



# ANNALES DE L'INSTITUT FOURIER

Fumihiko SANDA & YOTA SHAMOTO

**An analogue of Dubrovin's conjecture**

Tome 70, n° 2 (2020), p. 621-682.

<[http://aif.centre-mersenne.org/item/AIF\\_2020\\_\\_70\\_2\\_621\\_0](http://aif.centre-mersenne.org/item/AIF_2020__70_2_621_0)>

---

© Association des Annales de l'institut Fourier, 2020,

*Certains droits réservés.*



Cet article est mis à disposition selon les termes de la licence  
CREATIVE COMMONS ATTRIBUTION – PAS DE MODIFICATION 3.0 FRANCE.  
<http://creativecommons.org/licenses/by-nd/3.0/fr/>



# AN ANALOGUE OF DUBROVIN’S CONJECTURE

by Fumihiko SANDA & Yota SHAMOTO

---

ABSTRACT. — We propose an analogue of Dubrovin’s conjecture for the case where Fano manifolds have quantum connections of exponential type. It includes the case where the quantum cohomology rings are not necessarily semisimple. The conjecture is described as an isomorphism of two linear algebraic structures, which we call “mutation systems”. Given such a Fano manifold  $X$ , one of the structures is given by the Stokes structure of the quantum connection of  $X$ , and the other is given by a semiorthogonal decomposition of the derived category of coherent sheaves on  $X$ . We also prove the conjecture for a class of smooth Fano complete intersections in a projective space.

RÉSUMÉ. — Nous proposons un analogue de la conjecture de Dubrovin pour le cas où les variétés de Fano ont des connexions quantiques de type exponentiel. Cela inclut le cas où les cohomologies quantiques ne sont pas nécessairement semi-simples. La conjecture est décrite comme un isomorphisme de structures algébriques linéaires, que nous appelons systèmes de mutation. Étant donné une telle variété de Fano  $X$ , l’une des structures est donnée par la structure de Stokes de la connexion quantique de  $X$ , et l’autre est donnée par une décomposition semi-orthogonale de la catégorie dérivées des faisceaux cohérents sur  $X$ . De plus, nous prouvons la conjecture pour une classe d’intersections complètes lisses de Fano dans un espace projectif.

## 1. Introduction

The purpose of this paper is to propose an analogue of Dubrovin’s conjecture [15] in a more general setting. We shall firstly recall the original conjecture of Dubrovin in Section 1.1. We also review Gamma conjecture of Galkin–Golyshev–Iritani [17], [18] in Section 1.2 since their result plays a key role in this paper. Then we explain the outline of the formulation and the main result in Section 1.3–1.6.

---

*Keywords:* mirror symmetry, Fano manifolds, quantum cohomologies, Stokes matrix.  
*2020 Mathematics Subject Classification:* 14J33.

### 1.1. Dubrovin's conjecture

Let  $X$  be a Fano manifold. B. Dubrovin predicted some relations between the derived category  $D^b(X)$  of bounded complexes of coherent sheaves on  $X$  and the quantum cohomology ring of  $X$ . More precisely, he conjectured that  $D^b(X)$  has a full exceptional collection if and only if the quantum cohomology ring of  $X$  is (generically) semisimple. This conjecture is proved for many examples [2], [9], [10], [19], [22], [40], etc.

In the case where these two conjecturally equivalent conditions hold, he also predicted a relationship between full exceptional collections of  $D^b(X)$  and the quantum connection associated to the quantum cup product. To be more precise, let us roughly recall the definition of the quantum connection. For simplicity, we take the quantum parameter  $\tau$  to be 0.

