Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

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Tome 70, nº 5 (2020), p. 1847-1869.
<http://aif.centre-mersenne.org/item/AIF_2020__70_5_1847_0>

Ann. Inst. Fourier, Grenoble
70, 5 (2020) 1847-1869

SCATTERING FOR NLS WITH A SUM OF TWO REPULSIVE POTENTIALS

by David LAFONTAINE

Abstract. — We prove the scattering for a defocusing nonlinear Schrödinger equation with a sum of two repulsive potentials with strictly convex level surfaces, thus providing a scattering result in a trapped setting similar to the exterior of two strictly convex obstacles.

Résumé. — Nous montrons la diffusion pour une équation de Schrödinger non linéaire défocalisante avec une somme de deux potentiels répulsifs dont les surfaces de niveau sont strictement convexes. Il s'agit d'un résultat dans une géométrie captante similaire à l'extérieur de deux obstacles strictement convexes.

1. Introduction

We are concerned by the following defocusing non-linear Schrödinger equation with a potential

$$i\partial_t u + \Delta u - Vu = u|u|^\alpha,\ u(0) = \varphi \in H^1. \tag{1.1}$$

in arbitray spatial dimension $d \geqslant 1$. Once good dispersive properties of the linear flow, such as Strichartz estimates described below in the paper, are established, the local well-posedness of (1.1) follows by usual fixed point arguments. Because of the energy conservation law,

$$E(u(t)) := \frac{1}{2}\int |\nabla u(t)|^2 + \int V|u(t)|^2 + \frac{1}{\alpha+2}\int |u(t)|^{\alpha+2} = E(u(0))$$

this result extends to global well-posedness. Thus, it is natural to investigate the asymptotic behavior of solutions of (1.1).

Keywords: nonlinear Schrödinger equation, scattering, trapped trajectories, Morawetz estimates, concentration-compactness/rigidity.
2020 *Mathematics Subject Classification:* 35Q55, 35B40.

It is well-known since Nakanishi's paper [14] that for $V = 0$, in the intercritical regime

$$\frac{4}{d} < \alpha < \begin{cases} +\infty & d = 1, 2, \\ \frac{4}{d-2} & d \geqslant 3, \end{cases} \tag{1.2}$$

the solutions *scatter* in $H^1(\mathbb{R}^d)$, that is, for every solution $u \in C(\mathbb{R}, H^1(\mathbb{R}^d))$ of (1.1), there exists a unique couple of data $\psi_\pm \in H^1(\mathbb{R}^d)$ such that

$$\|u(t) - \mathrm{e}^{-\mathrm{i}\, t\Delta}\, \psi_\pm\|_{H^1(\mathbb{R}^d)} \underset{t\to\pm\infty}{\longrightarrow} 0.$$

The inhomogeneous setting $V \neq 0$ was investigated more recently, for example in [1, 4, 7, 11]. However, all these scattering results rely on a non-trapping assumption, namely, that the potential is *repulsive:*

$$x \cdot \nabla V \leqslant 0,$$

or, as in [2], that its non-repulsive part is sufficently small. The aim of this paper is to establish a scattering result in a trapping situation. More precisely, we are interested in one of the simplest unstable trapping framework, that is, the case where V is the sum of two positive, repulsive potentials with strictly convex level surfaces. It is the potential-analog of the homogeneous problem outside two strictly convex obstacles, and this note can be seen as a proxy for the scattering outside two strictly convex obstacles. This more intricate problem, where reflexions at the boundary have to be dealt with, will be treated in [13] using ideas developped here. Note that the case of the exterior of a star-shaped obstacle was treated by [8, 15, 16].

Let us precise our setting. Let V_1 and V_2 be two positive, smooth potentials. We will denote by $V = V_1 + V_2$ the total potential. We make the following geometrical assumptions:

- (G1) V_1 and V_2 are repulsive, that is, there exists a_1 and a_2 in $\mathbb{R}^d$ such that
$$(x - a_{1,2}) \cdot \nabla V_{1,2} \leqslant 0.$$
Without loss of generality, we assume that $0 \in [a_1, a_2]$.
- (G2) The level surfaces of V_1 and V_2 are convex, and uniformly strictly convex in the non-repulsive region: the eigenvalues of their second fundamental forms are uniformly bounded below by a strictly positive universal constant in $\{x \cdot \nabla V > 0\}$.
- (G3) All the trapped trajectories of the Hamiltonian flow associated with $-\Delta + V$ belong to a same line $\mathcal{R} \subset \{x \cdot \nabla V > 0\}$: for any pair Θ_1, Θ_2 of level surfaces of V_1 and V_2, the unique trapped ray of the geometrical optics of $\mathbb{R}^d \backslash \Theta_1 \cup \Theta_2$ is included in $\mathcal{R}$.

A non-trivial example of potential $V = V_1 + V_2$ verifying (G1)-(G2)-(G3) is given by $V_1(x) := \mathrm{e}^{-|x-a|^2}$, $V_2(x) := \mathrm{e}^{-|x+a|^2}$, the uniformity in the convexity assumption coming from the fact that this potential has a bounded non-repulsive region.

We will, in addition, assume the following decay assumption

$$V \in L^{\frac{d}{2}}((1+|x|^\beta)\mathrm{d}x);\ \nabla V_1, \nabla V_2 \in L^{\frac{d}{2}};\ \nabla V \in L^{\frac{d}{2}}(|x|^\beta \mathrm{d}x), \tag{1.3}$$

with $\beta > \frac{4}{3}$. It is the (improved) multi-dimensional analog of the decay assumption arising in [11]. And finally, that the pointwise dispersive estimate

$$\| \mathrm{e}^{\mathrm{i}\, t(-\Delta+V)} \|_{L^1 \longrightarrow L^\infty} \lesssim \frac{1}{|t|^{d/2}} \tag{1.4}$$

holds. Note that, in the same way as remarked in [11] for the one dimensional case, this last assumption is automaticaly verified using Goldberg and Schlag's result [6] under the non-negativity and decay assumptions with $\beta \geqslant 2$ in dimension $d = 3$. Our main result reads

Theorem 1.1. — *Assume that $d \geqslant 3$. Let V_1 and V_2 be two positive, repulsive* (G1) *smooth potentials, with convex and uniformly strictly convex in the non-repulsive region level surfaces* (G2), *and colinear trapped trajectories* (G3). *Assume moreover that $V = V_1 + V_2$ verifies the decay assumption* (1.3), *and the dispersive estimate* (1.4). *Then, in the intercritical regime* (1.2), *every solutions of* (1.1) *with potential $V = V_1 + V_2$ scatter in $H^1(\mathbb{R}^d)$.*

As in the aforementioned papers, we use the strategy of concentration, compactness and rigidity first introduced by Kenig and Merle in [10]: assuming that there exists a finite energy above which solutions do not scatter, one constructs a compact-flow solution and eliminates it. Notice that in the case of a repulsive potential, this last rigidity part is immediate by classical Morawetz estimates. It will be here the main difficulty to overcome and the novelty of this note. After some preliminaries, we construct a critical solution in the second section, following [11] and generalizing it to any spatial dimension. In the last section, we eliminate it using a family of Morawetz mulipliers for which the gradient almost vanishes on the trapped trajectory.

Remark 1.2. — We assume that $d \neq 2$ because our proof relies on endpoint Strichartz estimates that are not true in dimension two, and the convexity assumption we make on the potentials have no sense in the one dimensional case.

Remark 1.3. — The first two sections of this paper generalize in particular the one-dimensional result of [11], to any spatial dimension $d \geqslant 3$.

Remark 1.4. — As mentionned earlier, the geometrical framework (G1)-(G2)-(G3) is in many aspects the potential-analog of the homogeneous problem outside two strictly convex obstacles. This is the subject of a work in progress [13]. A rigidity argument in the particular case of two balls for the energy critical wave equation can be found in [12].

Remark 1.5. — It is straightforward from the last section that the result is still valid for an arbitrary finite sum of convex repulsive potentials $V = V_1 + \cdots + V_N$ for which all trapped trajectories are included in the same line. However, we present the proof for only two potentials in the seek of simplicity.

