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# ON QUARTIC DOUBLE FIVEFOLDS AND THE MATRIX FACTORIZATIONS OF EXCEPTIONAL QUATERNIONIC REPRESENTATIONS

by Roland ABUAF

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ABSTRACT. — We study quartic double fivefolds from the perspective of Fano manifolds of Calabi–Yau type and that of exceptional quaternionic representations. We first prove that the generic quartic double fivefold can be represented, in a finite number of ways, as a double cover of  $\mathbb{P}^5$  ramified along a linear section of the  $\text{Spin}_{12}$ -invariant quartic in  $\mathbb{P}^{31}$ . Then, using the geometry of the Vinberg’s type II decomposition of some exceptional quaternionic representations, and backed by some cohomological computations performed by Macaulay2, we prove the existence of a spherical rank 6 vector bundle on such a generic quartic double fivefold. We finally use the existence of this vector bundle to prove that the homological unit of the CY-3 category associated by Kuznetsov to the derived category of a generic quartic double fivefold is  $\mathbb{C} \oplus \mathbb{C}[3]$ .

RÉSUMÉ. — On étudie les quartiques doubles de dimension cinq du point de vue des variétés Fano de type Calabi–Yau et des représentations quaternioniques exceptionnelles. On démontre tout d’abord qu’une quartique double de dimension cinq générique peut être représentée comme un recouvrement double de  $\mathbb{P}^5$  ramifié le long d’une section linéaire de la quartique  $\text{Spin}_{12}$ -invariante dans  $\mathbb{P}^{31}$ . Le nombre de telles représentations est fini. Ensuite, en utilisant la géométrie des décompositions de Vinberg de type II de certaines représentations quaternioniques exceptionnelles et en se basant sur des calculs effectués grâce au logiciel Macaulay2, on démontre l’existence d’un fibré vectoriel sphérique de rang 6 sur de telles quartiques doubles. On déduit finalement de l’existence de ces fibrés que l’unité homologique des catégories Calabi–Yau de dimension trois associées par Kuznetsov aux quartiques doubles de dimension cinq est  $\mathbb{C} \oplus \mathbb{C}[3]$ .

## 1. Introduction

### 1.1. Manifolds of Calabi–Yau type

Manifolds of Calabi–Yau type were defined by Iliev and Manivel [17] as compact complex manifolds of odd dimension whose middle dimensional

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Hodge structure is similar to that of a Calabi–Yau threefold. More precisely, following [17]:

DEFINITION 1.1. — *Let  $X$  be a smooth complex compact algebraic variety of odd dimension  $2n + 1$ , for  $n \geq 1$ . We say that  $X$  is of Calabi–Yau type if the following three conditions hold:*

- (1) *The middle dimensional Hodge structure of  $X$  is numerically similar to that of Calabi–Yau threefold, that is:*

$$h^{n+2, n-1}(X) = 1, \quad \text{and} \quad h^{n+p+1, n-p}(X) = 0, \quad \text{for } p \geq 2.$$

- (2) *For any generator  $\tau \in H^{n+2, n-1}(X)$ , the contraction map:*

$$H^1(X, T_X) \xrightarrow{\lrcorner \tau} H^n(X, \Omega_X^{n+1})$$

*is an isomorphism.*

- (3) *The Hodge numbers  $h^{k,0}(X)$  vanish, for all  $1 \leq k \leq 2n$ .*

Remark 1.2. — By Serre duality, a smooth threefold with trivial canonical bundle automatically satisfies condition (2) in the above definition. On the other hand, for smooth manifolds of dimension bigger than four, it seems highly non-trivial to check this condition.

Potential examples of Fano manifolds of Calabi–Yau type (namely the cubic sevenfold and the quartic double fivefold) appeared some time ago in the Physics literature (see [6, 7, 25]). They were used to describe the mirrors of some rigid Calabi–Yau threefolds obtained as crepant resolutions of product of elliptic curves divided by some well-chosen finite groups (we refer to [6, 7, 25] for more details). These examples have been put into a more systematic mathematical treatment in [17]. Physicists were however not too far from exhausting all possible examples of complete intersections in (weighted) projective spaces which should be of Calabi–Yau type. Indeed, an inspection of Hodge numbers for smooth complete intersections in weighted projective spaces reveals the following:

PROPOSITION 1.3 ([17, Section 3.1]). — *Let  $X$  be a smooth complete intersection of Calabi–Yau type in a weighted projective space. Assume that  $\dim X \geq 4$ , then  $X$  is necessarily one of the following:*

- (1) *a smooth cubic sevenfold in  $\mathbb{P}^8$ ,*
- (2) *a smooth quartic double fivefold in  $\mathbb{P}(1, 1, 1, 1, 1, 1, 2)$ ,*
- (3) *a transverse intersection of a smooth cubic and a smooth quadric in  $\mathbb{P}^7$ .*

*Remark 1.4.* — We do not assert that all examples appearing in the above proposition are of Calabi–Yau type. Indeed, as mentioned in Remark 1.2, it is non-trivial to check that condition (2) in Definition 1.1 holds for these examples. It is still an open question to prove that these varieties are Fano manifolds of Calabi–Yau type.

The manifolds exhibited in Proposition 1.3 have a lot in common with the archetypal Calabi–Yau threefold : the quintic threefold. We will enumerate some interesting geometric and cohomological properties of the quintic threefold which have been (even partially) shown to be true for the above manifolds.

- (1) Clemens proved that the Griffiths group of smooth quintic in  $\mathbb{P}^4$  is not finitely generated [8]. Favero, Iliev and Katzarkov defined a notion of Griffiths group for the manifolds which appear in Proposition 1.3. Using some earlier work of Voisin [29], they showed that the Griffiths group of these manifolds have an infinitely generated Griffiths group [11]. We refer to the earlier work of Albano and Collino for the case of the cubic sevenfold [4].
- (2) Beauville proved that a generic quintic in  $\mathbb{P}^4$  has a finite number of determinantal representations [5]. In [16], Iliev and Manivel extended Beauville’s result to the case of cubic sevenfolds: a generic cubic in  $\mathbb{P}^8$  can be represented, in a finite number of ways, as a linear sections of the  $E_6$ -invariant cubic hypersurface in  $\mathbb{P}^{26}$ , the Cartan cubic.
- (3) Any line bundle on the quintic threefold is *spherical* (i.e. its Ext-algebra is isomorphic to  $\mathbb{C} \oplus \mathbb{C}[3]$ ) and spherical objects provide non-trivial auto-equivalences of the derived category of the quintic threefold (see [26]). Kuznetsov proved that the derived categories of the manifolds appearing in Proposition 1.3 always contain a semi-orthogonal component which is a Calabi–Yau category of dimension 3 [20]. Furthermore, Iliev and Manivel exhibited examples of spherical vector bundles contained in the CY-3 category associated to the derived category of a generic cubic sevenfold [16].
- (4) Generic quintic threefolds are endowed with a so-called Yukawa coupling which satisfies very interesting equations (see [22] for instance). It is explained in [7] that similar properties hold for the Yukawa coupling constructed on generic cubic sevenfolds. [17] ask if this could be true for any manifold of Calabi–Yau type.

Obviously, we do not claim that this enumeration is exhaustive in any sense. In fact, this is quite the opposite : we hope that many other remarkable

features of the quintic threefold will be shared by the complete intersection manifolds of Calabi–Yau type.

### 1.2. Generic quartic double fivefolds

In this paper, we will focus on quartic double fivefolds. Any such manifold is the zero locus of a weighted homogeneous polynomial of the form  $f_4(z_1, \dots, z_6) + x^2$  in  $\mathbb{P}(1, 1, 1, 1, 1, 2)$  where  $f_4$  is an element of  $S^4\mathbb{C}^6$ . Our main results are the following:

**THEOREM 1.5** (see Theorem 2.2). — *The generic quartic double fivefold can be represented, in a finite number of ways, as a double cover of  $\mathbb{P}^5$  ramified along a linear section of the  $\text{Spin}_{12}$ -invariant quartic  $\mathcal{Q}_{\text{Spin}_{12}} \subset \mathbb{P}^{31}$  (the Igusa quartic).*

**THEOREM 1.6** (see Theorem 3.8). — *The 3 dimensional Calabi–Yau category associated to the derived category of the generic quartic double fivefold contains a rank 6 spherical vector bundle.*

Our proof of Theorem 1.5 uses the strategy already highlighted in [3, 5, 16]. Namely, if  $P_{\text{Spin}_{12}}$  is an equation for the  $\text{Spin}_{12}$  invariant quartic  $\mathcal{Q}_{\text{Spin}_{12}} \subset \mathbb{P}^{31}$ , we prove that the pull-backs of the partial derivatives of  $P_{\text{Spin}_{12}}$  to  $\mathbb{C}[z_1, \dots, z_6]$  by a generic  $32 \times 6$  matrix generate  $S^4\mathbb{C}^6$ . We then deduce that the natural map:

$$G(6, \Delta) // \text{Spin}_{12} \longrightarrow S^4\mathbb{C}^6 // \text{GL}_6$$

which associates to  $L \in G(6, \Delta)$  its intersection with  $\mathcal{Q}_{\text{Spin}_{12}} \subset \mathbb{P}(\Delta)$  is generically étale. The computation of the dimension of the space generated by the pull-backs of the partial derivatives of  $P_{\text{Spin}_{12}}$  to  $\mathbb{C}[z_1, \dots, z_6]$  is done using Macaulay2 [12].

In order to demonstrate Theorem 1.6, we use the basic geometry of some exceptional quaternionic representations ([9, 13, 27]). We first start with the Vinberg type II decomposition of the Lie algebra  $\mathfrak{e}_6$ :

$$(1.1) \quad \mathfrak{e}_6 = \mathbb{C}^* \oplus \left( \bigwedge^3 \mathbb{C}^6 \right)^* \oplus \mathfrak{gl}_6 \oplus \bigwedge^3 \mathbb{C}^6 \oplus \mathbb{C},$$

The properties of type II grading for exceptional quaternionic representations entail that for any  $y \in \bigwedge^3 \mathbb{C}^6$ , we have:

$$(\text{ad}_y^{\mathfrak{e}_6})^4(X_{-\beta}) = P_{\text{SL}_6}(y).X_\beta,$$

where  $X_{-\beta}$  and  $X_\beta$  are generators of the one-dimensional factors appearing in degree  $-2$  and  $2$  in the decomposition (1.1) and  $P_{\text{SL}_6}$  is the equation

of the  $SL_6$  invariant quartic  $\mathcal{Q}_{SL_6} \subset \mathbb{P}(\wedge^3 \mathbb{C}^6)$ . As  $(\text{ad}_y^{\epsilon_6})^2$  is an element of  $\mathfrak{gl}_6^* \simeq \mathfrak{gl}_6$ , we deduce that the pair  $((\text{ad}_y^{\epsilon_6})^2, (\text{ad}_y^{\epsilon_6})^2)$  is a matrix factorization of  $P_{SL_6}$  (see Lemma 3.6). The matrices  $B = (\text{ad}_y^{\epsilon_6})^2 + ix.I_6$  and  $C = (\text{ad}_y^{\epsilon_6})^2 - ix.I_6$  form therefore a matrix factorization of  $P_{SL_6}(y) + x^2$ . Restricting  $B$  and  $C$  to a generic  $\mathbb{P}^5 \subset \mathbb{P}(\wedge^3 \mathbb{C}^6)$ , we deduce the existence of a specific matrix factorization for the quartic double fivefold ramified along a generic fourfold linear section of  $\mathcal{Q}_{SL_6}$ . Cohomological properties of the restriction of the cokernel of  $B$  to the quartic double fivefold determined by the choice of this  $\mathbb{P}^5$  will play a crucial role in the proof of Theorem 1.6.

In [16], Iliev and Manivel used similar ideas in order to construct a spherical rank 9 vector bundle on the generic cubic sevenfold. Their proof that this vector bundle is spherical highlights the impressive virtuosity of the authors in manipulating the Borel–Bott–Weil Theorem in type  $E_6$ . Our cohomological study of the cokernel of  $B$  is certainly less elegant but it has the merit to be more accessible to the layman : we compute the necessary Ext-groups using Macaulay2 [12]. The implementation of the matrix representing  $(\text{ad}_y^{\epsilon_6})^2 + ix.I_6$  follows an explicit description given in [18].

