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RIGIDITY OF OELJEKLAUS-TOMA MANIFOLDS

by Daniele ANGELLA, Maurizio PARTON & Victor VULETESCU (*)

 $\rm ABSTRACT.$ — We prove that Oeljeklaus–Toma manifolds of simple type are rigid, and that any line bundle on an Oeljeklaus–Toma manifold is flat.

RÉSUMÉ. — Nous montrons que les variétés de Oeljeklaus–Toma de type simple sont rigides, et que tous les fibrés en droites sur une variété de Oeljeklaus–Toma sont plats.

Introduction

Oeljeklaus–Toma manifolds are complex non-Kähler manifolds. They have been introduced in [11] as counterexamples to a conjecture by I. Vaisman concerning locally conformally Kähler metrics. Because of their construction using number fields techniques, many of their properties are encoded in the algebraic structure [5, 11, 22], and their class is well-behaved under such properties [19, 20]. They generalize Inoue–Bombieri surfaces in class VII [6, 18], and they are in fact solvmanifolds [8].

For example, K. Oeljeklaus and M. Toma proved in [11, Proposition 2.5], among other results, that the line bundles $K_X^{\otimes k}$ for $k \in \mathbb{Z}$ are flat. In this note, we use tools both from the number theoretic construction and from complex analysis and analytic geometry to prove more generally that:

Keywords: Oeljeklaus-Toma manifold, flat line bundle, deformation, rigidity.

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THEOREM 2.1. — Any line bundle on an Oeljeklaus–Toma manifold is flat.

Recently, A. Otiman and M. Toma [14] performed a precise and complete study of the Dolbeault cohomology of certain domains contained in Cousin groups, that includes our analysis below. We give here a self-contained proof in our specific context, that actually fits in the same perspective as [14, 21].

With techniques very similar to those developed in [14], we get the following vanishing result:

THEOREM 3.1. — Let X(K,U) be an Oeljeklaus–Toma manifold, and $\rho: U \to \mathbb{C}^*$ be a faithful representation. Consider L_{ρ} its associated flat holomorphic line bundle. Then one has $H^1(X; L_{\rho}) = 0$ unless $\rho = \bar{\sigma}_i^{-1}$ for some $i \in \{t + 1, \ldots, t + s\}$.

As a corollary, we get rigidity, in the sense of the theory of deformations of complex structures of Kodaira–Spencer–Nirenberg–Kuranishi, for Oeljeklaus–Toma manifolds of simple type. Note that for the Inoue– Bombieri surface S_M , this is proven by Inoue in [6, Proposition 2]. Here, by saying that the Oeljeklaus–Toma manifold X(K, U) associated to the algebraic number field K and to the admissible group U is of simple type, we understand that there exists no proper intermediate field extension $\mathbb{Q} \subset K' \subset K$ with $U \subseteq \mathcal{O}_{K'}^{*,+}$, that is, there exists no holomorphic foliation of X(K, U) with a leaf isomorphic to X(K', U) [11, Remark 1.7].

COROLLARY 3.3. — Oeljeklaus–Toma manifolds of simple type are rigid.

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1. Oeljeklaus–Toma manifolds

Oeljeklaus–Toma manifolds [11] provide a beautiful family of examples of compact complex non-Kähler manifolds, generalizing Inoue–Bombieri surfaces [6]. In this section, we briefly recall the construction and main properties of Oeljeklaus–Toma manifolds from [11]. See [13] and [15, Section 6 of arXiv version] for more details and algebraic number theory background.

Let K be an algebraic number field, namely, a finite extension of \mathbb{Q} . Then $K \simeq \mathbb{Q}[X]/(f)$ as \mathbb{Q} -algebras, where $f \in \mathbb{Q}[X]$ is a monic irreducible polynomial of degree $n = [K : \mathbb{Q}]$. By mapping X mod (f) to a root of f, the field K admits n = s + 2t embeddings in \mathbb{C} , more precisely, s real embeddings $\sigma_1, \ldots, \sigma_s \colon K \to \mathbb{R}$, and 2t complex embeddings $\sigma_{s+1}, \ldots, \sigma_{s+t}, \sigma_{s+t+1} = \overline{\sigma}_{s+1}, \ldots, \sigma_{s+2t} = \overline{\sigma}_{s+t} \colon K \to \mathbb{C}$. Note that, for any choice of natural numbers s and t, there is an algebraic number field with s real embeddings and 2t complex embeddings, [11, Remark 1.1].