Let  $\mathcal{H}_X := H^\bullet(X) \otimes \mathcal{O}_{\mathbb{C}_z}$  be the trivial  $(\mathbb{Z}/2\mathbb{Z}$ -graded)  $\mathcal{O}_{\mathbb{C}_z}$ -module. Here,  $\mathbb{C}_z$  denotes the complex plane whose coordinate function is  $z$ , and  $\mathcal{O}_{\mathbb{C}_z}$  denotes the sheaf of algebraic functions on  $\mathbb{C}_z$ . The quantum connection  $\nabla : \mathcal{H}_X \rightarrow \mathcal{H}_X \otimes \Omega_{\mathbb{C}_z}^1(\log\{0\}) \otimes \mathcal{O}_{\mathbb{C}_z}(\{0\})$  is defined by

$$\nabla := d - \left( \frac{1}{z} (c_1(X) *_0) - \mu \right) \frac{dz}{z},$$

where  $c_1(X)$  is the first Chern class of  $X$ ,  $*_0$  is the quantum cup product with respect to the quantum parameter  $\tau = 0$ , and  $\mu$  is the grading operator (see Definition 3.1 for details).

The quantum connection has a regular singularity at  $z = \infty$ , and an irregular singularity at  $z = 0$ . If the quantum cohomology ring is semisimple, we have a matrix called a Stokes matrix. We consider the following form of Dubrovin's conjecture:

**CONJECTURE 1.1.** — *If the quantum cohomology ring of  $X$  is semisimple, then there exists a full exceptional collection  $E_1, \dots, E_m$  of  $D^b(X)$  such that the Stokes matrix of  $(\mathcal{H}_X, \nabla)$  at  $z = 0$  is equal to the matrix  $(\chi(E_i, E_j))_{i,j}$ , where  $\chi(E, F) := \sum_k (-1)^k \dim \text{Hom}(E, F[k])$  for objects  $E, F \in D^b(X)$ .*

Conjecture 1.1 has been proved for some  $X$  with semisimple quantum cohomology rings [13], [21], [29], [54], [55], etc.

### 1.2. Gamma conjectures

Gamma conjecture II proposed by Galkin–Golyshev–Iritani [17] refines Conjecture 1.1. The conjecture is described as a coincidence of two kinds of cohomology classes of  $X$ : Asymptotic classes and Gamma classes.

Let  $C_X$  be the set of eigenvalues of the linear operator  $c_1(X)_{*0}$  on  $H^\bullet(X)$ . Fix a real number  $\theta_o$  (for simplicity of notation, we assume  $0 \leq \theta_o < 2\pi$ ). We assume that  $\theta_o$  is generic with respect to  $-C_X$  in a certain sense (see Section 2.1). Then we have an ordering  $\tau_{\theta_o} : -C_X \xrightarrow{\sim} \{1, \dots, m\}$ . If the quantum cohomology ring of  $X$  is semisimple, the space of solutions on the sector  $\{z \mid |\arg z - \theta_o| < \pi/2 + \varepsilon\}$  with sufficiently small  $\varepsilon > 0$  have a basis  $(y_i)_{i=1}^m$ . The basis  $(y_i)_{i=1}^m$  is characterized by their asymptotic growth as  $z \rightarrow 0$ , and  $y_i$  corresponds to  $\exp(c_i/z)$  where  $c_i := \tau_{\theta_o}^{-1}(i)$ . Since the fundamental solution at infinity identifies this space of solutions with  $H^\bullet(X)$  (see Proposition 3.3 or [17, Proposition 2.3.1]), the basis  $(y_i)_i$  gives a basis  $(A_i)_{i=1}^m$  of  $H^\bullet(X)$ . The classes  $A_i$  are called asymptotic classes (see [17, §4.5] for more precise).

The Gamma class [26], [30], [36] of  $X$  is defined by

$$\hat{\Gamma}_X := \prod_{j=1}^{\dim X} \Gamma(1 + \delta_j)$$

where  $\delta_1, \dots, \delta_{\dim X}$  are the Chern roots of the tangent bundle of  $X$  and  $\Gamma(x)$  is the Gamma function. Gamma conjecture II states that there exists a full exceptional collection  $E_1, \dots, E_m$  of  $D^b(X)$  such that

$$A_i = \hat{\Gamma}_X \operatorname{Ch}(E_i)$$

for all  $i$  (under the semisimplicity of the quantum cohomology ring and the existence of at least one full exceptional collection of  $D^b(X)$ ).