In order to prove Theorem 1.6, we consider the type II decomposition:

$$(1.2) \quad \epsilon_7 = \mathbb{C}^* \oplus \Delta^* \oplus \mathfrak{so}_{12} \oplus \mathbb{C} \oplus \Delta \oplus \mathbb{C}.$$

Once again, the properties of type II grading for quaternionic representations yield that for any  $z \in \Delta$ , we have:

$$(\text{ad}_z^{\epsilon_7})^4(X_{-\beta}) = P_{Spin_{12}}(z).X_{\beta},$$

where  $X_{-\beta}$  and  $X_{\beta}$  are generators of the one-dimensional factors appearing in degree  $-2$  and  $2$  in the decomposition (1.2) and  $P_{Spin_{12}}$  is the equation of the  $Spin_{12}$ -invariant quartic  $\mathcal{Q}_{Spin_{12}} \subset \mathbb{P}(\Delta)$ .

Let  $L \subset \mathbb{P}(\Delta)$  be a generic  $\mathbb{P}^5$  and denote by  $X_L$  the quartic double fivefold ramified over  $L \cap \mathcal{Q}_{Spin_{12}}$ . The restrictions to  $L$  of the matrices  $\tilde{B} = (\text{ad}_z^{\epsilon_7})^2 + ix.I_{12}$  and  $\tilde{C} = (\text{ad}_z^{\epsilon_7})^2 - ix.I_{12}$  provide a matrix factorization of the equation of  $X_L$  in  $\mathbb{P}(1, \dots, 1, 2)$ . If  $L_0$  is a generic  $\mathbb{P}^5$  in  $\mathbb{P}(\wedge^3 \mathbb{C}^6)$ , we are able to relate the cohomological properties of the restriction to  $X_L$  of cokernel of  $\tilde{B}(\text{ad}_z^{\epsilon_7})^2 + ix.I_{12}$  to those of the restriction to  $X_{L_0}$  of the cokernel of  $B(\text{ad}_y^{\epsilon_6})^2 + ix.I_6$ . We then deduce that a twist of the restriction to  $X_L$  of the cokernel of  $\tilde{B}$  is the spherical rank 6 vector bundle whose existence is claimed in Theorem 1.6.

In the last section of this paper, we discuss a “topological” application of the existence of a spherical vector bundle on the generic quartic double fivefold. In [2], the concept of *homological unit* was introduced for a large class of triangulated categories as a replacement for the algebra  $H^\bullet(\mathcal{O}_X)$

when the category under study is not (necessarily) the derived category of a projective variety. We prove here (see Section 3.3) that the homological unit of the 3 dimensional Calabi–Yau category associated to the derived category of a generic quartic double fivefold is  $\mathbb{C} \oplus \mathbb{C}[3]$ . This computation shows that the 3-dimensional Calabi–Yau category associated to the derived category of a generic double quartic fivefold is really a non-commutative analogue of a Calabi–Yau threefold, and not just a triangulated category whose Serre functor is the shift by [3].

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## 2. Linear sections of the $\text{Spin}_{12}$ invariant quartic in $\mathbb{P}^{31}$

### 2.1. The $\text{Spin}_{12}$ quartic invariant in $\mathbb{P}^{31}$

We will start this section with a quick reminder of the work of Igusa [15] on spinors of dimension 12. Let  $\Delta$  be one of the half-spin representation of dimension 32 for the group  $\text{Spin}_{12}$ . Recall that, as a vector space, we have:

$$\Delta \simeq \mathbb{C} \oplus \bigwedge^2 \mathbb{C}^6 \oplus \bigwedge^4 \mathbb{C}^6 \oplus \bigwedge^6 \mathbb{C}^6$$

Igusa proves that  $\mathbb{P}(\Delta)$  is a prehomogeneous space for  $\text{Spin}_{12}$  and that this space has a relative invariant which is of degree 4. Let us denote by  $P_{\text{Spin}_{12}}$  the equation of this relative invariant and by  $\mathcal{Q}_{\text{Spin}_{12}} \subset \mathbb{P}^{31}$  the corresponding quartic hypersurface. Igusa gives an explicit description for  $P_{\text{Spin}_{12}}$ , namely :

$$P_{\text{Spin}_{12}}(x) = x_0 \text{Pff}((x_{i,j})) + y_0 \text{Pff}((y_{i,j})) + \sum_{i < j} \text{Pff}((X_{i,j})) \text{Pff}((Y_{i,j})) - \frac{1}{4} \left( x_0 y_0 - \sum_{i < j} x_{i,j} y_{i,j} \right)^2,$$

for  $x = x_0 + \sum_{i < j} x_{i,j} \cdot e_i \wedge e_j + \sum_{i < j} y_{i,j} \cdot (e_i \wedge e_j)^* + y_0 \cdot e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6$  in which  $(x_{i,j})$  (resp.  $(y_{i,j})$ ) is the alternating matrix determined by the  $x_{i,j}$  (resp.  $y_{i,j}$ ) and  $(X_{i,j})$  (resp.  $(Y_{i,j})$ ) is the alternating matrix obtained from  $(x_{i,j})$  (resp.  $(y_{i,j})$ ) by crossing out its  $i$ -th and  $j$ -th lines and columns.

*Remark 2.1.* — The quartic hypersurface in  $\mathbb{P}^{31}$  which equation is given above is the tangent variety of the spinor variety  $\mathbb{S}_{12} \subset \mathbb{P}^{31}$ . This spinor variety appears in the third line of the Tits–Freudenthal magic square as the symplectic Grassmannian  $G_w(\mathbb{H}^3, \mathbb{H}^6)$  where  $\mathbb{H}$  is the algebra of complexified quaternions. In [21], Landsberg and Manivel found a general formula for the equations of the tangent varieties of the homogeneous spaces which appear in the third line of the magic square. Namely, this formula gives a uniform presentation for the equation of the tangent variety of  $v_3(\mathbb{P}^1) \subset \mathbb{P}^3$ ,  $G_w(3, 6) \subset \mathbb{P}^{13}$ ,  $G(3, 6) \subset \mathbb{P}^{19}$ ,  $\mathbb{S}_{12} \subset \mathbb{P}^{31}$  and  $E_7/P_7 \subset \mathbb{P}^{55}$ .

**2.2. Four-dimensional linear sections of the quartic  $\mathcal{Q}_{\text{Spin}_{12}} \subset \mathbb{P}^{31}$**

In [16], Iliev and Manivel proved that a generic cubic hypersurface in  $\mathbb{P}^8$  can be written, in a finite number of ways, as a linear section of the  $E_6$  invariant cubic in  $\mathbb{P}^{26}$ . They observed a similar numerical coincidence in the case of quartic hypersurfaces in  $\mathbb{P}^5$ . More precisely, let us denote by  $\mathcal{M}_4^4$  the moduli space of quartic hyperurfaces in  $\mathbb{P}^5$ . There is a natural map  $\Phi : G(6, \Delta) // \text{Spin}_{12} \rightarrow \mathcal{M}_4^4$ , which is given by restriction of the quartic  $\mathcal{Q}_{\text{Spin}_{12}}$  to a given  $\mathbb{P}^5 \subset \mathbb{P}^{31}$ . Note that the singular locus of  $\mathcal{Q}_{\text{Spin}_{12}}$  has dimension 24, so that a generic 4-dimensional linear section of  $\mathcal{Q}_{\text{Spin}_{12}}$  is smooth.

Iliev and Manivel noted that both  $\mathcal{M}_4^4$  and  $G(6, \Delta) // \text{Spin}_{12}$  have expected dimension 90 and they ask if the map  $\Phi$  is dominant. We answer positively to their question :

**THEOREM 2.2.** — *Let  $\Phi : G(6, \Delta) // \text{Spin}_{12} \rightarrow \mathcal{M}_4^4$  be the map which associates to  $L$  the quartic  $L \cap \mathcal{Q}_{\text{Spin}_{12}}$ . The map  $\Phi$  is generically étale and dominant. In particular a generic quartic hypersurface in  $\mathbb{P}^5$  can be represented as a linear section of  $\mathcal{Q}_{\text{Spin}_{12}} \subset \mathbb{P}(\Delta)$  in a finite number of ways.*

*Proof.* — First of all, we prove that the dimension of  $G(6, \Delta) // \text{Spin}_{12}$  is equal to its expected dimension, that is  $\dim G(6, \Delta) - \dim \text{Spin}_{12} = 90$ . We only have to show that the generic stabilizer of the action of  $\text{Spin}_{12}$  on  $G(6, \Delta)$  is finite. For generic  $L \in G(6, \Delta)$ , we denote by  $G_L$  the stabilizer of  $L$  in  $\text{Spin}_{12}$ . In the proof of Theorem 3.8 (which is independent of the Theorem 2.2), we will exhibit a  $\text{Spin}_{12}$ -equivariant matrix factorization of  $\mathcal{Q}_{\text{Spin}_{12}}$ , say  $(X, Y)$  <sup>(1)</sup>, such that  $\text{Hom}((X|_L, Y|_L), (X|_L, Y|_L)) = \mathbb{C}$ . The

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<sup>(1)</sup> In fact, as will be explained in the proof of Theorem 3.8, we have  $X = Y$ .

restriction  $(X|_L, Y|_L)$  is  $G_L$ -equivariant. Furthermore, the map  $G_L \rightarrow \text{Aut}((X|_L, Y|_L))$  is injective because  $(X|_L, Y|_L)$  is a matrix factorization of  $\mathcal{Q}_{\text{Spin}_{12}} \cap L$  and  $G_L$  injects into  $\text{Aut}(\mathcal{Q}_{\text{Spin}_{12}} \cap L)$ . We deduce that  $G_L$  is a finite subgroup of  $\text{Aut}((X|_L, Y|_L)) = \mathbb{C}^*$ .

We then notice that  $G(6, \Delta) = M_{6 \times \Delta} // \text{GL}_6$  and that  $\mathcal{M}_4^4 = S^4 \mathbb{C}^6 // \text{GL}_6$ , where  $M_{6 \times \Delta}$  is the space of linear maps from  $\mathbb{C}^6$  to  $\Delta$ . Furthermore, the map  $\Phi$  is the descent of the natural map:

$$\phi : M_{6 \times \Delta} \rightarrow S^4 \mathbb{C}^6$$

$$(m)_{i,j} \rightarrow P_{\text{Spin}_{12}}(m_{1,1} \cdot z_1 + \dots + m_{1,6} \cdot z_6, \dots, m_{32,1} \cdot z_1 + \dots + m_{32,6} \cdot z_6),$$

where  $z_1, \dots, z_6$  is a basis for  $(\mathbb{C}^6)^*$ . Since  $\dim S^4 \mathbb{C}^6 // \text{GL}_6 \geq \dim S^4 \mathbb{C}^6 - \dim \text{GL}_6 = 90$ , in order to prove Theorem 2.2, we just need to prove that the differential of  $\phi$  is generically surjective. The differential of  $\phi$  at a point  $(m)_{i,j} \in M_{6 \times \Delta}$  is given by :

$$d\phi_{(m_{i,j})} : M_{6 \times \Delta} \rightarrow S^4 \mathbb{C}^6$$

$$(q)_{i,j} \rightarrow \sum_{i=1}^{32} \left( \sum_{j=1}^6 q_{i,j} \cdot z_j \right) \times \frac{\partial P_{\text{Spin}_{12}}}{\partial x_i} (m_{1,1} \cdot z_1 + \dots + m_{1,6} \cdot z_6, \dots, m_{32,1} \cdot z_1 + \dots + m_{32,6} \cdot z_6),$$

where  $x_1, \dots, x_{32}$  is a basis of  $\Delta^*$  in which the equation of  $P_{\text{Spin}_{12}}$  is given. As a consequence, in order to prove that  $d\phi$  is generically surjective, we only have to prove the lemma below.