2. Preliminaries

2.1. Usefull exponents

From now on, we will fix the three following Strichartz exponents

$$r = \alpha + 2, \; q = \frac{2\alpha(\alpha+2)}{d\alpha^2 - (d-2)\alpha - 4}, \; p = \frac{2(\alpha+2)}{4-(d-2)\alpha}.$$

Moreover, let η be the conjugate of the critical exponent $2^\star$:

$$\frac{1}{2^\star} + \frac{1}{\eta} = 1. \tag{2.1}$$

Notice, for the sequel, the following two identities

$$\frac{2}{d} + \frac{1}{2^\star} = \frac{1}{\eta}, \tag{2.2}$$

and

$$\frac{2}{d} + \frac{2}{2^\star} = 1. \tag{2.3}$$

Finally, let γ be such that (γ, η') follows the admissibility condition of Theorem 1.4 of Foshi's inhomogeneous Strichartz estimates [5]. Note that, in the intercritical regime (1.2), all these exponents are well defined and larger than one.

2.2. Strichartz estimates

Let us recall that $\mathrm{e}^{-\mathrm{i}\,t(-\Delta+V)}$ verifies the pointwise dispersive estimates (1.4), by [6] in dimension $d=3$ for $\beta \geqslant 2$, or by assumption in other cases. Interpolating it with the mass conservation law, we obtain immediatly for all $a \in [2,\infty]$

$$\|\,\mathrm{e}^{\mathrm{i}\,t(-\Delta+V)}\,\psi\|_{L^a} \lesssim \frac{1}{|t|^{\frac{d}{2}(\frac{1}{a'}-\frac{1}{a})}}\|\psi\|_{L^{a'}}. \tag{2.4}$$

Moreover, it leads by the classical $TT^\star$ method (see for example [9]) to the Strichartz estimates

$$\|\,\mathrm{e}^{-\mathrm{i}\,t(-\Delta+V)}\,\varphi\|_{L^{q_1}L^{r_1}} + \left\|\int_0^t \exp^{-\mathrm{i}(t-s)(-\Delta+V)}\,F(s)\mathrm{d}s\right\|_{L^{q_2}L^{r_2}} \lesssim \|\varphi\|_{L^2} + \|F\|_{L^{q_3'}L^{r_3'}} \tag{2.5}$$

for all pairs (q_i,r_i) satisfying the admissibility condition

$$\frac{2}{q_i}+\frac{d}{r_i}=\frac{d}{2},\ (q_i,r_i,d)\neq(2,\infty,2).$$

We will use moreover the following Strichartz estimates associated to non admissible pairs:

Proposition 2.1 (Strichartz estimates). — *For all $\varphi \in H^1$, all $F \in L^{q'}L^{r'}$, all $G \in L^{q'}L^{r'}$ and all $H \in L^{\gamma'}L^{\eta}$*

$$\|\,\mathrm{e}^{-\mathrm{i}\,t(-\Delta+V)}\,\varphi\|_{L^pL^r} \lesssim \|\varphi\|_{H^1} \tag{2.6}$$

$$\left\|\int_0^t \mathrm{e}^{-\mathrm{i}(t-s)(-\Delta+V)}\,F(s)\mathrm{d}s\right\|_{L^\alpha L^\infty} \lesssim \|F\|_{L^{q'}L^{r'}} \tag{2.7}$$

$$\left\|\int_0^t \mathrm{e}^{-\mathrm{i}(t-s)(-\Delta+V)}\,G(s) \gg s\right\|_{L^pL^r} \lesssim \|G\|_{L^{q'}L^{r'}} \tag{2.8}$$

$$\left\|\int_0^t \mathrm{e}^{-\mathrm{i}(t-s)(-\Delta+V)}\,H(s)\mathrm{d}s\right\|_{L^pL^r} \lesssim \|H\|_{L^{\gamma'}L^{\eta}}. \tag{2.9}$$

Proof. — The estimate (2.6) follows from admissible Strichartz estimate

$$\|\,\mathrm{e}^{-\mathrm{i}\,t(-\Delta+V)}\,\varphi\|_{L^pL^{\frac{2dp}{dp-4}}} \lesssim \|\varphi\|_{L^2}$$

together with a Sobolev embedding. The estimate (2.8) is contained in Lemma 2.1 of [3]. Finally, (2.7) and (2.9) enters on the frame of the non-admissible inhomogheneous Strichartz estimates of Theorem 1.4 of Foschi's paper [5]. □

2.3. Perturbative results

The three following classical perturbative results, follow immediatly from the previous Strichartz inequalities with exact same proof as in [11].

Proposition 2.2. — *Let $u \in C(H^1)$ be a solution of* (1.1)*. If $u \in L^pL^r$, then u scatters in H^1.*

Proposition 2.3. — *There exists $\epsilon_0 > 0$, such that, for every data $\varphi \in H^1$ such that $\|\varphi\|_{H^1} \leqslant \epsilon_0$, the corresponding maximal solution of* (1.1) *scatters in H^1.*

Proposition 2.4. — *For every $M > 0$ there exists $\epsilon > 0$ and $C > 0$ such that the following occurs. Let $v \in C(H^1) \cap L^pL^r$ be a solution of the following integral equation with source term $e(t,x)$*

$$v(t) = \mathrm{e}^{-\mathrm{i}\,t(\Delta-V)}\,\varphi - \mathrm{i}\int_0^t \mathrm{e}^{-\mathrm{i}(t-s)(\Delta-V)}(v(s)|v(s)|^\alpha)\mathrm{d}s + e(t)$$

with $\|v\|_{L^pL^r} < M$ and $\|e\|_{L^pL^r} < \epsilon$. Assume moreover that $\varphi_0 \in H^1$ is such that $\|\,\mathrm{e}^{-\mathrm{i}\,t(\Delta-V)}\,\varphi_0\|_{L^pL^r} < \epsilon$. Then, the solution $u \in C(H^1)$ to (1.1) *with initial condition $\varphi + \varphi_0$ satisfies*

$$u \in L^pL^r, \quad \|u - v\|_{L^pL^r} < C.$$

3. Construction of a critical solution

The aim of this section is to extend the construction of a critical element of [11] to any dimension $d \neq 2$ – no repulsivity assumption is used in this first part of this work. This previous paper follows itself [1] which deals with a Dirac potential, which is more singular but for which explicit formulas are at hand. More precisely, let

$$E_c = \sup\left\{E>0 \;\middle|\; \begin{array}{l}\forall\,\varphi \in H^1,\ E(\varphi) < E\\ \Rightarrow \text{the solution of (1.1) with data } \varphi \text{ is in } L^pL^r\end{array}\right\}. \tag{3.1}$$

We will prove

Theorem 3.1. — *If $E_c < \infty$, then there exists $\varphi_c \in H^1$, $\varphi_c \neq 0$, such that the corresponding solution u_c of* (1.1) *has a relatively compact flow $\{u_c(t),\ t \geqslant 0\}$ in H^1 and does not scatter.*

We assume all along this section that $d \geqslant 3$.

3.1. Profile decomposition

We first show, with the same method as in [11], extended to any dimension, that we can use the abstract profile decomposition obtained by [1]:

THEOREM (Abstract profile decomposition, [1]). — *Let $A : L^2 \supset D(A) \to L^2$ be a self adjoint operator such that:*

- *for some positive constants c, C and for all $u \in D(A)$,*

$$c\|u\|_{H^1}^2 \leqslant (Au, u) + \|u\|_{L^2}^2 \leqslant C\|u\|_{H^1}^2, \tag{3.2}$$