This conjecture refines Conjecture 1.1 in the following sense. Let

$$[\cdot, \cdot]_X : H^\bullet(X) \otimes H^\bullet(X) \rightarrow \mathbb{C}$$

be a non-symmetric linear map defined by

$$[\alpha, \beta]_X := \frac{1}{(2\pi)^{\dim X}} \int_X e^{\pi i \mu} e^{-\pi i \rho_X} \alpha \cup \beta$$

where  $\rho_X$  is the cup product  $c_1(X) \cup$ . Then, on the one hand, the Stokes matrix coincides with  $([A_i, A_j]_X)_{i,j}$ . On the other hand, we have

$$\chi(E_i, E_j) = [\hat{\Gamma}_X \operatorname{Ch}(E_i), \hat{\Gamma}_X \operatorname{Ch}(E_j)]_X.$$

### 1.3. Mutation systems

The goal of this paper is to give an analogue of Conjecture 1.1 in the case where the quantum cohomology ring is not necessarily semisimple. To do this, we introduce a notion of mutation systems. For a finite dimensional

vector space  $V$  over a field  $\mathbf{k}$ , we call a pairing  $[\cdot, \cdot] : V \otimes V \rightarrow \mathbf{k}$  of  $V$  non-degenerate if the induced map  $V \ni v \mapsto (w \rightarrow [v, w]) \in V^\vee = \text{Hom}(V, \mathbf{k})$  is isomorphic. We often assume non-degenerateness of the pairings without a mention.

A mutation system is a tuple of a (finite dimensional) vector space  $V$  with a pairing  $[\cdot, \cdot]$ , a family of vector spaces with pairings  $(V_c, [\cdot, \cdot]_c)_{c \in C}$  indexed by  $C$ , a bijection  $\tau : C \rightarrow \{1, \dots, m\}$ , and an isomorphism of vector spaces  $f : \bigoplus_c V_c \xrightarrow{\sim} V$  with some conditions (see Definition 2.30 for details).

Mutation systems admit a kind of mutation, that is, we have a series of functors between categories of mutation systems with braid relations.

In the case where  $\dim V_c = 1$  for all  $c \in C$ , similar structure has been investigated in various contexts [3], [17], [34], [43], [56].

#### 1.4. A-mutation systems

We can construct a mutation system from the quantum connection of  $X$  under an assumption (Assumption 3.3). The construction has the following steps.

- (1) Apply the Riemann–Hilbert correspondence [38], [51] for the quantum connection (around  $z = 0$ ), we get a local system on  $S^1$  with a filtration called a Stokes filtration. We use the assumption here.
- (2) A reformulation of Stokes data by Hertling and Sabbah [23] gives a mutation system (we also use the Poincaré pairing on  $H^\bullet(X)$ ). Here, we need to fix a real number  $\theta_\circ$  with a genericity condition. We give this reformulation in Section 2.
- (3) By using the fundamental solution at  $z = \infty$ , the vector space  $V$  with a pairing  $[\cdot, \cdot]$  underlying the mutation system constructed in (2) is identified with  $(H^\bullet(X), [\cdot, \cdot]_X)$ .

The resulting mutation system is called an A-mutation system. Each step is closely related to the construction of asymptotic classes in the semisimple case. Although the resulting mutation system depends on the choice of  $\theta_\circ$ , they are all equivalent by mutations.