LEMMA 2.3. — *For generic  $m \in M_{6 \times \Delta}$ , the dimension of the subspace of  $S^4 \mathbb{C}^6$  generated by the:*

$$\left\{ z_j \cdot \frac{\partial P_{\text{Spin}_{12}}}{\partial x_i} (m_{1,1} \cdot z_1 + \dots + m_{1,6} \cdot z_6, \dots, m_{32,1} \cdot z_1 + \dots + m_{32,6} \cdot z_6) \right\}_{i=1 \dots 32, j=1 \dots 6}$$

has dimension 126 and is equal to  $S^4 \mathbb{C}^6$ .

*Proof.* — We use Macaulay2 to prove that the lemma. A similar algorithm is provided in [5] in order to prove that a general threefold in  $\mathbb{P}^4$  of degree less than 5 is Pfaffian. □

### 3. Matrix factorizations on generic quartic double fivefolds

In this section we prove the existence of a spherical rank 6 vector bundle on the double cover of  $\mathbb{P}^5$  ramified over a general fourfold linear section of

$\mathcal{Q}_{\text{Spin}_{12}}$ . In fact, we will derive the vanishing properties of this vector bundle from those of a rank 3 coherent sheaf defined on the double quartic fivefold ramified over a general section of  $\mathcal{Q}_{\text{SL}_6} \subset \mathbb{P}(\wedge^3 \mathbb{C}^6)$ . Thus, we start our study with a particular matrix factorization on the  $\text{SL}_6$ -invariant quartic  $\mathcal{Q}_{\text{SL}_6} \subset \mathbb{P}(\wedge^3 \mathbb{C}^6)$ , whose cokernel will be the sheaf we are interested in.

**3.1. Foretaste : matrix factorizations on quartic double fivefolds ramified over linear sections of  $\mathcal{Q}_{\text{SL}_6} \subset \mathbb{P}(\wedge^3 \mathbb{C}^6)$**

Let  $u_1, \dots, u_6$  be a basis of  $\mathbb{C}^6$  and let  $y_1, \dots, y_{20}$  be coordinates on  $\wedge^3 \mathbb{C}^6$  such that any  $y \in \wedge^3 \mathbb{C}^6$  can be written  $y = y_1 \cdot u_1 \wedge u_2 \wedge u_3 + \dots + y_{20} \cdot u_5 \wedge u_5 \wedge u_6$ . Denote by  $P_{\text{SL}_6}$  the equation of  $\mathcal{Q}_{\text{SL}_6}$ . The explicit formula for  $\mathcal{Q}_{\text{SL}_6}$  is given, for instance, on page 83 of [18]. Denote by:

$$Y^{(a)} := \begin{pmatrix} y_{11} & -y_5 & y_2 \\ y_{12} & -y_6 & y_3 \\ y_{13} & -y_7 & y_4 \end{pmatrix} \quad Y^{(b)} := \begin{pmatrix} y_{10} & -y_9 & y_8 \\ y_{16} & -y_{15} & y_{14} \\ y_{19} & -y_{18} & y_{17} \end{pmatrix},$$

then we have:

$$P_{\text{SL}_6}(y) = \left( y_1 y_{20} - \text{Tr}(Y^{(a)} Y^{(b)}) \right)^2 + 4y_1 \det(Y^{(b)}) + 4y_{20} \det(Y^{(a)}) - 4 \sum_{i,j} \det(Y_{i,j}^{(a)}) \det(Y_{i,j}^{(b)}),$$

where  $Y_{i,j}^{(a)}$  (resp.  $Y_{i,j}^{(b)}$ ) is the matrix obtained from  $Y^{(a)}$  (resp.  $Y^{(b)}$ ) by crossing its  $i$ -th line and  $j$ -th column. We recall the expression of a special matrix factorization of  $P_{\text{SL}_6}$  that was explicited by Kimura and Sato (see [18] page 80 and 81). For  $k = 1 \dots 6$ , define the operators:

$$D_k : \wedge^k \mathbb{C}^6 \longrightarrow \wedge^{k-1} \mathbb{C}^6 \otimes \mathbb{C}^6$$

$$u_{i_1} \wedge \dots \wedge u_{i_k} \longrightarrow \sum_{r=1}^k (-1)^{k-r} (u_{i_1} \wedge \dots \wedge u_{i_{r-1}} \wedge u_{i_{r+1}} \wedge \dots \wedge u_{i_k}) \otimes u_{i_r}.$$

For each  $y \in \wedge^3 \mathbb{C}^6$  and each  $z \in \wedge^4 \mathbb{C}^6$ , we have  $(z \otimes 1) \wedge D_3(y) \in \wedge^6 \mathbb{C}^6 \otimes \mathbb{C}^6 = \tau \otimes \mathbb{C}^6$ , where  $\tau = u_1 \wedge \dots \wedge u_6$  is the canonical volume form on  $\wedge^6 \mathbb{C}^6$ . Hence, there exists a bilinear map  $L : \wedge^4 \mathbb{C}^6 \times \wedge^3 \mathbb{C}^6 \longrightarrow \mathbb{C}^6$  such that  $(z \otimes 1) \wedge D_3(y) = \tau \otimes L(z, y)$  for all  $y \in \wedge^3 \mathbb{C}^6$  and  $z \in \wedge^4 \mathbb{C}^6$ . Now, for each  $y \in \wedge^3 \mathbb{C}^6$ , we define an operator:

$$S_y : \mathbb{C}^6 \longrightarrow \mathbb{C}^6$$

$$\theta \longrightarrow L(\theta \wedge y, y)$$

Kimura and Sato proves the following ([18], Proposition 7 page 81):

PROPOSITION 3.1. — *For all  $y \in \bigwedge^3 \mathbb{C}^6$ , we have  $S_y^2 = P_{\text{SL}_6}(y).I_6$ , where  $I_6$  is the  $6 \times 6$  identity.*

We will give the explicit form of the matrix  $S_y$ , when  $y = y_1.u_1 \wedge u_2 \wedge u_3 + \dots + y_{20}.u_5 \wedge u_5 \wedge u_6$ . Since it is too big to be reasonably displayed in  $\text{\LaTeX}$ , we write it in Macaulay2 code. Furthermore, writing this in Macaulay2 code will allow anyone willing to use the matrix for further Macaulay2 computations to just copy and paste it.

PROPOSITION 3.2. — *The matrix of  $S_y$  in the basis  $u_1, \dots, u_6$  is :*

```
S_y = matrix{{y_10*y_11-y_9*y_12+y_8*y_13+y_7*y_14-y_6*y_15+
y_5*y_16-y_4*y_17+y_3*y_18-y_2*y_19+y_1*y_20,
2*(y_13*y_14-y_12*y_15+y_11*y_16),
2*(y_13*y_17-y_12*y_18+y_11*y_19),
2*(y_15*y_17-y_14*y_18+y_11*y_20),
2*(y_16*y_17-y_14*y_19+y_12*y_20),
2*(y_16*y_18-y_15*y_19+y_13*y_20)},
{2*(-y_7*y_8+y_6*y_9-y_5*y_10),
-y_10*y_11+y_9*y_12-y_8*y_13-y_7*y_14+y_6*y_15-y_5*y_16-y_4*y_17+
y_3*y_18-y_2*y_19+y_1*y_20,
2*(-y_7*y_17+y_6*y_18-y_5*y_19),
2*(-y_9*y_17+y_8*y_18-y_5*y_20),
2*(-y_10*y_17+y_8*y_19-y_6*y_20),
2*(-y_10*y_18+y_9*y_19-y_7*y_20)},
{2*(y_4*y_8-y_3*y_9+y_2*y_10),
2*(y_4*y_14-y_3*y_15+y_2*y_16),
-y_10*y_11+y_9*y_12-y_8*y_13+y_7*y_14-y_6*y_15+y_5*y_16+y_4*y_17-
y_3*y_18+y_2*y_19+y_1*y_20,
2*(y_9*y_14-y_8*y_15+y_2*y_20),
2*(y_10*y_14-y_8*y_16+y_3*y_20),
2*(y_10*y_15-y_9*y_16+y_4*y_20)},
{2*(-y_4*y_6+y_3*y_7-y_1*y_10),
2*(-y_4*y_12+y_3*y_13-y_1*y_16),
2*(-y_7*y_12+y_6*y_13-y_1*y_19),
-y_10*y_11-y_9*y_12+y_8*y_13-y_7*y_14+y_6*y_15+y_5*y_16+y_4*y_17-
y_3*y_18-y_2*y_19-y_1*y_20,
2*(-y_10*y_12+y_6*y_16-y_3*y_19),
2*(-y_10*y_13+y_7*y_16-y_4*y_19)},
{2*(y_4*y_5-y_2*y_7+y_1*y_9),
2*(y_4*y_11-y_2*y_13+y_1*y_15),
2*(y_7*y_11-y_5*y_13+y_1*y_18),
```

$$\begin{aligned} &2*(y_9*y_{11}-y_5*y_{15}+y_2*y_{18}), \\ &y_{10}*y_{11}+y_9*y_{12}+y_8*y_{13}-y_7*y_{14}-y_6*y_{15}-y_5*y_{16}+y_4*y_{17}+ \\ &y_3*y_{18}+y_2*y_{19}-y_1*y_{20}, \\ &2*(y_9*y_{13}-y_7*y_{15}+y_4*y_{18}), \end{aligned}$$

$$\begin{aligned} &\{2*(-y_3*y_5+y_2*y_6-y_1*y_8), \\ &2*(-y_3*y_{11}+y_2*y_{12}-y_1*y_{14}), \\ &2*(-y_6*y_{11}+y_5*y_{12}-y_1*y_{17}), \\ &2*(-y_8*y_{11}+y_5*y_{14}-y_2*y_{17}), \\ &2*(-y_8*y_{12}+y_6*y_{14}-y_3*y_{17}), \\ &y_{10}*y_{11}-y_9*y_{12}-y_8*y_{13}+y_7*y_{14}+y_6*y_{15}-y_5*y_{16}-y_4*y_{17}- \\ &y_3*y_{18}+y_2*y_{19}-y_1*y_{20}\} \end{aligned}$$

*Proof.* — Trivial (though lengthy) handmade computations. One checks with Macaulay2 that  $S_y^2 = P_{SL_6}(y).I_6$ . □

As a consequence of Proposition 3.1, we have a exact sequence in  $\mathbb{P}(\wedge^3 \mathbb{C}^6)$ :

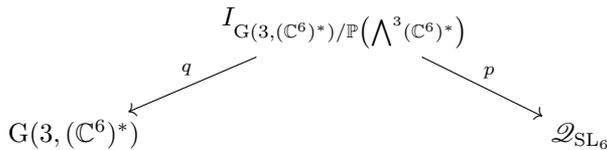
$$0 \longrightarrow \mathbb{C}^6 \otimes \mathcal{O}_{\mathbb{P}(\wedge^3 \mathbb{C}^6)}(-2) \xrightarrow{S_y} \mathbb{C}^6 \otimes \mathcal{O}_{\mathbb{P}(\wedge^3 \mathbb{C}^6)} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where  $\mathcal{F}$  is a sheaf scheme-theoretically supported on  $\mathcal{Q}_{SL_6}$ . Let  $L \subset \mathbb{P}(\wedge^3 \mathbb{C}^6)$  be a generic  $\mathbb{P}^5$ . By restricting the above sequence on  $L$ , we get a sequence:

$$0 \longrightarrow \mathbb{C}^6 \otimes \mathcal{O}_L(-2) \xrightarrow{S_y|_L} \mathbb{C}^6 \otimes \mathcal{O}_L \longrightarrow \mathcal{F}|_L \longrightarrow 0.$$

The matrix  $S_L := S_y|_L$  has full rank at a generic point of  $L$  and drops rank on  $\mathcal{Q}_{SL_6} \cap L$ . We deduce that  $\mathcal{F}_L$  is the push-forward of a pure sheaf (say  $F_L$ ) living on  $\mathcal{Q}_{SL_6} \cap L$ . Since  $\text{deg}(\mathcal{Q}_{SL_6} \cap L) = 4$  and  $\text{deg}(\det(S_L)) = 12$ , we find that  $\text{rank}(F_L) = 3$ .