Denote by \mathcal{O}_K the ring of algebraic integers of K, namely, elements of K satisfying monic polynomial equations with integer coefficients. Note that, as a \mathbb{Z} -module, \mathcal{O}_K is free of rank n. Denote by \mathcal{O}_K^* the multiplicative group of units of \mathcal{O}_K , namely, invertible elements in \mathcal{O}_K . By the Dirichlet's unit theorem, \mathcal{O}_K^* is a finitely generated Abelian group of rank s + t - 1. Denote by $\mathcal{O}_K^{*,+}$ the subgroup of finite index of \mathcal{O}_K^* whose elements are totally positive units, namely, units being positive in any real embedding: $u \in \mathcal{O}_K^*$ such that $\sigma_j(u) > 0$ for any $j \in \{1, \ldots, s\}$.

Let $\mathbb{H} := \{z \in \mathbb{C} : \text{Im } z > 0\}$ denote the upper half-plane. On $\mathbb{H}^s \times \mathbb{C}^t$, consider the following actions:

(1.1)
$$T: \mathcal{O}_K \circlearrowleft \mathbb{H}^s \times \mathbb{C}^t,$$
$$T_a(w_1, \dots, w_s, z_{s+1}, \dots, z_{s+t}) := (w_1 + \sigma_1(a), \dots, z_{s+t} + \sigma_{s+t}(a)),$$
and

(1.2)
$$R\colon \mathcal{O}_K^{*,+} \circlearrowleft \mathbb{H}^s \times \mathbb{C}^t,$$

$$R_u(w_1, \dots, w_s, z_{s+1}, \dots, z_{s+t}) := (w_1 \cdot \sigma_1(u), \dots, z_{s+t} \cdot \sigma_{s+t}(u)).$$

For any subgroup $U \subset \mathcal{O}_K^{*,+}$, one has the fixed-point-free action $\mathcal{O}_K \rtimes U \oslash \mathbb{H}^s \times \mathbb{C}^t$. One can always choose an admissible subgroup [11, p. 162], namely, a subgroup such that the above action is also properly discontinuous and cocompact. In particular, the rank of admissible subgroups is s. Conversely, when either s = 1 or t = 1, every subgroup U of $\mathcal{O}_K^{*,+}$ of rank s is admissible.

One defines the *Oeljeklaus–Toma manifold* associated to the algebraic number field K and to the admissible subgroup U of $\mathcal{O}_{K}^{*,+}$ as

$$X(K,U) := \mathbb{H}^s \times \mathbb{C}^t / \mathcal{O}_K \rtimes U$$

In particular, for an algebraic number field K with s = 1 real embeddings and 2t = 2 complex embeddings, choosing $U = \mathcal{O}_K^{*,+}$ we obtain that X(K,U) is an Inoue–Bombieri surface of type S_M [6].

The Oeljeklaus–Toma manifold X(K,U) is called of simple type when there exists no proper intermediate field extension $\mathbb{Q} \subset K' \subset K$ with $U \subseteq \mathcal{O}_{K'}^{*,+}$, that is, there exists no holomorphic foliation of X(K,U) with a leaf isomorphic to X(K',U) [11, Remark 1.7].

Oeljeklaus–Toma manifolds are non-Kähler solvmanifolds [8, §6], with Kodaira dimension $\kappa(X) = -\infty$ [11, Proposition 2.5]. Their first Betti number is $b_1 = s$, and their second Betti number in the case of simple type is $b_2 = \binom{s}{2}$ [11, Proposition 2.3]. Their group of holomorphic automorphisms is discrete [11, Corollary 2.7]. The vector bundles Ω_X^1 , Θ_X , $K_X^{\otimes k}$ for $k \in \mathbb{Z}$ are flat and admit no non-trivial global holomorphic sections [11, Proposition 2.5]. Other invariants are computed in [11, Proposition 2.5] and [17]. Recently, their Dolbeault cohomology is described in [14]. Oeljeklaus–Toma manifolds do not contain either any compact complex curve [19, Theorem 3.9], or any compact complex surface except Inoue– Bombieri surfaces [20, Theorem 3.5]. When t = 1, they admit a locally conformally Kähler structure [11, p. 169], with locally conformally Kähler rank either $\frac{b_1}{2}$ or b_1 [15, Theorem 5.4]. This is the Tricerri metric [18] in case s = 1 and t = 1.

In the case $t \ge 2$, no locally conformally Kähler metrics are known to exist, so far. The fact that such an Oeljeklaus–Toma manifold does not carry a locally conformally Kähler metric was proven for s = 1 already in the original paper [11, Proposition 2.9], later extended to the case s < tby [22, Theorem 3.1], and eventually widely extended to almost all cases by [5, Theorem 2]. Most likely, in the case $t \ge 2$, no Oeljeklaus–Toma manifold carries a locally conformally Kähler metric. However, note that Oeljeklaus–Toma manifolds admit no Vaisman metrics [8, Corollary 6.2].

2. Flatness of line bundles on Oeljeklaus–Toma manifolds

Let X = X(K, U) be the Oeljeklaus–Toma manifold associated to the algebraic number field K and to the admissible subgroup $U \subseteq \mathcal{O}_{K}^{*,+}$. Let s denote the number of real embeddings of K and 2t the number of complex embeddings of K.