#### 1.5. B-mutation systems

We can construct a mutation system for a semiorthogonal decomposition of  $D^b(X)$  (see, for example, [32, Definition 2.3] for the definition of

semiorthogonal decompositions). Put  $\mathrm{HH}_k(X) := \bigoplus_{p-q=k} H^q(X, \Omega_X^p)$  and  $\mathrm{HH}_\bullet(X) := \bigoplus_{k \in \mathbb{Z}} \mathrm{HH}_k(X)$ . We have a pairing  $[\cdot, \cdot]_X$  defined by

$$[\alpha, \beta]_X := \frac{1}{(2\pi i)^{\dim X}} \int_X e^{\pi i \rho_X} W(\alpha) \cup \beta,$$

where  $W(\alpha) = \mathbf{i}^{p+q} \alpha$  for  $\alpha \in H^q(X, \Omega_X^p)$  (cf. [7], [44], [46], [47], [52]). Let  $D^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$  be a semiorthogonal decomposition. Then, by using a theorem of Kuznetsov [32], [33], we can define the subspace  $\mathrm{HH}_\bullet(\mathcal{A}_i)$  of  $\mathrm{HH}_\bullet(X)$ , which we call the Hochschild homology of  $\mathcal{A}_i$  (cf. [7], [32], [45], [52]). They satisfies  $\mathrm{HH}_\bullet(X) = \bigoplus_i \mathrm{HH}_\bullet(\mathcal{A}_i)$ . Roughly speaking, this decomposition defines a mutation system, which we call a B-mutation system. The braid group action on the set of semiorthogonal decompositions of  $D^b(X)$  is compatible with the mutation on the B-mutation systems.

### 1.6. The formulation of the analogue and the main theorem

Using the Gamma class, we define an isomorphism

$$\Gamma : \mathrm{HH}_\bullet(X) \xrightarrow{\sim} H^\bullet(X)$$

with  $[\alpha, \beta]_X = [\Gamma(\alpha), \Gamma(\beta)]_X$ . We define an analogue of Dubrovin's conjecture by the existence of a semiorthogonal decomposition such that the B-mutation system for the semiorthogonal decomposition is isomorphic to the A-mutation system via  $\Gamma$ . We call the analogue "Dubrovin type conjecture". For more precise, see Definition 5.2. In the case where the quantum cohomology ring is semisimple, Gamma conjecture II implies this conjecture.

The main result of this paper is to give a class of examples such that this Dubrovin type conjecture holds. More precisely, we have the following.

**THEOREM 1.2** (Theorem 7.9). — *Let  $X$  be a smooth Fano complete intersection in a projective space. If Fano index of  $X$  is larger than 1, then  $X$  satisfies Dubrovin type conjecture.*

### 1.7. Plan of the paper

In Section 2, we introduce the notion of mutation systems, and define the mutation on mutation systems. We also relate it with Stokes filtered local systems, and recall some general facts on the Riemann–Hilbert correspondence for meromorphic connections on a germ of complex plane at

the origin. In Section 3, we give a definition of A-mutation systems. In Section 4, we give a definition of B-mutation systems. In Section 5, we formulate an analogue of Dubrovin’s conjecture, which we call “Dubrovin type conjecture”. In Section 6, we show some properties, which are used to prove the main theorem, of quantum connections of Fano manifolds. In Section 7, we show the main theorem, that is, we give a class of examples which satisfies Dubrovin type conjecture.

## 2. Preliminary

### 2.1. Stokes filtrations and Stokes data

Recall the definition of Stokes filtrations on local systems over  $S^1 = \{t \in \mathbb{C} \mid |t| = 1\}$  in the sense of [23] (see also [51]). Let  $\mathbf{k}$  be a field. Let  $\text{Loc}_{\mathbf{k}} := \text{Loc}_{\mathbf{k}}(S^1)$  denote the category of (finite rank)  $\mathbf{k}$ -local systems on  $S^1$ . We identify  $S^1$  with  $\mathbb{R}/2\pi\mathbb{Z}$  in the standard way and denote them simply by  $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ . For  $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ , let  $\leq_{\theta}$  be the partial order on  $\mathbb{C}$  defined by the following relation:

$$(2.1) \quad c \leq_{\theta} c' \Leftrightarrow \Re(e^{-i\theta}c) < \Re(e^{-i\theta}c') \text{ or } c = c'.$$

We also set  $c <_{\theta} c'$  if and only if  $c \leq_{\theta} c'$  and  $c \neq c'$ .

