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A GINZBURG–LANDAU MODEL WITH TOPOLOGICALLY INDUCED FREE DISCONTINUITIES

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Abstract. — We study a variational model which combines features of the Ginzburg–Landau model in 2D and of the Mumford–Shah functional. As in the classical Ginzburg–Landau theory, a prescribed number of point vortices appear in the small energy regime; the model allows for discontinuities, and the energy penalizes their length. The novel phenomenon here is that the vortices have a fractional degree $1/m$ with $m \geqslant 2$ prescribed. Those vortices must be connected by line discontinuities to form clusters of total integer degrees. The vortices and line discontinuities are therefore coupled through a topological constraint. As in the Ginzburg–Landau model, the energy is parameterized by a small length scale $\varepsilon > 0$. We perform a complete Γ-convergence analysis of the model as $\varepsilon \downarrow 0$ in the small energy regime. We then study the structure of minimizers of the limit problem. In particular, we show that the line discontinuities of a minimizer solve a variant of the Steiner problem. We finally prove that for small $\varepsilon > 0$, the minimizers of the original problem have the same structure away from the limiting vortices.

Résumé. — Nous étudions un modèle variationnel en deux dimensions qui combine les caractéristiques des fonctionnelles de Ginzburg–Landau et de Mumford–Shah. Comme dans la théorie classique de Ginzburg–Landau (et dans le régime de faible énergie) un nombre prescrit de vortex apparaît ; le modèle autorise aussi la formation de lignes de discontinuité dont l'énergie pénalise la longueur. Le phénomène nouveau est que les vortex ont un degé fractionnaire $1/m$ prescrit et qu'ils doivent être connectés par les lignes de discontinuité pour former des agrégats de degré total entier. Vortex et discontinuités sont donc couplés par une contrainte topologique. Comme dans le modèle de Ginzburg–Landau, l'énergie contient une échelle de longueur $\varepsilon > 0$. Nous faisons une analyse complète de la $\Gamma-$convergence de ce modèle lorsque $\varepsilon \downarrow 0$ dans le régime de faible énergie. Nous étudions ensuite la structure des minimiseurs du problème limite et montrons en particulier que les lignes de saut d'un tel minimiseur sont solutions d'une variante du problème de Steiner. Enfin, nous établissons que pour $\varepsilon > 0$ petit, les minimiseurs du problème initial possèdent la même structure, du moins loin des vortex.

Keywords: Free discontinuities, Ginzburg–Landau, Steiner problem, Calculus of Variations.
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1. Introduction

The purpose of this article is to study the asymptotic behavior of a family of functionals combining aspects of both Ginzburg–Landau [13, 54] and Mumford–Shah [4, 33, 42] functionals in dimension two. Those extend the standard Ginzburg–Landau energy, and give rise to the formation of vortex points connected by line defects in the small energy regime. Interestingly, vortices and line defects are coupled through topological constraints.

To be more specific, let us introduce the mathematical context. We consider for $m \in \mathbf{N}$, $m \geqslant 2$, the group of m-th roots of unity $\mathbf{G}_m = \{1, \mathbf{a}, \mathbf{a}^2, \ldots, \mathbf{a}^{m-1}\}$ with $\mathbf{a} := e^{2i\pi/m}$. We are interested in maps taking values in the quotient space $\mathbf{C}/\mathbf{G}_m$. We identify $\mathbf{C}/\mathbf{G}_m$ with the round cone

$$\mathcal{N} := \left\{(z,t) \in \mathbb{C} \times \mathbb{R} : t = |z|\sqrt{m^2-1}\right\} \subseteq \mathbf{R}^3$$

by means of the map $\mathrm{P} : \mathbf{C} \to \mathcal{N}$ defined as

$$\mathrm{P}(z) := \frac{1}{m}\left(\mathrm{p}(z), |z|\sqrt{m^2-1}\right) \quad \text{with } \mathrm{p}(z) := \frac{z^m}{|z|^{m-1}}\,.$$

The map P induces an isometry between $\mathbf{C}/\mathbf{G}_m$ and $\mathcal{N}$, and restricted to $\mathbf{C} \setminus \{0\}$ it defines a covering map of $\mathcal{N} \setminus \{0\}$ of degree m. For a given open set Ω and $p \geqslant 1$ we can thus say that $u \in W^{1,p}(\Omega, \mathbf{C}/\mathbf{G}_m)$ if $\mathrm{P}(u) \in W^{1,p}(\Omega, \mathcal{N})$ (where we say that a map $v \in W^{1,p}(\Omega, \mathcal{N})$ if v takes values in $\mathcal{N}$ and $v \in W^{1,p}(\Omega, \mathbf{R}^3)$).

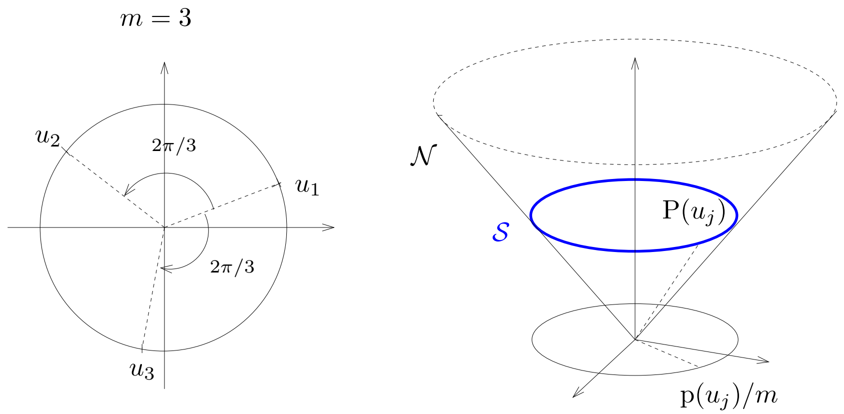

Figure 1.1. The cone $\mathcal{N}$ and the projection P. $\mathrm{P}(u_1) = \mathrm{P}(u_2) = \mathrm{P}(u_3)$.

For a simply connected smooth bounded domain $\Omega \subseteq \mathbf{R}^2$ and a "small" parameter $\varepsilon > 0$, the standard Ginzburg–Landau energy over Ω of a vector valued $W^{1,2}$-map reads

$$E_\varepsilon(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\,\mathrm{d}x\,.$$

Here, the main functional under investigation is defined for $u \in SBV^2(\Omega)$ satisfying the constraint $\mathrm{P}(u) \in W^{1,2}(\Omega;\mathcal{N})$ by

$$F^0_\varepsilon(u) := E_\varepsilon\big(\mathrm{P}(u)\big) + \mathcal{H}^1(J_u)\,, \tag{1.1}$$

where J_u denotes the jump set of u (see [4] and Section 2.3 below). We stress that F^0_ε extends E_ε, that is $F^0_\varepsilon(u) = E_\varepsilon(u)$ whenever $u \in W^{1,2}(\Omega)$, which comes from the isometric character of P. In the same way F^0_ε appears as a Mumford–Shah type functional since

$$F^0_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\,\mathrm{d}x + \mathcal{H}^1(J_u)\,,$$

where ∇u denotes the absolutely continuous part of the measure Du. The constraint $\mathrm{P}(u) \in W^{1,2}(\Omega;\mathcal{N})$ rephrases the fact that the functional is restricted to the class $\big\{u \in SBV^2(\Omega) : u^+/u^- \in \mathbf{G}_m \text{ on } J_u\big\}$. In particular, only specific discontinuities in the orientation are allowed. The case $m = 2$, which consists in identifying u and $-u$, is of special interest as it appears in many physical models, see Section 1.2 below.

We also consider an Ambrosio–Tortorelli regularization of (1.1) where the jump set J_u is (formally) replaced by the zero set $\{\psi \sim 0\}$ of some scalar phase field function ψ, and the length $\mathcal{H}^1(J_u)$ by a suitable energy of ψ. We introduce a second small parameter η and consider for $u \in L^2(\Omega)$ and $\psi \in W^{1,2}(\Omega;[0,1])$ satisfying $\mathrm{P}(u) \in W^{1,2}(\Omega;\mathcal{N})$ and $u\psi \in W^{1,2}(\Omega)$, the functional

$$F^\eta_\varepsilon(u,\psi) := E_\varepsilon\big(\mathrm{P}(u)\big) + \frac{1}{2}\int_\Omega \eta|\nabla\psi|^2 + \frac{1}{\eta}(1-\psi)^2\,\mathrm{d}x\,. \tag{1.2}$$

Compared to the original Ambrosio–Tortorelli functional [5, 6], u and ψ are only coupled through the constraint $u\psi \in W^{1,2}(\Omega)$, and not in the functional itself. As for F^0_ε, the functional F^η_ε extends E_ε in the sense that $F^\eta_\varepsilon(u,1) = E_\varepsilon(u)$ whenever $u \in W^{1,2}(\Omega)$.

We aim to study low energy states (in particular minimizers) of the functionals F^0_ε and F^η_ε under Dirichlet boundary conditions of the form $u = g$ on $\partial\Omega$ for a prescribed smooth $g \in C^\infty(\partial\Omega;\mathbf{S}^1)$. Concerning F^0_ε, we work in the class $\mathcal{G}_g(\Omega)$ of maps satisfying $\mathrm{P}(u) = \mathrm{P}(g)$ on $\partial\Omega$. Then, we penalize possible deviations from g on $\partial\Omega$ by considering the modified energy

$$F^0_{\varepsilon,g}(u) := F^0_\varepsilon(u) + \mathcal{H}^1\big(\{u \neq g\} \cap \partial\Omega\big)\,.$$

Notice that such a penalization is necessary in order to have lower semicontinuity of the functional (see for instance [34]). For the functional F_ε^η, we restrict ourselves to admissible pairs (u,ψ) satisfying $u\psi = g$ and $\psi = 1$ on $\partial\Omega$, and write $\mathcal{H}_g(\Omega)$ the corresponding class. In this setting, the functionals $F^0_{\varepsilon,g}$ and F_ε^η still extend E_ε restricted to $W^{1,2}_g(\Omega)$, so that

$$\min_{\mathcal{G}_g(\Omega)} F^0_{\varepsilon,g} \leqslant \min_{W^{1,2}_g(\Omega)} E_\varepsilon \quad \text{and} \quad \min_{\mathcal{H}_g(\Omega)} F_\varepsilon^\eta \leqslant \min_{W^{1,2}_g(\Omega)} E_\varepsilon\,. \tag{1.3}$$

As in the classical Ginzburg–Landau theory [13], we assume that the winding number (or degree) is strictly positive, i.e.,

$$d := \deg(g,\partial\Omega) > 0\,.$$

In this way, g does not admit a continuous $\mathbf{S}^1$-valued extension to Ω. This topological obstruction is responsible for the formation of *vortices* (point singularities) in any configuration of small energy E_ε as $\varepsilon \to 0$, and the minimum value of E_ε over $W^{1,2}_g$ is given by $\pi d|\log\varepsilon|$ at first order. In view of (1.3), creating discontinuities in the orientation may lead to configurations of smaller energy. Indeed, direct constructions of competitors show that the minimum value of $F^0_{\varepsilon,g}$ or F_ε^η is less than $\frac{\pi d}{m}|\log\varepsilon|$ at first order, and thus (almost) minimizers must have line singularities (or "diffuse" line singularities for F_ε^η), at least for ε (and η) small enough.

1.1. Heuristics

The starting point is the identity

$$E_\varepsilon\big(\mathrm{P}(u)\big) = \frac{1}{m^2}E_\varepsilon\big(\mathrm{p}(u)\big) + \frac{m^2-1}{m^2}E_\varepsilon\big(|\mathrm{p}(u)|\big)\,.$$

Following the standard theory of the Ginzburg–Landau functional [13, 54], one may expect that for configurations u of small energy, the leading term is $\frac{1}{m^2}E_\varepsilon\big(\mathrm{p}(u)\big)$, and that $\mathrm{p}(u)$ has (classical) Ginzburg–Landau energy E_ε close to the one of the minimizers under the boundary condition $\mathrm{p}(u) = \mathrm{p}(g)$ on $\partial\Omega$. Since $\mathrm{p}(g) = g^m$, its topological degree equals md, and $\mathrm{p}(u)$ should have md distinct vortices of degree $+1$, i.e., md distinct points x_k in Ω such that $\mathrm{p}(u)(x_k) = 0$ and

$$\mathrm{p}(u)(x) \sim \alpha_k \frac{x-x_k}{|x-x_k|} \quad \text{for } \varepsilon \ll |x-x_k| \ll 1 \text{ and some constant } \alpha_k \in \mathbf{S}^1.$$

In terms of E_ε, the energetic cost of each vortex is $\pi|\log\varepsilon|$ at leading order, and therefore $E_\varepsilon\big(\mathrm{P}(u)\big)$ should be less than $\frac{\pi d}{m}|\log\varepsilon|$, again at leading order.

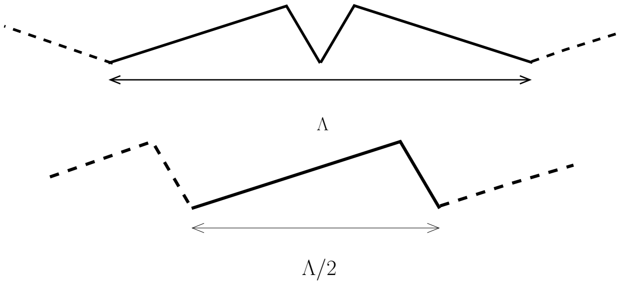

Figure 1.2. Top: profile of the $\Lambda-$phase. Bottom: profile of the $\Lambda/2-$phase.

This discussion led us to consider the energy regimes

$$F^0_{\varepsilon,g}(u) \leqslant \frac{\pi d}{m}|\log\varepsilon| + O(1) \quad \text{and} \quad F^\eta_\varepsilon(u,\psi) \leqslant \frac{\pi d}{m}|\log\varepsilon| + O(1) \tag{1.4}$$

for $u \in \mathcal{G}_g(\Omega)$ or $(u,\psi) \in \mathcal{H}_g(\Omega)$, respectively. Once again, it corresponds to the energy regime of md vortices of degree $+1$ in the variable $\mathrm{p}(u)$. By an elementary topological argument, one can see that any pre-image by p of $\frac{x-x_k}{|x-x_k|}$ must have at least one discontinuity line departing from x_k, and has a (formal) winding number around x_k equal to $1/m$ (in other words, the phase has a jump of $2\pi/m$ around x_k). For this reason, any configuration u satisfying (1.4) must be discontinuous. In the sharp interface case (1.1), we actually expect that each connected component of the jump set J_u connects mk vortices for some $k \in \{1,\dots,d\}$, since the winding number around any such connected component must be an integer. A similar picture should hold in the diffuse case (1.2) with J_u replaced by the zero set $\{\psi = 0\}$. The energy associated with discontinuities is their length (or diffuse length), and there should be a competition between this term which favors clustered vortices and the so-called renormalized energy from Ginzburg–Landau theory which is a repulsive (logarithmic) point interaction.

1.2. Motivation

Our original motivation for studying the functionals (1.2) and (1.1) stemmed from the analysis of the defect patterns observed in the so-called ripple or $P_{\beta'}$ phase in biological membranes such as lipid bilayers [10, 43, 46, 52, 53]. In this phase, which is intermediate between the gel and the liquid phase, periodic corrugations are observed at the surface of the membranes (see [52] for instance).

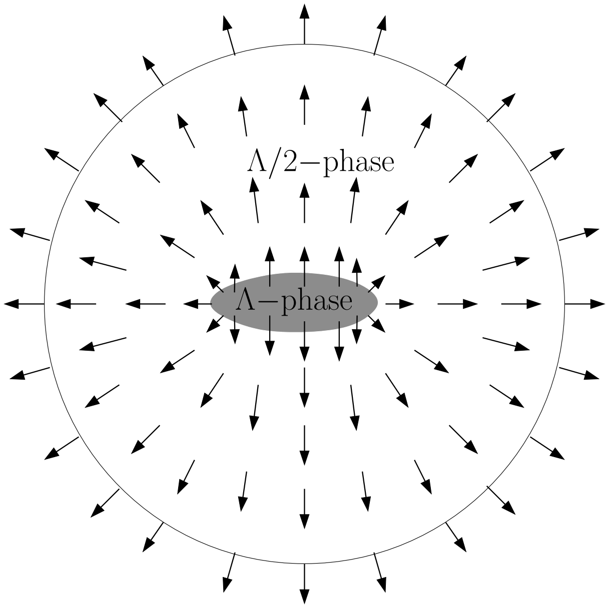

Figure 1.3. Creation of two vortices of degree $1/2$.

Two different kinds of periodic sawtooth profiles are observed. A symmetric one and an asymmetric one respectively called Λ and $\Lambda/2-$phases (see Figure 1.2 for a schematic representation of a cross-section). In the asymmetric phase, only defects of integer degree are allowed while in the symmetric phase half integer degree vortices are also permitted. Since two vortices of degree $1/2$ have an energetic cost of order $\frac{\pi}{2}|\log\varepsilon|$ (where ε is the lengthscale of the vortex) while a vortex of degree 1 has a cost of order $\pi|\log\varepsilon|$, it is expected that even in the regime where the $\Lambda/2-$phase is favored (which happens for nearly flat membranes), a phase transition occurs around the defects with the nucleation of a small island of $\Lambda-$phase leading to the formation of two vortices of degree $1/2$ (see Figure 1.3). In the model proposed by [10], the order parameter is given by $f(\varphi)$, where f is a fixed profile (corresponding to the one on the right part of Figure 1.2) and φ is the phase modulation. Their functional corresponds to F_ε^η, for $\varepsilon = \eta$, $m = 2$ and $u = \nabla\varphi$ (so that u represents the local speed at which the profile f is modulated). In [10], the authors further argue that the constraint of u being a gradient can be relaxed so that we recover completely our model.

We also point out that (1.1) and (1.2) have connections with many other models appearing in the literature. As an example, we can mention the issue of orientability of Sobolev vector fields into $\mathbf{RP}^2$, see [8]. More generally, our functionals resemble the ones suggested recently to model liquid crystals where both points and lines singularities appear, see [9]. Similarly to [8], a central issue here is to find square roots (and more generally $m-$th roots) of $W^{1,p}$-functions into $\mathbf{S}^1$ (see [39]), and this is intimately related to the question of lifting of Sobolev functions into $\mathbf{S}^1$, see [14, 21, 28, 30, 48].

While completing this article, we have been aware of the work [7], where the authors perform an analogous Γ-convergence analysis for a discrete model, obtaining in the continuous limit almost the same functional as ours. These authors were motivated by applications to liquid crystals, micromagnetics, and crystal plasticity, and we refer to their introduction for more references on the physical literature.

1.3. Main results

Our first main theorem is a Γ-convergence result in the energy regime (1.4) (we refer to [16, 26] for a complete exposition on Γ-convergence theory). To describe the limiting functional, we need to introduce the following objects. First, set $\mathcal{A}_d$ to be the family of all atomic measures of the form $\mu = 2\pi \sum_{k=1}^{md} \delta_{x_k}$, for some md distinct points $x_k \in \Omega$. To $\mu \in \mathcal{A}_d$, we associate the so-called *canonical harmonic map* v_μ defined by

$$v_\mu(x) := e^{i\varphi_\mu(x)} \prod_{k=1}^{md} \frac{x - x_k}{|x - x_k|} \quad \text{with} \quad \begin{cases} \Delta\varphi_\mu = 0 & \text{in } \Omega\,, \\ v_\mu = g^m & \text{on } \partial\Omega\,. \end{cases}$$

In turn, the *renormalized energy* $\mathbb{W}(\mu)$ can be defined as the finite part of the energy of v_μ, i.e.,

$$\mathbb{W}(\mu) := \lim_{r\downarrow 0} \left\{ \frac{1}{2} \int_{\Omega\setminus B_r(\mu)} |\nabla v_\mu|^2 \,\mathrm{d}x - \pi md|\log r| \right\} ,$$

and we refer to (2.15) for its explicit expression.

We provide below a concise version of the Γ-convergence result, complete statements can be found in Theorem 3.1 and Theorem 3.2.

Theorem 1.1. — *The functionals* $\{F^0_{\varepsilon,g} - \frac{\pi d}{m}|\log\varepsilon|\}$ *and* $\{F^\eta_\varepsilon - \frac{\pi d}{m}|\log\varepsilon|\}$ *(respectively restricted to* $\mathcal{G}_g(\Omega)$ *and* $\mathcal{H}_g(\Omega)$*)* Γ*-converge in the strong* L^1*-topology as* $\varepsilon \to 0$ *and* $\eta \to 0$ *to the functional*

$$F_{0,g}(u) := \frac{1}{2m^2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \frac{1}{m^2}\mathbb{W}(\mu) + md\boldsymbol{\gamma}_m \\ + \mathcal{H}^1\big(J_u\big) + \mathcal{H}^1\big(\{u \neq g\} \cap \partial\Omega\big)$$

defined for $u \in SBV(\Omega;\mathbf{S}^1)$ *such that* $u^m = e^{i\varphi}v_\mu$ *for some* $\mu \in \mathcal{A}_d$ *and* $\varphi \in W^{1,2}(\Omega)$ *satisfying* $e^{i\varphi} = 1$ *on* $\partial\Omega$*. The constant* $\boldsymbol{\gamma}_m$*, referred to as* core energy *(see* (3.1)*), only depends on* m*.*

We point out that there is of course a compactness result companion to Theorem 1.1. Namely, if a sequence $\{u_\varepsilon\}$ satisfies the energy bound (1.4), and is uniformly bounded in $L^\infty(\Omega)$, then $\{u_\varepsilon\}$ converges up to a subsequence in $L^1(\Omega)$, and $\{\mathrm{p}(u_\varepsilon)\}$ converges (again up to a subsequence) in the weak $W^{1,p}$-topology for every $p < 2$. As can be expected, the proof of Theorem 1.1 combines ideas coming from the study of the Ginzburg–Landau functional [2, 13, 24, 40, 45, 54], together with ideas from free discontinuities problems [4, 6, 15, 17]. Concerning the compactness part, we have included complete proofs to provide a rather self-contained exposition. Although some estimates (such as the $W^{1,p}$ bound, see Lemma 2.12) are certainly known to the Ginzburg–Landau community (see for instance [24, 45]), they have never been used in the context of Γ-convergence. The Γ-lim inf inequality is a relatively standard combination of techniques developed in [2, 17, 24], while the construction of recovery sequences is a much more delicate issue. The main difficulty comes from the constraint $u^m = e^{i\varphi}v_\mu$, which prevents us from applying directly the existing approximation results by functions with a smooth jump set, see e.g. [11, 18, 25, 29]). Our approach uses a (new) regularization technique (see Lemma 3.17) which is somehow reminiscent of [5] and could be of independent interest. Another difficulty comes from the optimal profile problem defining the core energy $\boldsymbol{\gamma}_m$. The underlying minimization problem involves the Ginzburg–Landau energy of $\mathcal{N}$-valued maps, and one has to find almost minimizers which can be lifted into $\mathbf{C}$-valued maps in SBV^2, see Section 3.2.

The Γ-convergence result applies to minimizers of either $F^0_{\varepsilon,g}$ or F^η_ε (whose existence is proven in Theorems 2.7 and 2.8). It shows that they converge in $L^1(\Omega)$ to a minimizer u of $F_{0,g}$. Our second main result deals with the characterization of such minimizer u. It is based on the following observations. First, from the explicit form of $F_{0,g}$, it follows that $\varphi = 0$ in the representation $u^m = e^{i\varphi}v_\mu$. In particular, u can be characterized as a solution of the minimization problem

$$\min\left\{\frac{1}{m^2}\mathbb{W}(\mu)+\mathcal{H}^1\big(J_u\big)+\mathcal{H}^1\big(\{u\neq g\}\cap\partial\Omega\big):\right.$$
$$\left. u\in SBV(\Omega;\mathbf{S}^1),\ u^m=v_\mu \text{ for some } \mu\in\mathcal{A}_d\right\}.$$

In turn, this later can be equivalently rewritten as

$$\min_{\mu\in\mathcal{A}_d}\min\left\{\frac{1}{m^2}\mathbb{W}(\mu)+\mathcal{H}^1\big(J_u\big)+\mathcal{H}^1\big(\{u\neq g\}\cap\partial\Omega\big):\right.$$
$$\left. u\in SBV(\Omega;\mathbf{S}^1)\,,\ u^m=v_\mu\right\}.$$

As a consequence, fixing $\mu\in\mathcal{A}_d$ and solving

$$L(\mu):=\min\left\{\mathcal{H}^1\big(J_u\big)+\mathcal{H}^1\big(\{u\neq g\}\cap\partial\Omega\big):u\in SBV(\Omega;\mathbf{S}^1)\,,\ u^m=v_\mu\right\},$$

we are left with a finite dimensional problem to recover the minimizers of $F_{0,g}$.

Given $\mu\in\mathcal{A}_d$, we compare in Theorem 1.2 below the minimization problem $L(\mu)$ with the following variant of the Steiner problem (see e.g. [35]):

$$\Lambda(\mu):=\min\Big\{\mathcal{H}^1(\Gamma):\Gamma\subseteq\overline{\Omega}\text{ compact with }\operatorname{spt}\mu\subseteq\Gamma$$
$$\text{and every connected component }\Sigma\text{ satisfies}$$
$$\operatorname{Card}(\Sigma\cap\operatorname{spt}\mu)\in m\mathbb{N}\Big\}.$$

We shall see that any minimizer Γ of $\Lambda(\mu)$ is made of at most d disjoint Steiner trees, i.e., connected trees made of a finite union of segments meeting either at points of $\operatorname{spt}\mu$, or at triple junction making a $120°$ angle. From now on, when talking about triple junctions we always implicitly include this condition on the angles.

Our second main result is the following theorem, in which we assume Ω to be convex (to avoid issues at the boundary).

Theorem 1.2. — *Assume that Ω is convex. For every $\mu\in\mathcal{A}_d$, $L(\mu)=\Lambda(\mu)$. Moreover, if u is a minimizer for $L(\mu)$, then its jump set J_u is a minimizer for $\Lambda(\mu)$, $u\in C^\infty(\overline{\Omega}\setminus J_u)$, and $u=g$ on $\partial\Omega$. Vice-versa, if Γ is a minimizer for $\Lambda(\mu)$, then there exists a minimizer u for $L(\mu)$ such that $J_u=\Gamma$.*

To complete the picture, we shall give several examples illustrating the fact that the geometry of minimizers for $\Lambda(\mu)$ strongly depends on m, d, and the location of $\operatorname{spt}\mu$. In the case $m = 2$, a minimizer for $\Lambda(\mu)$ is always given by a disjoint union of d segments connecting the points of $\operatorname{spt}\mu$ (see Proposition 4.7). However, for $m \geqslant 3$ and $d \geqslant 2$, minimizers are not always the disjoint union of d Steiner trees containing exactly m vortices (see Proposition 4.8 and Proposition 4.10).

In our third and last main result, we use the characterization of the minimizers of $F_{0,g}$ provided by Theorem 1.2 to show that for $\varepsilon > 0$ small enough, minimizers of $F^0_{\varepsilon,g}$ have essentially the same structure away from the limiting vortices.

Theorem 1.3. — *Assume that Ω is convex. Let $\varepsilon_h \to 0$, and let u_h be a minimizer of $F^0_{\varepsilon_h,g}$ over $\mathcal{G}_g(\Omega)$. Assume that $u_h \to u$ in $L^1(\Omega)$ as $h \to \infty$ for some minimizer u of $F_{0,g}$. Letting $\mu \in \mathcal{A}_d$ be such that $u^m = v_\mu$, for every $\sigma > 0$ small enough, if h is large enough (depending on σ), the following holds:*

(i) *$J_{u_h} \setminus B_\sigma(\mu)$ is a compact subset of $\Omega \setminus B_\sigma(\mu)$ made of finitely many segments, meeting by three at an angle of $120°$ (i.e., triple junctions).*

(ii) *$u_h \in C^\infty\big(\overline{\Omega} \setminus (B_\sigma(\mu) \cup J_{u_h})\big)$ and $u_h = g$ on $\partial\Omega$.*

In addition,

(iv) *J_{u_h} converges in the Hausdorff distance to J_u.*

(v) *$u_h \to u$ in $C^k_{\mathrm{loc}}(\Omega \setminus J_u) \cap C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega} \setminus J_u)$ for every $k \in \mathbb{N}$ and $\alpha \in (0,1)$.*

In proving Theorem 1.3, we actually show a stronger result that we now briefly describe (see Theorem 5.1, Remark 5.2, and Section 5.1). In each (sufficiently small) ball $B_r(x) \subseteq \Omega \setminus B_\sigma(\mu)$ and ε small enough, u_ε is bounded away from zero, and it can be decomposed as $u_\varepsilon = \phi_\varepsilon w_\varepsilon$ where $\phi_\varepsilon \in SBV^2(B_r(x))$ and w_ε is minimizing the classical Ginzburg–Landau energy $E_\varepsilon(\,\cdot\,, B_r(x))$ with respect to its own boundary condition (and as a consequence, w_ε is smooth). The proof of this decomposition relies on the energy splitting discovered by Lassoued and Mironescu [41]. Combined with the classical Wente estimate [19, 55], it leads to a lower expansion of the energy of the form

$$F^0_\varepsilon(u_\varepsilon, B_r(x)) \geqslant E_\varepsilon(w_\varepsilon, B_r(x)) + \frac{1}{\alpha}\left(\int_{B_r(x)} |\nabla\phi_\varepsilon|^2\,\mathrm{d}x + \alpha\mathcal{H}^1\big(J_{\phi_\varepsilon} \cap B_r(x)\big)\right),$$

for some constant $\alpha > 0$ (see Proposition 5.11). Using suitable competitors, we deduce that ϕ_ε is a Dirichlet minimizer of the Mumford–Shah functional in $B_r(x)$. Applying the calibration results of [1, 49], we infer that ϕ_ε takes values into the finite set $\mathbf{G}_m$, reducing the problem to a minimal partition problem in $B_r(x)$. The classical regularity results on two dimensional minimal clusters then yield the announced geometry of the jump set.

The paper is organized as follows. Section 2 is devoted to a full set of preliminary results. First, we present some fine properties of the BV-functions under investigation, and then we prove existence of minimizers for F^η_ε and $F^0_{\varepsilon,g}$. In a third part, we provide all the material and results concerning the Ginzburg–Landau energy that we shall use. The Γ-convergence result of Theorem 1.1 is the object of Section 3. In Section 4, we prove Theorem 1.2 and give the aforementioned examples of $\Lambda(\mu)$-minimizers. In the last Section 5, we return to the analysis of minimizers of $F^0_{\varepsilon,g}$, and prove Theorem 1.3.

2. Preliminaries

2.1. Conventions and notation

Throughout the paper we identify the complex plane $\mathbf{C}$ with $\mathbf{R}^2$. We say that a property holds a.e. if it holds outside a set of Lebesgue measure zero.

- For $a, b \in \mathbf{R}^2$, we write $a \wedge b := \det(a, b)$;
- For $a \in \mathbf{R}^2$ and $M = (b_1, \dots, b_n) \in \mathcal{M}_{2\times n}(\mathbf{R})$, we write

$$a \wedge M := {}^t(a \wedge b_1, \dots, a \wedge b_n) \in \mathbf{R}^n\,;$$

- For $M \in \mathcal{M}_{d\times n}(\mathbf{R})$, we write $|M| := |\mathrm{tr}(M\,{}^tM)|^{1/2}$;
- For $a = (a_1, a_2) \in \mathbf{R}^2$ we let $a^\perp := (-a_2, a_1)$
- for a set $\Omega \subseteq \mathbf{R}^2$, we call ν its external normal and τ its tangent chosen so that (ν, τ) is a direct basis (in particular $\nu^\perp = \tau$ and $\partial\Omega$ is oriented counterclockwise);
- The distributional derivative is denoted by Df;
- For $v \in \mathbf{R}^n$, we let $\partial_v f := Df(v)$ be the partial derivative of f in the direction v and if $v = e_l$ is a vector of the canonical basis of $\mathbf{R}^n$ then we simply write $\partial_l f := \partial_{e_l} f$;
- $\nabla f = (\partial_l f_k)_{k,l}$ is the Jacobian matrix of the vector valued function f;

- For $j = (j_1, j_2)$, we denote by $\operatorname{curl} j := \partial_1 j_2 - \partial_2 j_1$ the rotational of j;
- For $A \subseteq \mathbf{R}^n$, we denote by $B_r(A)$ the tubular neighborhood of A of radius r. For a measure μ, we simply write $B_r(\mu) := B_r(\operatorname{spt} \mu)$;
- In most of the paper, we work with Ω a given bounded open and simply connected set. Nevertheless, since in Sections 4 and 5 we will require that Ω is convex, we will repeat at the beginning of each section the hypothesis we are making on Ω;
- We shall not relabel subsequences if no confusion arises.

2.2. Finite subgroups of $\mathbf{S}^1$ and isometric cones.

Given an integer $m \geqslant 2$, we denote by $\mathbf{G}_m$ the subgroup of $\mathbf{S}^1$ made of all m-th roots of unity, i.e.,

$$\mathbf{G}_m = \left\{1, \mathbf{a}, \mathbf{a}^2, \dots, \mathbf{a}^{m-1}\right\} \quad \text{with } \mathbf{a} := e^{2i\pi/m}\,.$$

We consider the quotient space $\mathbf{C}/\mathbf{G}_m$ endowed with the canonical distance

$$\operatorname{dist}\big([z_1],[z_2]\big) := \min_{z_1\in[z_1],\, z_2\in[z_2]} |z_1 - z_2| = \min_{k=0,\dots,m-1} |z_1 - \mathbf{a}^k z_2|\,,$$

where $[z]$ is the equivalence class of $z \in \mathbf{C}$. We note that $\mathbf{C}/\mathbf{G}_m$ is isometrically embedded into $\mathbf{R}^3 \simeq \mathbf{C}\times\mathbf{R}$ by means of the Lipschitz mapping $\mathrm{P} : \mathbf{C} \to \mathbf{R}^3$ given by

$$\mathrm{P}(z) := \frac{1}{m}\Big(\mathrm{p}(z), |z|\sqrt{m^2-1}\Big) \quad \text{where } \mathrm{p}(z) := \frac{z^m}{|z|^{m-1}}\,.$$

In this way we identify $\mathbf{C}/\mathbf{G}_m$ with the round cone of $\mathbf{R}^3$,

$$\mathcal{N} := \mathrm{P}(\mathbf{C}) = \Big\{(x,t) \in \mathbf{R}^2\times\mathbf{R} : t = |x|\sqrt{m^2-1}\Big\}\,,$$

and one has $\operatorname{dist}\big([z_1],[z_2]\big) = \mathrm{d}_{\mathcal{N}}\big(\mathrm{P}(z_1),\mathrm{P}(z_2)\big)$ for every $z_1, z_2 \in \mathbf{C}$, where $\mathrm{d}_{\mathcal{N}}$ denotes the geodesic distance on $\mathcal{N}$ induced by the Euclidean metric (in particular, $|\mathrm{P}(z)| = |z|$ for every $z \in \mathbf{C}$). Similarly, $\mathbf{S}^1/\mathbf{G}_m$ coincides with the horizontal circle

$$\mathcal{S} := \big\{(x,t) \in \mathcal{N} : |x| = 1/m, t = \sqrt{1-1/m^2}\big\} = \mathrm{P}(\mathbf{S}^1)\,.$$

Note that in the case $m = 2$, $\mathcal{S} \simeq \mathbf{S}^1/\{\pm1\}$ is the real projective line $\mathbf{RP^1}$. Finally, we point out that P is smooth away from the origin, and since P is isometric,

$$|\nabla\mathrm{P}(z)v| = |v| \quad \text{for every } v \in \mathbf{R}^2 \text{ and every } z \in \mathbf{C}\setminus\{0\}\,, \tag{2.1}$$

where $\nabla \mathrm{P}(z) \in \mathcal{M}_{3\times 2}(\mathbf{R})$ is the differential of P at z represented in real coordinates. Similarly, we write $\nabla \mathrm{p}(z) \in \mathcal{M}_{2\times 2}(\mathbf{R})$ for the differential of p at z.

2.3. BV and SBV functions, weak Jacobians

Concerning functions of bounded variations, their fine properties, and standard notations, we refer to [4]. Let us briefly introduce the main properties and definitions used in the paper. For an open subset Ω of $\mathbf{R}^2$, we first recall that $BV(\Omega, \mathbf{R}^q)$ is the space of functions of bounded variation in Ω, i.e., functions $u \in L^1(\Omega, \mathbf{R}^q)$ for which the distributional derivative Du is a finite (matrix valued) Radon measure on Ω. We recall that for a function $u \in BV(\Omega, \mathbf{R}^q)$, we have the following decomposition

$$Du = \nabla u \mathrm{d}x + D^j u + D^c u\,,$$

where

$$D^j u := (u^+ - u^-) \otimes \nu_u \, \mathcal{H}^1 \llcorner J_u\,. \tag{2.2}$$

The functions $u^\pm$ denote the traces of u on the jump set J_u which is a countably $\mathcal{H}^1$-rectifiable set. Since all the properties we will consider are oblivious to modifications of J_u on sets of zero $\mathcal{H}^1$ measure, we shall not distinguish between J_u and the singular set of u (usually denoted as S_u). In particular, when J_u is regular or a finite union of polygonal curves, we will also not distinguish between J_u and its closure so that we shall often consider it as a compact set. Analogously, for sets E of finite perimeter, i.e., such that $\chi_E \in BV(\Omega)$, we simply denote by ∂E the reduced boundary.

The space $SBV(\Omega, \mathbf{R}^q)$ is defined as the subspace of $BV(\Omega, \mathbf{R}^q)$ made of functions u satisfying $D^c u \equiv 0$. For a finite exponent $p \geqslant 1$, the subspace $SBV^p(\Omega, \mathbf{R}^q) \subseteq SBV(\Omega, \mathbf{R}^q)$ is defined as

$$SBV^p(\Omega, \mathbf{R}^q) := \Big\{ u \in SBV(\Omega, \mathbf{R}^q) : \nabla u \in L^p(\Omega) \text{ and } \mathcal{H}^1(J_u) < \infty \Big\}.$$

Remark 2.1 (pre-Jacobian). — For a smooth function u, we define the pre-Jacobian of u as

$$j(u) := u \wedge \nabla u,$$

which also writes $j(u) = \mathrm{Re}(iu\nabla \overline{u})$ in complex notation. Notice that if $u = \rho e^{i\theta}$ for some smooth functions ρ and θ, then $j(u) = \rho^2 \nabla \theta$ so that $j(u)$ measures the variation of the phase. In particular, if Ω is simply connected and u takes values into $\mathbf{S}^1$, then $\operatorname{curl} j(u) = 0$ and we can write $j(u) = \nabla \theta$, hence recovering the phase θ.

To our purposes, we need to extend the notion of pre-Jacobian to BV-maps.

Definition 2.2. — *For $u \in BV(\Omega)$, we define the pre-Jacobian of u to be the measurable vector field*

$$j(u) := u \wedge \nabla u\,,$$

where ∇u is the absolutely continuous part of Du. It belongs to $L^1(\Omega)$ whenever $u \in L^\infty(\Omega)$ or $\nabla u \in L^2(\Omega)$ (since $BV(\Omega)$ is continuously embedded in $L^2(\Omega)$).

Lemma 2.3. — *Let $u \in BV(\Omega)$. Then $V := \mathrm{P}(u)$ and $v := \mathrm{p}(u)$ are of bounded variation in Ω, and*

(i) $V(x) \in \mathcal{N}$ *for a.e.* $x \in \Omega$*;*
(ii) $J_V \subseteq J_u$*;*
(iii) $\big(V^+, V^-, \nu_V\big) = \big(\mathrm{P}(u^+), \mathrm{P}(u^-), \nu_u\big)$ *on* J_V*;*
(iv) $\mathrm{P}\big(u^+(x)\big) = \mathrm{P}\big(u^-(x)\big)$ *for every* $x \in J_u \setminus J_V$*;*
(v) $|\nabla V| = |\nabla u|$ *a.e. in* Ω*;*
(vi) $|D^c V| = |D^c u|$*;*
(vii) $j(v) = m j(u)$ *a.e. in* Ω.

Proof. — The fact that $V \in BV(\Omega; \mathbf{R}^3)$, as well as items (i), (ii), (iii), and (iv), is a direct consequence of the 1-Lipschitz property of P, see [4, proof of Theorem 3.96]. Moreover, $|DV| \leqslant |Du|$. It remains to prove (v), (vi), and (vii). Recall that, by [4, Proposition 3.92], we have $|Du|(Z_u) = 0$ where

$$Z_u := \big\{x \in \Omega \setminus J_u : u(x) = 0\big\}\,.$$

For $k \in \mathbf{N}$, we set

$$\begin{aligned} A_0 &:= \big\{x \in \Omega \setminus J_u : |u(x)| > 1\big\}\,,\\ A_k &:= \big\{x \in \Omega \setminus J_u : 2^{-k} < |u(x)| \leqslant 2^{-k+1}\big\}\,, \end{aligned}$$

so that $\Omega \setminus Z_u = \bigcup_k A_k$ with a disjoint union. Then, for each $k \in \mathbf{N}$, we consider $\mathrm{P}_k \in C^1(\mathbf{C}; \mathbf{R}^3)$ such that $\mathrm{P}_k(z) = \mathrm{P}(z)$ whenever $|z| > 2^{-k}$. Using the chain-rule formula in BV (see [4, Theorem 3.96]), for $\mathrm{P}_k(u)$ and the locality of the derivative of a BV function (see [4, Remark 3.93]), we readily obtain (v) and (vi).