For a better understanding of the cohomology of X, we start from its very definition, in the form of the following diagram of regular coverings:



THEOREM 2.1. — Any line bundle on an Oeljeklaus–Toma manifold is flat.

Proof. — Equivalence classes of line bundles on X are given by $H^1(X; \mathcal{O}_X^*)$, and the flat ones are given by the image of the map $n : H^1(X; \mathbb{C}_X^*) \to H^1(X; \mathcal{O}_X^*)$ induced by $\mathbb{C}_X \hookrightarrow \mathcal{O}_X$. The statement is then equivalent to prove that the map

$$n: H^1(X; \mathbb{C}^*_X) \to H^1(X; \mathcal{O}^*_X)$$

is an isomorphism.

The map n appears naturally from the following morphism of short exact sequences of sheaves:



and the corresponding induced morphism of long exact sequences in cohomology:

By the Five Lemma, it suffices to prove that, in diagram (2.2):

(H1) m is an isomorphism;

(H2) q is injective.

Remark 2.2. — Notice that both of the claims are now proven in [14, Corollary 4.6, Corollary 4.9] as a consequence of a more general description of Dolbeault cohomology of certain domains contained in Cousin groups.

For our aim, we will need a description of $H^1(X^{ab}; \mathcal{O}_{X^{ab}})$ as in [14, Theorem 3.1]: for the sake of completeness, we give here below a self-contained argument in our simpler case, in the same line of thought. Compare also previous partial results by A. Tomassini and S. Torelli [17] for the case s = 2 real places and 2t = 2 complex places.

Proof of Claim (H1). — To prove that m is an isomorphism, consider the following exact sequence of sheaves:

 $0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{O}_X \longrightarrow d\mathcal{O}_X \longrightarrow 0$

and the induced exact sequence in cohomology:

$$H^0(X; d\mathcal{O}_X) \longrightarrow H^1(X; \mathbb{C}_X) \xrightarrow{m} H^1(X; \mathcal{O}_X).$$

Note that $H^0(X; d\mathcal{O}_X) = 0$, since $H^0(X; \Omega^1_X) = 0$ by [11, Proposition 2.5]. Therefore *m* is injective. Using the fact that $\dim_{\mathbb{C}} H^1(X; \mathbb{C}_X) = s$ [11, Proposition 2.3], it suffices to prove that $\dim_{\mathbb{C}} H^1(X; \mathcal{O}_X) = s$.

In order to describe the cohomology of X, we use diagram (2.1): we would like to relate the cohomology of X with the U-invariant cohomology of X^{ab} . In what follows, we use group cohomology and the Lyndon–Hochschild– Serre spectral sequence to accomplish this task.

In general, whenever one has a map $\pi: \widetilde{X} \to X = \widetilde{X}/G$, for a free and properly discontinuous action of a group G on \widetilde{X} , and a sheaf \mathcal{F} on X, there is an induced map

(2.3)
$$H^p(G; H^0(\widetilde{X}; \pi^* \mathcal{F})) \to H^p(X; \mathcal{F}),$$

where the first is the group cohomology of G with coefficients in the Gmodule $H^0(\widetilde{X}; \pi^* \mathcal{F})$, see for instance [10, Appendix at p. 22]. If, moreover, $\pi^* \mathcal{F}$ is acyclic over \widetilde{X} , then the map (2.3) is an isomorphism.

Using the previous argument on the $\mathcal{O}_K \rtimes U$ and the \mathcal{O}_K maps in diagram (2.1), with $\mathcal{F} = \mathcal{O}_X$ and $\mathcal{F} = \mathcal{O}_{X^{\mathrm{ab}}}$ respectively, and noting that $\mathcal{O}_{\tilde{X}}$ is acyclic over \tilde{X} , we obtain the isomorphisms

$$H^{p}(\mathcal{O}_{K} \rtimes U; H^{0}(\widetilde{X}; \mathcal{O}_{\widetilde{X}})) \simeq H^{p}(X; \mathcal{O}_{X})$$

and $H^{p}(\mathcal{O}_{K}; H^{0}(\widetilde{X}; \mathcal{O}_{\widetilde{X}})) \simeq H^{p}(X^{\mathrm{ab}}; \mathcal{O}_{X^{\mathrm{ab}}}).$

Hereafter, for the sake of clearness of notation, we denote the $\mathcal{O}_K \rtimes U$ module $R := H^0(\widetilde{X}; \mathcal{O}_{\widetilde{X}})$. The previous isomorphisms are then written as

(2.4)
$$H^p(\mathcal{O}_K \rtimes U; R) \simeq H^p(X; \mathcal{O}_X)$$
 and $H^p(\mathcal{O}_K; R) \simeq H^p(X^{\mathrm{ab}}; \mathcal{O}_{X^{\mathrm{ab}}}).$