- *let $B : D(A) \times D(A) \ni (u, v) \to (Au, v) + (u, v)_{L^2} - (u, v)_{H^1} \in \mathbb{C}$. Then, as n goes to infinity*

$$B(\tau_{x_n}\psi, \tau_{x_n}h_n) \to 0 \quad \forall\, \psi \in H^1 \tag{3.3}$$

as soon as

$$x_n \to \pm\infty, \quad \sup \|h_n\|_{H^1} < \infty$$

or

$$x_n \to \bar{x} \in \mathbb{R}, \quad h_n \underset{H^1}{\rightharpoonup} 0,$$

- *let $(t_n)_{n\geqslant 1}$, $(x_n)_{n\geqslant 1}$ be sequences of real numbers, and $\bar{t}, \bar{x} \in \mathbb{R}$. Then*

$$|t_n| \to \infty \Longrightarrow \|\,\mathrm{e}^{\mathrm{i}\,t_n A}\,\tau_{x_n}\psi\|_{L^p} \to 0, \quad \forall\, 2 < p < \infty,\ \forall\, \psi \in H^1 \tag{3.4}$$

$$t_n \to \bar{t},\ x_n \to \pm\infty \Longrightarrow \forall\, \psi \in H^1,\ \exists\, \varphi \in H^1,\ \ \tau_{-x_n}\,\mathrm{e}^{\mathrm{i}\,t_n A}\,\tau_{x_n}\psi \overset{H^1}{\to} \varphi \tag{3.5}$$

$$t_n \to \bar{t},\ x_n \to \bar{x} \Longrightarrow \forall\, \psi \in H^1, \quad \mathrm{e}^{\mathrm{i}\,t_n A}\,\tau_{x_n}\psi \overset{H^1}{\to} \mathrm{e}^{\mathrm{i}\,\bar{t}A}\,\tau_{\bar{x}}\psi. \tag{3.6}$$

And let $(u_n)_{n\geqslant 1}$ be a bounded sequence in H^1. Then, up to a subsequence, the following decomposition holds

$$u_n = \sum_{j=1}^{J} \mathrm{e}^{\mathrm{i}\,t_j^n A}\,\tau_{x_n^j}\psi_j + R_n^J \quad \forall\, J \in \mathbb{N}$$

where

$$t_j^n \in \mathbb{R},\ x_j^n \in \mathbb{R},\ \psi_j \in H^1$$

are such that

- *for any fixed j,*

$$t_j^n = 0\ \ \forall\, n, \quad \text{or} \quad t_n^j \overset{n\to\infty}{\to} \pm\infty \tag{3.7}$$

$$x_j^n = 0\ \ \forall\, n, \quad \text{or} \quad x_n^j \overset{n\to\infty}{\to} \pm\infty, \tag{3.8}$$

- *orthogonality of the parameters:*

$$|t_j^n - t_k^n| + |x_j^n - x_k^n| \overset{n\to\infty}{\to} \infty, \quad \forall\, j \neq k, \tag{3.9}$$

- *decay of the reminder:*

$$\forall\, \epsilon > 0, \exists\, J \in \mathbb{N}, \quad \limsup_{n\to\infty} \| \,\mathrm{e}^{-\,\mathrm{i}\, tA}\, R_n^J\|_{L^\infty L^\infty} \leqslant \epsilon, \tag{3.10}$$

- *orthogonality of the Hilbert norm:*

$$\|u_n\|_{L^2}^2 = \sum_{j=1}^{J} \|\psi_j\|_{L^2}^2 + \|R_n^J\|_{L^2}^2 + o_n(1), \quad \forall\, J \in \mathbb{N} \tag{3.11}$$

$$\|u_n\|_H^2 = \sum_{j=1}^{J} \|\tau_{x_n^j}\psi_j\|_H^2 + \|R_n^J\|_H^2 + o_n(1), \quad \forall\, J \in \mathbb{N} \tag{3.12}$$

where $(u,v)_H = (Au, v)$, *and*

$$\|u_n\|_{L^p}^p = \sum_{j=1}^{J} \| \,\mathrm{e}^{\mathrm{i}\, t_j^n A}\, \tau_{x_n^j}\psi_j\|_{L^p}^p + \|R_n^J\|_{L^p}^p + o_n(1), \qquad \forall\, 2 < p < 2^\star, \quad \forall\, J \in \mathbb{N}. \tag{3.13}$$

Let us show that the self-adjoint operator $A := -\Delta + V$ verifies the hypothesis of the previous theorem.

Proposition 3.2. — *Let $A := -\Delta + V$. Then A satisfies the assumptions* (3.2), (3.3), (3.4), (3.5), (3.6).

Proof. — *Assumption* (3.2). Because V is non-negative, by the Hölder inequality, (2.3), and the Sobolev embedding $H^1 \hookrightarrow L^{2^\star}$,

$$\begin{aligned}\|u\|_{H^1}^2 \leqslant (Au,u) + \|u\|_{L^2} &= \int |\nabla u|^2 + \int V|u|^2 + \int |u|^2 \\ &\leqslant \|u\|_{H^1}^2 + \|V\|_{L^{d/2}}\|u\|_{L^{2^\star}}^2 \leqslant (1 + C_{\mathrm{Sobolev}}\|V\|_{L^{d/2}})\|u\|_{H^1}^2.\end{aligned}$$

and (3.2) holds.

Assumption (3.3). We have

$$B(\tau_{x_n}\psi, \tau_{x_n}h_n) = \int V\tau_{x_n}\psi\overline{\tau_{x_n}h_n}.$$

Assume that $x_n \to \bar{x} \in \mathbb{R}$ and $h_n \underset{H^1}{\rightharpoonup} 0$. Notice that B can also be written

$$B(\tau_{x_n}\psi, \tau_{x_n}h_n) = \int (\tau_{-x_n}V)\psi\overline{h_n}.$$

By Sobolev embedding, $h_n \rightharpoonup 0$ weakly in $L^{2^\star}$. Moreover, $\tau_{-x_n}V \to \tau_{-\bar{x}}V$ strongly in $L^{d/2}$. Therefore, because $\psi \in L^{2^\star}$ by Sobolev embedding again, it follows from (2.3) that $B(\tau_{x_n}\psi, \tau_{x_n}h_n) \to 0$.

Now, let us assume that

$$x_n \to +\infty,\ \sup \|h_n\|_{H^1} < \infty.$$

We fix $\epsilon > 0$. By the Sobolev embedding $H^1 \hookrightarrow L^{2^\star}$, we can choose $\Lambda > 0$ large enough so that

$$\|\psi\|_{L^{2^\star}(|x|\geqslant\Lambda)} \leqslant \epsilon. \tag{3.14}$$

Because $V \in L^{d/2}$, Λ can also be choosen large enough so that

$$\|V\|_{L^{d/2}(|x|\geqslant\Lambda)} \leqslant \epsilon. \tag{3.15}$$

Then, by the Hölder inequality – recall that η is defined in (2.1) as the conjugate of $2^\star$ – by Sobolev embedding and the Minkoswski inequality

$$\begin{aligned}
&|B(\tau_{x_n}\psi, \tau_{x_n}h_n)| \leqslant \|h_n\|_{L^{2^\star}} \|V\tau_{x_n}\|_{L^\eta} \\
&\lesssim \sup_{j\geqslant 1} \|h_j\|_{H^1} \left(\|V\psi(\cdot - x_n)\|_{L^\eta(|x-x_n|\geqslant\Lambda)} + \|V\psi(\cdot - x_n)\|_{L^\eta(|x-x_n|\leqslant\Lambda)}\right).
\end{aligned}$$

Thus, by the Hölder inequality again, using this time (2.2), we have

$$\begin{aligned}
&|B(\tau_{x_n}\psi, \tau_{x_n}h_n)| \\
&\qquad \lesssim \|V\|_{L^{d/2}} \|\psi \mathbf{1}_{|x|\geqslant\Lambda}\|_{L^{2^\star}} + \|V\mathbf{1}_{|x-x_n|\leqslant\Lambda}\|_{L^{d/2}} \|\psi\|_{L^{2^\star}}.
\end{aligned} \tag{3.16}$$

Now, let n_0 be large enough so that for all $n \geqslant n_0$, $x_n \geqslant 2\Lambda$. Then, for all $n \geqslant n_0$

$$|x - x_n| \leqslant \Lambda \Rightarrow |x| \geqslant \Lambda$$

and, for all $n \geqslant n_0$ we get by (3.14), (3.15), (3.16)

$$|B(\tau_{x_n}\psi, \tau_{x_n}h_n)| \lesssim (\epsilon\|V\|_{L^\delta} + \epsilon\|\psi\|_{L^{2^\star}})$$

so (3.3) holds.