DEFINITION 2.1 ([23, 2.a]). — Let  $\mathcal{L}$  be a  $\mathbf{k}$ -local system on  $S^1$ . A family  $\mathcal{L}_{\bullet}$  of subsheaves  $\mathcal{L}_{\leq c} \subset \mathcal{L}$  ( $c \in \mathbb{C}$ ) is called a Stokes filtration if the following conditions are satisfied:

- (1) For each  $\theta$ , the germs form an exhaustive increasing filtration of  $\mathcal{L}_{\theta}$  with respect to the order defined by the equation (2.1) above.
- (2) Set  $\mathcal{L}_{<c,\theta} := \sum_{c' <_c} \mathcal{L}_{\leq c',\theta}$ . It defines a subsheaf  $\mathcal{L}_{<c}$  in  $\mathcal{L}$ . The second condition is that the sheaves  $\text{gr}_c \mathcal{L} := \mathcal{L}_{\leq c} / \mathcal{L}_{<c}$  are in  $\text{Loc}_{\mathbf{k}}(S^1)$ .
- (3) Set  $\text{gr} \mathcal{L} := \bigoplus_c \text{gr}_c \mathcal{L}$ . It has a natural filtration given by

$$(\text{gr} \mathcal{L}_{\leq c})_{\theta} := \bigoplus_{c' \leq_{\theta} c} \text{gr}_{c'} \mathcal{L}_{\theta}.$$

The last condition is that near any point  $\theta \in S^1$ , there are local isomorphisms  $\eta : \mathcal{L} \xrightarrow{\sim} \text{gr} \mathcal{L}$  such that  $\eta(\mathcal{L}_{\leq c}) \subset \text{gr} \mathcal{L}_{\leq c}$  for all  $c \in \mathbb{C}$ , and that the induced graded morphism is identity.

We call the pair  $(\mathcal{L}, \mathcal{L}_{\bullet})$  a Stokes filtered local system. Let  $(\mathcal{L}^i, \mathcal{L}_{\bullet}^i)$  be two Stokes filtered local systems ( $i = 0, 1$ ). A morphism of local systems  $\lambda : \mathcal{L}^0 \rightarrow \mathcal{L}^1$  is called a morphism of Stokes filtered local systems if it preserves Stokes filtrations, i.e.  $\lambda \mathcal{L}_{\leq c}^0 \subset \mathcal{L}_{\leq c}^1$  for all  $c \in \mathbb{C}$ .

For example, we define a Stokes filtration on  $\mathbf{k}_{S^1}$  by  $\mathbf{k}_{S^1 < 0, \theta} = 0$  and  $\mathbf{k}_{S^1 \leq 0, \theta} = \mathbf{k}_{S^1}$  for all  $\theta \in S^1$ . For two Stokes structures  $(\mathcal{L}^i, \mathcal{L}^\bullet_i)$  ( $i = 0, 1$ ), the tensor product  $\mathcal{L}^0 \otimes \mathcal{L}^1$  is equipped with a natural Stokes filtration by

$$(2.2) \quad (\mathcal{L}^0 \otimes \mathcal{L}^1)_{\leq c, \theta} := \sum_{c^0 + c^1 \leq_{\theta} c} \mathcal{L}^0_{\leq c^0, \theta} \otimes \mathcal{L}^1_{\leq c^1, \theta}.$$

Let  $\iota$  be an involution on  $S^1 = \mathbb{R}/2\pi\mathbb{Z}$  defined by  $\iota(\theta) = \theta + \pi$ . If  $(\mathcal{L}, \mathcal{L}_\bullet)$  is a Stokes structure,  $\iota^{-1}\mathcal{L}$  has a natural Stokes filtration by

$$(2.3) \quad (\iota^{-1}\mathcal{L})_{\leq c, \theta} := \mathcal{L}_{\leq -c, \theta + \pi} \quad (c \in \mathbb{C}, \theta \in S^1).$$