The extension $\mathcal{O}_K \hookrightarrow \mathcal{O}_K \rtimes U \twoheadrightarrow U$ gives the associated Lyndon– Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(U; H^q(\mathcal{O}_K; R)) \Rightarrow H^{p+q}(\mathcal{O}_K \rtimes U; R),$$

and the cohomology five-term exact sequence yields

$$\begin{array}{c} 0\\ H^{1}(U;H^{0}(\mathcal{O}_{K};R)) \longrightarrow H^{1}(\mathcal{O}_{K} \rtimes U;R)\\ \end{array}$$

$$H^{1}(\mathcal{O}_{K};R)^{U} \longrightarrow H^{2}(U;H^{0}(\mathcal{O}_{K};R)) \longrightarrow H^{2}(\mathcal{O}_{K} \rtimes U;R). \end{array}$$

From (2.4), we get $H^0(\mathcal{O}_K; R) \simeq H^0(X^{\mathrm{ab}}; \mathcal{O}_{X^{\mathrm{ab}}}) = \mathbb{C}$, see [11, Lemma 2.4], whence $H^1(U; H^0(\mathcal{O}_K; R)) = \mathbb{C}^{\mathrm{rk}(U)} = \mathbb{C}^s$. Applying again (2.4), the cohomology five-term exact sequence becomes

$$(2.5) \qquad 0 \longrightarrow \mathbb{C}^{s} \longrightarrow H^{1}(X; \mathcal{O}_{X}) \longrightarrow H^{1}(X^{\mathrm{ab}}; \mathcal{O}_{X^{\mathrm{ab}}})^{U}$$
$$H^{2}(U; \mathbb{C}_{U}) \xrightarrow{\checkmark} H^{2}(X; \mathcal{O}_{X}).$$

Therefore, the statement will follow by proving that

(H1') $H^1(X^{\rm ab}; \mathcal{O}_{X^{\rm ab}})^U = 0.$

This is a consequence of the more general result in [14, Theorem 3.1], and we give here below an argument.

More precisely, we first claim that any class in $[\alpha] \in H^1(X^{\mathrm{ab}}; \mathcal{O}_{X^{\mathrm{ab}}})$ has a unique flat representative, $[\alpha] \ni \sum_j c_j d\bar{z}_j$, where $c_j \in \mathbb{C}$ are constant, and (z_1, \ldots, z_t) denote the coordinates in \mathbb{C}^t . Since $U \ni u$ acts on $[\alpha]$ by (1.2), namely, $R_u^*[\alpha] = [\sum_j c_j \cdot \bar{\sigma}_j(u) d\bar{z}_j] \neq [\alpha]$ unless $[\alpha] = 0$, then it follows that $H^1(X^{\mathrm{ab}}; \mathcal{O}_{X^{\mathrm{ab}}})^U = 0$.

We prove now the claim. As suggested in [14, Proof of Lemma 3.2 at p. 5], we look at X^{ab} as a holomorphic fibre bundle over a complex torus $B = \mathbb{C}^t/T$ with fibres F being logarithmically convex Reinhardt domains in $(\mathbb{C}^*)^s$, whence Stein. Recall that the matrix T of periods of B is obtained by putting the matrix P of $\Lambda = \{(\sigma_1(a), \ldots, \sigma_{s+t}(a)) | a \in \mathcal{O}_K\}$ into a more convenient form, see [1, p. 4]. We consider the Borel–Serre spectral sequence for the Dolbeault cohomology of the holomorphic fibre bundle

 $F \hookrightarrow X^{\mathrm{ab}} \twoheadrightarrow B$ with Stein fibres [9]:

$${}^{p,q}E_2^{s,p+q-s} \simeq \bigoplus_{\ell} H^{\ell,s-\ell}_{\overline{\partial}}(B; H^{p-\ell,q-s+\ell}_{\overline{\partial}}(F)) \Rightarrow H^{p,q}_{\overline{\partial}}(X^{\mathrm{ab}}).$$

Since

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$$0 = {}^{0,0}E_2^{-1,1} \xrightarrow{d_2} {}^{0,1}E_2^{1,0} \xrightarrow{d_2} {}^{0,2}E_2^{3,-1} = 0,$$

we compute

$$^{0,1}E^{1,0}_{\infty} = {}^{0,1}E^{1,0}_2 \simeq H^{0,1}_{\overline{\partial}}(B;\mathcal{O}_F);$$

moreover,

$${}^{0,1}E_2^{0,1} \simeq H^{0,0}_{\overline{\partial}}(B; H^{0,1}_{\overline{\partial}}(F)) = 0.$$

Therefore, we can compute

$$H^{0,1}_{\overline{\partial}}(X^{\mathrm{ab}}) \simeq \bigoplus_{s+t=1}^{0,1} E^{s,t}_{\infty} = {}^{0,1}E^{1,0}_2 \simeq H^{0,1}_{\overline{\partial}}(B;\mathcal{O}_F).$$