*Remark 3.3.* — The above argument shows that  $\mathcal{F}$  is the push-forward of a rank 3 sheaf on  $\mathcal{Q}_{SL_6}$  (which we denote by  $F$ ). Remember that  $\mathcal{Q}_{SL_6} \subset \mathbb{P}(\wedge^3 \mathbb{C}^6)$  is the projective dual of  $G(3, (\mathbb{C}^6)^*) \subset \mathbb{P}(\wedge^3 (\mathbb{C}^6)^*)$  and we have a diagram:



where  $I_{G(3, (\mathbb{C}^6)^*)/\mathbb{P}(\wedge^3 (\mathbb{C}^6)^*)}$  is the projectivization of the conormal bundle of  $G(3, (\mathbb{C}^6)^*)$  in  $\mathbb{P}(\wedge^3 (\mathbb{C}^6)^*)$ . It is very likely that  $F = p_*q^* \mathcal{Q}(m)|_L$ , where  $\mathcal{Q}(m)$  is an appropriate twist of the quotient bundle on  $G(3, 6)$ . See Sections 3.3 and 3.4 of [16], where a similar phenomenon is shown to be true for the  $E_6$ -invariant cubic in  $\mathbb{P}^{26}$ .

If  $L \subset \mathbb{P}(\wedge^3 \mathbb{C}^6)$  is a generic  $\mathbb{P}^5$  with coordinates  $z_1, \dots, z_6$ , we denote by  $P_{\text{SL}_6}^{(L)}$  the restriction of  $P_{\text{SL}_6}$  to  $L$ . Let  $X_L$  be the projective subvariety of  $\mathbb{P}(1, 1, 1, 1, 1, 2)$  determined by  $P_{\text{SL}_6}^{(L)}(z_1, \dots, z_6) + x^2 = 0$ . This is a quartic double fivefold ramified over a  $L \cap \mathcal{Q}_{\text{SL}_6}$ . We denote by  $W^L$  the weighted homogeneous polynomial  $P_{\text{SL}_6}^{(L)}(z_1, \dots, z_6) + x^2$ . A result of Orlov (see Theorem 3.11 and remark 3.12 of [23]) shows that there is a semi-orthogonal decomposition:

$$D^b(X_L) = \langle D^b(\text{Gr } W^L), \mathcal{O}_{X_L}, \mathcal{O}_{X_L}(1), \mathcal{O}_{X_L}(2), \mathcal{O}_{X_L}(3) \rangle,$$

where  $D^b(\text{Gr } W^L)$  is the derived category of graded matrix factorization of the weighted homogeneous polynomial  $W^L$ . If  $X_L$  was smooth then, as explained in [17] and [20], the category  $D^b(\text{Gr } W^L)$  would be a Calabi–Yau category of dimension 3. The singular locus of  $\mathcal{Q}_{\text{SL}_6} \subset \mathbb{P}(\wedge^3 \mathbb{C}^6)$  has dimension 14. Hence, for any linear space  $L \subset \mathbb{P}(\wedge^3 \mathbb{C}^6)$  of dimension 5, the variety  $X_L$  is singular. Thus, if  $E_L$  is a coherent sheaf in  $D^b(\text{Gr } W^L)$  whose jumping locus coincide with the singular locus of  $X_L$ , then one can not expect that  $\text{Ext}^3(E_L, E_L) \simeq \text{Hom}(E_L, E_L)^*$  and  $\text{Ext}^2(E_L, E_L) \simeq \text{Ext}^1(E_L, E_L)^*$ .

It will be however very helpful to keep this idea in mind when we will extend our study to linear sections of  $\mathcal{Q}_{\text{Spin}_{12}} \subset \mathbb{P}(\Delta)$ . For now, we focus on the cohomological properties of the cokernel of a matrix factorization in  $D^b(\text{Gr } W^L)$  constructed from  $S_L$ . Consider the matrices :

$$B_L = S_L + ix.I_6 \qquad C_L = S_L - ix.I_6$$

where  $S_L$  is the restriction to  $L$  of the matrix  $S_y$  defined in Proposition 3.2 and  $I_6$  is the  $6 * 6$  identity. We observe that  $B_L \times C_L = C_L \times B_L = W^L.I_6$  and that  $B_L$  is weighted homogeneous of degree 2. We deduce that  $(B_L, C_L) \in D^b(\text{Gr } W^L)$ . The following is the main technical result of this subsection:

**THEOREM 3.4.** — *Let  $L \subset \mathbb{P}(\wedge^3 \mathbb{C}^6)$  be a generic  $\mathbb{P}^5$  with coordinates  $z_1, \dots, z_6$ . Let  $B_L = S_L + ix.I_6$  and  $C_L = S_L - ix.I_6$ , where  $S_L$  is the restriction to  $L$  of the matrix  $S$  explicited in Proposition 3.2. Denote by  $E_L$  the restriction to  $X_L$  of the cokernel of  $B_L$ . The coherent sheaf  $E_L$  has rank 3 and we have:*

$$\text{Hom}_{X_L}(E_L, E_L) = \mathbb{C} \quad \text{and} \quad \text{Ext}_{X_L}^1(E_L, E_L) = 0.$$

*Proof.* — The sheaf  $E_L$  has rank 3 since it is the cokernel of  $6 * 6$  homogeneous matrix with quadratic entries whose degenerating locus is a quartic

hypersurface in  $\mathbb{P}(1, \dots, 1, 2)$ . We provide a Macaulay2 code to prove that  $\text{Hom}(E_L, E_L) = \mathbb{C}$  and that  $\text{Ext}^1(E_L, E_L) = 0$ .

We first recall the expression of the matrix  $S_y$  we found in Proposition 3.2 defined over the ring  $\mathbb{Z}/313.\mathbb{Z}[i, y_1, \dots, y_{20}, z_1, \dots, z_6, x]$ , where  $i$  is the square root of  $-1$ ,  $y_1, \dots, y_{20}$  and  $z_1, \dots, z_6$  have degree 1 and  $x$  has degree 2.

```
i_1 : kk = ZZ/313
i_2 : VV = kk[y_1..y_20]
i_3 : R = kk[w,z_1,z_2,z_3,z_4,z_5,z_6,x,Degrees=>{0,1,1,1,1,1,2}]
i_4 : J = ideal(w^2+1)
i_5 : T = R/J
i_6 : U = VV**T
i_7 : S_y = \dots
```

We then create a random  $6 * 20$  matrix with integer coefficients. This matrix represents the equations defining  $L$ . We also create a matrix (denoted  $K1$ ) which is the restriction of  $S_y$  to  $L$ . For technical reasons while working with Macaulay2, we must first define it over

$$\mathbb{Z}/313.\mathbb{Z}[i, y_1, \dots, y_{20}, z_1, \dots, z_6, x]$$

and then replicate it over  $\mathbb{Z}/313.\mathbb{Z}[i, z_1, \dots, z_6, x]$ .

```
i_8 : L = random(ZZ^20,ZZ^6,Height=>50)
i_9 : L2 = L**U
i_10 : L3 = L2 * matrix{{z_1},{z_2},{z_3},{z_4},{z_5},{z_6}}
i_11 : K = mutableMatrix(S_y)
i_12 : for k from 0 to 5 do (for i from 0 to 5 do
(for j from 1 to 20 do K_(k,i) = sub(K_(k,i), y_j => L3_(j-1,0))))
i_13 : K = matrix(K)
i_14 : K1 = mutableMatrix(T,6,6)
i_15 : for k from 0 to 5 do
(for j from 0 to 5 do K1_(k,j) = sub(K_(k,j),T))
i_16 : K1 = matrix(K1)
```

We finally create the matrix  $B_L$  and its cokernel  $E_L$ . We finally compute  $\text{Hom}(E_L, E_L)$  and  $\text{Ext}^1(E_L, E_L)$ .

```
i_17 :use T
i_18 : A = matrix{{w*x,0,0,0,0,0},{0,w*x,0,0,0,0},{0,0,w*x,0,0,0},
{0,0,0,w*x,0,0},{0,0,0,0,w*x,0},{0,0,0,0,0,w*x}}
i_19 : B = K1 + A
i_20 : C = K1-A
i_21 : D = B*C
i_22 : P = D_(0,0)
i_23 : I = ideal(P)
i_24 : T1 = T/I
i_25 : BL = B**T1
i_26 : FL = coker BL
```

```

i_27 : X = Proj T1
i_28 : EL = sheaf FL
i_29 : Hom(EL,EL)
i_30 : Ext^1(EL,EL)

```

In about one hour and a half on a portable workstation, Macaulay2 gives the expected answer:

```

o_29 : kk^1
o_30 : 0

```

□

*Remarks 3.5.*

- (1) Computations over finite fields are certified by Macaulay2. The computation above shows that  $\text{Ext}^1(E_L, E_L) = 0$  and  $\text{Hom}(E_L, E_L) = k$ , for a specific  $L \in \text{Gr}(6, \bigwedge^3 \mathbb{C}^6)$  over  $\mathbb{Z}/313\mathbb{Z}$ . Hence, by semi-continuity, we have  $\text{Ext}^1(E_L, E_L) = 0$  and  $\text{Hom}(E_L, E_L) = k$ , for generic  $L \in \text{Gr}(6, \bigwedge^3 \mathbb{C}^6)$  over  $\mathbb{Z}/313\mathbb{Z}$ . A second application of the semi-continuity Theorem implies that  $\text{Ext}^1(E_L, E_L) = 0$  and  $\text{Hom}(E_L, E_L) = k$  for generic  $L \in \text{Gr}(6, \bigwedge^3 \mathbb{C}^6)$  over  $\mathbb{Q}$ . And then, the same holds over  $\mathbb{C}$ .
- (2) The vanishing of the  $\text{Ext}^1$  for the matrix factorization  $B_L$  is somehow quite remarkable. Indeed, one computes with Macaulay2 that  $\text{Ext}^1_{\mathcal{Q}_{\text{SL}_6} \cap L}(F_L, F_L) = \mathbb{C}^{21}$ . Hence, going to the double cover of  $\mathbb{P}^5$  ramified over  $\mathcal{Q}_{\text{SL}_6} \cap L$  kills the 21 deformation directions of the cokernel of the initial matrix factorization. We have no explanation for this phenomenon.

### 3.2. Matrix factorizations on quartic double fivefolds ramified over linear sections of $\mathcal{Q}_{\text{Spin}_{12}} \subset \mathbb{P}(\Delta)$

The goal of this subsection is to prove the existence of rank 6 spherical vector bundles on quartic double fivefolds ramified over general linear sections of  $\mathcal{Q}_{\text{Spin}_{12}} \subset \mathbb{P}(\Delta)$ . Since we proved in the first section of this paper that a general quartic in  $\mathbb{P}^5$  is a linear section of  $\mathcal{Q}_{\text{Spin}_{12}} \subset \mathbb{P}(\Delta)$ , this implies that our results hold for a generic quartic double fivefold.

We will use the vanishing result we obtained in Section 3.1 for the rank 3 sheaves  $E_L$  which are defined on quartic double fivefolds ramified over general linear sections of  $\mathcal{Q}_{\text{SL}_6} \subset \mathbb{P}(\bigwedge^3 \mathbb{C}^6)$  and a representation theoretic reduction to go from quartic double fivefolds ramified over linear sections of  $\mathcal{Q}_{\text{SL}_6}$  to quartic double fivefolds ramified over general linear sections of  $\mathcal{Q}_{\text{Spin}_{12}}$ . We start with a representation theoretic description of the matrix factorization we exhibited in the previous subsection.

Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{C}$  and denote by  $\beta$  the highest root of  $\mathfrak{g}$  (we have chosen a fixed Cartan subalgebra of  $\mathfrak{g}$ ). We say that  $\mathfrak{g}$  has a type II decomposition if there exists a graded decomposition of  $\mathfrak{g}$  :

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

such that

$$[\mathfrak{g}_i, \mathfrak{g}_k] \subset \mathfrak{g}_{i+k}, \quad \mathfrak{g}_{-2} = \mathbb{C}.X_{-\beta} \quad \text{and} \quad \mathfrak{g}_2 = \mathbb{C}.X_{\beta}.$$

If such a decomposition happens, Vinberg [28] proves that  $\mathfrak{g}_1$  is a prehomogeneous space under the restricted adjoint action of  $G_0$ , where  $G_0$  is a simply connected group algebraic group with Lie algebra  $\mathfrak{g}_0$ .

