-S. Lin defined Milnor numbers for *string links* and proved that Milnor numbers are well-defined invariants without taking modulo. These numbers are called *Milnor  $\mu$ -invariants*. It is remarkable that  $\mu$ -invariants for non-repeated sequences classify string links up to link-homotopy [13] (whereas  $\bar{\mu}$ -invariants are not enough strong to classify links with four or more components up to link-homotopy [18]). Here the *link-homotopy*, introduced by Milnor in [22], is the equivalence relation on (string) links generated by self-crossing changes and ambient isotopies. In addition to link-homotopy, there are various “geometric” equivalence relations on (string) links that are related to Milnor invariants, e.g. concordance [3, 28], (self)  $C_k$ -equivalence [11, 14, 19, 30, 31], Whitney tower concordance [5, 6, 7], etc.

A  $2n$ -move is a local move illustrated in Figure 1.1, and the  $2n$ -move equivalence is the equivalence relation generated by  $2n$ -moves and ambient isotopies. The  $2n$ -moves were probably first studied by S. Kinoshita in 1957 [15]. It is known that several  $2n$ -move equivalence invariants are derived from polynomial invariants, the Alexander [16], Jones, Kauffman and HOMFLYPT polynomials [27]. Besides polynomial invariants, Fox colorings and Burnside groups give  $2n$ -move equivalence invariants [8, 9].

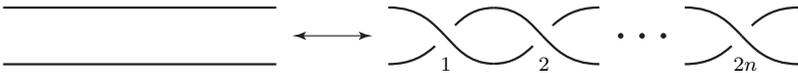


Figure 1.1.  $2n$ -move

Both Milnor invariants and  $2n$ -moves are well-studied in Knot Theory. However, to the best of the authors’ knowledge, there are no research articles relating Milnor invariants and  $2n$ -moves (except for the easily observed fact that the linking numbers modulo  $n$  are  $2n$ -move equivalence invariants). In this paper, we show the following theorem that establishes an unexpected relationship between Milnor link-homotopy invariants and  $2n$ -moves.

**THEOREM 1.1.** — *Let  $n$  be a positive integer. Two string links  $\sigma$  and  $\sigma'$  are  $(2n + \text{lh})$ -equivalent if and only if  $\mu_\sigma(I) \equiv \mu_{\sigma'}(I) \pmod{n}$  for any non-repeated sequence  $I$ .*

Here, the  $(2n + \text{lh})$ -equivalence is the equivalence relation generated by  $2n$ -moves, self-crossing changes and ambient isotopies. Note that “ $2n + \text{lh}$ ”

stands for the combination of  $2n$ -move equivalence and link-homotopy. In order to prove Theorem 1.1, we give a complete list of representatives for string links up to  $(2n + \text{lh})$ -equivalence (Proposition 5.3).

Let  $\mathcal{SL}(m)$  denote the set of  $m$ -component string links. Since the set of link-homotopy classes of  $\mathcal{SL}(m)$  forms a group [13], it can be seen that the set of  $(2n + \text{lh})$ -equivalence classes is also a group. Moreover, we have the following.

**COROLLARY 1.2.** — *The set of  $(2n + \text{lh})$ -equivalence classes of  $\mathcal{SL}(m)$  forms a finite group generated by elements of order  $n$ , and the order of the group is  $n^{s(m)}$ , where*

$$s(m) = \sum_{r=2}^m (r-2)! \binom{m}{r}.$$

The link-homotopy, concordance and  $C_k$ -equivalence give group structures on those equivalence classes of  $\mathcal{SL}(m)$ , respectively [13, 14]. The set of link-homotopy classes is a torsion free group of rank  $s(m)$  ([13, Section 3]), and the concordance classes contain elements of order 2. It is still open if the concordance classes contain elements of order  $\geq 3$ , and if the  $C_k$ -equivalence classes have torsion elements. In contrast to these facts, Corollary 1.2 implies that, for any integer  $n \geq 2$ , the  $(2n + \text{lh})$ -equivalence classes contain elements of order  $n$ .

As a consequence of Theorem 1.1, we obtain a necessary and sufficient condition for a link in  $S^3$  to be  $(2n + \text{lh})$ -equivalent to the trivial link by means of Milnor *numbers*.

**COROLLARY 1.3** (Corollary 6.4). — *Let  $n$  be a positive integer. An  $m$ -component link  $L$  in  $S^3$  is  $(2n + \text{lh})$ -equivalent to the trivial link if and only if  $\mu_L(I) \equiv 0 \pmod{n}$  for any non-repeated sequence  $I$ .*

In [12], R. H. Fox introduced the notion of *congruence classes modulo  $(n, q)$*  of knots in  $S^3$  for integers  $n > 0$  and  $q \geq 0$ , and asked whether the set of congruence classes of a knot determines the knot type. More precisely, he asked the following question: If two knots are congruent modulo  $(n, q)$  for every  $n$  and  $q$ , then are they ambient isotopic? We note that the notion of congruences and the question can be extended to (string) links. It is known in [12, 17, 25, 26] that the Alexander and Jones polynomials restrict the possible congruence classes. In particular, M. Lackenby proved that if two links are congruent modulo  $(n, 2)$  for every  $n$ , then they have the same Jones polynomial [17, Corollary 2.4].

Since the  $2n$ -move equivalence implies the congruence modulo  $(n, 2)$ , it would be interesting to ask whether the set of  $2n$ -move equivalence classes

of a (string) link determines the link type. Theorem 1.1 implies that if two string links are  $2n$ -move equivalent for every  $n$ , then they share all Milnor invariants for non-repeated sequences. Combining this and the classification of string links up to link-homotopy [13], we have the following corollary.

**COROLLARY 1.4.** — *If two string links are  $2n$ -move equivalent for every  $n$ , then they are link-homotopic. In particular, if a (string) link  $L$  is  $2n$ -move equivalent to the trivial one for every  $n$ , then  $L$  is link-homotopically trivial.*

## 2. Preliminaries

In this section, we summarize the definitions of string links and their Milnor invariants from [10, 13, 23, 30].

### 2.1. String links and Milnor $\mu$ -invariants

Let  $\mathbb{D}^2$  be the unit disk in the plane equipped with  $m$  points  $x_1, \dots, x_m$  in its interior, lying in order on the  $x$ -axis. Let  $I_1, \dots, I_m$  be  $m$  copies of  $[0, 1]$ . An  $m$ -component string link is the image of a proper embedding

$$\bigsqcup_{i=1}^m I_i \longrightarrow \mathbb{D}^2 \times [0, 1]$$

such that the image of each  $I_i$  runs from  $(x_i, 0)$  to  $(x_i, 1)$ . Each strand of a string link inherits an orientation from the usual orientation of  $[0, 1]$ . The  $m$ -component string link  $\{x_1, \dots, x_m\} \times [0, 1]$  in  $\mathbb{D}^2 \times [0, 1]$  is called the *trivial  $m$ -component string link*, and denoted by  $\mathbf{1}_m$ .

Given an  $m$ -component string link  $\sigma$ , let  $G(\sigma)$  denote the fundamental group of the complement  $(\mathbb{D}^2 \times [0, 1]) \setminus \sigma$  with a base point on the boundary of  $\mathbb{D}^2 \times \{0\}$ , and let  $G(\sigma)_q$  denote the  $q^{\text{th}}$  term of the lower central series of  $G(\sigma)$  ( $q = 1, 2, \dots$ ). Let  $\alpha_i$  and  $l_i$  be the  $i^{\text{th}}$  meridian and the  $i^{\text{th}}$  longitude of  $\sigma$ , respectively, illustrated in Figure 2.1. Abusing notation, we still denote by  $\alpha_i$  the image of  $\alpha_i$  in the  $q^{\text{th}}$  nilpotent quotient  $G(\sigma)/G(\sigma)_q$ . We assume that each  $l_i$  is the *preferred* longitude, i.e. the zero-framed parallel copy of the  $i^{\text{th}}$  component of  $\sigma$ . Since  $G(\sigma)/G(\sigma)_q$  is generated by  $\alpha_1, \dots, \alpha_m$  ([4, 28]), the  $i^{\text{th}}$  longitude  $l_i$  is expressed modulo  $G(\sigma)_q$  as a word in  $\alpha_1, \dots, \alpha_m$  for each  $i \in \{1, \dots, m\}$ . We denote by  $\lambda_i$  this word.

