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p^r-SELMER COMPANION MODULAR FORMS

by Somnath JHA, Dipramit MAJUMDAR & Sudhanshu SHEKHAR (*)

ABSTRACT. — The study of *n*-Selmer groups of elliptic curves over number fields in recent past has led to the discovery of some deep results in the arithmetic of elliptic curves. Given two elliptic curves E_1 and E_2 over a number field K, Mazur– Rubin have defined them to be *n*-Selmer companion if for every quadratic character χ of K, the *n*-Selmer groups of E_1^{χ} and E_2^{χ} over K are isomorphic. Given a prime p, they have given sufficient conditions for two elliptic curves to be p^r -Selmer companion in terms of mod- p^r congruences between the curves. We discuss an analogue of this for Bloch–Kato p^r -Selmer groups of modular forms. We compare the Bloch–Kato Selmer group of a modular form respectively with the Greenberg Selmer group when the modular form is *p*-ordinary and with the signed Selmer groups of Lei–Loeffler–Zerbes when the modular form is non-ordinary at p. We also indicate the relation between our results and the well-known congruence results for the special values of the corresponding L-functions due to Vatsal.

RÉSUMÉ. — L'étude de groupes n-Selmer de courbes elliptiques sur des corps de nombres algébriques dans un passé récent a conduit à la découverte de certains résultats profonds en arithmétique des courbes elliptiques. Étant données deux courbes elliptiques E_1 et E_2 sur un corps de nombres algébriques K, Mazur–Rubin les a définies comme n-Selmer compagnon si pour chaque caractère quadratique χ de K, les groupes n-Selmer de E_1^{χ} et E_2^{χ} sur K sont isomorphes. Étant donné un nombre premier p, ils ont donné des conditions suffisantes pour que deux courbes elliptiques soient des compagnons p^r -Selmer en termes de congruences mod- p^r entre les courbes. Nous discutons d'un analogue de ce résultat pour les groupes p^r -Bloch–Kato Selmer de formes modulaire. Nous comparons le groupe Bloch–Kato Selmer d'une forme modulaire respectivement avec le groupe Greenberg Selmer lorsque la forme modulaire est p-ordinaire et avec les groupes de Selmer signés de Lei–Loeffler–Zerbes lorsque la forme modulaire source bien connus pour les valeurs spéciales des fonctions L correspondantes dues à Vatsal.

Keywords: residual Bloch-Kato Selmer group, congruence of modular forms.

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1. Introduction

The study of *n*-Selmer groups of elliptic curves over number fields has been of considerable interest in recent past. For example, for certain *n*, some striking results on the bounds on the average rank of *n*-Selmer groups of elliptic curves over \mathbb{Q} has been established by Bhargava et al. On the other hand, some deep results related to the rank distribution of *n*-Selmer groups for certain *n*, of a family consisting of all quadratic twists of an elliptic curve, has been studied by Mazur–Rubin and others (cf. [20]). In [20], instead of the rank distribution of the *n*-Selmer group over the family, they formulate the inverse question: given a prime *p* and a number field *K*, what information about *E* is encoded in the *p*-Selmer group of an elliptic curve *E* over *K*? Motivated by this question, they define the following:

DEFINITION 1.1 ([20, Definition 1.2]). — Let K be a number field and $n \in \mathbb{N}$ be fixed. Two elliptic curves E_1 , E_2 are said to be n-Selmer companion, if for every quadratic character χ of K, there is an isomorphism of n-Selmer groups of E_1^{χ} and E_2^{χ} over K i.e. $S_n(E_1^{\chi}/K) \cong S_n(E_2^{\chi}/K)$.

That naturally led them to study p^r -Selmer companion elliptic curves for a (fixed) prime p with $r \in \mathbb{N}$. They gave sufficient conditions for two elliptic curves to be p^r -Selmer companion in terms of various conditions related to mod p^r -congruences between the curves (see main theorem [20, Theorem 3.1]). They point out in [20, §1] that it would be interesting to investigate this phenomenon more generally for p^r Bloch–Kato Selmer group of motives instead of elliptic curves and that has led us to study this.

In this article, we fix a prime p and study Bloch-Kato p^r -Selmer groups associated to modular forms and discuss p^r Selmer companion modular forms (see Definition 2.4). To avoid various technical difficulties which naturally arise in our case of Selmer groups of modular forms, throughout the paper we make the restrictive hypothesis that the prime p is odd. Also, we state all our results for Selmer groups defined over \mathbb{Q} (see Section 2.3). However, it can be seen from our proofs that, when the modular forms are p-ordinary, our results can be extended to a general number field K at the cost of notation and hypothesis becoming more cumbersome but essentially the same in nature. On the other hand, when the modular forms are nonordinary at p, for a general number field K, the theory of signed Selmer groups is not properly developed yet. Hence our main theorem can not be generalized to an arbitrary number field in the non-ordinary reduction case.

For $i \in \{1,2\}$ and for a positive integer N coprime to p, let $f_i \in S_k(\Gamma_0(Np^{t_i}), \epsilon_i)$ be a normalized cuspidal Hecke eigenform and let

 $K_{f_1,f_2,\epsilon_1,\epsilon_2}$ be the number field generated by the Fourier coefficients of f_1, f_2 and values of ϵ_1, ϵ_2 . Let π be a uniformizer of the ring of integers of the completion of $K_{f_1,f_2,\epsilon_1,\epsilon_2}$ at a prime above p. A sample of our results is given in the following theorem which is a special case of our main theorem (Theorem 4.10) for trivial nebentypus i.e. when $\epsilon_1 = \epsilon_2 = 1$.

THEOREM 1.2. — Let p be an odd prime and for i = 1, 2, let f_i be a normalized cuspidal Hecke eigenform in $S_k(\Gamma_0(Np^{t_i}))$, where (N, p) = 1, $k \ge 2$ and $t_i \in \mathbb{N} \cup \{0\}$. Let $r \in \mathbb{N}$ and $\phi : A_{f_1}[\pi^r] \longrightarrow A_{f_2}[\pi^r]$ be a $G_{\mathbb{Q}}$ linear isomorphism. We assume the following:

- (1) N is square-free and $\forall \ \ell \in \{\ell \text{ prime} : \ell \mid N\}, \ \operatorname{cond}_{\ell}(\bar{\rho}_{f_1}) = \ell = \operatorname{cond}_{\ell}(\bar{\rho}_{f_2}).$
- (2) Either (a) or (b) holds.
 - (a) $p \ge k$, f_1 and f_2 are non-ordinary at p, and $t_1 = t_2 = 0$.
 - (b) p > 2k 3, f_1 and f_2 are ordinary at p, and either (i) or (ii) holds.

(i)
$$t_1 = t_2 = 0$$
.

(ii) $a_p(f_i) \neq \pm 1 \pmod{\pi}$.

Then for every quadratic character χ of $G_{\mathbb{Q}}$ and for every fixed j with $0 \leq j \leq k-2$, we have an isomorphism of the π^r - Bloch-Kato Selmer groups

$$S_{\mathrm{BK}}(A_{f_1\chi(-j)}[\pi^r]/\mathbb{Q}) \cong S_{\mathrm{BK}}(A_{f_2\chi(-j)}[\pi^r]/\mathbb{Q}).$$

The basic strategy of the proof is to compare each local factor which arises in the definition of the Selmer groups. Specially, comparing the local factors of Selmer groups at the prime p requires most careful analysis. Note that in our case of studying Bloch–Kato Selmer groups of modular forms, we do not have the advantage of considering the Kummer map on abelian variety. Also, we do not have an obvious analogue of fppf cohomology on Néron model of elliptic curve, used by Mazur-Rubin for treating the case of local factors of Selmer group at the prime p. Thus we adopt a different strategy. Let f_1 and f_2 be two weight k normalized cuspforms which are congruent mod π^r . If f_1 and f_2 are *p*-ordinary, then we compare π^r -Bloch–Kato Selmer local condition at p with the π^r -Greenberg Selmer local condition at p. On the other hand, if f_1 and f_2 are non-ordinary at p, then we compare π^r -Bloch-Kato Selmer local conditions at p with the π^r -signed Selmer local conditions at p ([12]). The theory of signed Selmer groups for non-ordinary modular forms are developed using works of Lei-Loeffler–Zerbes ([16, 17]). These signed Selmer groups can be viewed as generalizations of the \pm Selmer groups of supersingular elliptic curves originally developed by S. Kobayashi (cf. [14]). The works of F. Sprung on Iwasawa main conjecture for elliptic curves at supersingular primes are also relevant (cf. [27]).

On the other hand, to compare the local factors of the Selmer groups at primes ℓ with $\ell \neq p$, we impose some condition on the conductor of the residual mod- ℓ Galois representation and use some Iwasawa theoretic techniques. When the modular form has weight 2 and corresponds to an elliptic curve, we have shown in Remark 3.2, that the hypothesis of Mazur–Rubin (see [20, Lemma 2.5]) at a prime $\ell \neq p$ of multiplicative reduction implies our hypothesis on the conductor of the residual mod- ℓ Galois representation. Moreover, we have illustrated in Section 6, this condition at $\ell \neq p$ can be verified in many cases; for example using *level lowering* results of Ribet, Serre et al.

We stress that there are some important differences in the notion of being p^r -Selmer companion for elliptic curves and modular forms. We demonstrate this in Remark 2.5, where we give an example of a weight 2 cuspform f such that f and f_r are π -Selmer companion for infinitely many distinct weight 2 cuspforms f_r . This is in contrast with elliptic curves, where given an elliptic curve E, Mazur–Rubin suggests there would be only finitely many elliptic curves E', such that the pair (E, E') are p^r -Selmer companion.

In Definition 4.14, we take the liberty to extend the definition of Selmer companion for two cuspforms of different weights. Using this definition in Corollary 4.15, we give sufficient conditions for two forms of two different weights (different *p*-power level and different nebentypus) to be π^r -Selmer companion. We give such examples of extended π^r -Selmer companion forms within an ordinary Hida family in Example I(4) in Section 6.

We also discuss π^r Selmer companion forms over \mathbb{Q}_{cyc} , when the congruent modular forms are good, ordinary at p. The proof over \mathbb{Q} easily adapts to this case. However, there is a well known congruence result of Vatsal [28] which shows if $f_1 \equiv f_2 \pmod{\pi^r}$ are of weight $k \ge 2$, level N and are good, ordinary at p, then for any Dirichlet character χ of conductor prime to N, the p-adic L-functions of $f_1 \otimes \chi$ and $f_2 \otimes \chi$ are congruent mod π^r in the Iwasawa algebra. Thus via the Iwasawa Main Conjecture, our result can be viewed as an algebraic counterpart of Vatsal's result. However, in case of quadratic characters, our congruence result on Selmer groups works for all possible quadratic characters. The structure of the article is as follows. After fixing the notation and basic set-up in Section 2.1, we recall the Galois representation attached to a newform in Sections 2.2 and 2.3 contains the definitions of Greenberg, Bloch–Kato, signed π^r -Selmer groups of a twisted nomalized cuspidal eigenform. Congruence results on Greenberg Selmer groups are discussed in Section 3. In Section 4, we study π^r Bloch–Kato Selmer companion forms and establish our main result (Theorem 4.10). We discuss π^r Selmer companion forms over \mathbb{Q}_{cyc} in Section 5, and point out its relation with the well known congruences of the corresponding *p*-adic *L*-functions. We compute several numerical examples verifying our results (at weight 2 as well as at higher weights) in Section 6. In particular, in Section 6 we give explicit numerical examples of π^r -Bloch–Kato Selmer companion modular forms which are (i) *p*-ordinary and (ii) those which are non-ordinary at *p* as well.

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2. Preliminaries

2.1. Notation and set up:

Throughout we fix an embedding ι_{∞} of a fixed algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} into \mathbb{C} and also an embedding ι_{ℓ} of $\overline{\mathbb{Q}}$ into a fixed algebraic closure $\overline{\mathbb{Q}}_{\ell}$ of the field \mathbb{Q}_{ℓ} of the ℓ -adic numbers, for every prime ℓ . Fix an odd prime p and a positive integer N with (N, p) = 1. Let $i \in \{1, 2\}$ and $f_i \in S_k(\Gamma_0(Np^{t_i}), \epsilon_i)$ where $t_i \in \mathbb{N} \cup \{0\}$, be a normalized cuspidal Hecke eigenform which is a newform of conductor Np^{t_i} and nebentypus ϵ_i . We assume that ϵ_i is a primitive Dirichlet character of conductor C_i . Then $C_i \mid Np^{t_i}$. We can write $\epsilon_i = \prod_{\ell} \epsilon_{i,\ell}$ where $\epsilon_{i,\ell}$ is a primitive Dirichlet character of conductor $= \ell^{n(i,\ell)}$ with $n(i,\ell) \in \mathbb{N}$, for every prime divisor ℓ of C_i . In particular, we can write $\epsilon_i = \epsilon_{i,p}\epsilon'_i$, where ϵ'_i is a Dirichlet character of conductor, say C'_i with $(C'_i, p) = 1$ and $C'_i \mid N$. The order of a Dirichlet character $\tau :=$ the order of the subgroup of the roots of unity in \mathbb{C}^* generated by the image of τ . We define a condition $(C'_{i,\ell})$ to be used later:

(2.1) $(C'_{i,\ell})$: For every prime $\ell \mid C'_i$, the order of $\epsilon^2_{i,\ell} \neq p^n$ for any $n \in \mathbb{N}$.

Remark 2.1. — For example, if the order of ϵ'_i is co-prime to p, then $(C'_{i,\ell})$ is satisfied.

Let $K = K_{f_1,f_2,\epsilon_1,\epsilon_2}$ be the number field generated by the Fourier coefficients of f_1, f_2 and the values of ϵ_1, ϵ_2 . Let π_K be a uniformizer of the ring of integers O_{K_p} of completion of K at a prime \mathfrak{p} lying above p induced by the embedding ι_p . To ease the notation, we often write $O = O_{K_p}$ and $\pi = \pi_K$. Put

$$S = S_{f_1, f_2} := \{ \ell \text{ prime} : \ell \mid | N \}.$$

Note that, by definition $p \notin S$. For any O module M, $M[\pi^r]$ will denote the set of π^r torsion points of M. For any separable field L, G_L will denote the Galois group $\operatorname{Gal}(\overline{L}/L)$. For a group G acting on a module M, we denote by $M^G = \{m \in M | gm = m \quad \forall g \in G\}$. Also for a number field or a p-adic field F, and a discrete $\operatorname{Gal}(\overline{F}/F)$ module M, $H^i(F, M)$ will denote the Galois cohomology group $H^i(\operatorname{Gal}(\overline{F}/F), M)$.

2.2. Galois representation of a modular form

Let p be a fixed odd prime and $A \in \mathbb{N}$ with (A, p) = 1. Let $h = \sum a_n(h)q^n \in S_k(\Gamma_0(Ap^t), \psi)$ be a normalized eigenform of weight $k \ge 2$ and nebentypus ψ . Then h is p-ordinary if $\iota_p(a_p(h))$ is a p-adic unit. Let K_h be the number field generated by the Fourier coefficients of h and the values of ψ . Let L be a number field containing K_h and L_p denote the completion of this number field at a prime \mathfrak{p} lying above p induced by the embedding ι_p . Let O_{L_p} denote the ring of integers of L_p and π_L be a uniformizer of O_{L_p} . We denote by $\omega_p : G_{\mathbb{Q}} \longrightarrow \mathbb{Z}_p^{\times}$ the p-adic cyclotomic character. theo

THEOREM 2.2 (Eichler, Shimura, Deligne, Mazur–Wiles, Wiles, etc.). Let $h = \sum a_n(h)q^n \in S_k(\Gamma_0(Ap^t), \psi)$ be a newform of weight $k \ge 2$ where (A, p) = 1. Then there exists a Galois representation, $\rho_h : G_{\mathbb{Q}} \longrightarrow GL_2(L_{\mathfrak{p}})$ such that

(1) at all primes $\ell \nmid Ap$, ρ_h is unramified with the characteristic polynomial of the (arithmetic) Frobenius is given by

$$\operatorname{trace}(\rho_h(\operatorname{Frob}_{\ell})) = a_{\ell}(h),$$
$$\operatorname{det}(\rho_h(\operatorname{Frob}_{\ell})) = \psi(\ell)\omega_p(\operatorname{Frob}_{\ell})^{k-1} = \psi(\ell)\ell^{k-1}.$$

It follows (by the Chebotarev Density Theorem) that $det(\rho_h) = \psi \omega_p^{k-1}$.