Assumption (3.4). The same proof as in [11] holds: it is an immediate consequence of the pointwise dispersive estimate (2.4) and the translation invariance of the L^p norms. Notice that the estimate

$$\|\mathrm{e}^{\mathrm{i}\,tA} f\|_{H^1} \lesssim \|f\|_{H^1}, \tag{3.17}$$

which is usefull to close the density argument of this previous paper, generalizes to dimensions $d \geqslant 2$ because, as V is positive and in $L^{d/2}$, by the Hölder inequality together with the Sobolev embedding $H^1 \hookrightarrow L^{2^\star}$ we get

$$\begin{aligned}
\|\nabla f\|_{L^2}^2 \leqslant \|(-\Delta + V)^{\frac{1}{2}} f\|_{L^2}^2 &= \int |\nabla u|^2 + \int V|u|^2 \\
&\leqslant \|f\|_{H^1}^2 + \|V\|_{L^{d/2}} \|u\|_{L^{2^\star}}^2 \lesssim \|f\|_{H^1}^2,
\end{aligned} \tag{3.18}$$

from which (3.17) follows because $\mathrm{e}^{\mathrm{i}\,tA}$ commute with $(-\Delta + V)^{\frac{1}{2}}$ and is an isometry on L^2.

Assumption (3.5). We will show that

$$t_n \to \bar{t},\ x_n \to +\infty \ \Rightarrow\ \|\tau_{-x_n} \mathrm{e}^{\mathrm{i}\,t_n(-\Delta+V)} \tau_{x_n}\psi - \mathrm{e}^{-\mathrm{i}\,\bar{t}\Delta} \psi\|_{H^1} \to 0$$

and (3.5) will hold with $\varphi = \mathrm{e}^{-\mathrm{i}\,\bar{t}\Delta}\,\psi$. As remarked in [11], it is sufficent to show that

$$\|\,\mathrm{e}^{\mathrm{i}\,t_n(-\Delta+V)}\,\tau_{x_n}\psi - \mathrm{e}^{-\mathrm{i}\,t_n\Delta}\,\tau_{x_n}\psi\|_{H^1} \to 0. \tag{3.19}$$

Notice $\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\tau_{x_n}\psi - \mathrm{e}^{\mathrm{i}\,t(-\Delta+V)}\,\tau_{x_n}\psi$ is a solution of the following linear Schrödinger equation with zero initial data

$$\mathrm{i}\,\partial_t u - \Delta u + Vu = V\,\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\tau_{x_n}\psi.$$

Therefore, by the inhomogenous Strichartz estimates, as $(2, 2^\star)$ is admissible in dimension $d \geqslant 3$ with dual exponent $(2, \eta)$, and because the translation operator commutes with $\mathrm{e}^{-\mathrm{i}\,t\Delta}$, we have for n large enough so that $t_n \in (0, \bar{t}+1)$

$$\begin{aligned}
&\|\,\mathrm{e}^{\mathrm{i}\,t_n(-\Delta+V)}\,\tau_{x_n}\psi - \mathrm{e}^{-\mathrm{i}\,t_n\Delta}\,\tau_{x_n}\psi\|_{L^2} \\
&\qquad\leqslant \|\,\mathrm{e}^{\mathrm{i}\,t(-\Delta+V)}\,\tau_{x_n}\psi - \mathrm{e}^{-\mathrm{i}\,t\Delta}\,\tau_{x_n}\psi\|_{L^\infty(0,\bar{t}+1)L^2} \\
&\qquad\leqslant \|V\,\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\tau_{x_n}\psi\|_{L^2(0,\bar{t}+1)L^\eta} \\
&\qquad= \|(\tau_{-x_n}V)\,\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\psi\|_{L^2(0,\bar{t}+1)L^\eta} \\
&\qquad\leqslant (\bar{t}+1)^{1/2}\|(\tau_{-x_n}V)\,\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\psi\|_{L^\infty(0,\bar{t}+1)L^\eta}.
\end{aligned}$$

Hence, estimating in the same manner the gradient of these quantities, it is sufficient to obtain (3.19) to show that, as n goes to infinity

$$\|(\tau_{-x_n}V)\,\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\psi\|_{L^\infty(0,\bar{t}+1)W^{1,\eta}} \to 0. \tag{3.20}$$

Let us fix $\epsilon > 0$. By Sobolev embedding in $L^{2^\star}$, because $\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\psi \in C([0, \bar{t}+1], H^1)$ and using the compacity in time, there exists $\Lambda > 0$ such that

$$\|\,\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\psi\|_{L^\infty(0,\bar{t}+1)L^{2^\star}(|x|\geqslant\Lambda)} \leqslant \epsilon. \tag{3.21}$$

On the other hand, as $V \in L^{d/2}$, Λ can also be taken large enough so that

$$\|V\|_{L^{d/2}(|x|\geqslant\Lambda)} \leqslant \epsilon. \tag{3.22}$$

Let n_0 be large enough so that for all $n \geqslant n_0$, $x_n \geqslant 2\Lambda$. Then, for $n \geqslant n_0$

$$|x + x_n| \leqslant \Lambda \Rightarrow |x| \geqslant \Lambda$$

and for all $t \in (0, \bar{t}+1)$ and all $n \geqslant n_0$ we obtain, by Minkowski inequality, Hölder inequality together with (3.21) and (3.22), and Sobolev embedding

$$\begin{aligned}
&\|(\tau_{-x_n} V)\, \mathrm{e}^{-\mathrm{i}\, t\Delta}\, \psi\|_{L^\eta} \\
&\leqslant \|V(\cdot + x_n)\, \mathrm{e}^{-\mathrm{i}\, t\Delta}\, \psi\|_{L^\eta(|x+x_n| \geqslant \Lambda)} + \|V(\cdot + x_n)\, \mathrm{e}^{-\mathrm{i}\, t\Delta}\, \psi\|_{L^\eta(|x+x_n| \leqslant \Lambda)} \\
&\leqslant \epsilon \|\, \mathrm{e}^{-\mathrm{i}\, t\Delta}\, \psi\|_{L^\infty(0,\bar{t}+1) L^{2^\star}} + \epsilon \|V\|_{L^\delta} \\
&\lesssim \epsilon (\|\, \mathrm{e}^{-\mathrm{i}\, t\Delta}\, \psi\|_{L^\infty(0,\bar{t}+1) H^1} + \|V\|_{L^\delta}),
\end{aligned}$$

thus

$$\|(\tau_{-x_n} V)\, \mathrm{e}^{-\mathrm{i}\, t\Delta}\, \psi\|_{L^\infty(0,\bar{t}+1) L^\eta} \to 0.$$

With the same argument, because $\nabla V \in L^{d/2}$, we have

$$\|\nabla(\tau_{-x_n} V)\, \mathrm{e}^{-\mathrm{i}\, t\Delta}\, \psi\|_{L^\infty(0,\bar{t}+1) L^\eta} \to 0.$$

Hence, to obtain (3.20), it only remains to show that

$$\|\tau_{-x_n} V \nabla(\mathrm{e}^{-\mathrm{i}\, t\Delta}\, \psi)\|_{L^\infty(0,\bar{t}+1) L^\eta} \to 0. \tag{3.23}$$

To this purpose, let $\widetilde{\psi}$ be a C^∞, compactly supported function such that

$$\|\psi - \widetilde{\psi}\|_{H^1} \leqslant \epsilon.$$

Notice that, by (2.1) we have

$$\frac{1}{\eta} = \frac{1}{2} + \frac{1}{d},$$

hence, by Minkowski and Hölder inequalities,

$$\begin{aligned}
\|\tau_{-x_n} V \nabla(\mathrm{e}^{-\mathrm{i}\, t\Delta}\, \psi)\|_{L^\eta}
&\leqslant \|\tau_{-x_n} V \nabla(\mathrm{e}^{-\mathrm{i}\, t\Delta}\, \widetilde{\psi})\|_{L^\eta} + \|\tau_{-x_n} V \nabla(\mathrm{e}^{-\mathrm{i}\, t\Delta} (\psi - \widetilde{\psi}))\|_{L^\eta} \\
&\leqslant \|\tau_{-x_n} V \nabla(\mathrm{e}^{-\mathrm{i}\, t\Delta}\, \widetilde{\psi})\|_{L^\eta} + \|V\|_{L^d} \|\nabla(\mathrm{e}^{-\mathrm{i}\, t\Delta} (\psi - \widetilde{\psi}))\|_{L^2} \\
&\leqslant \|\tau_{-x_n} V \nabla(\mathrm{e}^{-\mathrm{i}\, t\Delta}\, \widetilde{\psi})\|_{L^\eta} + \epsilon \|V\|_{L^d},
\end{aligned} \tag{3.24}$$

where $V \in L^d$ because of the (critical) Sobolev embedding $W^{1,d/2}(\mathbb{R}^d) \hookrightarrow L^d(\mathbb{R}^d)$.