Let  $x \in \mathfrak{g}_1$  and let  $\text{ad}_x = [x, \cdot]$  be the adjoint operator associated to  $x$ . By the grading property, we know that  $\text{ad}_x$  is of degree 1,  $(\text{ad}_x)^2$  is of degree 2 and  $(\text{ad}_x)^4$  is of degree 4. The operator  $(\text{ad}_x)^4$  can be then seen as a map:

$$(\text{ad}_x)^4 : \mathfrak{g}_{-2} \longrightarrow \mathfrak{g}_2.$$

Since  $\mathfrak{g}_{-2} = \mathbb{C}.X_{-\beta}$  and  $\mathfrak{g}_2 = \mathbb{C}.X_{\beta}$ , we have  $(\text{ad}_x)^4(X_{-\beta}) = P(x).X_{\beta}$ , where  $P$  is a  $G_0$ -invariant polynomial. We say that  $\mathfrak{g}$  has *maximal rank 4* if the polynomial  $P$  is non-identically zero (see [13]). In this case, since  $\text{ad}_x$  is linear in  $x$ , we deduce that  $P$  must have degree 4. The polynomial  $P$  is thus a degree 4 relative invariant for the prehomogeneous space  $(G_0, \mathfrak{g}_1)$ . The maximal rank 4 representation have been tabulated in [13] and they are the following:

$\mathfrak{g}$	$G_0$	$\mathfrak{g}_1$	$\dim \mathfrak{g}_1$
$B$	$\text{SL}_2 \times \mathbb{C}^* \times \text{SO}_d$	$\mathbb{C}^2 \otimes \mathbb{C}^d$	$2d$ ( $d$ odd)
$D$	$\text{SL}_2 \times \mathbb{C}^* \times \text{SO}_d$	$\mathbb{C}^2 \otimes \mathbb{C}^d$	$2d$ ( $d$ even)
$F_4$	$\mathbb{C}^* \times \text{Sp}_6$	$(\bigwedge^3 \mathbb{C}^6)_0$	14
$E_6$	$\mathbb{C}^* \times \text{SL}_6$	$\bigwedge^3 \mathbb{C}^6$	20
$E_7$	$\mathbb{C}^* \times \text{Spin}_{12}$	$\Delta$	32
$E_8$	$\mathbb{C}^* \times E_7$	minuscule	56

We focus on the type II decomposition

$$\mathfrak{e}_6 = \mathbb{C} \oplus \left( \bigwedge^3 \mathbb{C}^6 \right)^* \oplus \mathfrak{gl}_6 \oplus \bigwedge^3 \mathbb{C}^6 \oplus \mathbb{C}.$$

For  $y \in \bigwedge^3 \mathbb{C}^6$ , the graded decomposition above shows that  $(\text{ad}_y)^2 : \mathfrak{gl}_6 \longrightarrow \mathbb{C}$ . Hence, we can identify  $(\text{ad}_y)^2$  as a  $6 \times 6$  matrix. Furthermore, we have the following:

LEMMA 3.6. — *The pair  $((\text{ad}_y)^2, (\text{ad}_y)^2)$  is a matrix factorization of  $P_{\text{SL}_6}$ .*

*Proof.* — Consider the  $SL_6$ -equivariant map:

$$\begin{aligned} \Pi_{SL_6} : \bigwedge^3 \mathbb{C}^6 &\longrightarrow \text{End}(\mathbb{C}^6) \\ y &\longrightarrow (\text{ad}_y)^2 \circ (\text{ad}_y)^2. \end{aligned}$$

Since  $(\text{ad}_y)^4 \neq 0$ , we know that the map  $\Pi$  is not identically zero. The map  $\Pi$  is polynomial of degree 4, so that we get a non-zero  $SL_6$ -equivariant linear map which lifts  $\Pi_{SL_6}$ :

$$\widetilde{\Pi}_{SL_6} : S^4 \left( \bigwedge^3 \mathbb{C}^6 \right) \longrightarrow \text{End}(\mathbb{C}^6)$$

Furthermore, a computation with Lie [10] shows that the decomposition into  $SL_6$  irreducible representations of  $S^4(\bigwedge^3 \mathbb{C}^6)$  is :

$$\begin{aligned} S^4 \left( \bigwedge^3 \mathbb{C}^6 \right) &= [0, 0, 4, 0, 0] + [1, 0, 2, 0, 1] + [2, 0, 0, 0, 2] \\ &\quad + [0, 0, 2, 0, 0] + [0, 1, 0, 1, 0] + [0, 0, 0, 0, 0], \end{aligned}$$

where  $[0, 0, 0, 0, 0]$  represents  $P_{SL_6} \cdot \mathbb{C}$ . On the other hand, the decomposition of  $\text{End}(\mathbb{C}^6)$  into  $SL_6$  irreducible representations is:

$$\text{End}(\mathbb{C}^6) = [1, 0, 0, 0, 1] + [0, 0, 0, 0, 0],$$

where  $[0, 0, 0, 0, 0]$  is  $\text{id}_{\mathbb{C}^6} \cdot \mathbb{C}$ . By Schur's lemma, we deduce that  $\widetilde{\Pi}_{SL_6} = c \cdot P_{SL_6} \otimes \text{id}_{\mathbb{C}^6}$ , with  $c$  a non-zero scalar. Thus, we find that:

$$(\text{ad}_y)^2 \circ (\text{ad}_y)^2 = c \cdot P_{SL_6} \cdot \text{id}_{\mathbb{C}^6},$$

with  $c \neq 0$ . This proves the lemma. □

We will identify this matrix factorization with the one exhibited in the previous section.

PROPOSITION 3.7. — For all  $y \in \bigwedge^3 \mathbb{C}^6$ , we have:

$$(\text{ad}_y)^2 = S_y,$$

where  $S_y$  is the endomorphism of  $\mathbb{C}^6$  defined in Proposition 3.1.

*Proof.* — Following [14, Section 2], one observes that for all  $y \in \bigwedge^3 \mathbb{C}^6 \simeq (\bigwedge^3 \mathbb{C}^6)^*$  and for all  $\theta \in \mathbb{C}^6$ , we have:

$$\tau \otimes S_y(\theta) = A(\iota(\theta, y) \wedge y),$$

where  $\iota : \mathbb{C}^6 \times (\bigwedge^3 \mathbb{C}^6)^* \longrightarrow (\bigwedge^2 \mathbb{C}^6)^*$  is the interior product and  $A$  is the identification  $(\bigwedge^5 \mathbb{C}^6)^* \simeq \tau \otimes \mathbb{C}^6$  given by the wedge-pairing (here  $\tau$  is a

fixed generator of  $\bigwedge^6 \mathbb{C}^6$ ). Furthermore, following remark 2.17 of [27], we notice that  $y \in \bigwedge^3 \mathbb{C}^6$  and for all  $\theta \in \mathbb{C}^6$ , we have:

$$(\text{ad}_y)^2(\theta) = A(\iota(\theta, y) \wedge y).$$

This concludes the proof of the proposition. □

We are now in position to prove the main result of this section:

**THEOREM 3.8.** — *The generic quartic double fivefold is endowed with a spherical rank 6 vector bundle.*

The proof of this result is based on the construction of a  $\text{Spin}_{12}$ -equivariant matrix factorization of  $\mathcal{Q}_{\text{Spin}_{12}}$ . We will exhibit the pair of matrices at the beginning of the proof and show that they form indeed such a matrix factorization in Lemma 3.9, which is embedded in the proof of Theorem 3.8.

*Proof.* — Let  $X$  be a generic quartic double fivefold and let  $Y$  be the double cover of  $\mathbb{P}(\Delta)$  ramified along  $\mathcal{Q}_{\text{Spin}_{12}}$ . By the results of Section 2, we know that  $X$  is the fiber product of  $Y$  with a generic  $L \in G(6, \Delta)$ . As a consequence, we have to prove the existence of such a matrix factorization for  $X_L = Y \times_{\mathbb{P}(\Delta)} L$ , when  $L \in G(6, \Delta)$  is generic.

Consider the type II decomposition:

$$\mathfrak{e}_7 = \mathbb{C}^* \oplus \Delta^* \oplus (\mathfrak{so}_{12} \oplus \mathbb{C}) \oplus \Delta \oplus \mathbb{C}.$$

As  $\mathfrak{e}_7$  endowed with this decomposition has maximal rank 4, we know that:

$$(\text{ad}_z)^4(X_{-\beta}) = P_{\text{Spin}_{12}}(z).X_{\beta},$$

for all  $z \in \Delta$ . Since  $(\text{ad}_z)^2 \in \mathfrak{so}_{12}^* \simeq \mathfrak{so}_{12} \subset \mathfrak{gl}_{12}$ , we can see  $(\text{ad}_z)^2$  as a  $12 * 12$  matrix. We have the following :

**LEMMA 3.9.** — *The pair  $((\text{ad}_z)^2, (\text{ad}_z)^2)_{z \in \Delta}$  is a matrix factorization of  $P_{\text{Spin}_{12}}$ .*

*Proof.* — The proof is similar to that of Lemma 3.6. We check with Lie [10] that the decomposition into  $\text{Spin}_{12}$  irreducible representations of  $S^4 \Delta$  is:

$$\begin{aligned} S^4 \Delta = & [0, 0, 0, 0, 0, 4] + [0, 1, 0, 0, 0, 2] + [0, 2, 0, 0, 0, 0] \\ & + [0, 0, 0, 0, 0, 2] + [0, 0, 0, 1, 0, 0] + [0, 0, 0, 0, 0, 0], \end{aligned}$$

where  $[0, 0, 0, 0, 0]$  represents  $P_{\text{Spin}_{12}}.\mathbb{C}$ . Furthermore the decomposition of  $\text{End}(\mathbb{C}^{12})$  into  $\text{Spin}_{12}$  irreducible representations is:

$$\text{End}(\mathbb{C}^{12}) = [2, 0, 0, 0, 0, 0] + [0, 1, 0, 0, 0, 0] + [0, 0, 0, 0, 0, 0]$$

where  $[0, 0, 0, 0, 0, 0]$  is  $\text{id}_{\mathbb{C}^{12}}.\mathbb{C}$  □

The matrix factorization that will be of chief interest for us is closely related to the pair  $((\text{ad}_z)^2, (\text{ad}_z)^2)_{z \in \Delta}$ . In the following we write  $\text{ad}_z^{\mathfrak{g}}$  when we want to specify with which Lie algebra we work.