(2) Let G_p denote the decomposition subgroup of $G_{\mathbb{Q}}$ at p. In addition, let us assume h is p-ordinary and denote by $\alpha_p(h)$ (respectively $\beta_p(h)$) the p-adic unit (respectively non p-adic unit) root of the polynomial $X^2 - a_p(h)X + \psi(p)p^{k-1}$. Let λ_h be the unramified character with $\lambda_h(\text{Frob}_p) = \alpha_p(h)$. Then by Mazur–Wiles, Wiles

$$\rho_h|_{G_p} \sim \begin{pmatrix} \lambda_h^{-1} \psi \omega_p^{k-1} & * \\ 0 & \lambda_h \end{pmatrix},$$

Let $V_h \cong L_{\mathfrak{p}}^{\oplus 2}$ denotes the representation space of ρ_h . By compactness of $G_{\mathbb{Q}}$, choose an $O_{L_{\mathfrak{p}}}$ lattice $T_h \cong O_{L_{\mathfrak{p}}}^{\oplus 2}$ of V_h which is invariant under ρ_h . Let

$$\bar{\rho}_h: \mathcal{G}_{\mathbb{Q}} \longrightarrow \mathcal{GL}_2\left(\frac{O_{L_{\mathfrak{p}}}}{\pi_L}\right)$$

be the residual representation of ρ_h .

2.3. Definition of the Selmer groups

We choose and fix a quadratic character χ and set $M := \operatorname{cond}(\chi)$. Recall for $i = 1, 2, f_i \in S_k(\Gamma_0(Np^{t_i}), \epsilon_i)$ are two (fixed) normalized cuspidal eigneforms with (N, p) = 1 and $t_i \in \mathbb{N} \cup \{0\}$. Also recall $S = S_{f_1, f_2} := \{\ell \text{ prime} : \ell \mid N\}$. Let Σ be a finite set of primes of \mathbb{Q} such that $\Sigma \supset S \cup \{p\} \cup M$. Let \mathbb{Q}_{Σ} be the maximum algebraic extension of \mathbb{Q} unramified outside Σ and set $G_{\Sigma}(\mathbb{Q}) = \operatorname{Gal}(\mathbb{Q}_{\Sigma}/\mathbb{Q})$. In this subsection, we use h as a notation for f_1 or f_2 i.e. $h \in \{f_1, f_2\}$. Hence we can take $L = K_{f_1, f_2, \epsilon_1, \epsilon_2}, O_{L_p} = O$, $\pi_L = \pi$. With the O lattice T_h as above, we have an induced $G_{\mathbb{Q}}$ action on the discrete module $A_h := V_h/T_h$. We also have the canonical maps

$$(2.2) 0 \longrightarrow T_h \longrightarrow V_h \xrightarrow{p_h} A_h \longrightarrow 0.$$

For $j \in \mathbb{Z}$, set $V_{h\chi(-j)} = V_h \otimes \chi \omega_p^{-j} = V_h \otimes \mathbb{Q}_p(\chi \omega_p^{-j})$ with the diagonal action of $G_{\mathbb{Q}}$. We further define $T_{h\chi(-j)} = T_h \otimes \chi \omega_p^{-j}$ and put $A_{h\chi(-j)} = \frac{V_{h\chi(-j)}}{T_{h\chi(-j)}}$.

Let $K \subset \mathbb{Q}_{\Sigma}$ be a number field. We define Σ_K to be the set primes in K lying over the primes in Σ . In particular for $K = \mathbb{Q}$, $\Sigma_{\mathbb{Q}} = \Sigma$. For every prime $v \in \Sigma_K$, let us choose a subset $H^1_{\dagger}(K_v, A_{h\chi(-j)}) \subset H^1(K_v, A_{h\chi(-j)})$. For this choice, we define \dagger -Selmer group $S_{\dagger}(A_{h\chi(-j)}/K)$ as

(2.3)
$$S_{\dagger}(A_{h\chi(-j)}/K)$$
$$:= \operatorname{Ker}\left(H^{1}(\mathbb{Q}_{\Sigma}/K, A_{h\chi(-j)}) \longrightarrow \prod_{v \in \Sigma_{K}} \frac{H^{1}(K_{v}, A_{h\chi(-j)})}{H^{1}_{\dagger}(K_{v}, A_{h\chi(-j)})}\right).$$

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For any $r \in \mathbb{N}$, there is a canonical map

(2.4)
$$0 \longrightarrow A_{h\chi(-j)}[\pi^r] \xrightarrow{i_{h\chi(-j),r}} A_{h\chi(-j)}$$

Next we define π^r †-Selmer group $S_{\dagger}(A_{h\chi(-j)}[\pi^r]/K)$ as

$$(2.5) \quad S_{\dagger}(A_{h\chi(-j)}[\pi^r]/K)$$
$$:= \operatorname{Ker}\left(H^1(\mathbb{Q}_{\Sigma}/K, A_{h\chi(-j)}[\pi^r]) \longrightarrow \prod_{v \in \Sigma_K} \frac{H^1(K_v, A_{h\chi(-j)}[\pi^r])}{H^1_{\dagger}(K_v, A_{h\chi(-j)}[\pi^r])}\right),$$

where $H^1_{\dagger}(K_v, A_{h\chi(-j)}[\pi^r]) := i^{*^{-1}}_{h\chi(-j),r}(H^1_{\dagger}(K_v, A_{h\chi(-j)}))$ for every $v \in \Sigma_K$. Here

$$i_{h\chi(-j),r}^*: H^1(K_v, A_{h\chi(-j)}[\pi^r]) \longrightarrow H^1(K_v, A_{h\chi(-j)})$$

is induced from $i_{h\chi(-j),r}$ in (2.4). When there is no confusion, we may denote $i_r^* := i_{h\chi(-j),r}^*$.

Remark 2.3. — For $K = \mathbb{Q}$, note that if $A_h[\pi]$ is irreducible as a $G_{\mathbb{Q}}$ module then the natural map from $H^1(G_{\Sigma}(\mathbb{Q}), A_{h\chi(-j)}[\pi^r]) \longrightarrow H^1(G_{\Sigma}(\mathbb{Q}), A_{h\chi(-j)})[\pi^r]$, induced from the multiplication by π^r map on $A_{h\chi(-j)}$, is an isomorphism. Also it follows from the definition of the Selmer group above that if $A_{h\chi(-j)}[\pi]$ is irreducible, then the natural map $S_{\dagger}(A_{h\chi(-j)}[\pi^r]/\mathbb{Q}) \longrightarrow S_{\dagger}(A_{h\chi(-j)}/\mathbb{Q})[\pi^r]$ is an isomorphism as well.

For a prime $v \in \Sigma_K$, let I_v be the inertia subgroup of $\operatorname{Gal}(\overline{\mathbb{Q}}/K)$ at vand $p_{h\chi(-j)}^* : H^1(K_v, V_{h\chi(-j)}) \longrightarrow H^1(K_v, A_{h\chi(-j)})$ is induced from the map $p_{h\chi(-j)}$ in (2.2). Now we will make special choice of $H^1_{\dagger}(K_v, A_{h\chi(-j)})$ for every $v \in \Sigma_K$, to respectively define Bloch–Kato Selmer group and under suitable condition we will also define Greenberg, signed 1 and signed 2 Selmer groups.

First $\dagger = BK$ is defined as follows: For a $v \in \Sigma$, set

$$H^{1}_{\mathrm{BK}}(K_{v}, A_{h\chi(-j)}) := \begin{cases} p^{*}_{h\chi(-j)}(H^{1}_{\mathrm{unr}}(K_{v}, V_{h})) & \text{if } v \in \Sigma_{K}, v \nmid p \\ p^{*}_{h\chi(-j)}(H^{1}_{f}(K_{v}, V_{h})) & \text{if } v \mid p. \end{cases}$$

where,

(2.6)
$$H^{1}_{unr}(K_{v}, V_{h\chi(-j)})$$
$$= \operatorname{Ker}\Big(H^{1}(K_{v}, V_{h\chi(-j)}) \longrightarrow H^{1}(I_{v}, V_{h\chi(-j)})\Big), \quad v \in \Sigma_{K}, v \nmid p$$

and

(2.7)
$$H^1_{\mathrm{f}}(K_v, V_{h\chi(-j)}) = \mathrm{Ker}\Big(H^1(K_v, V_{h\chi(-j)}) \longrightarrow H^1(K_v, V_{h\chi(-j)} \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})\Big), \quad v \mid p$$

where $B_{\rm crys}$ is the ring of periods, as defined by Fontaine in [7] (see also [2, §1]). This completes the definition of Bloch–Kato Selmer groups.

For $v \mid p$, we also recall the definition of a subgroup $H_g^1(K_v, V_{h\chi(-j)})$ of $H^1(K_v, V_{h\chi(-j)})$, to be used later

$$H^1_g(K_v, V_{h\chi(-j)})$$

:= Ker $\Big(H^1(K_v, V_{h\chi(-j)}) \longrightarrow H^1(K_v, V_{h\chi(-j)} \otimes_{\mathbb{Q}_p} B_{dR})\Big), \quad v \mid p$

where B_{dR} is as defined by Fontaine in [7].

Next we take $\dagger = \text{Gr}$ and define Greenberg Selmer group. To define this, it is necessary to assume that *h* is ordinary at *p*. Then by the *p*-ordinary property of ρ_h , by Theorem 2.2(2), V_h has a filtration as a $G_{\mathbb{Q}_p}$ module

$$(2.8) 0 \longrightarrow V'_h \longrightarrow V_h \longrightarrow V''_h \longrightarrow 0,$$

where both V'_h and V''_h are free $K_{\mathfrak{p}}$ vector space of rank 1 and the action of $G_{\mathbb{Q}}$ on V''_h is unramified at p. We define the image of V'_h in A to be A'_h and set $A''_h := A/A'_h$. Hence we get a $G_{\mathbb{Q}_p}$ module filtration on A_h

$$(2.9) 0 \longrightarrow A'_h \longrightarrow A_h \longrightarrow A''_h \longrightarrow 0,$$

where both A'_{h}^{\vee} and A''_{h}^{\vee} are free O module of rank 1 and the action of $G_{\mathbb{Q}}$ on A''_{h} is unramified at p. We can also get a similar filtration on $A_{h\chi(-j)}$. We now define

$$H^{1}_{\mathrm{Gr}}(K_{v}, A_{h\chi(-j)})$$

$$:= \begin{cases} \operatorname{Ker}(H^{1}(K_{v}, A_{h\chi(-j)}) \longrightarrow H^{1}(I_{v}, A_{h\chi(-j)})) & \text{if } v \in \Sigma_{K}, v \nmid p \\ \operatorname{Ker}(H^{1}(K_{v}, A_{h\chi(-j)}) \longrightarrow H^{1}(I_{v}, A_{h\chi(-j)}^{''}))) & \text{if } v \mid p. \end{cases}$$

Now we take $\dagger = i$ and define signed *i* Selmer group for $i \in \{1, 2\}$ (cf. [12, 16]). To define this, it is necessary to assume that *h* is "good" at *p* and also non-ordinary at *p* i.e. $t_1 = t_2 = 0$, (N, p) = 1 and $v_p(a_p(h)) \neq 0$. We also assume that the quadratic character χ is trivial i.e. we only define signed Selmer group for $f \otimes \omega_p^{-j}$. Further, we assume $K \subset \mathbb{Q}(\mu_{p^{\infty}}) := \bigcup_n \mathbb{Q}(\mu_{p^n})$. For a prime $v \in \Sigma, v \nmid p$, we simply define

$$H_{i}^{1}(K_{v}, A_{h(-j)}) := H_{BK}^{1}(K_{v}, A_{h(-j)}), \quad i = 1, 2.$$

For $v \mid p$, we will now define $H_i^1(K_v, A_{h(-j)})$ for i = 1, 2 respectively.

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As $v_p(a_p(h)) \neq 0$, a pair of Coleman maps

$$\operatorname{Col}_{h,i}: H^1_{\operatorname{Iw}}(\mathbb{Q}_p, T_h) \cong \varprojlim_n H^1(\mathbb{Q}_p(\mu_{p^n}), T_h) \longrightarrow O[\Delta] \llbracket \Gamma \rrbracket \cong O[\Delta] \llbracket T \rrbracket$$

are defined (see [12, §2] [16] for details). Here $\Delta = \text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$. Let \Pr_{K_v} be the natural projection map

(2.10)
$$H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_{h(-j)}) \xrightarrow{P_{T_{K_v}}} H^1(K_v, T_{h(-j)}).$$

For i = 1, 2, we first define a subset $H_i^1(K_v, T_{h(-j)}) \subset H^1(K_v, T_{h(-j)})$ to be

$$H_i^1(K_v, T_{h(-j)}) := \Pr_{K_v}(\operatorname{Ker}(\operatorname{Col}_{h,i}) \otimes \omega_p^{-j}), \quad i = 1, 2$$

Now consider the natural map $\iota : H^1(K_v, T_{h(-j)}) \longrightarrow H^1(K_v, V_{h(-j)})$ induced by the inclusion of T_h in V_h and denote by V_i the subspace of $H^1(K_v, V_{h(-j)})$ generated by the image of $H^1_i(K_v, T_{h(-j)})$ under ι i.e. $V_i = \langle \iota(H^1_i(K_v, T_{h(-j)})) \rangle$. Finally, define $H^1_i(K_v, A_{h(-j)}) := \operatorname{Proj}(V_i)$ where Proj is the natural map $\operatorname{Proj} : H^1(K_v, V_{h(-j)}) \longrightarrow H^1(K_v, A_{h(-j)})$ induced by the projection of V_h to A_h . This completes the definition of signed 1 and 2 Selmer groups of $h \otimes \omega_p^{-j}$.

Finally, we define π^r Selmer companion modular forms.

DEFINITION 2.4. — Let $i \in \{1,2\}$ and $f_i \in S_k(\Gamma_0(N_i), \epsilon_i)$ be a normalized cuspidal eigenform where $N_i \in \mathbb{N}$. Let $r \in \mathbb{N}$. We say f_1 and f_2 are π^r (Bloch–Kato) Selmer companion, if for each critical twist j with $0 \leq j \leq k-2$ and for every quadratic character χ of $G_{\mathbb{Q}}$, we have an isomorphism of π^r Bloch–Kato Selmer groups of $f_1 \otimes \chi \omega_p^{-j}$ and $f_2 \otimes \chi \omega_p^{-j}$ over \mathbb{Q} i.e.

$$S_{\mathrm{BK}}(A_{f_1\chi(-j)}[\pi^r]/\mathbb{Q}) \cong S_{\mathrm{BK}}(A_{f_2\chi(-j)}[\pi^r]/\mathbb{Q}).$$

Note that for weight $k \ge 2$, corresponding to k-1 critical values, there are k-1 many Selmer groups associated to a cuspidal eigenform; and for f_1 and f_2 to be π^r -Selmer companion, we need each j with $0 \le j \le k-2$ and every χ , $S_{\rm BK}(A_{f_1\chi(-j)}[\pi^r]/\mathbb{Q}) \cong S_{\rm BK}(A_{f_2\chi(-j)}[\pi^r]/\mathbb{Q})$.

Remark 2.5. — Mazur and Rubin [20, §1] have stated that given an elliptic curve E over a number field K, they expect there are only finitely many elliptic curves E' over K such that E and E' are Selmer companion. Moreover they have shown in [20, Proposition 7.1] that given an elliptic curve E over K there are only finitely many elliptic curves E' over K such that the pair (E, E') satisfy all the conditions of their main theorem ([20, Theorem 3.1]).

Our situation is different. Let $f \in S_2(\Gamma_0(N))$ be a *p*-ordinary normalized Hecke eigen newform with $a_p(f) \neq \pm 1 \pmod{\pi}$, N is squarefree and coprime to p. We further assume that $\operatorname{cond}_{\ell}(\bar{\rho}_f) = \ell$ for every prime $\ell \parallel N$. (In Example I(1), we have given explicit example of such an f.)

Then by Hida theory (see [29]) there exists infinitely many $f_r \in S_2(\Gamma_0(Np^r), \psi_r)$ such that $f_r \equiv f \pmod{\pi}$, where $\pi = \pi_{f,f_r,\psi_r}$ and ψ_r is certain dirichlet character of conductor p^r satisfying ψ_r is trivial mod π . Then for each r, the pair (f, f_r) satisfies conditions (1), (2) and (3) of Theorem 4.10. In particular, f and f_r are π Selmer companion for infinitely many f_r .