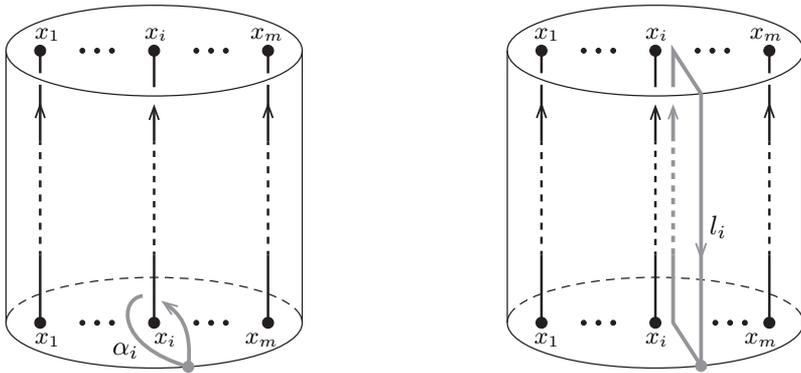


Figure 2.1. The  $i^{\text{th}}$  meridian  $\alpha_i$  and the  $i^{\text{th}}$  longitude  $l_i$

Let  $\langle \alpha_1, \dots, \alpha_m \rangle$  denote the free group on the set  $\{\alpha_1, \dots, \alpha_m\}$ , and let  $\mathbb{Z}\langle\langle X_1, \dots, X_m \rangle\rangle$  denote the ring of formal power series in non-commutative variables  $X_1, \dots, X_m$  with integer coefficients. The Magnus expansion is a homomorphism

$$E : \langle \alpha_1, \dots, \alpha_m \rangle \longrightarrow \mathbb{Z}\langle\langle X_1, \dots, X_m \rangle\rangle$$

defined, for  $1 \leq i \leq m$ , by

$$E(\alpha_i) = 1 + X_i, \quad E(\alpha_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \dots$$

Let  $I = j_1 j_2 \dots j_k i$  ( $k < q$ ) be a sequence of elements in  $\{1, \dots, m\}$ . The coefficient of  $X_{j_1} \dots X_{j_k}$  in the Magnus expansion  $E(\lambda_i)$  is called the Milnor  $\mu$ -invariant for the sequence  $I$  and denoted by  $\mu_\sigma(I)$  [13]. In particular, we define  $\mu_\sigma(i) = 0$ . The length  $|I|$  ( $= k + 1$ ) of  $I$  is called the length of  $\mu_\sigma(I)$ .

### 2.2. Milnor’s algorithm

To compute  $\mu_\sigma(I)$  we need to obtain explicitly the word  $\lambda_i$  in  $\alpha_1, \dots, \alpha_m$ . In [23], Milnor introduced an algorithm to give  $\lambda_i$  by using the Wirtinger presentation of  $G(\sigma)$  and a sequence of homomorphisms  $\eta_q$  as follows. (Although this algorithm was actually given for Milnor invariants of links in  $S^3$ , it can be applied to those of string links.)

Given an  $m$ -component string link  $\sigma$ , consider its diagram  $D_1 \cup \dots \cup D_m$ . Let  $a_{i1}$  be the arc which is incident to the endpoint  $\{x_i\} \times \{0\}$ . Successively label the other arcs of the  $i^{\text{th}}$  component  $D_i$  as  $a_{i2}, a_{i3}, \dots, a_{i r(i)}$  when we

go along  $D_i$  with respect to the orientation by starting from the arc  $a_{i1}$ , where  $r(i)$  denotes the number of arcs of  $D_i$  ( $i = 1, \dots, m$ ). Then the Wirtinger presentation of  $G(\sigma)$  has the form

$$\langle a_{ij} \ (1 \leq i \leq m, 1 \leq j \leq r(i)) \\ | a_{ij+1}^{-1} u_{ij}^{-1} a_{ij} u_{ij} \ (1 \leq i \leq m, 1 \leq j \leq r(i) - 1) \rangle,$$

where the  $u_{ij}$  are generators or inverses of generators which depend on the signs of the crossings. Here we put

$$v_{ij} = u_{i1} u_{i2} \dots u_{ij}.$$

Let  $\bar{A}$  denote the free group on the Wirtinger generators  $\{a_{ij}\}$ , and let  $A$  denote the free subgroup generated by  $a_{11}, a_{21}, \dots, a_{m1}$ . A sequence of homomorphisms  $\eta_q : \bar{A} \rightarrow A$  is defined inductively by

$$\eta_1(a_{ij}) = a_{i1}, \\ \eta_{q+1}(a_{i1}) = a_{i1}, \ \eta_{q+1}(a_{ij+1}) = \eta_q(v_{ij}^{-1} a_{i1} v_{ij}).$$

Let  $\bar{A}_q$  denote the  $q^{\text{th}}$  term of the lower central series of  $\bar{A}$ , and let  $N$  denote the normal subgroup of  $\bar{A}$  generated by the Wirtinger relations  $\{a_{ij+1}^{-1} u_{ij}^{-1} a_{ij} u_{ij}\}$ . In [23, page 290], Milnor proved that

$$(2.1) \quad \eta_q(a_{ij}) \equiv a_{ij} \pmod{\bar{A}_q N}.$$

By the construction of the Wirtinger presentation,  $a_{i1}$  represents the  $i^{\text{th}}$  meridian of  $\sigma$ . Hence, we have the natural homomorphism

$$\phi : A \longrightarrow \langle \alpha_1, \dots, \alpha_m \rangle$$

defined by  $\phi(a_{i1}) = \alpha_i$  ( $i = 1, \dots, m$ ). Since  $v_{ir(i)-1} = u_{i1} \dots u_{ir(i)-1}$  represents an  $i^{\text{th}}$  longitude, for the preferred longitude  $l_i$  we regard that  $l_i = a_{i1}^s v_{ir(i)-1}$  for some  $s \in \mathbb{Z}$ . Moreover, we can identify  $\phi \circ \eta_q(l_i)$  with  $\lambda_i$  by Congruence (2.1).

### 3. Milnor invariants and $2n$ -moves

In this section, we discuss the invariance of Milnor invariants under  $2n$ -moves.

**3.1. Milnor link-homotopy invariants and  $2n$ -moves**

The following theorem reveals how Milnor link-homotopy invariants, i.e.  $\mu$ -invariants for non-repeated sequences, behave under  $2n$ -moves.

**THEOREM 3.1.** — *Let  $n$  be a positive integer. If two string links  $\sigma$  and  $\sigma'$  are  $(2n + \text{lh})$ -equivalent, then  $\mu_\sigma(I) \equiv \mu_{\sigma'}(I) \pmod{n}$  for any non-repeated sequence  $I$ .*

For  $P, Q \in \mathbb{Z}\langle\langle X_1, \dots, X_m \rangle\rangle$ , we use the notation  $P \stackrel{(n)}{\equiv} Q$  if  $P - Q$  is contained in the ideal generated by  $n$ . To show Theorem 3.1, we need the following lemma.