Then, because $\nabla(\mathrm{e}^{-\mathrm{i}\, t\Delta}\, \widetilde{\psi}) \in H^1$,

$$\|\tau_{-x_n} V \nabla(\mathrm{e}^{-\mathrm{i}\, t\Delta}\, \widetilde{\psi})\|_{L^\infty(0,\bar{t}+1) L^\eta}$$

can be estimated as $\|(\tau_{-x_n} V)\, \mathrm{e}^{-\mathrm{i}\, t\Delta}\, \psi\|_{L^\infty(0,\bar{t}+1) L^\eta}$, hence (3.23) follows from (3.24) and the assumption is verified.

Assumption (3.6). It is a consequence of (3.17), the Lebesgue's dominated convergence theorem and the continuity of $t \in \mathbb{R} \longrightarrow \mathrm{e}^{\mathrm{i}\, tA}\, \tau_{\bar{x}} \psi \in H^1$ with the exact same proof as in [11]. □

3.2. Non linear profiles

Similarly to [11], we will now see that for a data which escapes to infinity, the solutions of (1.1) are the same as these of the homogeneous equation ($V=0$), in the sense given by the three next Propositions:

Proposition 3.3. — *Let $\psi \in H^1$, $(x_n)_{n\geqslant 1} \in \mathbb{R}^{\mathbb{N}}$ be such that $|x_n| \to \infty$. Then, up to a subsequence*

$$\|\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\tau_{x_n}\psi - \mathrm{e}^{-\mathrm{i}\,t(\Delta-V)}\,\tau_{x_n}\psi\|_{L^pL^r} \to 0 \tag{3.25}$$

as $n\to\infty$.

Proof. — We assume for example $x_n \to +\infty$.

By the dispersive estimate and a density argument, the same proof as in [11] gives

$$\sup_{n\in\mathbb{N}} \|\mathrm{e}^{\mathrm{i}\,t(-\Delta+V)}\,\tau_{x_n}\psi\|_{L^p(T,\infty)L^r} \to 0 \tag{3.26}$$

as $T\to\infty$. We are therefore reduced to show that for $T>0$ fixed

$$\|\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\tau_{x_n}\psi - \mathrm{e}^{\mathrm{i}\,t(-\Delta+V)}\,\tau_{x_n}\psi\|_{L^p(0,T)L^r} \to 0$$

as $n\to\infty$. Let us pick $\epsilon>0$. The differene $\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\tau_{x_n}\psi - \mathrm{e}^{\mathrm{i}\,t(-\Delta+V)}\,\tau_{x_n}\psi$ is a solution of the following linear Schrödinger equation with zero initial data

$$\mathrm{i}\,\partial_t u - \Delta u + Vu = V\,\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\tau_{x_n}\psi.$$

So, by the inhomogenous Strichartz estimate (2.9)

$$\begin{aligned}
\|\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\tau_{x_n}\psi - \mathrm{e}^{\mathrm{i}\,t(-\Delta+V)}\,\tau_{x_n}\psi\|_{L^p_t(0,T)L^r} &\\
&\lesssim \|V\,\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\tau_{x_n}\psi\|_{L_t^{\gamma'}(0,T)L^\eta}\\
&\lesssim T^{\frac{1}{\gamma'}}\|V\,\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\tau_{x_n}\psi\|_{L^\infty(0,T)L^\eta}\\
&= T^{\frac{1}{\gamma'}}\|(\tau_{-x_n}V)\,\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\psi\|_{L^\infty(0,T)L^\eta}
\end{aligned}$$

because the translation operator τ_{x_n} commutes with the propagator $\mathrm{e}^{-\mathrm{i}\,t\Delta}$. But

$$\|(\tau_{-x_n}V)\,\mathrm{e}^{-\mathrm{i}\,t\Delta}\,\psi\|_{L^\infty(0,T)L^\eta} \underset{n\to\infty}{\longrightarrow} 0$$

as seen in the proof of Proposition 3.2, point (3.5). ☐

Proposition 3.4. — *Let $\psi \in H^1$, $(x_n)_{n\geqslant 1} \in \mathbb{R}^{\mathbb{N}}$ be such that $|x_n| \to \infty$, $U \in C(H^1)\cap L^pL^r$ be the unique solution to the homogeneous equation*

$$\mathrm{i}\,\partial_t u + \Delta u = u|u|^\alpha$$

with initial data ψ, and $U_n(t,x) := U(t,x-x_n)$. Then, up to a subsequence

$$\left\| \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)\Delta} \left(U_n|U_n|^\alpha\right)(s)\mathrm{d}s - \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right)(s)\mathrm{d}s \right\|_{L^pL^r} \to 0 \tag{3.27}$$

as $n \to \infty$.

Proof. — In the exact same way as in [11], inhomogenous Strichartz estimates, and the pointwise dispersive estimate together with Hardy–Littlewood–Sobolev inequality leads

$$\sup_{n\in\mathbb{N}} \left\| \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right)(s)\mathrm{d}s \right\|_{L^p([T,\infty))L^r} \to 0 \tag{3.28}$$

as T goes to infinity. Thus; it remains to show that for $T>0$ fixed,

$$\left\| \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)\Delta} \left(U_n|U_n|^\alpha\right)\mathrm{d}s - \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right)\mathrm{d}s \right\|_{L^p(0,T)L^r} \to 0$$

as $n \to \infty$. The difference

$$\int_0^t \mathrm{e}^{-\mathrm{i}(t-s)\Delta} \left(U_n|U_n|^\alpha\right)\mathrm{d}s - \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right)\mathrm{d}s$$

is the solution of the following linear Schrödinger equation, with zero initial data

$$\mathrm{i}\,\partial_t u - \Delta u + Vu = V \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)\Delta} \left(U_n|U_n|^\alpha\right)\mathrm{d}s.$$

Hence, by the Strichartz estimate (2.9)

$$\begin{aligned}
&\left\| \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)\Delta} \left(U_n|U_n|^\alpha\right)\mathrm{d}s - \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)(\Delta-V)} \left(U_n|U_n|^\alpha\right)\mathrm{d}s \right\|_{L^p(0,T)L^r} \\
&\qquad\lesssim \left\| V \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)\Delta} \left(U_n|U_n|^\alpha\right)\mathrm{d}s \right\|_{L^{\gamma'}(0,T)L^\eta} \\
&\qquad\lesssim T^{\frac{1}{\gamma'}} \left\| (\tau_{-x_n}V) \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)\Delta} \left(U|U|^\alpha\right)\mathrm{d}s \right\|_{L^\infty(0,T)L^\eta}.
\end{aligned}$$