To prove (vii), we first notice that for $z \in \mathbf{C} \setminus \{0\}$ and $X \in \mathbf{R}^2$, we have

$$\mathrm{p}(z) \wedge (\nabla \mathrm{p}(z) X) = m z \wedge X\,.$$

Therefore, if $x \in \Omega \setminus Z_u$ is a Lebesgue point for ∇u and ∇V, we have for each $l \in \{1,2\}$,

$$v(x) \wedge \partial_l v(x) = \mathrm{p}(u(x)) \wedge \Big(\nabla \mathrm{p}(u(x))\partial_l u(x)\Big) = mu(x) \wedge \partial_l u(x)\,,$$

and the proof is complete. □

Corollary 2.4. — *If $u \in BV(\Omega)$ is such that $\mathrm{P}(u) \in W^{1,p}(\Omega;\mathcal{N})$ for some $p \geqslant 1$, then $u \in SBV(\Omega)$ and $\nabla u \in L^p(\Omega)$. Moreover, $u^\pm(x) \neq 0$ for every $x \in J_u$, and $u^+(x)/u^-(x) \in \mathbf{G}_m$. If, in addition, $|u| \geqslant \delta$ a.e. in Ω for some $\delta > 0$, then $u \in SBV^p(\Omega)$ and $|D^j u| \geqslant \delta|\mathbf{a}-1|\mathcal{H}^1 \llcorner J_u$.*

Proof. — The fact that $u \in SBV(\Omega)$ and $\nabla u \in L^p(\Omega)$ is a direct consequence of (vi) and (v) in Lemma 2.3, respectively. Next, assume that $u^+(x) = 0$ for some $x \in J_u$. Then (iv) in Lemma 2.3 yields $u^-(x) = 0$, so that $x \notin J_u$. Hence $u^\pm$ does not vanish on J_u. Moreover from (iv) in Lemma 2.3, we directly infer that $u^+/u^- \in \mathbf{G}_m \setminus \{1\}$ on J_u.

Finally, if $|u| \geqslant \delta > 0$ a.e. in Ω, then $|u^\pm| \geqslant \delta$ on J_u. Therefore, for every $x \in J_u$ we have

$$|u^+(x) - u^-(x)| \geqslant \delta|u^+(x)/u^-(x) - 1| \geqslant \delta \min_{k=1,\ldots,m-1} |\mathbf{a}^k - 1| = \delta|\mathbf{a}-1|\,,$$

and thus $|D^j u| \geqslant \delta|\mathbf{a}-1|\mathcal{H}^1 \llcorner J_u$ by (2.2). In particular, $\mathcal{H}^1(J_u) < \infty$ and $u \in SBV^p(\Omega)$. □

Definition 2.5 (weak Jacobian). — *For an open set $\Omega \subseteq \mathbf{R}^2$ and $u \in BV(\Omega)$ such that $j(u) \in L^1(\Omega)$, the weak Jacobian of u is defined as the distributional curl in Ω of the vector field $j(u)$. It belongs to $(C_0^{0,1}(\Omega))^*$, and its action on a Lipschitz function $\phi \in C_0^{0,1}(\Omega)$ that vanishes on the boundary is*

$$\langle \operatorname{curl} j(u), \phi\rangle = -\int_\Omega j(u) \cdot \nabla^\perp \phi \,\mathrm{d}x\,.$$

Lemma 2.6. — *Assume that $\Omega \subseteq \mathbf{R}^2$ is simply connected. Let $u_1, u_2 \in SBV(\Omega;\mathbf{S}^1)$ be such that $\mathrm{p}(u_k) \in W^{1,1}(\Omega;\mathbf{S}^1)$ for $k = 1,2$. Then, the following properties are equivalent*

(i) curl $j(u_1) =$ curl $j(u_2)$ *in $\mathcal{D}'(\Omega)$;*

(ii) *there exist $\varphi \in W^{1,1}(\Omega)$ and a Caccioppoli partition $\{E_k\}_{k=1}^m$ of Ω (see e.g.* [4, Chapter 4, Section 4.4]*) such that*

$$u_2 = \left(\sum_{k=1}^m \mathbf{a}^k \chi_{E_k}\right) e^{i\varphi} u_1\,. \tag{2.3}$$

In addition, if $\mathrm{P}(u_1) = \mathrm{P}(u_2)$ and (i) *holds, then φ is a multiple constant of $2\pi/m$.*

Proof. — Define $\widetilde{u} := u_2\bar{u}_1 \in SBV(\Omega;\mathbf{S}^1)$ and $\widetilde{v} := \mathrm{p}(\widetilde{u}) \in W^{1,1}(\Omega;\mathbf{S}^1)$. By Corollary 2.4, we have $\mathcal{H}^1(J_{\tilde{u}}) < \infty$. Then Lemma 2.3, together with the fact that $\mathrm{p}(\widetilde{u}) = \mathrm{p}(u_2)\overline{\mathrm{p}(\mathrm{u}_1)}$, leads to

$$j(\widetilde{v}) = j\big(\mathrm{p}(u_2)\,\overline{\mathrm{p}(\mathrm{u}_1)}\,\big) = j(\mathrm{p}(u_2)) - j(\mathrm{p}(u_1)) = m\big(j(u_2) - j(u_1)\big)\,.$$

If (i) holds, then $\operatorname{curl} j(\widetilde{v}) = 0$ in $\mathcal{D}'(\Omega)$. By [30] (see also [21, Theorem 7]) there exists $\varphi \in W^{1,1}(\Omega)$ such that $\widetilde{v} = e^{im\varphi}$. Consequently, $\mathrm{p}(e^{-i\varphi}\widetilde{u}) = 1$ and thus $e^{-i\varphi}\widetilde{u} \in BV(\Omega;\mathbf{G}_m)$, so that $e^{-i\varphi}\widetilde{u} = \sum_{k=0}^{m-1}\mathbf{a}^k\chi_{E_k}$ for some Caccioppoli partition $\{E_k\}_{k=0}^{m-1}$ of Ω. This proves (2.3). When $\mathrm{p}(u_1) = \mathrm{p}(u_2)$, then $\widetilde{v} = 1$, and we infer that $\varphi(x) \in \frac{2\pi}{m}\mathbf{Z}$ for a.e. $x \in \Omega$. Since $\varphi \in W^{1,1}(\Omega)$ we conclude that φ is constant.

If (ii) holds, then for each $l \in \{1,2\}$,

$$\partial_l u_2 = \left(\sum_{k=0}^{m-1}\mathbf{a}^k\chi_{E_k}\right)e^{i\varphi}\big(\partial_l u_1 + i\partial_l\varphi u_1\big) \quad \text{a.e. in } \Omega\,.$$

Consequently, $j(u_2) = j(u_1) + \nabla\varphi$ a.e. in Ω, and (ii) follows. □

2.4. Energies, functional classes, and existence of minimizers

Throughout this section, we assume that $\Omega \subseteq \mathbf{R}^2$ is a smooth, bounded, and *simply connected* domain. For $q \in \{2,3\}$ and $\varepsilon > 0$, we consider the Ginzburg–Landau functional $E_\varepsilon : W^{1,2}(\Omega;\mathbf{R}^q) \to [0,\infty)$ defined by

$$E_\varepsilon(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{1}{4\varepsilon^2}\int_\Omega (1-|u|^2)^2\,\mathrm{d}x\,.$$

For any Borel set $A \subseteq \Omega$, we let

$$E_\varepsilon(u,A) := \frac{1}{2}\int_A |\nabla u|^2\,\mathrm{d}x + \frac{1}{4\varepsilon^2}\int_A (1-|u|^2)^2\,\mathrm{d}x\,.$$

We shall use the analogous notation for the localized version of most of the energies under consideration.

For $u \in SBV^2(\Omega)$ such that $v := \mathrm{p}(u) = u^m/|u|^{m-1} \in W^{1,2}(\Omega)$, we have (by Lemma 2.3)

$$E_\varepsilon\left(\mathrm{P}(u)\right) = \frac{1}{m^2}E_\varepsilon(v) + \frac{m^2-1}{m^2}E_\varepsilon(|v|) =: G_\varepsilon(v)\,. \tag{2.4}$$

Equivalently, the functional $G_\varepsilon : W^{1,2}(\Omega) \to [0,\infty)$ can be defined by

$$G_\varepsilon(v) = \frac{1}{2m^2}\int_\Omega |\nabla v|^2 + (m^2-1)\big|\nabla|v|\big|^2\,\mathrm{d}x + \frac{1}{4\varepsilon^2}\int_\Omega (1-|v|^2)^2\,\mathrm{d}x\,. \tag{2.5}$$

For the phase field we consider the functionals $I_\eta : W^{1,2}(\Omega;[0,1]) \to [0,\infty)$ defined for $\eta > 0$ as

$$I_\eta(\psi) := \frac{\eta}{2}\int_\Omega |\nabla\psi|^2\,\mathrm{d}x + \frac{1}{2\eta}\int_\Omega (1-\psi)^2\,\mathrm{d}x\,.$$

The classes of functions we are interested in are the following

$$\mathcal{H}(\Omega) := \Big\{(u,\psi) \in L^2(\Omega)\times W^{1,2}(\Omega;[0,1]) :\\ \mathrm{P}(u) \in W^{1,2}(\Omega;\mathcal{N}) \text{ and } \psi u \in W^{1,2}(\Omega)\Big\},$$

and

$$\mathcal{G}(\Omega) := \Big\{u \in SBV^2(\Omega) : \mathrm{P}(u) \in W^{1,2}(\Omega;\mathcal{N})\Big\}\,.$$

Notice that in the definition of $\mathcal{H}(\Omega)$, the condition $\psi u \in W^{1,2}(\Omega)$ degenerates on the set $\{\psi = 0\}$ allowing for discontinuities of u. Typically, u may jump through lines where ψ vanishes and (since $\mathrm{P}(u)$ does not jump) the jump satisfies formally the constraint $\mathrm{P}(u^+) = \mathrm{P}(u^-)$ in the spirit of Lemma 2.3 (iv).

On $\mathcal{H}(\Omega)$ and $\mathcal{G}(\Omega)$, we define the functionals $F^\eta_\varepsilon : \mathcal{H}(\Omega) \to [0,\infty)$ and $F^0_\varepsilon : \mathcal{G}(\Omega) \to [0,\infty)$ by

$$F^\eta_\varepsilon(u,\psi) := E_\varepsilon\big(\mathrm{P}(u)\big) + I_\eta(\psi) \quad \text{and} \quad F^0_\varepsilon(u) := E_\varepsilon\big(\mathrm{P}(u)\big) + \mathcal{H}^1(J_u)\,. \tag{2.6}$$

Note that F^0_ε is a functional of the type "Mumford–Shah". Indeed, by Lemma 2.3 we have

$$F^0_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\,\mathrm{d}x + \mathcal{H}^1(J_u)\,.$$

As already pointed out in the introduction, F^η_ε can be seen as an "Ambrosio–Tortorelli" regularization of F^0_ε (with a coupling between u and ψ in the class $\mathcal{H}(\Omega)$ rather than in the functional itself).

We aim to minimize F^η_ε and F^0_ε under a given Dirichlet condition on the boundary. We fix a smooth map $g : \partial\Omega \simeq \mathbf{S}^1 \to \mathbf{S}^1$ of topological degree $d \geqslant 0$. Accordingly, we introduce the subclasses

$$\mathcal{H}_g(\Omega) := \Big\{(u,\psi) \in \mathcal{H}(\Omega) : \psi = 1 \text{ and } \psi u = g \text{ on } \partial\Omega\Big\}\,, \tag{2.7}$$

and

$$\mathcal{G}_g(\Omega) := \Big\{u \in \mathcal{G}(\Omega) : \mathrm{P}(u) = \mathrm{P}(g) \text{ on } \partial\Omega\Big\}\,.$$

Note that in $\mathcal{G}_g(\Omega)$ we do not impose the condition $u = g$ on $\partial\Omega$. Instead we penalize deviations from g minimizing over $\mathcal{G}_g(\Omega)$ the functional

$$F^0_{\varepsilon,g}(u) := F^0_\varepsilon(u) + \mathcal{H}^1\big(\{u \neq g\} \cap \partial\Omega\big) \tag{2.8}$$

in place of F^0_ε. As already mentioned in the introduction, this penalization is necessary to ensure lower semi-continuity (since $F^0_{\varepsilon,g}$ is precisely the $L^1(\Omega)$-relaxation of F^0_ε).

As a warm-up, let us prove that the functionals F^η_ε and $F^0_{\varepsilon,g}$ admit minimizers.

Theorem 2.7. — *The functional F^η_ε admits a minimizing pair $(u_\varepsilon,\psi_\varepsilon)$ in $\mathcal{H}_g(\Omega)$. In addition, any such minimizer satisfies $\|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant 1$.*

Theorem 2.8. — *The functional $F^0_{\varepsilon,g}$ has a minimizer u_ε in $\mathcal{G}_g(\Omega)$. In addition, any such minimizer satisfies $\|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant 1$.*

The proof of Theorem 2.7 rests on the following compactness result.

Proposition 2.9. — *Let $\Omega \subseteq \mathbf{R}^2$ be a bounded open subset. Let $\{(u_h,\psi_h)\}_{h\in\mathbf{N}} \subseteq \mathcal{H}(\Omega)$ be such that*

$$\sup_h \Big\{ \|u_h\|_{L^\infty(\Omega)} + \|\nabla\psi_h\|_{L^2(\Omega)} + \big\|\nabla(\mathrm{P}(u_h))\big\|_{L^2(\Omega)} \Big\} < \infty\,.$$

Then, there exist a subsequence and $(u,\psi)\in\mathcal{H}(\Omega)$ such that

$$\big(\psi_h,\psi_h u_h,\mathrm{P}(u_h)\big) \rightharpoonup \big(\psi,\psi u,\mathrm{P}(u)\big) \quad \textit{weakly in } W^{1,2}(\Omega)\,.$$

Proof. — Set $\phi_h := \psi_h u_h$, $V_h := \mathrm{P}(u_h)$, and notice that $\mathrm{P}(\phi_h) = \psi_h V_h$. Therefore,

$$\nabla(\mathrm{P}(\phi_h)) = \psi_h \nabla V_h + \nabla\psi_h \otimes V_h\,.$$

By (2.1), $|\nabla\phi_h| = \big|\nabla(\mathrm{P}(\phi_h))\big|$ a.e. in Ω, and since $0 \leqslant \psi_h \leqslant 1$, we infer that

$$\int_\Omega |\nabla\phi_h|^2\,\mathrm{d}x \leqslant 2\int_\Omega |\nabla V_h|^2\,\mathrm{d}x + 2\|u_h\|^2_{L^\infty(\Omega)} \int_\Omega |\nabla\psi_h|^2\,\mathrm{d}x\,.$$

Hence $\{\psi_h\}$, $\{V_h\}$, and $\{\phi_h\}$ are bounded in $W^{1,2}(\Omega)$. Thus, we can find a subsequence such that $(\psi_h,\phi_h,V_h) \rightharpoonup (\psi,\phi,V)$ weakly in $W^{1,2}(\Omega)$ and a.e. in Ω, for some $(\psi,\phi,V)\in W^{1,2}(\Omega;[0,1])\times W^{1,2}(\Omega)\times W^{1,2}(\Omega;\mathcal{N})$. On the one hand, since $\{u_h\}$ is bounded in $L^\infty(\Omega)$, the sequence $\{u_h(x)\}$ is bounded for a.e. $x\in\Omega$, and we deduce that $\phi(x) = \lim_h \psi_h(x)u_h(x) = 0$ for a.e. $x\in\{\psi=0\}$. On the other hand, one has $\lim_h u_h(x) = \phi(x)/\psi(x)$ and $V(x) = \lim_h V_h(x) = \mathrm{P}(\phi(x)/\psi(x))$ for a.e. $x\in\{\psi\neq 0\}$. Now, we define $u\in L^\infty(\Omega)$ by setting

$$u := \frac{\phi}{\psi}\chi_{\{\psi\neq 0\}} + \boldsymbol{\alpha}(V)\chi_{\{\psi=0\}}\,,$$

where $\boldsymbol{\alpha}:\mathcal{N}\to\mathbf{C}$ is the "unrolling map" of the cone $\mathcal{N}$, i.e.,

$$\boldsymbol{\alpha}(z,t) := \begin{cases} m|z|e^{i\theta/m} & \text{for } z = |z|e^{i\theta}\in\mathbf{C}\setminus\{0\} \text{ with } \theta\in[0,2\pi)\,,\\ 0 & \text{otherwise}\,. \end{cases}$$

By construction, we have $\phi = \psi u$ and $V = \mathrm{P}(u)$, and the proof is complete. $\square$

Proof of Theorem 2.7. — Let $\{(u_h, \psi_h)\} \subseteq \mathcal{H}_g(\Omega)$ be a minimizing sequence for F^η_ε in $\mathcal{H}_g(\Omega)$. Since $\mathrm{P}(u_h) \in W^{1,2}(\Omega; \mathbf{R}^3)$, we have $|u_h| \in W^{1,2}(\Omega)$ and thus also $[\max(1, |u_h|)]^{-1} \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$. Since $\psi_h = 1$ and $\psi_h u_h = g$ on $\partial\Omega$, we have $|u_h| = 1$ on $\partial\Omega$ and then also $[\max(1, |u_h|)]^{-1} = 1$ on $\partial\Omega$. Consequently, setting

$$\widehat{u}_h := \frac{u_h}{\max(1, |u_h|)}\,, \tag{2.9}$$

it quickly follows that $\psi_h \widehat{u}_h \in W^{1,2}(\Omega)$, $\psi_h \widehat{u}_h = g$ on $\partial\Omega$, and $\|\widehat{u}_h\|_{L^\infty(\Omega)} \leqslant 1$. Since also,

$$\mathrm{P}(\widehat{u}_h) = \frac{\mathrm{P}(u_h)}{\max(1, |u_h|)} = \frac{\mathrm{P}(u_h)}{\max(1, |\mathrm{P}(u_h)|)} \in W^{1,2}(\Omega; \mathcal{N})\,, \tag{2.10}$$

we have $(\psi_h, \widehat{u}_h) \in \mathcal{H}_g(\Omega)$. Moreover, (2.10) implies that $E_\varepsilon(\mathrm{P}(\widehat{u}_h)) \leqslant E_\varepsilon(\mathrm{P}(u_h))$ with equality if and only if $|\mathrm{P}(u_h)| \leqslant 1$ a.e. in Ω (and since $|\mathrm{P}(u_h)| = |u_h|$, equality holds if and only if $|u_h| \leqslant 1$ a.e. in Ω).

As a consequence, $F^\eta_\varepsilon(\widehat{u}_h, \psi_h) \leqslant F^\eta_\varepsilon(u_h, \psi_h)$, and thus $\{(\widehat{u}_h, \psi_h)\}$ is also a minimizing sequence for F^η_ε in $\mathcal{H}_g(\Omega)$. Since $\|\widehat{u}_h\|_{L^\infty(\Omega)} \leqslant 1$, we can apply Proposition 2.9 to find a subsequence such that $(\psi_h, \psi_h \widehat{u}_h, \mathrm{P}(\widehat{u}_h)) \rightharpoonup (\psi_\varepsilon, \psi_\varepsilon u_\varepsilon, \mathrm{P}(u_\varepsilon))$ weakly in $W^{1,2}(\Omega)$ for some $(u_\varepsilon, \psi_\varepsilon) \in \mathcal{H}(\Omega)$ with $|\mathrm{P}(u_\varepsilon)| = |u_\varepsilon| \leqslant 1$ a.e. in Ω. From the continuity of the trace operator, we deduce that $\psi_\varepsilon = 1$ and $\psi_\varepsilon u_\varepsilon = g$ on $\partial\Omega$, that is $(u_\varepsilon, \psi_\varepsilon) \in \mathcal{H}_g(\Omega)$. Finally, the functional F^η_ε being clearly lower semi-continuous with respect to the weak convergence in $W^{1,2}(\Omega)$, we conclude that $(u_\varepsilon, \psi_\varepsilon)$ minimizes F^η_ε over $\mathcal{H}_g(\Omega)$. Since the truncation argument above shows that any minimizer satisfies the announced L^∞ bound, the proof is complete. $\square$

Proof of Theorem 2.8. — The truncation argument is identical to the one above so we may reduce ourselves to the class of functions u satisfying $\|u\|_{L^\infty(\Omega)} \leqslant 1$. Let $\{u_h\} \subseteq \mathcal{G}_g(\Omega)$ be a minimizing sequence for $F^0_{\varepsilon,g}$.

We fix some $r_0 > 0$ small enough in such a way that

$$\widetilde{\Omega} := \left\{x \in \mathbf{R}^2 : \mathrm{dist}(x, \Omega) < r_0\right\} \tag{2.11}$$

defines a smooth domain, and that the nearest point projection on $\partial\Omega$, denoted by Π, is well defined and smooth in $\left\{x \in \mathbf{R}^2 : \mathrm{dist}(x, \partial\Omega) < 2r_0\right\}$. We extend each u_h to $\widetilde{\Omega}$ by setting $u_h(x) = g\big(\Pi(x)\big)$ for $x \in \widetilde{\Omega} \setminus \Omega$. Then we have $J_{u_h} \cap \widetilde{\Omega} = (J_{u_h} \cap \Omega) \cup (\{u_h \neq g\} \cap \partial\Omega)$, so that

$$F^0_\varepsilon(u_h, \widetilde{\Omega}) = F^0_{\varepsilon,g}(u_h) + C_g\,,$$

for a constant C_g depending only on g, r_0, and Ω. Since $|\nabla u_h| = |\nabla(\mathrm{P}(u_h))|$ by Lemma 2.3, we deduce that $\{\nabla u_h\}$ is bounded in $L^2(\widetilde{\Omega})$. Hence we can apply [4, Theorem 4.7 and 4.8] to find a subsequence such that $u_h \rightharpoonup u_\varepsilon$ weakly* in $BV(\widetilde{\Omega})$ and a.e. in Ω to some $u_\varepsilon \in SBV^2(\widetilde{\Omega})$. From the a.e. convergence, we deduce that $u_\varepsilon(x) = g\big(\Pi(x)\big)$ for $x \in \widetilde{\Omega} \setminus \Omega$. Then, still by [4, Theorem 4.7],

$$\begin{aligned}(2.12)\quad \liminf_{h\to\infty} \mathcal{H}^1(J_{u_h}\cap\Omega) + \mathcal{H}^1\big(\{u_h \neq g\}\cap\partial\Omega\big) &= \liminf_{h\to\infty} \mathcal{H}^1(J_{u_h}\cap\widetilde{\Omega}) \\ &\geqslant \mathcal{H}^1(J_{u_\varepsilon}\cap\widetilde{\Omega}) = \mathcal{H}^1(J_{u_\varepsilon}\cap\Omega) + \mathcal{H}^1\big(\{u_\varepsilon \neq g\}\cap\partial\Omega\big).\end{aligned}$$

Since $\{\mathrm{P}(u_h)\}$ is bounded in $W^{1,2}(\Omega)$ and $\mathrm{P}(u_h) \to \mathrm{P}(u_\varepsilon)$ a.e. in Ω, we infer that $\mathrm{P}(u_h) \rightharpoonup \mathrm{P}(u_\varepsilon)$ weakly in $W^{1,2}(\Omega)$. As a consequence, $u_\varepsilon \in \mathcal{G}_g(\Omega)$. Finally, the lower semi-continuity of E_ε with respect to the weak $W^{1,2}$-convergence, together with (2.12), leads to $F^0_{\varepsilon,g}(u_\varepsilon) \leqslant \liminf_h F^0_{\varepsilon,g}(u_h)$. Hence u_ε is a minimizer of $F^0_{\varepsilon,g}$ in $\mathcal{G}_g(\Omega)$. □

2.5. Asymptotic for the Ginzburg–Landau functional

The aim of this subsection is to recall some classical facts about the asymptotic limit as $\varepsilon \downarrow 0$ of low energy states for the Ginzburg–Landau functional E_ε. In this section we still assume that $\Omega \subseteq \mathbf{R}^2$ is a smooth, bounded, and *simply connected* domain. Some of the material below can be found with greater details in [2, 13, 54] and the references therein. We start with the notion of renormalized energy originally introduced in [13].

2.5.1. The renormalized energy and canonical harmonic maps

Let us denote by $\mathcal{A}_d$ the set of all finite positive measures μ of the form

$$(2.13)\qquad \mu = 2\pi \sum_{k=1}^{md} \delta_{x_k}\,,$$

for some md distinct points $\{x_1, \dots, x_{md}\} \subseteq \Omega$.

Given $\mu \in \mathcal{A}_d$, the canonical harmonic map $v_\mu : \Omega \setminus \operatorname{spt}\mu \to \mathbf{C}$ associated to μ is the map defined by

$$(2.14)\qquad v_\mu(x) := e^{i\varphi_\mu(x)} \prod_{k=1}^{md} \frac{x - x_k}{|x - x_k|}\,,$$

with

$$\begin{cases} \Delta\varphi_\mu = 0 & \text{in } \Omega\,,\\ v_\mu = g^m & \text{on } \partial\Omega\,. \end{cases}$$

Note that φ_μ is a smooth function in $\overline{\Omega}$ uniquely determined up to constant multiple of 2π. The canonical map v_μ is a smooth harmonic map from $\Omega \setminus \operatorname{spt}\mu$ into $\mathbf{S}^1$. It satisfies

$$\begin{cases} \operatorname{div} j(v_\mu) = 0 \\ \operatorname{curl} j(v_\mu) = \mu \end{cases} \qquad \text{in } \mathcal{D}'(\Omega)\,.$$

It turns out that $v_\mu \in W^{1,p}(\Omega)$ for every $p \in [1,2)$, but fails to be in $W^{1,2}(\Omega)$. However, the Dirichlet energy of v_μ still have a well defined finite part called *the renormalized energy* given by

$$\begin{aligned}(2.15)\quad \mathbb{W}(\mu) := -\pi \sum_{k\neq l} \log|x_k - x_l| \\ + \frac{1}{2}\int_{\partial\Omega} g^m \wedge \frac{\partial(g^m)}{\partial\tau}\,\Phi_\mu \,\mathrm{d}\mathcal{H}^1 - \pi\sum_{k=1}^{md} R_\mu(x_k)\,,\end{aligned}$$

where Φ_μ is the solution of

$$\begin{cases} \Delta\Phi_\mu = \mu & \text{in } \Omega\,,\\ \frac{\partial\Phi_\mu}{\partial\nu} = g^m \wedge \frac{\partial(g^m)}{\partial\tau} & \text{on } \partial\Omega\,,\\ \int_{\partial\Omega}\Phi_\mu\,\mathrm{d}\mathcal{H}^1 = 0\,, \end{cases}$$

and $R_\mu(x) := \Phi_\mu(x) - \sum_k \log|x - x_k|$. Note that R_μ is an harmonic function in Ω, smooth up to $\partial\Omega$. The function Φ_μ is related to the harmonic map v_μ through the relation

$$(2.16)\qquad j(v_\mu) = \nabla^\perp\Phi_\mu\,,$$

and $\mathbb{W}(\mu)$ is the finite part of the Dirichlet energy of v_μ in the sense that

$$(2.17)\qquad \lim_{r\downarrow 0}\left\{\frac{1}{2}\int_{\Omega\setminus B_r(\mu)} |\nabla v_\mu|^2\,\mathrm{d}x - \pi m d|\log r|\right\} = \mathbb{W}(\mu)\,.$$

2.5.2. Asymptotic for low energy states

We are now ready to state the following compactness result, which is a slight improvement of [2, Theorem 6.1]. The proof is postponed to the end of this section.

Theorem 2.10. — *For a sequence $\varepsilon_h \downarrow 0$, let $\{v_h\} \subseteq W^{1,2}_{g^m}(\Omega)$ be such that $\{v_h\}$ is bounded in $L^\infty(\Omega)$, and*

$$E_{\varepsilon_h}(v_h) \leqslant \pi m d|\log \varepsilon_h| + O(1) \quad \text{as } h \to \infty\,. \tag{2.18}$$

There exist a subsequence, a measure $\mu \in \mathcal{A}_d$, and a phase $\varphi \in W^{1,2}(\Omega)$ such that

(i) $v_h \rightharpoonup e^{i\varphi} v_\mu$ *weakly in* $W^{1,p}(\Omega)$ *for every* $p \in [1,2)$*;*
(ii) $v_h \rightharpoonup e^{i\varphi} v_\mu$ *weakly in* $W^{1,2}_{\mathrm{loc}}(\overline{\Omega} \setminus \operatorname{spt}\mu)$*;*
(iii) $e^{i\varphi} = 1$ *on* $\partial\Omega$*;*
(iv) *for* $r > 0$ *small enough,*

$$\liminf_{h\to\infty} \left\{ E_{\varepsilon_h}\big(v_h, B_r(\mu)\big) - \pi m d \log \frac{r}{\varepsilon_h} \right\} \geqslant C_*\,, \tag{2.19}$$

for a constant C_* independent of r, and

$$\liminf_{r\downarrow 0} \liminf_{h\to\infty} \left\{ \frac{1}{2} \int_{\Omega\setminus B_r(\mu)} |\nabla v_h|^2 \,\mathrm{d}x - \pi m d |\log r| \right\} \geqslant \frac{1}{2}\int_\Omega |\nabla \varphi|^2 \,\mathrm{d}x + \mathbb{W}(\mu)\,.$$

Moreover, $\mu_h := \operatorname{curl}\, j(v_h) \in L^1(\Omega)$ converges to $\mu = \operatorname{curl}\, j(e^{i\varphi} v_\mu)$ in the weak topology of $(C^{0,1}_0(\Omega))^*$.*

The proof of this theorem relies on the following two auxiliary results. In particular, Lemma 2.12 provides an a priori $W^{1,p}$-bound for sequences of low Ginzburg–Landau energy. We believe that Proposition 2.11 and Lemma 2.12 are already well known to experts (see in particular [24, Theorem 1.4.4]). Since we did not find clear statements and proofs in the existing literature, we have decided to provide here (mostly) self-contained proofs.

Proposition 2.11. — *Let $v \in W^{1,1}(\Omega; \mathbf{S}^1)$ and $\mu \in \mathcal{A}_d$ be such that*

$$\begin{cases} \operatorname{curl}\, j(v) = \mu & \text{in } \mathcal{D}'(\Omega)\,, \\ v = g^m & \text{on } \partial\Omega\,. \end{cases}$$

If $v \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega}\setminus \operatorname{spt}\mu)$ and

$$\liminf_{r\downarrow 0} \left\{ \frac{1}{2} \int_{\Omega\setminus B_r(\mu)} |\nabla v|^2 \,\mathrm{d}x - \pi m d |\log r| \right\} < \infty\,, \tag{2.20}$$

then $v = e^{i\varphi} v_\mu$ for some $\varphi \in W^{1,2}(\Omega)$ such that $e^{i\varphi} = 1$ on $\partial\Omega$. In addition,

$$\lim_{r\downarrow 0} \left\{ \frac{1}{2} \int_{\Omega\setminus B_r(\mu)} |\nabla v|^2 \,\mathrm{d}x - \pi m d |\log r| \right\} = \frac{1}{2}\int_\Omega |\nabla \varphi|^2 \,\mathrm{d}x + \mathbb{W}(\mu)\,.$$

Proof. — The fact that $v = e^{i\varphi}v_\mu$ for some $\varphi \in W^{1,1}(\Omega)$ with $e^{i\varphi} = 1$ on $\partial\Omega$ follows as in the proof of Lemma 2.6. Moreover, $v \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega} \setminus \operatorname{spt}\mu; \mathbf{S}^1)$ yields $\varphi \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega} \setminus \operatorname{spt}\mu)$. Let us prove that in fact $\varphi \in W^{1,2}(\Omega)$. First notice that

$$\begin{aligned}(2.21)\quad |\nabla v|^2 = |j(v)|^2 &= |\nabla\varphi|^2 + |j(v_\mu)|^2 + 2\nabla\varphi \cdot j(v_\mu) \\ &= |\nabla\varphi|^2 + |\nabla v_\mu|^2 + 2\nabla\varphi \cdot \nabla^\perp\Phi_\mu\,,\end{aligned}$$

where the last identity follows from (2.16).

For each $k \in \{1, \dots, md\}$, we set

$$R^k_\mu(x) := \Phi_\mu(x) - \log|x - x_k|\,,$$

so that R^k_μ is a smooth harmonic function in $\overline{\Omega} \setminus \bigcup_{l\neq k}\{x_l\}$. Notice in particular that

$$\partial_\tau\Phi_\mu = \partial_\tau R^k_\mu \qquad \text{on } \partial B_r(x_k).$$

Integrating by parts (2.21) in $\Omega_r := \Omega \setminus B_r(\mu)$ with $r > 0$ small enough, leads to

$$\begin{aligned}\int_{\Omega_r} |\nabla v|^2\,\mathrm{d}x &= \int_{\Omega_r} |\nabla\varphi|^2\,\mathrm{d}x + \int_{\Omega_r} |\nabla v_\mu|^2\,\mathrm{d}x + 2\sum_{k=1}^{md}\int_{\partial B_r(x_k)} \varphi\partial_\tau\Phi_\mu\,\mathrm{d}\mathcal{H}^1 \\ (2.22)\qquad &= \int_{\Omega_r} |\nabla\varphi|^2\,\mathrm{d}x + \int_{\Omega_r} |\nabla v_\mu|^2\,\mathrm{d}x + 2\sum_{k=1}^{md}\int_{\partial B_r(x_k)} \varphi\partial_\tau R^k_\mu\,\mathrm{d}\mathcal{H}^1\,.\end{aligned}$$

By the boundary trace theorem for BV functions [4, Theorem 3.87], and the embedding of $W^{1,1}$ into L^2,

$$\begin{aligned}(2.23)\quad \int_{\partial B_r(x_k)} |\varphi|\,\mathrm{d}\mathcal{H}^1 &\lesssim \int_{B_r(x_k)} |\nabla\varphi| + \frac{1}{r}|\varphi|\,\mathrm{d}x \\ &\lesssim \int_{B_r(x_k)} |\nabla\varphi|\mathrm{d}x + \left(\int_{B_r(x_k)} |\varphi|^2\mathrm{d}x\right)^{1/2} \underset{r\to 0}{\longrightarrow} 0\,.\end{aligned}$$

Using the smoothness of R^k_μ near x_k, we can combine (2.17), (2.20), (2.22), and (2.23) to deduce that

$$\int_{\Omega_r} |\nabla\varphi|^2\,\mathrm{d}x = O(1) \quad \text{as } r \to 0\,.$$

Therefore $\varphi \in W^{1,2}(\Omega)$. Going back to (2.22), we subtract $\pi md|\log r|$ from both sides of this identity, and we let $r \to 0$ to reach the conclusion. □

Lemma 2.12. — *For a sequence $\varepsilon_h \downarrow 0$, let $\{v_h\} \subseteq W^{1,2}_{g^m}(\Omega, \mathbf{C})$ be such that $\{v_h\}$ is bounded in $L^\infty(\Omega)$, (2.18) holds, and for which $\mu_h := j(v_h)$ weakly* converges in $(C^{0,1}_0(\Omega))^*$ to some measure $\mu \in \mathcal{A}_d$ as $h \to \infty$. Then $\{v_h\}$ is bounded in $W^{1,p}(\Omega)$ for every $1 \leqslant p < 2$.*

The proof of Lemma 2.12 rests on the so-called "ball construction" in [54, Theorem 4.1] that we now recall.

Theorem 2.13 ([54]). — *For any $\alpha \in (0,1)$ there exists $\varepsilon_0(\alpha) > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0(\alpha))$ and any $v \in C^\infty(\Omega)$ satisfying $E_\varepsilon(v) \leqslant \varepsilon^{\alpha-1}$, the following holds for some universal constants c_0, c_1, and c_2: for any $r \in \left[c_0\varepsilon^{\alpha/2}, 1\right)$ there exists a finite collection $\mathcal{B}_r = \left\{B_j\right\}_{j\in J}$ of disjoint closed balls such that*

(i) $r = \sum_j r_j$;

(ii) *setting* $\Omega_\varepsilon := \left\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\right\}$ *and* $V^\varepsilon_r := \Omega_\varepsilon \cap \left(\bigcup_j B_j\right)$,

$$\left\{x \in \Omega_\varepsilon : \big||v(x)| - 1\big| \geqslant \varepsilon^{\alpha/4}\right\} \subseteq V^\varepsilon_r \,;$$

(iii) *setting* $d_j = \deg(v, \partial B_j)$ *if* $B_j \subseteq \Omega_\varepsilon$, *and* $d_j = 0$ *otherwise,*

$$E_\varepsilon(v, V^\varepsilon_r) \geqslant \pi D_r \left(\log \frac{r}{D_r\varepsilon} - c_1\right) \tag{2.24}$$

whenever $D_r := \sum_j |d_j| \neq 0$;

(iv) *the following estimate holds*

$$D_r \leqslant c_2 \frac{E_\varepsilon(v)}{\alpha|\log\varepsilon|}\,. \tag{2.25}$$

Finally, if $r_1 < r_2$, then every ball of $\mathcal{B}_{r_1}$ is contained in a ball of $\mathcal{B}_{r_2}$.

Proof of Lemma 2.12. — Since Ω is a smooth bounded domain and g is smooth, any map in $W^{1,2}_{g^m}(\Omega) \cap L^\infty(\Omega)$ can be (strongly) approximated in the $W^{1,2}$-sense by a sequence in $\{v \in C^\infty(\Omega) : v = g^m \text{ on } \partial\Omega\}$ which also remains bounded in $L^\infty(\Omega)$. Hence, we can assume $v_h \in C^\infty(\Omega)$ for each h. Recall that μ writes $\mu = 2\pi\sum_{k=1}^{md} \delta_{x_k}$. Setting

$$\sigma_0 := \frac{1}{4}\min\left\{1, \min_k \operatorname{dist}(x_k, \partial\Omega), \min_{k\neq l}|x_k - x_l|\right\},$$

we may assume without loss of generality that $\sigma_0 = 1$. We choose $\alpha = 1/2$ in Theorem 2.13 (this choice of α is arbitrary). By (2.18), we have $E_{\varepsilon_h}(v_h) \leqslant \varepsilon_h^{-1/2}$ for ε_h small enough, and we can therefore apply Theorem 2.13 to v_h.

We claim that for ε_h sufficiently small,

$$D_r \geqslant md \quad \text{for every } r \in \left[c_0\varepsilon_h^{1/4}, 1/6\right]. \tag{2.26}$$

Let us introduce the modified function

$$\widetilde{v}_h := \min\left\{\frac{|v_h|}{1-\varepsilon_h^{1/8}}, 1\right\}\frac{v_h}{|v_h|} \in W^{1,2}(\Omega)\,.$$

Noticing that

$$j(\widetilde{v}_h) = \min\left\{\frac{1}{\big(1-\varepsilon_h^{1/8}\big)^2}, \frac{1}{|v_h|^2}\right\} j(v_h)\,,$$

and setting $\widetilde{\mu}_h := \operatorname{curl} j(\widetilde{v}_h)$, we estimate

$$\begin{aligned}
&\|\widetilde{\mu}_h - \mu_h\|_{(C_0^{0,1}(\Omega))^*} \\
&\quad = \sup_{\|\phi\|_{C_0^{0,1}(\Omega))}\leqslant 1} \int_\Omega \left(\min\left\{\frac{1}{\big(1-\varepsilon_h^{1/8}\big)^2}, \frac{1}{|v_h|^2}\right\} - 1\right) j(v_h)\cdot\nabla^\perp\phi\,\mathrm{d}x \\
&\quad \leqslant \int_\Omega \left|\min\left\{\frac{1}{\big(1-\varepsilon_h^{1/8}\big)^2}, \frac{1}{|v_h|^2}\right\} - 1\right| |j(v_h)|\mathrm{d}x \\
&\quad \lesssim \varepsilon_h^{1/8}\|j(v_h)\|_{L^1(\Omega)} \underset{h\to\infty}{\longrightarrow} 0\,, \qquad (2.27)
\end{aligned}$$

where in the last step we have used that since $j(v_h) = v_h \wedge \nabla v_h$,

$$\begin{aligned}
\|j(v_h)\|_{L^1(\Omega)} &\lesssim \|v_h\|_{L^\infty(\Omega)}\|\nabla v_h\|_{L^\infty(\Omega)} \\
&\lesssim \|v_h\|_{L^\infty(\Omega)} E_{\varepsilon_h}^{1/2}(v_h) \overset{(2.18)}{\lesssim} \|v_h\|_{L^\infty(\Omega)}|\log\varepsilon_h|^{1/2}\,.
\end{aligned}$$

Given $r \in \big[c_0\varepsilon_h^{1/4}, 1/6\big]$, we set

$$A_r^{\varepsilon_h} := \big\{t \in [1/2, 1-r] : \partial B_t(\mu)\cap V_r^{\varepsilon_h} = \emptyset\big\}\,.$$

By item (i) in Theorem 2.13, we have $|A_r^{\varepsilon_h}| \geqslant 1/2 - 2r \geqslant 1/6$. Then, for each $k = 1, \dots, md$, we define a function $\zeta_k \in C^{0,1}(\Omega)$ compactly supported in Ω by setting

$$\zeta_k(x) := \int_{\min(1,|x-x_k|)}^1 \chi_{A_r^{\varepsilon_h}}(t)\,\mathrm{d}t\,.$$

Notice that $\|\zeta_k\|_{C^{0,1}(\Omega)} \leqslant 2$. Since $\widetilde{\mu}_h \to \mu$ in $(C_0^{0,1}(\Omega))^*$ by (2.27), we have $\|\widetilde{\mu}_h - \mu\|_{(C_0^{0,1}(\Omega))^*} \leqslant \pi/12$ for ε_h small enough. Consequently, by definition of σ_0 and for ε_h small,

$$\langle\widetilde{\mu}_h, \zeta_k\rangle \geqslant \langle\mu, \zeta_k\rangle - 2\|\widetilde{\mu}_h - \mu\|_{(C_0^{0,1}(\Omega))^*} \geqslant 2\pi|A_r^{\varepsilon_h}| - \pi/6 \geqslant \pi/6\,,$$

for each $k=1,\dots,md$. Moreover, using that for $t\in A_r^{\varepsilon_h}$ and $x\in\partial B_t(x_k)$, $\widetilde{v}_h=\frac{v_h}{|v_h|}$, we have

$$\begin{aligned}\langle\widetilde{\mu}_h,\zeta_k\rangle&=-\int_{B_1(x_k)}j(\widetilde{v}_h)\cdot\nabla^\perp\zeta_k\,\mathrm{d}x\\&=\int_0^1\chi_{A_r^{\varepsilon_h}}(t)\left(\int_{\partial B_t(x_k)}j(\widetilde{v}_h)\cdot\tau\,\mathrm{d}\mathcal{H}^1\right)\mathrm{d}t\\&=\int_{A_r^{\varepsilon_h}}\left(\int_{\partial B_t(x_k)}j\left(\frac{v_h}{|v_h|}\right)\cdot\tau\,\mathrm{d}\mathcal{H}^1\right)\mathrm{d}t\\&=2\pi\int_{A_r^{\varepsilon_h}}\deg\big(v_h,\partial B_t(x_k)\big)\,\mathrm{d}t\,,\end{aligned}$$

and we conclude that for ε_h sufficiently small (independently of r),

$$\int_{A_r^{\varepsilon_h}}\deg\big(v_h,\partial B_t(x_k)\big)\,\mathrm{d}t\geqslant 1/12\quad\text{for each }k=1,\dots,md\,.$$

Hence, for each $k=1,\dots,md$, there exists a radius $\rho_h^k\in A_r^{\varepsilon_h}$ such that $\deg\big(v_h,\partial B_{\rho_h^k}(x_k)\big)\neq 0$ whenever ε_h is small enough (independently of r). In turn, it implies the existence, for each $k=1,\dots,md$, of an element $B_h^k(r)\in\mathcal{B}_r$ such that $B_h^k(r)\subseteq B_{\rho_h^k}(x_k)\subseteq\Omega_{\varepsilon_h}$ and $\deg\big(v_h,\partial B_h^k(r)\big)\neq 0$, whenever ε_h is small. By the very definition of D_r, we infer that (2.26) holds for ε_h small (independently of r).