**LEMMA 3.2.** — *Let  $n \geq 2$  be an integer and let  $\sigma$  be an  $m$ -component string link. For any Wirtinger generators  $a_{ij}$  and  $a_{kl}$  of  $G(\sigma)$ , there exists  $R(X_i, X_k) \in \mathbb{Z}\langle\langle X_1, \dots, X_m \rangle\rangle$  such that each term of  $R(X_i, X_k)$  contains  $X_i$  and  $X_k$ , and*

$$E\left(\phi \circ \eta_q\left(\left(a_{ij}^\varepsilon a_{kl}^\delta\right)^{\pm n}\right)\right) \stackrel{(n)}{\equiv} 1 + \binom{n}{2} R(X_i, X_k) + \mathcal{O}(2),$$

where  $\varepsilon, \delta \in \{1, -1\}$  and  $\mathcal{O}(2)$  is the ideal generated by monomials containing at least twice  $X_r$  for some  $r (= 1, \dots, m)$ .

*Proof.* — By the definition of  $\eta_q$ ,  $\phi \circ \eta_q(a_{ij}^\varepsilon) = w^{-1} \alpha_i^\varepsilon w$  for some word  $w$  in  $\alpha_1, \dots, \alpha_m$ . Put  $E(w) = 1 + W$  and  $E(w^{-1}) = 1 + \overline{W}$ , where  $W$  and  $\overline{W}$  denote the terms of degree  $\geq 1$ . Note that  $(1 + \overline{W})(1 + W) = 1$ . Then it follows that

$$\begin{aligned} E\left(\phi \circ \eta_q\left(a_{ij}^\varepsilon\right)\right) &= E\left(w^{-1} \alpha_i^\varepsilon w\right) \\ &= (1 + \overline{W})(1 + \varepsilon X_i)(1 + W) + \mathcal{O}(2) \\ &= 1 + \varepsilon X_i + \varepsilon X_i W + \varepsilon \overline{W} X_i + \varepsilon \overline{W} X_i W + \mathcal{O}(2) \\ &= 1 + \varepsilon P(X_i) + \mathcal{O}(2), \end{aligned}$$

where  $P(X_i) = X_i + X_i W + \overline{W} X_i + \overline{W} X_i W$ . Note that each term in  $P(X_i)$  contains  $X_i$ . Similarly, we have

$$E\left(\phi \circ \eta_q\left(a_{kl}^\delta\right)\right) = 1 + \delta Q(X_k) + \mathcal{O}(2),$$

where  $Q(X_k)$  denotes the terms of degree  $\geq 1$ , each of which contains  $X_k$ . Therefore, we have the following.

$$\begin{aligned}
 & E\left(\phi \circ \eta_q \left( (a_{ij}^\varepsilon a_{kl}^\delta)^n \right)\right) \\
 &= \left( (1 + \varepsilon P(X_i) + \mathcal{O}(2)) (1 + \delta Q(X_k) + \mathcal{O}(2)) \right)^n \\
 &= \left( 1 + \varepsilon P(X_i) + \delta Q(X_k) + \varepsilon \delta P(X_i) Q(X_k) + \mathcal{O}(2) \right)^n \\
 &= 1 + \sum_{r=1}^n \binom{n}{r} \left( \varepsilon P(X_i) + \delta Q(X_k) + \varepsilon \delta P(X_i) Q(X_k) + \mathcal{O}(2) \right)^r \\
 &= 1 + \sum_{r=1}^2 \binom{n}{r} \left( \varepsilon P(X_i) + \delta Q(X_k) + \varepsilon \delta P(X_i) Q(X_k) + \mathcal{O}(2) \right)^r + \mathcal{O}(2) \\
 &\stackrel{(n)}{\equiv} 1 + \binom{n}{2} \left( \varepsilon P(X_i) + \delta Q(X_k) + \varepsilon \delta P(X_i) Q(X_k) + \mathcal{O}(2) \right)^2 + \mathcal{O}(2) \\
 &\stackrel{(n)}{\equiv} 1 + \binom{n}{2} \left( P(X_i) + Q(X_k) + P(X_i) Q(X_k) + \mathcal{O}(2) \right)^2 + \mathcal{O}(2) \\
 &= 1 + \binom{n}{2} \left( P(X_i) Q(X_k) + Q(X_k) P(X_i) \right) + \mathcal{O}(2).
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 & E\left(\phi \circ \eta_q \left( (a_{ij}^\varepsilon a_{kl}^\delta)^{-n} \right)\right) \\
 &= E\left(\phi \circ \eta_q \left( (a_{kl}^{-\delta} a_{ij}^{-\varepsilon})^n \right)\right) \\
 &\stackrel{(n)}{\equiv} 1 + \binom{n}{2} \left( Q(X_k) P(X_i) + P(X_i) Q(X_k) \right) + \mathcal{O}(2) \\
 &= 1 + \binom{n}{2} \left( P(X_i) Q(X_k) + Q(X_k) P(X_i) \right) + \mathcal{O}(2).
 \end{aligned}$$

Putting  $R(X_i, X_k) = P(X_i)Q(X_k) + Q(X_k)P(X_i)$ , we obtain the desired congruence. □

*Proof of Theorem 3.1.* — Since it is obvious for  $n = 1$ , we consider the case  $n \geq 2$ . Since  $\mu$ -invariants for non-repeated sequences are link-homotopy invariants, we show that their residue classes modulo  $n$  are preserved under  $2n$ -moves.

Assume that two  $m$ -component string links  $\sigma$  and  $\sigma'$  are related by a single  $2n$ -move. A  $2n$ -move involving two strands of a single component is realized by link-homotopy. Furthermore, a  $2n$ -move whose two strands are oriented antiparallel is generated by link-homotopy and a  $2n$ -move whose strands are oriented parallel; see Figure 3.1. The link-homotopy in the figure is given by the fact that two 2-component string links having the

same linking number are link-homotopic. Thus, we may assume that two strands performing the  $2n$ -move, which relates  $\sigma$  to  $\sigma'$ , are oriented parallel and belong to different components. Moreover, the  $2n$ -move is realized by the move in the disk  $\Delta$  of Figure 3.2. It follows that there are diagrams  $D$  and  $D'$  of  $\sigma$  and  $\sigma'$ , respectively, which are identical except in  $\Delta$  where they differ as illustrated in Figure 3.2.

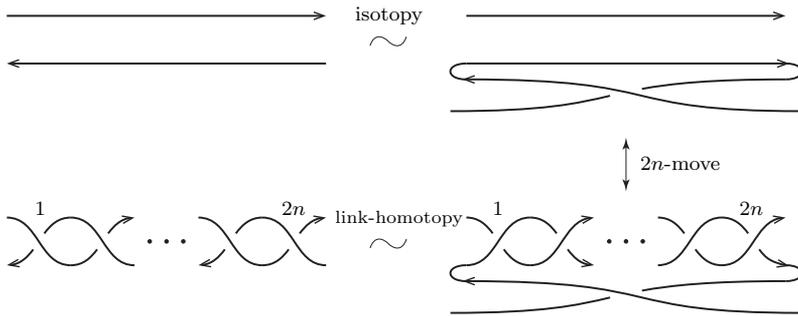


Figure 3.1.

Put labels  $a_{ij}$  ( $1 \leq i \leq m, 1 \leq j \leq r(i)$ ) on all arcs of  $D$  as described in Section 2.2, and put labels  $a'_{ij}$  on all arcs in  $D' \setminus \Delta$  which correspond to the arcs labeled  $a_{ij}$  in  $D \setminus \Delta$ . Also put labels  $b'_1, \dots, b'_{2n}, c'_1, \dots, c'_{2n}$  on the arcs of  $D'$  in  $\Delta$  as illustrated in Figure 3.2. Let  $\overline{A'}$  be the free group on  $\{a'_{ij}\} \cup \{b'_1, \dots, b'_{2n}, c'_1, \dots, c'_{2n}\}$  and  $A'$  the free subgroup on  $\{a'_{11}, a'_{21}, \dots, a'_{m1}\}$ . Let  $\eta'_q : \overline{A'} \rightarrow A'$  denote the sequence of homomorphisms associated with  $D'$  given in Section 2.2, and define a homomorphism  $\phi' : A' \rightarrow \langle \alpha_1, \dots, \alpha_m \rangle$  by  $\phi'(a'_{i1}) = \alpha_i$  ( $i = 1, \dots, m$ ).