But $\int_0^t \mathrm{e}^{-\mathrm{i}(t-s)\Delta}(U|U|^\alpha)\mathrm{d}s \in C([0,T],H^1)$, so by Sobolev embedding in $L^{2^\star}$ and compacity in time there exists $\Lambda>0$ such that

$$\left\| \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)\Delta} \left(U|U|^\alpha\right)\mathrm{d}s \right\|_{L^\infty(0,T)L^{2^\star}(|x|\geqslant\Lambda)} \leqslant \epsilon$$

therefore

$$\left\|(\tau_{-x_n}V)\int_0^t \mathrm{e}^{-\,\mathrm{i}(t-s)\Delta}\,(U|U|^{\alpha})\,\mathrm{d}s\right\|_{L^\infty(0,T)L^\eta}\underset{n\to\infty}{\longrightarrow}0$$

in the same way as in the proof of Proposition 3.2, point (3.5). □

Proposition 3.5. — *Let $\psi\in H^1$, $(x_n)_{n\geqslant 1}$, $(t_n)_{n\geqslant 1}\in\mathbb{R}^{\mathbb{N}}$ be such that $|x_n|\to\infty$ and $t_n\to\pm\infty$, U be a solution to the homogeneous equationsuch that*

$$\|U(t)-\mathrm{e}^{-\,\mathrm{i}\,t\Delta}\,\psi\|_{H^1}\underset{t\to\pm\infty}{\longrightarrow}0$$

and $U_n(t,x):=U(t-t_n,x-x_n)$. Then, up to a subsequence

$$\|\,\mathrm{e}^{-\,\mathrm{i}(t-t_n)\Delta}\,\tau_{x_n}\psi-\mathrm{e}^{-\,\mathrm{i}(t-t_n)(\Delta-V)}\,\tau_{x_n}\psi\|_{L^pL^r}\to 0 \tag{3.29}$$

and

$$\left\|\int_0^t \mathrm{e}^{-\,\mathrm{i}(t-s)\Delta}(U_n|U_n|^{\alpha})\,\mathrm{d}s-\int_0^t \mathrm{e}^{-\,\mathrm{i}(t-s)(\Delta-V)}(U_n|U_n|^{\alpha})\,\mathrm{d}s\right\|_{L^pL^r}\to 0 \tag{3.30}$$

as $n\to\infty$.

Proof. — The proof is the same as for Proposition 3.3 and Proposition 3.4, decomposing the time interval in $\{|t-t_n|>T\}$ and his complementary. □

Finaly, we will need the following Proposition of non linear scattering:

Proposition 3.6. — *Let $\varphi\in H^1$. Then there exists $W_\pm\in C(H^1)\cap L^p_{\mathbb{R}^\pm}L^r$, solution of (1.1) such that*

$$\|W_\pm(t,\cdot)-\mathrm{e}^{-\,\mathrm{i}\,t(\Delta-V)}\,\varphi\|_{H^1}\underset{t\to\pm\infty}{\longrightarrow}0 \tag{3.31}$$

moreover, if $t_n\to\mp\infty$ and

$$\varphi_n=\mathrm{e}^{-\,\mathrm{i}\,t_n(\Delta-V)}\,\varphi,\quad W_{\pm,n}(t)=W_\pm(t-t_n) \tag{3.32}$$

then

$$W_{\pm,n}(t)=\mathrm{e}^{-\,\mathrm{i}\,t(\Delta-V)}\,\varphi_n+\int_0^t \mathrm{e}^{-\,\mathrm{i}(t-s)(\Delta-V)}(W_{\pm,n}|W_{\pm,n}|^{\alpha})(s)\mathrm{d}s+f_{\pm,n}(t) \tag{3.33}$$

where

$$\|f_{\pm,n}\|_{L^p_{\mathbb{R}^\pm}L^r}\underset{n\to\infty}{\longrightarrow}0\,. \tag{3.34}$$

Proof. — The same proof as [1, Proposition 3.5] holds, as it involves only the analogous Strichartz estimates. □

3.3. Conclusion

Theorem 3.1 is now a consequence of the linear profile decomposition together with the nonlinear profiles results of Propositions 3.3, 3.4, 3.5, 3.6, perturbative result of Proposition 2.4 and Strichartz inequalities of Proposition 2.1, in the exact same way as in [11, Section 5].

4. Extinction of the critical solution

The aim of this section is to prove the following ridity theorem

THEOREM 4.1. — *There is no non-trivial compact-flow solution of* (1.1).

By compact flow solution, we mean here a solution u with a relatively compact flow $\{u_c(t),\ t \geqslant 0\}$ in H^1. Our key tool will be the following Morawetz identity – or virial computation:

LEMMA 4.2. — *Let* $u \in C(H^1)$ *be a solution of* (1.1) *and* $\chi \in C^\infty(\mathbb{R}^d)$ *be a smooth function. Then*

$$\partial_t \int \chi |u|^2 = 2 \operatorname{Im} \int \nabla\chi \cdot \nabla u \bar{u} \tag{4.1}$$

$$\partial_t^2 \int \chi |u|^2 = 4 \int (D^2\chi \nabla u, \nabla u) + \frac{2}{\alpha+2} \int \Delta\chi |u|^{\alpha+2} - 2 \int \nabla\chi \cdot \nabla V |u|^2 - \int \Delta^2 \chi |u|^2. \tag{4.2}$$

In the case of a repulsive potential, taking the weight $\chi = |x|^2$ gives the result by a classical argument, as all the terms, and in particular

$$\int \nabla\chi \cdot \nabla V |u|^2 \tag{4.3}$$

have the right sign. However, with a non-repulsive potential, this straightforward choose of weight does not permit to conclude because (4.3) is no signed anymore.

However, in our framework of the sum of two repulsive potentials verifying the convexity assumptions (G1)-(G2)-(G3), we are able to construct a family of weights that have the right behavior and for which the non-negative part of (4.3) can be made small enough. The idea is to construct it in such a way that $\nabla\chi$ is almost orthogonal to the line $\mathcal{R}$ containing the trapped trajectories. More precisely, we would like to take as a weight

$$|x - \boldsymbol{c}| + |x + \boldsymbol{c}|,$$

where $\boldsymbol{c}$ is such that $(-\boldsymbol{c}, \boldsymbol{c}) \subset \mathcal{R}$ and will be sent to infinity.

The smallness of the non-negative part (4.3) will be a consequence of the following lemma, where Θ_1 and Θ_2 have to be tought as level surfaces of V_1 and V_2. The assumptions (2) and (3) of the lemma correponds to assumptions (G2) and (G3). In the following, n is choosen as the outward-pointing normal to Θ_1 and Θ_2.

Lemma 4.3. — *Let $\alpha > 0$, $R \in C^0([A, +\infty[, \mathbb{R}_+)$ be such that $R(c)/c \longrightarrow 0$ as $c \longrightarrow +\infty$ and, for all $c \geqslant A$, $(\Theta_1)(c)$, $(\Theta_2)(c)$ be two families of smooth convex subsets of $\mathbb{R}^d$. We assume that, for all $c \geqslant A$ and any elements Θ_1, Θ_2 of $(\Theta_1)(c), (\Theta_2)(c)$*

1. *Θ_1 and Θ_2 are contained in $B(0, R(c))$,*
2. *in the non star-shaped region $\{x \in \partial(\Theta_1 \cup \Theta_2),\ x \cdot n(x) < 0\}$, the eigenvalues of the second fundamental forms of $\partial\Theta_1$ and $\partial\Theta_2$ are bounded below by α,*
3. *the trapped ray associated with $\mathbb{R}^d \backslash (\Theta_1 \cup \Theta_2)$ is a segment of the line $\{x_2 = \cdots = x_d = 0\}$.*

Let $\boldsymbol{c} := (c, 0, \ldots, 0)$. Then, for any elements Θ_1, Θ_2 of $(\Theta_1)(c), (\Theta_2)(c)$ and $x \in \partial(\Theta_1 \cup \Theta_2)$, we have as $c \longrightarrow +\infty$

$$\left(\frac{x - \boldsymbol{c}}{|x - \boldsymbol{c}|} + \frac{x + \boldsymbol{c}}{|x + \boldsymbol{c}|} \right) \cdot n(x) \geqslant O\left(\frac{R(c)^4}{c^4} \right).$$

Proof. — For $x \in B(0, R)$, let us denote $x = (x_1, \widetilde{x})$ with $\widetilde{x} \in \mathbb{R}^{d-1}$. Remark that

$$|x + \boldsymbol{c}| = c + x_1 + \frac{1}{2c} |\widetilde{x}|^2 + O\left(\frac{R^4}{c^3} \right)$$

and therefore

$$\frac{x - \boldsymbol{c}}{|x - \boldsymbol{c}|} + \frac{x + \boldsymbol{c}}{|x + \boldsymbol{c}|} = \frac{1}{|x - \boldsymbol{c}||x + \boldsymbol{c}|} \left(2c(0, \widetilde{x}) + x \frac{|\widetilde{x}|^2}{c} + O\left(\frac{R^4}{c^2} \right) \right). \tag{4.4}$$

Notice that, in the star-shaped region $\{x \in \partial(\Theta_1 \cup \Theta_2)(c),\ x \cdot n(x) \geqslant 0\}$, (4.4) together with the fact that $\widetilde{x} \cdot n \geqslant 0$ by convexity of the obstacles, and noticing that

$$|x - \boldsymbol{c}||x + \boldsymbol{c}| \gtrsim c^2 \tag{4.5}$$

by the hypothesis $R(c)/c \longrightarrow 0$, gives the result.