Combining (2.18) and (2.25), we deduce that $D_r\leqslant C$ for some constant C independent of ε_h and r. Then, (2.24) yields for ε_h small enough,

$$E_{\varepsilon_h}(v_h,V_r^{\varepsilon_h})\geqslant\pi md\log\left(\frac{r}{\varepsilon_h}\right)-C\quad\text{for every }r\in\left[c_0\varepsilon_h^{1/4},1/6\right],$$

where C is still a constant independent of r and ε_h. In view of (2.18), we thus have

$$E_{\varepsilon_h}(v_h,\Omega\setminus V_r^{\varepsilon_h})\leqslant\pi md|\log r|+C\quad\text{for every }r\in\left[c_0\varepsilon_h^{1/4},1/6\right].\tag{2.28}$$

Now we define on the set $\{|v_h|>0\}$ the map $\widehat{v}_h:=v_h/|v_h|$. Given $r\in\left[c_0\varepsilon_h^{1/4},1/12\right]$, we have $\big||v_h|-1\big|\leqslant\varepsilon_h^{1/8}$ on $V_{2r}^{\varepsilon_h}\setminus V_r^{\varepsilon_h}$, and we can apply [54, Proposition 4.2] to deduce that for ε_h sufficiently small (independently of r),

$$\frac12\int_{V_{2r}^{\varepsilon_h}\setminus V_r^{\varepsilon_h}}|\nabla\widehat{v}_h|^2\,\mathrm{d}x\geqslant\sum_{k=1}^{md}\frac12\int_{B_h^k(2r)\setminus V_r^{\varepsilon_h}}|\nabla\widehat{v}_h|^2\,\mathrm{d}x\geqslant\pi md\log 2\,.$$

Therefore, if ε_h is small, using that

$$|\nabla v_h|^2=|\nabla|v_h||^2+|v_h|^2|\nabla\widehat{v}_h|^2\geqslant|v_h|^2|\nabla\widehat{v}_h|^2;$$

we obtain

$$\begin{aligned}(2.29)\quad \frac{1}{2}\int_{V^{\varepsilon_h}_{2r}\setminus V^{\varepsilon_h}_{r}} |\nabla v_h|^2\,\mathrm{d}x \\ \geqslant \frac{1}{2}\int_{V^{\varepsilon_h}_{2r}\setminus V^{\varepsilon_h}_{r}} |v_h|^2|\nabla \widehat{v}_h|^2\,\mathrm{d}x \geqslant \pi m d\log 2 - C\varepsilon_h^{1/8}\,,\end{aligned}$$

for some constant C independent of r and ε_h. Then set for $j\in\mathbf{N}$, $r_j := 2^{-j}/6$ and define

$$J_h := \max\left\{j\in\mathbf{N} : r_j \geqslant c_0\varepsilon_h^{1/4}\right\}.$$

Using the fact that $V^{\varepsilon_h}_{r_{j+1}}\subseteq V^{\varepsilon_h}_{r_j}$, estimate (2.28) leads to

$$\begin{aligned}(2.30)\quad \sum_{j=0}^{J_h-1}\frac{1}{2}\int_{V^{\varepsilon_h}_{r_j}\setminus V^{\varepsilon_h}_{r_{j+1}}} |\nabla v_h|^2\,\mathrm{d}x \\ \leqslant E_{\varepsilon_h}(v_h,\Omega\setminus V^{\varepsilon_h}_{r_{J_h}}) \leqslant \big(\pi m d\log 2\big)J_h + C\,.\end{aligned}$$

Since $J_h = O(|\log\varepsilon_h|)$, we infer from (2.29) and (2.30) that

$$\begin{aligned}(2.31)\quad \int_{V^{\varepsilon_h}_{r_j}\setminus V^{\varepsilon_h}_{r_{j+1}}} |\nabla v_h|^2\,\mathrm{d}x \leqslant C(1+J_h\varepsilon_h^{1/8}) \leqslant C \\ \text{for every } j=0,\ldots,J_h-1\,,\end{aligned}$$

for a constant C independent of ε_h.

Finally, fix an arbitrary $p\in[1,2)$. Noticing that $|V^{\varepsilon_h}_{r_j}| = O(r_j^2)$, we estimate by means of (2.28), (2.31), and Hölder's inequality,

$$\begin{aligned}\int_{\Omega\setminus V^{\varepsilon_h}_{r_{J_h}}} |\nabla v_h|^p\,\mathrm{d}x &\leqslant \int_{\Omega\setminus V^{\varepsilon_h}_{r_0}} |\nabla v_h|^p\,\mathrm{d}x + \sum_{k=0}^{J_h-1}\int_{V^{\varepsilon_h}_{r_j}\setminus V^{\varepsilon_h}_{r_{j+1}}} |\nabla v_h|^p\,\mathrm{d}x \\ &\leqslant C\left(1+\sum_{k=0}^{J_h-1} r_j^{2-p}\left(\int_{V^{\varepsilon_h}_{r_j}\setminus V^{\varepsilon_h}_{r_{j+1}}} |\nabla v_h|^2\,\mathrm{d}x\right)^{p/2}\right) \leqslant \frac{C}{2^{2-p}-1}\,,\end{aligned}$$

for some constant C independent of ε_h (and p). Since,

$$\int_{V^{\varepsilon_h}_{r_{J_h}}} |\nabla v_h|^p\,\mathrm{d}x \leqslant C r_{J_h}^{2-p}\left(\int_\Omega |\nabla v_h|^2\,\mathrm{d}x\right)^{p/2} \leqslant C\varepsilon_h^{\frac{2-p}{4}}|\log\varepsilon_h|^{p/2} \leqslant C\,,$$

we conclude that $\{v_h\}$ is indeed bounded in $W^{1,p}(\Omega)$. $\square$

Proof of Theorem 2.10. — In view of (2.18), we can apply [2, Theorem 6.1] to find a subsequence such that $\mu_h \stackrel{*}{\rightharpoonup} \mu$ weakly* in $(C^{0,1}_0(\Omega))^*$ for some measure $\mu = 2\pi \sum_{k=1}^{md} \delta_{x_k} \in \mathcal{A}_d$. Moreover, for a radius r satisfying

$$0 < r \leqslant \sigma_0 := \frac{1}{4} \min \left\{ 1, \min_k \operatorname{dist}(x_k, \partial\Omega), \min_{k \neq l} |x_k - x_l| \right\},$$

estimate (2.19) holds by [2, Theorem 4.1]. Consequently,

$$E_{\varepsilon_h}\big(v_h, \Omega \setminus B_r(\mu)\big) \leqslant \pi md |\log r| + C\,, \tag{2.32}$$

for a constant C independent of r and ε_h. As a consequence of (2.32), we can extract a further subsequence such that $v_h \rightharpoonup v_0$ weakly in $W^{1,2}_{\mathrm{loc}}(\Omega \setminus \operatorname{spt} \mu)$ for some $v_0 \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus \operatorname{spt} \mu; \mathbf{S}^1)$. By lower semi-continuity we have

$$\liminf_{h \to \infty} E_{\varepsilon_h}\big(v_h, \Omega \setminus B_r(\mu)\big) \geqslant \frac{1}{2} \int_{\Omega \setminus B_r(\mu)} |\nabla v_0|^2 \, \mathrm{d}x\,. \tag{2.33}$$

In addition, from the continuity of the trace operator we deduce that $v_0 = g^m$ on $\partial\Omega$. Thanks to Lemma 2.12, $\{v_h\}$ is also bounded in $W^{1,p}(\Omega)$ for every $1 \leqslant p < 2$ so that $v_h \rightharpoonup v_0$ weakly in $W^{1,p}(\Omega)$ for every $p \in [1, 2)$. From this convergence, we easily derive

$$\langle \mu_h, \zeta \rangle = -\int_\Omega j(v_h) \cdot \nabla^\perp \zeta \, \mathrm{d}x \underset{h \to \infty}{\longrightarrow} -\int_\Omega j(v_0) \cdot \nabla^\perp \zeta \, \mathrm{d}x = \langle \operatorname{curl} j(v_0), \zeta \rangle\,,$$

for every $\zeta \in \mathcal{D}(\Omega)$, and thus $\operatorname{curl} j(v_0) = \mu$ in $\mathcal{D}'(\Omega)$. Combining (2.32) with (2.33) yields

$$\limsup_{r \downarrow 0} \left\{ \frac{1}{2} \int_{\Omega \setminus B_r(\mu)} |\nabla v_0|^2 \, \mathrm{d}x - \pi md |\log r| \right\} < \infty\,.$$

Hence, we are now in position to apply Proposition 2.11 to conclude that $v_0 = e^{i\varphi} v_\mu$ for some $\varphi \in W^{1,2}(\Omega)$ satisfying $e^{i\varphi} = 1$ on $\partial\Omega$, and the proof is complete. □

3. The Γ-convergence results

In this section, our main objective is to determine the Γ-limit of the functional F_ε^η defined in (2.6) as $\eta \downarrow 0$ and $\varepsilon \downarrow 0$. We introduce $\widetilde{F}_\varepsilon^\eta : L^1(\Omega) \times L^1(\Omega) \to (-\infty, \infty]$ given as

$$\widetilde{F}_\varepsilon^\eta(u, \psi) := \begin{cases} F_\varepsilon^\eta(u, \psi) - \frac{\pi d}{m} |\log \varepsilon| & \text{if } (u, \psi) \in \mathcal{H}_g(\Omega)\,, \\ \infty & \text{otherwise}\,, \end{cases}$$

where F_ε^η and the class $\mathcal{H}_g(\Omega)$ are defined in (2.6), (2.7) respectively. Throughout this section $\Omega \subseteq \mathbf{R}^2$ denotes a smooth, bounded, and *simply connected* domain.

In a first part, we shall prove that the domain of the Γ-limit is determined by the class of functions

$$\mathcal{L}_g(\Omega) := \Big\{ u \in SBV(\Omega;\mathbf{S}^1) : u^m = e^{i\varphi} v_\mu \text{ for some } \mu \in \mathcal{A}_d \\ \text{and } \varphi \in W^{1,2}(\Omega) \text{ satisfying } e^{i\varphi} = 1 \text{ on } \partial\Omega \Big\},$$

where $\mathcal{A}_d$ is the family of measures defined in (2.13), and v_μ is the canonical harmonic map associated to μ through (2.14). We emphasize that $\mathcal{L}_g(\Omega) \subseteq SBV^p(\Omega;\mathbf{S}^1)$ for every $p \in [1,2)$ by Corollary 2.4. In turn, the Γ-limit is given by the functional $F_{0,g} : \mathcal{L}_g(\Omega) \to \mathbf{R}$ defined by

$$F_{0,g}(u) := E_0(u) + \mathcal{H}^1(J_u) + \mathcal{H}^1\big(\{u \neq g\} \cap \partial\Omega\big)\,,$$

where we have set for $u^m =: e^{i\varphi} v_\mu$,

$$E_0(u) := \frac{1}{2m^2}\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x + \frac{1}{m^2}\mathbb{W}(\mu) + md\boldsymbol{\gamma}_m\,.$$

In the expression above, $\boldsymbol{\gamma}_m$ is a structural constant which is usually interpreted as the core energy of a singularity. In our context, it is defined as

$$\boldsymbol{\gamma}_m := \lim_{R\to\infty} \min\Bigg\{ E_1(w,B_R) - \frac{\pi}{m^2}\log R : w \in W^{1,2}(B_R;\mathcal{N})\,, \\ w(z) = \frac{1}{m}\left(\frac{z}{|z|}, \sqrt{m^2-1}\right) \text{ on } \partial B_R \Bigg\}. \tag{3.1}$$

Existence and finiteness of this limit follows from a classical comparison argument (see Lemma 3.9, and [13, Lemma III.1]). We also note that the value of $F_{0,g}(u)$ only depends on u and not on a particular representation $u^m = e^{i\varphi}v_\mu$. Indeed, one always has $\mu = \operatorname{curl} j(u^m)$ and $|\nabla\varphi| = |\nabla(\bar{v}_\mu u^m)|$.

To properly state the Γ-convergence result, it is now convenient to introduce $\widetilde{F}_0 : L^1(\Omega)\times L^1(\Omega) \to (-\infty,\infty]$ given by

$$\widetilde{F}_0(u,\psi) := \begin{cases} F_{0,g}(u) & \text{if } u \in \mathcal{L}_g(\Omega) \text{ and } \psi \equiv 1\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

Theorem 3.1. — *Let $\varepsilon_h \downarrow 0$ and $\eta_h \downarrow 0$ be arbitrary sequences. The sequence of functionals $\{\widetilde{F}^{\eta_h}_{\varepsilon_h}\}$ Γ-converges in the strong $\big[L^1(\Omega)\big]^2$-topology to $\widetilde{F}_0$ as $h \to \infty$. More precisely:*

(i) *If $\{(u_h,\psi_h)\} \subseteq \mathcal{H}_g(\Omega)$, $\{u_h\}$ is a bounded sequence in $L^\infty(\Omega)$, and $\sup_h \widetilde{F}^{\eta_h}_{\varepsilon_h}(u_h,\psi_h) < \infty$, then there exist a subsequence and $u \in \mathcal{L}_g(\Omega)$ with $u^m =: e^{i\varphi}v_\mu$ such that $(u_h,\psi_h) \to (u,1)$ in $L^1(\Omega)$, $v_h := \mathrm{p}(u_h) \rightharpoonup u^m$ weakly in $W^{1,p}(\Omega)$ for every $p<2$ and weakly in*

$W^{1,2}_{\mathrm{loc}}(\overline{\Omega}\backslash\operatorname{spt}\mu)$, *and the measures* $\mu_h := \operatorname{curl} j(v_h)$ *weakly* converge to* $\mu = \operatorname{curl} j(u^m)$ *in the* $(C^{0,1}_0(\Omega))^*$ *topology.*

(ii) *Under the conclusions of* (i),

$$\liminf_{h\to\infty} \left\{ E_{\varepsilon_h}\big(\mathrm{P}(u_h)\big) - \frac{\pi d}{m}|\log\varepsilon_h| \right\} \geqslant E_0(u)\,, \tag{3.2}$$

and

$$\liminf_{h\to\infty} I_{\eta_h}(\psi_h) \geqslant \mathcal{H}^1(J_u) + \mathcal{H}^1\big(\{u\neq g\}\cap\partial\Omega\big)\,. \tag{3.3}$$

Moreover, if $\widetilde{F}_0(u,1) = \lim_h \widetilde{F}^{\eta_h}_{\varepsilon_h}(u_h,\psi_h) < \infty$, *then* $\mathrm{p}(u_h) \to u^m$ *strongly in* $W^{1,p}(\Omega)$ *for every* $p<2$ *and strongly in* $W^{1,2}_{\mathrm{loc}}(\overline{\Omega}\backslash\operatorname{spt}\mu)$,

$$\lim_{h\to\infty} E_{\varepsilon_h}\big(\mathrm{P}(u_h),\Omega\setminus B_r(\mu)\big) = \frac{1}{2m^2}\int_{\Omega\setminus B_r(\mu)} |\nabla(u^m)|^2\,\mathrm{d}x \qquad \textit{for every } r>0\,, \tag{3.4}$$

and

$$\lim_{h\to\infty} I_{\eta_h}\big(\psi_h\big) = \mathcal{H}^1(J_u) + \mathcal{H}^1\big(\{u\neq g\}\cap\partial\Omega\big)\,. \tag{3.5}$$

(iii) *For every* $u\in\mathcal{L}_g(\Omega)$, *there exists a sequence* $\{(u_h,\psi_h)\}\subseteq\mathcal{H}_g(\Omega)$ *such that* $u_h = g$ *on* $\partial\Omega$, $(u_h,\psi_h)\to(u,1)$ *in* $L^1(\Omega)$, $\mathrm{p}(u_h)\to u^m$ *strongly in* $W^{1,p}(\Omega)$ *for every* $p<2$ *and strongly in* $W^{1,2}_{\mathrm{loc}}(\overline{\Omega}\backslash\operatorname{spt}\mu)$, *and satisfying*

$$\lim_{h\to\infty} \left\{ E_{\varepsilon_h}\big(\mathrm{P}(u_h)\big) - \frac{\pi d}{m}|\log\varepsilon_h| \right\} = E_0(u)\,, \tag{3.6}$$

$$\lim_{h\to\infty} I_{\eta_h}(\psi_h) = \mathcal{H}^1(J_u) + \mathcal{H}^1\big(\{u\neq g\}\cap\Omega\big)\,. \tag{3.7}$$

We proceed analogously with the sharp interface functionals $F^0_{\varepsilon,g}$ defined in (2.8), and introduce $\widetilde{F}^0_\varepsilon : L^1(\Omega)\to(-\infty,\infty]$ and $\widetilde{F}_0 : L^1(\Omega)\to(-\infty,\infty]$ defined as

$$\widetilde{F}^0_\varepsilon(u) := \begin{cases} F^0_{\varepsilon,g}(u) - \frac{\pi d}{m}|\log\varepsilon| & \text{if } u\in\mathcal{G}_g(\Omega)\,,\\ \infty & \text{otherwise}\,,\end{cases}$$

and

$$\widetilde{F}_0(u) := \begin{cases} F_{0,g}(u) & \text{if } u\in\mathcal{L}_g(\Omega)\,,\\ \infty & \text{otherwise}\,.\end{cases}$$

Theorem 3.2. — *Let* $\varepsilon_h\downarrow 0$ *be an arbitrary sequence. The sequence of functionals* $\big\{\widetilde{F}^0_{\varepsilon_h}\big\}_{h\in\mathbf{N}}$ Γ*-converges in the strong* $L^1(\Omega)$*-topology to* $\widetilde{F}_0$ *as* $h\to\infty$. *More precisely:*

(i) *If* $\{u_h\} \subseteq \mathcal{G}_g(\Omega)$, $\{u_h\}$ *is bounded in* $L^\infty(\Omega)$, *and* $\sup_h \widehat{F}^0_{\varepsilon_h}(u_h) < \infty$, *then there exist a subsequence and* $u \in \mathcal{L}_g(\Omega)$ *with* $u^m =: e^{i\varphi} v_\mu$ *such that* $u_h \to u$ *in* $L^1(\Omega)$, $v_h := \mathrm{p}(u_h) \rightharpoonup u^m$ *weakly in* $W^{1,p}(\Omega)$ *for every* $p < 2$ *and weakly in* $W^{1,2}_{\mathrm{loc}}(\overline{\Omega} \setminus \operatorname{spt}\mu)$, *and the measures* $\mu_h := \operatorname{curl} j(v_h)$ *weakly* converge to* $\mu = \operatorname{curl} j(u^m)$ *in the* $(C^{0,1}_0(\Omega))^*$ *topology.*

(ii) *If* $\{u_h\} \subseteq \mathcal{G}_g(\Omega)$ *is such that* $u_h \to u$ *in* $L^1(\Omega)$, *then*

$$\liminf_{h\to\infty} \widetilde{F}^0_{\varepsilon_h}(u_h) \geqslant \widetilde{F}_0(u)\,. \tag{3.8}$$

Moreover, if $\widetilde{F}_0(u) = \lim_h \widetilde{F}^0_{\varepsilon_h}(u_h) < \infty$, *then* $\mathrm{p}(u_h) \to u^m$ *strongly in* $W^{1,p}(\Omega)$ *for every* $p < 2$ *and strongly in* $W^{1,2}_{\mathrm{loc}}(\overline{\Omega} \setminus \operatorname{spt}\mu)$, *identity* (3.4) *holds, and for every open set* $A \subseteq \mathbf{R}^2$ *such that* $\mathcal{H}^1\big(J_u \cap (\Omega \cap \partial A)\big) + \mathcal{H}^1\big(\{u \neq g\} \cap \partial\Omega \cap \partial A\big) = 0$,

$$\begin{aligned} \lim_{h\to\infty} \mathcal{H}^1\big(J_{u_h} \cap (\Omega \cap A)\big) &+ \mathcal{H}^1\big(\{u_h \neq g\} \cap \partial\Omega \cap A\big) \\ &= \mathcal{H}^1\big(J_u \cap (\Omega \cap A)\big) + \mathcal{H}^1\big(\{u \neq g\} \cap \partial\Omega \cap A\big). \end{aligned} \tag{3.9}$$

(iii) *For every* $u \in \mathcal{L}_g(\Omega)$, *there exists a sequence* $\{u_h\} \subseteq \mathcal{G}_g(\Omega)$ *such that* $u_h \to u$ *in* $L^1(\Omega)$ *and*

$$\lim_{h\to\infty} \widetilde{F}^0_{\varepsilon_h}(u_h) = \widetilde{F}_0(u)\,. \tag{3.10}$$

As a standard consequence of these Γ-convergence results, we have the following corollaries concerning the minimizers of F^η_ε and $F^0_{\varepsilon,g}$, whose existence was proved in Theorems 2.7 and 2.8 respectively (together with the uniform L^∞-bound allowing for compactness).

Corollary 3.3. — *Let* $\varepsilon_h \downarrow 0$ *and* $\eta_h \downarrow 0$ *be arbitrary sequences. For each* $h \in \mathbf{N}$, *let* (u_h, ψ_h) *be a minimizer of* $F^{\eta_h}_{\varepsilon_h}$ *in* $\mathcal{H}_g(\Omega)$. *There exists a subsequence and a map* u *minimizing* $F_{0,g}$ *over* $\mathcal{L}_g(\Omega)$ *such that* $(u_h, \psi_h) \to (u, 1)$ *in* $L^1(\Omega)$, $\mathrm{p}(u_h) \to u^m$ *strongly in* $W^{1,p}(\Omega)$ *for every* $p < 2$ *and strongly in* $W^{1,2}_{\mathrm{loc}}(\overline{\Omega} \setminus \operatorname{spt}\mu)$ *where* $\mu := \operatorname{curl} j(u^m)$. *In addition,*

$$F^{\eta_h}_{\varepsilon_h}(u_h, \psi_h) = \frac{\pi d}{m}|\log \varepsilon_h| + F_{0,g}(u) + o(1) \quad \text{as } h \to \infty\,.$$

Corollary 3.4. — *Let* $\varepsilon_h \downarrow 0$ *be an arbitrary sequence. For each* $h \in \mathbf{N}$, *let* u_h *be a minimizer of* $F^0_{\varepsilon_h,g}$ *in* $\mathcal{G}_g(\Omega)$. *There exists a subsequence and a map* u *minimizing* $F_{0,g}$ *over* $\mathcal{L}_g(\Omega)$ *such that* $u_h \to u$ *in* $L^1(\Omega)$, $\mathrm{p}(u_h) \to u^m$ *strongly in* $W^{1,p}(\Omega)$ *for every* $p < 2$ *and strongly in* $W^{1,2}_{\mathrm{loc}}(\overline{\Omega} \setminus \operatorname{spt}\mu)$ *where* $\mu := \operatorname{curl} j(u^m)$. *In addition,*

$$F^0_{\varepsilon_h,g}(u_h) = \frac{\pi d}{m}|\log \varepsilon_h| + F_{0,g}(u) + o(1) \quad \text{as } h \to \infty\,.$$

Remark 3.5. — From the definition of $F_{0,g}$, any minimizer u of $F_{0,g}$ over $\mathcal{L}_g(\Omega)$ satisfies $u^m = v_\mu$ where $\mu := \operatorname{curl} j(u^m)$ (i.e., in any representation $u^m = e^{i\varphi} v_\mu$, the phase φ is a constant multiple of 2π). As a consequence,

$$F_{0,g}(u) = \frac{1}{m^2}\mathbb{W}(\mu) + md\boldsymbol{\gamma}_m + \mathcal{H}^1(J_u) + \mathcal{H}^1\big(\{u \neq g\} \cap \partial\Omega\big)\,.$$

The rest of this section if devoted to the proofs of Theorems 3.1 and 3.2. Starting with Theorem 3.1, compactness, Γ-lim inf, and Γ-lim sup parts are proved in Subsections 3.1, 3.3, and 3.4 respectively. The proof of Theorem 3.2 is the object of Subsection 3.5.

3.1. Proof of Theorem 3.1(i): Compactness

Proposition 3.6. — *Let $\varepsilon_h \downarrow 0$ and $\eta_h \downarrow 0$ be arbitrary sequences. Let $\{(u_h, \psi_h)\} \subseteq \mathcal{H}_g(\Omega)$ be such that $\{u_h\}$ is bounded in $L^\infty(\Omega)$, and*

$$\sup_h \left\{ F^{\eta_h}_{\varepsilon_h}(u_h, \psi_h) - \frac{\pi d}{m}|\log \varepsilon_h| \right\} < \infty\,. \tag{3.11}$$

Then, there exist a subsequence and $u \in \mathcal{L}_g(\Omega)$ with $u^m =: e^{i\varphi} v_\mu$ such that

(i) $(u_h, \psi_h) \to (u, 1)$ *strongly in* $L^1(\Omega)$*;*
(ii) $v_h := \mathrm{p}(u_h) \rightharpoonup u^m$ *weakly in* $W^{1,p}(\Omega)$ *for every* $p \in [1, 2)$*;*
(iii) $v_h \rightharpoonup u^m$ *weakly in* $W^{1,2}_{\mathrm{loc}}(\overline{\Omega} \setminus \operatorname{spt}\mu)$

Moreover, $\mu_h := \operatorname{curl} j(v_h) \in L^1(\Omega)$ converges to $\mu = \operatorname{curl} j(u^m)$ in the weak topology of $(C^{0,1}_0(\Omega))^*$.*

The proposition above partially rests on the following preliminary lemma.

Lemma 3.7. — *Let $\{(u_h, \psi_h)\} \subseteq \mathcal{H}(\Omega)$ be such that $\psi_h \to 1$ a.e. in Ω. Assume that for some $p \in (1, 2]$,*

$$\sup_h \left\{ \|u_h\|_{L^\infty(\Omega)} + \|\psi_h - 1\|_{L^2(\Omega)} \|\nabla \psi_h\|_{L^2(\Omega)} + \big\|\nabla \mathrm{p}(u_h)\big\|_{L^p(\Omega)} \right\} < \infty\,.$$

Then there exist a subsequence and $u \in SBV^p(\Omega)$ such that $\mathrm{P}(u) \in W^{1,p}(\Omega; \mathcal{N})$, $u_h \to u$ strongly in $L^1(\Omega)$, and $\mathrm{P}(u_h) \rightharpoonup \mathrm{P}(u)$ weakly in $W^{1,p}(\Omega)$.

Proof. — By assumption and Cauchy–Schwarz inequality, we have

$$\int_\Omega (1 - \psi_h)|\nabla \psi_h| \,\mathrm{d}x \leqslant C\,,$$

for some constant C independent of h. According to the co-area formula (see [4, Theorem 3.40]),

$$\int_\Omega (1-\psi_h)|\nabla\psi_h|\,\mathrm{d}x = \int_0^1 (1-t)\mathcal{H}^1\big(\partial\{\psi_h<t\}\cap\Omega\big)\,\mathrm{d}t \geqslant \int_{1/4}^{3/4}(1-t)\mathcal{H}^1\big(\partial\{\psi_h<t\}\cap\Omega\big)\,\mathrm{d}t\,.$$

Therefore, we can find a level $t_h\in(1/4,3/4)$ such that

$$\int_\Omega (1-\psi_h)|\nabla\psi_h|\,\mathrm{d}x \geqslant \frac{1}{4}\mathcal{H}^1(\partial E_h\cap\Omega)\,,\quad \text{with } E_h:=\{\psi_h<t_h\}\,.$$

Notice that $|E_h|\to 0$ since $\psi_h\to 1$ a.e. in Ω.

Let us now define

$$\widetilde{u}_h := (1-\chi_{E_h})u_h\,.$$

With our choice of E_h, we have that $(1-\chi_{E_h})/\psi_h\in SBV^p(\Omega)\cap L^\infty(\Omega)$. Since $\psi_h u_h\in W^{1,2}(\Omega)\cap L^\infty(\Omega)$, we deduce that $\widetilde{u}_h=(\psi_h u_h)(1-\chi_{E_h})/\psi_n\in SBV^p(\Omega)\cap L^\infty(\Omega)$ with $J_{\tilde{u}_h}\subseteq\partial E_h$. Since $\mathrm{P}(\widetilde{u}_h)=(1-\chi_{E_h})\mathrm{P}(u_h)\in SBV(\Omega;\mathbf{R}^3)$, we infer that

$$|\nabla\widetilde{u}_h| = \big|\nabla(\mathrm{P}(\widetilde{u}_h))\big| = (1-\chi_{E_h})\big|\nabla(\mathrm{P}(u_h))\big| \leqslant \big|\nabla(\mathrm{P}(u_h))\big| \quad \text{a.e. in } \Omega\,.$$

Consequently,

$$\sup_h\big\{\|\widetilde{u}_h\|_{L^\infty(\Omega)}+\|\nabla\widetilde{u}_h\|_{L^p(\Omega)}+\mathcal{H}^1(J_{\tilde{u}_h})\big\}<\infty\,. \tag{3.12}$$

Now select a subsequence such that $\mathrm{P}(u_h)\rightharpoonup w$ weakly in $W^{1,p}(\Omega)$. In view of (3.12), we can apply Ambrosio's compactness theorem in SBV (see e.g. [4, Theorem 4.8 and Remark 4.9]) to find a further subsequence such that $\widetilde{u}_h\to u$ strongly in $L^1(\Omega)$ for some $u\in SBV^p(\Omega)\cap L^\infty(\Omega)$. Then,

$$\|u_h-u\|_{L^1(\Omega)} \leqslant \|\widetilde{u}_h-u\|_{L^1(\Omega)}+\|u_h\|_{L^\infty(\Omega)}|E_h| \underset{h\to\infty}{\longrightarrow} 0\,.$$

Since P is 1-Lipschitz, we have $\|\mathrm{P}(u_h)-\mathrm{P}(u)\|_{L^1(\Omega)}\leqslant\|u_h-u\|_{L^1(\Omega)}$, and thus $w=\mathrm{P}(u)$. □

Proof of Proposition 3.6. — Let us first recall (2.4), that is $E_{\varepsilon_h}(\mathrm{P}(u_h))=G_{\varepsilon_h}(v_h)$ with $v_h:=\mathrm{p}(u_h)$. In view of (2.4), assumption (3.11) implies

$$E_{\varepsilon_h}(v_h)\leqslant \pi md|\log\varepsilon_h|+C\,, \tag{3.13}$$

that is (2.18) holds. In turn, we deduce from (3.11) that

$$\|\psi_h-1\|_{L^2(\Omega)}\|\nabla\psi_h\|_{L^2(\Omega)} \leqslant \frac{\eta}{2}\int_\Omega|\nabla\psi_h|^2\mathrm{d}x+\frac{1}{2\eta}\int_\Omega(1-\psi_h)^2\mathrm{d}x = I_{\eta_h}(\psi_h)\leqslant C\,,$$

for a constant C independent of η_h. Clearly, it implies that $\psi_h \to 1$ a.e. Ω, at least for a suitable subsequence. We are thus in position to apply first Theorem 2.10 to $\{v_h\}$, and then Lemma 3.7 to $\{(u_h, \psi_h)\}$ to conclude the proof. □

Remark 3.8. — We emphasize that, in addition to the conclusions of Proposition 3.6, assumption (3.11) implies $\sup_h I_{\eta_h}(\psi_h) < \infty$.

3.2. Optimal profiles and the constant γ_m

In this subsection, we first study the core energy associated with one vortex in the Ginzburg–Landau energies $\{E_\varepsilon\}$. We consider for $R > 0$ and $\varepsilon > 0$, the minimum value

$$\gamma_m(\varepsilon, R) := \min\left\{ E_\varepsilon(w, B_R) - \frac{\pi}{m^2}\log\frac{R}{\varepsilon} : w \in W^{1,2}(B_R;\mathcal{N})\,,\right.$$
$$\left. w(z) = \frac{1}{m}\left(\frac{z}{|z|}, \sqrt{m^2-1}\right) \text{ on } \partial B_R \right\}. \tag{3.14}$$

In view of identity (2.5) defining the functional G_ε, the value $\boldsymbol{\gamma}_m(\varepsilon, R)$ can be written as

$$\gamma_m(\varepsilon, R) = \min\left\{ G_\varepsilon(v, B_R) - \frac{\pi}{m^2}\log\frac{R}{\varepsilon} : \right.$$
$$\left. v \in W^{1,2}(B_R)\,,\ v(z) = \frac{z}{|z|} \text{ on } \partial B_R \right\}.$$

Notice that, by homogeneity,

$$\gamma_m(\varepsilon, R) = \gamma_m(1, R/\varepsilon) =: \gamma_m(R/\varepsilon)\,. \tag{3.15}$$

We start by proving that γ_m admits a limit as $R \to \infty$. This will be needed for the lower bounds (3.2) and (3.8).

LEMMA 3.9. — *The function $R \mapsto \gamma_m(R)$ is non increasing and the limit*

$$\boldsymbol{\gamma}_m := \lim_{R\to\infty} \gamma_m(R) \tag{3.16}$$

is finite.

Proof. — The proof of this lemma closely follows the proof of [13, Lemma III.1]. Let us first show that γ_m is non increasing. Let $0 < R_1 < R_2$, and consider an admissible competitor w for the minimization problem defining

$\gamma_m(R_1)$. We extend w by 0-homogeneity in the annulus $B_{R_2} \setminus B_{R_1}$, i.e., $w(x) = w(R_1 x/|x|)$ for $x \in B_{R_2} \setminus B_{R_1}$. By construction $w \in W^{1,2}(B_{R_2}; \mathcal{N})$, and it is an admissible competitor for $\gamma_m(R_2)$. Elementary computations then yield

$$E_1(w, B_{R_2}) = E_1(w; B_{R_1}) + \frac{\pi}{m^2} \log \frac{R_2}{R_1} \,.$$

Hence, $\gamma_m(R_2) \leqslant E_1(w; B_{R_1}) - \frac{\pi}{m^2} \log R_1$. Taking the infimum with respect to w yields $\gamma_m(R_2) \leqslant \gamma_m(R_1)$, so that γ_m is indeed non increasing. Next, for an arbitrary $w \in W^{1,2}(B_1; \mathcal{N})$ satisfying $w(x) = (1/m)(x, \sqrt{m^2-1})$ on ∂B_1, we have $w = (1/m)(v, \sqrt{m^2-1}|v|)$ with $v \in W^{1,2}(B_1)$ satisfying $v(x) = x$ on ∂B_1. Consequently, for $\varepsilon > 0$ we have

$$E_\varepsilon(w, B_1) = G_\varepsilon(v, B_1) \geqslant \frac{1}{m^2} E_\varepsilon(v, B_1) \geqslant \frac{\pi}{m^2} \log \frac{1}{\varepsilon} - C \,,$$

for some universal constant C by [13]. In view of (3.15), we infer that $\gamma_m(1/\varepsilon)$ is bounded from below, and thus $\boldsymbol{\gamma}_m > -\infty$. □

The following lemma and its subsequent corollary will allow us to construct a recovery sequence close to the vortices.

Lemma 3.10. — *For every $v \in W^{1,2}(B_1)$ satisfying $v(x) = x$ on ∂B_1, there exists a sequence $\{u_k\} \subseteq SBV^2(B_1)$ such that*

(i) *$\mathrm{p}(u_k) \in W^{1,2}(B_1)$ and $\mathrm{p}(u_k)(x) = x$ in a neighborhood of ∂B_1;*

(ii) *$J_{u_k} \subseteq \Sigma_k$ where Σ_k is a smooth simple curve (i.e., a smooth image of $[0,1]$) contained in $\overline{B_1}$;*

(iii) *$\mathrm{p}(u_k) \to v$ strongly in $W^{1,2}(B_1)$.*

Proof.

Step 1. — Since the function $\widetilde{v} : z \mapsto v(x) - x$ belongs to $W_0^{1,2}(B_1)$, for each $k \in \mathbf{N}$ we can find $\phi_k \in C_c^\infty(B_1)$ such that $\|\widetilde{v} - \phi_k\|_{W^{1,2}(B_1)} \leqslant 2^{-k}$. Implicitly, we extend ϕ_k by 0 outside B_1. Then, we select a sequence of radii $\{r_k\} \subseteq (3/4, 1)$ such that $r_k \to 1$ as $k \to \infty$, and $\operatorname{spt} \phi_k \subseteq B_{r_k}$. By Morse–Sard Theorem, we can find $\{c_k\} \subseteq \mathbf{C}$ with $|c_k| < (1-r_k)^2$ such that c_k is a regular value of the mapping $x \mapsto x + \phi_k(x)$ for each $k \in \mathbf{N}$. Next, we consider for each $k \in \mathbf{N}$ a cut-off function $\chi_k \in C^\infty(\mathbf{R}^2; [0,1])$ satisfying $\chi_k(x) = 1$ for $|x| \leqslant r_k$, $\chi_k(x) = 0$ for $|x| \geqslant (1+r_k)/2$, and $(1-r_k)|\nabla \chi_k| \leqslant C$ for a constant C independent of k. Now we define the smooth function

$$v_k(x) := x + \phi_k(x) - c_k \chi_k(x) \,,$$

which satisfies $v_k(x) = x$ in a neighborhood of ∂B_1. We estimate

$$\|v_k - v\|_{W^{1,2}(B_1)} \leqslant \|\widetilde{v} - \phi_k\|_{W^{1,2}(B_1)} + |c_k| \|\chi_k\|_{W^{1,2}(B_1)} \leqslant 2^{-k} + C(1-r_k) \,.$$

Therefore $v_k \to v$ strongly in $W^{1,2}(B_1)$ as $k \to \infty$.