Note that for a non-repeated sequence  $I = j_1 \dots j_k i$ , the Milnor invariant  $\mu(I)$  is the coefficient of  $X_{j_1} \dots X_{j_k}$  which does not involve  $X_i$ . Hence, for the  $i^{\text{th}}$  preferred longitudes  $l_i$  and  $l'_i$  associated with  $D$  and  $D'$ , respectively, it is enough to show that

$$(3.1) \quad E(\phi \circ \eta_q(l_i)) \stackrel{(n)}{\equiv} E(\phi' \circ \eta'_q(l'_i)) + \mathcal{O}(2) + \mathcal{P}(X_i)$$

for any  $1 \leq i \leq m$ , where  $\mathcal{P}(X_i)$  is the ideal generated by monomials containing  $X_i$ . To show this congruence, we use the following claim.

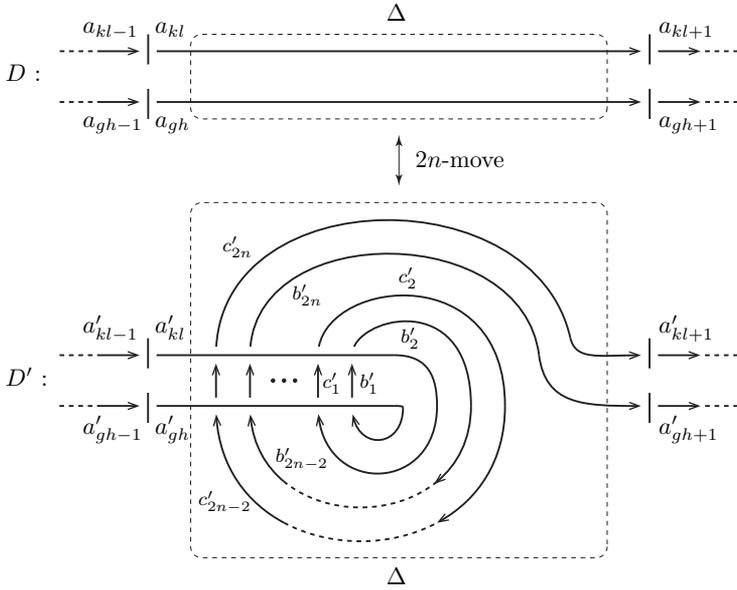


Figure 3.2.  $D$  and  $D'$  are related by a single  $2n$ -move.

CLAIM 3.3. — For any  $1 \leq i \leq m$  and  $1 \leq j \leq r(i)$ , we have

$$E(\phi \circ \eta_q(a_{ij})) \stackrel{(n)}{\equiv} E(\phi' \circ \eta'_q(a'_{ij})) + \mathcal{O}(2).$$

Before showing Claim 3.3, we observe that it implies Congruence (3.1). Without loss of generality, we may assume that  $i = 1$ , i.e. we compare the preferred longitudes  $l_1 = a_{11}^s v_{1r(1)-1}$  and  $l'_1 = a_{11}^t v'_{1r(1)-1}$  ( $s, t \in \mathbb{Z}$ ). Since the two strands in  $\Delta$  belong to different components, we only need to consider two cases.

If both of the two strands in  $\Delta$  do not belong to the 1<sup>st</sup> component, then  $s = t$  and  $l'_1$  is obtained from  $l_1$  by replacing  $u_{1j}$  with  $u'_{1j}$  ( $j = 1, \dots, r(1) - 1$ ) and  $a_{11}$  with  $a'_{11}$ . Therefore, Congruence (3.1) follows from Claim 3.3.

If one of the two strands in  $\Delta$  belongs to the 1<sup>st</sup> component, then Figure 3.2 indicates that  $l_1$  and  $l'_1$  can be written respectively in the forms

$$l_1 = a_{11}^s u_{11} \dots u_{1h-1} u_{1h} \dots u_{1r(1)-1}$$

and

$$l'_1 = a'^{s-n}_{11} u'_{11} \dots u'_{1h-1} (a'_{1h} a'_{kl})^n u'_{1h} \dots u'_{1r(1)-1}.$$

Both  $E(\phi \circ \eta_q(a^s_{11}))$  and  $E(\phi' \circ \eta'_q(a'^{s-n}_{11}))$  have the form  $1 + \mathcal{P}(X_1)$ . Furthermore, by Lemma 3.2 we have

$$E\left(\phi' \circ \eta'_q\left((a'_{1h} a'_{kl})^n\right)\right) \equiv^{(n)} 1 + \binom{n}{2} R(X_1, X_k) + \mathcal{O}(2) = 1 + \mathcal{P}(X_1) + \mathcal{O}(2).$$

Therefore, this together with Claim 3.3 proves Congruence (3.1).

Now, we turn to the proof of Claim 3.3. The proof is done by induction on  $q$ . The assertion certainly holds for  $q = 1$ . Recall that

$$\phi \circ \eta_{q+1}(a_{ij+1}) = \phi \circ \eta_q(v_{ij}^{-1} a_{i1} v_{ij})$$

and

$$\phi' \circ \eta'_{q+1}(a'_{ij+1}) = \phi' \circ \eta'_q\left(v'_{ij}{}^{-1} a'_{i1} v'_{ij}\right).$$

If  $v_{ij}$  does not pass through  $\Delta$ , then it is clear that  $v'_{ij}$  is obtained from  $v_{ij}$  by replacing  $a_{ij}$  with  $a'_{ij}$ . Therefore,

$$E(\phi \circ \eta_q(v_{ij})) \equiv^{(n)} E(\phi' \circ \eta'_q(v'_{ij})) + \mathcal{O}(2)$$

by the induction hypothesis. This implies that

$$\begin{aligned} E(\phi \circ \eta_{q+1}(a_{ij+1})) &= E(\phi \circ \eta_q(v_{ij}^{-1} a_{i1} v_{ij})) \\ &\equiv^{(n)} E\left(\phi' \circ \eta'_q\left(v'_{ij}{}^{-1} a'_{i1} v'_{ij}\right)\right) + \mathcal{O}(2) \\ &= E(\phi' \circ \eta'_{q+1}(a'_{ij+1})) + \mathcal{O}(2). \end{aligned}$$

If  $v_{ij}$  passes through  $\Delta$ , then  $v_{ij}$  and  $v'_{ij}$  can be written respectively in the forms

$$v_{ij} = u_{i1} \dots u_{ih-1} u_{ih} \dots u_{ij}$$

and

$$v'_{ij} = u'_{i1} \dots u'_{ih-1} (a'_{ih} a'_{kl})^n u'_{ih} \dots u'_{ij}.$$

Put

$$\begin{aligned} E(\phi \circ \eta_q(u_{i1} \dots u_{ih-1})) &= 1 + F, \\ E\left(\phi \circ \eta_q\left((u_{i1} \dots u_{ih-1})^{-1}\right)\right) &= 1 + \bar{F}, \\ E(\phi \circ \eta_q(u_{ih} \dots u_{ij})) &= 1 + G \end{aligned}$$

and

$$E\left(\phi \circ \eta_q\left((u_{ih} \dots u_{ij})^{-1}\right)\right) = 1 + \bar{G},$$

where  $F, \bar{F}, G$  and  $\bar{G}$  denote the terms of degree  $\geq 1$ . Then we have

$$E(\phi \circ \eta_{q+1}(a_{ij+1})) = (1 + \bar{G})(1 + \bar{F})(1 + X_i)(1 + F)(1 + G).$$