Remark 2.6. — In the converse direction, Mazur and Rubin have asked if two elliptic curves E and E' over K are p^r -Selmer companion, then can we say that $E[p^r] \cong E'[p^r]$ as $\operatorname{Gal}(\overline{K}/K)$ -modules (see [20, Conjecture 7.14 in arxiv version])? It would be interesting to investigate if π^r -Selmer companion modular forms of weight $k \ge 2$ are congruent mod π^r .

3. "Greenberg Selmer companion forms"

In this section, we compare twisted π^r -Greenberg Selmer groups of two *p*ordinary π^r congruent cuspforms. We use these results in the next section to study Bloch–Kato Selmer companion forms. Our main result in this section is the following:

THEOREM 3.1. — Let p be an odd prime and for i = 1, 2, let f_i be a p-ordinary normalized eigenform in $S_k(\Gamma_0(Np^{t_i}), \epsilon_i)$, where (N, p) = 1, $k \ge 2, t_i \in \mathbb{N} \cup \{0\}$. Recall from Section 2.1, C'_i is the tame conductor of ϵ_i . Let $r \in \mathbb{N}$ and $\phi : A_{f_1}[\pi^r] \longrightarrow A_{f_2}[\pi^r]$ be a $G_{\mathbb{Q}}$ linear isomorphism. We assume the following:

- (1) N is square-free and $\forall \ell \in S$, $\operatorname{cond}_{\ell}(\bar{\rho}_{f_1}) = \ell = \operatorname{cond}_{\ell}(\bar{\rho}_{f_2})$.
- (2) The condition $(C'_{i,\ell})$, defined in equation (2.1), is satisfied for i = 1, 2.
- (3) Assume $\omega_p^{k-1} \epsilon_{i,p} \neq 1 \pmod{\pi}$ for i = 1, 2.

Then for each fixed j with $0 \leq j \leq k-2$, and for every quadratic character χ of $G_{\mathbb{Q}}$ there is an isomorphism

$$S_{\mathrm{Gr}}(A_{f_1\chi(-j)}[\pi^r]/\mathbb{Q}) \cong S_{\mathrm{Gr}}(A_{f_2\chi(-j)}[\pi^r]/\mathbb{Q}).$$

Here $\operatorname{cond}_{\ell}(\bar{\rho}_{f_1})$ and $\operatorname{cond}_{\ell}(\rho_{f_2})$ are defined following Serre and can be found in [18, §1, p. 135]. The proof of the theorem is divided into several

lemmas and propositions. Before going into the proof, we make the following remark to compare it with the corresponding hypothesis at $\ell \neq p$ of Mazur–Rubin.

Remark 3.2. — Let E be an elliptic curve over \mathbb{Q} with squarefree conductor N with $p \nmid N$. Let $f_E \in S_2(\Gamma_0(N))$ associated with E via modularity and $\rho_E = \rho_{f_E}$ be the corresponding Galois representation (Section 2.2). Let $\ell \mid N$ be a prime of multiplicative reduction for E. Then note that Ehas split-multiplicative reduction over an unramified quadratic extension of \mathbb{Q}_ℓ and hence E has split multiplicative reduction over $\mathbb{Q}_\ell(\mu_p)$. Therefore $E_{p^{\infty}}(\mathbb{Q}_\ell(\mu_{p^{\infty}}))$ is isomorphic to a subgroup of $\mathbb{Q}_\ell(\mu_{p^{\infty}})^{\times}/q^{\mathbb{Z}}$ generated by $\mu_{p^{\infty}}$ and q^{1/p^m} for some $m \in \mathbb{N} \cup \{0\}$ (see [11, Proposition 5.1(ii)]). Here $q \in \mathbb{Q}_\ell(\mu_p)^{\times}$ is the Tate period of the elliptic curve. The assumption $p \nmid \operatorname{ord}_\ell(j(E))$ of [20, Lemma 2.5] together with the fact $\mathbb{Q}_\ell(\mu_{p^{\infty}})$ is unramified at ℓ implies that m = 0 above. Thus $E_{p^{\infty}}(\mathbb{Q}_\ell(\mu_{p^{\infty}}))[p] \cong \mathbb{Z}/p\mathbb{Z}$. In other words, $E[p]^{I_\ell} \cong \mathbb{Z}/p\mathbb{Z}$. Recall by the definition of conductor at ℓ (see [18, §1]),

$$\operatorname{cond}_{\ell}(\rho_E) = \ell^{\operatorname{codim} \rho_E^{l_{\ell}} + sw(\rho_E^{ss})}$$
 and $\operatorname{cond}_{\ell}(\bar{\rho}_E) = \ell^{\operatorname{codim} \bar{\rho}_E^{l_{\ell}} + sw(\bar{\rho}_E)}$.

Also note that using [18, Proposition 1.1], we have $sw(\rho_E^{ss}) = sw(\bar{\rho}_E)$. Now as E has multiplicative reduction at ℓ , using the above facts, we deduce that $E[p]^{I_\ell} \cong \mathbb{Z}/p\mathbb{Z}$ if and only if $\operatorname{cond}_\ell(\bar{\rho}_E) = \ell = \operatorname{cond}_\ell(\rho_E)$. Thus when f is of weight 2 and corresponds to an elliptic curve, we have shown that the condition of [20, Lemma 2.5] implies the hypothesis on conductor given in condition (1) of our Theorem 3.1. Note that [20, Lemma 2.5] is used to show that the condition (iii) of the their main theorem ([20, Theorem 3.1]) holds.

PROPOSITION 3.3. — Let $i \in \{1,2\}$ and $f_i \in S_k(\Gamma_0(Np^{t_i}), \epsilon_i)$ with (N,p) = 1. Assume N is square-free and $\forall \ell \in S$, $\operatorname{cond}_\ell(\bar{\rho}_{f_i}) = \ell$. Also assume the hypothesis $(C'_{i,\ell})$ defined in equation (2.1). Then for every quadratic character χ of $G_{\mathbb{Q}}$, for any $j \in \mathbb{Z}$ and for every prime $\ell \in \Sigma \setminus \{p\}$, $\operatorname{cond}_\ell(\rho_{f_i\chi(-j)}) = \operatorname{cond}_\ell(\bar{\rho}_{f_i\chi(-j)})$.

Proof. — Let $i \in \{1,2\}$. Recall, $M = \text{conductor of } \chi \text{ and } C_i = \text{conductor of } \epsilon_i$. Then M = D or 4D with D square-free. Note as ω_p is unramified at ℓ , $\operatorname{cond}_{\ell}(\rho_{f_i\chi(-j)}) = \operatorname{cond}_{\ell}(\rho_{f_i\chi})$ and $\operatorname{cond}_{\ell}(\bar{\rho}_{f_i\chi(-j)}) = \operatorname{cond}_{\ell}(\bar{\rho}_{f_i\chi})$. Thus it suffices to show that for every quadratic character χ of $G_{\mathbb{Q}}$, $\operatorname{cond}_{\ell}(\rho_{f_i\chi}) = \operatorname{cond}_{\ell}(\bar{\rho}_{f_i\chi})$. We prove this by considering various cases.

Case 1: $\ell \parallel N$ and $\ell \nmid M$. — Since $\ell \nmid M$, χ is unramified at ℓ and we have $\operatorname{cond}_{\ell}(\rho_{f_i\chi}) = \operatorname{cond}_{\ell}(\rho_{f_i})$ and $\operatorname{cond}_{\ell}(\bar{\rho}_{f_i\chi}) = \operatorname{cond}_{\ell}(\bar{\rho}_{f_i})$. Now from

the first assumption of the proposition, we have $\operatorname{cond}_{\ell}(\rho_{f_i}) = \operatorname{cond}_{\ell}(\bar{\rho}_{f_i})$. Therefore $\operatorname{cond}_{\ell}(\rho_{f_i\chi}) = \operatorname{cond}_{\ell}(\bar{\rho}_{f_i\chi})$.

Case 2: $\ell \nmid N$ and $\ell \mid M$. — Since $\ell \nmid N$, ρ_{f_i} is unramified at ℓ . Therefore $\operatorname{cond}_{\ell}(\rho_{f_i\chi}) = \operatorname{cond}_{\ell}(\chi)$ and $\operatorname{cond}_{\ell}(\bar{\rho}_{f_i\chi}) = \operatorname{cond}_{\ell}(\bar{\chi})$ where $\bar{\chi}$ denote the residual character mod π associated to χ . Since χ is a quadratic character and $p \neq 2$, $\operatorname{cond}_{\ell}(\chi) = \operatorname{cond}_{\ell}(\bar{\chi})$. This implies that $\operatorname{cond}_{\ell}(\rho_{f_i\chi}) = \operatorname{cond}_{\ell}(\bar{\rho}_{f_i\chi})$.

Case 3: $\ell \mid | N, \ell \nmid C'_i$ and $\ell | M.$ — By our assumption, $\operatorname{cond}_{\ell}(\rho_{f_i}) = \operatorname{cond}_{\ell}(\bar{\rho}_{f_i}) = \ell$. Then from [18, p. 135], we get

$$\operatorname{cond}_{\ell}(\rho_{f_i}) = \ell^{\operatorname{codim} \rho_{f_i}^{I_{\ell}} + sw(\rho_{f_i}^{ss})} \quad \text{and} \quad \operatorname{cond}_{\ell}(\bar{\rho}_{f_i}) = \ell^{\operatorname{codim} \bar{\rho}_{f_i}^{I_{\ell}} + sw(\bar{\rho}_{f_i})}$$

In particular, codim $\bar{\rho}_{f_i}^{I_\ell} \leq 1$ and therefore dim $\bar{\rho}_{f_i}^{I_\ell} \geq 1$. Since $\bar{\rho}_{f_i}$ is ramified at ℓ , dim $\bar{\rho}_{f_i}^{I_\ell} = 1$. This implies that $\bar{\rho}_{f_i}$ has an unramified submodule, say \overline{V}_1 . Put $\overline{V}_2 := \overline{V}_{f_i}/\overline{V}_1$ where \overline{V}_{f_i} denote the vector space corresponding to $\bar{\rho}_{f_i}$. Let $\bar{\chi}_1$ and $\bar{\chi}_2$ be the characters associated to \overline{V}_1 and \overline{V}_2 respectively. As $\ell \nmid C'_i$, we get $\bar{\chi}_1 \bar{\chi}_2 = \bar{\omega}_p^{k-1} \bar{\epsilon}_i$. This implies that $\bar{\chi}_2$ is also unramified. Thus we have $\bar{\rho}_{f_i} \sim \begin{pmatrix} \bar{\chi}_1 & \bar{\chi}_1 \\ 0 & \bar{\chi}_2 \end{pmatrix}$ and consequently

$$\bar{\rho}_{f_i\chi} = \bar{\rho}_{f_i} \otimes \bar{\chi} \sim \begin{pmatrix} \bar{\chi}_1 \bar{\chi} & * \\ 0 & \bar{\chi}_2 \bar{\chi} \end{pmatrix}.$$

Now χ being quadratic and $\ell \mid M$, both χ and $\overline{\chi}$ are ramified at ℓ . In particular, $(\overline{V}_{f_i} \otimes \overline{\chi})^{I_{\ell}} = 0$ i.e. $\operatorname{codim}(\overline{V}_{f_i} \otimes \overline{\chi})^{I_{\ell}} = 2$. On the other hand,

$$\rho_{f_i\chi}|_{G_\ell} \sim \begin{pmatrix} \eta_i \omega_p \chi & * \\ 0 & \eta_i \chi \end{pmatrix},$$

where η_i is an unramified character (see [13, Theorem 3.26(3)]). Again as χ is ramified at ℓ , we deduce $(V_{f_i\chi})^{I_\ell} = 0$; in other words, $\operatorname{codim}(V_{f_i\chi})^{I_\ell} = 2$ as well. Further by [18, Proposition 1.1], $sw(\overline{V}_{f_i}\otimes\overline{\chi}) = sw((V_{f_i\chi})^{ss})$. Hence $\operatorname{cond}_{\ell}(\rho_{f_i\chi}) = \operatorname{cond}_{\ell}(\bar{\rho}_{f_i\chi})$ holds true in this case.

Case 4: $\ell \mid \mid N, \ell \mid C'_i \text{ and } \ell \mid M.$ — In this case, $\rho_{f_i|_{I_\ell}} \sim \begin{pmatrix} \epsilon_{i,\ell} & 0 \\ 0 & 1 \end{pmatrix}$ and $\bar{\rho}_{f_i|_{I_\ell}} \sim \begin{pmatrix} \bar{\epsilon}_{i,\ell} & 0 \\ 0 & 1 \end{pmatrix}$ (see [13, Theorem 3.26(3)]). Similarly,

$$\rho_{f_i\chi_{|_{I_\ell}}} \sim \begin{pmatrix} \epsilon_{i,\ell}\chi & 0\\ 0 & \chi \end{pmatrix} \quad \text{ and } \quad \bar{\rho}_{f_i\chi_{|_{I_\ell}}} \sim \begin{pmatrix} \bar{\epsilon}_{i,\ell}\bar{\chi} & 0\\ 0 & \bar{\chi} \end{pmatrix}.$$

Note that as $\ell \mid M$, both χ and $\overline{\chi}$ are ramified at ℓ as in the previous case.

First we consider the subcase when $\bar{\epsilon}_i \bar{\chi}$ is ramified at ℓ . This implies that $\epsilon_i \chi$ is also ramified. Thus $\operatorname{codim}(\overline{V}_{f_i} \otimes \bar{\chi})^{I_\ell} = 2 = \operatorname{codim}(V_{f_i \chi})^{I_\ell}$ holds.

Next we assume $\bar{\epsilon}_i \bar{\chi}$ is unramified i.e. $\bar{\epsilon}_{i,\ell} \bar{\chi}_{\ell}$ is trivial. Then we have $\bar{\epsilon}_{i,\ell}^2$ is trivial. Now by the second assumption of the proposition, the order

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of $\epsilon_{i,\ell}^2$ is not a positive power of p. Hence $\bar{\epsilon}_{i,\ell}^2$ is trivial gives $\epsilon_{i,\ell}^2$ is also trivial. Thus $\epsilon_{i,\ell}$ and $\epsilon_{i,\ell}\chi_\ell$ are both quadratic characters. Hence $\bar{\epsilon}_{i,\ell}\overline{\chi}_\ell$ is trivial implies $\epsilon_{i,\ell}\chi_\ell$ is trivial. Then using $\rho_{f_i\chi_{|I_\ell}} \sim \epsilon_{i,\ell}\chi_\ell \oplus \chi_\ell$ and χ is ramified at ℓ , we deduce $\operatorname{codim}(\overline{V}_{f_i} \otimes \overline{\chi})^{I_\ell} = 1 = \operatorname{codim}(V_{f_i\chi})^{I_\ell}$. Again from [18, Proposition 1.1], $sw(\overline{V}_{f_i} \otimes \overline{\chi}) = sw((V_{f_i\chi})^{ss})$, hence we obtain $\operatorname{cond}_\ell(\rho_{f_i\chi}) = \operatorname{cond}_\ell(\overline{\rho}_{f_i\chi})$, as required.

COROLLARY 3.4. — We keep the hypotheses of Proposition 3.3. Then for every quadratic character χ of $G_{\mathbb{Q}}$ and for the $(-j)^{th}$ Tate twist of $f_i \otimes \chi$, we have $A_{f_i\chi(-j)}^{I_\ell}$ is divisible.

Proof. — This is a direct consequence of Proposition 3.3 and the proof of [6, Lemma 4.1.2]. \Box

LEMMA 3.5. — Let us keep the hypotheses of Proposition 3.3. Then for every $q \in \Sigma \setminus \{p\}$, every χ and every j, we have

$$H^{1}_{\mathrm{Gr}}(\mathbb{Q}_{q}, A_{f_{i}\chi(-j)}[\pi^{r}]) = \mathrm{Ker}\Big(H^{1}(\mathbb{Q}_{q}, A_{f_{i}\chi(-j)}[\pi^{r}]) \longrightarrow H^{1}(I_{q}, A_{f_{i}\chi(-j)}[\pi^{r}])\Big).$$

Proof. — Consider the commutative diagram

$$(3.1) H^{1}(\mathbb{Q}_{q}, A_{f_{i}\chi(-j)}[\pi^{r}]) \xrightarrow{\phi_{r}} H^{1}(I_{q}, A_{f_{i}\chi(-j)}[\pi^{r}]) \xrightarrow{\phi_{r}} H^{1}(I_{q}, A_{f_{i}\chi(-j)}) \xrightarrow{\phi_{r}} H^{1}(I_{q}, A_{f_{i}\chi(-j)})$$

Now $b \in H^1_{\mathrm{Gr}}(\mathbb{Q}_q, A_{f_i\chi(-j)}[\pi^r])$ if and only if $b \in \mathrm{Ker}(\phi \circ i_r^*) = \mathrm{Ker}(s_r \circ \phi_r)$. As s_r is induced by the Kummer map, $\mathrm{Ker}(s_r) \cong \frac{(A_{f_i\chi(-j)})^{I_q}}{\pi^r (A_{f_i\chi(-j)})^{I_q}}$. It follows from Corollary 3.4 that $\mathrm{ker}(s_r) = 0$ which proves the lemma. \Box

Using Lemma 3.5, the following expression of $S_{\text{Gr}}(A_{f_i\chi(-j)}[\pi^r]/\mathbb{Q})$ is immediate.