Step 2. — Let us fix an index $k \in \mathbf{N}$. To complete the proof, we shall produce a map $u_k \in SBV^2(B_1)$ such that $\mathrm{p}(u_k) = v_k$ and J_{u_k} is contained in a closed and smooth simple curve. First notice that our choice of c_k and the fact that $\phi_k = 0$ on $B_1 \setminus B_{r_k}$ imply that

$$|v_k| \geqslant r_k - |c_k| \geqslant 11/16 \quad \text{in } B_1 \setminus B_{r_k}\,,$$

so that $\{v_k = 0\} = \{v_k = 0\} \cap \overline{B}_{r_k} = \{x + \phi_k(x) = c_k\} \cap \overline{B}_{r_k}$ is a finite set. Hence we can find a smooth simple curve Σ_k contained in $\overline{B}_1$ such that $\Sigma_k \cap \partial B_1 = \{a_k, b_k\}$ with a_k, b_k the distinct endpoints of Σ_k, Σ_k meets ∂B_1 orthogonally at a_k and b_k, and $\{v_k = 0\} \subseteq \Sigma_k$. In this way, $B_1 \setminus \Sigma_k = A_k^1 \cup A_k^2$ where A_k^1 and A_k^2 are disjoint simply connected open sets with Lipschitz boundary. Since v_k does not vanish in each A_k^j, it admits a smooth m-th root u_k^j in each A_k^j in the sense that $\mathrm{p}(u_k^j) = v_k$ in A_k^j, which is continuous up to ∂A_k^j. We define

$$u_k(x) := u_k^j(x) \quad \text{if } x \in A_k^j\,.$$

It is then elementary to check that $u_k \in SBV^2(B_1)$. By construction, items (i), (ii), and (iii) hold. $\square$

Corollary 3.11. — *Let $\varepsilon_h \downarrow 0$ be an arbitrary sequence. There exist $\{u_h\} \subseteq SBV^2(B_1) \cap L^\infty(\Omega)$ with $\|u_h\|_{L^\infty(\Omega)} \leqslant 1$, and a sequence of smooth simple curves $\{\Sigma_h\} \subseteq \overline{B_1}$ such that $J_{u_h} \subseteq \Sigma_h$ for every $h \in \mathbf{N}$, $\mathrm{p}(u_h) \in W^{1,2}(\Omega)$, $\mathrm{p}(u_h)(x) = x$ in a neighborhood of ∂B_1, and*

$$\lim_{h\to\infty} \left\{ E_{\varepsilon_h}\big(\mathrm{P}(u_h), B_1\big) - \frac{\pi}{m^2}|\log \varepsilon_h| \right\} = \boldsymbol{\gamma}_m\,.$$

Proof. — Let us fix an arbitrary $h \in \mathbf{N}$. Consider $w_h \in W^{1,2}(B_1, \mathcal{N})$ a solution of the minimization problem (3.14) defining $\gamma_m(\varepsilon_h, 1)$ (existence easily follows from the direct method of calculus of variations), and write $w_h = (1/m)(v_h, |v_h|\sqrt{m^2-1})$ with $v_h \in W^{1,2}(B_1)$. We apply Lemma 3.10 to v_h to produce a sequence $\{u_k\}$ and curves $\{\Sigma_k\}$. The convergence property (iii) in Lemma 3.10 implies that $E_{\varepsilon_h}\big(\mathrm{P}(u_k)\big) \to E_{\varepsilon_h}(w_h)$ as $k \to \infty$. Hence, we can find $k_h \in \mathbf{N}$ such that, setting $\widetilde{u}_h := u_{k_h}$ and $\Sigma_h := \Sigma_{k_h}$, one has $J_{\tilde{u}_h} \subseteq \Sigma_h$ and $E_{\varepsilon_h}\big(\mathrm{P}(\widetilde{u}_h)\big) \leqslant E_{\varepsilon_h}(w_h) + \varepsilon_h$. Setting $u_h := \widetilde{u}_h / \max\big(1, |\widetilde{u}_h|\big)$, we observe that $u_h \in SBV^2(B_1)$, $\|u_h\|_{L^\infty(\Omega)} \leqslant 1$, and $J_{u_h} \subseteq J_{\tilde{u}_h} \subseteq \Sigma_h$. As in (2.10) we have $\mathrm{p}(u_h) = \mathrm{p}(\widetilde{u}_h)/\max\big(1, |\mathrm{p}(\widetilde{u}_h)|\big)$, and we infer that $\mathrm{p}(u_h) \in W^{1,2}(\Omega)$, $\mathrm{p}(u_h)(x) = x$ in a neighborhood of ∂B_1, and

$$\gamma_m(\varepsilon_h, 1) \leqslant E_{\varepsilon_h}\big(\mathrm{P}(u_h)\big) \leqslant E_{\varepsilon_h}\big(\mathrm{P}(\widetilde{u}_h)\big) \leqslant E_{\varepsilon_h}(w_h) + \varepsilon_h\,.$$

Since $E_{\varepsilon_h}(w_h) = \gamma_m(\varepsilon_h, 1) \to \boldsymbol{\gamma}_m$ as $h \to \infty$ by (3.15) and Lemma 3.9, the conclusion follows. □

To close this section, we characterize the optimal one dimensional profile related to the energy I_η. This will be important to go from a recovery sequence for the sharp interface functional $F^0_{\varepsilon,g}$ to a recovery sequence of the diffuse interface functional F^η_ε.

Lemma 3.12. — *For every $\eta > 0$,*

$$\min\left\{ I^{1D}_\eta(\psi) := \frac{\eta}{2}\int_{\mathbf{R}} |\psi'(s)|^2\,\mathrm{d}s + \frac{1}{2\eta}\int_{\mathbf{R}} (1-\psi(s))^2\,\mathrm{d}s \,:\, \psi(0)=0 \right\} = 1\,.$$

The minimum is uniquely achieved by $\psi_\eta(s) := \psi_\star(s/\eta)$ with $\psi_\star(s) = 1 - e^{-|s|}$.

Proof. — By rescaling, we may assume without loss of generality that $\eta = 1$. Write

$$\Phi(t) := (1-|t|)^2/2 = \int_{|t|}^1 (1-s)\,\mathrm{d}s\,.$$

Let $\psi : \mathbf{R} \to \mathbf{R}$ be such that $\psi(0) = 0$ and $I^{1D}_1(\psi) < \infty$. First, notice that the condition $I^{1D}_1(\psi) < \infty$ implies that $\lim_{s\to\pm\infty}\psi(s) = 1$. Then, by Cauchy–Schwarz and Young inequalities, we have

$$\begin{aligned} I^{1D}_1(\psi) \geqslant \int_{\mathbf{R}} |1-\psi(s)|\,|\psi'(s)|\,\mathrm{d}s &= \int_{-\infty}^0 \left|(\Phi\circ\psi)'\right| ds + \int_0^\infty \left|(\Phi\circ\psi)'\right| ds \\ &\geqslant 2(\Phi(0)-\Phi(1)) = 1\,. \end{aligned}$$

We have equality in the above chain of inequalities if and only if $|\psi'| = |1-\psi|$. Using $\psi(0) = 0$ and the condition $\int_{\mathbf{R}} |\psi'|^2\,ds < \infty$ leads to the optimal profile $\psi_\star$. □

3.3. Proof of Theorem 3.1(ii): The Γ-$\liminf$ inequality

Proposition 3.13. — *Let $\varepsilon_h \downarrow 0$ and $\eta_h \downarrow 0$ be arbitrary sequences. If $\{(u_h,\psi_h)\} \subseteq \mathcal{H}_g(\Omega)$ is such that $(u_h,\psi_h) \to (u,\psi)$ in $L^1(\Omega)$, then*

$$\widetilde{F}_0(u,\psi) \leqslant \liminf_{h\to\infty} \widetilde{F}^{\eta_h}_{\varepsilon_h}(u_h,\psi_h)\,. \tag{3.17}$$

In addition,

(i) *if the liminf is finite, then $u \in \mathcal{L}_g(\Omega)$, $\psi \equiv 1$, and (3.2)–(3.3) hold;*
(ii) *if equality holds and the liminf is a finite limit, then $\mathrm{p}(u_h)$ converges to $u^m =: e^{i\varphi}v_\mu$ strongly in $W^{1,p}(\Omega)$ for every $p < 2$ and strongly in $W^{1,2}_{\mathrm{loc}}(\overline{\Omega}\setminus \operatorname{spt}\mu)$, and (3.4)–(3.5) hold.*

Proof. — Without loss of generality, we may assume that

$$\liminf_{h\to\infty} \widetilde{F}^{\eta_h}_{\varepsilon_h}(u_h,\psi_h) = \lim_{h\to\infty} \widetilde{F}^{\eta_h}_{\varepsilon_h}(u_h,\psi_h) < \infty\,. \tag{3.18}$$

We may also assume that $\|u_h\|_{L^\infty(\Omega)} \leqslant 1$. Indeed, on the first hand (3.18) clearly implies that $|u| = 1$. On the other hand, the truncation argument used in the proof of Theorem 2.7 shows that replacing u_h by $\widehat{u}_h$ given by (2.9) does not increase the energy. Moreover,

$$\|u_h - \widehat{u}_h\|_{L^1(\Omega)} \leqslant \big\||u_h| - 1\big\|_{L^1(\Omega)} \leqslant \|u_h - u\|_{L^1(\Omega)}\,, \tag{3.19}$$

so that $\widehat{u}_h \to u$ in $L^1(\Omega)$. Hence (u_h,ψ_h) can be replaced by $(\widehat{u}_h,\psi_h)$.

Next, we apply Theorem 3.6 to extract a further subsequence to obtain all the conclusions of that theorem. As a consequence, $\psi = 1$ and $u \in \mathcal{L}_g(\Omega)$ with $u^m = e^{i\varphi}v_\mu$. We have to show that

$$F_{0,g}(u) \leqslant \lim_{h\to\infty} \widetilde{F}^{\eta_h}_{\varepsilon_h}(u_h,\psi_h)\,. \tag{3.20}$$

We shall prove this inequality in several steps.

Step 1. — We first claim that (3.3) holds. By Remark 3.8, $\sup_h I_{\eta_h}(\psi_h) < \infty$. We consider the larger domain $\widetilde{\Omega}$ defined in (2.11), as

$$\widetilde{\Omega} := \big\{x \in \mathbf{R}^2 : \operatorname{dist}(x,\Omega) < r_0\big\},$$

for r_0 small enough and we recall that the nearest point projection Π on $\partial\Omega$ is well defined and smooth in $\widetilde{\Omega}\setminus\Omega$. We extend (u_h,ψ_h) and u to $\widetilde{\Omega}$ by setting for $x \in \widetilde{\Omega}\setminus\Omega$, $\psi_h(x) = 1$, $u_h(x) = g\big(\Pi(x)\big)$, and $u(x) = g\big(\Pi(x)\big)$. Then, it is elementary to check that $(u_h,\psi_h) \in \mathcal{H}(\widetilde{\Omega})$ and $u \in SBV^p(\widetilde{\Omega})$ for every $p < 2$. In addition, $\mathrm{P}(u_h) \rightharpoonup \mathrm{P}(u)$ weakly in $W^{1,p}(\widetilde{\Omega})$ for every $p < 2$, and

$$\begin{aligned}\int_{\tilde\Omega} (1-\psi_h)|\nabla\psi_h|\,\mathrm{d}x \leqslant \frac{\eta_h}{2}\int_{\tilde\Omega} |\nabla\psi_h|^2\,\mathrm{d}x + \frac{1}{2\eta_h}\int_{\tilde\Omega}(1-\psi_h)^2\,\mathrm{d}x\\ = I_{\eta_h}(\psi_h) \leqslant C\,.\end{aligned}$$

Next, consider some arbitrary $\delta \in (0,1/2)$. Arguing as in the proof of Lemma 3.7,

$$\int_\delta^{1-\delta} (1-t)\mathcal{H}^1\big(\partial\{\psi_h < t\}\cap\widetilde{\Omega}\big)\,\mathrm{d}t \leqslant I_{\eta_h}(\psi_h)\,,$$

so that we can find a level $t_h \in (\delta, 1-\delta)$ such that

$$\frac{1-2\delta}{2}\mathcal{H}^1(\partial E_h \cap \widetilde{\Omega}) \leqslant I_{\eta_h}(\psi_h) \quad \text{with } E_h := \{\psi_h < t_h\}\,. \tag{3.21}$$

Notice that $E_h \subseteq \overline{\Omega}$, and $|E_h| \to 0$ since $\psi_h \to 1$ in $L^1(\Omega)$.

Fix some $p \in (1,2)$. Defining

$$\widetilde{u}_h := (1 - \chi_{E_h})u_h \,,$$

we argue as in the proof of Lemma 3.7 to show that $\widetilde{u}_h \in SBV^p(\widetilde{\Omega}) \cap L^\infty(\widetilde{\Omega})$ with $J_{\tilde{u}_h} \subseteq \partial E_h$, and that $\widetilde{u}_h \to u$ in $L^1(\widetilde{\Omega})$. Moreover, $|\nabla \widetilde{u}_h| \leqslant \big|\nabla(\mathrm{P}(u_h))\big|$ a.e. in $\widetilde{\Omega}$, so that $\{\nabla \widetilde{u}_h\}$ is bounded in $L^p(\widetilde{\Omega})$. By [17, Theorem 1], we have

$$\liminf_{h\to\infty} \left\{ \delta \int_{\widetilde{\Omega}} |\nabla \widetilde{u}_h|^p \,\mathrm{d}x + \frac{1-2\delta}{2}\mathcal{H}^1(\partial E_h \cap \widetilde{\Omega}) \right\} \geqslant \delta \int_{\tilde{\Omega}} |\nabla \widetilde{u}|^p \,\mathrm{d}x + (1-2\delta)\mathcal{H}^1(J_u \cap \widetilde{\Omega})\,,$$

and thus

$$\mathcal{H}^1(J_u \cap \widetilde{\Omega}) \leqslant \liminf_{h\to\infty} \frac{1}{2}\mathcal{H}^1(\partial E_h \cap \widetilde{\Omega}) + C\delta\,.$$

Inserting this inequality in (3.21) and letting $\delta \downarrow 0$, we conclude that

$$\liminf_{h\to\infty} I_{\eta_h}(\psi_h) \geqslant \mathcal{H}^1(J_u \cap \widetilde{\Omega})\,.$$

Since $u = g \circ \Pi$ in $\widetilde{\Omega} \setminus \overline{\Omega}$, we have $J_u \subseteq \overline{\Omega}$, and thus $\mathcal{H}^1(J_u \cap \widetilde{\Omega}) = \mathcal{H}^1(J_u \cap \Omega) + \mathcal{H}^1\big(\{u \neq g\} \cap \partial\Omega\big)$, and inequality (3.3) follows.

Step 2. — We now prove the lower bound (3.2). Note that putting (3.2) and (3.3) together leads to (3.20). First, we recall that (3.18) implies (2.18) with $v_h := \mathrm{p}(u_h)$ (by means of (2.4)). Then the proof of (3.2) follows very closely the ones of [2, Theorem 5.3] and [24, Lemma 4.1.1] for the classical Ginzburg–Landau functional. We provide a quite detailed proof for the reader's convenience. Write $\mu = 2\pi \sum_{k=1}^{md} x_k$, and choose $\sigma > 0$ in such a way that the balls $B_\sigma(x_k)$ are contained inside Ω and are pairwise disjoint. Set

$$\mathcal{K} := \left\{ v_\alpha : \mathbf{C} \setminus \{0\} \to \mathbf{S}^1 \text{ defined as } \; v(z) = \alpha \frac{z}{|z|} : \alpha \in \mathbf{S}^1 \right\}.$$

For each $k \in \{1, \dots, md\}$, our aim is to prove that either v_h is $W^{1,2}$-close to $\mathcal{K}$ on $\partial B_\sigma(x_k)$, or it has "large" energy. We define for $t \in (0,\sigma]$ and $w \in W^{1,2}(B_t \setminus \overline{B_{t/2}})$,

$$d_t(w, \mathcal{K}) := \min \left\{ \|w - v\|_{W^{1,2}(B_t \setminus B_{t/2})} : v \in \mathcal{K} \right\}.$$

It is proven in [2] that for a given $\delta \in (0,1)$, there exists a constant $c_\delta > 0$ independent of t such that the condition $\liminf_h d_t(v_h(\,\cdot\, + x_k), \mathcal{K}) \geqslant \delta$ implies

$$\liminf_{h\to\infty} \frac{1}{2} \int_{B_t(x_k) \setminus B_{t/2}(x_k)} |\nabla v_h|^2 \,\mathrm{d}x \geqslant \pi \log 2 + c_\delta\,. \tag{3.22}$$

Now let $L \in \mathbf{N}$ be such that

$$\frac{Lc_\delta}{m^2} \geqslant \frac{1}{2m^2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \frac{1}{m^2}\mathbb{W}(\mu) + md\boldsymbol{\gamma}_m - \frac{\pi d}{m}\log\sigma - \frac{C_*}{m^2}\,,$$

where c_δ is the constant from (3.22), and C_* is the constant from (2.19). For $l \in \{1,\dots,L\}$ we write $C_l(x_k) := B_{2^{1-l}\sigma}(x_k) \setminus B_{2^{-l}\sigma}(x_k)$. By the weak $W^{1,2}_{\mathrm{loc}}(\overline{\Omega}\setminus \mathrm{spt}\mu)$ convergence of v_h to $u^m = e^{i\varphi}v_\mu$, we have for each k and l,

$$\liminf_{h\to\infty} \frac{1}{2}\int_{C_l(x_k)} |\nabla v_h|^2\,\mathrm{d}x \geqslant \frac{1}{2}\int_{C_l(x_k)} |\nabla\varphi|^2 + |\nabla v_\mu|^2\,\mathrm{d}x \geqslant \pi\log 2\,. \tag{3.23}$$

We now have to distinguish two different cases.

Case 1. — For h large enough, and for each $1 \leqslant l \leqslant L$, there exists at least one $k_l \in \{1,\dots,md\}$ such that $d_{2^{1-l}\sigma}(u_h(\cdot + x_{k_l}),\mathcal{K}) \geqslant \delta$. Then, we estimate

$$\begin{aligned} E_{\varepsilon_h}\big(P(u_h)\big) - \frac{\pi d}{m}|\log\varepsilon_h| &\geqslant \frac{1}{m^2}E_{\varepsilon_h}(v_h) - \frac{\pi d}{m}|\log\varepsilon_h| \\ &\geqslant \frac{1}{m^2}\Big\{E_{\varepsilon_h}(v_h, B_{2^{-L}\sigma}(\mu)) - \pi dm|\log\varepsilon_h|\Big\} \\ &\qquad + \frac{1}{m^2}\sum_{l=1}^{L}\sum_{k=1}^{md}\frac{1}{2}\int_{C_l(x_k)}|\nabla v_h|^2\,\mathrm{d}x\,. \end{aligned}$$

Taking the liminf in h, and combining (2.19), (3.22), and (3.23) yields

$$\begin{aligned} \liminf_{h\to\infty}\Big\{E_{\varepsilon_h}\big(P(u_h)\big) - \frac{\pi d}{m}|\log\varepsilon_h|\Big\} \\ &\geqslant \frac{\pi d}{m}\log\frac{\sigma}{2^L} + \frac{C_*}{m^2} + \frac{L}{m^2}(\pi dm\log 2 + c_\delta) \\ &= \frac{\pi d}{m}\log\sigma + \frac{C_*}{m^2} + \frac{Lc_\delta}{m^2} \\ &\geqslant \frac{1}{2m^2}\int_\Omega|\nabla\varphi|^2\,\mathrm{d}x + \frac{1}{m^2}\mathbb{W}(\mu) + md\boldsymbol{\gamma}_m\,, \end{aligned} \tag{3.24}$$

and thus (3.2) holds.

Case 2. — For a subsequence there exists $\bar{l} \in \{1,\dots,L\}$ such that, setting $\overline{\sigma} := 2^{1-\bar{l}}\sigma$, $d_{\overline{\sigma}}(u_h(\cdot + x_k),\mathcal{K}) < \delta$ for every $k \in \{1,\dots,md\}$. Let us prove that for ε_h small enough,

$$G_{\varepsilon_h}\big(v_h, B_{\overline{\sigma}}(\mu)\big) - \frac{\pi d}{m}\log\frac{\overline{\sigma}}{\varepsilon_h} \geqslant md\gamma_m(\varepsilon_h,\overline{\sigma}) - C_{\overline{\sigma}}\delta\,, \tag{3.25}$$

where, here and below, $C_{\overline{\sigma}}$ denotes a nonnegative number depending on $\overline{\sigma}$ but not on h or δ. To establish this inequality, we shall modify v_h in $B_{\overline{\sigma}}(\mu)$ without increasing the energy too much, and in such a way that it is admissible for (3.16). We can proceed on each ball $B_{\overline{\sigma}}(x_k)$ separately, and we may assume without loss of generality that $x_k = 0$. Up to a rotation, we can even assume that

$$\int_{B_{\overline{\sigma}}\setminus B_{\overline{\sigma}/2}} \left|\nabla v_h - \nabla(e^{i\theta})\right|^2 \,\mathrm{d}x \leqslant \delta^2 \,, \tag{3.26}$$

where $\theta(x)$ denotes the argument of $x/|x|$. As (2.18) holds, we infer from (2.19) in Theorem 2.10 that

$$\int_{B_{3\overline{\sigma}/4}\setminus B_{\overline{\sigma}/2}} |\nabla v_h|^2 + \frac{1}{\varepsilon_h^2}(1-|v_h|^2)^2 \,\mathrm{d}x \leqslant C_{\overline{\sigma}} \,.$$

Therefore, for every h we can find $\widetilde{\sigma}_h \in [\overline{\sigma}/2, 3\overline{\sigma}/4]$ for which

$$\int_{\partial B_{\tilde{\sigma}_h}} |\nabla v_h|^2 + \frac{1}{\varepsilon_h^2}(1-|v_h|^2)^2 \,\mathrm{d}\mathcal{H}^1 \leqslant C_{\overline{\sigma}} \,, \tag{3.27}$$

and

$$\int_{\partial B_{\tilde{\sigma}_h}} \left|\nabla v_h - \nabla(e^{i\theta})\right|^2 \,\mathrm{d}\mathcal{H}^1 \leqslant C_{\overline{\sigma}}\delta^2 \,. \tag{3.28}$$

From (3.28) we first derive

$$\begin{aligned}
\int_{\partial B_{\tilde{\sigma}_h}} |\nabla v_h|^2 \,\mathrm{d}\mathcal{H}^1 &= \int_{\partial B_{\tilde{\sigma}_h}} \left|\nabla v_h - \nabla(e^{i\theta}) + \nabla(e^{i\theta})\right|^2 \,\mathrm{d}\mathcal{H}^1 \\
&\leqslant \frac{1+\delta}{\delta} \int_{\partial B_{\tilde{\sigma}_h}} \left|\nabla v_h - \nabla(e^{i\theta})\right|^2 \,\mathrm{d}\mathcal{H}^1 \\
&\qquad + (1+\delta) \int_{\partial B_{\tilde{\sigma}_h}} \left|\nabla(e^{i\theta})\right|^2 \,\mathrm{d}\mathcal{H}^1 \\
&\leqslant C_{\overline{\sigma}}\delta + \frac{2\pi}{\widetilde{\sigma}_h}(1+\delta) \leqslant C_{\overline{\sigma}}\delta + \frac{2\pi}{\widetilde{\sigma}_h} \,. \qquad (3.29)
\end{aligned}$$

Next, by a scaling argument, one obtains

$$\||u|-1\|^2_{L^\infty(\partial B_r)} \lesssim C\varepsilon_h \int_{\partial B_r} |\nabla u|^2 + \frac{1}{\varepsilon_h^2}(1-|u|^2)^2 \,\mathrm{d}\mathcal{H}^1$$

for every $\varepsilon_h \leqslant r$ and $u \in W^{1,2}(\partial B_r)$. Hence, (3.27) yields

$$\||v_h|-1\|_{L^\infty(\partial B_{\tilde{\sigma}_h})} \leqslant C_{\overline{\sigma}}\varepsilon_h^{1/2} \,. \tag{3.30}$$

We can thus write $v_h = \rho_h e^{i\theta_h}$ on $\partial B_{\tilde{\sigma}_h}$. Moreover, we have $\deg(v_h, \partial B_{\tilde{\sigma}_h}) = 1$ by (3.28), so that $\theta_h - \theta$ can be chosen to be single valued. Let us extend ρ_h and θ_h by zero homogeneity outside $B_{\tilde{\sigma}_h}$. For ε_h small enough, we set $\widehat{\sigma}_h := \widetilde{\sigma}_h + \varepsilon_h^{1/2}$, and we define $\widetilde{v}_h$ in $B_{\hat{\sigma}_h}$ as

$$\widetilde{v}_h(x) := \begin{cases} v_h(x) & \text{in } B_{\tilde{\sigma}_h} \,, \\ \left(\rho_h(x)\frac{\hat{\sigma}_h - |x|}{\hat{\sigma}_h - \tilde{\sigma}_h} + \frac{|x| - \tilde{\sigma}_h}{\hat{\sigma}_h - \tilde{\sigma}_h}\right) e^{i\theta_h(x)} & \text{in } B_{\hat{\sigma}_h} \setminus B_{\tilde{\sigma}_h} \,. \end{cases}$$

From (3.27) and (3.30) we infer that

$$G_{\varepsilon_h}(\widetilde{v}_h, B_{\hat{\sigma}_h} \setminus B_{\tilde{\sigma}_h}) \leqslant C_{\overline{\sigma}}\varepsilon_h^{1/2}\,. \tag{3.31}$$

In turn, (3.29) and (3.30) yield, for ε_h small enough,

$$\int_{\partial B_{\hat{\sigma}_h}} |\nabla \widetilde{v}_h|^2 \,\mathrm{d}\mathcal{H}^1 \leqslant C_{\overline{\sigma}}\delta + \frac{2\pi}{\widehat{\sigma}_h}\,. \tag{3.32}$$

We now are left to define $\widetilde{v}_h$ in $B_{\overline{\sigma}} \setminus B_{\hat{\sigma}_h}$. We define for $x \in B_{\overline{\sigma}} \setminus B_{\hat{\sigma}_h}$,

$$\widehat{\theta}_h(x) := \frac{\overline{\sigma} - |x|}{\overline{\sigma} - \widehat{\sigma}_h}\theta_h + \frac{|x| - \widehat{\sigma}_h}{\overline{\sigma} - \widehat{\sigma}_h}\theta\,,$$

i.e., the linear interpolation between $\theta_h(x)$ and $\theta(x)$. Setting $\widetilde{v}_h(x) := e^{i\hat{\theta}_h(x)}$ in $B_{\overline{\sigma}} \setminus B_{\hat{\sigma}_h}$, it can be proven by means of (3.32) and elementary computations (see [24, Lemma 4.1.1]) that

$$\int_{B_{\overline{\sigma}} \setminus B_{\hat{\sigma}_h}} |\nabla \widetilde{v}_h|^2 \,\mathrm{d}x \leqslant 2\pi \log \frac{\overline{\sigma}}{\widehat{\sigma}_h} + C_{\overline{\sigma}}\delta\,.$$

Combining this last inequality with (3.31), we first derive

$$\begin{aligned}
&G_{\varepsilon_h}(v_h, B_{\overline{\sigma}})\\
&= G_{\varepsilon_h}(\widetilde{v}_h, B_{\overline{\sigma}}) + G_{\varepsilon_h}(v_h, B_{\overline{\sigma}} \setminus B_{\tilde{\sigma}_h}) - G_{\varepsilon_h}(\widetilde{v}_h, B_{\overline{\sigma}} \setminus B_{\tilde{\sigma}_h})\\
&\geqslant G_{\varepsilon_h}(\widetilde{v}_h, B_{\overline{\sigma}}) + \frac{1}{2m^2}\int_{B_{\overline{\sigma}} \setminus B_{\hat{\sigma}_h}} |\nabla v_h|^2 - |\nabla \widetilde{v}_h|^2 \,\mathrm{d}x - G_{\varepsilon_h}(\widetilde{v}_h, B_{\hat{\sigma}_h} \setminus B_{\tilde{\sigma}_h})\\
&\geqslant G_{\varepsilon_h}(\widetilde{v}_h, B_{\overline{\sigma}}) + \frac{1}{2m^2}\left(\int_{B_{\overline{\sigma}} \setminus B_{\hat{\sigma}_h}} |\nabla v_h|^2 \,\mathrm{d}x - 2\pi \log \frac{\overline{\sigma}}{\widehat{\sigma}_h}\right) - C_{\overline{\sigma}}(\delta + \varepsilon_h^{1/2})\,.
\end{aligned}$$

From (3.26) we then obtain

$$\begin{aligned}
&\int_{B_{\overline{\sigma}} \setminus B_{\hat{\sigma}_h}} |\nabla v_h|^2 \,\mathrm{d}x\\
&\quad\geqslant \int_{B_{\overline{\sigma}} \setminus B_{\hat{\sigma}_h}} \left|\nabla(e^{i\theta})\right|^2 \,\mathrm{d}x + \int_{B_{\overline{\sigma}} \setminus B_{\hat{\sigma}_h}} \left|\nabla v_h - \nabla(e^{i\theta})\right|^2 \,\mathrm{d}x\\
&\qquad\quad - 2\left(\int_{B_{\overline{\sigma}} \setminus B_{\hat{\sigma}_h}} \left|\nabla(e^{i\theta})\right|^2 \,\mathrm{d}x\right)^{1/2}\left(\int_{B_{\overline{\sigma}} \setminus B_{\hat{\sigma}_h}} \left|\nabla v_h - \nabla(e^{i\theta})\right|^2 \,\mathrm{d}x\right)^{1/2}\\
&\quad\geqslant 2\pi \log \frac{\overline{\sigma}}{\widehat{\sigma}_h} - C_{\overline{\sigma}}\delta\,,
\end{aligned}$$

and we deduce that for ε_h small enough,

$$G_{\varepsilon_h}(v_h, B_{\overline{\sigma}}) \geqslant G_{\varepsilon_h}(\widetilde{v}_h, B_{\overline{\sigma}}) - C_{\overline{\sigma}}(\delta + \varepsilon_h^{1/2})\,.$$

By the definition of $\gamma_m(\varepsilon,\overline{\sigma})$ (see (3.14)), we conclude that for ε_h small,

$$G_{\varepsilon_h}(v_h,B_{\overline{\sigma}})-\frac{\pi}{m^2}\log\frac{\overline{\sigma}}{\varepsilon_h}\geqslant\gamma_m(\varepsilon_h,\overline{\sigma})-C_{\overline{\sigma}}(\delta+\varepsilon_h^{1/2})\,,$$

which proves (3.25).

We can now complete the proof of (3.2). Indeed, by (3.25) and the convergence of v_h towards $e^{i\varphi}v_\mu$ in the weak $W^{1,2}_{\mathrm{loc}}(\Omega\setminus\operatorname{spt}\mu)$ topology,

$$\begin{aligned}(3.33)\quad &\liminf_{h\to\infty}\left\{G_{\varepsilon_h}(v_h)-\frac{\pi d}{m}|\log\varepsilon_h|\right\}\\
&=\liminf_{h\to\infty}\left\{G_{\varepsilon_h}\big(v_h,\Omega\setminus B_{\overline{\sigma}}(\mu)\big)-\frac{\pi d}{m}|\log\overline{\sigma}|+G_{\varepsilon_h}(v_h,B_{\overline{\sigma}}(\mu))-\frac{\pi d}{m}\log\frac{\overline{\sigma}}{\varepsilon_h}\right\}\\
&\geqslant\liminf_{h\to\infty}\left\{G_{\varepsilon_h}\big(v_h,\Omega\setminus B_{\overline{\sigma}}(\mu)\big)-\frac{\pi d}{m}|\log\overline{\sigma}|\right\}+md\lim_{h\to\infty}\gamma_m(\varepsilon_h,\overline{\sigma})-C_{\overline{\sigma}}\delta\\
&\geqslant\frac{1}{2m^2}\int_{\Omega\setminus B_{\overline{\sigma}}(\mu)}|\nabla(e^{i\varphi}v_\mu)|^2\,\mathrm{d}x-\frac{\pi d}{m}|\log\overline{\sigma}|+md\boldsymbol{\gamma}_m-C_{\overline{\sigma}}\delta\,,\end{aligned}$$

where the last inequality follows from (3.15) and Lemma 3.9. In view of Proposition 2.11, letting first $\delta\downarrow 0$ and then $\sigma\downarrow 0$ leads to (3.2).

Step 3. — In order to complete the proof, let us show that if equality holds in (3.17), then (3.4) and (3.5) hold. Note that (3.4) rewrites as

$$(3.34)\quad \lim_{h\to\infty}G_{\varepsilon_h}\big(v_h,\Omega\setminus B_r(\mu)\big)=\frac{1}{2m^2}\int_{\Omega\setminus B_r(\mu)}|\nabla(e^{i\varphi}v_\mu)|^2\,\mathrm{d}x\qquad\forall\, r>0\,,$$

which, combined with the weak convergence of $\{v_h\}$ in $W^{1,2}_{\mathrm{loc}}(\overline{\Omega}\setminus\operatorname{spt}\mu)$ to $e^{i\varphi}v_\mu$, classically leads to its strong convergence in $W^{1,2}_{\mathrm{loc}}(\overline{\Omega}\setminus\operatorname{spt}\mu)$.

From (3.2) and (3.3) we first infer that

$$\lim_{h\to\infty}I_{\eta_h}(\psi_h)=\mathcal{H}^1(J_u)+\mathcal{H}^1\big(\{u\neq g\}\cap\partial\Omega\big)\,,$$

and

$$\lim_{h\to\infty}\left\{G_{\varepsilon_h}(v_h)-\frac{\pi d}{m}|\log\varepsilon_h|\right\}=\frac{1}{2m^2}\int_\Omega|\nabla\varphi|^2\,\mathrm{d}x+\frac{1}{m^2}\mathbb{W}(\mu)+md\boldsymbol{\gamma}_m\,.$$

In view of (3.24), Case 2 in Step 2 above must hold. We argue by contradiction assuming that (3.34) does not hold. Then we can find a subsequence, $\sigma_0>0$, and $\eta_0>0$ such that

$$\liminf_{h\to\infty}G_{\varepsilon_h}\big(v_h,\Omega\setminus B_{\sigma_0}(\mu)\big)\geqslant\frac{1}{2m^2}\int_{\Omega\setminus B_{\sigma_0}(\mu)}|\nabla(e^{i\varphi}v_\mu)|^2\,\mathrm{d}x+\eta_0\,.$$

By lower semi-continuity of the Dirichlet energy, the same inequality holds for every $\sigma \in (0,\sigma_0)$. Then, for σ and δ small enough, we have by (3.33),

$$\begin{aligned}\frac{1}{2m^2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \frac{1}{m^2}\mathbb{W}(\mu) + md\boldsymbol{\gamma}_m &= \lim_{h\to\infty}\left\{G_{\varepsilon_h}(v_h) - \frac{\pi d}{m}|\log\varepsilon_h|\right\}\\ &\geqslant \frac{1}{2m^2}\int_{\Omega\setminus B_{\overline{\sigma}}(\mu)} |\nabla(e^{i\varphi}v_\mu)|^2\,\mathrm{d}x - \frac{\pi d}{m}|\log\overline{\sigma}| + md\boldsymbol{\gamma}_m - C_{\overline{\sigma}}\delta + \eta_0\,,\end{aligned}$$

where $\overline{\sigma}$ is determined from σ as in Case 2 above. Using Proposition 2.11 again, we let $\delta\to 0$ and then $\sigma\to 0$ to reach a contradiction.

To conclude, it only remains to prove that $v_h \to e^{i\varphi}v_\mu$ strongly in $W^{1,p}(\Omega)$ for every $p<2$. Fix an arbitrary $p\in(1,2)$. Since $\{|\nabla v_h|^p\}$ is bounded in $L^1(\Omega)$, we can extract a further subsequence such that $|\nabla v_h|^p \rightharpoonup |\nabla(e^{i\varphi}v_\mu)|^p + \nu_*$ weakly* as measures for some non negative finite measure ν_* on Ω. From the strong convergence of v_h in $W^{1,2}_{\mathrm{loc}}(\Omega\setminus\operatorname{spt}\mu)$, we infer that $\operatorname{spt}\nu_*\subseteq\operatorname{spt}\mu$. Since $\{\nabla v_h\}$ is also bounded in $L^q(\Omega)$ for every $q\in(p,2)$, we have by Hölder's inequality

$$\nu_*(\Omega) \leqslant \liminf_{h\to\infty}\int_{B_r(\mu)}|\nabla v_h|^p\,\mathrm{d}x \leqslant Cr^{2(1-p/q)} \qquad \text{for every } r>0 \text{ and } q\in(p,2)\,.$$

Letting $r\downarrow 0$ we deduce that $\nu_*\equiv 0$ which together with the weak convergence in $W^{1,p}(\Omega)$ of v_h towards $e^{i\varphi}v_\mu$ concludes the proof. $\square$

3.4. Proof of Theorem 3.1(iii): Construction of recovery sequences

In this section we prove the $\Gamma-$limsup inequality.

Proposition 3.14. — *Let $\varepsilon_h\downarrow 0$ and $\eta_h\downarrow 0$ be arbitrary sequences. For every $u\in\mathcal{L}_g(\Omega)$ with $u^m =: e^{i\varphi}v_\mu$, there exists a sequence $\{(u_h,\psi_h)\}\subseteq\mathcal{H}_g(\Omega)$ such that $\{u_h\}\subseteq SBV^2(\Omega)$, and*

- (i) *$(u_h,\psi_h)\to(u,1)$ strongly in $L^1(\Omega)$;*
- (ii) *$\mathrm{p}(u_h)\to u^m$ strongly in $W^{1,2}_{\mathrm{loc}}(\overline{\Omega}\setminus\operatorname{spt}\mu)$ and $W^{1,p}(\Omega)$ for every $p<2$;*
- (iii) *(3.6) and (3.7) hold.*

Moreover, we can choose u_h such that

$$\lim_{h\to\infty}\mathcal{H}^1(J_{u_h}) = \mathcal{H}^1(J_u) + \mathcal{H}^1\big(\{u\neq g\}\cap\partial\Omega\big)\,. \tag{3.35}$$

The proof of Proposition 3.14 relies on a suitable approximation procedure showing that maps in $\mathcal{L}_g(\Omega)$ having a compact jump set lying at a positive distance from the boundary are dense in energy. This is the purpose of the following section. The proof of Proposition 3.14 is then performed in Section 3.4.2.

3.4.1. Some density results

LEMMA 3.15. — *Let $u \in \mathcal{L}_g(\Omega)$ with $u^m =: e^{i\varphi} v_\mu$ and $\mu = 2\pi \sum_{k=1}^{md} \delta_{x_k}$. For every $\delta > 0$, there exists $u_* \in \mathcal{L}_g(\Omega)$ and constants $\xi_1, \dots, \xi_{md} \in \mathbf{S}^1$ such that*

(i) $\operatorname{dist}(J_{u_*}, \partial\Omega) > 0$, $u_* = g$ *on* $\partial\Omega$, *and* $\operatorname{curl} j(u_*^m) = \mu$*;*
(ii) $u_*^m(x) = \xi_k \frac{x - x_k}{|x - x_k|}$ *in a neighborhood of* x_k *for each* $k \in \{1, \dots, md\}$*;*
(iii) $\|u - u_*\|_{L^1(\Omega)} < \delta$ *and* $E_0(u_*) < E_0(u) + \delta$*;*
(iv) $\mathcal{H}^1(J_{u_*}) < \mathcal{H}^1(J_u) + \mathcal{H}^1(\{u \neq g\} \cap \partial\Omega) + \delta$*.*

Remark 3.16. — If one considers $\delta_k \downarrow 0$ and $\{u_k\} \subseteq \mathcal{L}_g(\Omega)$ the corresponding sequence provided by Lemma 3.15, then $u_k^m \to u^m$ strongly in $W^{1,p}(\Omega)$ for every $p < 2$. Indeed, writing $u_k^m = e^{i\varphi_k} v_\mu$, item (iii) implies the strong convergence of $\{\nabla\varphi_k\}$ in $L^2(\Omega)$, which in turn yields the strong convergence of $\{u_k^m\}$ in $W^{1,p}(\Omega)$.

Proof of Lemma 3.15.

Step 1. — We start by modifying u in such a way that (i), (iii), and (iv) hold. We proceed as follows. We consider the larger domain $\widetilde{\Omega}$ defined in (2.11), and we recall that the nearest point projection Π on $\partial\Omega$ is well defined and smooth in $\{x : \operatorname{dist}(x, \partial\Omega) < 2r_0\}$. Denote by $d_\Omega : \mathbf{R}^2 \to \mathbf{R}$ the signed distance to $\partial\Omega$, i.e., $d_\Omega(x) := \operatorname{dist}(x, \partial\Omega)$ for $x \in \mathbf{R}^2 \setminus \Omega$, and $d_\Omega(x) := -\operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$. Note that d_Ω is smooth in $\{x : \operatorname{dist}(x, \partial\Omega) < 2r_0\}$. Next we consider a smooth vector field $X \in C_c^\infty(\widetilde{\Omega} \setminus \operatorname{spt}\mu; \mathbf{R}^2)$ satisfying $X = \nabla d_\Omega$ in $\{x : \operatorname{dist}(x, \partial\Omega) < r_0/2\}$, and we fix $\sigma \in (0, r_0/2)$ such that $\operatorname{spt} X \subseteq \widetilde{\Omega} \setminus \overline{B}_\sigma(\mu)$. We denote by $\{\phi_t\}_{t \in \mathbf{R}}$ the integral flow on $\mathbf{R}^2$ generated by X. Then $\operatorname{spt}(\phi_t - \mathrm{id}) \subseteq \widetilde{\Omega} \setminus \overline{B}_\sigma(\mu)$ for every $t \in \mathbf{R}$, so that $\{\phi_t\}_{t \in \mathbf{R}}$ defines a one parameter family of smooth diffeomorphisms from $\widetilde{\Omega} \setminus \overline{B}_\sigma(\mu)$ onto $\widetilde{\Omega} \setminus \overline{B}_\sigma(\mu)$.

We extend u to $\widetilde{\Omega}$ by setting $u(x) := g\big(\Pi(x)\big)$ for $x \in \widetilde{\Omega} \setminus \Omega$. Then we set $u_t := u \circ \phi_t$, and we consider the restriction of u_t to Ω. By construction, for every $t > 0$ we have $u_t \in \mathcal{L}_g(\Omega)$, $u_t = g$ on $\partial\Omega$, $u_t = u$ in $B_\sigma(\mu)$, and $\operatorname{dist}(J_{u_t}, \partial\Omega) > 0$. Moreover, as $t \downarrow 0$, we have $u_t \to u$ weakly* in $BV(\Omega)$, $\nabla u_t \to \nabla u$ strongly in $L^2(\Omega \setminus B_\sigma(\mu))$, and

$$\mathcal{H}^1(J_{u_t}) \to \mathcal{H}^1(J_u) + \mathcal{H}^1(\{u \neq g\} \cap \partial\Omega) \,.$$

Given $\delta > 0$, we can then choose $t > 0$ small enough so that u_t satisfies (i), (iii), and (iv) of Lemma 3.15.