The graded quotient  $\text{gr}_c(\iota^{-1}\mathcal{L})$  is naturally identified with  $\iota^{-1}\text{gr}_{-c}\mathcal{L}$  ([23, §3.a]). The dual local system  $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathbf{k}_{S^1})$  also has a natural filtration defined by

$$(2.4) \quad (\mathcal{L}^\vee)_{\leq c} := (\mathcal{L}_{< -c})^\perp$$

where  $(\mathcal{L}_{< c})^\perp$  consists of local morphisms  $\mathcal{L} \rightarrow \mathbf{k}_{S^1}$  which send  $\mathcal{L}_{< c}$  to 0. The graded quotient  $\text{gr}_c(\mathcal{L}^\vee)$  is naturally identified with  $(\text{gr}_{-c}\mathcal{L})^\vee$ . Hence  $\mathbb{D}\mathcal{L} = \iota^{-1}\mathcal{L}^\vee$  has natural Stokes filtration by

$$(2.5) \quad (\mathbb{D}\mathcal{L})_{\leq c} = (\mathcal{L}_{< c, \theta + \pi})^\perp \quad (c \in \mathbb{C}, \theta \in S^1),$$

whose graded quotient  $\text{gr}_c(\mathbb{D}\mathcal{L})$  is naturally identified with  $\mathbb{D}(\text{gr}_c\mathcal{L})$ .

The category of Stokes filtered local systems on  $S^1$  defined above is denoted by  $\text{St}_{\mathbf{k}}(S^1)$ . In this paper we only consider Stokes filtered local systems on  $S^1$ . We abbreviate  $S^1$  and simply denote it by  $\text{St}_{\mathbf{k}}$ . For each  $(\mathcal{L}, \mathcal{L}_\bullet) \in \text{St}_{\mathbf{k}}$ , the set  $\{c \in \mathbb{C} \mid \text{gr}_c\mathcal{L} \neq 0\}$  is called exponents of  $(\mathcal{L}, \mathcal{L}_\bullet)$ . For a finite subset  $C$  in  $\mathbb{C}$ ,  $\text{St}_{\mathbf{k}}^C$  denote the full subcategory of Stokes filtered local systems whose exponents are contained in  $C$ . We remark that  $\mathbb{D}$  defines a contravariant functor from  $\text{St}_{\mathbf{k}}^C$  to itself.

For two distinct points  $c \neq c'$  in  $\mathbb{C}$ , the Stokes direction of the pair is the set of points  $\theta$  in  $S^1$  such that  $\Re(e^{-i\theta}(c - c')) = 0$ . For a finite set  $C \subset \mathbb{C}$ , an open interval  $I \subset \mathbb{R}$  is called  $C$ -good if the image in  $\mathbb{R}/2\pi\mathbb{Z}$  of  $I$  contains exactly one Stokes direction for each pair  $c \neq c'$  in  $C$ . We put  $I_{\theta_0} := ]\theta_0 - \frac{\pi}{2} - \varepsilon, \theta_0 + \frac{\pi}{2} + \varepsilon[$  for fixed  $\theta_0 \in \mathbb{R}$  where  $\varepsilon$  is a sufficiently small positive number. If  $I_{\theta_0}$  is  $C$ -good for sufficiently small  $\varepsilon > 0$ ,  $\theta_0$  is called  $C$ -generic. It is equivalent to the condition that  $\theta_0 + \pi/2$  is not the Stokes direction of any pair  $c \neq c'$  in  $C$ .

PROPOSITION 2.2 ([23, Proposition 2.2]).