LEMMA 3.6. — We keep the hypotheses of Proposition 3.3. Then we have an exact sequence

$$0 \to S_{\mathrm{Gr}}(A_{f_i\chi(-j)}[\pi^r]/\mathbb{Q}) \longrightarrow H^1(G_{\Sigma}(\mathbb{Q}), A_{f_i\chi(-j)}[\pi^r]) \\ \longrightarrow \prod_{q \in \Sigma, q \neq p} H^1(I_q, A_{f_i\chi(-j)}[\pi^r]) \times \frac{H^1(\mathbb{Q}_p, A_{f_i\chi(-j)}[\pi^r])}{H^1_{G_r}(\mathbb{Q}_p, A_{f_i\chi(-j)}[\pi^r])}.$$

Next we study the *p*-part of the local terms defining $S_{\text{Gr}}(A_{f_i\chi(-j)}[\pi^r]/\mathbb{Q})$.

PROPOSITION 3.7. — Set $I'_p = \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\operatorname{unr}}(\mu_{p^{\infty}}))$ and $i \in \{f_1, f_2\}$. Then

$$\begin{aligned} H^{1}_{\mathrm{Gr}}(\mathbb{Q}_{p}, A_{f_{i}\chi(-j)}[\pi^{r}]) &= \begin{cases} \mathrm{Ker}\big(H^{1}(\mathbb{Q}_{p}, A_{f_{i}\chi(-j)}[\pi^{r}]) \longrightarrow H^{1}(I_{p}, A_{f_{i}\chi(-j)}^{\prime\prime}[\pi^{r}])\big) \\ & \text{if } (\omega_{p}^{-j}\chi)_{|_{I_{p}}} \neq 1 \pmod{\pi} \text{ or } j = 0, \\ \mathrm{Ker}\big(H^{1}(\mathbb{Q}_{p}, A_{f_{i}\chi(-j)}[\pi^{r}]) \longrightarrow H^{1}(I_{p}^{\prime}, A_{f_{i}\chi(-j)}^{\prime\prime}[\pi^{r}])\big) & \text{otherwise.} \end{cases}$$

Proof.

Case 1: Either $(\omega^{-j}\chi)|_{I_p} \neq 1 \pmod{\pi}$ or j = 0. — Consider the commutative diagram

It suffices to show in Case 1 that $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A_{f_i\chi(-j)}[\pi^r]) = \mathrm{Ker}(\theta_r)$. From the above diagram in (3.2), we observe that $b \in H^1_{\mathrm{Gr}}(\mathbb{Q}_q, A_{f_i\chi(-j)}[\pi^r])$ if and only if $b \in \mathrm{Ker}(\theta \circ i_r^*) = \mathrm{Ker}(i_r''^* \circ \theta_r)$. Thus it further reduces to show $\mathrm{Ker}(i_r''^*) = 0$ in this case. Note that $\mathrm{Ker}(i_r''^*) \cong (A_{f_i\chi(-j)}'')^{I_p}/\pi^r (A_{f_i\chi(-j)}'')^{I_p}$. We divide the proof in three subcases.

First, let j = 0 and χ is unramified at p: Then $A_{f_i\chi(-j)}^{\prime\prime I_p} = A_{f_i\chi}^{\prime\prime}$ is divisible, whence $\operatorname{Ker}(i_x^{\prime\prime*}) = 0$.

Second, let j = 0 and χ is ramified: Here being a quadratic character $\chi \pmod{\pi}$ is also ramified (here we use p is odd). Thus I_p acts non-trivially on $A''_{f_i\chi}[\pi]$ and hence $(A''_{f_i\chi}[\pi])^{I_p} = 0 = (A''_{f_i\chi})^{I_p}$ and consequently $\operatorname{Ker}(i''_r^*) = 0$.

Finally, let j > 0 and $(\omega^{-j}\chi)_{|_{I_p}} \neq 1 \pmod{\pi}$: In this scenario, $(\omega_p^{-j}\chi)_{|_{I_p}} \pmod{\pi}$ (mod π) is nontrivial. Hence I_p again acts non-trivially on $A''_{f_i\chi(-j)}[\pi]$ and hence $(A''_{f_i\chi(-j)}[\pi])^{I_p} = 0 = (A''_{f_i\chi(-j)})^{I_p}$ and again we conclude $\operatorname{Ker}(i''_r) = 0$.

Case 2: j > 0 and simultaneously $(\omega_p^{-j}\chi)|_{I_p} = 1 \pmod{\pi}$. We now consider the following commutative diagram

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From the diagram (3.3), it suffices to show that $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A_{f_i\chi(-j)}[\pi^r]) = \mathrm{Ker}(\psi_r)$ to complete the proof of the Lemma. As j > 0, we have $(A''_{f_i\chi(-j)})^{I_p}$ is finite. Moreover, as $I_p/I'_p = \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{unr}}(\mu_{p^{\infty}})/\mathbb{Q}_p^{\mathrm{unr}})$ is pro-cyclic, $H^1(I_p/I'_p, A''_{f_i\chi(-j)})$ is finite as well. Now the second assumption $(\omega_p^{-j}\chi)|_{I_p} = 1 \pmod{\pi}$ implies that I'_p acts trivially on $A''_{f_i\chi(-j)}$. Hence $H^1(I_p/I'_p, A''_{f_i\chi(-j)}) = H^1(I_p/I'_p, A''_{f_i\chi(-j)})$ is divisible also. Hence $H^1(I_p/I'_p, A''_{f_i\chi(-j)}) = 0$. Therefore the natural restriction map $H^1(I_p, A''_{f_i\chi(-j)}) \longrightarrow H^1(I'_p, A''_{f_i\chi(-j)})$ is injective. Thus we have shown that

$$H^{1}_{\mathrm{Gr}}(\mathbb{Q}_{p}, A_{f_{i}\chi(-j)})$$

:= Ker $\left(H^{1}(\mathbb{Q}_{p}, A_{f_{i}\chi(-j)}) \longrightarrow H^{1}(I_{p}, A_{f_{i}\chi(-j)}'')\right) = \mathrm{Ker}(\psi).$

On the other hand, divisibility of $A_{f_i\chi(-j)}^{''I_p'} = A_{f_i\chi(-j)}^{''}$ implies that $i_r^{''*}$ is injective. Now by an argument similar to Case 1, we get that $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A_{f_i\chi(-j)}[\pi^r]) = \mathrm{Ker}(\psi_r)$.

From the discussions in Case 1 of Proposition 3.7, we deduce the following corollary.

COROLLARY 3.8. — Assume that either (i) j = 0 or (ii) j > 0 and $\omega_p^{-j}\chi_{|_{I_p}} \neq 1 \pmod{\pi}$. Then for $i = 1, 2, A_{f_i\chi(-j)}^{\prime\prime I_p}$ is π -divisible.

Remark 3.9. — We have $0 \leq j \leq k-2$. If we choose $\frac{p-1}{2} > k-2$ i.e. p > 2k-3, then $j < \frac{p-1}{2}$ and $\omega_p^j \pmod{\pi}$ is not a quadratic character. In particular, we have $(\omega_p^{-j}\chi)_{|_{I_p}} \neq 1 \pmod{\pi}$ and conditions (i) and (ii) of Corollary 3.8 are satisfied.

COROLLARY 3.10. — We keep the hypotheses of Proposition 3.3. Then it follows from Lemma 3.5, Lemma 3.6 and Proposition 3.7 that: If either $(\omega_p^{-j}\chi)|_{I_p} \neq 1 \pmod{\pi}$ or when j = 0, then

$$\begin{split} S_{\mathrm{Gr}}(A_{f_i\chi(-j)}[\pi^r]/\mathbb{Q}) \\ &= \mathrm{Ker}\Bigg(H^1(G_{\Sigma}(\mathbb{Q}), A_{f_i\chi(-j)}[\pi^r]) \\ &\longrightarrow \prod_{q\in\Sigma, q\neq p} H^1(I_q, A_{f_i\chi(-j)}[\pi^r]) \times H^1(I_p, A_{f_i\chi(-j)}'[\pi^r]) \Bigg). \end{split}$$

In other case i.e. when $(\omega_p^{-j}\chi)|_{I_p} = 1 \pmod{\pi}$ as well as j > 0,

$$\begin{aligned} S_{\mathrm{Gr}}(A_{f_i\chi(-j)}[\pi^r]/\mathbb{Q}) \\ &= \mathrm{Ker}\Bigg(H^1(G_{\Sigma}(\mathbb{Q}), A_{f_i\chi(-j)}[\pi^r]) \\ &\longrightarrow \prod_{q\in\Sigma, q\neq p} H^1(I_q, A_{f_i\chi(-j)}[\pi^r]) \times H^1(I'_p, A''_{f_i\chi(-j)}[\pi^r]) \Bigg) \end{aligned}$$

LEMMA 3.11. — Recall $\phi: A_{f_1}[\pi^r] \longrightarrow A_{f_2}[\pi^r]$ is the given $G_{\mathbb{Q}}$ linear isomorphism in Theorem 3.1. Let k be such that $\omega_p^{k-1}\epsilon_{i,p} \neq 1 \pmod{\pi}$. Then $\phi|_{G_p}$ induces an isomorphism $A''_{f_1}[\pi^r] \cong A''_{f_2}[\pi^r]$. Consequently, for all quadratic character χ and all $0 \leq j \leq k-2$, $\phi|_{G_p}$ induces isomorphisms $A''_{f_1\chi(-j)}[\pi^r] \cong A''_{f_2\chi(-j)}[\pi^r]$.

Proof. — Let $i \in \{1, 2\}$. First we claim that

(3.4)
$$H_0(I_p, A'_{f_i}[\pi^r]) = 0.$$

This is proved by induction on r. The case r = 1 is proved as follows: the action of I_p on the one dimensional O_f/π vector space $A'_{f_i}[\pi]$ is via $\omega_p^{k-1}\epsilon_{i,p} \pmod{\pi}$ and thus by given condition on k in the hypothesis, this action is non-trivial. Hence $H_0(I_p, A'_{f_i}[\pi]) = 0$. Then we apply induction using the short exact sequence $0 \to A'_{f_i}[\pi] \longrightarrow A'_{f_i}[\pi^r] \longrightarrow A'_{f_i}[\pi^{r-1}] \to 0$ to establish the claim for a general r.

Next we consider the exact sequence $0 \to A'_{f_i}[\pi^r] \longrightarrow A_{f_i}[\pi^r] \longrightarrow A'_{f_i}[\pi^r] \to 0$. Then using equation (3.4), we deduce

(3.5)
$$H_0(I_p, A_{f_i}^{''}[\pi^r]) \cong H_0(I_p, A_{f_i}[\pi^r]).$$

Notice that as I_p acts on A''_{f_i} trivially, we have the identification

(3.6)
$$A_{f_i}^{''}[\pi^r] = H_0(I_p, A_{f_i}^{''}[\pi^r]).$$

Using (3.5) and (3.6), we finally get the required G_p linear isomorphism

$$A_{f_1}^{''}[\pi^r] \cong H_0(I_p, A_{f_1}[\pi^r]) \cong H_0(I_p, A_{f_2}[\pi^r]) \cong A_{f_2}^{''}[\pi^r].$$

Proof of Theorem 3.1. — It is now plain from Corollary 3.10 and Lemma 3.11. $\hfill \Box$

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4. Bloch–Kato Selmer Companion forms

In this section we study Bloch–Kato Selmer companion forms and establish our main result (Theorem 4.10). We shall begin by comparing Greenberg and Bloch–Kato Selmer groups. Recall from Theorem 2.2(2), λ_{f_i} is the unramified character of G_p with $\lambda_{f_i}(\operatorname{Frob}_p) = \alpha_p(f_i)$.

PROPOSITION 4.1. — Let $i \in \{1, 2\}$. We assume all of the following hypotheses.

- (1) Let f_i be p-ordinary.
- (2) The tame level N of f_i is square-free and $\forall \ell \in S$, $\operatorname{cond}_{\ell}(\bar{\rho}_{f_i}) = \ell$.
- (3) The condition $(C'_{i,\ell})$, defined in equation (2.1), is satisfied.
- (4) $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A_{f_i\chi(-i)})$ is π -divisible.
- (5) $\alpha_p(f_i) \neq \pm 1$ and $\lambda_{f_i} \neq \pm \epsilon'_i$. Then,

$$S_{\mathrm{Gr}}(A_{f_i\chi(-j)}[\pi^r]/\mathbb{Q}) \cong S_{\mathrm{BK}}(A_{f_i\chi(-j)}[\pi^r]/\mathbb{Q}).$$

Proof. — It suffices to show

$$H^1_{\mathrm{BK}}(\mathbb{Q}_q, A_{f_i\chi(-j)}[\pi^r]) = H^1_{\mathrm{Gr}}(\mathbb{Q}_q, A_{f_i\chi(-j)}[\pi^r])$$

for every $q \in \Sigma$.

By assumptions (1) and (2), it follows from Corollary 3.4 that $A_{f_i\chi(-j)}^{I_q}$ is divisible for every $q \in \Sigma \setminus \{p\}$. Thus for such a q, $H_{\mathrm{Gr}}^1(\mathbb{Q}_q, A_{f_i\chi(-j)}) = H^1(G_q/I_q, A_{f_i\chi(-j)}^{I_q})$ is divisible. Note that $H_{\mathrm{BK}}^1(\mathbb{Q}_q, A_{f_i\chi(-j)})$ is the maximal divisible subgroup of $H^1(G_q/I_q, A_{f_i\chi(-j)}^{I_q})$ (see for example [23, Proposition 4.2]). Hence we deduce for every $q \in \Sigma \setminus \{p\}, H_{\mathrm{BK}}^1(\mathbb{Q}_q, A_{f_i\chi(-j)})$ $= H_{\mathrm{Gr}}^1(\mathbb{Q}_q, A_{f_i\chi(-j)})$ and consequently $H_{\mathrm{BK}}^1(\mathbb{Q}_q, A_{f_i\chi(-j)}[\pi^r]) = H_{\mathrm{Gr}}^1(\mathbb{Q}_q, A_{f_i\chi(-j)})$

Next we consider the case of prime p. Note that in this case, the image of $H_g^1(\mathbb{Q}_p, V_{f_i\chi(-j)})$ in $H^1(\mathbb{Q}_p, A_{f_i\chi(-j)})$ is the maximum divisible subgroup of $H_{\mathrm{Gr}}^1(\mathbb{Q}_p, A_{f_i\chi(-j)})$ [23, Proof of Proposition 4.2]. We first claim, $H_{\mathrm{BK}}^1(\mathbb{Q}_p, V_{f_i\chi(-j)}) = H_g^1(\mathbb{Q}_p, V_{f_i\chi(-j)})$. Assume this claim to be true at the moment. Then from the assumption (3), we get that $H_{\mathrm{BK}}^1(\mathbb{Q}_p, A_{f_i\chi(-j)}) = H_{\mathrm{Gr}}^1(\mathbb{Q}_p, A_{f_i\chi(-j)})$. This implies that $H_{\mathrm{BK}}^1(\mathbb{Q}_p, A_{f_i\chi(-j)}[\pi^r]) = H_{\mathrm{Gr}}^1(\mathbb{Q}_p, A_{f_i\chi(-j)}[\pi^r])$.