There is a type I decomposition (that is a graded decomposition in degree  $-1, 0$  and  $1$ ):

$$(3.1) \quad \mathfrak{so}_{12} = \bigwedge^2 \mathbb{C}^6 \oplus \mathfrak{gl}_6 \oplus \bigwedge^4 \mathbb{C}^6.$$

If one chooses the quadratic form on  $\mathbb{C}^{12}$  defined on a fixed basis by:

$$J_{12} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

then the decomposition of  $\mathfrak{so}_{12}$  as in equation (3.1) can be restated in the matrix context as follows. Any  $P \in \mathfrak{so}_{12}$  can be written:

$$P = \begin{pmatrix} A & K \\ M & -A^t \end{pmatrix},$$

where  $A \in \mathfrak{gl}_6$ ,  $A^t$  is the transpose of  $A$  with respect to the anti-diagonal and  $K, M$  are skew-symmetric matrices with respect to the anti-diagonal. Hence, for any  $z \in \Delta$ , we have:

$$(\text{ad}_z^{\mathfrak{e}_7})^2 = \begin{pmatrix} A_z & K_z \\ M_z & -A_z^t \end{pmatrix},$$

where  $A_z \in \mathfrak{gl}_6$ ,  $A_z^t$  is the transpose of  $A_z$  with respect to the anti-diagonal and  $K_z, M_z$  are skew-symmetric matrices with respect to the anti-diagonal. Furthermore, we observe that the type II decomposition:

$$\mathfrak{e}_7 = \mathbb{C}^* \oplus \Delta^* \oplus (\mathfrak{so}_{12} \oplus \mathbb{C}) \oplus \Delta \oplus \mathbb{C},$$

combined with the type I decompositions:

$$\mathfrak{so}_{12} = \bigwedge^2 \mathbb{C}^6 \oplus \mathfrak{gl}_6 \oplus \bigwedge^4 \mathbb{C}^6 \quad \Delta = (\mathbb{C}^6)^* \oplus \bigwedge^3 \mathbb{C}^6 \oplus \mathbb{C}^6$$

gives a bigraded decomposition:

$$\mathfrak{e}_7 = \left( \begin{array}{cccccc} \mathbb{C}^* & \oplus & (\mathbb{C}^6)^* & \oplus & (\bigwedge^3 \mathbb{C}^6)^* & \oplus & \mathbb{C}^6 & \oplus \\ (-2, 0) & & (-1, -1) & & (-1, 0) & & (-1, 1) & \\ & & \bigwedge^2 \mathbb{C}^6 & \oplus & (\mathfrak{gl}_6 \oplus \mathbb{C}) & \oplus & \bigwedge^4 \mathbb{C}^6 & \oplus \\ & & (0, -1) & & (0, 0) & & (0, 1) & \\ (\mathbb{C}^6)^* & \oplus & \bigwedge^3 \mathbb{C}^6 & \oplus & \mathbb{C}^6 & \oplus & \mathbb{C} & \\ (1, -1) & & (1, 0) & & (1, 1) & & (2, 0) & \end{array} \right)$$

This means that for  $z \in \mathfrak{e}_7$  of bidegree  $(a, b)$ , we have  $\text{ad}_z : \mathfrak{e}_{7(i,k)} \rightarrow \mathfrak{e}_{7(i+a,k+b)}$ . In particular, if  $y \in \bigwedge^3 \mathbb{C}^6$  is of bidegree  $(1, 0)$ , the bigraded decomposition above implies:

$$(\text{ad}_y^{\mathfrak{e}_7})^2 : \bigwedge^2 \mathbb{C}^6 \rightarrow \{0\}$$

and

$$(\text{ad}_y^{\mathfrak{e}_7})^2 : \bigwedge^4 \mathbb{C}^6 \rightarrow \{0\}.$$

As a consequence, for  $y \in \bigwedge^3 \mathbb{C}^6$  of bidegree  $(1, 0)$ , the matrix representation of  $(\text{ad}_y^{\mathfrak{e}_7})^2$  is:

$$(\text{ad}_y^{\mathfrak{e}_7})^2 = \begin{pmatrix} A_y & 0 \\ 0 & -A_y^t \end{pmatrix}$$

Furthermore, the above bigraded decomposition of  $\mathfrak{e}_7$  can also be obtained (but with swapped grading) from the type I decompositions:

$$\begin{aligned} \mathfrak{e}_7 &= (V(\omega_6, E_6))^* \oplus (\mathfrak{e}_6 \oplus \mathbb{C}) \oplus V(\omega_6, E_6) \\ V(\omega_6, E_6) &= (\mathbb{C}^6)^* \oplus \bigwedge^2 \mathbb{C}^6 \oplus \mathbb{C}^6 \end{aligned}$$

and the type II decomposition:

$$\mathfrak{e}_6 = \mathbb{C} \oplus \left( \bigwedge^3 \mathbb{C}^6 \right)^* \oplus \mathfrak{gl}_6 \oplus \bigwedge^3 \mathbb{C}^6 \oplus \mathbb{C}.$$

This establishes that  $A_y$  is the matrix representation of  $(\text{ad}_y^{\mathfrak{e}_6})^2$  for any  $y \in \bigwedge^3 \mathbb{C}^6 \subset \Delta$ . Hence, thanks to Proposition 3.7, we have for all  $y \in \bigwedge^3 \mathbb{C}^6$ :

$$(\text{ad}_y^{\mathfrak{e}_7})^2 = \begin{pmatrix} S_y & 0 \\ 0 & -S_y^t \end{pmatrix}$$

where  $S_y$  is the matrix we studied in Section 3.1.

Let  $L$  be a  $\mathbb{P}^5$  in  $\mathbb{P}(\Delta)$  and denote by  $\tilde{B}_L = (\text{ad}_z^{\mathfrak{e}_7})^2|_L + ix.I_{12}$  and  $\tilde{C}_L = (\text{ad}_z^{\mathfrak{e}_7})^2|_L - ix.I_{12}$ . We observe that:

$$\tilde{B}_L \times \tilde{C}_L = \tilde{C}_L \times \tilde{B}_L = (x^2 + P_{\text{Spin}_{12}}^{(L)}(z)).I_{12}.$$

The pair  $(\tilde{B}_L, \tilde{C}_L)$  is thus a matrix factorization of the polynomial defining  $X_L$  in  $\mathbb{P}(1, \dots, 1, 2)$ . Denote by  $\tilde{E}_L$  the cokernel of the restriction to  $X_L$  of  $\tilde{B}_L$ . If  $L$  is not included in  $\mathcal{Q}_{\text{Spin}_{12}}$ , then  $X_L$  is the degenerating locus of  $\tilde{B}_L$ . Since  $\deg(X_L) = 4$ , we find that  $\tilde{E}_L$  has generically rank 6.

Let  $L_0$  generic inside  $\mathbb{P}(\wedge^3 \mathbb{C}^6)$ . By the above discussion on the bigraded decomposition of  $\mathfrak{e}_7$ , we know that:

$$\tilde{B}_{L_0} = \begin{pmatrix} B_{L_0} & 0 \\ 0 & -B_{L_0}^t \end{pmatrix}$$

where  $B_{L_0}$  is the matrix factorization of  $x^2 + P_{\text{SL}_6}^{(L_0)}(z)$  we studied in Section 3.1. As a consequence,  $\tilde{E}_{L_0} = E_{L_0} \oplus G_{L_0}$ , where  $G_{L_0}$  is the restriction to  $X_{L_0}$  of the cokernel of  $-B_{L_0}^t$ . Since  $G_{L_0}$  is the cokernel of  $-B_{L_0}^t$ , we have  $G_{L_0} \simeq E_{L_0}^*(2)$ . Using a similar Macaulay2 algorithm to the one used for the proof of Theorem 3.4, we find that:

$$\text{Ext}_{X_{L_0}}^1(E_{L_0}, G_{L_0}) = \text{Ext}_{X_{L_0}}^1(G_{L_0}, E_{L_0}) = 0$$

and

$$\text{Hom}_{X_{L_0}}(E_{L_0}, G_{L_0}) = \text{Hom}_{X_{L_0}}(G_{L_0}, E_{L_0}) = 0.$$

By Theorem 3.4, we deduce that:

$$\text{Ext}_{X_{L_0}}^1(\tilde{E}_{L_0}, \tilde{E}_{L_0}) = 0$$

and

$$\text{Hom}_{X_{L_0}}(\tilde{E}_{L_0}, \tilde{E}_{L_0}) = \left\{ \begin{pmatrix} \lambda \cdot I_6 & 0 \\ 0 & \mu \cdot I_6 \end{pmatrix}, \lambda, \mu \in \mathbb{C} \right\}.$$

Let  $L \subset \mathbb{P}(\Delta)$  a generic  $\mathbb{P}^5$ . We will describe the algebra  $\text{Ext}^\bullet(\tilde{E}_L, \tilde{E}_L)$ . Consider  $\mathcal{X}$  the family of quartic double fivefolds obtained as a double cover of  $\mathbb{P}^5$  ramified along linear sections of  $\mathcal{Q}_{\text{Spin}_{12}} \subset \mathbb{P}(\Delta)$ . We have:

$$\mathcal{X} = Y \times_{\mathbb{P}(\Delta)} \mathbb{P}(\mathcal{R}),$$

where  $\mathbb{P}(\mathcal{R})$  is the projectivization of the tautological bundle over  $G(6, \Delta)$  and  $Y$  is the double cover of  $\mathbb{P}(\Delta)$  ramified along  $\mathcal{Q}_{\text{Spin}_{12}}$ . The natural projection  $\pi : \mathcal{X} \rightarrow G(6, \Delta)$  is a proper and flat morphism. Let  $\tilde{E}$  be the rank 6 sheaf defined on  $Y$  as the cokernel of the restriction to  $Y$  of the matrix representing  $(\text{ad}_z^{\mathfrak{e}_7})^2 + ix \cdot I_{12}$ , for  $z \in \mathbb{P}(\Delta)$  and  $x$  of degree 2. We denote by  $\tilde{\mathcal{E}}$  the pull-back of  $\tilde{E}$  to  $\mathcal{X}$ . We observe that for any  $L \in G(6, \Delta)$  which intersects  $\mathcal{Q}_{\text{Spin}_{12}}$  properly, we have:

$$\tilde{\mathcal{E}}|_{\pi^{-1}(L)} = \tilde{E}_L.$$

By the discussion above, we know that:

$$\text{Ext}_{X_{L_0}}^1(\tilde{E}_{L_0}, \tilde{E}_{L_0}) = 0$$

and

$$\text{Hom}_{X_{L_0}}(\tilde{E}_{L_0}, \tilde{E}_{L_0}) = \left\{ \begin{pmatrix} \lambda \cdot I_6 & 0 \\ 0 & \mu \cdot I_6 \end{pmatrix}, \lambda, \mu \in \mathbb{C} \right\},$$

for generic  $L_0 \in G(6, \wedge^3 \mathbb{C}^6)$ . Since the morphism  $\pi : \mathcal{X} \rightarrow G(6, \Delta)$  is proper and flat and the base  $G(6, \Delta)$  is smooth, the semi-continuity Theorem implies that:

$$\text{Ext}_{X_L}^1(\tilde{E}_L, \tilde{E}_L) = 0$$

Furthermore, for generic  $L$ , we have:

$$\tilde{B}_L = \begin{pmatrix} A_L & K_L \\ M_L & -A_L^t \end{pmatrix},$$

with  $K_L, M_L \neq 0$ . Let us prove that  $\text{Hom}_{X_L}(\tilde{E}_L, \tilde{E}_L) = \mathbb{C} \cdot I_{12}$ , for generic  $L \in \text{Gr}(6, \Delta)$ . Let us fix  $L \in \text{Gr}(6, \Delta)$ , generic. Any  $f \in \text{Hom}_{X_L}(\tilde{E}_L, \tilde{E}_L)$  can be represented as a pair of (constant) linear maps  $a_L, b_L : \mathbb{C}^{12} \rightarrow \mathbb{C}^{12}$  which makes the following diagram commutes:

$$\begin{CD} \mathbb{C}^{12} \otimes \mathcal{O}_{\mathbb{P}(1, \dots, 1, 2)}(-2) @>\tilde{B}_L>> \mathbb{C}^{12} \otimes \mathcal{O}_{\mathbb{P}(1, \dots, 1, 2)} @>\tilde{C}_L>> \mathbb{C}^{12} \otimes \mathcal{O}_{\mathbb{P}(1, \dots, 1, 2)}(2) \\ @V a_L VV @VV b_L V @VV a_L V \\ \mathbb{C}^{12} \otimes \mathcal{O}_{\mathbb{P}(1, \dots, 1, 2)}(-2) @>\tilde{B}_L>> \mathbb{C}^{12} \otimes \mathcal{O}_{\mathbb{P}(1, \dots, 1, 2)} @>\tilde{C}_L>> \mathbb{C}^{12} \otimes \mathcal{O}_{\mathbb{P}(1, \dots, 1, 2)}(2) \end{CD}$$

Since we have chosen a splitting  $\mathbb{C}^{12} = \mathbb{C}^6 \oplus \mathbb{C}^6$ , we can write:

$$a_L = \begin{pmatrix} a_L^{(1)} & a_L^{(2)} \\ a_L^{(3)} & a_L^{(4)} \end{pmatrix} \quad \text{and} \quad b_L = \begin{pmatrix} b_L^{(1)} & b_L^{(2)} \\ b_L^{(3)} & b_L^{(4)} \end{pmatrix}.$$