From now on, for simplicity of calculations, we will denote the elements of \mathbb{H}^s by $w = (w_1, \ldots, w_s)$ and respectively those in \mathbb{C}^t by $z = (z_1, \ldots, z_t)$. We then notice that a class in $H^{0,1}_{\overline{\partial}}(X^{ab})$ is represented by

$$\alpha = \sum_{j} f_j(w, z) \mathrm{d}\bar{z}_j$$

where $f_j(w, z)$ are smooth functions in $(w, z) \in \mathbb{H}^s \times \mathbb{C}^t$, periodic with respect to T (as defined in (1.1)), and such that $\overline{\partial}\alpha = 0$; that is, f_j are holomorphic in w and satisfy $\frac{\partial f_j}{\partial \overline{z}_k} = \frac{\partial f_k}{\partial \overline{z}_j}$ for any $j \neq k$. We claim that we can find $c_j(\operatorname{Im} w) \in \mathcal{C}^{\infty}(\operatorname{Im} \mathbb{H}^s)$ and $g(w, z) \in \mathcal{C}^{\infty}(\mathbb{H}^s \times \mathbb{C}^t)$, holomorphic in w and periodic with respect to T, such that

$$\frac{\partial g}{\partial \bar{z}_j}(w,z) + c_j(\operatorname{Im} w) = f_j(w,z).$$

Since $c_j(\operatorname{Im} w)$ are holomorphic and periodic with respect to T, they are constant, and we can take $c_j = f_j(\operatorname{Im} w, 0)$. We are then reduced to find g such that

$$\frac{\partial g}{\partial \bar{z}_j} = f_j - c_j.$$

We name coordinates $(v, a) := (\operatorname{Im} w, (\operatorname{Re} w, \operatorname{Re} z, \operatorname{Im} z)) \in \mathbb{R}^s \times \mathbb{R}^{s+2t}$. By using Fourier expansion, we can write

$$f_j(v,a) - c_j = \sum_{L \in \mathbb{Z}^{s+2t} \setminus 0} f_{j,L}(v) \exp\left(2\pi\sqrt{-1}\langle AL|a \rangle\right)$$

where A is the matrix whose columns are the coefficients of the lattice \mathcal{O}_K with respect to the standard basis of \mathbb{R}^{s+2t} , so, A has algebraic coefficients. Here $L \neq 0$ because of $f_j(\operatorname{Im} w, 0) - c_j = 0$.

The condition $\frac{\partial f_j}{\partial \bar{z}_k} - \frac{\partial f_k}{\partial \bar{z}_j} = 0$ rewrites as: for any L, for any $j \neq k$,

(2.6)
$$f_{j,L} \cdot \left((AL)_{s+k} + \sqrt{-1} (AL)_{s+t+k} \right) = f_{k,L} \cdot \left((AL)_{s+j} + \sqrt{-1} (AL)_{s+t+j} \right).$$

Analogously, we expand

$$g(v,a) = \sum_{L \in \mathbb{Z}^{s+2t}} g_L(v) \exp\left(2\pi\sqrt{-1}\langle AL|a\rangle\right).$$

Denoting z_j by $x_j + \sqrt{-1}y_j$, we compute

$$\begin{aligned} \frac{\partial g}{\partial \bar{z}_j} &= \frac{1}{2} \left(\frac{\partial g}{\partial x_j} + \sqrt{-1} \frac{\partial g}{\partial y_j} \right) = \frac{1}{2} \left(\frac{\partial g}{\partial a_{s+j}} + \sqrt{-1} \frac{\partial g}{\partial a_{s+t+j}} \right) \\ &= \pi \sqrt{-1} \cdot \sum_{L \in \mathbb{Z}^{s+2t}} g_L(v) \exp\left(2\pi \sqrt{-1} \langle A \cdot L | a \rangle\right) \\ &\quad \cdot \left((AL)_{s+j} + \sqrt{-1} (AL)_{s+t+j} \right) \end{aligned}$$

We notice that, for any L, there is at least one j such that $(AL)_{s+j} + \sqrt{-1}(AL)_{s+t+j} \neq 0$, since the columns of A are linearly independent over \mathbb{Q} . Therefore we can set $g_0 = 0$ (up to an additive constant) and, for $L \in \mathbb{Z}^{s+2t} \setminus 0$,

$$g_L := \frac{1}{\pi \sqrt{-1}} \left((AL)_{s+j} + \sqrt{-1} (AL)_{s+t+j} \right)^{-1} f_{j,L},$$

and there is no ambiguity in the choice of such a j because of (2.6).