The proposition follows from the above discussion once we establish the claim. We calculate, $\dim_{\mathbb{Q}_p} H^1_{\mathrm{BK}}(\mathbb{Q}_p, V_{f_i\chi(-j)}) - \dim_{\mathbb{Q}_p} H^1_g(\mathbb{Q}_p, V_{f_i\chi(-j)})$. By [1, 2.3.5, p. 35],

$$\dim_{\mathbb{Q}_p} \frac{H^1_g(\mathbb{Q}_p, V_{f_i\chi(-j)})}{H^1_{\mathrm{BK}}(\mathbb{Q}_p, V_{f_i\chi(-j)})} = \dim_{\mathbb{Q}_p} D_{\mathrm{crys}}(V^*_{f_i\chi(-j)}(1))^{\varphi=1}$$

where $V^* = \text{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p)$ and $D_{\text{crys}}(\cdot) := (\cdot \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{G_{\mathbb{Q}_p}}$ as defined in [2, (3.5), p. 353]. (Note in [2]'s notation $\text{crys}(\cdot) = D_{\text{crys}}(\cdot)$ in our notation.)

We shall first show $D_{\text{crys}}(V''_{f_i\chi(-j)}(1))^{\varphi=1} = 0$. If χ is ramified then it follows from ([8, §7.2.4, §7.2.5, Proposition 7.20]) that $V''_{f_i\chi(-j)}(1)$ is not crystalline. Therefore $D_{\text{crys}}(V''_{f_i\chi(-j)}(1)) = 0$. Now suppose that χ is unramified. In this case $V''_{f_i\chi(-j)}(1)$ is crystalline. Then using [25, §4.2.3]) and the assumption $\lambda_{f_i}(\text{Frob}_p) = \alpha_p(f_i) \neq \pm 1$ implies that $D_{\text{crys}}(V''_{f_i\chi(-j)}(1))^{\varphi=1} = 0$.

Next, we show $D_{\text{crys}}(V'_{f_i\chi(-j)}^*(1))^{\varphi=1} = 0$. The Galois group G_p acts on $V'_{f_i\chi(-j)}(1)$ via $\lambda_{f_i}^{-1}\epsilon'_i\epsilon_{i,p}\chi\omega_p^{k-2-j}$. Now if $\epsilon_{i,p}\chi$ is ramified, then $V'_{f_i\chi(-j)}^*(1)$ is not crystalline and therefore $D_{\text{crys}}(V'_{f_i\chi(-j)}^*(1)) = 0$.

On the other hand, assume that $\epsilon_{i,p}\chi$ is unramified. Moreover if χ is also unramified, we necessarily have $\epsilon_{i,p} = 1$. Further the assumption that $\lambda_{f_i} \neq \pm \epsilon'_i$ implies that $\lambda_{f_i}^{-1} \epsilon'_i \chi \neq 1$. Then from [25, §4.2.3] we have $D_{\text{crys}}(V''_{f_i\chi(-j)}(1))^{\varphi=1} = 0$. Finally, if χ is ramified then we get $(\epsilon_{i,p}\chi)_{|_{I_p}} =$ 1. This shows that $\epsilon_{i,p}$ is a quadratic character. Therefore $\epsilon_{i,p}\chi$ is an unramified quadratic character. Now again from [25, §4.2.3]) and the assumption $\lambda_{f_i} \neq \pm \epsilon'_i$ we have that $D_{\text{crys}}(V''_{f_i\chi(-j)}(1))^{\varphi=1} = 0$. This completes the proof of the proposition.

Remark 4.2. — We note that the condition (4) in Proposition 4.1 is satisfied if either weight k > 2 or if p is a good prime for f_i i.e. $t_1 = t_2 = 0$ holds [22, Theorem 4.6.17(3)].

LEMMA 4.3. — Let f_i be p-ordinary. If $\omega_p^{k-2-j}\epsilon_i\chi \neq \lambda_{f_i} \pmod{\pi}$ then $H^2(\mathbb{Q}_p, A'_{f_i\chi(-j)}[\pi]) = 0$ and in particular this implies that $H^1(\mathbb{Q}_p, A'_{f_i\chi(-j)})$ is π -divisible.

Proof. — We write $\overline{\omega}_p := \omega_p \pmod{\pi}$. By Tate duality, the Pontryagin dual of $H^2(\mathbb{Q}_p, A'_{f_i\chi(-j)}[\pi])$ is equal to $H^0(\mathbb{Q}_p, (A'_{f_i\chi(-j)}[\pi])^*(1))$, where $A'_{f_i\chi(-j)}[\pi])^*(1)$ is defined as $\operatorname{Hom}(A'_{f_i\chi(-j)}[\pi], \overline{\omega}_p)$. Thus it suffices to show $H^0(\mathbb{Q}_p, (A'_{f_i\chi(-j)}[\pi])^*(1)) = 0$ or equivalently G_p acts non-trivially on $\operatorname{Hom}(A'_{f_i\chi(-j)}[\pi], \overline{\omega}_p)$. On the other hand, $\operatorname{Hom}(A'_{f_i\chi(-j)}[\pi], \overline{\omega}_p)$ is equal to the Pontryagin dual of $A'_{f_i\chi(-j-1)}[\pi]$. Thus G_p acts non-trivially on $\operatorname{Hom}(A'_{f_i\chi(-j)}[\pi], \overline{\omega}_p)$ if and only if G_p acts non-trivially on $A'_{f_i\chi(-j-1)}[\pi]$. This action of G_p is via $\omega_p^{k-2-j}\epsilon_i\chi\lambda_{f_i}^{-1} \pmod{\pi}$. Thus using the given hypothesis $\omega_p^{k-2-j}\epsilon_i\chi \neq \lambda_{f_i} \pmod{\pi}$, it follows that $H^2(\mathbb{Q}_p, A'_{f_i\chi(-j)})[\pi]) = 0$. To show $H^1(\mathbb{Q}_p, A'_{f_i\chi(-j)})$ is π -divisible, note that there is an exact sequence via Kummer theory

$$H^{1}(\mathbb{Q}_{p}, A'_{f_{i}\chi(-j)}) \longrightarrow H^{1}(\mathbb{Q}_{p}, A'_{f_{i}\chi(-j)}) \longrightarrow H^{2}(\mathbb{Q}_{p}, A'_{f_{i}\chi(-j)})[\pi])$$

where the first map is given by multiplication by π . Hence if $H^2(\mathbb{Q}_p, A'_{f_i\chi(-j)})[\pi]) = 0$ then $H^1(\mathbb{Q}_p, A'_{f_i\chi(-j)})$ is π -divisible. \Box

LEMMA 4.4. — Let i = 1, 2 and f_i be p-ordinary. Assume the following conditions.

- (1) $H^2(\mathbb{Q}_p, A'_{f_i\chi(-i)}[\pi]) = 0.$
- (2) $A_{f_i \gamma(-i)}^{\prime \prime I_p}$ is π -divisible.

Then $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A_{f_i\chi(-j)})$ is π -divisible.

Proof. — Consider the natural restriction map $g: H^1(\mathbb{Q}_p, A''_{f_i\chi(-j)}) \longrightarrow H^1(I_p, A''_{f_i\chi(-j)})$. Then $\operatorname{Ker}(g) \cong H^1(G_p/I_p, A''_{f_i\chi(-j)}) \cong H^1(\langle \operatorname{Frob}_p \rangle, A''_{f_i\chi(-j)})$, is divisible by assumption (2). On the other hand, the exact sequence $0 \longrightarrow A'_{f_i\chi(-j)} \longrightarrow A_{f_i\chi(-j)} \longrightarrow A''_{f_i\chi(-j)} \longrightarrow 0$ induces the natural maps on cohomology

$$H^{1}(\mathbb{Q}_{p}, A'_{f_{i}\chi(-j)}) \xrightarrow{\psi} H^{1}(\mathbb{Q}_{p}, A_{f_{i}\chi(-j)}) \xrightarrow{f} H^{1}(\mathbb{Q}_{p}, A''_{f_{i}\chi(-j)})$$

Then f is surjective by assumption (1). Also $\text{Ker}(f) \cong \text{Img}(\psi)$ is divisible by our assumption (1) together with Lemma 4.3. Now we consider the exact sequence

$$0 \longrightarrow \operatorname{Ker}(f) \longrightarrow \operatorname{Ker}(g \circ f) = H^{1}_{\operatorname{Gr}}(\mathbb{Q}_{p}, A_{f_{i}\chi(-j)})$$
$$\longrightarrow \operatorname{Ker}(g) \longrightarrow \operatorname{Coker}(f) = 0$$

The divisibility of $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A_{f_i\chi(-j)})$ follows from the divisibility of $\mathrm{Ker}(f)$ and $\mathrm{Ker}(g)$.

LEMMA 4.5. — If either one of the following conditions hold

- (i) p > 2k 3 and $\omega_p^{k-2-j} \epsilon_{i,p} \pmod{\pi}$ is not a quadratic character.
- (ii) p > 2k 3 and $a_p(f_i) \neq \pm \epsilon'_i(\operatorname{Frob}_p) \pmod{\pi}$,

then the assumptions of Lemma 4.4 hold and $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A_{f_i\chi(-j)})$ is π -divisible.

In particular, if $t_1 = t_2 = 0$ and either condition (a) or condition (b) stated below holds, then $H^1_{Gr}(\mathbb{Q}_p, A_{f_i\chi(-j)})$ is π -divisible.

- (a) p > 2k 3 and $j \neq k 2$.
- (b) p > 2k 3 and $a_p(f_i) \neq \pm \epsilon_i(\operatorname{Frob}_p) \pmod{\pi}$,

Proof. — Recall $\lambda_{f_i}(\operatorname{Frob}_p) = \alpha_p(f_i) \equiv a_p(f_i) \pmod{\pi}$. If p > 2k - 3then $A_{f_i\chi(-j)}^{\prime\prime I_p}$ is π -divisible (by Corollary 3.8 and Remark 3.9). In addition, if $\omega_p^{k-2-j}\epsilon_{i,p} \pmod{\pi}$ is not a quadratic character then $(\omega_p^{k-2-j}\epsilon_{i,p})|_{I_p} \neq \pm 1 \pmod{\pi}$ as ω_p and $\epsilon_{i,p}$ are determined by their values on I_p . Hence $(\omega_p^{k-2-j}\epsilon_{i,p}\chi)|_{I_p} \neq 1 \pmod{\pi}$ for any quadratic character χ . As ϵ'_i and λ_{f_i} are unramified at p, we deduce that $\omega_p^{k-2-j}\epsilon_{i,p}\chi \neq \epsilon'_i^{-1}\lambda_{f_i} \pmod{\pi}$ or equivalently $\omega_p^{k-2-j}\epsilon_i\chi \neq \lambda_{f_i} \pmod{\pi}$. Using this, we get $H^2(\mathbb{Q}_p, A_{f_i\chi(-j)})$ is π divisible.

In the second case, assume p > 2k - 3, $a_p(f_i) \neq \pm \epsilon'_i(\operatorname{Frob}_p) \pmod{\pi}$ and $\omega_p^{k-2-j}\epsilon_{i,p} \pmod{\pi}$ is a quadratic character. Then $\omega_p^{k-2-j}\epsilon_{i,p}\chi \pmod{\pi}$ is quadratic and $\epsilon'_i^{-1}\lambda_{f_i}$ is not a quadratic character. Therefore $\omega_p^{k-2-j}\epsilon_{i,p}\chi \neq \epsilon'_i^{-1}\lambda_{f_i} \pmod{\pi}$. Thus again by Lemma 4.3, $H^2(\mathbb{Q}_p, A'_{f_i\chi(-j)}[\pi]) = 0$ and $H^1_{\operatorname{Gr}}(\mathbb{Q}_p, A_{f_i\chi(-j)})$ is π -divisible in this case as well.

LEMMA 4.6. — Recall $f_i \in S_k(\Gamma_0(Np^{t_i}), \epsilon_i)$ be p-ordinary, where i = 1, 2. Let us assume $k \neq 3$ and p > 2k - 3. Also assume $t_1 = t_2 = 0$ i.e. $f_i \in S_k(\Gamma_0(N), \epsilon_i)$ with $p \nmid N$. Consider the isomorphism $H^1(\mathbb{Q}_p, A_{f_1\chi(-j)}[\pi^r]) \xrightarrow{[\phi]} H^1(\mathbb{Q}_p, A_{f_2\chi(-j)}[\pi^r])$ induced from the $G_{\mathbb{Q}}$ linear isomorphism $\phi : A_{f_1\chi(-j)}[\pi^r] \longrightarrow A_{f_2\chi(-j)}[\pi^r]$ with j = k - 2. Then for j = k - 2, $[\phi]$ induces an isomorphism $H^1_{\mathrm{BK}}(\mathbb{Q}_p, A_{f_1\chi(-j)}[\pi^r]) \xrightarrow{[\phi]} H^1_{\mathrm{BK}}(\mathbb{Q}_p, A_{f_1\chi(-j)}[\pi^r])$ for every quadratic character χ of $G_{\mathbb{Q}}$.

Proof. — We first consider the case when χ is a ramified character at p. Since j = k - 2 and ϵ_i , λ_{f_i} are unramified at p, we deduce $\epsilon_i \chi \neq \lambda_{f_i} \pmod{\pi}$ and hence by Lemma 4.3, $H^2(\mathbb{Q}_p, A'_{f_i\chi(-j)})[\pi^r]) = 0$ for i = 1, 2. Also note that $(A''_{f_i\chi(-j)})^{I_p}$ is divisible by Remark 3.9 and thus by Lemma 4.4, $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A_{f_i\chi(-j)})$ is π -divisible. Further, by Remark 4.2 and from the proof of Proposition 4.1, we obtain $H^1_{\mathrm{BK}}(\mathbb{Q}_p, A_{f_i\chi(-j)}[\pi^r]) \cong H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A_{f_i\chi(-j)}[\pi^r])$ for i = 1, 2. Finally, applying Proposition 3.7, we deduce the lemma in this case.

Next we consider the case when χ is unramified at p. In this case as $t_i = 0$, $V_{f_i\chi}$ is crystalline at p for i = 1, 2. Then for any $j \neq \frac{k-1}{2}$, under the $G_{\mathbb{Q}}$ linear isomorphism $\phi : A_{f_1\chi(-j)}[\pi^r] \longrightarrow A_{f_2\chi(-j)}[\pi^r]$, it is shown in [5, Theorem 6.1, Case (3), p. 10] that $[\phi]$ induces an isomorphism $H^1_{\text{BK}}(\mathbb{Q}_p, A_{f_1\chi(-j)}[\pi^r]) \xrightarrow{[\phi]} H^1_{\text{BK}}(\mathbb{Q}_p, A_{f_2\chi(-j)}[\pi^r])$. Note that in the proof of [5, Theorem 6.1, Case (3), p. 10] the case r = 1 is covered; however from their proof we can see that the result hold for a general r as well. Now as

 $k \neq 3, k-2 \neq \frac{k-1}{2}$ and the assertion in the lemma follows from the above mentioned result of [5].

Remark 4.7. — In fact, the proof of [5, Theorem 6.1, Case (3), p. 10] identifying $H^1_{BK}(\mathbb{Q}_p, A_{f_1\chi(-j)}[\pi^r])$ with $H^1_{BK}(\mathbb{Q}_p, A_{f_2\chi(-j)}[\pi^r])$ works for the pair $V_{f_1\chi(-j)}$ and $V_{f_2\chi(-j)}$ in both the cases when f_1 and f_2 are either ordinary or non-ordinary at p; however they require $V_{f_i\chi(-j)}$ to be crystalline for i = 1, 2 as well as $j \neq \frac{k-1}{2}$. Thus the conclusion of Lemma 4.6 can not be deduced only from the results of [5].

Recall, if p does not divide level of h and $v_p(a_p(h)) \neq 0$, then for any K with $\mathbb{Q}_p \subset K \subset \mathbb{Q}_p(\mu_{p^{\infty}})$ we have defined the signed i Selmer local condition $H_i^1(K, A_{h(-j)}[\pi^r])$ in Section 2.3.