- (1) On any  $C$ -good open interval  $I \subset \mathbb{R}$ , there exists a unique splitting  $\eta|_I : \mathcal{L}|_I \xrightarrow{\sim} \text{gr}\mathcal{L}|_I$  compatible with the Stokes filtrations where  $I$  is the image of  $I$  in  $\mathbb{R}/2\pi\mathbb{Z}$ .
- (2) Let  $\lambda : (\mathcal{L}, \mathcal{L}_\bullet) \rightarrow (\mathcal{L}', \mathcal{L}'_\bullet)$  be a morphism in  $\text{St}_{\mathbf{k}}^C$ . Then, for any  $C$ -good open interval  $I \subset \mathbb{R}$ , the restriction  $\lambda|_I$  is graded with respect to the splittings in (1). In other words, the induced morphism  $\lambda_{c,c'} : \text{gr}_c \mathcal{L} \rightarrow \text{gr}_{c'} \mathcal{L}$  is zero for any pair  $c \neq c'$  in  $C$ .

Let  $\text{Rep}_{\mathbb{Z}}(\mathbf{k})$  be the category of finite dimensional representations of the fundamental group  $\pi_1(S^1) \simeq \mathbb{Z}$  over  $\mathbf{k}$ . This category is considered as the category of pairs  $(V, T)$  of a finite dimensional  $\mathbf{k}$ -vector space  $V$  and an automorphism  $T$  on  $V$ . Let  $C$  be a finite set and  $\tau : C \xrightarrow{\sim} \{1, 2, \dots, m\}$  be a bijection.

DEFINITION 2.3. — Let  $(V, T)$  be an object in  $\text{Rep}_{\mathbb{Z}}(\mathbf{k})$ . Stokes data on  $(V, T)$  of type  $(C, \tau)$  are a family of objects  $(V_c, T_c)_{c \in C}$  in  $\text{Rep}_{\mathbb{Z}}(\mathbf{k})$ , isomorphisms  $f : \bigoplus_{c \in C} V_c \rightarrow V$ , and  $f^* : V \xrightarrow{\sim} \bigoplus_{c \in C} V_c$  of vector spaces with the following properties:

- (1) Let  $f_c : V_c \rightarrow V$  be the composition of  $f$  and the natural inclusion  $V_c \hookrightarrow \bigoplus_{c' \in C} V_{c'}$ . Let  $f_c^* : V \rightarrow V_c$  be the composition of the projection  $\bigoplus_{c' \in C} V_{c'} \twoheadrightarrow V_c$  and  $f^*$ . Then,  $f_{c'}^* \circ f_c = 0$  if  $\tau(c') < \tau(c)$ , and  $f_c^* \circ f_c = \text{id}_{V_c}$  for all  $c \in C$ .
- (2) Set  $f_c^! := T_c^{-1} \circ f_c^* \circ T$  for  $c \in C$ . Then  $f_{c'}^! \circ f_c = 0$  for  $\tau(c) < \tau(c')$ , and  $f_c^! \circ f_c = \text{id}_{V_c}$  for all  $c \in C$ .

Let  $((V^{(a)}, T^{(a)}), (V_c^{(a)}, T_c^{(a)})_{c \in C}, f^{(a)}, f^{*(a)})$  ( $a = 0, 1$ ) be two objects in  $\text{Rep}_{\mathbb{Z}}(\mathbf{k})$  with Stokes data of type  $(C, \tau)$ . A morphism

$$g : (V^0, T^{(0)}) \rightarrow (V^{(1)}, T^{(1)})$$

is compatible with the Stokes data when the induced maps  $(f^{(1)})^{-1} \circ g \circ f^{(0)}$  and  $f^{*(1)} \circ g \circ (f^{*(0)})^{-1}$  are graded.

Remark 2.4. — We do not need to assume that  $f^*$  is an isomorphism since it is deduced from the condition (1) and the condition that  $f$  is an isomorphism. Similarly, we can show that  $f^! := \prod_c f_c^! : V \rightarrow \bigoplus_c V_c$  is an isomorphism.