The existence of the above diagram is equivalent to the commutation relations:

$$\begin{aligned} a_L^{(1)} A_L + a_L^{(2)} M_L &= A_L b_L^{(1)} + K_L b_L^{(3)} \\ a_L^{(1)} K_L + a_L^{(2)} (-A_L^t) &= A_L b_L^{(2)} + K_L b_L^{(4)} \\ a_L^{(3)} A_L + a_L^{(4)} M_L &= M_L b_L^{(1)} + (-A_L^t) b_L^{(3)} \\ a_L^{(3)} K_L + a_L^{(4)} (-A_L^t) &= M_L b_L^{(2)} + (-A_L^t) b_L^{(4)}. \end{aligned}$$

We know that for  $L = L_0$ , these relations imply  $a_{L_0}^{(1)} = b_{L_0}^{(1)} = \lambda \cdot I_6$ ,  $a_{L_0}^{(4)} = b_{L_0}^{(4)} = \mu \cdot I_6$ ,  $a_{L_0}^{(2)} = b_{L_0}^{(2)} = a_{L_0}^{(3)} = b_{L_0}^{(3)} = 0$ . As a consequence, by semi-continuity, for generic  $L \in \text{Gr}(6, \Delta)$ , we have:

$$\text{Hom}_{X_L}(\tilde{E}_L, \tilde{E}_L) \subset \left\{ \begin{pmatrix} \lambda \cdot I_6 & 0 \\ 0 & \mu \cdot I_6 \end{pmatrix}, \lambda, \mu \in \mathbb{C} \right\}.$$

Furthermore, the non-vanishing of  $K_L$  or  $M_L$  for generic  $L \in \text{Gr}(6, \Delta)$  and the above commutation relations necessarily imply that  $\text{Hom}_{X_L}(\tilde{E}_L, \tilde{E}_L) = \mathbb{C} \cdot I_{12}$ .

We can now conclude the proof of Theorem 3.8. For generic  $L \in G(6, \Delta)$ , the section  $L \cap \mathcal{Q}_{\text{Spin}_{12}}$  is smooth and by [20], we have a decomposition:

$$D^b(X_L) = \langle \mathcal{A}_{X_L}, \mathcal{O}_{X_L}, \mathcal{O}_{X_L}(1), \mathcal{O}_{X_L}(2), \mathcal{O}_{X_L}(3) \rangle,$$

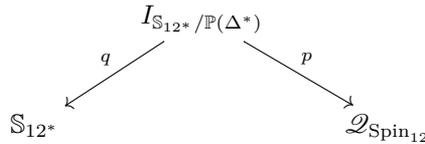
where  $\mathcal{A}_{X_L}$  is a CY-3 category. Using the resolution:

$$0 \longrightarrow \mathbb{C}^{12} \otimes \mathcal{O}_{\mathbb{P}(1, \dots, 1, 2)}(-2) \longrightarrow \mathbb{C}^{12} \otimes \mathcal{O}_{\mathbb{P}(1, \dots, 1, 2)} \longrightarrow i_*(\tilde{E}_L) \longrightarrow 0,$$

we easily prove that  $\tilde{E}_L(-1) \in \mathcal{A}_{X_L}$  and  $\tilde{E}_L(-2) \in \mathcal{A}_{X_L}$ . Since  $\mathcal{A}_{X_L}$  is a CY-3 category, we deduce that  $\text{Ext}^3(\tilde{E}_L, \tilde{E}_L) = \text{Hom}(\tilde{E}_L, \tilde{E}_L)^* = \mathbb{C}$  and that  $\text{Ext}^2(\tilde{E}_L, \tilde{E}_L) = \text{Ext}^1(\tilde{E}_L, \tilde{E}_L) = 0$ . This establishes that  $\tilde{E}_L(-1)$  and  $\tilde{E}_L(-2)$  are spherical rank 6 vector bundles contained in  $\mathcal{A}_{X_L}$  and finishes the proof of Theorem 3.8.  $\square$

The following is analogous to Remark 3.3.

*Remark 3.10.* — It is certainly worth noting that the cokernel of the matrix factorization  $((\text{ad}_z^{\epsilon_7})^2, (\text{ad}_z^{\epsilon_7})^2)$  is a rank 6 coherent sheaf living on  $\mathcal{Q}_{\text{Spin}_{12}} \in \mathbb{P}(\Delta)$ . We have a diagram:



where  $\mathbb{S}_{12^*} \subset \mathbb{P}(\Delta^*)$  is one of the connected component of the orthogonal Grassmannian  $\text{OG}(6, 12^*)$  in the (dual) spinor embedding and  $I_{\mathbb{S}_{12^*}/\mathbb{P}(\Delta^*)}$  is the projectivization of the conormal bundle of  $\mathbb{S}_{12}$  in  $\mathbb{P}(\Delta^*)$ . It is very likely that this rank 6 coherent sheaf is  $p_*q^*\mathcal{Q}(m)|_L$ , where  $\mathcal{Q}(m)$  is an appropriate twist of the quotient bundle on  $\mathbb{S}_{12^*}$ .

### 3.3. Homological unit of the CY-3 category associated to a generic quartic double fivefold

In [2], the concept of *homological unit* was introduced for a large class of triangulated categories as a replacement for the algebra  $H^\bullet(\mathcal{O}_X)$  when the category under study is not (necessarily) the derived category of a projective variety. This notion has been further explored in [19], where intriguing examples of units have been constructed and in [1], where it was

used to define hyper-Kähler categories. We give a quick reminder on the definition and basic properties of the homological units.

In the following, we only consider  $\mathbb{C}$ -linear triangulated categories which can be realized as derived categories of  $DG$ -modules over a smooth and proper  $DG$ -algebra (call them *smooth* and *proper*). In particular, for any triangulated category  $\mathcal{T}$  that we may consider, and any  $\mathcal{F}, \mathcal{G} \in \mathcal{T}$ , the graded vector space  $\bigoplus_{k \in \mathbb{Z}} \text{Ext}^k(\mathcal{F}, \mathcal{G})$  is finite dimensional over  $\mathbb{C}$ .

DEFINITION 3.11. — *Let  $\mathcal{T}$  be a smooth proper triangulated category. A rank function on  $\mathcal{T}$  is a function  $\text{rk} : \mathcal{T} \rightarrow \mathbb{Z}$  which is additive with respect to exact triangles and such that  $\text{rk}(\mathcal{F}[1]) = -\text{rk}(\mathcal{F})$ , for any  $\mathcal{F} \in \mathcal{T}$ . We say that the rank function is trivial if it is the zero function.*

DEFINITION 3.12. — *Let  $\mathcal{T}$  be a smooth proper triangulated category endowed with a non-trivial rank function. A graded algebra  $\mathfrak{T}^\bullet_{\mathcal{T}}$  is called a homological unit for  $\mathcal{T}$ , if  $\mathfrak{T}^\bullet_{\mathcal{T}}$  is maximal, with respect to inclusion, for the following property. For any object  $\mathcal{F} \in \mathcal{T}$ , there exists a pair of morphisms  $i_{\mathcal{F}} : \mathfrak{T}^\bullet_{\mathcal{T}} \rightarrow \text{Hom}^\bullet_{\mathcal{T}}(\mathcal{F}, \mathcal{F})$  and  $t_{\mathcal{F}} : \text{Hom}^\bullet_{\mathcal{T}}(\mathcal{F}, \mathcal{F}) \rightarrow \mathfrak{T}^\bullet_{\mathcal{T}}$  such that:*

- *the morphism  $i_{\mathcal{F}} : \mathfrak{T}^\bullet_{\mathcal{T}} \rightarrow \text{Hom}^\bullet_{\mathcal{T}}(\mathcal{F}, \mathcal{F})$  is a graded algebra morphism which is functorial in the following sense. Let  $\mathcal{F}, \mathcal{G} \in \mathcal{T}$  and let  $a \in \mathfrak{T}^k_{\mathcal{T}}$  for some  $k$ . Then, for any morphism  $\psi : \mathcal{F} \rightarrow \mathcal{G}$ , there is a commutative diagram:*

$$\begin{array}{ccc}
 \mathcal{F} & \xrightarrow{i_{\mathcal{F}}} & \mathcal{F}[k] \\
 \downarrow \psi & & \downarrow \psi[k] \\
 \mathcal{G} & \xrightarrow{i_{\mathcal{G}}} & \mathcal{G}[k]
 \end{array}$$

- *the morphism  $t_{\mathcal{F}} : \text{Hom}^\bullet_{\mathcal{T}}(\mathcal{F}, \mathcal{F}) \rightarrow \mathfrak{T}^\bullet_{\mathcal{T}}$  is a graded vector spaces morphism such that for any  $\mathcal{F} \in \mathcal{T}$  and any  $a \in \mathfrak{T}^\bullet_{\mathcal{T}}$ , we have  $t_{\mathcal{F}}(i_{\mathcal{F}}(a)) = \text{rank}(\mathcal{F}).a$ .*

With hypotheses as above, an object  $\mathcal{F} \in \mathcal{T}$  is said to be *unitary*, if  $\text{Hom}^\bullet_{\mathcal{T}}(\mathcal{F}, \mathcal{F}) \simeq \mathfrak{T}^\bullet_{\mathcal{T}}$  as graded rings, where  $\mathfrak{T}^\bullet_{\mathcal{T}}$  is a homological unit for  $\mathcal{T}$ .

Remarks 3.13.

- (1) By “maximal with respect to inclusion”, we (obviously) mean that for any algebra  $\mathfrak{B}^\bullet$  satisfying the conditions in the above definition, a graded algebra monomorphism  $r^\bullet : \mathfrak{T}^\bullet_{\mathcal{T}} \hookrightarrow \mathfrak{B}^\bullet$  which makes the

following diagram commutative for any  $\mathcal{F} \in \mathcal{T}$ :

$$\begin{array}{ccc}
 \mathfrak{T}_{\mathcal{T}}^{\bullet} & \xrightarrow{r^{\bullet}} & \mathfrak{B}^{\bullet} \\
 \searrow^{i_{\mathcal{F}}^{\bullet}} & & \swarrow_{i_{\mathcal{F}}^{\bullet}} \\
 & \text{Hom}_{\mathcal{T}}^{\bullet}(\mathcal{F}, \mathcal{F}) &
 \end{array}$$

is necessarily an isomorphism

- (2) Let  $\mathcal{X}$  be a projective Deligne–Mumford stack which can be written as a global quotient  $[X/G]$  where  $X$  is a smooth projective variety and  $G$  is a reductive group acting linearly on  $X$ . Let  $\mathcal{O}_X(1)$  be a  $G$ -equivariant line bundle. A minor modification of the arguments in Theorem 4 of [24] shows that there is an equivalence:

$$D^{\text{perf}}(\mathcal{X}) \simeq D^{\text{perf}}(\mathcal{C}),$$

where

$$\mathcal{C} = \text{RHom}_X^G \left( \bigoplus_{i=0}^{\dim X} \mathcal{O}_X(i), \bigoplus_{i=0}^{\dim X} \mathcal{O}_X(i) \right).$$

Let us consider the rank of an  $\mathcal{O}_x$ -module as a rank function on  $D^{\text{perf}}(\mathcal{X})$ . In such a case, we have  $\mathfrak{T}_{D^{\text{perf}}(\mathcal{X})}^{\bullet} = H^{\bullet}(\mathcal{O}_x)$ . Furthermore, for any  $\mathcal{F} \in \mathcal{D}^{\text{perf}}(\mathcal{X})$ , the morphism  $i_{\mathcal{F}}$  is the tensor product (over  $\mathcal{O}_x$ ) with the identity map of  $\mathcal{F}$  and the morphism  $t_{\mathcal{F}}$  is the trace map  $\text{Hom}_x^{\bullet}(\mathcal{F}, \mathcal{F}) \rightarrow H^{\bullet}(\mathcal{O}_x)$ .