Step 2. — To complete the proof, we shall modify further u_t in $B_\sigma(\mu)$ to achieve property (ii). Fix $\sigma' \in (0,\sigma)$ such that $2\sigma' < \min\{|x_k - x_l| : 1 \leqslant k < l \leqslant md\}$. From the specific form of v_μ (see (2.14)), for each $k \in \{1,\dots,dm\}$ there exists $\psi_k \in W^{1,2}(B_{\sigma'}(x_k))$ such that

$$u^m(x) = e^{im\psi_k(x)}\,\frac{x - x_k}{|x - x_k|} \qquad \text{for } x \in B_{\sigma'}(x_k)\,.$$

For $\rho \geqslant 0$ and $\theta \in [0,2\pi)$ we set $\mathrm{q}_m(\rho e^{i\theta}) := \rho e^{i\theta/m}$, and define $\vartheta_k(x) := \mathrm{q}_m((x - x_k)/|x - x_k|)$ for $x \in B_{\sigma'}(x_k)$. Since $\operatorname{curl} j(\vartheta_k^m) = \mu = \operatorname{curl} j(u^m)$ in $B_{\sigma'}(x_k)$ and $\mathrm{p}(\vartheta_k e^{i\psi_k}) = \mathrm{p}(u)$, we can invoke Lemma 2.6 to infer that there exists $\widehat{u}_k \in SBV(B_{\sigma'}(x_k),\mathbf{G}_m)$ such that

$$u = \widehat{u}_k \vartheta_k e^{i\psi_k} \qquad \text{in } B_{\sigma'}(x_k)\,.$$

We now fix a cut-off function $\chi \in C_c^\infty(B_1,[0,1])$ such that, $\chi \equiv 1$ in $B_{1/2}$, and we set $\chi_{k,r}(x) := \chi((x - x_k)/r)$ for $r > 0$. We define for $k \in \{1,\dots,md\}$ and $r \in (0,\sigma')$,

$$u_{k,r}(x) := \widehat{u}_k(x)\vartheta_k(x)\exp\Big(i\psi_k(x) + i\chi_{k,r}(x)\big(\overline{\psi}_{k,r} - \psi_k(x)\big)\Big) \qquad \text{for } x \in B_{\sigma'}(x_k)\,,$$

where $\overline{\psi}_{k,r}$ denotes the mean value

$$\overline{\psi}_{k,r} := \frac{1}{\pi r^2}\int_{B_r(x_k)} \psi_k(x)\,\mathrm{d}x\,.$$

By construction, we have

$$u_{k,r}^m(x) = \xi_{k,r}\,\frac{x - x_k}{|x - x_k|} \qquad \text{for } x \in B_{r/2}(x_k)\,,$$

with $\xi_{k,r} := \exp(im\overline{\psi}_{k,r})$, and $u_{k,r} = u = u_t$ in $B_{\sigma'}(x_k) \setminus B_r(x_k)$. Finally, we set

$$u_*(x) := \begin{cases} u_{k,r}(x) & \text{if } x \in B_{\sigma'}(x_k)\,,\\ u_t(x) & \text{if } x \in \Omega \setminus B_{\sigma'}(\mu)\,. \end{cases}$$

By means of Poincaré's inequality (and Step 1), it is standard to check that for r small enough, u_* complies to all the requirements of the lemma. □

Now we show that we can substitute to u_* a mapping $u_\sharp$ with a compact jump set.

Lemma 3.17. — *Let $u_* \in \mathcal{L}_g(\Omega)$ be such that $\operatorname{dist}(J_{u_*}, \partial\Omega) > 0$. For every $\delta > 0$, there exist $u_\sharp \in \mathcal{L}_g(\Omega)$, a compact set $K \subseteq \Omega$, and $\sigma > 0$ such that*

(i) $\|u_\sharp - u_*\|_{L^1(\Omega)} \leqslant \delta$, $u_\sharp^m = u_*^m$, *and* $\mathcal{H}^1(J_{u_\sharp} \setminus J_{u_*}) = 0$;
(ii) $\mathcal{H}^1(K \triangle J_{u_\sharp}) = 0$;
(iii) $\mathcal{H}^1\big(K \cap B_r(x)\big) \geqslant r$ *for every $x \in K$ and every $r \in (0, \sigma)$.*

Proof.

Step 1. — Given a parameter $\lambda \geqslant 1$, we consider the class of maps

$$\mathscr{A} := \Big\{ u \in \mathcal{L}_g(\Omega) : u^m = u_*^m \,,\ \mathcal{H}^1(J_u \setminus J_{u_*}) = 0 \Big\}\,,$$

and the functional $A_\lambda : \mathscr{A} \to [0, \infty)$ defined by

$$A_\lambda(u) := \mathcal{H}^1(J_u) + \lambda \int_\Omega |u - u_*|^2 \, \mathrm{d}x\,.$$

We claim that A_λ admits at least one minimizer $u_\lambda \in \mathscr{A}$. Indeed, $|\nabla u| = \big|\nabla(\mathrm{P}(u_*)\big| = |\nabla u_*|$ for every $u \in \mathscr{A}$, so that $\{\nabla u : u \in \mathscr{A}\}$ is bounded in $L^p(\Omega)$ for every $p < 2$. Obviously, if $\{u_k\} \subseteq \mathscr{A}$ is a minimizing sequence, then $A_\lambda(u_k)$ is bounded. By the compactness theorem for SBV functions [4, Theorems 4.7 and 4.8, and Remark 4.9], we can find $u_\lambda \in SBV(\Omega)$ and a subsequence such that $u_k \to u_\lambda$ in $L^1(\Omega)$ and a.e. in Ω, and satisfying $A_\lambda(u_\lambda) \leqslant \liminf_k A_\lambda(u_k)$. From the pointwise convergence we infer that u_λ is $\mathbf{S}^1$-valued, and $u_\lambda^m = u_*^m$. Moreover, the sequence of positive measures $\{\mathcal{H}^1 \llcorner J_{u_k}\}$ weakly* converges towards $\mathcal{H}^1 \llcorner J_{u_\lambda}$. Since $\mathcal{H}^1 \llcorner J_{u_k} \leqslant \mathcal{H}^1 \llcorner J_{u_*}$, we have $\mathcal{H}^1 \llcorner J_{u_\lambda}(U) \leqslant \liminf_k \mathcal{H}^1 \llcorner J_{u_k}(U) \leqslant \mathcal{H}^1 \llcorner J_{u_*}(U)$ for every open set $U \subseteq \Omega$. By outer regularity, we deduce that $\mathcal{H}^1 \llcorner J_{u_\lambda} \leqslant \mathcal{H}^1 \llcorner J_{u_*}$. Hence $u_\lambda \in \mathscr{A}$, and thus u_λ is a minimizer of A_λ.

Noticing that $A_\lambda(u_\lambda) \leqslant A_\lambda(u_*)$ and $\mathcal{H}^1(J_{u_\lambda}) \leqslant \mathcal{H}^1(J_{u_*})$, we now deduce that $u_\lambda \to u_*$ in $L^1(\Omega)$ as $\lambda \to \infty$. Given $\delta > 0$, we choose λ large enough so that (i) holds with $u_\sharp := u_\lambda$. To complete the proof, we have to find a compact set K and $\sigma > 0$ such that (ii) and (iii) hold. This is the purpose of the next steps.

Step 2. — Write $\operatorname{curl} j(u_*^m) =: \mu = 2\pi \sum_{k=1}^{md} \delta_{x_k}$, and set

$$\sigma := \min\left(\frac{1}{8\lambda}\,,\, \operatorname{dist}(J_{u_*}, \partial\Omega)\,,\, \frac{1}{2} \min\big\{ |x_k - x_l| : 1 \leqslant k < l \leqslant md \big\} \right).$$

We claim that for $\mathcal{H}^1$-a.e. $x \in J_{u_\sharp}$, there holds

$$\mathcal{H}^1\big(J_{u_\sharp} \cap B_r(x)\big) \geqslant r \quad \text{for every } r \in (0, \sigma)\,. \tag{3.36}$$

Let us first recall that for $\mathcal{H}^1$-a.e. $x \in J_{u_\sharp}$,

$$\mathcal{H}^1\big(J_{u_\sharp} \cap B_r(x)\big) > 0 \quad \text{for every } r \in (0, \sigma)\,. \tag{3.37}$$

Hence, it is enough to establish (3.36) at every $x \in J_{u_\sharp} \setminus \operatorname{spt}\mu$ such that (3.37) holds. Let us fix such a point x and without loss of generality let us assume $x = 0$. Setting $\sigma_0 := \min\big(\sigma, \operatorname{dist}(0, \operatorname{spt}\mu)\big)$, we shall distinguish the two cases $r \in (0, \sigma_0]$ and $r \in (\sigma_0, \sigma)$.

Case 1: $r \in (0,\sigma_0]$. — Since $B_{\sigma_0} \subseteq \Omega \setminus \operatorname{spt}\mu$ and $u_\sharp^m = u_*^m$, we have $\operatorname{curl} j(u_\sharp) = 0$ in $\mathscr{D}'(B_{\sigma_0})$ by Lemma 2.3. Applying Lemma 2.6 in the ball B_{σ_0} (with $u_1 = 1$ and $u_2 = u_\sharp$), we obtain a function $\varphi \in W^{1,1}(B_{\sigma_0})$ and a Caccioppoli partition $\{E_k\}_{k=0}^{m-1}$ of the ball B_{σ_0} such that

$$u_\sharp = \left(\sum_{k=0}^{m-1} \mathbf{a}^k \chi_{E_k}\right) e^{i\varphi} \quad \text{in } B_{\sigma_0}\,.$$

Since $\varphi \in W^{1,1}(B_{\sigma_0})$, we have

$$J_{u_\sharp} \cap B_{\sigma_0} = \bigcup_{k=0}^{m-1} \partial E_k \quad \text{up to an } \mathcal{H}^1\text{-null set}\,,$$

where ∂E_k denotes the reduced boundary of E_k in B_{σ_0}. Moreover (see [4, Remark 4.22]),

$$\mathcal{H}^1(J_{u_\sharp} \cap B_r) = \frac{1}{2}\sum_{k=0}^{m-1} \mathcal{H}^1(\partial E_k \cap B_r) \quad \text{for every } r \in (0,\sigma_0)\,. \tag{3.38}$$

Before going any further, let us recall a slicing property of sets of finite perimeter (see e.g. [4, Chapter 3, Section 3.11] for further details). For a Borel set $E \subseteq \mathbf{R}^2$, we first denote by E^1 the (Borel) set of points of density 1 for E, i.e.,

$$E^1 := \left\{ y \in \mathbf{R}^2 : \lim_{\rho \downarrow 0} \frac{|E \cap B_\rho(y)|}{\pi\rho^2} = 1 \right\}.$$

For $r > 0$, we write $E_r := E^1 \cap \partial B_r$, and we consider E_r as a subset of ∂B_r (in particular, ∂E_r and $\operatorname{int}(E_r)$ denote the relative boundary and relative interior of E_r in ∂B_r, respectively). If $E \subseteq \mathbf{R}^2$ has a finite perimeter, then for a.e. $r > 0$ the following properties hold:

$$\partial E_r \text{ is finite and is equal to } \partial E \cap \partial B_r\,. \tag{3.39}$$

We point out that since $\{E_k\}$ is a Caccioppoli partition of B_{σ_0}, by [4, Theorem 4.17]

$$B_{\sigma_0} = \bigcup_{k=0}^{m-1} E_k^1 \cup \bigcup_{k=0}^{m-1} \partial E_k \quad \text{up to an } \mathcal{H}^1\text{-null set}\,. \tag{3.40}$$

For almost all radii $r \in (0,\sigma_0)$, (3.39) holds for each E_k, $k \in \{0,\dots,m-1\}$. We claim that for such radii

$$\sum_{k=0}^{m-1} \mathcal{H}^0(\partial E_k \cap \partial B_r) \geqslant 2. \tag{3.41}$$

To prove (3.41), let us fix such a r and assume by contradiction that

$$\partial E_k \cap \partial B_r = \partial (E_k)_r = \emptyset \qquad \text{for every } k \in \{0,\dots,m-1\}\,.$$

Then each $(E_k)_r$ is either equal to ∂B_r or empty, in particular $\mathcal{H}^1(E_k \cap \partial B_r) \in \{0, 2\pi r\}$. Since $\mathcal{H}^1(\partial E_k \cap \partial B_r) = 0$ for each k by (3.39), we infer from (3.40) that $\mathcal{H}^1(E_{k_r} \cap \partial B_r) = 2\pi r$ for exactly one index $k_r \in \{0,\dots,m-1\}$. Consequently, every point of ∂B_r is a point of density 1 for E_{k_r}, and of density 0 for E_k with $k \neq k_r$. We then introduce the competitor

$$\widetilde{u} := \mathbf{a}^{k_r} e^{i\varphi} \quad \text{in } B_r\,, \quad \widetilde{u} := u_\sharp \quad \text{in } \Omega \setminus B_r\,.$$

By construction $J_{\tilde{u}} \setminus \overline{B}_r = J_{u_\sharp} \setminus \overline{B}_r$ and $J_{\tilde u} \cap \overline{B_r} = \emptyset$, so that $\widetilde{u}$ is an admissible competitor. By optimality of $u_\sharp$, we have $A_\lambda(\widetilde{u}) \geqslant A_\lambda(u_\sharp)$. Using that

$$|\widetilde{u}-u_*|^2 - |u_\sharp - u_*|^2 = \left(|\widetilde{u}-u_*| + |u_\sharp - u_*|\right)\left(|\widetilde{u}-u_*| - |u_\sharp - u_*|\right) \leqslant 4|\widetilde{u}-u_\sharp|$$

(since $|\widetilde{u}| = |u_*| = |u_\sharp| = 1$), we compute

$$\begin{aligned}
0 \leqslant A_\lambda(\widetilde{u}) - A_\lambda(u_\sharp) &= \lambda \int_{B_r} |\widetilde{u}-u_*|^2 - |u_\sharp - u_*|^2 \,\mathrm{d}x - \mathcal{H}^1(J_{u_\sharp} \cap B_r)\\
&\leqslant\; 4\lambda \int_{B_r} |\widetilde{u} - u_\sharp| \,\mathrm{d}x - \mathcal{H}^1(J_{u_\sharp} \cap B_r)\\
&= 4\lambda \sum_{k \neq k_r} |1 - \mathbf{a}^{k-k_r}||E_k \cap B_r| - \frac{1}{2}\sum_{k=0}^{m-1} \mathcal{H}^1(\partial E_k \cap B_r)\\
&\leqslant \frac{1}{2} \sum_{k\neq k_r} \Big\{ 16\lambda |E_k \cap B_r| - \mathcal{H}^1(\partial E_k \cap B_r)\Big\}\,.
\end{aligned}$$

Since $\mathcal{H}^1(\partial E_k \cap \partial B_r) = 0$ and $(E_k)^1 \cap \partial B_r = \emptyset$ for $k \neq k_r$, we infer that $E_k \cap B_r$ has finite perimeter and $\partial(E_k \cap B_r) = \partial E_k \cap B_r$ for $k \neq k_r$. Therefore,

$$\sum_{k \neq k_r} \Big\{ 16\lambda |E_k \cap B_r| - \mathcal{H}^1\big(\partial(E_k \cap B_r)\big)\Big\} \geqslant 0\,.$$

By the (two dimensional) isoperimetric inequality, we have

$$|E_k \cap B_r| \leqslant \sqrt{|B_r|}\sqrt{|E_k \cap B_r|} \leqslant \frac{r}{2}\mathcal{H}^1\big(\partial(E_k \cap B_r)\big)\,,$$

so that

$$(8\lambda r-1)\sum_{k\neq k_r}\mathcal{H}^1\big(\partial(E_k\cap B_r)\big)\geqslant 0\,.$$

Since $r<\sigma\leqslant(8\lambda)^{-1}$, the prefactor above is negative, and we deduce that $\mathcal{H}^1(\partial(E_k\cap B_r))=0$ for each $k\neq k_r$. As a consequence $B_r\subseteq E^1_{k_r}$, so that $u_\sharp=\widetilde{u}\,$. In particular, $\mathcal{H}^1(J_{u_\sharp}\cap B_r)=0$ which contradicts (3.37), and (3.41) is proved.

By (3.41), we can now infer from the coarea formula (see [47, Theorem II.7.7] for instance) that $r\in(0,\sigma_0]$

$$\begin{aligned}\sum_{k=0}^{m-1}\mathcal{H}^1(\partial E_k\cap B_r)&\geqslant\sum_{k=0}^{m-1}\int_0^r\mathcal{H}^0(\partial E_k\cap\partial B_t)\,\mathrm{d}t\\&=\int_0^r\sum_{k=0}^{m-1}\mathcal{H}^0(\partial E_k\cap\partial B_t)\,\mathrm{d}t\geqslant 2r.\end{aligned}$$

Combining this inequality with (3.38) yields (3.36) for every $r\in(0,\sigma_0]$.

Case 2: $r\in(\sigma_0,\sigma)$. — In this case, we must have $\sigma_0<\sigma$, so that $\sigma_0=\operatorname{dist}(0,\operatorname{spt}\mu)$ and $B_\sigma\cap\operatorname{spt}\mu\neq\emptyset$. Our choice of σ then implies that $B_\sigma\cap\operatorname{spt}\mu$ is reduced to a singleton, i.e., there exists $k_0\in\{0,\dots,m-1\}$ such that $B_\sigma\cap\operatorname{spt}\mu=\{x_{k_0}\}$. Moreover, $x_{k_0}\in\partial B_{\sigma_0}$.

By the definition of $\mathcal{L}_g(\Omega)$ and the slicing properties of BV-functions (see e.g. [4, Chapter 3, Section 3.11] for details), for a.e. $r\in(\sigma_0,\sigma)$ we have $\operatorname{spt}\mu\cap\partial B_r=\emptyset$, the trace $u_r:=u_{\sharp|\partial B_r}$ belongs to $SBV^2(\partial B_r;\mathbf{S}^1)$, and $J_{u_r}=J_{u_\sharp}\cap\partial B_r$ is finite. We claim that for every such r,

$$\mathcal{H}^0(J_{u_r})\geqslant 1. \tag{3.42}$$

Indeed, let r be such a radius and assume by contradiction that $J_{u_r}=\emptyset$. Then $u_r\in W^{1,2}(\partial B_r,\mathbf{S}^1)$ and thus u_r is continuous. In addition, the trace $(u_\sharp^m)_r:=u^m_{\sharp\,|\partial B_r}$ belongs to $W^{1,2}(\partial B_r,\mathbf{S}^1)$ and it satisfies $(u^m_\sharp)_r=(u_r)^m$. Hence the topological degree ℓ of $(u^m_\sharp)_r$ is equal to m times the topological degree of u_r. Moreover, we have $u^m_\sharp(x)=e^{i\psi}(x-x_{k_0})/|x-x_{k_0}|$ in B_r for some $\psi\in W^{1,2}(B_r)$ satisfying $\psi_{|\partial B_r}\in W^{1,2}(\partial B_r)$. Hence $\ell=1$ which contradicts $\ell\in m\mathbf{Z}$, and (3.42) is proved.

By (3.42), we can infer again from the coarea formula that

$$\mathcal{H}^1\big(J_{u_\sharp}\cap(B_r\setminus B_{\sigma_0})\big)\geqslant\int_{\sigma_0}^r\mathcal{H}^0(J_{u_\sharp}\cap\partial B_t)\,\mathrm{d}t=\int_{\sigma_0}^r\mathcal{H}^0(J_{u_t})\,\mathrm{d}t\geqslant r-\sigma_0$$

for every $r\in(\sigma_0,\sigma)$. Since $\mathcal{H}^1(J_{u_\sharp}\cap B_{\sigma_0})\geqslant\sigma_0$ by Case 1, we deduce that

$$\mathcal{H}^1(J_{u_\sharp}\cap B_r)=\mathcal{H}^1\big(J_{u_\sharp}\cap(B_r\setminus B_{\sigma_0})\big)+\mathcal{H}^1(J_{u_\sharp}\cap B_{\sigma_0})\geqslant(r-\sigma_0)+\sigma_0=r$$

for every $r \in (\sigma_0, \sigma)$, and (3.36) is proved for every $r \in (\sigma_0, \sigma)$.

Step 3. — We define

$$K := \left\{ x \in \Omega \ : \ \mathcal{H}^1\big(J_{u_\sharp} \cap B_r(x)\big) \geqslant r \text{ for every } r \in (0, \sigma) \right\} .$$

By definition, K is closed and since $\operatorname{dist}(J_{u_*}, \partial\Omega) > 0$, it is a compact subset of Ω. On the one hand, we can deduce from Step 2 that $\mathcal{H}^1(J_{u_\sharp} \setminus K) = 0$. On the other hand,

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^1(J_{u_\sharp} \cap B_r(x))}{r} = 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \Omega \setminus J_{u_\sharp} ,$$

see e.g. [4, Section 2.9, Theorem 2.56 and (2.42)]. In particular, $\mathcal{H}^1(K \setminus J_{u_\sharp}) = 0$, and therefore $\mathcal{H}^1(K \triangle J_{u_\sharp}) = 0$. □

3.4.2. Proof of Proposition 3.14

Proof. — Thanks to Lemma 3.15, Remark 3.16, and Lemma 3.17 and a diagonal argument, it is enough to make the construction for a map $u \in \mathcal{L}_g(\Omega)$ with $u^m =: e^{i\varphi} v_\mu$ and $\mu = 2\pi \sum_{k=1}^{md} \delta_{x_k}$ and such that there exists $\sigma > 0$ and a compact set $K \subseteq \Omega$ such that

(a) $\operatorname{dist}(J_u, \partial\Omega) \geqslant 2\sigma$ and $u = g$ on $\partial\Omega$;
(b) $\operatorname{dist}(\operatorname{spt} \mu, \partial\Omega) \geqslant 2\sigma$ and $|x_k - x_l| \geqslant 2\sigma$ for $k \neq l$;
(c) $u^m(x) = \xi_k (x - x_k)/|x - x_k|$ in each $B_\sigma(x_k)$ for some $\xi_k \in \mathbf{S}^1$;
(d) $\mathcal{H}^1(K \triangle J_u) = 0$;
(e) $\operatorname{dist}(K, \partial\Omega) \geqslant 2\sigma$ and $\mathcal{H}^1\big(K \cap B_r(x)\big) \geqslant r$ for every $x \in K$ and $r \in (0, \sigma)$.

By a further diagonal argument, it is enough to fix $\delta > 0$ and construct a sequence $\{(u_h, \psi_h)\} \subseteq \mathcal{H}_g$ such that $\{u_h\} \subseteq \mathcal{G}_g(\Omega) \cap L^\infty(\Omega)$ with $\|u_h\|_{L^\infty(\Omega)} \leqslant 1$, $(u_h, \psi_h) \to (u, 1)$ in $L^1(\Omega)$ as $h \to \infty$, $\mathrm{p}(u_h) \to u^m$ strongly in $W^{1,p}(\Omega)$ for every $p < 2$, and

$$\limsup_{h \to \infty} \left\{ E_{\varepsilon_h}\big(\mathrm{P}(u_h)\big) - \frac{\pi d}{m} |\log \varepsilon_h| \right\} \leqslant E_0(u) + \delta \,, \tag{3.43}$$

$$\limsup_{h \to \infty} \mathcal{H}^1(J_{u_h}) \leqslant \mathcal{H}^1(J_u) \,, \tag{3.44}$$

$$\limsup_{h \to \infty} I_{\eta_h}(\psi_h) \leqslant (1 + \delta) \mathcal{H}^1(J_u) \,. \tag{3.45}$$

First, by Corollary 3.11 we can find $\varepsilon \in (0,1)$, $\widetilde{u} \in SBV^2(B_1) \cap L^\infty(B_1)$ with $\|\widetilde{u}\|_{L^\infty(B_1)} \leqslant 1$, and a closed smooth curve $\Sigma \subseteq \overline{B_1}$ such that $\mathrm{p}(\widetilde{u})(x) = x$ in a neighborhood of ∂B_1, $J_{\tilde{u}} \subseteq \Sigma$, and

$$E_\varepsilon\big(\mathrm{P}(\widetilde{u}), B_1\big) - \frac{\pi}{m^2} \log \frac{1}{\varepsilon} \leqslant \boldsymbol{\gamma}_m + \frac{\delta}{md} \,.$$

Then, for each $k \in \{1, \dots, md\}$ we select $\zeta_k \in \mathbf{S}^1$ such that $\zeta_k^m = \xi_k$, and we set for $\varepsilon_h \in (0, \sigma\varepsilon)$,

$$u_h(x) := \begin{cases} u(x) & \text{if } x \in \Omega \setminus B_{\varepsilon_h/\varepsilon}(\mu)\,, \\ \zeta_k \widetilde{u}\big(\frac{\varepsilon}{\varepsilon_h}(x - x_k)\big) & \text{if } x \in B_{\varepsilon_h/\varepsilon}(x_k)\,,\ k \in \{1, \dots, dm\}\,. \end{cases}$$

By construction, we have $u_h \in \mathcal{G}_g(\Omega) \cap L^\infty(\Omega)$ with $\|u_h\|_{L^\infty(\Omega)} \leqslant 1$ (since $|u| = 1$), and

$$\|u_h - u\|_{L^1(\Omega)} \leqslant 2\pi md(\varepsilon_h/\varepsilon)^2 \underset{h\to\infty}{\longrightarrow} 0\,.$$

For $p < 2$, we estimate

$$\begin{aligned} &\|\mathrm{p}(u_h) - u^m\|_{W^{1,p}(\Omega)} \\ &\qquad \lesssim \|u^m\|_{W^{1,p}(B_{\varepsilon_h/\varepsilon}(\mu))} + (\varepsilon_h/\varepsilon)^{(2-p)/p} \|\mathrm{p}(\widetilde{u})\|_{W^{1,p}(B_1)} \underset{h\to\infty}{\longrightarrow} 0\,. \end{aligned}$$

Next, changing variables, we have for each $k \in \{1, \dots, md\}$,

$$E_{\varepsilon_h}\big(\mathrm{P}(u_h); B_{\varepsilon_h/\varepsilon}(x_k)\big) = E_\varepsilon\big(\mathrm{P}(\widetilde{u}); B_1\big) \leqslant \frac{\pi}{m^2} \log\frac{1}{\varepsilon} + \boldsymbol{\gamma}_m + \frac{\delta}{md}\,.$$

Consequently,

$$\begin{aligned} E_{\varepsilon_h}\big(\mathrm{P}(u_h)\big) - \frac{\pi d}{m}\log\frac{1}{\varepsilon_h} &\leqslant E_{\varepsilon_h}\big(\mathrm{P}(u), \Omega\setminus B_{\varepsilon_h/\varepsilon}(\mu)\big) - \frac{\pi d}{m}\log\frac{\varepsilon}{\varepsilon_h} + md\boldsymbol{\gamma}_m + \delta \\ &= \frac{1}{2m^2}\int_{\Omega\setminus B_{\varepsilon_h/\varepsilon}(\mu)} |\nabla(u^m)|^2\,\mathrm{d}x - \frac{\pi d}{m}\log\frac{\varepsilon}{\varepsilon_h} + md\boldsymbol{\gamma}_m + \delta\,, \end{aligned}$$

and (3.43) follows from Proposition 2.11.

By construction, we have

$$J_{u_h} \subseteq K_h := K \cup \Big(\operatorname{spt}\mu + \frac{\varepsilon_h}{\varepsilon}(\partial B_1 \cup \Sigma)\Big) \quad \text{up to an } \mathcal{H}^1\text{-null set}\,,$$

so that

$$\mathcal{H}^1(J_{u_h}) \leqslant \mathcal{H}^1(K) + md\frac{\varepsilon_h}{\varepsilon}\big(2\pi + \mathcal{H}^1(\Sigma)\big)\,.$$

Since $\mathcal{H}^1(K) = \mathcal{H}^1(J_u)$, (3.44) follows letting $h \to \infty$ in the inequality above.

To produce the sequence $\{\psi_h\}$, we argue in a way similar to [5, Section 5]. We start by introducing a family of smooth profiles approximating the optimal profile $\psi_\star(s) = 1 - e^{-|s|}$ from Lemma 3.12. For $\lambda > 0$ we define $\psi_\lambda : [0, \infty) \to [0, 1]$ as

$$\psi_\lambda(t) := \begin{cases} 1 - \exp\big(-\frac{\lambda t}{\lambda - t}\big) & \text{if } t < \lambda\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

Notice that $1-\psi_\lambda$ and ψ'_λ are supported on $[0,\lambda]$. Setting $e_\lambda(t):=\big(\psi'_\lambda(t)\big)^2+\big(1-\psi_\lambda(t)\big)^2$, elementary computations yield

$$s_\lambda := \int_0^\lambda e_\lambda(t)\,\mathrm{d}t \underset{\lambda\to\infty}{\longrightarrow} 1\,.$$

Hence we can find $\lambda>0$ such that $s_\lambda\leqslant 1+\delta$. Setting $d_h(x):=\operatorname{dist}(x,K_h)$, we define ψ_h as

$$\psi_h(x) := \psi_\lambda\left(\frac{d_h(x)}{\eta_h}\right).$$

The function ψ_h is Lipschitz continuous and for $\varepsilon_h\in(0,\sigma\varepsilon)$ we have $\operatorname{dist}(K_h,\partial\Omega)>\sigma$ so that $\psi_h=1$ on $\partial\Omega$ whenever $\eta_h\in(0,\sigma/\lambda)$. Since $u_h\in W^{1,2}(\Omega\setminus K_h)$ and $\psi_h=0$ on K_h, we infer that $(u_h,\psi_h)\in\mathcal{H}_g(\Omega)$.

To estimate $I_{\eta_h}(\psi_h)$, we first notice that d_h is 1-Lipschitz. This leads to

$$I_{\eta_h}(\psi_h)\leqslant \frac{1}{2\eta_h}\int_\Omega e_\lambda\big(d_h(x)/\eta_h\big)\,\mathrm{d}x = \frac{1}{2\eta_h}\int_{K_h+B_{\lambda\eta_h}} e_\lambda\big(d_h(x)/\eta_h\big)\,\mathrm{d}x\,.$$

We use Fubini's theorem to obtain

$$\begin{aligned}\int_{K_h+B_{\lambda\eta_h}} e_\lambda\big(d_h(x)/\eta_h\big)\,\mathrm{d}x &= -\int_{K_h+B_{\lambda\eta_h}}\left(\int_{d_h(x)/\eta_h}^\lambda e'_\lambda(t)\,\mathrm{d}t\right)\mathrm{d}x\\ &= -\int_0^\lambda e'_\lambda(t)|K_h+B_{t\eta_h}|\,\mathrm{d}t\,.\end{aligned}$$

Noticing that $e'_\lambda(t)\leqslant 0$ and

$$|K_h+B_{t\eta_h}|\leqslant |K+B_{t\eta_h}| + md\left|\frac{\varepsilon_h}{\varepsilon}(\partial B_1\cup\Sigma)+B_{t\eta_h}\right|,$$

we derive

$$\begin{aligned}I_{\eta_h}(\psi_h)\leqslant &-\int_0^\lambda te'_\lambda(t)\frac{|K+B_{t\eta_h}|}{2t\eta_h}\,\mathrm{d}t\\ &- md\int_0^\lambda te'_\lambda(t)\frac{\left|\frac{\varepsilon_h}{\varepsilon}(\partial B_1\cup\Sigma)+B_{t\eta_h}\right|}{2t\eta_h}\,\mathrm{d}t =: J_h^1+J_h^2\,.\end{aligned}\tag{3.46}$$

By the Ahlfors regularity assumption on K stated in (e), we have (see e.g. [4, Theorem 2.104])

$$\lim_{r\downarrow 0}\frac{|K+B_r|}{2r} = \mathcal{H}^1(K)\,.$$

Hence, by dominated convergence

$$\lim_{h\to\infty} J_h^1 = -\mathcal{H}^1(K)\int_0^\lambda te'_\lambda(t)\,\mathrm{d}t = s_\lambda\mathcal{H}^1(J_u)\leqslant (1+\delta)\mathcal{H}^1(J_u)\,.\tag{3.47}$$

We now claim that

$$\lim_{h\to\infty} J_h^2 = 0\,. \tag{3.48}$$

To prove (3.48) we may argue on subsequences if necessary. We distinguish the two complementary cases $\lim_h(\varepsilon_h/\eta_h) < \infty$ and $\lim_h(\varepsilon_h/\eta_h) = \infty$. If $\lim_h(\varepsilon_h/\eta_h) < \infty$, then we estimate for $t \in (0, \lambda)$

$$\left|\frac{\varepsilon_h}{\varepsilon}(\partial B_1 \cup \Sigma) + B_{t\eta_h}\right| \leqslant \left|B_{t\eta_h+\varepsilon_h/\varepsilon}\right| \lesssim t^2\eta_h^2 + (\varepsilon_h/\varepsilon)^2 \leqslant C\lambda^2\eta_h^2\,,$$

so that $J_h^2 = O(\eta_h)$ as $h \to \infty$. We now assume that $\lim_h(\varepsilon_h/\eta_h) = \infty$, and we write for $t > 0$,

$$\frac{\left|\frac{\varepsilon_h}{\varepsilon}(\partial B_1 \cup \Sigma) + B_{t\eta_h}\right|}{2t\eta_h} = \left(\frac{\varepsilon_h}{\varepsilon}\right)\frac{\left|(\partial B_1 \cup \Sigma) + B_{t\varepsilon\eta_h/\varepsilon_h}\right|}{2t\varepsilon\eta_h/\varepsilon_h}\,.$$

By smoothness of ∂B_1 and Σ, we have

$$\lim_{r\downarrow 0}\frac{|(\partial B_1 \cup \Sigma) + B_r|}{2r} = \mathcal{H}^1(\partial B_1 \cup \Sigma)\,,$$

and we conclude that $J_h^2 = O(\varepsilon_h)$ as $h \to \infty$. In both cases (3.48) holds true.

Eventually, putting together (3.46), (3.47), and (3.48) leads to (3.45). □

3.5. Proof of Theorem 3.2

The proof of Theorem 3.2 is very similar to the proof of Theorem 3.1 but we include a sketch of proof for the reader's convenience. It is divided as usual into three parts: compactness, the Γ-lim inf inequality, and the construction of recovery sequences. Concerning recovery sequences, we note that they are actually provided by Proposition 3.14 since (3.6) and (3.35) clearly lead to (3.10). We proceed with the proof of points (i) and (ii) of the theorem.

Compactness, proof of (i). — For the compactness part, we argue as in the proof of Proposition 3.6 and apply in particular Theorem 2.10. We can argue in particular that $\{\nabla u_h\}$ is bounded in $L^p(\Omega)$ and that $\mathcal{H}^1(J_{u_h})$ is bounded. Therefore, since we also know that $\|u_h\|_{L^\infty(\Omega)}$ is bounded, we may apply [4, Theorem 4.8] to conclude the proof as in Theorem 2.8. □

The Γ-lim inf *inequality, proof of* (ii). — Without loss of generality, we can assume that

$$\liminf_{h\to\infty} \widetilde{F}^0_{\varepsilon_h}(u_h) = \lim_{h\to\infty} \widetilde{F}^0_{\varepsilon_h}(u_h) < \infty\,.$$

Moreover, the truncation argument in the proof of Theorem 2.8 together with (3.19) shows that u_h can be replaced by $\widehat{u}_h$ defined in (2.9). Hence we can also assume that $\|u_h\|_{L^\infty(\Omega)} \leqslant 1$, and the proof of (i) above applies. In particular $v_h := \mathrm{p}(u_h)$ satisfies (2.18), and we reproduce verbatim the proof of Theorem 3.1 to show that (3.2) holds.

Moreover, up to extending u_h and u to a larger domain $\widetilde{\Omega}$ as in the proof of Theorem 2.8, we may assume that $\mathcal{H}^1(\{u_h \neq g\} \cap \partial\Omega) = \mathcal{H}^1(\{u \neq g\} \cap \partial\Omega) = 0$. We consider the sequence of non negative finite measures on Ω

$$\nu_h := \mathcal{H}^1 \llcorner (J_{u_h \cap \Omega}) .$$

Since $\nu_h(\Omega) = \mathcal{H}^1(J_{u_h})$ is bounded, we can extract a further subsequence such that $\nu_h \rightharpoonup \nu_*$ weakly* as measures for some non negative finite measure ν_* on Ω. Since $\operatorname{spt} \nu_h \subseteq \overline{\Omega}$, we have $\operatorname{spt} \nu_* \subseteq \overline{\Omega}$ and $\nu_h(\Omega) \to \nu_*(\Omega)$.

We claim that

$$\nu_* \geqslant \mathcal{H}^1 \llcorner (J_u \cap \Omega) . \tag{3.49}$$

Before proving (3.49), we observe that it implies

$$\lim_{h\to\infty} \mathcal{H}^1(J_{u_h}) = \lim_{h\to\infty} \nu_h(\Omega) = \nu_*(\Omega) \geqslant \mathcal{H}^1(J_u) = \mathcal{H}^1(J_u) , \tag{3.50}$$

which, combined with (3.2), leads to (3.8).

To prove (3.49), we fix an open set $A \subseteq \Omega$. Consider an arbitrary compact set $K \subseteq A$, and choose another open set B such that $K \subseteq B \subseteq \overline{B} \subseteq A$. By the proof of (i) above, we can apply [4, Theorem 4.7] in the open set B to derive

$$\nu_*(A) \geqslant \nu_*(\overline{B}) \geqslant \liminf_{h\to\infty} \nu_h(B) = \liminf_{h\to\infty} \mathcal{H}^1(J_{u_h} \cap B) \geqslant \mathcal{H}^1(J_u \cap B) .$$

Hence $\nu_*(A) \geqslant \mathcal{H}^1(J_u \cap K)$, and by inner regularity it implies that $\nu_*(A) \geqslant \mathcal{H}^1(J_u \cap A)$. By outer regularity, we conclude that (3.49) holds.

Let us now assume that $F_0(u) = \lim_h \widehat{F}^0_\varepsilon(u_h)$. In view of (3.2) and (3.50), we have

$$\lim_{h\to\infty} \left\{ G_{\varepsilon_h}(v_h) - \frac{\pi d}{m} |\log \varepsilon_h| \right\} = \frac{1}{2m^2} \int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x + \frac{1}{m^2} \mathbb{W}(\mu) + md\boldsymbol{\gamma}_m , \tag{3.51}$$

and

$$\nu_*(\Omega) = \lim_{h\to\infty} \mathcal{H}^1(J_{u_h}) = \mathcal{H}^1(J_u) . \tag{3.52}$$

From (3.51), we can argue exactly as in the proof of Proposition 3.13, Step 3, to deduce that $v_h \to u^m$ strongly in $W^{1,p}(\Omega)$ for every $p < 2$ and strongly in $W^{1,2}_{\rm loc}(\Omega\setminus\operatorname{spt}\mu)$, and that (3.4) holds. Then, we infer from (3.49) and (3.52) that $\nu_*\equiv\mathcal{H}^1\llcorner J_u$. As a consequence, if $A\subseteq\mathbf{R}^2$ is an open set such that $\nu_*(\partial A)=0$, then

$$\mathcal{H}^1(J_u\cap A)=\nu_*(A)=\lim_{h\to\infty}\nu_h(A)=\mathcal{H}^1(J_{u_h}\cap A)\,,$$

which proves (3.9) since $\nu_*(\partial A)=\mathcal{H}^1(J_u\cap\partial A)$. □

4. The limiting problem

The aim of this section is to study minimizers over $\mathcal{L}_g(\Omega)$ of the limiting functional $F_{0,g}$. To avoid some technical issues (at the boundary), we shall assume for simplicity that Ω is a smooth bounded *convex* set.

By Remark 3.5 and Lemma 2.3, minimizers of $F_{0,g}$ over $\mathcal{L}_g(\Omega)$ coincide with solutions of

$$\min\left\{\frac{1}{m^2}\mathbb{W}(\mu)+\mathcal{H}^1(J_u)+\mathcal{H}^1\big(\{u\neq g\}\cap\partial\Omega\big):\right.\\ \left. u\in SBV(\Omega;\mathbf{S}^1)\,,\ \mu:=m\operatorname{curl}j(u)\in\mathcal{A}_d\text{ and }u^m=v_\mu\right\}. \tag{4.1}$$

In turn, (4.1) amounts to solve for each $\mu\in\mathcal{A}_d$,

$$L(\mu):=\min\Big\{\mathcal{H}^1(J_u)+\mathcal{H}^1\big(\{u\neq g\}\cap\partial\Omega\big):u\in SBV(\Omega,\mathbf{S}^1)\,,\ u^m=v_\mu\Big\}, \tag{4.2}$$

and then minimize $\frac{1}{m^2}\mathbb{W}(\,\cdot\,)+L(\,\cdot\,)$ over $\mathcal{A}_d$ which is a finite dimensional optimization problem. Let us however point out that since on the one hand Steiner type problems are usually very hard to solve and since on the other hand, the minimization of $\mathbb{W}$ can be rarely explicitly done (see [38]), this finite dimensional problem does not seem easy to handle.

For the rest of this section we fix a measure $\mu\in\mathcal{A}_d$, and focus on problem (4.2). First, we notice that existence of minimizers in (4.2) follows as in the proof of Lemma 3.17 (Step 1) since $|\nabla v_\mu|\in L^p(\Omega)$ for every $p<2$ and $|\nabla u|=\frac{1}{m}|\nabla v_\mu|$ for any admissible competitor u (see Lemma 2.3). We will prove that minimizers of (4.2) are related to a variant of the Steiner problem that we now describe.

Write $\mu=:2\pi\sum_{k=1}^{md}\delta_{x_k}$, and recall that the x_k's are distinct points of the domain Ω. We let

$$\Lambda(\mu):=\min\Big\{\mathcal{H}^1(\Gamma)\,:\,\operatorname{spt}\mu\subseteq\Gamma,\text{ and for every connected}\\ \text{component }\Sigma\text{ of }\Gamma,\ \operatorname{Card}(\Sigma\cap\operatorname{spt}\mu)\in m\mathbf{N}\Big\}. \tag{4.3}$$

Notice that since we can always remove connected components which do not contain any vortex x_k, we can reduce the above minimization problem to sets Γ with the property that every connected component contains a positive number of vortices. Of course, this implies that Γ has at most d connected components. Each of these connected components Σ is a competitor for the Steiner problem related to $\mathrm{spt}\mu \cap \Sigma$. This shows that minimizers of (4.3) exist and are made of (at most d) Steiner trees i.e. finite union of segments meeting only at the vortices or at triple junctions (see [50, Prop. 2.2]).