Let us now consider x in the more intricate non star-shaped region

$$\{x \in \partial(\Theta_1 \cup \Theta_2)(c),\ x \cdot n(x) < 0\}.$$

On $\partial\Theta_i$, n is near the trapped ray associated with $\mathbb{R}^d\backslash(\Theta_1\cup\Theta_2)$ of the form

$$n(x)=\left(\pm\frac{x_1}{|x_1|},0,\dots,0\right)+(0,\lambda_2x_2,\dots,\lambda_dx_d)+O(|\widetilde{x}|^2)$$

with $\lambda_k>0$. And thus

$$\left(2c(0,\widetilde{x})+x\frac{|\widetilde{x}|^2}{c}\right)\cdot n(x)\geqslant\left(2c\min\lambda_k-\frac{C}{c}\right)|\widetilde{x}|^2+O(|\widetilde{x}|^2). \tag{4.6}$$

Because of the uniform convexity assumption of $\Theta_{1,2}$ in the non star-shaped region (assumption (2) of the lemma), $\min\lambda_k$ is bounded below, uniformly in c, by a strictly positive, universal constant. Hence, by (4.6), there exists $\rho\geqslant 0$ and $D_1>0$ such that, for every $c>D_1$ we have

$$|\widetilde{x}|\leqslant\rho\implies\left(2c(0,\widetilde{x})+x\frac{|\widetilde{x}|^2}{c}\right)\cdot n(x)\geqslant 0. \tag{4.7}$$

On the other hand, there exists $\epsilon_0>0$ such that, for all $x\in\partial(\Theta_1\cup\Theta_2)$,

$$|\widetilde{x}|\geqslant\rho\implies(0,\widetilde{x})\cdot n(x)\geqslant\epsilon_0.$$

Notice that the uniformity in c is a consequence of assumption (2) again. Hence, if $|\widetilde{x}|\geqslant\rho$

$$\left(2c(0,\widetilde{x})+x\frac{|\widetilde{x}|^2}{c}\right)\cdot n(x)\geqslant 2c\epsilon_0-\frac{C}{c},$$

and therefore, there exists $D_2>0$ such that, if $c>D_2$

$$|\widetilde{x}|\geqslant\rho\implies\left(2c(0,\widetilde{x})+x\frac{|\widetilde{x}|^2}{c}\right)\cdot n(x)\geqslant 0. \tag{4.8}$$

Combining (4.4), (4.7), (4.8), and (4.5) gives the result. □

We are now in position to prove the rigidity theorem:

Proof of Theorem 4.1. By contradiction, let $u\neq 0$ be a solution of (1.1) with a relatively compact flow $\{u(t),\,t\in\mathbb{R}\}$ in H^1.

We choose a system of coordinates such that $\mathcal{R}=\{x_2=\dots=x_d=0\}$. Let $c>0$ and $\boldsymbol{c}:=(c,0,\dots,0)$. We would like to take

$$|x-\boldsymbol{c}|+|x+\boldsymbol{c}| \tag{4.9}$$

as a weight. However, because of the singularities in $\pm\boldsymbol{c}$, it is not smooth and we cannot use it explicitly. Therefore, we take instead

$$\chi_c(x):=(|x-\boldsymbol{c}|+|x+\boldsymbol{c}|)\,\psi(\frac{x}{c/4}),$$

where $\psi\in C^\infty$ is such that $\psi(x)=1$ for $|x|\leqslant 1$ and $\psi(x)=0$ for $|x|\geqslant 2$. The idea is that now χ_c is smooth, and it coincides with (4.9) in $B(0,c/4)$, but as c will be sent to infinity, the part outside this ball will not be seen by compact flow solutions. Let us denote

$$z(t) = \int \chi_c |u|^2.$$

By (4.1), the Cauchy–Schwarz inequality and the conservation of mass and energy

$$|z'(t)| \leqslant \sqrt{CE(u)M(u)}. \tag{4.10}$$

Moreover, (4.2) writes

$$z''(t) = 4\int (D^2\chi_c \nabla u, \nabla u) + \frac{2}{\alpha+2}\int \Delta\chi_c |u|^{\alpha+2} - 2\int \nabla\chi_c \cdot \nabla V |u|^2 - \int \Delta^2\chi_c |u|^2. \tag{4.11}$$

Let us write down $D^2\chi_c$, $\Delta\chi_c$ and $\Delta^2\chi_c$. To this purpose, let

$$\chi_c^-(x) := |x - \boldsymbol{c}|\psi\left(\frac{x}{c/4}\right), \quad \chi_c^+(x) := |x + \boldsymbol{c}|\psi\left(\frac{x}{c/4}\right),$$

in such a way that $\chi_c = \chi_c^+ + \chi_c^-$. We have

$$\Delta\chi_c^\pm(x) = \frac{d-1}{|x\pm\boldsymbol{c}|}\psi\left(\frac{x}{c/4}\right) + \frac{8}{c}\frac{x\pm\boldsymbol{c}}{|x\pm\boldsymbol{c}|}\cdot\nabla\psi\left(\frac{x}{c/4}\right) + \frac{16}{c^2}|x\pm\boldsymbol{c}|\Delta\psi\left(\frac{x}{c/4}\right), \tag{4.12}$$

$$D^2\chi_c^\pm(x) = \frac{1}{|x\pm\boldsymbol{c}|}\left(\mathrm{Id} - \frac{(x\pm\boldsymbol{c})(x\pm\boldsymbol{c})^{\mathrm{t}}}{|x\pm\boldsymbol{c}|^2}\right)\psi\left(\frac{x}{c/4}\right) + \frac{4}{c}\frac{x\pm\boldsymbol{c}}{|x\pm\boldsymbol{c}|}\left(\nabla\psi\left(\frac{x}{c/4}\right)\right)^{\mathrm{t}} + \frac{4}{c}\nabla\psi\left(\frac{x}{c/4}\right)\left(\frac{x\pm\boldsymbol{c}}{|x\pm\boldsymbol{c}|}\right)^{\mathrm{t}} + \frac{16}{c^2}|x\pm\boldsymbol{c}|D^2\psi\left(\frac{x}{c/4}\right), \tag{4.13}$$

$$\begin{aligned}\Delta^2\chi_c^\pm(x) = &-\frac{(d-1)(d-3)}{|x\pm\boldsymbol{c}|^3}\psi\left(\frac{x}{c/4}\right) - \frac{16(d-1)}{c}\frac{x\pm\boldsymbol{c}}{|x\pm\boldsymbol{c}|}\cdot\nabla\psi\left(\frac{x}{c/4}\right)\\ &+\frac{32}{c^2}\frac{d+1}{|x\pm\boldsymbol{c}|}\Delta\psi\left(\frac{x}{c/4}\right) + \frac{256}{c^3}\frac{x\pm\boldsymbol{c}}{|x\pm\boldsymbol{c}|}\cdot\nabla\left(\Delta\psi\left(\frac{x}{c/4}\right)\right)\\ &-\frac{64}{c^2}\frac{1}{|x\pm\boldsymbol{c}|^3}\left(D^2\psi\left(\frac{x}{c/4}\right)(x\pm\boldsymbol{c}), x\pm\boldsymbol{c}\right)\\ &+\frac{256}{c^4}|x\pm\boldsymbol{c}|\Delta^2\psi\left(\frac{x}{c/4}\right),\end{aligned} \tag{4.14}$$

where X^{t} denote the transpose of the column vector $X \in \mathbb{R}^d$. Observe that

$$x \in \operatorname{Supp} \psi\left(\frac{\cdot}{c/4}\right) \implies |x \pm c| \geqslant \frac{c}{2},$$

therefore, by (4.12), (4.13) and (4.14)

$$|\Delta \chi_c| + |D^2 \chi_c| + |\Delta^2 \chi_c| \lesssim \frac{1}{c}. \tag{4.15}$$