- (3) In the above definition, the existence of the morphism  $i_{\mathcal{F}}$  for all  $\mathcal{F} \in \mathcal{T}$  and its functorial properties is equivalent to the existence of a morphism of graded algebras:

$$\mathfrak{T}_{\mathcal{T}}^{\bullet} \rightarrow \text{HH}^{\bullet}(\mathcal{T}),$$

where  $\text{HH}^{\bullet}(\mathcal{T})$  is the Hochschild cohomology of  $\mathcal{T}$ . If the rank function on  $\mathcal{T}$  is non-trivial, the splitting property of  $t^{\bullet}$  implies that the map  $\mathfrak{T}_{\mathcal{T}}^{\bullet} \rightarrow \text{HH}^{\bullet}(\mathcal{T})$  is injective.

- (4) On the other hand, the definition and the (splitting) properties of the morphisms  $t_{\mathcal{F}}$ , for  $\mathcal{F} \in \mathcal{T}$  with non-zero rank do not seem to be easily written using only the notion of graded morphisms between  $\text{HH}^{\bullet}(\mathcal{T})$  and  $\mathfrak{T}_{\mathcal{T}}^{\bullet}$ . It appears that there is no obvious way to write that  $t_{\mathcal{F}}$  splits  $i_{\mathcal{F}}$  whenever the rank of  $\mathcal{F}$  is not zero only in terms of Hochschild cohomology.
- (5) If  $\mathcal{T}$  contains a unitary object whose rank is not zero, then the homological unit of  $\mathcal{T}$  is necessarily unique (though the embedding

of the homological unit in  $\mathrm{HH}^\bullet(\mathcal{A})$  is certainly not unique). This follows from the maximality condition imposed in Definition 3.12.

- (6) Let  $X$  and  $Y$  be smooth projective varieties of dimension less or equal to 4 such that  $\mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$ . It is proved in [2] that the algebras  $H^\bullet(\mathcal{O}_X)$  and  $H^\bullet(\mathcal{O}_Y)$  are isomorphic. This suggests that the homological unit of a DG category of geometric origin could be independent of the embedding into the derived category of a smooth projective Deligne–Mumford stack (at least if the dimensions of the varieties are small enough). In the next subsection, we will investigate in more details the invariance properties of homological units attached to geometric Calabi–Yau categories of dimension 3.

In case  $\mathcal{T}$  contains a spherical object whose rank is non-zero, the homological unit is easily computed:

PROPOSITION 3.14. — *Let  $X$  be a smooth projective variety and  $\mathcal{T} \subset \mathrm{D}^b(X)$  be an admissible subcategory. Assume that  $\mathcal{T}$  is a Calabi–Yau category of dimension  $2p + 1$  and that it contains a  $2p + 1$ -spherical object whose rank (as a  $\mathcal{O}_X$ -module) is non-zero. Then, the homological unit of  $\mathcal{T}$  (with respect to the rank function coming from  $\mathrm{D}^b(X)$ ) is  $\mathbb{C} \oplus \mathbb{C}[2p + 1]$ .*

We recall that an object  $\mathcal{E} \in \mathcal{T}$  is  $2p + 1$ -spherical if we have a ring isomorphism:

$$\mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{E}) \simeq \mathbb{C} \oplus \mathbb{C}[2p + 1].$$

*Proof.* — Let  $\mathfrak{I}_{\mathcal{T}}^\bullet$  be a homological unit for  $\mathcal{T}$  with respect to the rank function coming from  $\mathrm{D}^b(X)$ . Let  $\mathcal{E}$  be a  $2p + 1$ -spherical object in  $\mathcal{T}$  which rank is not zero. By definition of homological unit, we must have  $\mathfrak{I}_{\mathcal{T}}^\bullet \hookrightarrow \mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{E}) = \mathbb{C} \oplus \mathbb{C}[2p + 1]$ .

We now prove that  $\mathbb{C} \oplus \mathbb{C}[2p + 1]$  is indeed a homological unit for  $\mathcal{T}$  with respect to the rank function coming from  $\mathrm{D}^b(X)$ . For any  $\mathcal{E} \in \mathcal{T}$ , any  $a \in \mathbb{C}$  and any  $f \in \mathrm{Hom}(\mathcal{E}, \mathcal{E})$ , we put:

$$i_{\mathcal{E}}^0(a) = a \cdot \mathrm{id}_{\mathcal{E}} \quad \text{and} \quad t_{\mathcal{E}}^0(f) = \mathrm{Trace}(f),$$

where  $\mathrm{Trace}$  is the trace map inherited from  $\mathrm{D}^b(X)$ . It is clear that  $t_{\mathcal{E}}^0(i_{\mathcal{E}}^0(a)) = \mathrm{rank}(\mathcal{E}) \cdot a$ , for any  $a \in \mathbb{C}$ . Let  $\omega$  be a generator of  $H^n(X, \omega_X)$ , where  $n = \dim X$ . By the Hochschild–Kostant–Rosenberg isomorphism, we can see  $\omega \in \mathrm{HH}_0(\mathrm{D}^b(X))$ . Let  $\delta : \mathcal{T} \hookrightarrow \mathrm{D}^b(X)$  be the admissible embedding of  $\mathcal{T}$  in  $\mathrm{D}^b(X)$ . We have  $\delta^! \omega \in \mathrm{HH}_0(\mathcal{T})$ . Note that for any  $\mathcal{F} \in \mathrm{D}^b(X)$ , we have:

$$\omega|_{\mathcal{F}} = \mathrm{id}_{\mathcal{F}} \otimes \omega : \mathcal{F} \longrightarrow \mathcal{F} \otimes \omega_X[n].$$

Furthermore, for any  $\mathcal{E} \in \mathcal{T}$ , we have:

$$\begin{aligned} \text{Hom}(\delta_*\mathcal{E}, \delta_*E \otimes \omega_X[n]) &= \text{Hom}(\mathcal{E}, \delta^!(\delta_*E \otimes \omega_X[n])), \text{ by adjunction,} \\ &= \text{Hom}(\mathcal{E}, \mathcal{E}[2p+1]), \text{ because } \mathcal{A} \text{ is Calabi–Yau} \\ &\hspace{15em} \text{of dimension } 2p+1. \end{aligned}$$

For any  $\mathcal{E} \in \mathcal{T}$  and any  $f \in \text{Ext}^{2p+1}(\mathcal{E}, \mathcal{E})$ , we then define  $t_{\mathcal{E}}^{2p+1}(f)$  as the trace of  $f$  seen as an element in  $\text{Hom}(\delta_*\mathcal{E}, \delta_*\mathcal{E} \otimes \omega_X[n])$ .

The category  $\mathcal{T}$  is Calabi–Yau of dimension  $2p+1$ , we thus have an isomorphism  $\text{HH}_0(\mathcal{T}) \simeq \text{HH}^{2p+1}(\mathcal{T})$ . We therefore see  $\delta^!\omega$  as an element in  $\text{HH}^{2p+1}(\mathcal{T})$  and for any  $\mathcal{E} \in \mathcal{T}$  and any  $a \in \mathbb{C}$ , we define:

$$i_{\mathcal{E}}^{2p+1}(a) = a \cdot (\delta^!\omega)|_{\mathcal{E}},$$

where  $(\delta^!\omega)|_{\mathcal{E}}$  is the restriction in  $\text{Ext}^{2p+1}(\mathcal{E}, \mathcal{E})$  of  $\delta^!\omega$ . Since for any  $\mathcal{E} \in \mathcal{T}$ , we have:

$$(\delta^!\omega)|_{\mathcal{E}} = \delta^!(\omega|_{\mathcal{E}}),$$

we deduce that, for all  $a \in \mathbb{C}$ :

$$t_{\mathcal{E}}^{2p+1}(i_{\mathcal{E}}^{2p+1}(a)) = t_{\mathcal{E}}^{2p+1}(a \cdot \delta^!(\omega|_{\mathcal{E}})) = \text{Tr}_{\text{D}^b(X)}(a \cdot \text{id}_{\mathcal{E}} \otimes \omega) = a \cdot \text{rk}(\mathcal{E}).$$

In order to show that  $\mathbb{C} \oplus \mathbb{C}[2p+1]$  is the homological unit of  $\mathcal{T}$  with respect to the rank function coming from  $\text{D}^b(X)$ , we are left to prove that the map  $i_{\mathcal{E}}^{\bullet} : \mathbb{C} \oplus \mathbb{C}[2p+1] \rightarrow \text{Hom}^{\bullet}(\mathcal{E}, \mathcal{E})$  is a ring morphism for any  $\mathcal{E} \in \mathcal{T}$ . This is equivalent to the vanishing of  $i_{\mathcal{E}}^{2p+1}(a) \circ i_{\mathcal{E}}^{2p+1}(a)$ , for any  $\mathcal{E} \in \mathcal{T}$  and any  $a \in \mathbb{C}$ . For such  $a$  and  $\mathcal{E}$ , we have:

$$\begin{aligned} i_{\mathcal{E}}^{2p+1}(a) \circ i_{\mathcal{E}}^{2p+1}(a) &= a^2 \delta^!(\omega)|_{\mathcal{E}} \circ \delta^!(\omega)|_{\mathcal{E}}, \text{ by definition,} \\ &= a^2 (\delta^!\omega \circ \delta^!\omega)|_{\mathcal{E}}, \text{ by functoriality.} \end{aligned}$$

But the algebra  $\text{HH}^{\bullet}(\mathcal{T})$  is graded commutative, since  $\delta^!\omega \in \text{HH}^{2p+1}(\mathcal{T})$ , we deduce that  $\delta^!\omega \circ \delta^!\omega = 0 \in \text{HH}^{4p+2}(\mathcal{T})$ . As a consequence, we have  $i_{\mathcal{E}}^{2p+1}(a) \circ i_{\mathcal{E}}^{2p+1}(a) = 0$ , for any  $\mathcal{E} \in \mathcal{T}$  and any  $a \in \mathbb{C}$ . This finally demonstrates that the ring  $\mathbb{C} \oplus \mathbb{C}[2p+1]$  is a homological unit for  $\mathcal{T}$ . □

*Remark 3.15.* — We stated our result only in the odd-dimensional case in order to benefit from the graded-commutativity of the algebra  $\text{HH}^{\bullet}(\mathcal{T})$  and therefore get a quick proof that the graded vector space morphism  $\mathbb{C} \oplus \mathbb{C}[2p+1] \hookrightarrow \text{Hom}^{\bullet}(\mathcal{E}, \mathcal{E})$  is indeed a ring morphism. We of course expect that Proposition 3.14 should be true also in the even dimensional case.

From this lemma and Theorem 3.8, we deduce the:

COROLLARY 3.16. — *Let  $X$  be a generic quartic double fivefold and let  $\mathcal{A}_X$  the CY-3 category associated to the derived category of this fivefold. The homological unit of  $\mathcal{A}_X$  is  $\mathbb{C} \oplus \mathbb{C}[3]$ .*

This contrasts sharply with the fact that  $H^\bullet(\mathcal{O}_X) = \mathbb{C}$ , when  $X$  is a quartic double fivefold. This corollary also suggests a natural question on the homological units of the CY-3 categories potentially associated to a manifolds of Calabi–Yau type, namely:

QUESTION 3.17. — *Let  $X$  be a manifold of Calabi–Yau type. Assume that a semi-orthogonal component of the derived category of  $X$  is a CY-3 category. Is the homological unit of this CY-3 category  $\mathbb{C} \oplus \mathbb{C}[3]$ ?*

It is proved in [16], that the CY-3 category associated to the derived category of a generic cubic sevenfold contains a spherical rank 9 vector bundle. Hence, by the above lemma, the homological unit of this CY-3 category is  $\mathbb{C} \oplus \mathbb{C}[3]$ . Thus, as far as manifolds of Calabi–Yau type obtained as generic complete intersections of dimension bigger than 4 in weighted projective spaces are concerned, in order to answer the above question we only need to prove that the CY-3 category associated to the transverse intersection of a generic cubic and a generic quadric in  $\mathbb{P}^7$  contains a spherical vector bundle.

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