It follows from the induction hypothesis that

$$\begin{aligned} &E(\phi' \circ \eta'_{q+1}(a'_{ij+1})) \\ &\stackrel{(n)}{\equiv} (1 + \bar{G}) E\left(\phi' \circ \eta'_q\left((a'_{ih} a'_{kl})^{-n}\right)\right) (1 + \bar{F})(1 + X_i) \\ &\quad \times (1 + F) E\left(\phi' \circ \eta'_q\left((a'_{ih} a'_{kl})^n\right)\right) (1 + G) + \mathcal{O}(2). \end{aligned}$$

Lemma 3.2 implies that

$$\begin{aligned} &E(\phi' \circ \eta'_{q+1}(a'_{ij+1})) \\ &\stackrel{(n)}{\equiv} (1 + \bar{G}) \left(1 + \binom{n}{2} R(X_i, X_k)\right) (1 + \bar{F})(1 + X_i) \\ &\quad \times (1 + F) \left(1 + \binom{n}{2} R(X_i, X_k)\right) (1 + G) + \mathcal{O}(2). \end{aligned}$$

In particular, we have the following.

$$\begin{aligned} &\left(1 + \binom{n}{2} R(X_i, X_k)\right) (1 + \bar{F})(1 + X_i)(1 + F) \left(1 + \binom{n}{2} R(X_i, X_k)\right) \\ &= \left(1 + \binom{n}{2} R(X_i, X_k)\right) (1 + (1 + \bar{F}) X_i (1 + F)) \left(1 + \binom{n}{2} R(X_i, X_k)\right) \\ &= 1 + (1 + \bar{F}) X_i (1 + F) + 2 \binom{n}{2} R(X_i, X_k) + \mathcal{O}(2) \\ &\stackrel{(n)}{\equiv} 1 + (1 + \bar{F}) X_i (1 + F) + \mathcal{O}(2) \\ &= (1 + \bar{F})(1 + X_i)(1 + F) + \mathcal{O}(2). \end{aligned}$$

This proves Claim 3.3, and hence completes the proof of Theorem 3.1.  $\square$

### 3.2. Milnor isotopy invariants and $2p$ -moves

For Milnor isotopy invariants, i.e.  $\mu$ -invariants possibly with repeated sequences, we have the following.

PROPOSITION 3.4. — *Let  $p$  be a prime number. If two string links  $\sigma$  and  $\sigma'$  are  $2p$ -move equivalent, then  $\mu_\sigma(I) \equiv \mu_{\sigma'}(I) \pmod{p}$  for any sequence  $I$  of length  $\leq p$ .*

Remark 3.5.

- (1) The restriction on the length of sequences in Proposition 3.4 must be necessary. In fact, there exists the following example: Let  $\sigma = \sigma_1^4$ , where  $\sigma_1$  is the generator of 2-braids. We can verify that  $\mu_\sigma(112) = 1$  by using a computer program written by Y. Takabatake, T. Kuboyama and H. Sakamoto [29].<sup>(1)</sup> While  $\sigma$  is 4-move equivalent to  $1_2$ ,  $\mu_\sigma(112)$  is not congruent to 0 modulo 2.
- (2) Proposition 3.4 cannot be extended to the  $2n$ -move equivalence classes of string links for a nonprime number  $n$ . For example, let  $\sigma = \sigma_1^8$  then the computer program of Takabatake–Kuboyama–Sakamoto gives us that  $\mu_\sigma(211) = 10$ . While  $\sigma$  is 8-move equivalent to  $1_2$ ,  $\mu_\sigma(211)$  is not congruent to 0 modulo 4.

*Proof of Proposition 3.4.* — Let  $D$  and  $D'$  be diagrams of  $m$ -component string links  $\sigma$  and  $\sigma'$ , respectively. Assume that  $D$  and  $D'$  are related by a single  $2p$ -move whose strands are oriented parallel. (In the case where the orientations of two strands of a  $2p$ -move are antiparallel, the proof is strictly similar. Therefore, we omit the case.) We use the same notation as in the proof of Theorem 3.1. It is enough to show that, for any  $1 \leq i \leq m$ ,

$$E(\phi \circ \eta_q(l_i)) \stackrel{(p)}{\equiv} E(\phi' \circ \eta'_q(l'_i)) + (\text{terms of degree } \geq p).$$

By arguments similar to those in the proof of Theorem 3.1,  $l'_i$  is obtained from  $l_i$  by replacing  $a_{kl}$  with  $\overline{a'_{kl}}$  for all  $k, l$  and inserting the  $p^{\text{th}}$  powers of elements in the free group  $\overline{A'}$  on the Wirtinger generators of  $G(\sigma')$ . The following claim completes the proof. □

CLAIM 3.6. —

- (1) For any word  $w$  in  $\alpha_1, \dots, \alpha_m$ , we have

$$E(w^p) \stackrel{(p)}{\equiv} 1 + (\text{terms of degree } \geq p).$$

---

<sup>(1)</sup> Using the technique of “grammar compression”, Takabatake, Kuboyama and Sakamoto [29] made a computer program in the program language C++, based on Milnor’s algorithm, which is able to give us  $\mu$ -invariants of length  $\leq 16$ .

(2) For any  $1 \leq i \leq m$  and  $1 \leq j \leq r(i)$ , we have

$$E(\phi \circ \eta_q(a_{ij})) \stackrel{(p)}{\equiv} E(\phi' \circ \eta'_q(a'_{ij})) + (\text{terms of degree } \geq p).$$

*Proof.* — Put  $E(w) = 1 + W$ , where  $W$  denotes the terms of degree  $\geq 1$ . Then

$$E(w^p) = (1 + W)^p \stackrel{(p)}{\equiv} 1 + W^p.$$

This proves Claim 3.6(1).

By arguments similar to those in the proof of Claim 3.3,  $\eta'_{q+1}(a'_{ij})$  is obtained from  $\eta_{q+1}(a_{ij})$  by replacing  $\eta_q(a_{kl})$  with  $\eta'_q(a'_{kl})$  for all  $k, l$  and inserting  $\eta'_q(w^p)$  for some elements  $w$  in  $\overline{A}$ . Hence, using Claim 3.6(1), we complete the proof of Claim 3.6(2) by induction on  $q$ .  $\square$

## 4. Claspers

To show Theorem 1.1, we will use the theory of claspers introduced by K. Habiro in [14]. In this section, we briefly recall the basic notions of clasper theory from [14]. We only need the notion of  $C_k$ -tree in this paper, and refer the reader to [14] for the general definition of claspers.

### 4.1. Definitions

DEFINITION 4.1. — Let  $\sigma$  be a string link in  $\mathbb{D}^2 \times [0, 1]$ . An embedded disk  $T$  in  $\mathbb{D}^2 \times [0, 1]$  is called a tree clasper for  $\sigma$  if it satisfies the following:

- (1)  $T$  decomposes into disks and bands.<sup>(2)</sup>
- (2) Bands are called edges and each of them connects two distinct disks.
- (3) Each disk has either one or three incident edges, and is then respectively called a disk-leaf or node.
- (4)  $\sigma$  intersects  $T$  transversely and the intersections are contained in the union of the interior of the disk-leaves.

We say that  $T$  is a  $C_k$ -tree if the number of disk-leaves of  $T$  is  $k + 1$ , and is simple if each disk-leaf of  $T$  intersects  $\sigma$  at a single point. (Note that a tree clasper is called a strict tree clasper in [14].)

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<sup>(2)</sup> More precisely,  $T$  has a handle decomposition with 0-handles (“disks”) and 1-handles (“bands”).

We will make use of the drawing convention for claspers of [14, Figure 7] except for the following: A  $\oplus$  (resp.  $\ominus$ ) on an edge represents a positive (resp. negative) half-twist. This replaces the circled  $S$  (resp.  $S^{-1}$ ) notation used in [14].