LEMMA 4.8. — Let $p \nmid N$ and for $i \in \{1, 2\}$, let $f_i \in S_k(\Gamma_0(N), \epsilon_i)$ be nonordinary at p with $p \ge k$. Let K be any field such that $\mathbb{Q}_p(\mu_p) \subset K \subset \mathbb{Q}_p(\mu_{p^{\infty}})$. Consider the isomorphism $H^1(K, A_{f_1(-j)}[\pi^r]) \xrightarrow{[\phi]} H^1(K, A_{f_2(-j)}[\pi^r])$ induced from the $G_{\mathbb{Q}}$ linear isomorphism $\phi = \phi^r : A_{f_1(-j)}[\pi^r] \longrightarrow A_{f_2(-j)}[\pi^r]$ where $j \in \mathbb{Z}$. Then $[\phi]$ induces an isomorphism $H_1^1(K, A_{f_1(-j)}[\pi^r]) \cong$ $H_1^1(K, A_{f_2(-j)}[\pi^r])$. In particular, for $0 \le j \le k - 2$, we have a canonical identification induced by ϕ ,

$$H_1^1(\mathbb{Q}_p(\mu_p), A_{f_1(-j)}[\pi^r]) \cong H_1^1(\mathbb{Q}_p(\mu_p), A_{f_2(-j)}[\pi^r])$$

Proof. — First of all, by an argument entirely similar to [12, Lemma 4.4], for every j, we have a canonical isomorphism induced by ϕ

(4.1)
$$H_1^1(\mathbb{Q}_p(\mu_{p^\infty}), A_{f_1(-j)}[\pi^r]) \cong H_1^1(\mathbb{Q}_p(\mu_{p^\infty}), A_{f_2(-j)}[\pi^r]).$$

Under the assumption $p \ge k$, it follows by [15, Lemma 4.4] that

$$A_{f_i(-j)}^{G_{\mathbb{Q}_p(\mu_p\infty)}} = 0$$

This is used in [12, Lemma 4.3] to deduce for i = 1, 2 and any j

(4.2)
$$H_1^1(\mathbb{Q}_p(\mu_{p^{\infty}}), A_{f_i(-j)}[\pi^r]) \cong H_1^1(\mathbb{Q}_p(\mu_{p^{\infty}}), A_{f_i(-j)})[\pi^r].$$

Further by [12, Remark 2.5] we see that

(4.3)
$$H_1^1(\mathbb{Q}_p(\mu_{p^{\infty}}), A_{f_i(-j)})^{\operatorname{Gal}(\mathbb{Q}_p(\mu_{p^{\infty}})/K)} \cong H_1^1(K, A_{f_i(-j)}).$$

From (4.2) and (4.3), we deduce

(4.4)
$$H_1^1(K, A_{f_i(-j)})[\pi^r] \cong H_1^1(\mathbb{Q}_p(\mu_{p^\infty}), A_{f_i(-j)}[\pi^r])^{\operatorname{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/K)}$$

On the other hand, from the definition of $H_1^1(K, A_{f_i(-j)}[\pi^r])$ and the canonical identification of $H^1(K, A_{f_i(-j)}[\pi^r]) \cong H^1(K, A_{f_i(-j)})[\pi^r]$ induced via the inclusion map, we have

(4.5)
$$H_1^1(K, A_{f_i(-j)}[\pi^r]) \cong H_1^1(K, A_{f_i(-j)})[\pi^r]$$

From (4.4) and (4.5), for every j, we have a canonical isomorphism

(4.6)
$$H_1^1(K, A_{f_1(-j)}[\pi^r]) \cong H_1^1(K, A_{f_2(-j)}[\pi^r])$$

which is induced from the isomorphism $H^1(K, A_{f_1(-j)}[\pi^r]) \xrightarrow{[\phi]} H^1(K, A_{f_2(-j)}[\pi^r])$ coming from the $G_{\mathbb{Q}}$ linear isomorphism ϕ .

PROPOSITION 4.9. — Let $p \nmid N$ and for $i \in \{1, 2\}$, let $f_i \in S_k(\Gamma_0(N), \epsilon_i)$ be non-ordinary at p with $p \ge k$. Then for every $0 \le j \le k-2$,

(4.7)
$$H^{1}_{\mathrm{BK}}(\mathbb{Q}_{p}(\mu_{p}), A_{f_{i}(-j)}[\pi^{r}]) = H^{1}_{1}(\mathbb{Q}_{p}(\mu_{p}), A_{f_{i}(-j)}[\pi^{r}])$$

= $H^{1}_{2}(\mathbb{Q}_{p}(\mu_{p}), A_{f_{i}(-j)}[\pi^{r}])$

Proof. — The main idea is that Bloch–Kato condition at p is given by the kernel of the Perrin–Riou dual exponential map and local condition at p for the signed i Selmer group is given by the kernel of the Colman_i map. One then deduces the result by looking at the relation between the dual exponential map and the Colman map. We thank Antonio Lei for discussion on this proposition.

We define $H_{BK}^1(\mathbb{Q}_p(\mu_p), T_{h(-j)})$ to be the preimage of $H_{BK}^1(\mathbb{Q}_p(\mu_p), V_{h(-j)})$ under the natural map induced by the inclusion $T_{h(-j)} \longrightarrow V_{h(-j)}$. Let $z \in H_{Iw}^1(\mathbb{Q}_p, T_h)$ and we denote by z_j its image in $H^1(\mathbb{Q}_p(\mu_p), T_{h(-j)})$ under the natural composite map $H_{Iw}^1(\mathbb{Q}_p, T_h) \longrightarrow H^1(\mathbb{Q}_p(\mu_p), T_{h(-j)})$. We choose two distinct u_1 and u_2 in \mathbb{Z}_p^{\times} in [17, Proposition 3.11] and use it in [17, Equation (3.6), p. 836] for n = 1 to deduce for $h \in \{f_1, f_2\}$,

$$z_{j} \in H^{1}_{BK}(\mathbb{Q}_{p}(\mu_{p}), T_{h(-j)}) \iff z_{j} \in \operatorname{Pr}_{\mathbb{Q}_{p}(\mu_{p})}(\operatorname{Ker}(\operatorname{Col}_{h,1}) \otimes \omega_{p}^{-j})$$
$$\iff z_{j} \in \operatorname{Pr}_{\mathbb{Q}_{p}(\mu_{p})}(\operatorname{Ker}(\operatorname{Col}_{h,2}) \otimes \omega_{p}^{-j}),$$

where $0 \leq j \leq k-2$. Then it follows from the definitions of Bloch-Kato and signed Selmer condition that for i = 1, 2

(4.8)
$$H_{BK}^1(\mathbb{Q}_p(\mu_p), T_{f_i(-j)}) = H_1^1(\mathbb{Q}_p(\mu_p), T_{f_i(-j)}) = H_2^1(\mathbb{Q}_p(\mu_p), T_{f_i(-j)})$$

Consequently, we obtain

$$(4.9) \quad H^1_{BK}(\mathbb{Q}_p(\mu_p), A_{f_i(-j)}) = H^1_1(\mathbb{Q}_p(\mu_p), A_{f_i(-j)}) = H^1_2(\mathbb{Q}_p(\mu_p), A_{f_i(-j)})$$

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Then from the definition of π^r -Selmer condition given in (2.5), it follows that for i = 1, 2,

(4.10)
$$H^1_{BK}(\mathbb{Q}_p(\mu_p), A_{f_i(-j)}[\pi^r]) = H^1_1(\mathbb{Q}_p(\mu_p), A_{f_i(-j)}[\pi^r])$$

= $H^1_2(\mathbb{Q}_p(\mu_p), A_{f_i(-j)}[\pi^r]).$
In particular, (4.7) holds,

In particular, (4.7) holds.

Now, we are ready to prove our main result of this article.

THEOREM 4.10. — Let p be an odd prime and for i = 1, 2, let f_i be a normalized cuspidal Hecke eigenform in $S_k(\Gamma_0(Np^{t_i}), \epsilon_i)$, where (N, p) = 1, $k \geq 2$ and $t_i \in \mathbb{N} \cup \{0\}$. Let $r \in \mathbb{N}$ and $\phi : A_{f_1}[\pi^r] \longrightarrow A_{f_2}[\pi^r]$ be a $G_{\mathbb{Q}}$ linear isomorphism. We assume the following:

- (1) N is square-free and $\forall \ell \in S$, $\operatorname{cond}_{\ell}(\bar{\rho}_{f_i}) = \ell$ for i = 1, 2.
- (2) The condition $(C'_{i\ell})$, defined in equation (2.1), is satisfied for i =1, 2.
- (3) Either (p-ss) or (p-ord) holds.
 - (p-ss) $p \ge k$, f_1 and f_2 are non-ordinary at p, and $t_1 = t_2 = 0$.
- (p-ord) p > 2k-3, f_1 and f_2 are ordinary at p, and either (p-good) or (p-general) holds.

 $(p-\text{good}) \ t_1 = t_2 = 0 \ and \ k \neq 3.$

- (p-general) All of (A), (B) and (C) are satisfied.
 - (A) $a_p(f_i) \neq \pm \epsilon'_i(\operatorname{Frob}_p) \pmod{\pi}$.
 - (B) $\omega_n^{k-1} \epsilon_{i,p} \neq 1 \pmod{\pi}$ for i = 1, 2.
 - (C) If both k = 2 and $t_i > 0$ holds, then in addition assume $\lambda_{f_i} \neq \pm 1$.

Then for every quadratic character χ of $G_{\mathbb{Q}}$ and for every fixed j with $0 \leq j \leq k-2$, we have an isomorphism of the π^r -Bloch-Kato Selmer groups

$$S_{\mathrm{BK}}(A_{f_1\chi(-j)}[\pi^r]/\mathbb{Q}) \cong S_{\mathrm{BK}}(A_{f_2\chi(-j)}[\pi^r]/\mathbb{Q}).$$

Proof. — First we consider the case (1), (2) and (p-ord) are satisfied. Note that by Theorem 2.2(2) and Lemma 3.11, $a_p(f_1) \equiv a_p(f_2) \pmod{\pi^r}$ and hence $a_p(f_1) \equiv a_p(f_2) \pmod{\pi}$ also. Now we consider a subcase where (*p*-ord) is satisfied via (*p*-general). Then by hypothesis (A) i.e. $a_p(f_i) \neq$ $\pm \epsilon'_i(\operatorname{Frob}_p) \pmod{\pi}$ together with Lemma 4.5 (ii), we get $H^1_{\operatorname{Gr}}(\mathbb{Q}_p, A_{f_i\chi(-j)})$ is π -divisible for i = 1, 2. Therefore using conditions (1), (2) and $\lambda_{f_i} \neq \pm 1$ (if necessary), Proposition 4.1 and Remark 4.2, we deduce for each fixed jwith $0 \leq j \leq k-2$ and for every χ ,

(4.11)
$$S_{\mathrm{BK}}(A_{f_i\chi(-j)}[\pi^r]/\mathbb{Q}) \cong S_{\mathrm{Gr}}(A_{f_i\chi(-j)}[\pi^r]/\mathbb{Q}).$$

Further given the hypothesis (B) i.e. $\omega_p^{k-1} \epsilon_{i,p} \neq 1 \pmod{\pi}$ for i = 1, 2, we apply (4.11) in Theorem 3.1 to deduce Theorem 4.10 in this case.

On the other hand if (*p*-ord) is satisfied via (*p*-good), then by Lemma 4.5 (i), we get $H^1_{\text{Gr}}(\mathbb{Q}_p, A_{f_i\chi(-j)})$ is π -divisible for i = 1, 2 as long as $j \neq k-2$. Therefore once again using conditions (1), (2), Proposition 4.1 and Remark 4.2, we deduce for $j \neq k-2$ and for every χ , the isomorphism in (4.11) between Greenberg and Bloch–Kato Selmer group continue to hold. Also as p > 2k - 3 and $t_1 = t_2 = 0$, $\omega_p^{k-1} \neq 1 \pmod{\pi}$ and $\epsilon_{i,p} = 1$ for i = 1, 2. Hence as in previous paragraph, we use (4.11) in Theorem 3.1 to obtain $0 \leq j \leq k-3$ case of Theorem 4.10. On the other hand, if j = k-2then as $k \neq 3$ by our assumption, we apply Lemma 4.6 to directly obtain the canonical isomorphism induced from ϕ ,

(4.12)
$$H^{1}_{\mathrm{BK}}(\mathbb{Q}_{p}, A_{f_{1}\chi(-j)}[\pi^{r}]) \xrightarrow{[\phi]} H^{1}_{\mathrm{BK}}(\mathbb{Q}_{p}, A_{f_{2}\chi(-j)}[\pi^{r}])$$

for every quadratic character χ of $G_{\mathbb{Q}}$. Moreover as explained in the proof of Proposition 4.1, for every $q \in \Sigma \setminus \{p\}$, by assumptions (1), (2) and Corollary 3.4 we have $H^1_{BK}(\mathbb{Q}_q, A_{f_i\chi(-j)}[\pi^r]) = H^1_{Gr}(\mathbb{Q}_q, A_{f_i\chi(-j)}[\pi^r])$ and thus for i = 1, 2 and for every j such that $0 \leq j \leq k-2$, we can canonically identify,

(4.13)
$$\frac{H^{1}(\mathbb{Q}_{q}, A_{f_{i}\chi(-j)}[\pi^{r}])}{H^{1}_{\mathrm{BK}}(\mathbb{Q}_{q}, A_{f_{i}\chi(-j)}[\pi^{r}])} \cong H^{1}(I_{q}, A_{f_{i}\chi(-j)}[\pi^{r}]), \quad q \in \Sigma \setminus \{p\}$$

From (4.12) and (4.13), by the definition of π^r Bloch–Kato Selmer group, the remaining j = k - 2 case of Theorem 4.10 in the (*p*-good) case is obtained. This completes the proof of Theorem 4.10 when assumptions (1), (2) and (*p*-ord) are satisfied.

Next we consider the case when assumptions (1), (2) and (*p*-ss) are satisfied. Then the definition of the local factor for $S_{\text{BK}}(A_{f_1\chi(-j)}[\pi^r]/\mathbb{Q})$ at a prime $q \in \Sigma \setminus \{p\}$ is the same as in the (*p*-ord) case. Hence using the same argument as in the (*p*-ord) case, we obtain the identification of (4.13) for every quadratic character χ of $G_{\mathbb{Q}}$ and every $0 \leq j \leq k-2$ thanks to our assumptions (1) and (2). Thus to prove Theorem 4.10, it suffices to establish a canonical identification induced by ϕ for every χ and $0 \leq j \leq k-2$, analogues to (4.12) under the assumption (*p*-ss).

Now as $p \nmid N$, $t_1 = t_2 = 0$ and $p \ge k$, using Lemma 4.8 and Proposition 4.9, we get a canonical isomorphism

(4.14)
$$H^{1}_{\mathrm{BK}}(\mathbb{Q}_{p}(\mu_{p}), A_{f_{1}(-j)}[\pi^{r}]) \cong H^{1}_{\mathrm{BK}}(\mathbb{Q}_{p}(\mu_{p}), A_{f_{2}(-j)}[\pi^{r}]).$$

for every $0 \leq j \leq k-2$ which is induced by ϕ . Now let χ be a quadratic character. Following the notation of Section 2.1, we can write $\chi = \chi_p \chi'$

where χ_p is quadratic character whose conductor is a power of p and χ' is unramified at p. Note that $f_i \otimes \chi'$ is good at p i.e. the level of $f_i \otimes \chi'$ is coprime to p. Thus from (4.14), we still have a canonical isomorphism

(4.15)
$$H^1_{\mathrm{BK}}(\mathbb{Q}_p(\mu_p), A_{f_1\chi'(-j)}[\pi^r]) \cong H^1_{\mathrm{BK}}(\mathbb{Q}_p(\mu_p), A_{f_2\chi'(-j)}[\pi^r]).$$

induced by ϕ for every j. Notice that χ_p is the unique quadratic character of $\Delta = \operatorname{Gal}(\mathbb{Q}_p(\mu_p)/\mathbb{Q}_p)$, whence $H^1_{\operatorname{BK}}(\mathbb{Q}_p(\mu_p), A_{f_i\chi'(-j)}[\pi^r]) \otimes \chi_p \cong H^1_{\operatorname{BK}}(\mathbb{Q}_p(\mu_p), A_{f_i\chi'\chi_p(-j)}[\pi^r])$ for i = 1, 2. Thus we have,

(4.16)
$$H^{1}_{\mathrm{BK}}(\mathbb{Q}_{p}(\mu_{p}), A_{f_{1}\chi(-j)}[\pi^{r}]) \cong H^{1}_{\mathrm{BK}}(\mathbb{Q}_{p}(\mu_{p}), A_{f_{2}\chi(-j)}[\pi^{r}]).$$

Now we take invariance by Δ in (4.16). Then using inflation-restriction sequence and the fact that order of Δ is co-prime to p, we deduce for quadratic every character χ of $G_{\mathbb{Q}}$ and $0 \leq j \leq k-2$, a canonical isomorphism

(4.17)
$$H^{1}_{\mathrm{BK}}(\mathbb{Q}_{p}, A_{f_{1}\chi(-j)}[\pi^{r}]) \cong H^{1}_{\mathrm{BK}}(\mathbb{Q}_{p}, A_{f_{2}\chi(-j)}[\pi^{r}]).$$

induced by ϕ . This completes the comparison of local π^r Bloch–Kato condition at p of f_1 and f_2 , under the condition (p-ss). This completes the proof of the Theorem 4.10 under the hypotheses (1), (2) and (p-ss), as required.