Definition 4.1. — *A compact set Γ is said to be a $\Lambda(\mu)$-minimizer if it solves* (4.3).

Remark 4.2. — Since we assumed that Ω is convex, any $\Lambda(\mu)$-minimizer Γ is contained in the convex hull of $\mathrm{spt}\,\mu$. More precisely, since projecting on convex sets reduces distances, if Σ is a connected component of Γ, then Σ is contained in the convex hull of $\Sigma \cap \mathrm{spt}\,\mu$. Since Σ is a tree, we also infer the following property: if $C \subseteq \Sigma$ is an open segment, such that $C \cap \mathrm{spt}\,\mu = \emptyset$, then $\Sigma \setminus C$ is made of two connected components A and B satisfying $\mathrm{Card}(A \cap \mathrm{spt}\,\mu) \notin m\mathbf{N} \setminus \{0\}$ and $\mathrm{Card}(B \cap \mathrm{spt}\,\mu) \notin m\mathbf{N} \setminus \{0\}$. Otherwise, $\Gamma \setminus C$ would be an admissible competitor for $\Lambda(\mu)$ with strictly lower length, contradicting minimality.

We are now ready to prove the main result of this section, which states that the jump set of any minimizer of (4.2) is a $\Lambda(\mu)-$minimizer.

Theorem 4.3. — *Assume that Ω is a smooth, bounded, and convex open set. For every $\mu \in \mathcal{A}_d$, it holds*

$$L(\mu) = \Lambda(\mu)\,. \tag{4.4}$$

Moreover, if u is a minimizer of $L(\mu)$, then J_u is a $\Lambda(\mu)$-minimizer, $u \in C^\infty(\overline{\Omega} \setminus J_u)$, and $u = g$ on $\partial\Omega$. Vice-versa, if Γ is a $\Lambda(\mu)$-minimizer, then there exists u minimizing $L(\mu)$ with $J_u = \Gamma$.

Proof.

Step 1. — Let $\mu = 2\pi \sum_{k=1}^{md} \delta_{x_k}$ be fixed. We first prove that

$$L(\mu) \leqslant \Lambda(\mu)\,.$$

Consider Γ a $\Lambda(\mu)$-minimizer. Treating each connected component of Γ separately, we may assume without loss of generality that Γ is connected. Since $\mathbf{R}^2 \setminus \Gamma$ is connected, we can find a smooth injective curve with arc-length parameterization $\gamma_1 : [0,\infty) \to \mathbf{R}^2$ satisfying $\gamma_1(0) = x_1$, $|\gamma_1(t)| \to \infty$ as $t \to \infty$, and $\gamma_1(0,\infty) \cap \Gamma = \emptyset$. Setting $D_1 := \gamma_1((0,\infty))$, we orient

D_1 according to its parameterization γ_1 (i.e., in the direction of increasing t's). Since $\mathbf{R}^2 \setminus \overline{D}_1$ is simply connected, we can find a smooth map $\varphi_1 : \mathbf{R}^2 \setminus \overline{D}_1 \to \mathbf{R}$ which is smooth up to D_1 from both sides, has a constant (oriented, pointwise defined) jump across D_1 equal to 2π, and such that $e^{i\varphi_1(x)} = (x - x_1)/|x - x_1|$. We then set $u_1 := e^{i\varphi_1/m}$.

Since Γ is a tree, for each $k \in \{2, \dots, md\}$ there is a unique injective polygonal curve $\gamma_k : [0,1] \to \Gamma$ such that $\gamma_k(0) = x_k$ and $\gamma_k(1) = x_1$. Setting $D_k := \gamma_k((0,1))$, we orient D_k according to the curve γ_k. Notice that for $k \neq l$, the orientation of D_k coincides with the orientation of D_l on $D_k \cap D_l$. Moreover, one has $\Gamma = \bigcup_{k \geqslant 2} \overline{D}_k$ by minimality of Γ. As a consequence, Γ inherits the orientation induced by the D_k's.

Since $\mathbf{R}^2 \setminus (\overline{D}_1 \cup \overline{D}_k)$ is simply connected, we can find a smooth map $\varphi_k : \mathbf{R}^2 \setminus (\overline{D}_1 \cup \overline{D}_k) \to \mathbf{R}$, smooth up to $\overline{D}_1 \cup D_k$ from both sides, with a constant (oriented) jump across $\overline{D}_1 \cup D_k$ equal to 2π, and such that $e^{i\varphi_k(x)} = (x - x_k)/|x - x_k|$. We consider

$$u_\Gamma := \exp\left(\frac{i}{m}\left[\varphi_\mu + \sum_{k=1}^{md} \varphi_k\right]\right),$$

where φ_μ is the map defined in (2.14). By construction, we have $u_\Gamma \in SBV(\Omega; \mathbf{S}^1)$ and $u_\Gamma^m = v_\mu$, i.e., u_Γ is an admissible competitor for $L(\mu)$. In addition, u_Γ is smooth outside $\Gamma \cup D_1$. Since $\frac{1}{m}\sum_k \varphi_k$ has a constant jump equal to $2\pi d$ across D_1, we infer that u_Γ is actually smooth in $\overline{\Omega} \setminus \Gamma$, and $u = \mathbf{a}^j g$ on $\partial\Omega$ for some $j \in \{0, \dots, m-1\}$. Replacing u_Γ by $\mathbf{a}^{-j} u_\Gamma$ if necessary, we can assume that $u_\Gamma = g$ on $\partial\Omega$. Now consider an arbitrary point $x \in \Gamma \setminus \operatorname{spt} \mu$. By Remark 4.2, the number of x_k's before x, according to the orientation of Γ, is not a multiple of m. This shows that the jump of $\frac{1}{m}\sum_k \varphi_k$ across Γ at x is not a multiple of 2π, and consequently $x \in J_{u_\Gamma}$. Therefore $J_{u_\Gamma} = \Gamma$, and $L(\mu) \leqslant \mathcal{H}^1(J_{u_\Gamma}) = \Lambda(\mu)$.

Step 2. — We now prove that

$$L(\mu) \geqslant \Lambda(\mu). \tag{4.5}$$

Let us consider an arbitrary minimizer u of $L(\mu)$, and assume without loss of generality that $0 \in \Omega$. We shall first prove in this step that

$$u = g \text{ on } \partial\Omega\,, \text{ and } \operatorname{dist}(J_u, \partial\Omega) > 0\,. \tag{4.6}$$

To show that (4.6) holds, we consider Γ a $\Lambda(\mu)$-minimizer, and u_Γ the map constructed in Step 1. We extend u and u_Γ to $\mathbf{R}^2$ by setting $\widetilde{u}(x) = g \circ \Pi(x)$ and $\widetilde{u}_\Gamma(x) = g \circ \Pi(x)$ for $x \in \mathbf{R}^2 \setminus \Omega$, where Π denotes the orthogonal projection on the convex set $\overline{\Omega}$. Then, $J_{\tilde{u}} \subseteq \overline{\Omega}$, $J_{\tilde{u}_\Gamma} \subseteq \Gamma$, and

$\mathcal{H}^1(J_{\tilde{u}}) = \mathcal{H}^1(J_u) + \mathcal{H}^1\big(\{u \neq g\}\big)$. We can thus from now on identify u (respectively u_Γ) and $\widetilde{u}$ (respectively $\widetilde{u}_\Gamma$). We define

$$w := u/u_\Gamma\,.$$

Since $u^m = v_\mu = u_\Gamma^m$, we have $w \in SBV_{\mathrm{loc}}(\mathbf{R}^2;\mathbf{G}_m)$ with $w = 1$ in $\mathbf{R}^2 \setminus \overline{\Omega}$ and we can find a Caccioppoli partition $\{E_k\}_{k=0}^{m-1}$ of $\mathbf{R}^2$ such that

$$w = \sum_{k=0}^{m-1} \mathbf{a}^k \chi_{E_k}\,,$$

with $E_k \subseteq \overline{\Omega}$ for $k = 1, \ldots, m$, and $\mathbf{R}^2 \setminus \overline{\Omega} \subseteq E_0$. Since u_Γ is smooth in $\overline{\Omega} \setminus \Gamma$, we deduce that

$$J_u \cap (\mathbf{R}^2 \setminus \Gamma) = \bigcup_{k=0}^{m-1} \partial E_k \setminus \Gamma \quad \text{up to an } \mathcal{H}^1\text{-null set}\,. \tag{4.7}$$

Let K be the convex envelope of $\operatorname{spt}\mu$. The set K is then a closed polygonal subset of Ω. By an elementary geometric construction, we can find a strictly convex open set ω with C^1-boundary satisfying $K \subseteq \omega$ and $\overline{\Omega} \subseteq \Omega$, and such that the Hausdorff distance between K and $\overline{\Omega}$ is arbitrarily small. Given such ω, we consider the mapping $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $\Phi(x) := \frac{1}{2}(x + \Pi_\omega(x))$, where Π_ω denotes the orthogonal projection on $\overline{\Omega}$. Then Φ is a (global) C^1-diffeomorphism of $\mathbf{R}^2$ satisfying $\Phi(x) = x$ for every $x \in \overline{\Omega}$.

Consider now the sets $\widehat{E}_k := \Phi(E_k)$ for $k = 1, \ldots, m-1$, so that $\{\widehat{E}_k\}_{k=0}^{m-1}$ defines a Caccioppoli partition of $\mathbf{R}^2$. As a consequence, the map

$$\widehat{u} := \left(\sum_{k=0}^{m-1} \mathbf{a}^k \chi_{\hat{E}_k}\right) u_\Gamma$$

is an admissible competitor for $L(\mu)$. By the chain rule formula for BV-functions and (4.7), we have $J_{\hat{u}} \setminus \overline{\Omega} = \Phi(J_u \setminus \overline{\Omega})$ and $J_{\hat{u}} \cap \overline{\Omega} = J_u \cap \overline{\Omega}$. The minimality of u together with the area formula (see [4, Theorem 2.91]) then leads to

$$\mathcal{H}^1(J_u \setminus \overline{\Omega}) \leqslant \mathcal{H}^1(J_{\hat{u}} \setminus \overline{\Omega}) \leqslant \int_{J_u \setminus \overline{\Omega}} |\nabla \Phi| \, \mathrm{d}\mathcal{H}^1\,. \tag{4.8}$$

Our assumption on ω implies that $|\nabla\Phi(x)| \leqslant 1 - c_\omega \operatorname{dist}(x, \overline{\Omega})$ for every $x \in \overline{\Omega}$ and some constant $c_\omega > 0$ depending only on ω and Ω. Inserting this estimate in (4.8) shows that $\mathcal{H}^1(J_u \setminus \overline{\Omega}) = 0$, which clearly implies (4.6).

Step 3. — In this final step we show (4.5). Let us fix an arbitrary ball $B_{2r}(y) \subseteq \Omega \setminus \operatorname{spt}\mu$. Since v_μ is smooth in $B_{2r}(y)$, we can find a smooth function φ on $\overline{B}_r(y)$ such that $v_\mu = e^{i\varphi}$ in $\overline{B}_r(y)$. The map $u_\star := e^{i\varphi/m}$ is then smooth on $\overline{B}_r(y)$, and satisfies $u_\star^m = v_\mu$ in $\overline{B}_r(y)$. Arguing as above, it implies that any competitor $u_{\rm comp}$ for $L(\mu)$ can be written as

$$u_{\rm comp} = \left(\sum_{k=0}^{m} \mathbf{a}^k \chi_{F_k}\right) u_\star \quad \text{in } B_r(y)\,,$$

for some Caccioppoli partition $\{F_k\}_{k=0}^{m-1}$ of $B_r(y)$, and

$$J_{u_{\rm comp}} \cap B_r(y) = \bigcup_{k=0}^{m-1} \partial F_k \cap B_r(y) \quad \text{up to an } \mathcal{H}^1\text{-null set}\,.$$

In addition,

$$\mathcal{H}^1\big(J_{u_{\rm comp}} \cap B_r(y)\big) = \frac{1}{2}\sum_{k=0}^{m-1} \mathcal{H}^1\big(\partial F_k \cap B_r(y)\big)\,. \tag{4.9}$$

As a consequence, the minimizer u of $L(\mu)$ that we consider can be written as $u = \big(\sum_k \mathbf{a}^k \chi_{E_k}\big)u_\star$, for some Caccioppoli partition $\{E_k\}_{k=0}^{m-1}$ of $B_r(y)$ minimizing the right-hand side of (4.9). By classical results on minimal planar clusters (see for instance [23, Theorem 5.2]), $\bigcup_k \partial E_k \cap B_r(y)$ is locally a finite union of segments meeting at triple junctions. Since $u_\star$ is smooth in $B_r(y)$, it implies that $J_u \cap B_{r/2}(y) = \bigcup_k \partial E_k \cap B_{r/2}(y)$, and $u \in C^\infty(B_{r/2}(y) \setminus J_u)$.

We are now ready to prove that J_u is a $\Lambda(\mu)$-minimizer. Let us fix for a moment $\sigma > 0$ such that $\overline{B}_\sigma(x_k) \cap \overline{B}_\sigma(x_l) = \emptyset$ for $k \neq l$, and $\overline{B}_\sigma(\mu) \subseteq \Omega$. By the discussion above, $u \in C^\infty\big(\overline{\Omega} \setminus (J_u \cup B_{\sigma/2}(\mu))\big)$, and $J_u \setminus B_{\sigma/2}(\mu)$ is a finite union of segments. Hence, $J_u \cup \overline{B}_\sigma(\mu)$ is a compact subset of Ω. Since $J_u \cup \overline{B}_\sigma(\mu)$ converges to $J_u \cup \operatorname{spt}\mu$ as $\sigma \to 0$ in the Hausdorff distance, we infer from Blaschke's theorem that $J_u \cup \operatorname{spt}\mu$ is a compact subset of Ω. Moreover, we have $J_u \supseteq \operatorname{spt}\mu$ since $u^m = v_\mu$. Therefore J_u is a compact subset of Ω, and $u \in C^\infty(\Omega \setminus J_u)$. To complete the proof, it now only remains to prove that any connected component of J_u contains a multiple of m vortices (possibly equal to zero). Indeed, this would lead to $L(\mu) = \mathcal{H}^1(J_u) \geqslant \Lambda(\mu)$, and (4.5) would be proven. Furthermore, by Step 1, we would also obtain that J_u is a minimizer of $\Lambda(\mu)$.

Let us consider Σ a connected component of J_u, and $A \subseteq \Omega$ a connected smooth open neighborhood of Σ such that $(J_u \setminus \Sigma) \cap \overline{A} = \emptyset$. We may write $A = A_0 \setminus \bigcup_{n=1}^N \overline{A}_n$ where the A_n are connected and simply connected smooth open sets satisfying $\overline{A}_n \subseteq A_0$ for $n = 1, \ldots, N$, and $\overline{A}_n$ are pairwise disjoint. Since v_μ and u are smooth on ∂A_n for $n = 0, \ldots, N$, and $u^m = v_\mu$,

$$\deg(v_\mu, \partial A_n) = m \deg(u, \partial A_n) \in m\mathbf{N}$$

and thus

$$\operatorname{Card}(\Sigma \cap \operatorname{spt} \mu) = \deg(v_\mu, \partial A_0) - \sum_{n=1}^{N} \deg(v_\mu, \partial A_n) \in m\mathbf{N},$$

concluding the proof. □

Remark 4.4. — The proof of Theorem 4.3 shows that every $L(\mu)$-minimizer u is smooth on both sides of J_u away from $\operatorname{spt}\mu$. More precisely, one can find a radius $r > 0$ such that if $\overline{B}_r(x) \subseteq \Omega \setminus \operatorname{spt}\mu$, then $B_r(x) \setminus J_u$ is made of at most three connected sets and $v_\mu = e^{i\varphi}$ in $B_r(x)$ for some smooth function φ. In each connected region of $B_r(x) \setminus J_u$, we have $u = \mathbf{a}^k e^{i\varphi/m}$ for some $k \in \{0, \dots, m-1\}$.

Remark 4.5. — When Ω is simply connected but not convex, (4.4) still holds true if one adds the condition $\Gamma \subseteq \overline{\Omega}$ for the admissible sets for $\Lambda(\mu)$. For minimizers, the set $\Gamma \cap \partial\Omega$ can then be non-empty. The proof would follow the same lines as in the convex case using boundary regularity of minimizers for the constrained Steiner and constrained minimal cluster problems.

Remark 4.6. — Given a reference map $u_\star$ which is an admissible competitor for $L(\mu)$, we have seen that any other competitor u can be written as $u = (\sum_{k=0}^{m-1} \mathbf{a}^k \chi_{E_k}) u_\star$ for some Caccioppoli partition $\{E_k\}_{k=0}^{m-1}$ of Ω. This allows to rephrase the minimization problem defining $L(\mu)$ as an optimal partition problem. Notice however that $\mathcal{H}^1(J_u)$ does not coincide in general with the boundary length of the partition plus $\mathcal{H}^1(J_{u_\star})$ since possible cancellations have to be taken into account (see Figure 4.1).

4.1. Structure of $\Lambda(\mu)$-minimizers.

We now move on to the study of the $\Lambda(\mu)$-minimizers. In the case $m = 2$, it reduces to a variant of the classical minimal connection problem (see for instance [20]). We recall that if $P := \{p_1, \dots, p_d\}$ and $Q := \{q_1, \dots, q_d\}$ are two sets of given points in $\mathbf{R}^2$, then the *length of a minimal connection* between P and Q is defined as

$$\min_\sigma \sum_{k=1}^{d} |p_k - q_{\sigma(k)}|,$$

where the minimum runs over all permutations σ of $\{1, \dots, d\}$.

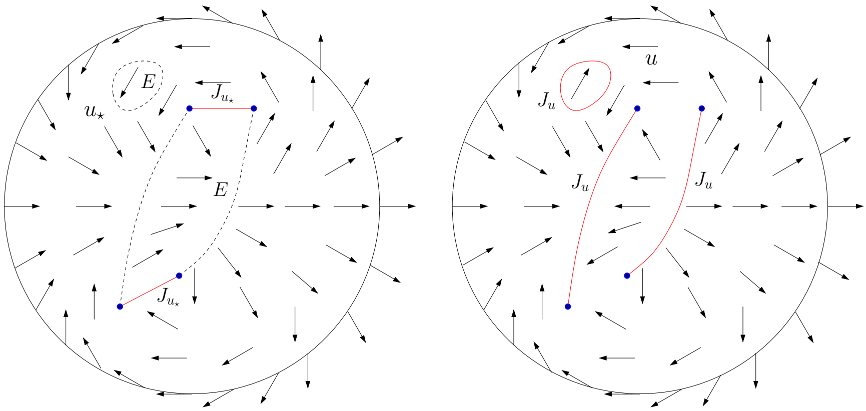

Figure 4.1. Cancellations in the case $m = 2$ ($E_1 = E$, $E_0 = \Omega \setminus E$).

Proposition 4.7. — *Assume that $m = 2$. Let $\mu \in \mathcal{A}_d$ and Γ be a $\Lambda(\mu)$-minimizer. Then, Γ is made of exactly d disjoint segments $\Gamma_1, \dots, \Gamma_d$, and each $\Gamma_k \cap \operatorname{spt} \mu$ contains exactly two points $\{p_k, q_k\}$. In particular, $\Gamma_k = [p_k, q_k]$ for each k, and $\mathcal{H}^1(\Gamma)$ is the length of a minimal connection between $P = \{p_1, \dots, p_d\}$ and $Q = \{n_1, \dots, q_d\}$.*

Proof. — Let Γ be a $\Lambda(\mu)$-minimizer, and let us prove that every connected component Γ_k of Γ contains exactly two points of $\operatorname{spt} \mu$. It would obviously imply that each Γ_k is a segment, and that $\mathcal{H}^1(\Gamma)$ is the length of minimal connection by the definition (4.3) of $\Lambda(\mu)$.

To prove the claim, we start with the following observation. By Theorem 4.3, we can find a map u achieving $L(\mu)$ and such that $J_u = \Gamma$. Then, consider an arbitrary open ball $B_{2r}(x) \subseteq \Omega \setminus \operatorname{spt} \mu$. Since $v_\mu = u^2$ is smooth in that ball and $\deg(v_\mu, \partial B_{2r}(x)) = 0$, we can find $u_\star \in C^\infty(B_{2r}(x); \mathbf{S}^1)$ such that $u_\star^2 = v_\mu$. Arguing as in the proof of Theorem 4.3, we infer that $u = (\chi_E - \chi_{E^c}) u_\star$ in $B_r(x)$ for some set E having a minimizing perimeter in $B_r(x)$. By minimality, $\partial E \cap B_{r/2}(x) = J_u \cap B_{r/2}(x) = \Gamma \cap B_{r/2}(x)$ is smooth, and thus $\Gamma \cap B_{r/2}(x)$ does not contain triple junctions. Hence Γ is a finite union of segments, only intersecting at points of $\operatorname{spt} \mu$.

Let us now consider Γ_k a connected component of Γ. Assume by contradiction that there is a point $x \in \Gamma_k \cap \operatorname{spt} \mu$ such that $J \geqslant 2$ segments meet at x (if $J = 1$ for every point of $\Gamma_k \cap \operatorname{spt} \mu$, then there is nothing to prove). For $j \in \{1, \dots, J\}$, let $C_j := [x, y_j]$ be the segments in Γ_k departing from x. Denote by n_j the number of points in $\operatorname{spt} \mu$ belonging to the connected component of $\Gamma_k \setminus \{x\}$ containing $C_j \setminus \{x\}$. Notice that each n_j

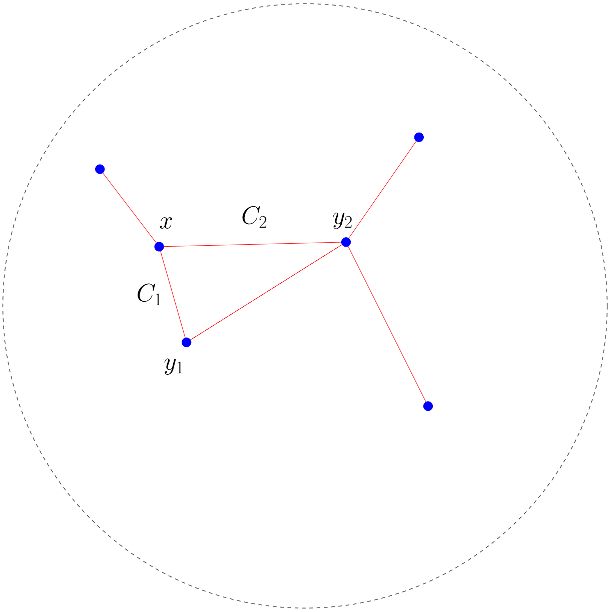

Figure 4.2. Construction of a competitor.

must be odd (otherwise one could remove the corresponding segment C_j from Γ, thus contradicting minimality). Moreover, the cardinal of $\operatorname{spt}\mu\cap\Gamma_k$ is even, and since it is equal to $1+\sum_{j=1}^{J} n_j$, we deduce that J is odd. Hence $J \geqslant 3$, and among the segments C_j, at least two of them are not collinear. Assume without loss of generality that C_1 and C_2 are not collinear. Then we can replace C_1 and C_2 by the segment $[y_1, y_2]$ to obtain a competitor with strictly lower length than Γ (see Figure 4.2), which again contradicts minimality. This establishes that $J = 1$, and concludes the proof. □

The case $m \geqslant 3$ is more involved, and it is no longer true that any $\Lambda(\mu)$-minimizer is a disjoint union of d Steiner trees.

Proposition 4.8. — *Assume that $d \in \{2, 3, 4\}$ and $m \geqslant d + 1$. There exists $\mu \in \mathcal{A}_d$ such that every $\Lambda(\mu)$-minimizer is connected.*

Proof. — For clarity reason, we shall start by giving full details of the proof for $d = 2$ and $m = 3$. We will then explain how to extend this construction to the other cases. Let Y_1, Y_2, and Y_3 be three equidistant points on the unit circle. The unique solution to the Steiner problem for connecting these three points is given by the triple junction Σ. Given $\varepsilon \ll 1$,

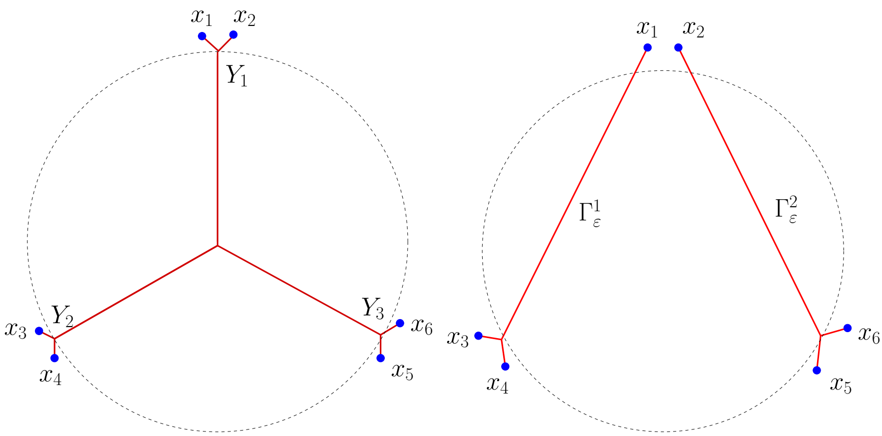

Figure 4.3. An example of a connected minimizer with six vortices of degree 1/3.

let $\{x_1, \ldots, x_6\}$ be such that (see Figure 4.3)

$$|x_1 - Y_1| = |x_2 - Y_1| = |x_3 - Y_2| = |x_4 - Y_2| = |x_5 - Y_3| = |x_6 - Y_3| = \varepsilon\,.$$

Consider the measure $\mu_\varepsilon := 2\pi \sum_{k=1}^{6} \delta_{x_k}$, and let Γ_ε be a $\Lambda(\mu_\varepsilon)$-minimizer. Set Σ_ε to be the union of Σ with the segments connecting each x_j to the closest Y_i. By minimality,

$$\mathcal{H}^1(\Gamma_\varepsilon) \leqslant \mathcal{H}^1(\Sigma_\varepsilon) \leqslant \mathcal{H}^1(\Sigma) + 6\varepsilon\,. \tag{4.10}$$

If Γ_ε is not connected, then it is has two connected components Γ_ε^1 and Γ_ε^2, each of them containing exactly three points among the x_j's. Then, at least one of the pairs $\{x_1, x_2\}$, $\{x_3, x_4\}$ and $\{x_5, x_6\}$, intersects both Γ_ε^1 and Γ_ε^2, say $\{x_1, x_2\}$. Up to a subsequence, we have that each Γ_ε^i converges to a connected set Γ^i with $\Gamma := \Gamma^1 \cup \Gamma^2$ admissible for the Steiner problem related to Y_1, Y_2 and Y_3. Therefore, by (4.10)

$$\mathcal{H}^1(\Sigma) \geqslant \liminf_{\varepsilon \to 0} \mathcal{H}^1(\Gamma_\varepsilon) \geqslant \mathcal{H}^1(\Gamma) \geqslant \mathcal{H}^1(\Sigma).$$

Hence, the above inequalities are actually equalities and since

$$\liminf_{\varepsilon \to 0} \mathcal{H}^1(\Gamma_\varepsilon) = \mathcal{H}^1(\Gamma^1) + \mathcal{H}^1(\Gamma^2),$$

we have $\mathcal{H}^1(\Gamma^1 \cup \Gamma^2) = \mathcal{H}^1(\Gamma^1) + \mathcal{H}^1(\Gamma^2)$ so that Γ^1 and Γ^2 only intersect at Y_1. We have thus obtained a connected graph Γ containing Y_1, Y_2 and Y_3 but for which the degree of Y_1 is two. Since Σ is the only minimizer of the Steiner problem for (Y_1, Y_2, Y_3),

$$\mathcal{H}^1(\Gamma) > \mathcal{H}^1(\Sigma)\,,$$

contradicting (4.10) for ε small enough.

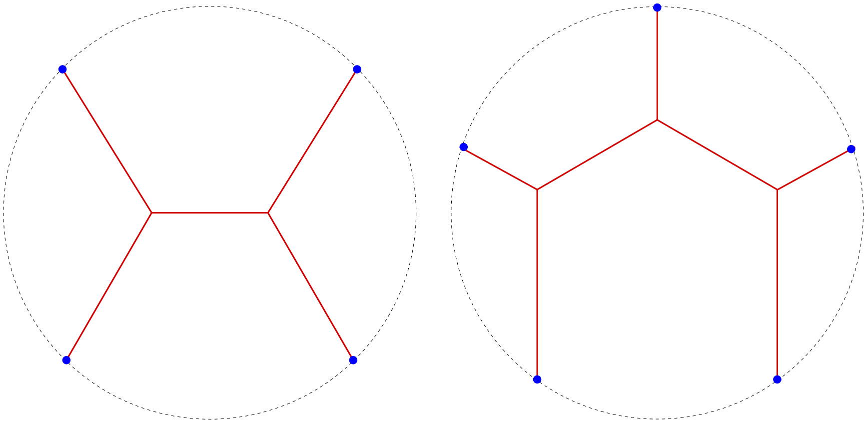

Figure 4.4. Minimizers of the Steiner problem for the vertices of the square on the left and of the regular pentagon on the right.

If now $m \geqslant 3$, we can repeat the same construction placing $m-1$ points close to Y_1, $m-1$ close to Y_2 and two close to Y_3 to construct an example where the minimizer is connected for $2m$ points.

For $d = 3, 4$ and $m = d+1$, the construction is similar to the case $d = 2$. For this we let $(Y_1, \dots, Y_m)$ be the vertices of a regular m−gone[(1)] inscribed in the unit circle and consider $(x_1, \dots, x_{md})$ points such that for every $k \in [1, m]$, there are exactly d of these points at distance ε from Y_k. Let Σ be the minimizer of the Steiner problem for $(Y_1, \dots, Y_m)$. Then, as for the case $d = 2$,

$$\Lambda(\mu) \leqslant \mathcal{H}^1(\Sigma) + md\varepsilon\,. \tag{4.11}$$

As above, let Γ_ε be a minimizer for $\Lambda(\mu)$ and assume that it is not connected. Let $\widehat{\Gamma}_\varepsilon$ be the set made of Γ_ε and the union of the segments joining the points x_j to the nearest Y_k. Since every connected component of Γ_ε contains a multiple of m points among the points x_j, it must also be the same for the connected components of $\widehat{\Gamma}_\varepsilon$. However, at the same time, it should also be a multiple of $m-1$ since each Y_k is connected to the $m-1$ closest points x_j. Therefore, $\widehat{\Gamma}_\varepsilon$ must be connected. Letting $\varepsilon \to 0$, we obtain a set $\widehat{\Gamma}$ which is admissible for the Steiner problem for $(Y_1, \dots, Y_m)$

(1) That is a square if $m = 4$ and a regular pentagon if $m = 5$.

but for which at least one of the points Y_k has degree at least two. Since all the points Y_k have degree one for the minimizer of the Steiner problem for the $m-$gone with $m = 4, 5$ (see for instance [31] and Figure 4.4), we reach a contradiction with (4.11). The extension to $d = 4, 5$ and $m > d+1$ is obtained as before by placing $m-1$ points x_j at distance ε from d points Y_k and d points x_j at distance ε from the last Y_k. □

Remark 4.9. — For $m \geqslant 6$, the solution of the Steiner problem for the vertices of a regular $m-$gone is known to be the $m-$gone itself minus one of its side [31]. For this reason our construction does not work for $d \geqslant 5$. It would be interesting to understand if it is possible to find another construction which works for every $d \in \mathbf{N}$.

In light of Proposition 4.8, one could conjecture that the maximum number of points that a $\Lambda(\mu)$-minimizer can carry is equal to $m(m-1)$. However, as the following example shows, this is again not the case.

Proposition 4.10. — *Assume that $m = d = 3$. There exists $\mu \in \mathcal{A}_d$ such that every $\Lambda(\mu)$-minimizer is connected.*

Proof. — The idea is to iterate the construction made above (see Figure 4.5). Let $1 \gg \varepsilon \gg \delta$. We first fix the points (Y_1, Y_2, Y_3) as before and then choose the points (X_1, X_2, X_3, X_4) so that

$$|X_1 - Y_1| = |X_2 - Y_1| = |X_3 - Y_2| = |X_4 - Y_2| = \varepsilon\,,$$

and that all the angles are of 120°. Let $(x_1, \ldots, x_8)$ be such that each X_k is at distance δ of exactly two x_j's and let finally $x_9 = Y_3$. Let Γ be a minimizer of the corresponding $\Lambda(\mu)$. As above, by comparing with a connected competitor, it holds

$$\mathcal{H}^1(\Gamma) \leqslant 3 + 4\varepsilon + 8\delta\,. \tag{4.12}$$

If Γ is not connected, then it can have either two or three connected components. Arguing as in the proof of Proposition 4.8, we see that the connected component containing x_9 must also contain at least one of the vortices close to Y_1, say x_1 and one of the vortices close to Y_2 say x_8 (otherwise Γ would be very close to a non-optimal competitor for the Steiner problem for (Y_1, Y_2, Y_3)). Let Γ_1 be this connected component. If Γ has three connected components, then each of them must contain exactly three points. Up to relabeling, this means that Γ_2 contains x_2, x_3 and x_4 and Γ_3 contains x_5, x_6 and x_7. Letting Σ_1 be the triple junction connecting x_9, X_1 and X_4 (see Figure 4.5), we obtain that,

$$\mathcal{H}^1(\Gamma) \geqslant \mathcal{H}^1(\Sigma_1) + |X_1 - X_2| + |X_3 - X_4| + O(\delta)\,.$$

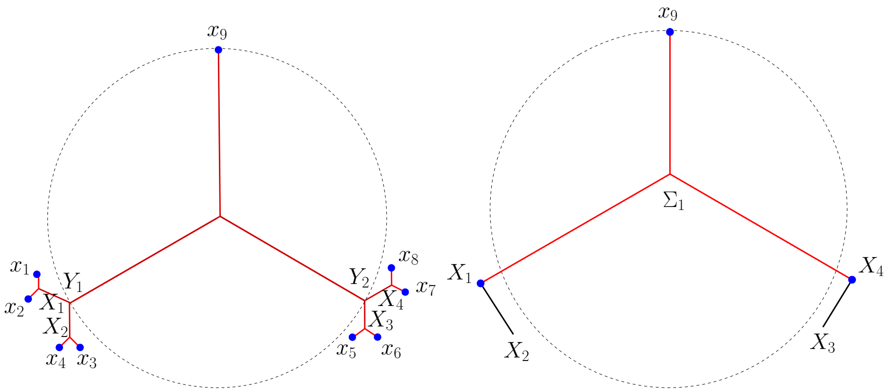

Figure 4.5. An example of a connected minimizer with nine vortices of degree 1/3.

A simple computation gives $|X_1 - X_2| = |X_3 - X_4| = \sqrt{3}\varepsilon$ and $\mathcal{H}^1(\Sigma_1) = 3 + \varepsilon$ so that $\mathcal{H}^1(\Gamma) \geqslant 3 + (1 + 2\sqrt{3})\varepsilon - O(\delta)$, which contradicts (4.12) for δ and ε small enough. The cases when Σ_1 must be the triple junction connecting X_2, x_9 and X_3 or X_1, x_9 and X_3 can be treated analogously.

If now Γ is made of only two components, then Γ_1 must contain six points and the other connected component Γ_2 must contain the remaining three points. Without loss of generality, we can assume that $\{x_1, x_2, x_3, x_4, x_8, x_9\} \subseteq \Gamma_1$ and $\{x_5, x_6, x_7\} \subseteq \Gamma_2$. Let Σ_2 be the optimal Steiner tree connecting X_1, X_2 x_9 and X_4 (see Figure 4.6). We then have

$$\mathcal{H}^1(\Gamma) \geqslant \mathcal{H}^1(\Sigma_2) + |X_4 - X_3| - O(\delta)\,.$$

In order to compute $\mathcal{H}^1(\Sigma_2)$, we notice that since at first order, it must have length 3, it must have at least one triple junction. If it has only one, then we are basically back to the situation of Figure 4.5. Otherwise, it has exactly two triple junctions and we can obtain $\mathcal{H}^1(\Sigma_2)$ by constructing the two equilateral triangles $X_1X_2S_1$ and $X_4x_9S_2$ (see Figure 4.6) and computing the distance S_1S_2 (see [51]). After some computations using for instance complex numbers, we find that $\mathcal{H}^1(\Sigma_2) = 3 + \frac{5}{2}\varepsilon + o(\varepsilon)$ so that

$$\mathcal{H}^1(\Gamma) \geqslant 3 + \left(\frac{5}{2} + \sqrt{3}\right)\varepsilon + o(\varepsilon) - O(\delta)\,,$$

contradicting (4.12) again. □

Remark 4.11. — In light of these examples, it would be interesting to understand what is the maximal number of vortices which can be carried by a single tree, given $m \geqslant 3$.

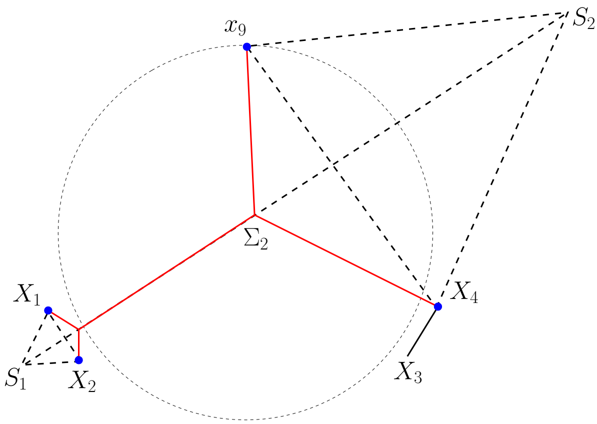

Figure 4.6. The set Σ_2.

5. Structure of minimizers at small $\varepsilon > 0$

The aim of this final section is to use the structure of the minimizers of the limiting functional $F_{0,g}$ given by Theorem 4.3 to prove that minimizers of $F^0_{\varepsilon,g}$ have the same structure for $\varepsilon > 0$ small enough. In turn, this gives an improved convergence result for minimizers as $\varepsilon \downarrow 0$ (compare to Corollary 3.4). Since we will use some tools developed for the analysis of the Mumford–Shah functional, we will only focus on the sharp interface functional $F^0_{\varepsilon,g}$. It would be interesting to understand if similar results can be obtained for the "phase field approximation" F^η_ε.

As in the previous section, we shall assume that Ω is a convex domain. The main results of this section can be summarized in the following theorem. We recall that the L^1-convergence of minimizers of $F^0_{\varepsilon_h,g}$ towards minimizers of $F_{0,g}$ is given by Corollary 3.4.

Theorem 5.1. — *Let $\varepsilon_h \to 0$, and let u_h be a minimizer of $F^0_{\varepsilon_h,g}$ over $\mathcal{G}_g(\Omega)$. Assume that $u_h \to u$ in $L^1(\Omega)$ as $h \to \infty$ for some minimizer u of $F_{0,g}$. Setting $\mu := \operatorname{curl} j(u^m)$, for every $\sigma > 0$ small enough, and every h large enough (depending on σ), the following holds:*

- (i) *$J_{u_h} \setminus B_\sigma(\mu)$ is a compact subset of $\Omega \setminus B_\sigma(\mu)$ made of finitely many segments, meeting at triple junctions.*
- (ii) *$u_h \in C^\infty\big(\overline{\Omega} \setminus (B_\sigma(\mu) \cup J_{u_h})\big)$ and $u_h = g$ on $\partial\Omega$.*
- (iii) *If $B_r(x) \subseteq \Omega \setminus B_\sigma(\mu)$, then there exists $\phi_h \in BV(B_r(x); \mathbf{G}_m)$ such that $u_h/\phi_h \in C^\infty(B_r(x))$.*

In addition,

(iv) *J_{u_h} converges in the Hausdorff distance to J_u.*
(v) *$u_h \to u$ in $C^k_{\mathrm{loc}}(\Omega \setminus J_u) \cap C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega} \setminus J_u)$ for every $k \in \mathbb{N}$ and $\alpha \in (0,1)$.*
(vi) *If $B_r(x) \subseteq \Omega \setminus B_\sigma(\mu)$, then there exists $\phi \in BV(B_r(x); \mathbf{G}_m)$ such that $u_h/\phi_h \to u/\phi$ in $C^k_{\mathrm{loc}}(B_r(x))$ for every $k \in \mathbb{N}$.*

Remark 5.2. — In the proof of Theorem 5.1, we are actually going to prove a stronger result on the structure of u_h (see Section 5.1). As a consequence, it solves in $\Omega \setminus B_\sigma(\mu)$ the Ginzburg–Landau system with free discontinuities

$$\begin{cases} -\Delta u_h = \frac{1}{\varepsilon_h^2}(1-|u_h|^2)u_h & \text{in } \Omega \setminus (B_\sigma(\mu) \cup J_{u_h})\,, \\ (u_h^+)^m = (u_h^-)^m & \text{on } J_{u_h} \setminus B_\sigma(\mu)\,, \\ u_h = g & \text{on } \partial\Omega\,. \end{cases}$$

Remark 5.3. — As a consequence of items (iii) and (vi) above, we have $\mathrm{p}(u_h) \in C^\infty(\overline{\Omega} \setminus B_\sigma(\mu))$ for h large enough, and $\mathrm{p}(u_h) \to u^m$ in $C^k_{\mathrm{loc}}(\Omega \setminus B_\sigma(\mu)) \cap C^{1,\alpha}(\overline{\Omega} \setminus B_\sigma(\mu))$.