In addition, as $\{u(t),\, t \in \mathbb{R}\}$ is relatively compact in H^1 and by Sobolev embedding in $L^{\alpha+2}$ – recall that, by assumption (1.2), we are in particular in the subcritical regime –, we have

$$\sup_{t\in\mathbb{R}} \left(\|u\|_{L^{\alpha+2}(|x|\geqslant c/4)} + \|u\|_{L^2(|x|\geqslant c/4)} + \|\nabla u\|_{L^2(|x|\geqslant c/4)}\right) = \epsilon(c), \tag{4.16}$$

where $\epsilon(c) \longrightarrow 0$ as $c \longrightarrow +\infty$. Therefore (4.11) together with (4.15) and (4.16) yields

$$\begin{aligned} z''(t) = \int_{B(0,c/4)} & 4(D^2\chi_c \nabla u, \nabla u) + \frac{2}{\alpha+2}\Delta\chi_c |u|^{\alpha+2} \\ & - \Delta^2 \chi_c |u|^2 - 2\int_{\mathbb{R}^d} \nabla\chi_c \cdot \nabla V |u|^2 + \frac{1}{c}\epsilon(c). \end{aligned} \tag{4.17}$$

Now, in $B(0, c/4)$, χ_c coincides with (4.9) and:

$$\Delta\chi_c^{\pm} = \frac{d-1}{|x \pm \boldsymbol{c}|},\ D^2\chi_c^{\pm} = \frac{1}{|x\pm\boldsymbol{c}|}\left(\mathrm{Id} - \frac{(x\pm\boldsymbol{c})(x\pm\boldsymbol{c})^{\mathrm{t}}}{|x\pm\boldsymbol{c}|^2}\right),$$
$$\Delta^2\chi_c^{\pm} = -\frac{(d-1)(d-3)}{|x\pm\boldsymbol{c}|^3}, \quad \text{in } B(0,c/4).$$

In particular, it verifies there

$$D^2\chi_c \geqslant 0,\ \Delta\chi_c \gtrsim \frac{1}{c}, \Delta^2\chi_c \leqslant 0 \quad \text{in } B(0,c/4),$$

where the sign of $\Delta^2\chi_c$ is due to $d \geqslant 3$, and therefore, by (4.17), we have, for all $A \leqslant c/4$

$$z''(t) \gtrsim \frac{1}{c}\int_{B(0,A)} |u|^{\alpha+2} - \int_{\mathbb{R}^d} \nabla\chi_c \cdot \nabla V |u|^2 + \frac{1}{c}\epsilon(c). \tag{4.18}$$

Now, as $u \neq 0$ and $\{u(t),\, t \in \mathbb{R}\}$ is relatively compact in H^1, there exists $\mu > 0$ and $A > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_{B(0,A)} |u|^{\alpha+2} \geqslant 2\mu.$$

We fix such an $A>0$ and take $c>0$ large enough so that $A \leqslant c/4$ and $|\epsilon(c)| \leqslant \mu$. Then (4.18) gives

$$z''(t) \gtrsim \frac{1}{c}\mu - \int_{\mathbb{R}^d} \nabla\chi_c \cdot \nabla V |u|^2. \tag{4.19}$$

Let R be a continuous function of c such that $R(c)/c \longrightarrow 0$ as $c \longrightarrow +\infty$. In the seek of readability, we will write R for $R(c)$ in the sequel. Note that, by the Hölder inequality and because $\nabla\chi_c$ is bounded and $|x|^\beta \nabla V \in L^{d/2}$

$$\begin{aligned}\left|\int_{|x|\geqslant R} \nabla\chi_c \cdot \nabla V |u|^2\right| &\leqslant \|\nabla V\|_{L^{d/2}(|x|\geqslant R)} \|u\|^2_{L^{2^\star}(|x|\geqslant R)} \\ &\leqslant \frac{1}{R^\beta} \||x|^\beta \nabla V\|_{L^{d/2}} \|u\|^2_{L^{2^\star}(|x|\geqslant R)},\end{aligned} \tag{4.20}$$

but, because $\{u(t),\, t\in\mathbb{R}\}$ is relatively compact in H^1 and by Sobolev embedding in $L^{2^\star}$,

$$\sup_{t\in\mathbb{R}} \|u\|_{L^{2^\star}(|x|\geqslant R)} = \epsilon(R), \tag{4.21}$$

where $\epsilon(R) \longrightarrow 0$ when $R \longrightarrow +\infty$ and thus, using (4.19), (4.20) and (4.21)

$$z''(t) \gtrsim \mu/c - \int_{B(0,R)} \nabla\chi_c \cdot \nabla V |u|^2 + \frac{1}{R^\beta}\epsilon(R). \tag{4.22}$$

Now, notice that as $R(c) \ll c$, χ_c coincides with (4.9) in $B(0,R)$ and in particular

$$\nabla\chi_c(c) = \frac{x-\boldsymbol{c}}{|x-\boldsymbol{c}|} + \frac{x+\boldsymbol{c}}{|x+\boldsymbol{c}|} \text{ in } B(0,R).$$

Because V_1 and V_2 are repulsive (assumption (G1)), the outward-pointing normal to their level surfaces is

$$-\frac{\nabla V_{1,2}}{|\nabla V_{1,2}|}.$$

Thus, by Lemma 4.3 applied to the level surfaces of V_1 and V_2, together with assumptions (G2) and (G3), we get, in $B(0,R)$

$$-\nabla\chi_c \cdot \frac{\nabla V_{1,2}}{|\nabla V_{1,2}|} \geqslant O\left(\frac{R^4}{c^4}\right).$$

Therefore, by Hölder inequality, Sobolev embedding and conservation of energy

$$\begin{aligned}(4.23)\quad \left|\int_{|x|\leqslant R, -\nabla\chi\cdot\nabla V(x)<0} -\nabla\chi_c\cdot\nabla V|u|^2\right| &\lesssim \frac{R^4}{c^4}\int \big(|\nabla V_1|+|\nabla V_2|\big)|u|^2 \\ &\leqslant \frac{R^4}{c^4}\big(\|\nabla V_1\|_{L^{d/2}}+\|\nabla V_2\|_{L^{d/2}}\big)\|u\|^2_{L^{2^\star}} \\ &\leqslant \frac{R^4}{c^4}\big(\|\nabla V_1\|_{L^{d/2}}+\|\nabla V_2\|_{L^{d/2}}\big)\|u\|^2_{H^1} \lesssim \frac{R^4}{c^4}E(u_0)^2.\end{aligned}$$

Hence, (4.22) together with (4.23) gives

$$z''(t) \gtrsim \frac{\mu}{c} + O\left(\frac{R^4}{c^4}\right) + \frac{1}{R^\beta}\epsilon(R).$$

Let us take $R(c) = c^\nu$. Then we get

$$z''(t) \gtrsim \frac{1}{c}(\mu + O(c^{4\nu-3}) + c^{1-\beta\nu}\epsilon(c^\nu)).$$

Thus, taking

$$\nu = \frac{1}{\beta},$$

and assuming

$$\nu < \frac{3}{4} \iff \beta > \frac{4}{3},$$

in such a way that $R(c)/c \longrightarrow 0$ and, in particular, $4\nu - 3 < 0$, we get, for $c > 0$ fixed large enough

$$z''(t) \gtrsim \frac{\mu}{2c},$$

and (4.10) is contradicted. □

Our main result now follows:

Proof of Theorem 1.1. If $E_c < \infty$, then Theorem 3.1 allows us to extract a critical element $\varphi_c \in H^1$, $\varphi_c \neq 0$, such that the corresponding solution u_c of (1.1) verifies that $\{u_c(t),\ t \geqslant 0\}$ is relatively compact in H^1. By Theorem 4.1, such a solution cannot exist, so $E_c = \infty$ and by Proposition 2.2, all the solutions of (1.1) scatter in H^1. □

Aknowledgments

The author is grateful to Fabrice Planchon for many enlightening discussions about multipliers methods, and to the anonymous referee for their valuable remarks and suggestions.

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Manuscrit reçu le 13 décembre 2018,
révisé le 2 juillet 2019,
accepté le 11 juillet 2019.

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