It remains to prove that the formal solution $g = \sum g_L \exp(2\pi \sqrt{-1} \langle AL|a \rangle)$ is actually smooth: that is, that the Fourier coefficients g_L decay faster than any power $||L||^{-N}$ for N > 0, as $||L|| \to \infty$.

We make use of the following application of the Subspace Theorem [16], generalizing the Roth Theorem for n = 1:

THEOREM 2.3 (Schmidt [16], see e.g. [3, Thm. 7.3.2]). — Let $\alpha_0, \ldots, \alpha_n$ be algebraic numbers. Then, for every $\varepsilon > 0$, the inequality

$$0 < |\alpha_0 q_0 + \dots + \alpha_n q_n| < (\max\{|q_0|, \dots, |q_n|\})^{-n-\varepsilon}$$

has only finitely-many solution $(q_0, \ldots, q_n) \in \mathbb{Z}^{n+1}$.

We recall that A_k^h , for $k \in \{s+1, \ldots, s+2t\}$ and $h \in \{1, \ldots, s+2t\}$, are algebraic numbers. Moreover, once fixed $L \in \mathbb{Z}^{s+2t} \setminus 0$, there exists $j \in \{1, \ldots, t\}$ such that $(AL)_{s+j} + \sqrt{-1}(AL)_{s+t+j} \neq 0$, namely, there exists $k \in \{1, \ldots, 2t\}$ such that $(AL)_{s+k} \neq 0$, and we can take any such

j's for defining g_L thanks to the compatibility condition (2.6). Then, for $L \in \mathbb{Z}^{s+2t} \setminus 0$:

$$\begin{aligned} |g_L| &= \frac{1}{\pi} \left| (AL)_{s+j} + \sqrt{-1} (AL)_{s+t+j} \right|^{-1} |f_{j,L}| \\ &\leqslant \frac{1}{\pi} \frac{1}{|(AL)_{s+k}|} |f_{j,L}| = \frac{1}{\pi} \frac{1}{|A_{s+k}^1 L_1 + \dots + A_{s+k}^{s+2t} L_{s+2t}|} |f_{j,L}| \\ &< \frac{c}{\pi} \max\{|L_1|, \dots, |L_{s+2t}|\}^{n+\varepsilon} |f_{j,L}| \\ &\leqslant \frac{c}{\pi} \|L\|^{n+\varepsilon} |f_{j,L}|, \end{aligned}$$

where c is a positive constant depending just on A and on the fixed $\varepsilon > 0$, and independent of L and of the chosen j and k. We have then proven that the Fourier coefficients g_L decay as

$$(2.7) |g_L| < \frac{c}{\pi} \cdot \|L\|^{n+\varepsilon} \cdot |f_{j,L}|.$$

Since the form α is smooth, the fast decay is satisfied by the Fourier coefficients $f_{j,L}$ of the f_j 's, and together with formula (2.7) this implies the fast decay also for the Fourier coefficients g_L of g. Whence, g is a smooth solution.

This finally proves that any class in $H^1(X^{ab}; \mathcal{O}_{X^{ab}})$ has a unique flat representative, and by the above argument we get that $H^1(X^{ab}; \mathcal{O}_{X^{ab}})^U = 0$. \Box

Proof of Claim (H2). — First of all, we argue as we did for diagram (2.5), the only difference being that this time we forget the holomorphic structure. Namely, we use $\mathcal{F} = \mathbb{C}_X$ instead of $\mathcal{F} = \mathcal{O}_X$. Everything works the same way, thanks to $H^j(\widetilde{X}; \mathbb{C}_{\widetilde{X}}) = 0$ for any $j \ge 1$. Denoting by $S := H^0(\widetilde{X}; \mathbb{C}_{\widetilde{X}})$, the Lyndon–Hochschild–Serre spectral sequence reads

$$E_2^{p,q} = H^p(U; H^q(\mathcal{O}_K; S)) \Rightarrow H^{p+q}(\pi_1(X); S) ,$$

and the associated cohomology five-term exact sequence yields

$$0 \longrightarrow \mathbb{C}^{s} \longrightarrow H^{1}(X; \mathbb{C}_{X}) \longrightarrow H^{1}(X^{\mathrm{ab}}; \mathbb{C}_{X^{\mathrm{ab}}})^{U}$$
$$H^{2}(U; \mathbb{C}_{U}) \xrightarrow{\longleftarrow} H^{2}(X; \mathbb{C}_{X}) .$$

The map $\mathbb{C}_{\tilde{X}} \to \mathcal{O}_{\tilde{X}}$ induces a map $S \to R$, and hence a morphism of exact sequences