Remark 5.4. — In the case $m = 2$, J_u is made of d disjoint segments connecting points of $\operatorname{spt}\mu$, see Proposition 4.7. In particular, J_u contains no triple junctions. Concerning the minimizer u_h of $F^0_{\varepsilon,g}$, it implies that (for h large enough), the set $J_{u_h} \setminus B_\sigma(\mu)$ is made of d disjoint segments connecting components of $\overline{B}_\sigma(\mu)$.

Remark 5.5. — Let us notice that in the case $\deg(g, \partial\Omega) = 0$, Theorem 5.1 shows that for ε small enough, the minimizer of $F^0_{\varepsilon,g}$ is unique and smooth, i.e. there is no jump set, and it coincides with the unique minimizer of E_ε over $W^{1,2}_g(\Omega)$ (see [56]). For the classical Mumford–Shah functional, similar results were obtained using calibration methods [1] (see also [33, Theorem 3.1] for a simple proof originally due to Chambolle).

Remark 5.6. — Theorem 5.1 does not provide regularity results near $\operatorname{spt}\mu$. Nevertheless, repeating verbatim the proof of [22, Theorem 3.1], one can prove that for every $\varepsilon > 0$ and every minimizer u of $F^0_{\varepsilon,g}$, the jump set is essentially closed, that is $\mathcal{H}^1(\overline{J_u} \setminus J_u) = 0$. Since $\Omega \subseteq \mathbf{R}^2$, the proof of this result only requires the simplest forms of [22, Lemma 2.3 and Lemma 2.4].

Remark 5.7. — It would be interesting to study the behavior of the minimizers u_h close to the vortices, i.e., in $B_\sigma(\mu)$. One could expect that there is only one point in each component of $B_\sigma(\mu)$ where u_h vanishes, and that the jump set of u_h is a union of Steiner trees connecting those zeroes in the spirit of the $\Lambda(\mu)$ minimization problem. In this direction, a first step may consist in understanding the optimal profile problem

$$\gamma_m^{\#}(\varepsilon,R) := \min\Big\{E_\varepsilon\big(\mathrm{P}(u),B_R\big)+\mathcal{H}^1(J_u\cap B_R)-\frac{\pi}{m^2}\log\frac{R}{\varepsilon}:\\ u\in\mathcal{G}(B_R)\,,\ \mathrm{P}(u)(z)=\frac{1}{m}\left(\frac{z}{|z|},\sqrt{m^2-1}\right)\text{ on }\partial B_R\Big\}.$$

Considering a solution u of this problem, one may ask if $|u|$ is radial, increasing, vanishing at the origin, and if J_u is just a segment joining the origin to the boundary. It seems to be a difficult question since it combines both issues related to the presence of an expected singularity in the jump set in the spirit of the so called crack tip (see for instance [27]) for the Mumford–Shah functional, with the fact that $\mathrm{P}(u)$ should have the same regularity as minimizing harmonic maps with values into the singular cone $\mathcal{N}$. Such harmonic maps satisfy non standard elliptic equations, and are usually more singular than minimizing harmonic maps with values into a smooth target [36, 44, 37, 3][(2)].

5.1. Sketch of the proof of Theorem 5.1.

Before starting the proof of Theorem 5.1, let us explain the strategy. Away from $\operatorname{spt}\mu$, the limiting function v_μ is smooth. Therefore, if we consider a small enough ball $B_r(x)$ outside $B_\sigma(\mu)$, then the oscillation of v_μ on this ball is very small. By the strong convergence in $W^{1,2}(B_r(x))$ of v_h to v_μ (recall Theorem 3.2), this will still be true for v_h on $\partial B_r(x)$ (actually, on $\partial B_{\rho_h}(x)$ for some $\rho_h\sim r$). Hence[(3)], we can find $g_h\in W^{1,2}(\partial B_r(x))$ such that $\mathrm{p}(g_h)=v_h$ on $\partial B_r(x)$. Considering w_h a solution of the Ginzburg–Landau equation

$$\begin{cases}-\Delta w_h=\frac{1}{{\varepsilon_h}^2}(1-|w_h|^2)w_h & \text{in } B_r(x)\\ \qquad w_h=g_h & \text{in }\partial B_r(x),\end{cases}\tag{5.1}$$

we aim at proving that in $B_r(x)$, $u_h=\left(\sum_{k=0}^{m-1}\mathbf{a}^k\chi_{E_h^k}\right)w_h$ where the E_h^k are pairwise disjoint and satisfy (up to a relabeling)

(i) if $B_r(x)\cap J_u=\emptyset$, then $E_h^0=B_r(x)$ i.e. $u_h=w_h$ in $B_r(x)$;

(ii) if $B_r(x)\cap J_u$ is a segment then $u_h=(\chi_{E_h^0}+\mathbf{a}^k(1-\chi_{E_h^0}))w_h$ for some $k\neq 0$ with $\partial E_h^0\cap B_r(x)$ a segment;

(2) quoting [44]: "Unfortunately, the equations satisfied by s and u are so bad that no existing result can be applied".

(3) Notice that actually, some care is needed in the choice of g_h to guarantee that no jump is created at the boundary.

(iii) if $B_r(x) \cap J_u$ contains a triple point then $u_h = (\chi_{E_h^0} + \mathbf{a}^{k_1}\chi_{E_h^{k_1}} + \mathbf{a}^{k_2}\chi_{E_h^2})w_h$ with $0 < k_1 < k_2 \leqslant m-1$ and $\partial E_h^0 \cup \partial E_h^{k_1} \cup \partial E_h^{k_2}$ a triple junction in $B_r(x)$.

In order to show that indeed u_h is of this form, a powerful tool that we introduce in the next section is the Lassoued–Mironescu decomposition argument [41] which allows to conveniently split the energy into a Ginzburg–Landau term and a Mumford–Shah type energy.

5.2. Ginzburg–Landau minimizers and energy splitting

In this section, we consider a radius $r > 0$, a sequence $\varepsilon_h \to 0$, and a sequence of boundary conditions $\{g_h\} \subseteq W^{1,2}(\partial B_r) \cap L^\infty(\partial B_r)$ satisfying

$$\|g_h\|_{L^\infty(\partial B_r)} \leqslant 1\,, \tag{5.2}$$

$$\int_{\partial B_r} |\partial_\tau g_h|^2 + \frac{1}{\varepsilon_h^2}(1-|g_h|^2)^2\,\mathrm{d}\mathcal{H}^1 \leqslant C\,, \tag{5.3}$$

for some constant $C > 0$ independent of ε_h. We further assume that

$$g_h \to g_\star \quad \text{uniformly on } \partial B_r \text{ as } h \to \infty\,, \tag{5.4}$$

for some $g_\star \in W^{1,2}(\partial B_r; \mathbf{S}^1)$ satisfying

$$\deg(g_\star, \partial B_r) = 0\,. \tag{5.5}$$

From this last assumption, we can write $g_\star = e^{i\varphi_\star}$ for some harmonic function $\varphi_\star \in W^{1,2}(B_r)$ (which is unique up to a constant multiple of 2π). As in [12], the map

$$w_\star := e^{i\varphi_\star} \in W^{1,2}_{g_\star}(B_r; \mathbf{S}^1)$$

is the unique solution of the minimization problem

$$\min_{W^{1,2}_{g_\star}(B_r;\mathbf{S}^1)} \int_{B_r} |\nabla w|^2\,\mathrm{d}x\,. \tag{5.6}$$

We are now interested in the minimization problem

$$\min_{w \in W^{1,2}_{g_h}(B_r)} E_{\varepsilon_h}(w, B_r)\,. \tag{5.7}$$

We recall that minimizers of (5.7) are in particular solutions of (5.1). We shall make an essential use of the following proposition. It constitutes a slight extension of [12, Theorem 2] to the case of a boundary condition which merely belongs to $W^{1,2}(\partial B_r)$. Since the estimates obtained in [12, Theorem 2] only depend on the $W^{1,2}(\partial B_r)$ bounds satisfied by g_h, the proof of Proposition 5.8 readily follows from [12, Theorem 2] together with an approximation argument (to regularize the boundary condition).

Proposition 5.8. — *Assume that* (5.2), (5.3), (5.4), *and* (5.5) *hold. There exists* $\{w_h\} \subseteq W^{1,2}(B_r) \cap C^0(\overline{B}_r) \cap C^\infty(B_r)$ *such that* w_h *solves* (5.7), *and*

$$w_h \to w_\star \quad \text{strongly in } W^{1,2}(B_r)\,,$$
$$|w_h| \to 1 \quad \text{uniformly in } \overline{B}_r\,,$$
$$w_h \to w_\star \quad \text{in } C^k_{\mathrm{loc}}(B_r) \text{ for every } k \in \mathbb{N}\,.$$

For the rest of this subsection, we still denote by w_h a solution of (5.7) obtained from Proposition 5.8. We continue with a very useful energy decomposition, originally introduced in [41].

Lemma 5.9. — *Let* $u \in \mathcal{G}(B_r) \cap L^\infty(B_r)$ *be such that* $\mathrm{p}(u) = \mathrm{p}(g_h)$ *on* ∂B_r. *For* ε_h *small enough, we have* $u = w_h\phi$ *for some* $\phi \in \mathcal{G}(B_r) \cap L^\infty(B_r)$ *satisfying* $\mathrm{p}(\phi) = 1$ *on* ∂B_r,

$$(5.8)\quad E_{\varepsilon_h}\big(\mathrm{P}(u), B_r\big) = E_{\varepsilon_h}(w_h, B_r) + \frac{1}{2}\int_{B_r} |w_h|^2|\nabla\phi|^2 + \frac{|w_h|^4}{2\varepsilon_h^2}(1-|\phi|^2)^2 + \frac{2}{m}j(\mathrm{p}(\phi))\cdot j(w_h)\,\mathrm{d}x\,,$$

and $\mathcal{H}^1(J_u \cap B_r) = \mathcal{H}^1(J_\phi \cap B_r)$.

Proof. — By Proposition 5.8, we have $|w_h|^2 \geqslant 1/2$ for ε_h small enough. Setting $\phi := u/w_h$, we have $\phi \in SBV^2(B_r)\cap L^\infty(B_r)$ and $\mathrm{P}(\phi) \in W^{1,2}(B_r)$, thus $\phi \in \mathcal{G}(B_r)\cap L^\infty(B_r)$. Since $|w_h|^2$ and $|\phi|^2$ belong to $W^{1,2}(B_r)$, by chain rule we have

$$|\nabla u|^2 = |\phi|^2|\nabla w_h|^2 + |w_h|^2|\nabla\phi|^2 + \frac{1}{2}\nabla(|\phi|^2)\cdot\nabla(|w_h|^2) + 2j(\phi)\cdot j(w_h) \quad \text{a.e. in } B_r\,.$$

Recalling that $mj(\phi) = j(\mathrm{p}(\phi))$ (see Lemma 2.3), and since $|\phi|^2 = |\mathrm{p}(\phi)|^2$, we obtain

$$(5.9)\quad \int_{B_r} |\nabla u|^2\,\mathrm{d}x = \int_{B_r} |\mathrm{p}(\phi)|^2|\nabla w_h|^2 + |w_h|^2|\nabla\phi|^2 + \frac{1}{2}\nabla(|\mathrm{p}(\phi)|^2)\cdot\nabla(|w_h|^2) + \frac{2}{m}j(\mathrm{p}(\phi))\cdot j(w_h)\,\mathrm{d}x\,.$$

Testing equation (5.1) with $|\mathrm{p}(\phi)|^2 w_h \in W^{1,2}(B_r)$, we derive

$$(5.10)\quad \int_{B_r} |\mathrm{p}(\phi)|^2|\nabla w_h|^2 + \frac{1}{2}\nabla(|\mathrm{p}(\phi)|^2)\cdot\nabla(|w_h|^2)\,\mathrm{d}x = \int_{\partial B_r} \partial_\nu w_h \cdot w_h\,\mathrm{d}\mathcal{H}^1 + \int_{B_r} \frac{1}{\varepsilon_h^2}(1-|w_h|^2)|w_h|^2|\mathrm{p}(\phi)|^2\,\mathrm{d}x\,,$$

where the first integral in the right hand side is understood in the $W^{-1/2,2}$–$W^{1/2,2}$ sense. Here we have also used the fact $|\mathrm{p}(\phi)|^2 = 1$ on ∂B_r (so that $|\mathrm{p}(\phi)|^2 w_h = w_h$ on ∂B_r). Testing now (5.1) with w_h yields

$$\int_{\partial B_r} \partial_\nu w_h \cdot w_h \,\mathrm{d}\mathcal{H}^1 = \int_{B_r} |\nabla w_h|^2 - \frac{1}{\varepsilon_h^2}(1-|w_h|^2)|w_h|^2 \,\mathrm{d}x\,.$$

Putting together this identity with (5.9) and (5.10) leads to

$$\begin{aligned}\int_{B_r} |\nabla u|^2 + \frac{1}{2\varepsilon_h^2}(1-|u|^2)^2 \,\mathrm{d}x \\ = \int_{B_r} |\nabla w_h|^2 + |w_h|^2|\nabla\phi|^2 + \frac{2}{m} j(\mathrm{p}(\phi))\cdot j(w_h) \\ + \frac{1}{\varepsilon_h^2}\Bigg[\frac{1}{2}(1-|w_h|^2|\mathrm{p}(\phi)|^2)^2 - (1-|w_h|^2)|w_h|^2 \\ + (1-|w_h|^2)|w_h|^2|\mathrm{p}(\phi)|^2\Bigg]\,\mathrm{d}x\,.\end{aligned}$$

In view of the algebraic identity

$$\begin{aligned}\frac{1}{2}(1-a^2b^2)^2 - (1-a^2)a^2 + (1-a^2)a^2b^2 \\ = \frac{1}{2}(1-a^2)^2 + \frac{a^4}{2}(1-b^2)^2 \qquad \text{for } a,b \geqslant 0\,,\end{aligned}$$

we have obtained

$$\begin{aligned}\frac{1}{2}\int_{B_r} |\nabla u|^2 + \frac{1}{2\varepsilon_h^2}(1-|u|^2)^2 \,\mathrm{d}x = E_{\varepsilon_h}(w_h, B_r) \\ + \frac{1}{2}\int_{B_r} \left\{|w_h|^2|\nabla\phi|^2 + \frac{|w_h|^4}{2\varepsilon_h^2}(1-|\phi|^2)^2 + \frac{2}{m} j(\mathrm{p}(\phi))\cdot j(w_h)\right\}\mathrm{d}x\,.\end{aligned}$$

Finally, since $w_h \in W^{1,2}(B_r)$ we have $J_u \cap B_r = J_\phi \cap B_r$ (up to an $\mathcal{H}^1$-null set), and the conclusion follows. □

We now use Lemma 5.9 to derive a lower bound on the energy. In particular, we want to be able to control the last term in (5.8), which is the purpose of the following lemma.

Lemma 5.10. — *There exists a universal constant $C_\star > 0$ such that if*

$$\int_{B_r} |\nabla w_h|^2 \,\mathrm{d}x \leqslant \delta\,, \tag{5.11}$$

and

$$r^{1/2}\left(\int_{\partial B_r} |\partial_\tau g_h|^2 + \frac{(1-|g_h|^2)^2}{2\varepsilon_h^2}\,\mathrm{d}\mathcal{H}^1\right)^{1/2} \leqslant \delta, \tag{5.12}$$

for some constant $\delta \geqslant 0$, then for every $\Phi \in W^{1,2}(B_r)$ satisfying $\Phi = 1$ on ∂B_r, there holds

$$\left| \int_{B_r} j(\Phi) \cdot j(w_h) \, \mathrm{d}x \right| \leqslant C_\star \delta \int_{B_r} |\nabla \Phi|^2 \, \mathrm{d}x.$$

Proof. — Rescaling variables we may assume that $r = 1$. Arguing by approximation as in Proposition 5.8, we may assume that w_h is smooth.

First, using equation (5.1) we derive that $\operatorname{div} j(w_h) = w_h \wedge \Delta w_h = 0$. By Hodge decomposition, we can find a smooth scalar function H such that $j(w_h) = \nabla^\perp H$. Notice that H is defined up to an additive constant that we shall fix later on.

By approximation, we may assume that the test function Φ is smooth. Since Φ is constant on ∂B_1, the vector field $j(\Phi)$ satisfies $j^\perp(\Phi) \cdot \nu = 0$ on ∂B_1. Since $\operatorname{curl} j(\Phi) = 2 \det \nabla \Phi$,

$$\int_{B_1} j(\Phi) \cdot j(w_h) \, \mathrm{d}x = - \int_{B_1} j^\perp(\Phi) \cdot \nabla H \, \mathrm{d}x = - \int_{B_1} 2H \det(\nabla \Phi) \, \mathrm{d}x \, .$$

We may now estimate

$$\left| \int_{B_1} j(\Phi) \cdot j(w_h) \, \mathrm{d}x \right| \leqslant \int_{B_1} |H| |\nabla \Phi|^2 \, \mathrm{d}x \leqslant \|H\|_{L^\infty(B_1)} \int_{B_1} |\nabla \Phi|^2 \, \mathrm{d}x \, ,$$

and we are left to prove that $\|H\|_{L^\infty(B_1)}$ is controlled by δ. We consider the function H_1 solving

$$\begin{cases} \Delta H_1 = 2 \det(\nabla w_h) & \text{in } B_1 \, , \\ H_1 = 0 & \text{on } \partial B_1 \, , \end{cases}$$

and set $H_2 := H - H_1$. Then, H_2 is harmonic in B_1 since

$$\Delta H_2 = - \operatorname{div} j^\perp(w_h) - 2 \det(\nabla w_h) = \operatorname{curl} j(w_h) - 2 \det(\nabla w_h) = 0 \, .$$

In addition,

$$\partial_\tau H_2 = \partial_\tau H = -\nu \cdot j(w_h) \quad \text{on } \partial B_1 \, . \tag{5.13}$$

Thanks to Wente's estimate (see [55] or [19, Lemma A.1]), there exists a universal constant $C_\sharp > 0$ such that

$$\|H_1\|_{L^\infty(B_1)} \leqslant C_\sharp \int_{B_1} |\nabla w_h|^2 \, \mathrm{d}x \leqslant C_\sharp \delta \, .$$

Moreover, by the maximum principle,

$$\inf_{B_1} H_2 = \inf_{\partial B_1} H_2 \quad \text{and} \quad \sup_{B_1} H_2 = \sup_{\partial B_1} H_2 \, .$$

We now fix the additive constant for H so that $\sup_{B_1} H_2 + \inf_{B_1} H_2 = 0$. This yields

$$\|H_2\|_{L^\infty(B_1)} = \frac{1}{2}\left[\sup_{\partial B_1} H_2 - \inf_{\partial B_1} H_2\right] \leqslant \frac{1}{4}\int_{\partial B_1} |\partial_\tau H_2| \,\mathrm{d}\mathcal{H}^1 .$$

Recalling that $|w_h| \leqslant 1$, we have by (5.13) $|\partial_\tau H_2| \leqslant |w_h||\partial_\nu w_h| \leqslant |\partial_\nu w_h|$, so that by Cauchy–Schwarz inequality,

$$\|H_2\|_{L^\infty(B_1)} \leqslant \frac{\sqrt{2\pi}}{4}\left(\int_{\partial B_1} |\partial_\nu w_h|^2 \,\mathrm{d}\mathcal{H}^1\right)^{1/2} .$$

Let us now recall that the Pohozaev identity applied to equation (5.1) (see e.g. [54, (5.2)]) leads to

$$\int_{B_1} \frac{(1-|w_h|^2)^2}{\varepsilon_h^2}\,\mathrm{d}x = \int_{\partial B_1} |\partial_\tau w_h|^2 - |\partial_\nu w_h|^2 + \frac{(1-|w_h|^2)^2}{2\varepsilon_h^2}\,\mathrm{d}\mathcal{H}^1 .$$

Hence,

$$\int_{\partial B_1} |\partial_\nu w_h|^2 \,\mathrm{d}\mathcal{H}^1 \leqslant \int_{\partial B_1} |\partial_\tau w_h|^2 + \frac{(1-|w_h|^2)^2}{2\varepsilon_h^2}\,\mathrm{d}\mathcal{H}^1 ,$$

which then implies

$$\|H_2\|_{L^\infty(B_1)} \leqslant \frac{\sqrt{2\pi}}{4}\left(\int_{\partial B_1} |\partial_\tau w_h|^2 + \frac{(1-|w_h|^2)^2}{2\varepsilon_h^2}\,\mathrm{d}\mathcal{H}^1\right)^{1/2} \overset{(5.12)}{\leqslant} \frac{\sqrt{2\pi}}{4}\delta .$$

The conclusion now follows with $C_\star := C_\sharp + \sqrt{2\pi}/4$. ☐

Combining Lemma 5.9 and Lemma 5.10 yields the following lower bound for the energy.

Proposition 5.11. — *Let $C_\star$ be the constant given by Proposition 5.10, and let $\delta > 0$ be such that $C_\star\delta \leqslant \frac{1}{16m}$. For ε_h small enough, if w_h satisfies* (5.11) *and* (5.12), *then*

$$F^0_\varepsilon(u, B_r) \geqslant E_{\varepsilon_h}(w_h, B_r) + \frac{1}{8}\int_{B_r} |\nabla\phi|^2 \,\mathrm{d}x + \mathcal{H}^1(J_\phi \cap B_r) \tag{5.14}$$

for every $u \in \mathcal{G}(B_r) \cap L^\infty(B_r)$ satisfying $\mathrm{p}(u) = \mathrm{p}(g_h)$ on ∂B_r, where $\phi := u/w_h \in \mathcal{G}(B_r) \cap L^\infty(B_r)$.

Proof. — By Proposition 5.8 and Lemma 5.9, identity (5.8) holds and $|w_h|^2 \geqslant 1/2$ for ε_h small enough. Applying Lemma 5.10 with $\Phi := \mathrm{p}(\phi)$, we derive that

$$E_{\varepsilon_h}\big(\mathrm{P}(u), B_r\big) \geqslant E_{\varepsilon_h}(w_h, B_r) + \frac{1}{4}\int_{B_r} |\nabla\phi|^2 + \frac{1}{4\varepsilon_h^2}(1-|\phi|^2)^2 \,\mathrm{d}x - \frac{1}{8m^2}\int_{B_r} |\nabla \mathrm{p}(\phi)|^2 \,\mathrm{d}x .$$

By Lemma 2.3 we have

$$\frac{1}{m^2}\int_{B_r}|\nabla \mathrm{p}(\phi)|^2\,\mathrm{d}x \leqslant \int_{B_r}|\nabla \mathrm{P}(\phi)|^2\,\mathrm{d}x = \int_{B_r}|\nabla\phi|^2\,\mathrm{d}x\,,$$

and the conclusion follows. □

Remark 5.12. — Under the assumptions of Proposition 5.11, we can obtain that the minimizer of (5.7) is unique. Indeed, any solution $\widetilde{w}_h$ of (5.7) satisfies $\|\widetilde{w}_h\|_{L^\infty(B_r)} \leqslant 1$ by minimality. Applying (5.14) to $\widetilde{w}_h$ then yields

$$E_{\varepsilon_h}(\widetilde{w}_h, B_r) \geqslant E_{\varepsilon_h}(w_h, B_r) + \frac{1}{8}\int_{B_r}|\nabla\phi|^2\,\mathrm{d}x$$

with $\phi := \widetilde{w}_h/w_h$. Since $E_{\varepsilon_h}(\widetilde{w}_h, B_r) = E_{\varepsilon_h}(w_h, B_r)$, we deduce that $\phi \equiv 1$, that is $\widetilde{w}_h = w_h$. A similar idea was used in [32] to prove uniqueness results, extending those from [56].

5.3. Proof of Theorem 5.1

This section is devoted to the proof of Theorem 5.1. We fix a sequence $\varepsilon_h \to 0$, and minimizers u_h of $F^0_{\varepsilon_h,g}$ over $\mathcal{G}_g(\Omega)$. We assume that $u_h \to u$ strongly in $L^1(\Omega)$ as $h \to \infty$, where u is a minimizer of $F_{0,g}$ over $\mathcal{L}_g(\Omega)$. We recall that by Theorem 3.2 $v_h := \mathrm{p}(u_h) \to u^m$ strongly in $W^{1,p}(\Omega)$ for every $p < 2$ and in $W^{1,2}_{\mathrm{loc}}(\overline{\Omega}\setminus \operatorname{spt}\mu)$, where $\mu := \operatorname{curl} j(u^m) \in \mathcal{A}_d$. According to Section 4, the compact set $\Gamma := J_u \subseteq \Omega$ is a $\Lambda(\mu)$-minimizer in the sense of Definition 4.1, and thus a union of at most d Steiner trees. We denote by $T \subseteq \Omega$ the (finite) set of Steiner points of $\Gamma\setminus \operatorname{spt}\mu$, i.e., the triple junctions of Γ away from $\operatorname{spt}\mu$. We finally recall that $u \in C^\infty(\overline{\Omega}\setminus\Gamma)$, $u = g$ on $\partial\Omega$, and that $u^m = v_\mu \in C^\infty(\overline{\Omega}\setminus \operatorname{spt}\mu)$.

Writing $\operatorname{spt}\mu =: \{x_1, \dots, x_{md}\}$ and $T := \{y_1, \dots, y_q\}$, we now fix $\sigma_0 > 0$ satisfying

$$\sigma_0 < \frac{1}{2}\min\left\{\min_{k\neq l}|x_k - x_l|\,, \min_{k\neq l}|y_k - y_l|\,, \operatorname{dist}(\Gamma, \partial\Omega)\,, \operatorname{dist}(T, \operatorname{spt}\mu)\right\},$$

and we set for $\sigma \in (0, \sigma_0)$,

$$K_\sigma := \|\nabla v_\mu\|_{L^\infty(\Omega\setminus B_{\sigma/4}(\mu))}\,. \tag{5.15}$$

Moreover we fix the positive constant δ to be

$$\delta := \min\left\{1/(4\sqrt{\pi}m), 1/(16mC_\star)\right\},$$

$C_\star$ being the constant given by Proposition 5.10. For $\sigma \in (0, \sigma_0)$, we finally set

$$r_\sigma := \min\left\{\frac{\sigma}{10}, \frac{\delta}{8\sqrt{\pi}K_\sigma}\right\}.$$

Theorem 5.1 is a consequence of a covering argument combined with Proposition 5.13, Proposition 5.16 and Proposition 5.18 which respectively give the structure of u_h away from Γ, close to Γ but away from the triple junctions and at the triple junctions.

5.3.1. Smoothness and convergence away from Γ

Proposition 5.13. — *Let $\sigma \in (0, \sigma_0)$. For h large enough, $u_h \in W^{1,2}_g(\Omega \setminus B_\sigma(\Gamma))$ and u_h minimizes $E_{\varepsilon_h}(\,\cdot\,, \Omega \setminus B_\sigma(\Gamma))$ under its own boundary condition. In addition, $u_h \in C^\infty(\overline{\Omega} \setminus B_\sigma(\Gamma))$ and $u_h \to u$ in $C^{1,\alpha}(\overline{\Omega} \setminus B_\sigma(\Gamma))$ and $C^k_{\mathrm{loc}}(\Omega \setminus B_\sigma(\Gamma))$ for every $\alpha \in (0,1)$ and $k \in \mathbb{N}$.*

The proof of Proposition 5.13 is a direct consequence of Lemma 5.14 and Lemma 5.15 below, together with a suitable covering argument.

Lemma 5.14. — *For $\sigma \in (0, \sigma_0)$, let $r \in (0, r_\sigma)$ and $x_0 \in \Omega$ be such that $\overline{B}_{2r}(x_0) \subseteq \Omega \setminus B_\sigma(\Gamma)$. For h large enough, $u_h \in W^{1,2}(B_r(x_0))$, and u_h minimizes $E_{\varepsilon_h}(\,\cdot\,, B_r(x_0))$ under its own boundary condition. In addition, $u_h \in C^\infty(B_r(x_0))$ and $u_h \to u$ in $C^k_{\mathrm{loc}}(B_r(x_0))$ for every $k \in \mathbb{N}$.*

Proof.

Step 1. — Without loss of generality, we may assume that $x_0 = 0$. Set

$$\gamma_h := \int_{B_{2r}} |\nabla v_h - \nabla v_\mu|^2 + |v_h - v_\mu|^2 \,\mathrm{d}x + \frac{1}{2{\varepsilon_h}^2}\int_{B_{2r}} (1 - |u_h|^2)^2 \,\mathrm{d}x\,.$$

By Theorem 3.2 and Corollary 3.4, $\gamma_h \to 0$ as $h \to \infty$. Since $\Gamma \cap \overline{B}_{2r} = \emptyset$, (3.9) shows that

$$\mathcal{H}^1(J_{u_h} \cap B_{2r}) \leqslant \frac{r}{2} \tag{5.16}$$

for h large enough. From now on, we assume that (5.16) holds.

By the coarea formula (see [47, Theorem II.7.7]), we have

$$\int_r^{2r} \mathcal{H}^0(J_{u_h} \cap \partial B_t)\,\mathrm{d}t \leqslant \mathcal{H}^1(J_{u_h} \cap B_{2r}) \leqslant \frac{r}{2}\,.$$

Setting

$$A_h := \Big\{t \in (r, 2r) : J_{u_h} \cap \partial B_t = \emptyset\Big\}\,,$$

we deduce that $\mathcal{H}^1(A_h) \geqslant r/2$. Notice that $u_h \llcorner \partial B_t \in W^{1,2}(\partial B_t)$ for a.e. $t \in A_h$. Since

$$\int_{A_h} \left[\int_{\partial B_t} |\nabla v_h - \nabla v_\mu|^2 + |v_h - v_\mu|^2 + \frac{1}{2{\varepsilon_h}^2}(1-|u_h|^2)^2 \,\mathrm{d}\mathcal{H}^1 \right] \mathrm{d}t \leqslant \gamma_h \,,$$

we can find a radius $\rho_h \in A_h$ such that $u_h \llcorner \partial B_{\rho_h} \in W^{1,2}(\partial B_{\rho_h}; \mathbf{C})$ and

$$\int_{\partial B_{\rho_h}} |\nabla v_h - \nabla v_\mu|^2 + |v_h - v_\mu|^2 + \frac{1}{2{\varepsilon_h}^2}(1-|u_h|^2)^2 \,\mathrm{d}\mathcal{H}^1 \leqslant \frac{2\gamma_h}{r} \,. \tag{5.17}$$

By definition of p, $|\nabla u_h| \leqslant |\nabla v_h|$ and thus

$$\begin{aligned}
\int_{\partial B_{\rho_h}} |\nabla u_h|^2 \,\mathrm{d}\mathcal{H}^1 &\leqslant \int_{\partial B_{\rho_h}} |\nabla v_h|^2 \,\mathrm{d}\mathcal{H}^1 \\
&\leqslant 2\int_{\partial B_{\rho_h}} |\nabla v_h - \nabla v_\mu|^2 \,\mathrm{d}\mathcal{H}^1 + 4\pi\rho_h \|\nabla v_\mu\|^2_{L^\infty(B_{2r})} \\
&\leqslant \frac{4\gamma_h}{r} + 8\pi r K_\sigma^2 \,,
\end{aligned}$$

which leads to

$$\rho_h^{1/2} \left(\int_{\partial B_{\rho_h}} |\nabla u_h|^2 + \frac{1}{2{\varepsilon_h}^2}(1-|u_h|^2)^2 \,\mathrm{d}\mathcal{H}^1 \right)^{1/2} \leqslant \delta \tag{5.18}$$

for h large enough since $\rho_h \leqslant 2r_\sigma$.

Step 2. — We select a subsequence such that $\rho_h \to \rho \in [r, 2r]$. Define $g_h(x) := u_h(\rho_h x)$ for $x \in \partial B_1$. Then $g_h \in W^{1,2}(\partial B_1) \cap L^\infty(B_1)$ satisfies $|g_h| \leqslant 1$, and

$$\left(\int_{\partial B_1} |\partial_\tau g_h|^2 + \frac{1}{2\widetilde{\varepsilon}_h^2}(1-|g_h|^2)^2 \,\mathrm{d}\mathcal{H}^1 \right)^{1/2} \leqslant \delta \,, \tag{5.19}$$

where $\widetilde{\varepsilon}_h := \varepsilon_h/\rho_h$. Extracting a further subsequence if necessary, we may thus assume that $g_h \to g_\star$ uniformly of ∂B_1 for some $g_\star \in W^{1,2}(\partial B_1; \mathbf{S}^1)$. Estimate (5.17) yields

$$g_\star^m(x) = \lim_{h\to\infty} \mathrm{p}(g_h)(x) = \lim_{h\to\infty} v_h(\rho_h x) = v_\mu(\rho x) \quad \forall\, x \in \partial B_1 \,. \tag{5.20}$$

Since $\deg(v_\mu, \partial B_\rho) = 0$, we deduce that $\deg(g_\star, \partial B_1) = 0$. We are now in position to apply Proposition 5.8 to produce minimizers w_h of $E_{\widetilde{\varepsilon}_h}(\,\cdot\,, B_1)$ over $W^{1,2}_{g_h}(B_1)$. Then $w_h \to w_\star$ strongly in $W^{1,2}(B_1)$ where $w_\star$ is the unique solution of (5.6). We claim that

$$w_\star^m(x) = v_\mu(\rho x) \quad \forall\, x \in B_1 \,.$$

Indeed, recalling (2.14), $v_\mu(\rho x) = e^{i\psi_\mu(x)}$ for $x \in \overline{B}_1$ and a smooth harmonic function ψ_μ (which is unique up to a constant multiple of 2π).

Moreover, $w_\star = e^{i\varphi_\star}$ for some harmonic function $\varphi_\star \in W^{1,2}(B_1)$. In view of (5.20), we have $m\varphi_\star = \psi_\mu + 2k\pi$ on ∂B_1 for some constant $k \in \mathbb{N}$. By uniqueness of the harmonic extension, we infer that $m\varphi_\star = \psi_\mu + 2k\pi$ in B_1, and the claim follows.

As a consequence of this last identity, we deduce that

$$\int_{B_1} |\nabla w_\star|^2 \,\mathrm{d}x = \frac{1}{m^2}\int_{B_{\rho_h}} |\nabla v_\mu|^2 \,\mathrm{d}x \leqslant \frac{4\pi r^2 K_\sigma^2}{m^2} \leqslant \frac{\delta}{2}\,.$$

Since $w_h \to w_\star$ strongly in $W^{1,2}(B_1)$, we thus have for h large enough

$$\int_{B_1} |\nabla w_h|^2 \,\mathrm{d}x \leqslant \delta\,. \tag{5.21}$$

Step 3. — Let us define $\widehat{w}_h(x) := w_h(x/\rho_h)$, and consider the competitor $\widehat{u}_h \in \mathcal{G}_g(\Omega)$ given by

$$\widehat{u}_h := \begin{cases} u_h & \text{in } \Omega \setminus B_{\rho_h}\,, \\ \widehat{w}_h & \text{in } B_{\rho_h}\,. \end{cases}$$

By minimality we have $F^0_{\varepsilon_h,g}(u_h) \leqslant F^0_{\varepsilon_h,g}(\widehat{u}_h)$, and since $J_{u_h} \cap \partial B_{\rho_h} = \emptyset$, we deduce that

$$F^0_{\varepsilon_h}(u_h, B_{\rho_h}) \leqslant E_{\varepsilon_h}(\widehat{w}_h, B_{\rho_h})\,.$$

Setting $\widetilde{u}_h(x) := u_h(\rho_h x)$ and rescaling variables, we obtain

$$E_{\tilde{\varepsilon}_h}\big(\mathrm{P}(\widetilde{u}_h), B_1\big) + \rho_h \mathcal{H}^1(J_{\tilde{u}_h} \cap B_1) \leqslant E_{\tilde{\varepsilon}_h}(w_h, B_1)\,. \tag{5.22}$$

In view of (5.19) and (5.21) (and our choice of δ), we can apply Lemma 5.9 and Proposition 5.11 to derive that

$$\begin{aligned} E_{\tilde{\varepsilon}_h}\big(\mathrm{P}(\widetilde{u}_h), B_1\big) &+ \rho_h \mathcal{H}^1(J_{\tilde{u}_h} \cap B_1) \\ &\geqslant E_{\tilde{\varepsilon}_h}(w_h, B_1) + \frac{1}{8}\int_{B_1} |\nabla \phi_h|^2 \,\mathrm{d}x + \rho_h \mathcal{H}^1(J_{\phi_h} \cap B_1) \end{aligned} \tag{5.23}$$

for h large enough, where $\phi_h := \widetilde{u}_h / w_h$ satisfies $\phi_h = 1$ on ∂B_1. Putting (5.22) and (5.23) together leads to $\int_{B_1} |\nabla \phi_h|^2 \,\mathrm{d}x = 0 = \mathcal{H}^1(J_{\phi_h} \cap B_1)$, and thus $\phi_h \equiv 1$. In other words, $\widetilde{u}_h \equiv w_h$ for h large enough.

Scaling back to the original variables (and recalling that $u_h \to u$ in $L^1(\Omega)$), we conclude from Proposition 5.8 that for h large enough, u_h minimizes $E_{\varepsilon_h}(\,\cdot\,, B_r)$ in $W^{1,2}(B_r)$ under its own boundary condition, $u_h \in C^\infty(B_r)$ and $u_h \to u$ in $C^k_{\mathrm{loc}}(B_r)$ for every $k \in \mathbb{N}$. Since the limit is unique, we deduce that these facts actually hold for the full sequence (and not only for a subsequence). □

The next lemma is devoted to smoothness and convergence of u_h near the boundary of Ω. Since $\partial\Omega$ is assumed to be smooth, we can find a radius $r_\Omega > 0$ such that

$$\mathcal{H}^1(\partial\Omega \cap B_r(x)) \leqslant 3r \quad \forall\, r \in (0, r_\Omega)\,,\ \forall\, x \in \partial\Omega\,. \tag{5.24}$$

For the sake of variety, in the proof below, we do not use the energy splitting argument. Notice that, either way, it could be possible to adapt this alternative argument to prove Lemma 5.14, or to adapt the energy splitting approach to treat boundary points.

Lemma 5.15. — *For $\sigma \in (0, \sigma_0)$, let $r \in (0, \min\{r_\sigma, r_\Omega\})$ and $x_0 \in \partial\Omega$. For h large enough, $u_h \in W^{1,2}(B_r(x_0) \cap \Omega)$ with $u_h = g$ on $\partial\Omega \cap B_r(x_0)$ and u_h minimizes $E_{\varepsilon_h}(\,\cdot\,, B_r(x_0) \cap \Omega)$ under its own boundary conditions. In addition, $u_h \in C^\infty(B_r(x_0) \cap \overline{\Omega})$ and $u_h \to u$ in $C^{1,\alpha}_{\mathrm{loc}}(B_r(x_0) \cap \overline{\Omega})$ and $C^k_{\mathrm{loc}}(B_r(x_0) \cap \Omega)$ for every $\alpha \in (0,1)$ and $k \in \mathbb{N}$.*

Proof. — Without loss of generality, we may assume that $x_0 = 0$. As in the proof of Proposition 5.14, it is enough to find $\rho_h \in (r, 2r)$ such that (5.18) holds (with $\partial B_{\rho_h} \cap \Omega$ in place of ∂B_{ρ_h}), $J_{u_h} \cap (\overline{B}_{\rho_h} \cap \Omega) = \emptyset$ (so that $u_h \in W^{1,2}(B_{\rho_h} \cap \Omega)$), and $u_h = g$ on $\partial\Omega \cap B_{\rho_h}$, for h large enough. Indeed, for any $w \in W^{1,2}_{u_h}(B_{\rho_h})$, one can consider the competitor $\widehat{u}_h \in \mathcal{G}_g(\Omega)$ given by $\widehat{u}_h = w$ in $B_{\rho_h} \cap \Omega$, and $\widehat{u}_h = u_h$ in $\Omega \setminus B_{\rho_h}$. By minimality, $F^0_{\varepsilon_h,g}(u_h) \leqslant F^0_{\varepsilon_h,g}(\widehat{u}_h)$, which then leads to

$$E_{\varepsilon_h}(u_h, B_{\rho_h} \cap \Omega) = F^0_{\varepsilon_h}(u_h, B_{\rho_h} \cap \Omega) \leqslant F^0_{\varepsilon_h}(w, B_{\rho_h} \cap \Omega) = E_{\varepsilon_h}(w; B_{\rho_h} \cap \Omega)\,.$$

Hence u_h minimizes $E_{\varepsilon_h}(\,\cdot\,, B_{\rho_h})$ in $W^{1,2}(B_{\rho_h} \cap \Omega)$ under its own boundary condition. Then the remaining conclusions follow from [12] (see also [13, Theorem A.3]) together with the fact that $u_h \to u$ in $L^1(\Omega)$.