Given a  $C_k$ -tree  $T$  for a string link  $\sigma$ , there is a procedure to construct a zero-framed link  $\gamma(T)$  in the complement of  $\sigma$ . *Surgery along  $T$*  means surgery along  $\gamma(T)$ . Since surgery along  $\gamma(T)$  preserves the ambient space, surgery along the  $C_k$ -tree  $T$  can be regarded as a local move on  $\sigma$  in  $\mathbb{D}^2 \times [0, 1]$ . Denote by  $\sigma_T$  the string link in  $\mathbb{D}^2 \times [0, 1]$  which is obtained from  $\sigma$  by surgery along  $T$ . Similarly, we define the string link  $\sigma_{T_1 \cup \dots \cup T_r}$  obtained from  $\sigma$  by surgery along a disjoint union of tree claspers  $T_1 \cup \dots \cup T_r$ . A  $C_k$ -tree  $T$  having the shape of the tree clasper in Figure 4.1 (with possibly some half-twists on the edges of  $T$ ) is called a *linear  $C_k$ -tree*. As illustrated in Figure 4.1, surgery along a simple linear  $C_k$ -tree for  $\sigma$  is ambient isotopic to a band summing of  $\sigma$  and the  $(k + 1)$ -component Milnor link<sup>(3)</sup>.

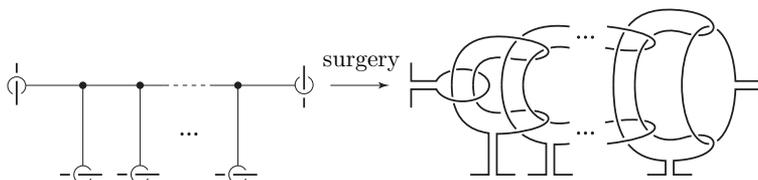


Figure 4.1. Surgery along a simple linear  $C_k$ -tree

The  $C_k$ -equivalence is the equivalence relation on string links generated by surgery along  $C_k$ -trees and ambient isotopies. Habiro proved that two string links  $\sigma$  and  $\sigma'$  are  $C_k$ -equivalent if and only if there exists a disjoint union of simple  $C_k$ -trees  $T_1 \cup \dots \cup T_r$  such that  $\sigma'$  is ambient isotopic to  $\sigma_{T_1 \cup \dots \cup T_r}$  [14, Theorem 3.17]. This implies that surgery along any  $C_k$ -tree can be replaced with surgery along a disjoint union of *simple  $C_k$ -trees*. Hereafter, by a  $C_k$ -tree we mean a simple  $C_k$ -tree.

### 4.2. Some technical lemmas

This subsection gives some lemmas, which will be used to show Theorem 1.1.

<sup>(3)</sup>Also referred to as the *Sutton Hoo link* because of a cauldron chain from the Sutton Hoo exhibited in the British Museum [10, page 222].

Given a  $C_k$ -tree  $T$  for an  $m$ -component string link  $\sigma = \sigma_1 \cup \cdots \cup \sigma_m$ , the set  $\{i \mid \sigma_i \cap T \neq \emptyset, 1 \leq i \leq m\}$  is called the *index* of  $T$  and denoted by  $\text{Ind}(T)$ . The following is a direct consequence of [11, Lemma 1.2].

LEMMA 4.2 (cf. [11, Lemma 1.2]). — *Let  $T$  be a  $C_k$ -tree for a string link  $\sigma$  with  $|\text{Ind}(T)| \leq k$ . Then  $\sigma_T$  is link-homotopic to  $\sigma$ .*

The set of ambient isotopy classes of  $m$ -component string links has a monoid structure under the *stacking product* “\*”, and with the trivial  $m$ -component string link  $\mathbf{1}_m$  as the unit element. Combining Lemma 4.2 and [31, Lemma 2.4], we have the following.

LEMMA 4.3 (cf. [31, Lemma 2.4]). — *Let  $T$  be a  $C_k$ -tree for  $\mathbf{1}_m$ , and let  $\overline{T}$  be a  $C_k$ -tree obtained from  $T$  by adding a half-twist on an edge. Then  $(\mathbf{1}_m)_T * (\mathbf{1}_m)_{\overline{T}}$  is link-homotopic to  $\mathbf{1}_m$ .*

By Lemma 4.2 together with [20, Lemma 2.2(2) and Remark 2.3], we have the following.

LEMMA 4.4 (cf. [20, Lemma 2.2(2) and Remark 2.3]). — *Let  $T_1$  be a  $C_k$ -tree for a string link  $\sigma$ , and  $T_2$  a  $C_1$ -tree for  $\sigma$ . Let  $T'_1 \cup T'_2$  be obtained from  $T_1 \cup T_2$  by changing a crossing of an edge of  $T_1$  and that of  $T_2$ . Then  $\sigma_{T_1 \cup T_2}$  is link-homotopic to  $\sigma_{T'_1 \cup T'_2}$ .*

By *parallel tree claspers* we mean a family of  $r$  parallel copies of a tree clasper  $T$  for some  $r \geq 1$ . We call  $r$  the *multiplicity* of the parallel clasper. The following can be proved by Lemma 4.2 and [20, Lemma 2.2(1) and Remark 2.3].

LEMMA 4.5 (cf. [20, Lemma 2.2(1) and Remark 2.3]). — *Let  $T_1$  be a  $C_k$ -tree for a string link  $\sigma$ , and  $T_2$  a parallel  $C_1$ -tree with multiplicity  $r$  for  $\sigma$ . Let  $T'_1 \cup T'_2$  be obtained from  $T_1 \cup T_2$  by sliding a leaf  $f$  of  $T_1$  over  $r$  parallel leaves of  $T_2$  (see Figure 4.2). Then  $\sigma_{T_1 \cup T_2}$  is link-homotopic to  $\sigma_{T'_1 \cup T'_2 \cup Y}$ , where  $Y$  denotes the parallel  $C_{k+1}$ -tree with multiplicity  $r$  obtained by inserting a vertex  $v$  in the edge  $e$  of  $T_2$  and connecting  $v$  to the edge incident to  $f$  as illustrated in Figure 4.2.*

## 5. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Habegger and Lin [13] proved that Milnor link-homotopy invariants classify string links up to link-homotopy. In [31], the third author gave an alternative proof for this by using clasper theory. Actually, he constructed

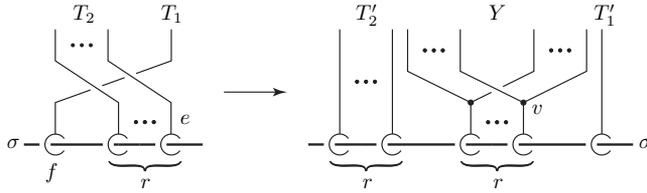


Figure 4.2.

explicit representatives, determined by Milnor link-homotopy invariants, for the link-homotopy classes as follows. Let

$$\pi : \{1, \dots, k\} \rightarrow \{1, \dots, m\} \quad (2 \leq k \leq m)$$

be an injection such that  $\pi(i) < \pi(k-1) < \pi(k)$  ( $i = 1, \dots, k-2$ ), and let  $\mathcal{F}_k$  be the set of such injections. Given  $\pi \in \mathcal{F}_k$ , let  $T_\pi$  and  $\bar{T}_\pi$  be linear  $C_{k-1}$ -trees with index  $\{\pi(1), \dots, \pi(k)\}$  illustrated in the left- and right-hand side of Figure 5.1, respectively. Here, Figure 5.1 describes the images of homeomorphisms from the neighborhoods of  $T_\pi$  and  $\bar{T}_\pi$  to the 3-ball. Putting  $V_\pi = (\mathbf{1}_m)_{T_\pi}$  and  $V_\pi^{-1} = (\mathbf{1}_m)_{\bar{T}_\pi}$ , we have the following theorem.