COROLLARY 4.11. — From the proof of Theorem 4.10, we can identify the Bloch–Kato Selmer group with the signed Selmer groups for $\mathbb{Q}(\mu_p)$. More precisely, let $h \in S_k(\Gamma_0(N), \psi)$ be a non-ordinary at $p \ge k$ with (p, N) = 1 and $p \ge k$. Then

$$S_{\mathrm{BK}}(A_{h(-j)}/\mathbb{Q}(\mu_p)) \cong S_1(A_{h(-j)}/\mathbb{Q}(\mu_p)) \cong S_2(A_{h(-j)}/\mathbb{Q}(\mu_p))$$

where $S_i(A_{h(-j)}/\mathbb{Q}(\mu_p))$ is the signed *i* Selmer group of *h* over $\mathbb{Q}(\mu_p)$ and $0 \leq j \leq k-2$.

COROLLARY 4.12. — Let $h \in S_k(\Gamma_0(N), \epsilon)$ be a p-ordinary newform with (p, N) = 1 and p > 2k - 3. Assume conditions (1), (2) and (p-general) of Theorem 4.10 holds for h. Then for $0 \leq j \leq k-2$, and for every quadratic character χ ,

$$S_{\mathrm{BK}}(A_{h\chi(-j)}[\pi^r]/\mathbb{Q}) \cong S_{\mathrm{Gr}}(A_{h\chi(-j)}[\pi^r]/\mathbb{Q}).$$

Remark 4.13. — When f is p-ordinary, it can be checked that $H^1_{BK}(\mathbb{Q}_p, A_{f\chi(-j)}[\pi^r]) = i_r^{*-1}(\psi(H^1(\mathbb{Q}_p, A_{f\chi(-j)}')_{div}))$, where $H^1(\mathbb{Q}_p, A_{f\chi(-j)}') \xrightarrow{\psi} H^1(\mathbb{Q}_p, A_{f\chi(-j)})$ is the natural map induced by inclusion and $i_r^* : H^1(\mathbb{Q}_p, A_{f\chi(-j)}) \longrightarrow H^1(\mathbb{Q}_p, A_{f\chi(-j)})$ is induced by the Kummer map. Further when f corresponds to an elliptic curve E over \mathbb{Q} , then the condition $a_p(f) \neq \pm \epsilon'_i(\operatorname{Frob}_p) \pmod{\pi}$ is $a_p(f) \neq \pm 1 \pmod{p}$. In particular,

 $a_p(f) \neq 1 \pmod{p}$ is precisely the condition $p \nmid \# \widetilde{E}(\mathbb{F}_p)$. Such a prime is called a non-anomalous prime (cf. [10]).

We now extend the notion of Selmer companion forms to cupsforms of two different weights.

DEFINITION 4.14. — Let $p \nmid N$ and $f_i \in S_{k_i}(\Gamma_0(Np^{t_i}), \epsilon_i)$ be a normalized cuspidal eigenform for i = 1, 2. Then f_1 and f_2 are π^r (Bloch–Kato) Selmer companion if for each critical twist j with $0 \leq j \leq \min\{k_1-2, k_2-2\}$ and for every quadratic character χ of $G_{\mathbb{Q}}$,

$$S_{\mathrm{BK}}(A_{f_1\chi(-j)}[\pi^r]/\mathbb{Q}) \cong S_{\mathrm{BK}}(A_{f_2\chi(-j)}[\pi^r]/\mathbb{Q}).$$

COROLLARY 4.15. — Let p be odd and $f_i \in S_{k_i}(\Gamma_0(Np^{t_i}), \epsilon_i)$ be a normalized cuspidal eigenform with $p \nmid N$, $k_i \ge 2$ and $t_i \in \mathbb{N} \cup \{0\}$, i = 1, 2. Let $\phi : A_{f_1}[\pi^r] \longrightarrow A_{f_2}[\pi^r]$ be a $G_{\mathbb{Q}}$ linear isomorphism. We assume the following:

- (1) N is square-free and $\forall \ell \in S$, $\operatorname{cond}_{\ell}(\bar{\rho}_{f_i}) = \ell$ for i = 1, 2.
- (2) The condition $(C'_{i,\ell})$, defined in equation (2.1), is satisfied for i = 1, 2.
- (3) $p > 2 \max\{k_1, k_2\} 3$, f_1 and f_2 are ordinary at p.
- (4) $a_p(f_i) \neq \pm \epsilon'_i(\operatorname{Frob}_p) \pmod{\pi}$ and $\omega_p^{k_i-1} \epsilon_{i,p} \neq 1 \pmod{\pi}$ for i = 1, 2.
- (5) If $k_i = 2$ and $t_i > 0$, then in addition assume $\lambda_{f_i} \neq \pm 1$.

Then for every quadratic character χ of $G_{\mathbb{Q}}$ and for every fixed j with $0 \leq j \leq \min\{k_1-2, k_2-2\}$, we have an isomorphism of the π^r -Bloch-Kato Selmer groups

$$S_{\mathrm{BK}}(A_{f_1\chi(-j)}[\pi^r]/\mathbb{Q}) \cong S_{\mathrm{BK}}(A_{f_2\chi(-j)}[\pi^r]/\mathbb{Q}).$$

Proof. — The proof is similar to the proof of Theorem 4.10 when conditions (1), (2), and (p-ord) (via (p-general)) conditions in Theorem 4.10 are satisfied. Hence the proof is omitted.

5. The cyclotomic case

Let \mathbb{Q}_{cyc} be the cyclotomic \mathbb{Z}_p extension of \mathbb{Q} . For a prime $q \in \mathbb{Q}$, let q_{∞} be a prime in \mathbb{Q}_{cyc} dividing q. Let $\mathbb{Q}_{\text{cyc},q_{\infty}}$ denote the completion of \mathbb{Q}_{cyc} at q_{∞} . Also $I_{q_{\infty}}$ and $G_{\mathbb{Q}_{\text{cyc},q_{\infty}}}$ will respectively denote the inertia and decomposition subgroup of $G_{\mathbb{Q}_{\text{cyc}}}$ at q_{∞} . Let Σ_{∞} be the set of all primes of \mathbb{Q}_{cyc} lying above the primes of Σ . Let $h \in S_k(\Gamma_0(N), \epsilon)$ be a p-ordinary and recall $p \nmid N$. For $\dagger \in \{\text{Gr}, \text{BK}\}$, we define

$$S_{\dagger}(A_{h\chi(-j)}[\pi^r]/\mathbb{Q}_{cyc}) = \varinjlim_n S_{\dagger}(A_{h\chi(-j)}[\pi^r]/\mathbb{Q}_n),$$

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where $\mathbb{Q}_n \subset \mathbb{Q}(\mu_{p^{n+1}})$ with $\operatorname{Gal}(\mathbb{Q}_n/\mathbb{Q}) \cong \frac{\mathbb{Z}}{p^n\mathbb{Z}}$ and $S_{\dagger}(A_{h\chi(-j)}[\pi^r]/\mathbb{Q}_n)$ was defined in Section 2.3. We can explicitly write,

(5.1)
$$S_{\mathrm{Gr}}(A_{h\chi(-j)}/\mathbb{Q}_{\mathrm{cyc}})$$
$$:= \mathrm{Ker}\left(H^{1}(\mathbb{Q}_{\Sigma}/\mathbb{Q}_{\mathrm{cyc}}, A_{h\chi(-j)}) \longrightarrow \prod_{q_{\infty} \in \Sigma_{\infty}} \frac{H^{1}(\mathbb{Q}_{\mathrm{cyc},q_{\infty}}, A_{h\chi(-j)})}{H^{1}_{\mathrm{Gr}}(\mathbb{Q}_{\mathrm{cyc},q_{\infty}}, A_{h\chi(-j)})}\right)$$

with

$$\begin{aligned} H^{1}_{\mathrm{Gr}}(\mathbb{Q}_{\mathrm{cyc},q_{\infty}}, A_{h\chi(-j)}) \\ &:= \begin{cases} \mathrm{Ker}\big(H^{1}(\mathbb{Q}_{\mathrm{cyc},q_{\infty}}, A_{h\chi(-j)}) \longrightarrow H^{1}(I_{q_{\infty}}, A_{h\chi(-j)})\big) & \text{if } q_{\infty} \nmid p \\ \mathrm{Ker}\big(H^{1}(\mathbb{Q}_{\mathrm{cyc},q_{\infty}}, A_{h\chi(-j)}) \longrightarrow H^{1}(I_{q_{\infty}}, A^{-}_{h\chi(-j)})\big) & \text{if } q_{\infty} \mid p. \end{cases} \end{aligned}$$

Then $S_{\text{Gr}}(A_{f_i\chi(-j)}[\pi^r]/\mathbb{Q}_{\text{cyc}})$ is defined from (5.1) using (2.5). We have the following analogue of Theorem 4.10.

THEOREM 5.1. — Let p be an odd prime and for i = 1, 2, let f_i be a p-ordinary normalized cuspidal Hecke eigenform in $S_k(\Gamma_0(N), \epsilon_i)$, where $(N, p) = 1, k \ge 2$. Let $r \in \mathbb{N}$ and $\phi : A_{f_1}[\pi^r] \longrightarrow A_{f_2}[\pi^r]$ be a $G_{\mathbb{Q}}$ linear isomorphism. We assume the following:

- (1) N is square-free and $\forall \ell \in S$, $\operatorname{cond}_{\ell}(\bar{\rho}_{f_i}) = \ell$ for i = 1, 2.
- (2) The condition $(C'_{i,\ell})$, defined in equation (2.1), is satisfied for i = 1, 2.
- (3) p > k.

Then for every quadratic character χ of $G_{\mathbb{Q}}$ and for every fixed j with $0 \leq j \leq k-2$, we have an isomorphism of the π^r \dagger -Selmer groups for $\dagger \in \{\text{BK}, \text{Gr}\},$

$$S_{\dagger}(A_{f_1\chi(-j)}[\pi^r]/\mathbb{Q}_{\text{cyc}}) \cong S_{\dagger}(A_{f_2\chi(-j)}[\pi^r]/\mathbb{Q}_{\text{cyc}}).$$

Proof. — First of all, we note that it suffices to show for every χ and $0 \leq j \leq k-2$,

(5.2)
$$S_{\mathrm{Gr}}(A_{f_1\chi(-j)}[\pi^r]/\mathbb{Q}_{\mathrm{cyc}}) \cong S_{\mathrm{Gr}}(A_{f_2\chi(-j)}[\pi^r]/\mathbb{Q}_{\mathrm{cyc}}).$$

Indeed, as $t_1 = t_2 = 0$ and (N, p) = 1, the conditions (i), (ii) and (iii)' of [9, Proposition 4.2.30] is verified. Thus for i = 1, 2,

$$H^{1}_{\mathrm{Gr}}(\mathbb{Q}_{\mathrm{cyc},q_{\infty}}, A_{f_{i}\chi(-j)}) \cong H^{1}_{\mathrm{BK}}(\mathbb{Q}_{\mathrm{cyc},q_{\infty}}, A_{f_{i}\chi(-j)})$$

by [9, Proposition 4.2.30] for $q_{\infty} \mid p$ as well as $q_{\infty} \mid q$ with $q \in \Sigma \setminus \{p\}$. Using the definition of π^r Selmer group in (2.5), we deduce for every $q_{\infty} \in \Sigma_{\infty}$,

$$H^{1}_{\mathrm{Gr}}(\mathbb{Q}_{\mathrm{cyc},q_{\infty}}, A_{f_{i}\chi(-j)}[\pi^{r}]) \cong H^{1}_{\mathrm{BK}}(\mathbb{Q}_{\mathrm{cyc},q_{\infty}}, A_{f_{i}\chi(-j)}[\pi^{r}]).$$

Hence Theorem 5.1 will follow once we establish (5.2). Note that in the proof of Theorem 3.1, we have assumed p > 2k - 3 to show that $A_{fi\chi(-j)}^{''I_p}$ is π -divisible (see Corollary 3.8 and Remark 3.9). In this case of \mathbb{Q}_{cyc} , $A_{fi\chi(-j)}^{''I_{p\infty}}$ is π -divisible even without the assumption p > 2k - 3, as we explain now. Note that $I_{p\infty}$ acts on $A_{fi\chi(-j)}^{''I_{p\infty}}$ via $\overline{\omega}_p^{-j}\chi$. From the proof of Proposition 3.7, Case 1, we can see that $A_{fi\chi(-j)}^{''I_{p\infty}} = 0$ unless j > 0 and $(\overline{\omega}_p^{-j}\chi)_{|_{I_{p\infty}}} = 1 \pmod{\pi}$. Since $\overline{\omega}_p^{-j}\chi$ has order prime to p, in the later case, we get $\overline{\omega}_p^{-j}\chi = 1$ as a character of $I_{p\infty}$. Consequently, $A_{fi\chi(-j)}^{''I_{p\infty}} = A_{fi\chi(-j)}^{''}$ is π -divisible.

Note the assumption p > k is needed in the proof of Lemma 3.11. Now the proof of (5.2) is very similar to the proof of Theorem 3.1 and hence omitted to avoid repetition.

We denote the Teichmüller character $G_{\mathbb{Q}} \longrightarrow \mathbb{F}_p^{\times} \subset \mathbb{Z}_p^{\times}$ by $\overline{\omega}_p$. Note that (5.2) is true if and only if

$$S_{\mathrm{Gr}}(A_{f_1\chi\bar{\omega}_p^{-j}}[\pi^r]/\mathbb{Q}_{\mathrm{cyc}}) \cong S_{\mathrm{Gr}}(A_{f_2\chi\bar{\omega}_p^{-j}}[\pi^r]/\mathbb{Q}_{\mathrm{cyc}}).$$

Remark 5.2. — If we assume $A_{f_i}[\pi]$ is an irreducible $G_{\mathbb{Q}}$ module, then as in Remark 2.3, we can deduce

(5.3)
$$S_{\dagger}(A_{f_i\chi\bar{\omega}_p^{-j}}/\mathbb{Q}_{\rm cyc})[\pi^r] \cong S_{\dagger}(A_{f_i\chi\bar{\omega}_p^{-j}}[\pi^r]/\mathbb{Q}_{\rm cyc})$$

for $\dagger \in \{\text{Gr}, \text{BK}\}$. Thus using (5.3) together with the hypotheses of Theorem 5.1, it follows that for every quadratic χ and every $0 \leq j \leq k-2$, we have an isomorphism

(5.4)
$$S_{\dagger}(A_{f_1\chi\bar{\omega}_p^{-j}}/\mathbb{Q}_{cyc})^{\vee}/\pi^r \cong S_{\dagger}(A_{f_2\chi\bar{\omega}_p^{-j}}/\mathbb{Q}_{cyc})^{\vee}/\pi^r, \quad \dagger \in \{\mathrm{BK}, \mathrm{Gr}\}.$$

Here for a discrete module $\Lambda := O[\![\Gamma]\!] \cong O[\![T]\!]$ module M, we denote by M^{\vee} the Pontryagin dual $\operatorname{Hom}_{\operatorname{cont}}(M, \mathbb{Q}_p/\mathbb{Z}_p)$. By a deep theorem of Kato, as f_i is *p*-ordinary we know $S_{\operatorname{Gr}}(A_{f_i\chi\bar{\omega}_p^{-j}}/\mathbb{Q}_{\operatorname{cyc}})^{\vee}$ is a finitely generated torsion Λ module. Moreover, in this case (cf. [6, Theorem 4.1.1], [10]) $S_{\operatorname{Gr}}(A_{f_i\chi\bar{\omega}_p^{-j}}/\mathbb{Q}_{\operatorname{cyc}})^{\vee}$ has no pseudonull (finite) Λ -submodule. It then follows from (5.4) that for every quadratic character χ of $G_{\mathbb{Q}}$ and for critical values $0 \leq j \leq k-2$,

(5.5)
$$C_{\Lambda}\left(S_{\mathrm{Gr}}(A_{f_1\chi\bar{\omega}_p^{-j}}/\mathbb{Q}_{\mathrm{cyc}})^{\vee}\right) \equiv C_{\Lambda}\left(S_{\mathrm{Gr}}(A_{f_2\chi\bar{\omega}_p^{-j}}/\mathbb{Q}_{\mathrm{cyc}})^{\vee}\right) \pmod{\pi^r}.$$

Here for a finitely generated torsion Λ module M, we denote $C_{\Lambda}(M)$, the characteristic ideal of M in Λ .