Here, we used that: by Claim (H1), we have that the map $H^1(X; \mathcal{O}_X) \to$ $H^1(X^{ab}; \mathcal{O}_{X^{ab}})^U$ is the zero map; by [11, Proposition 2.3], we have $b_1 = s$, so the map $H^1(X; \mathbb{C}_X) \to H^1(X^{\mathrm{ab}}; \mathbb{C}_{X^{\mathrm{ab}}})^U$ is the zero map, too; again by Claim (H1'), the map $H^1(X^{\mathrm{ab}}; \mathbb{C}_{X^{\mathrm{ab}}})^U \to H^1(X^{\mathrm{ab}}; \mathcal{O}_{X^{\mathrm{ab}}})^U$ is surjective. Finally, the map $H^2(U; \mathbb{C}_U) \to H^2(X; \mathbb{C}_X)$ is surjective: indeed, we claim that the map $H^2(U; \mathbb{C}_U) \to E_{\infty}^{2,0}$ is surjective and $E_2^{0,2} = 0 = E_2^{1,1}$. This follows by [11, p. 166–167] in the case when X is of simple type. In fact, thanks to [7, Theorem 3.1], we just need that there are no embeddings σ_i and σ_k , for $j,k \in \{1,\ldots,s+2t\}$ with $j \neq k$, such that $\sigma_i(u)\sigma_k(u) =$ 1 for any $u \in U$; Oeljeklaus–Toma manifolds of simple type satisfy this condition, see [7, p. 16]. We claim that this latter property always holds true, even when X is not of simple type. Indeed, consider $K' := \mathbb{Q}[U]$. Since U is still admissible for defining an Oeljeklaus–Toma manifold X' :=X'(K', U), see [11, Lemma 1.6], and X' is of simple type, therefore there are no embeddings σ_j and σ_k of K', with $j \neq k$, such that $\sigma_j(u)\sigma_k(u) = 1$ for any $u \in U$. Moreover, there is no embedding σ_j of K' such that $\sigma_j^2(u) = 1$ for any $u \in U$. Since the embeddings of K' are just the restrictions of the embeddings of K, we get the claim.

At the end, we get that q is injective by diagram chasing.

This completes the proof of Theorem 2.1 by proving that any line bundle on an Oeljeklaus–Toma manifold is flat. \Box

We note that the argument in the last lines of the previous proof shows that $\rho_2 = 0$ in the notation of [7, Theorem 3.1], therefore, thanks to Istrati and Otiman's result, we get the following, generalizing [11, Proposition 2.3]:

PROPOSITION 2.4. — Any Oeljeklaus–Toma manifold has $b_2 = {s \choose 2}$, where s is the number of real embeddings.

A well-known result by Ornea and Verbitsky [12] for t = 1 and, for any s, t, by Battisti and Oeljeklaus [2], states that Oeljeklaus–Toma manifolds have no divisors. In the following proposition, we show that this result follows from Theorem 2.1.

PROPOSITION 2.5 (Battisti and Oeljeklaus [2, Theorem 3.5]). — Let X = X(K, U) be an Oeljeklaus–Toma manifold. Then X has no divisors.

Proof. — Take any line bundle on X, which is then flat, and let ρ be the associated representation. But any representation $\rho: \pi_1(X) \to \mathbb{C}^*$ induces the identity on \mathcal{O}_K [4, Proposition 6]. Therefore the pull-back of L_ρ to X^{ab} is trivial, and its sections are constants. Therefore L_ρ has no non-trivial sections on X.

3. Rigidity of Oeljeklaus–Toma manifolds

In this section we extensively apply techniques similar to the ones used in Section 2, to prove the following vanishing result.

THEOREM 3.1. — Let X = X(K, U) be an Oeljeklaus–Toma manifold. Take any faithful representation $\rho: U \to \mathbb{C}^*$, and let L_{ρ} be its associated flat holomorphic line bundle on X. Then $H^1(X; L_{\rho}) = 0$ unless $\rho = \bar{\sigma}_i^{-1}$ for some $i\{t + 1, \ldots t + s\}$.

Proof. — We use group cohomology, with the action of $U \ni u$ on $R = H^0(\widetilde{X}; \mathcal{O}_{\widetilde{X}})$ given by

$$L_u(f) := \rho(u)f \circ R_u,$$

where R_u is the rotation given by equation (1.2), see Section 2. Since the pull-back of L_{ρ} to \widetilde{X} is trivial, we get

$$H^1(\mathcal{O}_K \rtimes U; R) \simeq H^1(X; L_\rho).$$

From the Lyndon–Hochschild–Serre spectral sequence and the cohomology five-term exact sequence we obtain, as in diagram (2.5), the exact sequence

$$H^1(U; H^0(\mathcal{O}_K; R)) \longrightarrow H^1(X; L_\rho) \longrightarrow H^1(\mathcal{O}_K; R)^U$$
.