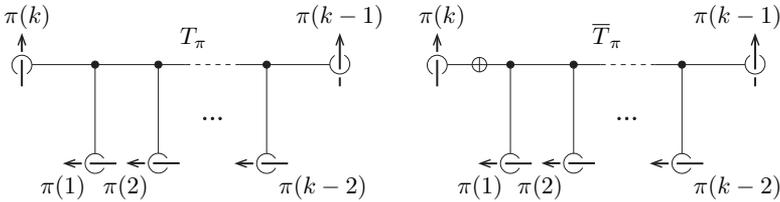


Figure 5.1. Linear  $C_{k-1}$ -trees  $T_\pi$  and  $\bar{T}_\pi$  with index  $\{\pi(1), \dots, \pi(k)\}$

**THEOREM 5.1** ([31, Theorem 4.3]). — *Let  $\sigma$  be an  $m$ -component string link. Then  $\sigma$  is link-homotopic to  $\sigma_1 * \dots * \sigma_{m-1}$ , where for each  $k$ ,*

$$\sigma_k = \prod_{\pi \in \mathcal{F}_{k+1}} V_\pi^{x_\pi},$$

with

$$x_\pi = \begin{cases} \mu_\sigma(\pi(1)\pi(2)) & (k = 1), \\ \mu_\sigma(\pi(1) \dots \pi(k+1)) - \mu_{\sigma_1 * \dots * \sigma_{k-1}}(\pi(1) \dots \pi(k+1)) & (k \geq 2). \end{cases}$$

The following is the key lemma to show Theorem 1.1.

LEMMA 5.2. — *Let  $n$  be a positive integer and  $\varepsilon \in \{1, -1\}$ . Then, for any  $\pi \in \mathcal{F}_{k+1}$  ( $1 \leq k \leq m - 1$ ),  $V_\pi^{\varepsilon n}$  is  $(2n + \text{lh})$ -equivalent to  $\mathbf{1}_m$ .*

*Proof.* — Since  $V_\pi^{-n} * V_\pi^n$  is link-homotopic to  $\mathbf{1}_m$  by Lemma 4.3, it is enough to show the case  $\varepsilon = 1$ , i.e. for any  $\pi \in \mathcal{F}_{k+1}$ ,  $V_\pi^n$  is  $(2n + \text{lh})$ -equivalent to  $\mathbf{1}_m$ . For the case  $k = 1$ , we see that  $V_\pi^n$  and  $\mathbf{1}_m$  are related by a single  $2n$ -move.

Assume that  $k \geq 2$ . Let  $T_1$  be the linear  $C_{k-1}$ -tree for  $\mathbf{1}_m$  of Figure 5.2(a) with index  $\{\pi(1), \dots, \pi(k)\}$ , and let  $\bar{T}_1$  be obtained from  $T_1$  by adding a positive half-twist on an edge. Then  $\mathbf{1}_m$  is link-homotopic to  $(\mathbf{1}_m)_{\bar{T}_1 \cup T_1}$  by Lemma 4.3. Let  $T_2$  be the parallel  $C_1$ -tree of Figure 5.2(b) with multiplicity  $n$ . Since surgery along  $T_2$  is realized by a  $2n$ -move,  $(\mathbf{1}_m)_{\bar{T}_1 \cup T_1}$  is  $2n$ -move equivalent to  $(\mathbf{1}_m)_{\bar{T}_1 \cup T_1 \cup T_2}$  in Figure 5.2(b). Let  $T'_1 \cup T'_2$  be obtained from  $T_1 \cup T_2$  by sliding a leaf of  $T_1$  over  $n$  parallel leaves of  $T_2$ , and let  $Y$  be the parallel  $C_k$ -tree with multiplicity  $n$  as illustrated in Figure 5.2(c). It follows from Lemmas 4.4 and 4.5 that  $(\mathbf{1}_m)_{\bar{T}_1 \cup T_1 \cup T_2}$  is link-homotopic to  $(\mathbf{1}_m)_{\bar{T}_1 \cup T'_1 \cup T'_2 \cup Y}$ . Furthermore, by Lemma 4.3,  $(\mathbf{1}_m)_{\bar{T}_1 \cup T'_1 \cup T'_2 \cup Y}$  is  $(2n + \text{lh})$ -equivalent to  $(\mathbf{1}_m)_Y = V_\pi^n$ .  $\square$

Combining Theorem 5.1 and Lemma 5.2, we give a complete list of representatives for string links up to  $(2n + \text{lh})$ -equivalence as follows.

PROPOSITION 5.3. — *Let  $\sigma$  be an  $m$ -component string link and  $x_\pi$  as in Theorem 5.1. Then  $\sigma$  is  $(2n + \text{lh})$ -equivalent to  $\tau_1 * \dots * \tau_{m-1}$ , where for each  $k$ ,*

$$\tau_k = \prod_{\pi \in \mathcal{F}_{k+1}} V_\pi^{y_\pi}$$

with  $0 \leq y_\pi < n$  and  $y_\pi \equiv x_\pi \pmod{n}$ .

*Proof.* — Theorem 5.1 implies that  $\sigma$  is link-homotopic to  $\sigma_1 * \dots * \sigma_{m-1}$ , where

$$\sigma_k = \prod_{\pi \in \mathcal{F}_{k+1}} V_\pi^{x_\pi}.$$

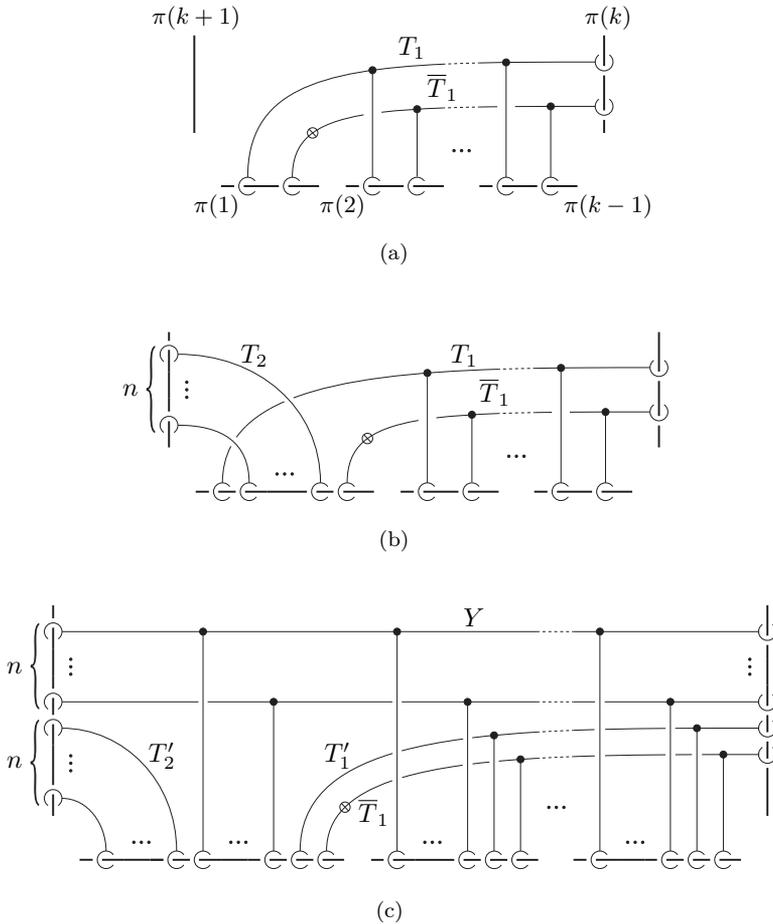


Figure 5.2.