Recall, $f_i \in S_k(\Gamma_0(N))$ is a *p*-ordinary newform. Let χ be a Dirichlet character whose conductor is coprime to N. Associated to $f_i \chi \omega_p^{-j}$, Mazur– Tate–Tidelbaum [21] have constructed a *p*-adic *L*-function $\mathcal{L}_{f_i \chi \bar{\omega}_p^{-j}}^p(T) \in$ $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Implicit in this construction of $\mathcal{L}_{f_i \chi \bar{\omega}_p^{-j}}^p(T)$ by [21], is choice of a period, which is defined up to an algebraic constant. Now we assume $A_{f_i}[\pi]$ is an absolutely irreducible $G_{\mathbb{Q}}$ module and $f_1 \equiv f_2 \pmod{\pi^r}$. Then Vatsal [28, §1.3, §2.2] has constructed canonical periods for f_1 and f_2 , which are well defined up to *p*-adic units. Moreover, he has shown the *p*-adic *L*functions $\mathcal{L}_{f_1 \chi \bar{\omega}_p^{-j}}^p(T)$ and $\mathcal{L}_{f_2 \chi \bar{\omega}_p^{-j}}^p(T)$, constructed with respect to these canonical periods, are in fact elements of Λ and they satisfy the following congruence [28, Theorem 1.10]:

(5.6)
$$\mathcal{L}^{p}_{f_{1}\chi\bar{\omega}_{p}^{-j}}(T) \equiv \mathcal{L}^{p}_{f_{2}\chi\bar{\omega}_{p}^{-j}}(T) \pmod{\pi^{r}\Lambda}.$$

Now under the assumption that $A_{f_i}[\pi]$ is an absolutely irreducible $G_{\mathbb{Q}}$ module, Iwasawa–Greenberg Main Conjecture states that [26, §1.1, p. 5],

(5.7)
$$C_{\Lambda}\left(S_{\mathrm{Gr}}(A_{f_{i}\chi\bar{\omega}_{p}^{-j}}/\mathbb{Q}_{\mathrm{cyc}})^{\vee}\right) = (\mathcal{L}_{f_{i}\chi\bar{\omega}_{p}^{-j}}^{p}(T)),$$

as ideals in Λ .

In view of (5.7), the congruence in (5.6) implies the congruence in (5.5) for any quadratic χ whose conductor is coprime to N. Thus our Theorem 5.1 can be thought of as an algebraic reflection of the congruence result of Vatsal via Iwasawa main conjecture. However, our Theorem 5.1 is valid for all possible quadratic character χ .

Remark 5.3. — Note that in the non-ordinary case i.e. when $a_p(f_i)$ is not a *p*-adic unit, $S_{\text{BK}}(A_{f_i\chi(-j)}/\mathbb{Q}_{\text{cyc}})^{\vee}$ is not a torsion Λ module i.e. it has positive Λ rank (cf. [10]). For weight k (> 2) congruent cuspforms which are good and non-ordinary at p, it is not clear how to establish Theorem 5.1 over \mathbb{Q}_{cyc} for the π^r Bloch–Kato Selmer groups.

6. Examples

In this section we give several numerical examples to illustrate all our main results.

Example I.

(1) We consider the example of elliptic curves 1246B, 1246C considered in [3, Table 1] and choose the prime p = 5. Let $f, g \in S_2(\Gamma_0(1246))$ be the primitive modular forms associated to 1246B and 1246Crespectively via modularity. We have $1246 = 2 \times 7 \times 89$ is squarefree and $5 \nmid 1246$. Note the Fourier expansions of f and g are given

by [19]

$$f(q) = q - q^{2} + 2q^{3} + q^{4} + 2q^{5} - 2q^{6} - q^{7} - q^{8} + q^{9} - 2q^{10} + 2q^{12} + O(q^{13}),$$

$$g(q) = q - q^{2} - 3q^{3} + q^{4} - 3q^{5} + 3q^{6} - q^{7} - q^{8} + 6q^{9} + 3q^{10} - 3q^{12} + O(q^{13})$$

By computing the minimal discriminant of 1246*B* and 1246*C* and using [4, Proposition 2.12 (c)] we can show that $\forall \ell \in S = \{2, 7, 89\}$, $\operatorname{cond}_{\ell}(\bar{\rho}_f) = \ell = \operatorname{cond}_{\ell}(\bar{\rho}_g)$. Also using [19] we get that $\bar{\rho}_f$ and $\bar{\rho}_g$ are irreducible and equivalent. Thus *f* and *g* satisfies all the hypotheses Theorem 4.10. Hence *f* and *g* are 5-Bloch–Kato Selmer companion forms.

- (2) Since $a_5(f) = 2$ and $a_5(g) = -3$, we see that 5 is a prime of ordinary reduction for f and g. Using Hida theory, corresponding to f and g, there exists primitive forms f_3 and g_3 of weight = 3, level 5N = 5×1246 and nebentypus $\overline{\omega}_5^{-1}$, where $\overline{\omega}_5$ is the Teichmüller character, such that $f_3 \equiv f \equiv g \equiv g_3 \pmod{\pi}$ (cf. [29]). Here π is a prime ideal of $K_{f_3,g_3,\overline{\omega}_5}$ lying above 5. As k = 3, the condition [*p*-good] of Theorem 4.10 does not apply although 5 is a prime of good reduction of f, g. However, $a_5(h) \neq \pm 1 \pmod{5}$ for $h \in \{f, g\}$ and hence also for $h \in \{f_3, g_3\}$. Thus the condition [*p*-general] of Theorem 4.10 hold. We deduce the weight 3 forms f_3 and g_3 are π -Bloch-Kato Selmer companion.
- (3) Again using Hida theory, by Remark 2.5, there are infinitely many cuspidal Hecke newforms $f_r \in S_2(\Gamma_0(1246 \times 5^r), \psi_r)$ such that (f, f_r) are π Selmer companion.
- (4) Using the extended definition of Selmer companion forms of different weights, from Corollary 4.15, we have that f and f_3 are π -Bloch-Kato Selmer companion and same is true for the pair gand g_3 .

Example II.

(1) Consider the pair of modular forms f = 127k4A and g = 127k4C of level 127, weight 4, trivial nebentypus and Galois orbits A and C respectively as appeared in [5, Table 1]. Here K_f = Q and [K_g: Q] = 17. As the nebentypus is trivial, in this case K_{f,g,ϵ} = K_g. Then there exists a prime p of K_g lying above the prime number p = 43 such that f ≡ g (mod p) [5, §7]. The level 127 is square free (a prime). Using [24] we have calculated, a₄₃(f) = 80 which is

coprime to 43. It follows that f has good and ordinary reduction at 43. Note that 43 > 2k - 3 = 5 Since $f \equiv g \pmod{\mathfrak{p}}$ and $43 \nmid 127$, the same holds for g. Thus the conditions [p-ord] and [p-good] of Theorem 4.10 holds. Note that there are no newforms of weight 4, level 1 and trivial nebentypus. Then from the level lowering results of modular forms (by Ribet, Serre *et.al*), we get that the prime to p conductor of $\bar{\rho}_f = \bar{\rho}_g$ is not 1. In particular, the condition (1) of Theorem 4.10 also holds. Also the nebentypus is trivial. Thus all the conditions Theorem 4.10 are satisfied and we deduce f and gare \mathfrak{p} Bloch–Kato-Selmer companion forms.

(2) Note that a₄₃(f) ≠ ±1 (mod 43). Using the Hida family passing through f and g, we can generate more examples of higher weight p Selmer companion modular forms as in Examples I(2), I(3).

Example III.

- (1) We take f = 159k4B and $g = 159k4E \in S_4(\Gamma_0(153))$ with trivial nebentypus. Note $N = 159 = 3 \times 53$ is square-free. The Fourier coefficients of f belongs to \mathbb{Q} , on the other hand, $K_{f,g} = K_g$ with $[K_g : \mathbb{Q}] = 16$. We take p = 5 and using [24] compute that $a_5(f) =$ 0. As p = 5 > k = 4, $5 \nmid 159$ and $a_5(f) = 0$; the condition [*p*-ss] of Theorem 4.10 is satisfied. It was shown in [5] that there exists a prime \mathfrak{p} of K_g lying above p = 5 such that $f \equiv g \pmod{\mathfrak{p}}$. Moreover, in [5, §7.2, paragraph 3] it is given that there is no congruences between f (respectively g) at p with a newform of level dividing N =159. In particular, using level lowering result of modular forms (by Ribet, Serre *et.al*) it follows that the hypothesis (1) on conductor of $\bar{\rho}_f = \bar{\rho}_g$ holds. Thus via Theorem 4.10, f and g are \mathfrak{p} Bloch–Kato Selmer companion forms.
- (2) We again take the same forms f = 159k4B and $g = 159k4E \in S_4(\Gamma_0(153))$. However we now take p = 23 and using [24] compute that $a_{23}(f) = -49$. It was shown in [5] that there exists a prime π of K_g lying above p = 23 such that $f \equiv g \pmod{\mathfrak{p}}$. As the nebentypus is trivial, and $a_{23}(f) = -49$, we can conclude that (p = good) and (p -ord) of Theorem 4.10 are satisfied. As before, by [5, §7.2, paragraph 3] there is no congruences between f(resp g) at p with a newform of level dividing N = 159. In particular, using level lowering result of modular forms condition (1) holds. Thus via Theorem 4.10, f and g are again π Bloch–Kato Selmer companion forms.

Example IV. — Next we consider the example of f = 365k4A and $g = 365k4E \in S_4(\Gamma_0(365))$ with $N = 365 = 5 \times 73$ and we choose p = 29. Then f has Fourier coefficients defined over \mathbb{Q} and we compute via [24] $a_{29}(f) = -123$. Also $[K_g : \mathbb{Q}] = 18$ and there exists a prime \mathfrak{p} of K_g lying above p = 29 such that $f \equiv g \pmod{\mathfrak{p}}$ [5]. It is given in [5, §7.2, paragraph 3] there is no congruence between f or g at \mathfrak{p} with a newform of level dividing 365. Thus using the same reasoning as in Example III, we conclude f and g are \mathfrak{p} Bloch–Kato Selmer companion forms.

Example V. — Consider $N = 453 = 3 \times 151$ and $f, g \in S_4(\Gamma_0(453))$ where f = 453k4A with Fourier coefficients in \mathbb{Q} and g = 453k4E with $[K_g : \mathbb{Q}] = 23$. For the prime 17 we compute using $[24] a_{17}(f) = -66$. Once again from [5], (i) \exists a prime \mathfrak{p} of K_g lying above 17 such that $f \equiv g \pmod{\mathfrak{p}}$ and (ii) there is no congruence between f (resp g) at \mathfrak{p} with a newform of level dividing 453. By same reasoning as in Example III, f and g are \mathfrak{p} Bloch–Kato Selmer companion.

BIBLIOGRAPHY

- J. BELLAÏCHE, "An introduction to the conjecture of Bloch and Kato", Lectures at the Clay Mathematical Institute summer School, Honolulu, Hawaii, available at https://www.claymath.org/sites/default/files/bellaiche.pdf, 2009.
- [2] S. BLOCH & K. KATO, "L-functions and Tamagawa numbers of motives", in The Grothendieck Festschrift, Vol. I, Progress in Mathematics, vol. 86, Birkhäuser, 1990, p. 333-400.
- [3] J. E. CREMONA & B. MAZUR, "Visualizing elements in the Shafarevich-Tate group", Exp. Math. 9 (2000), no. 1, p. 13-28.
- [4] H. DARMON, F. DIAMOND & R. TAYLOR, "Fermat's last theorem", in Elliptic curves, modular forms & Fermat's last theorem (Hong Kong, 1993), International Press, 1997, p. 2-140.
- [5] N. DUMMIGAN, W. STEIN & M. WATKINS, "Constructing elements in Shafarevich– Tate groups of modular motives", in *Number theory and algebraic geometry*, London Mathematical Society Lecture Note Series, vol. 303, Cambridge University Press, 2003, p. 91-118.
- [6] M. EMERTON, R. POLLACK & T. WESTON, "Variation of Iwasawa invariants in Hida families", Invent. Math. 163 (2006), no. 3, p. 523-580.
- [7] J.-M. FONTAINE, "Le corps des périodes p-adiques", in Périodes p-adiques (Buressur-Yvette, 1988), Astérisque, vol. 223, Société Mathématique de France, 1994, With an appendix by Pierre Colmez, p. 59-111.
- [8] J.-M. FONTAINE & Y. OUYANG, "Theory of p-adic Galois Representations", available at https://www.math.u-psud.fr/~fontaine/galoisrep.pdf.
- [9] T. FUKAYA & K. KATO, "A formulation of conjectures on p-adic zeta functions in noncommutative Iwasawa theory", in *Proceedings of the St. Petersburg Mathematical Society. Vol. XII*, Translations. Series 2, vol. 219, American Mathematical Society, 2006, p. 1-85.
- [10] R. GREENBERG, "Iwasawa theory for elliptic curves", in Arithmetic theory of elliptic curves (Cetraro, 1997), Lecture Notes in Mathematics, vol. 1716, Springer, 1999, p. 51-144.

- [11] Y. HACHIMORI & K. MATSUNO, "An analogue of Kida's formula for the Selmer groups of elliptic curves", J. Algebr. Geom. 8 (1999), no. 3, p. 581-601.
- [12] J. HATLEY & A. LEI, "Arithmetic properties of signed Selmer groups at nonordinary primes", Ann. Inst. Fourier 69 (2019), no. 3, p. 1259-1294.
- [13] H. HIDA, Modular forms and Galois cohomology, Cambridge Studies in Advanced Mathematics, vol. 69, Cambridge University Press, 2000, x+343 pages.
- [14] S.-I. KOBAYASHI, "Iwasawa theory for elliptic curves at supersingular primes", Invent. Math. 152 (2003), no. 1, p. 1-36.
- [15] A. LEI, "Iwasawa theory for modular forms at supersingular primes", Compos. Math. 147 (2011), no. 3, p. 803-838.
- [16] A. LEI, D. LOEFFLER & S. L. ZERBES, "Coleman maps and the p-adic regulator", Algebra Number Theory 5 (2011), no. 8, p. 1095-1131.
- [17] ——, "On the asymptotic growth of Bloch–Kato–Shafarevich–Tate groups of modular forms over cyclotomic extensions", Can. J. Math. 69 (2017), no. 4, p. 826-850.
- [18] R. LIVNÉ, "On the conductors of mod l Galois representations coming from modular forms", J. Number Theory **31** (1989), no. 2, p. 133-141.
- [19] THE LMFDB COLLABORATION, "The L-functions and Modular Forms Database", 2013, http://www.lmfdb.org.
- [20] B. MAZUR & K. RUBIN, "Selmer companion curves", Trans. Am. Math. Soc. 367 (2015), no. 1, p. 401-421.
- [21] B. MAZUR, J. TATE & J. TEITELBAUM, "On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer", Invent. Math. 84 (1986), no. 1, p. 1-48.
- [22] T. MIYAKE, Modular forms, Springer, 1989, Translated from the Japanese by Yoshitaka Maeda, x+335 pages.
- [23] T. OCHIAI, "Control theorem for Bloch-Kato's Selmer groups of p-adic representations", J. Number Theory 82 (2000), no. 1, p. 69-90.
- [24] THE SAGE DEVELOPERS, "SageMath, the Sage Mathematics Software System", 2016, http://www.sagemath.org/.
- [25] A. J. SCHOLL, "Motives for modular forms", Invent. Math. 100 (1990), no. 2, p. 419-430.
- [26] C. SKINNER & E. URBAN, "The Iwasawa main conjectures for GL₂", Invent. Math. 195 (2014), no. 1, p. 1-277.
- [27] F. SPRUNG, "The Iwasawa Main Conjecture for elliptic curves at odd supersingular primes", https://arxiv.org/abs/1610.10017, 2016.
- [28] V. VATSAL, "Canonical periods and congruence formulae", Duke Math. J. 98 (1999), no. 2, p. 397-419.
- [29] A. WILES, "On ordinary λ -adic representations associated to modular forms", Invent. Math. 94 (1988), no. 3, p. 529-573.

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Somnath JHA Department of Mathematics and Statistics IIT Kanpur Kanpur 208016, India jhasom@gmail.com Dipramit MAJUMDAR Department of Mathematics IIT Madras Chennai 600036, India dipramit@iitm.ac.in Sudhanshu SHEKHAR Department of Mathematics and Statistics IIT Kanpur Kanpur 208016, India sudhansh@iitk.ac.in