On the one side, $H^1(U; H^0(\mathcal{O}_K; R)) = 0$ since ρ is faithful and U is free Abelian (this follows easily from the fact that, for a free cyclic group U, one has $H^0(U; \mathbb{C}) = H^1(U; \mathbb{C}) = 0$ for any non-trivial representation ρ , and then performing induction on the rank of U using again the Lyndon–Hochschild–Serre spectral sequence). On the other side, we have $H^1(\mathcal{O}_K; R)^U = H^1(X^{ab}; \mathcal{O}_{X^{ab}})^U$. But for any $u \in U$ we have

$$L_u^*(\mathrm{d}\bar{z}_i) = \rho(u)\overline{\sigma}_i(u)\mathrm{d}\bar{z}_i,$$

hence the conclusion.

Remark 3.2. — Another possible argument for Theorem 3.1 may be found on elliptic Hodge theory, as suggested in [17]. We just notice that, if ϑ is the closed 1-form determined by ρ as $\rho(\gamma) = \exp \int_{\gamma} \vartheta$, then the (de Rham) cohomology of X with values in the complex line bundle L_{ρ} corresponds to the cohomology of the trivial bundle $X \times \mathbb{C}$ with respect to the flat connection $d_{\vartheta} := d + \vartheta \wedge _$. We split $d_{\vartheta} = \overline{\partial}_{\vartheta} + \partial_{\vartheta}$ where $\overline{\partial}_{\vartheta} := \overline{\partial} + \vartheta^{0,1} \wedge _$. Here, $\vartheta^{0,1}$ is the (0,1)-component of ϑ . The (Dolbeault) cohomology of X with values in the holomorphic line bundle L_{ρ} corresponds to the cohomology of the trivial bundle with respect to the flat connection $\overline{\partial}_{\vartheta}$. Elliptic Hodge theory applies with the operator $[\overline{\partial}_{\vartheta}, \overline{\partial}_{\vartheta}^*]$. Note indeed that the operator is elliptic, since the second-order part of it is equal to the second-order part of $[\overline{\partial}, \overline{\partial}^*]$. We claim that the zeroth-order part of $[\overline{\partial}_{\vartheta}, \overline{\partial}^*_{\vartheta}]$ is positive (with respect to the L^2 -pairing). Indeed, note that $\overline{\partial}^*_{\vartheta} = -*\overline{\partial}_{-\vartheta}*$. Therefore the zeroth-order term is given by $\vartheta^{0,1} \wedge *(\vartheta^{0,1} \wedge *_) + *(\vartheta^{0,1} \wedge *(\vartheta^{0,1} \wedge _))$. Note that, on 1-forms γ , it holds $\langle \vartheta^{0,1} \wedge *(\vartheta^{0,1} \wedge *\gamma) | \gamma \rangle = \| \vartheta^{0,1} \wedge *\gamma \|^2 \ge 0$, and, similarly, $\langle *(\vartheta^{0,1} \wedge *(\vartheta^{0,1} \wedge \gamma)) | \gamma \rangle = \left\| \vartheta^{0,1} \wedge \gamma \right\|^2 \ge 0.$

As a corollary, we get rigidity in the sense of the theory of deformations of complex structures of Kodaira–Spencer–Nirenberg–Kuranishi. See [6, Proposition 2] for rigidity in the case s = t = 1 of Inoue–Bombieri surfaces.

COROLLARY 3.3. — Oeljeklaus–Toma manifolds of simple type are rigid.

Proof. — Note that $\Theta_{\mathbb{H}^s \times \mathbb{C}^t} = \langle \frac{\partial}{\partial w^1}, \dots, \frac{\partial}{\partial w^s}, \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^t} \rangle$, and $\mathcal{O}_K \rtimes U \ni (a, u)$ acts on $\frac{\partial}{\partial w^h}$, respectively $\frac{\partial}{\partial z^k}$, as multiplication by $\sigma_h(u)$, respectively $\sigma_{s+k}(u)$. Whence the holomorphic tangent bundle of an Oeljeklaus–Toma manifold splits as

$$\Theta_X = \bigoplus_{j=1}^{s+t} L_{\sigma_j^{-1}},$$

where L_{σ_j} are the line bundle associated to the embeddings σ_j . By Theorem 3.1, we get $H^1(X; \Theta_X) = 0$, unless $\sigma_i^{-1}(u) = \overline{\sigma}_j^{-1}(u)$ for some $i \in \{1, \ldots s + t\}, j \in \{s + 1, \ldots, s + t\}$ and any $u \in U$. But this would imply that for all $u \in U$ we have $\sigma_i(u) = \sigma_{j+t}(u)$ hence all $u \in U$ live in a proper subfield of K, absurd since we assumed X to be of simple type. This proves the claim.

Remark 3.4. — For the case t = 1, a stronger result was obtained by O. Braunling. He proves in [4, Proposition 1] that, if two Oeljeklaus–Toma manifolds $X' = X(K'; \mathcal{O}_{K'}^{*,+})$ and $X'' = X(K''; \mathcal{O}_{K''}^{*,+})$, both having t = 1, are homotopy equivalent, then they are isomorphic.

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