By Lemmas 5.2 and 4.3, we can insert/delete  $V_{\pi}^{\pm n}$  and remove  $V_{\pi}^{\varepsilon} * V_{\pi}^{-\varepsilon}$  up to  $(2n + \text{lh})$ -equivalence ( $\varepsilon \in \{1, -1\}$ ). Hence,  $\sigma_k$  is  $(2n + \text{lh})$ -equivalent to  $\tau_k$  for each  $k$ . □

*Proof of Theorem 1.1.* — This follows from Theorem 3.1 and Proposition 5.3. □

*Proof of Corollary 1.2.* — By combining Theorem 1.1, Lemma 5.2 and Proposition 5.3, we have the corollary. □

*Remark 5.4.* — Theorem 1.1 characterizes Milnor link-homotopy invariants modulo  $n$  by two local moves, the  $2n$ -move and self-crossing change. In [1], B. Audoux, P. Bellingeri, J.-B. Meilhan and E. Wagner defined Milnor invariants, denoted by  $\mu^w$ , for *welded string links* and proved that  $\mu^w$ -invariants for non-repeated sequences classify welded string links up to *self-crossing virtualization*. (Later, this classification led to a link-homotopy classification of *2-dimensional string links* in 4-space [2]). For welded string links, we can show a similar result to Theorem 1.1 that characterizes  $\mu^w$ -invariants for non-repeated sequences modulo  $n$  in terms of the  $2n$ -move and self-crossing virtualization. While the idea of the proof is similar to that of Theorem 1.1, we need *arrow calculus* and representatives for welded string links up to self-crossing virtualization given in [21] instead of clasper calculus and representatives for string links up to link-homotopy. We give the details in [24].

## 6. Links in $S^3$

In the previous sections, we have studied *string links*. We now address the case of *links* in  $S^3$ .

Given an  $m$ -component string link  $\sigma$ , its *closure* is an  $m$ -component link in  $S^3$  obtained from  $\sigma$  by identifying points on the boundary of  $\mathbb{D}^2 \times [0, 1]$  with their images under the projection  $\mathbb{D}^2 \times [0, 1] \rightarrow \mathbb{D}^2$ . The link inherits an ordering and orientation from  $\sigma$ . Note that every link can be represented by the closure of some string link.

Habegger and Lin proved that for two link-homotopic links  $L$  and  $L'$ , and for a string link  $\sigma$  whose closure is  $L$ , there exists a string link  $\sigma'$  whose closure is  $L'$  such that  $\sigma'$  is link-homotopic to  $\sigma$  [13, Lemma 2.5]. Similarly, we have the following.

**LEMMA 6.1.** — *Let  $n$  be a positive integer. Let  $L$  and  $L'$  be  $(2n + \text{lh})$ -equivalent (resp.  $2n$ -move equivalent) links and  $\sigma$  a string link whose closure is  $L$ . Then there exists a string link  $\sigma'$  whose closure is  $L'$  such that  $\sigma'$  is  $(2n + \text{lh})$ -equivalent (resp.  $2n$ -move equivalent) to  $\sigma$ .*

The proof is strictly similar to that of [13, Lemma 2.5], and hence we omit it.

Let  $\sigma$  be a string link. We define  $\Delta_\sigma(I)$  to be the greatest common divisor of all  $\mu_\sigma(J)$  such that  $J$  is obtained from  $I$  by removing at least one index and permuting the remaining indices cyclically. It is known from [13] that the integer  $\Delta_\sigma(I)$  and the residue class of  $\mu_\sigma(I)$  modulo  $\Delta_\sigma(I)$  are invariants of the closure of  $\sigma$ . For a link  $L$ , we define  $\Delta_L^{(n)}(I)$  to be  $\gcd\{\Delta_\sigma(I), n\}$

and  $\overline{\mu}_L^{(n)}(I)$  to be the residue class of  $\mu_\sigma(I)$  modulo  $\Delta_L^{(n)}(I)$  for a string link  $\sigma$  whose closure is  $L$ . Obviously,  $\Delta_L^{(n)}(I)$  and  $\overline{\mu}_L^{(n)}(I)$  are invariants of  $L$ . Moreover, we have the following.

PROPOSITION 6.2. — *Let  $L$  and  $L'$  be links.*

- (1) *Let  $n$  be a positive integer. If  $L$  and  $L'$  are  $(2n + \text{lh})$ -equivalent, then  $\Delta_L^{(n)}(I) = \Delta_{L'}^{(n)}(I)$  and  $\overline{\mu}_L^{(n)}(I) = \overline{\mu}_{L'}^{(n)}(I)$  for any non-repeated sequence  $I$ .*
- (2) *Let  $p$  be a prime number. If  $L$  and  $L'$  are  $2p$ -move equivalent, then  $\Delta_L^{(p)}(I) = \Delta_{L'}^{(p)}(I)$  and  $\overline{\mu}_L^{(p)}(I) = \overline{\mu}_{L'}^{(p)}(I)$  for any sequence  $I$  of length  $\leq p$ .*

*Proof.* — Let  $\sigma$  be a string link whose closure is  $L$ . By Lemma 6.1, there exists a string link  $\sigma'$  whose closure is  $L'$  such that  $\sigma'$  is  $(2n + \text{lh})$ -equivalent to  $\sigma$ . By Theorem 3.1, for any non-repeated sequence  $I$ ,  $\mu_\sigma(I) \equiv \mu_{\sigma'}(I) \pmod n$ . Therefore,

$$\Delta_L^{(n)}(I) = \gcd \{ \Delta_\sigma(I), n \} = \gcd \{ \Delta_{\sigma'}(I), n \} = \Delta_{L'}^{(n)}(I).$$

Since  $\Delta_L^{(n)}(I)$  divides  $n$ , it follows that

$$\mu_\sigma(I) \equiv \mu_{\sigma'}(I) \pmod{\Delta_L^{(n)}(I)}.$$

This completes the proof of Proposition 6.2(1).

Using Proposition 3.4 instead of Theorem 3.1, Proposition 6.2(2) is shown similarly. □

Proposition 6.2(1) together with Theorem 1.1 implies the following.

THEOREM 6.3. — *Let  $n$  be a positive integer, and let  $L$  and  $L'$  be  $m$ -component links. Assume that  $\Delta_L^{(n)}(I) = \Delta_{L'}^{(n)}(I) = n$  for any non-repeated sequence  $I$  of length  $m$ . Then,  $L$  and  $L'$  are  $(2n + \text{lh})$ -equivalent if and only if  $\overline{\mu}_L^{(n)}(I) = \overline{\mu}_{L'}^{(n)}(I)$  for any non-repeated sequence  $I$  of length  $m$ .*

*Proof.* — Since the “only if” part directly follows from Proposition 6.2(1), it is enough to show the “if” part.

Let  $\sigma$  and  $\sigma'$  be string links whose closures are  $L$  and  $L'$ , respectively. Since  $\Delta_L^{(n)}(I) = \Delta_{L'}^{(n)}(I) = n$  for any non-repeated sequence  $I$  of length  $m$ , it follows that

$$\mu_\sigma(J) \equiv \mu_{\sigma'}(J) \equiv 0 \pmod n$$

for any non-repeated sequence  $J$  of length  $< m$ . Furthermore, since  $\overline{\mu}_L^{(n)}(I) = \overline{\mu}_{L'}^{(n)}(I)$  for any non-repeated sequence  $I$  of length  $m$ , we have

$$\mu_\sigma(I) \equiv \mu_{\sigma'}(I) \pmod n.$$

Hence,  $\sigma$  and  $\sigma'$  are  $(2n + \text{lh})$ -equivalent by Theorem 1.1. This completes the proof.  $\square$

As a consequence of Theorem 6.3, we have the following.

**COROLLARY 6.4.** — *Let  $n$  be a positive integer. An  $m$ -component link  $L$  is  $(2n + \text{lh})$ -equivalent to the trivial link if and only if  $\Delta_L^{(n)}(I) = n$  and  $\bar{\mu}_L^{(n)}(I) = 0$  for any non-repeated sequence  $I$  of length  $m$ .*

*Proof.* — This follows from Proposition 6.2(1) and Theorem 6.3.  $\square$

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