We select the radius ρ_h by repeating the Fubini type argument used in Step 1 of the proof of Proposition 5.14. The main additional point is to select ρ_h so that u_h belongs to $W^{1,2}(\partial B_{\rho_h} \cap \Omega)$ with $u_h = g$ on $\partial B_{\rho_h} \cap \partial\Omega$. This is possible via the coarea formula since (3.9) implies that for h large enough

$$\mathcal{H}^1(J_{u_h} \cap B_{2r} \cap \Omega) + \mathcal{H}^1(\{u_h \neq g\} \cap \partial\Omega \cap B_{2r}) \leqslant \frac{r}{2}.$$

By our choice of ρ_h, the map g_h defined by $g_h := u_h$ in $\partial B_{\rho_h} \cap \Omega$, and $g_h := g$ in $B_{\rho_h} \cap \partial\Omega$, belongs to $W^{1,2}(\partial(B_{\rho_h} \cap \Omega))$. In view of (5.18), for h

large enough we have $|g_h| \geqslant 1/2$ on $\partial(B_{\rho_h} \cap \Omega)$, and

$$\begin{aligned}
\operatorname{osc}_{\partial(B_{\rho_h}\cap\Omega)} g_h &\leqslant \left(\mathcal{H}^1(\partial(B_{\rho_h} \cap \Omega))\right)^{1/2} \left(\int_{\partial(B_{\rho_h}\cap\Omega)} |\partial_\tau g_h|^2 \,\mathrm{d}\mathcal{H}^1\right)^{1/2} \\
&\leqslant \sqrt{2\pi\rho_h} \left(\int_{B_{\rho_h}\cap\partial\Omega} |\partial_\tau g|^2 \,\mathrm{d}\mathcal{H}^1 + \int_{\partial B_{\rho_h}\cap\Omega} |\nabla u_h|^2 \,\mathrm{d}\mathcal{H}^1\right)^{1/2} \\
&\overset{(5.15),(5.24)\&(5.18)}{\leqslant} \sqrt{2\pi\rho_h} \left(3\rho_h K_\sigma^2 + \delta^2/\rho_h\right)^{1/2} \\
&\leqslant \sqrt{2\pi} \left(6r^2 K_\sigma^2 + \delta^2\right)^{1/2} \\
&\leqslant \frac{1}{2m}\,.
\end{aligned}$$

Rotating coordinates in the image if necessary, we may assume that $g_h(0) = 1$, which in turn yields

$$|\mathrm{p}(g_h(x)) - 1| \leqslant \frac{1}{2} \qquad \forall\, x \in \partial(B_{\rho_h} \cap \Omega)\,. \tag{5.25}$$

We claim that

$$\min_{v\in W^{1,2}_{\mathrm{p}(g_h)}(B_{\rho_h}\cap\Omega)} G_{\varepsilon_h}(v, B_{\rho_h} \cap \Omega) = \min_{u\in W^{1,2}_{g_h}(B_{\rho_h}\cap\Omega)} E_{\varepsilon_h}(u, B_{\rho_h} \cap \Omega)\,. \tag{5.26}$$

Before proving this claim, let us show how (5.26) leads to the conclusion. By minimality of u_h (and our choice of ρ_h), we have

$$\begin{aligned}
&\min_{u\in W^{1,2}_{g_h}(B_{\rho_h}\cap\Omega)} E_{\varepsilon_h}(u, B_{\rho_h} \cap \Omega) \\
&\qquad = \min_{u\in W^{1,2}_{g_h}(B_{\rho_h}\cap\Omega)} F^0_{\varepsilon_h}(u, B_{\rho_h} \cap \Omega) \\
&\qquad \geqslant \min_{u\in \mathcal{G}_{g_h}(B_{\rho_h}\cap\Omega)} \left\{F^0_{\varepsilon_h}(u, B_{\rho_h} \cap \Omega) + \mathcal{H}^1\big(\{u \neq g_h\} \cap \partial(B_{\rho_h} \cap \Omega)\big)\right\} \\
&\qquad = G_{\varepsilon_h}(v_h, B_{\rho_h} \cap \Omega) + \mathcal{H}^1(J_{u_h} \cap (B_{\rho_h} \cap \Omega)) \\
&\qquad\qquad\qquad + \mathcal{H}^1(\{u_h \neq g\} \cap (\partial\Omega \cap B_{\rho_h})) \\
&\qquad \geqslant \min_{v\in W^{1,2}_{\mathrm{p}(g_h)}(B_{\rho_h}\cap\Omega)} G_{\varepsilon_h}(v, B_{\rho_h} \cap \Omega)\,.
\end{aligned}$$

Then, (5.26) implies that all the inequalities above are in fact equalities and as a consequence

$$\mathcal{H}^1(J_{u_h} \cap (B_{\rho_h} \cap \Omega)) + \mathcal{H}^1(\{u_h \neq g\} \cap (\partial\Omega \cap B_{\rho_h})) = 0\,.$$

Hence $J_{u_h} \cap (B_{\rho_h} \cap \Omega)$ is empty, and $u_h = g$ on $\partial\Omega \cap B_{\rho_h}$.

In view of the above chain of inequalities, to prove (5.26) it is enough to show that

$$\min_{v\in W^{1,2}_{\mathrm{p}(g_h)}(B_{\rho_h}\cap\Omega)} G_{\varepsilon_h}(v,B_{\rho_h}\cap\Omega)\geqslant \min_{u\in W^{1,2}_{g_h}(B_{\rho_h}\cap\Omega)} E_{\varepsilon_h}(u,B_{\rho_h}\cap\Omega)\,. \tag{5.27}$$

We consider $\bar v$ a minimizer of the left-hand side. To establish (5.27), it is enough to construct $\bar u \in W^{1,2}_{g_h}(B_{\rho_h}\cap\Omega)$ satisfying $\mathrm{p}(\bar u)=\bar v$ since, in this case, $G_{\varepsilon_h}(\bar v,B_{\rho_h}\cap\Omega)=E_{\varepsilon_h}(\bar u,B_{\rho_h}\cap\Omega)$. Let $\Pi:\mathbf{C}\to\mathbf{C}$ the map defined by $\Pi(z):=|\mathrm{Re}(z)|+i\,\mathrm{Im}(z)$. By (5.25) we have $\Pi(\mathrm{p}(g_h))=\mathrm{p}(g_h)$ and $\Pi(\bar v)\in W^{1,2}_{\mathrm{p}(g_h)}(B_{\rho_h}\cap\Omega)$. In addition, $G_{\varepsilon_h}(\Pi(\bar v),B_{\rho_h}\cap\Omega)=G_{\varepsilon_h}(\bar v,,B_{\rho_h}\cap\Omega)$. Replacing $\bar v$ by $\Pi(\bar v)$ if necessary, we may thus assume that the real part of $\bar v$ is nonnegative in $B_{\rho_h}\cap\Omega$. Now, let us introduce the map $\mathrm{q}:\mathbf{C}\cap\{\mathrm{Re}(z)\geqslant 0\}\to\mathbf{C}$ defined by $\mathrm{q}(z)=|z|e^{i\theta/m}$ for $z=|z|e^{i\theta}$ with $\theta\in[-\pi/2,\pi/2]$. Then, q is Lipschitz continuous left inverse of p. In view of (5.25) we have $\mathrm{q}(\mathrm{p}(g_h))=g_h$, and as a consequence $\bar u:=\mathrm{q}(\bar v)\in W^{1,2}_{g_h}(B_{\rho_h}\cap\Omega)$ with $\mathrm{p}(\bar u)=\bar v$. ☐

5.3.2. Smoothness and convergence away from triple junctions

We continue our asymptotic analysis by considering the local behavior of u_h near Γ, but away from $T\cup\operatorname{spt}\mu$. In the statement below, we understand the convergence of half spaces in the sense of local Hausdorff convergence. Let us write

$$c_m:=|1-\mathbf{a}|^2.$$

Proposition 5.16. — *For $\sigma\in(0,\sigma_0)$, let $r\in(0,\min\{r_\sigma,c_m/32\})$ and $x_0\in\Gamma\setminus B_\sigma(T\cup\operatorname{spt}\mu)$. For h large enough, there exist a half space H_h and $k\in\{1,\dots,m-1\}$ such that $u_h=:(\chi_{H_h}+\mathbf{a}^k\chi_{H_h^c})w_h$ with $w_h\in W^{1,2}(B_r(x_0))$, and w_h minimizes $E_{\varepsilon_h}(\,\cdot\,,B_r(x_0))$ under its own boundary conditions. In addition, $w_h\in C^\infty(B_r(x_0))$, $H_h\to H$ for some half space H satisfying $\partial H\cap B_r(x_0)=\Gamma\cap B_r(x_0)$, and $w_h\to(\chi_H+\mathbf{a}^{-k}\chi_{H^c})u$ in $C^\ell_{\mathrm{loc}}(B_r(x_0))$ for every $\ell\in\mathbb{N}$.*

Proof.

Step 1. — Once again we may assume that $x_0=0$. We follow the strategy used in the proof of Lemma 5.14 considering

$$\gamma_h:=\int_{B_{5r}}|\nabla v_h-\nabla v_\mu|^2+|v_h-v_\mu|^2\,\mathrm{d}x+\frac{1}{2{\varepsilon_h}^2}\int_{B_{5r}}(1-|u_h|^2)^2\,\mathrm{d}x\underset{h\to\infty}{\longrightarrow}0\,.$$

By Theorem 3.2 and Corollary 3.4 we have (3.9), and thus $\mathcal{H}^1(J_{u_h}\cap B_{5r})\to\mathcal{H}^1(\Gamma\cap B_{5r})=10r$. As a consequence, $\mathcal{H}^1(J_{u_h}\cap B_{5r})\leqslant 11r$ for h large enough, which in turn leads to

$$\int_r^{5r} \mathcal{H}^0(J_{u_h}\cap\partial B_t)\,\mathrm{d}t \leqslant \mathcal{H}^1(J_{u_h}\cap B_{5r}) \leqslant 11r\,. \tag{5.28}$$

Setting

$$A_h := A_h^0\cup A_h^1\cup A_h^2 \quad\text{with}\quad A_h^k := \Big\{t\in(r,5r) : \mathcal{H}^0(J_{u_h}\cap\partial B_t)=k\Big\}\,,$$

we infer from (5.28) that $\mathcal{H}^1(A_h)\geqslant r/3$ for h large enough. Notice that $u_h\in W^{1,2}(\partial B_t)$ for a.e. $t\in A_h^0$, and $u_h\in SBV^2(\partial B_t)$ for a.e. $t\in A_h^1\cup A_h^2$.

We claim that $\mathcal{H}^1(A_h^0)\leqslant r/6$ for h large enough. Indeed, assume by contradiction that $\mathcal{H}^1(A_h^0)\geqslant r/6$ for some subsequence. Then, we could apply the proof of Lemma 5.14 (choosing a good radius $\rho_h\in A_h^0$) to infer that u_h is smooth in B_r for h large enough, and thus that $J_{u_h}\cap B_r=\emptyset$. However (3.9) tells us that $\mathcal{H}^1(J_{u_h}\cap B_r)\to 2r$ as $h\to\infty$, a contradiction. We have thus proved that $\mathcal{H}^1(A_h^1\cup A_h^2)\geqslant r/6$ for h large enough. Now we claim that for h even larger, we have $\mathcal{H}^1(A_h^1)\leqslant r/12$. By contradiction again, assume that $\mathcal{H}^1(A_h^1)\geqslant r/12$ for some subsequence. Then, we can find a good radius $\rho_h\in A_h^1$ such that $u_h\in SBV^2(\partial B_{\rho_h})$ and

$$\int_{\partial B_{\rho_h}} |\nabla v_h-\nabla v_\mu|^2+|v_h-v_\mu|^2+\frac{1}{2{\varepsilon_h}^2}(1-|u_h|^2)^2\,\mathrm{d}\mathcal{H}^1 \leqslant \frac{12\gamma_h}{r}\,. \tag{5.29}$$

By our choice of ρ_h, there is a single point $x_h\in\partial B_{\rho_h}$ such that $u_h\in W^{1,2}(\partial B_{\rho_h}\backslash\{x_h\})$. Rescaling variables if necessary, we may assume without too much loss of generality that the radius $\rho_h=\rho$ is independent of h. By (5.29), $v_h\to v_\mu$ uniformly on ∂B_ρ. As a consequence, $|u_h|=|v_h|\geqslant 1/2$ on ∂B_ρ, and $\deg(v_h,\partial B_\rho)=0$ for h large enough. In particular, we can write $u_h=|u_h|e^{i\varphi_h}$ on $\partial B_{\rho_b}\setminus\{x_h\}$ for some $\varphi_h\in W^{1,2}(\partial B_{\rho_b}\setminus\{x_h\})$. Let $\varphi_h^{\pm}$ be the traces of φ_h at x_h. Since $v_h=|u_h|e^{im\varphi_h}\in W^{1,2}(\partial B_\rho)$, we have $m(\varphi_h^+-\varphi_h^-)=2\pi\deg(v_h,\partial B_\rho)=0$. Hence $\varphi_h^+=\varphi_h^-$, which yields $\varphi_h\in W^{1,2}(\partial B_\rho)$. We obtain that $u_h\in W^{1,2}(\partial B_\rho)$ contradicting our choice $\rho\in A_h^1$.

We may now assume that h is sufficiently large so that $\mathcal{H}^1(A_h^2)\geqslant r/12$. Arguing as in the proof of Lemma 5.14, we select a good radius $\rho_h\in A_h^2$ so that $u_h\in SBV^2(\partial B_{\rho_h})$ and (5.18) holds together with (5.29). Here again we shall assume for simplicity $\rho_h=\rho$ is independent of h (otherwise we rescale variables as in the proof of Lemma 5.14). We write $\{x_h,y_h\}:=J_{u_h}\cap\partial B_\rho$, and then $\mathcal{C}_1^h$ and $\mathcal{C}_2^h$ the two (open) arcs in ∂B_ρ joining x_h and y_h. As above, we infer from (5.29) that $|u_h|=|v_h|\geqslant 1/2$ on ∂B_ρ, and $\deg(v_h,\partial B_\rho)=0$ for h large enough. Since $u_h\in W^{1,2}(\mathcal{C}_j^h)$ for $j=1,2$, we deduce that there exist $\varphi_h^j\in W^{1,2}(\mathcal{C}_j^h)$ such that $u_h=|u_h|e^{i\varphi_h^j}$ on $\mathcal{C}_j^h$. Denote by $\varphi_h^{j,1}$ the trace of φ_h^j at x_h, and $\varphi_h^{j,2}$ the trace of φ_h^j at y_h. Since $v_h \in W^{1,2}(\partial B_\rho)$, and $v_h = |u_h|e^{im\varphi_h^j}$ on $\mathcal{C}_j^h$, we obtain the relation

$$m(\varphi_h^{2,1} - \varphi_h^{1,1}) = 2\pi k_1^h \quad \text{and} \quad m(\varphi_h^{2,2} - \varphi_h^{1,2}) = 2\pi k_2^h$$

for some $k_1^h, k_2^h \in \mathbb{Z} \setminus \{0\}$. Define $k_h \in \{1, \dots, m-1\}$ to be such that $\mathbf{a}^{k_h} = e^{2i\pi k_2^h/m}$, and consider the map

$$g_h := \begin{cases} u_h & \text{in } \mathcal{C}_1^h\,, \\ \mathbf{a}^{-k_h} u_h & \text{in } \mathcal{C}_2^h\,. \end{cases}$$

By construction we have $\mathrm{p}(g_h) = v_h$ and $g_h \in W^{1,2}(\partial B_\rho \setminus \{x_h\})$. However, since $\deg(v_h, \partial B_\rho) = 0$, we can argue as above (when proving that $\mathcal{H}^1(A_h^1) \leqslant r/12$) to show that $g_h \in W^{1,2}(\partial B_\rho)$. In addition, (5.18) yields

$$\rho^{1/2}\left(\int_{\partial B_\rho} |\partial_\tau g_h|^2 + \frac{(1-|g_h|^2)^2}{2\varepsilon_h^2}\,\mathrm{d}\mathcal{H}^1\right)^{1/2} \leqslant \delta\,. \tag{5.30}$$

We also notice that $\|g_h\|_{L^\infty(B_\rho)} \leqslant 1$ since $|u_h| \leqslant 1$.

Step 2. — Define H_h to be the half space containing $\mathcal{C}_1^h$ and such that $x_h, y_h \in \partial H_h$. We claim that $k_h = k \in \{1, \dots, m-1\}$ is independent of h for h sufficiently large, that $H_h \to H$ for some half space H such that $\Gamma \cap \partial B_\rho = \partial H \cap \partial B_\rho$, and that $g_h \to g_\star$ uniformly on ∂B_ρ where $g_\star \in W^{1,2}(\partial B_\rho; \mathbf{S}^1)$ is given by

$$g_\star := \begin{cases} u & \text{in } \partial B_\rho \cap H\,, \\ \mathbf{a}^{-k} u & \text{in } \partial B_\rho \setminus H\,. \end{cases} \tag{5.31}$$

First observe that $\liminf_h |x_h - y_h| > 0$. Indeed, if for some subsequence we have $|x_h - y_h| \to 0$, then either $\chi_{H_h} \to 0$ in $L^1(\partial B_\rho)$ or $\chi_{H_h^c} \to 0$ in $L^1(\partial B_\rho)$. Assume that $\chi_{H_h^c} \to 0$ in $L^1(\partial B_\rho)$ (the other case being analogous). From Proposition 5.13, we infer that $u_h \to u$ in $L^1(\partial B_\rho)$, so that $g_h \to u$ in $L^1(\partial B_\rho)$. In view of (5.30), we deduce that u belongs to $W^{1,2}(\partial B_\rho)$, a contradiction. Next, by Proposition 5.13 again, $u_h \to u$ in $C^0_{\mathrm{loc}}(\partial B_\rho \setminus \Gamma)$, which now implies that $\{x_h, y_h\} \to \Gamma \cap \partial B_\rho$ as $h \to \infty$. Writing $\Gamma \cap B_\rho =: \{x_\star, y_\star\}$, we may assume that $x_h \to x_\star$ and $y_h \to y_\star$. In the same way, we may assume that $\mathcal{C}_1^h \to \mathcal{C}_1$ where $\mathcal{C}_1$ is an arc of ∂B_ρ joining $x_\star$ and $y_\star$. This clearly implies that $H_h \to H$ where H is the half space containing $\mathcal{C}_1$ and such that $x_\star, y_\star \in \partial H$. In view of Remark 4.4, there exists a unique $k \in \{1, \dots, m-1\}$ such that the map defined in (5.31) belongs to $W^{1,2}(\partial B_\rho)$. Combining this fact with (5.30) and the convergence of u_h toward u in $L^1(\partial B_\rho)$, we deduce that $g_h \to g_\star$ uniformly in ∂B_ρ, and that $k_h = k$ for h large enough.

Since $g_\star^m = u^m = v_\mu$ on ∂B_ρ, and $\deg(v_\mu, \partial B_\rho) = 0$, we derive that $\deg(g_\star, \partial B_\rho) = 0$. We can now apply Proposition 5.8 to produce minimizers w_h of $E_{\varepsilon_h}(\,\cdot\,, B_\rho)$ over $W^{1,2}_{g_h}(B_\rho)$. Then $w_h \to w_\star$ strongly in $W^{1,2}(B_1)$ where $w_\star$ is the unique solution of (5.6). Arguing as in the proof of Lemma 5.14 (Step 2), we obtain that $w_\star^m = v_\mu$ in B_ρ, which leads for h large enough to

$$\int_{B_\rho} |\nabla w_h|^2 \,\mathrm{d}x \leqslant \delta\,. \tag{5.32}$$

Step 3. — Consider the competitor $\widehat{u}_h \in \mathcal{G}_g(\Omega)$ given by

$$\widehat{u}_h := \begin{cases} u_h & \text{in } \Omega \setminus B_\rho\,,\\ (\chi_{H_h} + \mathbf{a}^k \chi_{H_h^c}) w_h & \text{in } B_\rho\,. \end{cases}$$

By minimality we have $F^0_{\varepsilon_h,g}(u_h) \leqslant F^0_{\varepsilon_h,g}(\widehat{u}_h)$, and since $J_{u_h} \cap \partial B_\rho = \{x_h, y_h\}$, we deduce that

$$F^0_{\varepsilon_h}(u_h, B_\rho) \leqslant F^0_{\varepsilon_h}(\widehat{u}_h, B_\rho) = E_{\varepsilon_h}(w_h, B_\rho) + L_h\,, \tag{5.33}$$

where $L_h := |x_h - y_h|$. Since $\mathrm{p}(u_h) = \mathrm{p}(g_h)$ on ∂B_ρ, and in view of (5.30) and (5.32) (and our choice of δ), we can apply Lemma 5.9 and Proposition 5.11 to derive that

$$F^0_{\varepsilon_h}(u_h, B_\rho) \geqslant E_{\varepsilon_h}(w_h, B_\rho) + \frac{1}{8}\int_{B_\rho} |\nabla \phi_h|^2 \,\mathrm{d}x + \mathcal{H}^1(J_{\phi_h} \cap B_\rho)\,, \tag{5.34}$$

where $\phi_h := u_h / w_h$ satisfies $\phi_h = \chi_{H_h} + \mathbf{a}^k \chi_{H_h^c}$ on ∂B_ρ (and thus $\mathrm{p}(\phi_h) = 1$ on ∂B_ρ). Putting (5.33) and (5.34) together leads to

$$L_h \geqslant \frac{1}{8}\int_{B_\rho} |\nabla \phi_h|^2 \,\mathrm{d}x + \mathcal{H}^1(J_{\phi_h} \cap B_\rho)\,. \tag{5.35}$$

Let us now prove that

$$\mathcal{H}^1(J_{\phi_h} \cap B_\rho) = L_h\,. \tag{5.36}$$

Up to a rotation, we assume that $x_h = (a,t)$ and $y_h = (b,t)$ with $b - a = L_h$. For $s \in (a,b)$, we write $V_s := \{s\} \times \mathbf{R}$. Now, assume by contradiction that $\mathcal{H}^1(J_{\phi_h} \cap B_\rho) = L_h - \gamma$ for some $\gamma > 0$. Then we infer from the coarea formula [47, Theorem II.7.7] that

$$L_h - \gamma = \mathcal{H}^1(J_{\phi_h} \cap B_\rho) \geqslant \int_a^b \mathcal{H}^0(J_{\phi_h} \cap B_\rho \cap V_s)\, ds\,. \tag{5.37}$$

Set $\widetilde{A}_h := \big\{s \in (a,b) : J_{\phi_h} \cap B_\rho \cap V_s = \emptyset\big\}$, and recall that $\phi_h \in W^{1,2}(B_\rho \cap V_s)$ for a.e. $s \in \widetilde{A}_h$. From (5.37) we deduce that $\mathcal{H}^1(\widetilde{A}_h) \geqslant \gamma$. Since, $\phi_h =$

$\chi_{H_h} + \mathbf{a}^k \chi_{H_h^c}$ on ∂B_ρ, we have $\phi_h = 1$ on $\mathcal{C}_1^h \cap V_s$ and $\phi_h = \mathbf{a}^k$ on $\mathcal{C}_2^h \cap V_s$ for a.e. $s \in (a,b)$. Therefore,

$$\int_{B_\rho \cap V_s} |\partial_\tau \phi_h|^2 \,\mathrm{d}\mathcal{H}^1 \geqslant \frac{|1-\mathbf{a}^k|^2}{2\rho} \geqslant \frac{c_m}{10r} \quad \text{for a.e. } s \in (a,b)\,.$$

Integrating with respect to s (and recalling that $r < c_m/80$) yields

$$\begin{aligned}\frac{1}{8}\int_{B_\rho} |\nabla \phi_h|^2 \,\mathrm{d}x + \mathcal{H}^1(J_{\phi_h} \cap B_\rho) \\ \geqslant \frac{1}{8}\int_{\tilde{A}_h}\int_{B_\rho \cap V_s} |\partial_\tau \phi_h|^2 \,\mathrm{d}\mathcal{H}^1 \,\mathrm{d}x + L_h - \gamma \\ \geqslant L_h + \left(\frac{c_m}{80r} - 1\right)\gamma > L_h\,,\end{aligned}$$

which contradicts (5.35).

By combining (5.35) and (5.36) we deduce that $|\nabla p(\phi_h)| \leqslant m|\nabla\phi_h| = 0$ in B_ρ. Since $\mathrm{p}(\phi_h) = 1$ on ∂B_ρ, we conclude that $\mathrm{p}(\phi_h) = 1$ in B_ρ. In other words, ϕ_h takes values in $\mathbf{G}_m$. Hence, there is a Caccioppoli partition $\{E_j\}_{j=0}^{m-1}$ of B_ρ such that

$$\phi_h = \sum_{j=0}^{m-1} \mathbf{a}^j \chi_{E_j}\,.$$

Recalling [4, Section 4.4], we have

$$\begin{aligned}(5.38)\quad L_h &= \mathcal{H}^1(J_{\phi_h} \cap B_\rho) \\ &= \mathcal{H}^1(\partial E_0 \cap B_\rho) + \frac{1}{2}\sum_{j,\ell=1,\, j\neq\ell}^{m-1} \mathcal{H}^1(\partial E_j \cap \partial E_\ell \cap B_\rho)\,.\end{aligned}$$

Using that $\chi_{E_0} = \chi_{H_h}$ on ∂B_ρ, we have that $\mathcal{H}^1(\partial E_0 \cap B_\rho) \geqslant L_h$ with equality if and only if $E_0 = H_h \cap B_\rho$. Therefore $E_0 = H_h \cap B_\rho$ and the sum on the right-hand side of (5.38) vanishes. Since $\chi_{E_k} = \chi_{H_h^c}$ on ∂B_ρ, we conclude that $E_k = H_h^c \cap B_\rho$ and $E_j = \emptyset$ for $j \notin \{0,k\}$. In other words, $\phi_h = \chi_{H_h} + \mathbf{a}^k \chi_{H_h^c}$ in B_ρ, and thus $u_h = (\chi_{H_h} + \mathbf{a}^k \chi_{H_h^c}) w_h$ in B_ρ.

To conclude, we observe that $w_h = (\chi_{H_h} + \mathbf{a}^{-k}\chi_{H_h^c})u_h$. Since $u_h \to u$ in $L^1(\Omega)$ and $H_h \to H$, Proposition 5.8 tells us that $w_h \to (\chi_{H_\star} + \mathbf{a}^{-k}\chi_{H_\star^c})u$ in $C^\ell_{\mathrm{loc}}(B_\rho)$ for every $\ell \in \mathbb{N}$. □

Remark 5.17. — In order to prove (5.36) one could also use a calibration argument (see [1, 49]). However since our proof is elementary, we have decided to keep it this way.

5.3.3. Smoothness and convergence near triple junctions

We now focus on the behavior of u_h near the points of T, i.e., triple junctions. It will be convenient to describe a triple junction in the following way. First write for $j=0,1,2$, $Y^j_{\mathrm{ref}} := \big\{z\in\mathbf{C}\setminus\{0\} : \arg(z)\in(2j\pi/3, 2(j+1)\pi/3)\big\}$. We say that an ordered triplet of open sets (Y^0,Y^1,Y^2) is a triple junction if there are $x_0\in\mathbf{C}$ and $\theta\in[0,2\pi)$ such that $Y^j = x_0 + e^{i\theta}Y^j_{\mathrm{ref}}$ for $j=0,1,2$. Then, we say that x_0 is the center of the triple junction (Y^0,Y^1,Y^2). In the statement below, we understand the convergence of triple junctions in the sense of local Hausdorff convergence.

Proposition 5.18. — *For $\sigma\in(0,\sigma_0)$, let $r\in(0,\min\{r_\sigma, c_m/128\})$ and $x_0\in T$. For h large enough, there exist a triple junction (Y^0_h,Y^1_h,Y^2_h) and distinct $k_1,k_2\in\{1,\dots,m-1\}$ such that $u_h =: (\chi_{Y^0_h} + \mathbf{a}^{k_1}\chi_{Y^1_h} + \mathbf{a}^{k_2}\chi_{Y^2_h})w_h$ with $w_h\in W^{1,2}(B_r(x_0))$, and w_h minimizes $E_{\varepsilon_h}(\,\cdot\,, B_r(x_0))$ under its own boundary conditions. In addition, $w_h\in C^\infty(B_r(x_0))$, $(Y^0_h,Y^1_h,Y^2_h)\to(Y^0,Y^1,Y^2)$ for some triple junction satisfying $\bigcup_j \partial Y^j\cap B_r(x_0) = \Gamma\cap B_r(x_0)$, and $w_h\to(\chi_{Y^0_h} + \mathbf{a}^{-k_1}\chi_{Y^1_h} + \mathbf{a}^{-k_2}\chi_{Y^2_h})u$ in $C^\ell_{\mathrm{loc}}(B_r(x_0))$ for every $\ell\in\mathbb{N}$.*

Proof.

Step 1. — Without loss of generality, we may assume that $x_0=0$. From Remark 4.4, we infer that there exist a triple junction (Y^0,Y^1,Y^2) centered in 0 and distinct $k_1,k_2\in\{1,\dots,m-1\}$ such that the map $(\chi_{Y^0} + \mathbf{a}^{-k_1}\chi_{Y^1} + \mathbf{a}^{-k_2}\chi_{Y^2})u$ is smooth in B_{2r}, and $\bigcup_j\partial Y^j\cap B_{2r} = \Gamma\cap B_{2r}$. Since the values of k_1 and k_2 play no role we will assume that $k_1=1$ and $k_2=2$ to keep notation simpler. We write $\{a\} := \partial Y^2\cap\partial Y^0\cap\partial B_r$, $\{b\} := \partial Y^0\cap\partial Y^1\cap\partial B_r$, and $\{c\} := \partial Y^1\cap\partial Y^2\cap\partial B_r$.

Choosing a sufficiently small radius $0<\kappa<r/2$, we can apply Proposition 5.16 in the balls $B_{2\kappa}(a)$, $B_{2\kappa}(b)$, and $B_{2\kappa}(c)$, and infer that there exist half spaces H^0_h, H^1_h, and H^2_h such that $H^0_h\cap B_{2\kappa}(a)\to Y^0\cap B_{2\kappa}(a)$, $H^1_h\cap B_{2\kappa}(b)\to Y^1\cap B_{2\kappa}(b)$, $H^2_h\cap B_{2\kappa}(c)\to Y^2\cap B_{2\kappa}(c)$, and

$$(\chi_{H^0_h} + \mathbf{a}^{-2}\chi_{(H^0_h)^c})u_h \to (\chi_{Y^0} + \mathbf{a}^{-2}\chi_{Y^2})u \quad \text{in } C^k_{\mathrm{loc}}(B_{2\kappa}(a))\,, \tag{5.39}$$

$$(\chi_{(H^1_h)^c} + \mathbf{a}^{-1}\chi_{H^1_h})u_h \to (\chi_{Y^0} + \mathbf{a}^{-1}\chi_{Y^1})u \quad \text{in } C^k_{\mathrm{loc}}(B_{2\kappa}(b))\,, \tag{5.40}$$

$$(\mathbf{a}^{-1}\chi_{(H^2_h)^c} + \mathbf{a}^{-2}\chi_{H^2_h})u_h \to (\mathbf{a}^{-1}\chi_{Y^1} + \mathbf{a}^{-2}\chi_{Y^2})u \quad \text{in } C^k_{\mathrm{loc}}(B_{2\kappa}(c))\,. \tag{5.41}$$

In view of Proposition 5.13, we deduce that for h large enough, $J_{u_h}\cap(B_{r+\kappa}\setminus\overline{B}_{r-\kappa})$ is made of three (disjoint) segments, each of them intersecting ∂B_t almost orthogonally (in particular at a single point) for every $t\in(r-\kappa, r+\kappa)$. As a consequence, for h large enough the open set $(B_{r+\kappa}\setminus\overline{B}_{r-\kappa})\setminus J_{u_h}$

has three connected components Z^0_h, Z^1_h, and Z^2_h satisfying $Z^j_h \to (B_{r+\kappa} \setminus \overline{B}_{r-\kappa}) \cap Y^j$. Combining (5.39)–(5.41) with Proposition 5.16, we derive that

$$(\chi_{Z^0_h} + \mathbf{a}^{-1}\chi_{Z^1_h} + \mathbf{a}^{-2}\chi_{Z^2_h})u_h \to (\chi_{Y^0} + \mathbf{a}^{-1}\chi_{Y^1} + \mathbf{a}^{-2}\chi_{Y^2})u \quad \text{in } C^k_{\mathrm{loc}}(B_{r+\kappa} \setminus \overline{B}_{r-\kappa})\,.$$

Step 2. — Arguing as the proof of Lemma 5.14 (Step 1), we find a good radius $\rho_h \in (r, r+\kappa/2)$ such that (5.18) holds (for h even larger). Rescaling variables if necessary, we may assume without too much loss of generality that $\rho_h = \rho$ is independent of h. To simplify, we will further assume that actually $\rho = r$. Setting

$$g_h := (\chi_{Z^0_h} + \mathbf{a}^{-1}\chi_{Z^1_h} + \mathbf{a}^{-2}\chi_{Z^2_h})u_h \in C^\infty(\partial B_r)\,,$$

estimate (5.30) holds, $\|g_h\|_{L^\infty(B_r)} \leqslant 1$, and $g_h \to g_\star := (\chi_{Y^0} + \mathbf{a}^{-1}\chi_{Y^1} + \mathbf{a}^{-2}\chi_{Y^2})u$ uniformly on ∂B_r. Once again, since $g^m_\star = v_\mu$ we have $\deg(g_\star, \partial B_r) = 0$. Then, we apply Proposition 5.8 to produce minimizers w_h of $E_{\varepsilon_h}(\,\cdot\,, B_r)$ over $W^{1,2}_{g_h}(B_r)$, and $w_h \to w_\star$ strongly in $W^{1,2}(B_r)$ where $w_\star$ is the unique solution of (5.6). Again, as in the proof of Lemma 5.14 (Step 2), we obtain that $w^m_\star = v_\mu$ in B_r, which leads to (5.32) for h large enough.

Step 3. — By Step 1, we have $J_{u_h} \cap \partial B_r = \{x_h, y_h, z_h\}$ for h large enough, with $x_h \to a$, $y_h \to b$, and $z_h \to c$. For h large enough, we can then find a triple junction (Y^0_h, Y^1_h, Y^2_h) (which might not be centered at the origin) such that $\{x_h\} = \partial Y^2_h \cap \partial Y^0_h \cap \partial B_r$, $\{y_h\} = \partial Y^0_h \cap \partial Y^1_h \cap \partial B_r$, and $\{z_h\} := \partial Y^1_h \cap \partial Y^2_h \cap \partial B_r$. Obviously, $Y^j_h \to Y^j$ as $h \to \infty$. Notice also that $g_h = (\chi_{Y^0_h} + \mathbf{a}^{-1}\chi_{Y^1_h} + \mathbf{a}^{-2}\chi_{Y^2_h})u_h$ on ∂B_r.

Next, we consider the competitor $\widehat{u}_h \in \mathcal{G}_g(\Omega)$ given by

$$\widehat{u}_h := \begin{cases} u_h & \text{in } \Omega \setminus B_r\,, \\ (\chi_{Y^0_h} + \mathbf{a}\chi_{Y^1_h} + \mathbf{a}^2\chi_{Y^2_h})w_h & \text{in } B_r\,. \end{cases}$$

By minimality we have $F^0_{\varepsilon_h,g}(u_h) \leqslant F^0_{\varepsilon_h,g}(\widehat{u}_h)$, and since $J_{u_h} \cap \partial B_r = \{x_h, y_h, z_h\}$, we deduce that

$$F^0_{\varepsilon_h}(u_h, B_r) \leqslant F^0_{\varepsilon_h}(\widehat{u}_h, B_r) = E_{\varepsilon_h}(w_h, B_r) + \mathcal{H}^1(\mathbf{Y}_h \cap B_r)\,, \tag{5.42}$$

where we have set $\mathbf{Y}_h := \bigcup_j \partial Y^j_h$. Once again $\mathrm{p}(u_h) = \mathrm{p}(g_h)$ on ∂B_r, and by (5.30) and (5.32), we can apply Lemma 5.9 and Proposition 5.11 to derive that (5.34) holds, where $\phi_h := u_h/w_h$ satisfies $\phi_h = \chi_{Y^0_h} + \mathbf{a}^{k_1}\chi_{Y^1_h} + \mathbf{a}^{k_2}\chi_{Y^2_h}$ on ∂B_r (and $\mathrm{p}(\phi_h) = 1$ on ∂B_r). Combining (5.34) with (5.42) leads to

$$\mathcal{H}^1(\mathbf{Y}_h \cap B_r) \geqslant \frac{1}{8}\int_{B_r} |\nabla\phi_h|^2 \,\mathrm{d}x + \mathcal{H}^1(J_{\phi_h} \cap B_r)\,. \tag{5.43}$$

Our choice of r (small compare to c_m) allows us to use the calibration in [49, Example 5.4] (with $\alpha = 16$) to deduce that for h large enough the map $\chi_{Y_h^0} + \mathbf{a}\chi_{Y_h^1} + \mathbf{a}^2\chi_{Y_h^2}$ is a Dirichlet minimizer of the Mumford–Shah functional(4) [49, Definition 3.1]. As a consequence,

$$\frac{1}{16}\int_{B_r} |\nabla\phi_h|^2 \,\mathrm{d}x + \mathcal{H}^1(J_{\phi_h} \cap B_r) \geqslant \mathcal{H}^1(\mathbf{Y}_h \cap B_r)\,. \tag{5.44}$$

Putting together (5.43) and (5.44) yields

$$\int_{B_r} |\nabla\phi_h|^2 \,\mathrm{d}x = 0 \quad \text{and} \quad \mathcal{H}^1(J_{\phi_h} \cap B_r) = \mathcal{H}^1(\mathbf{Y}_h \cap B_r)\,.$$

Arguing as in the proof of Proposition 5.16 (Step 3), we deduce that

$$\phi_h = \sum_{k=0}^{m-1} \mathbf{a}^k \chi_{E_k}$$

for a Caccioppoli partition $\{E_k\}_{k=0}^{m-1}$ of B_r satisfying

$$\begin{aligned} E_0 \cap (B_r \setminus \overline{B}_{r-\kappa}) = Z_0^h \cap B_r\,,\ E_1 \cap (B_r \setminus \overline{B}_{r-\kappa}) \\ = Z_1^h \cap B_r\,,\ E_2 \cap (B_r \setminus \overline{B}_{r-\kappa}) = Z_2^h \cap B_r\,, \end{aligned} \tag{5.45}$$

and $E_k \subseteq B_{r-\kappa}$ for $k \notin \{0,1,2\}$.

Let us now consider an arbitrary Caccioppoli partition $\{F_k\}_{k=0}^{m-1}$ of B_r such that each $F_k \triangle E_k$ is compactly contained in B_r, and define the competitor $\widetilde{u}_h \in \mathcal{G}_g(\Omega)$ by

$$\widetilde{u}_h := \begin{cases} u_h & \text{in } \Omega \setminus B_r\,, \\ \phi w_h & \text{in } B_r\,, \end{cases} \quad \text{with} \quad \phi := \sum_{k=0}^{m-1} \mathbf{a}^k \chi_{F_k}\,.$$

By minimality $F^0_{\varepsilon_h,g}(u_h) \leqslant F^0_{\varepsilon_h,g}(\widetilde{u}_h)$, which leads as before to

$$\mathcal{H}^1(J_{\phi_h} \cap B_r) \leqslant \mathcal{H}^1(J_\phi \cap B_r)\,.$$

As in the proof of Theorem 4.3 (Step 3), it implies that $\{E_k\}_{k=0}^{m-1}$ is a minimal partition of B_r, so that $J_{\phi_h} \cap B_r = \bigcup_k \partial E_k \cap B_r$ is locally a finite union of segments (see [23, Theorem 5.2]). Since we already know that J_{ϕ_h} is made of three segments in a neighborhood of ∂B_r, we conclude that $\bigcup_k \partial E_k \cap B_r$ is made of finitely many segments with $\bigcup_k \partial E_k \cap \partial B_r =$

(4) Even though the calibrations defined in [49] (see also [1]) are given for centered triple junctions, we can consider restrictions to B_r of calibrations defined on a larger ball centered at the center of $\mathbf{Y_h}$.

$\{x_h, y_h, z_h\}$(5). In view of (5.45), we have $\{x_h, y_h\} \subseteq \partial E_0 \cap B_r$, and the connected component of $\partial E_0 \cap B_r$ containing x_h is a polygonal curve joining x_h to y_h. Similarly, $\partial E_2 \cap B_r$ contains a polygonal curve connecting x_h to z_h. Set Γ_h to be the union of these two curves. Then Γ_h is a connected set containing $\{x_h, y_h, z_h\}$, and contained in $J_{\phi_h} \cap B_r$. Since $\mathbf{Y}_h \cap \overline{B}_r$ is the unique solution of the Steiner problem relative to the points $\{x_h, y_h, z_h\}$, we have

$$\mathcal{H}^1(\mathbf{Y}_h \cap B_r) = \mathcal{H}^1(J_{\phi_h} \cap B_r) \geqslant \mathcal{H}^1(\Gamma_h) \geqslant \mathcal{H}^1(\mathbf{Y}_h \cap B_r)\,,$$

and it follows that $J_{\phi_h} \cap B_r = \Gamma_h \cap B_r = \mathbf{Y}_h \cap B_r$. From (5.45) we conclude that

$$\phi_h = \chi_{Y_h^0} + \mathbf{a}\chi_{Y_h^1} + \mathbf{a}^2\chi_{Y_h^2}\,,$$

that is $u_h = (\chi_{Y_h^0} + \mathbf{a}\chi_{Y_h^1} + \mathbf{a}^2\chi_{Y_h^2})w_h$ in B_r.

Since $u_h \to u$ in $L^1(\Omega)$ and $Y_h^j \to Y^j$, Proposition 5.8 implies that $w_h \to (\chi_{Y^0} + \mathbf{a}^{-1}\chi_{Y^1} + \mathbf{a}^{-2}\chi_{Y^2})u$ in $C^k_{\mathrm{loc}}(B_r)$ for every $k \in \mathbb{N}$, and the proof is complete. ☐

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(5) Here and in the rest of the proof, by an abuse of notation we identify $\partial E_k \cap B_r$ and $\overline{\partial E_k \cap B_r}$.

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