Let  $\text{Std}_{\mathbf{k}}(C, \tau)$  denote the category of representations in  $\text{Rep}_{\mathbb{Z}}(\mathbf{k})$  with Stokes data of type  $(C, \tau)$  defined above (a morphism in this category is defined as a morphism in  $\text{Rep}_{\mathbb{Z}}(\mathbf{k})$  compatible with the Stokes data). Fix an  $m$ -point set  $C \subset \mathbb{C}$  and  $C$ -generic point  $\theta_\circ \in \mathbb{R}$ . Then we have a unique

bijection  $\tau_{\theta_0} : C \xrightarrow{\sim} \{1, 2, \dots, m\}$  with the following property: when we put  $c_i := \tau_{\theta_0}^{-1}(i)$  for  $i = 1, 2, \dots, m$ , we have

$$c_1 <_{\theta_0+\pi/2} c_2 <_{\theta_0+\pi/2} \dots <_{\theta_0+\pi/2} c_m.$$

In the rest of Section 2.1, we shall construct the functor  $\mathfrak{A}_{\theta_0} : \text{St}_{\mathbf{k}}^C \rightarrow \text{Std}_{\mathbf{k}}(C, \tau_{\theta_0})$  and show that  $\mathfrak{A}_{\theta_0}$  gives an equivalence of categories.

To prepare for the construction of the functor, we recall the equivalence  $\text{Loc}_{\mathbf{k}} \xrightarrow{\sim} \text{Rep}_{\mathbf{k}}$  of categories. Let  $p : \mathbb{R} \rightarrow S^1 := \mathbb{R}/2\pi\mathbb{Z}$  be the quotient map. This gives a universal covering of  $S^1$ . For  $\mathcal{L} \in \text{Loc}_{\mathbf{k}}$ , set  $V_{\mathcal{L}} := \Gamma(\mathbb{R}, p^{-1}\mathcal{L})$  and define the map  $T_{\mathcal{L}} : V_{\mathcal{L}} \rightarrow V_{\mathcal{L}}$  by  $T_{\mathcal{L}}v(\theta) := v(\theta + 2\pi)$ . The correspondence  $\mathcal{L} \mapsto (V_{\mathcal{L}}, T_{\mathcal{L}})$  is functorial and we can construct the quasi-inverse functor explicitly, this is an equivalence of categories.

By Proposition 2.2(1), we have a unique splitting

$$\eta_{[I_{\theta_0}]} : \mathcal{L}|_{[I_{\theta_0}]} \xrightarrow{\sim} \text{gr}\mathcal{L}|_{[I_{\theta_0}]}.$$

We put  $\eta_{\theta_0} := (p^{-1}\eta_{[I_{\theta_0}]})|_{I_{\theta_0}} : (p^{-1}\mathcal{L})|_{I_{\theta_0}} \xrightarrow{\sim} (p^{-1}\text{gr}\mathcal{L})|_{I_{\theta_0}}$ , which is an isomorphism of sheaves on  $I_{\theta_0} \subset \mathbb{R}$ .

For an object  $(\mathcal{L}, \mathcal{L}_{\bullet}) \in \text{St}_{\mathbf{k}}^C$ , we have  $(V_{\mathcal{L}}, T_{\mathcal{L}})$  and

$$(V_c, T_c) := (V_{\text{gr}_c\mathcal{L}}, T_{\text{gr}_c\mathcal{L}})$$

for  $c \in C$ . Let  $r_{\text{gr}\mathcal{L}}^{\theta} : \bigoplus_c V_c \xrightarrow{\sim} \Gamma(I_{\theta}, p^{-1}\text{gr}\mathcal{L})$  and  $r_{\mathcal{L}}^{\theta} : V_{\mathcal{L}} \xrightarrow{\sim} \Gamma(I_{\theta}, p^{-1}\mathcal{L})$  be the restriction maps for  $\theta \in \mathbb{R}$ . We define  $f_{\theta_0} : \bigoplus_{c \in C} V_c \xrightarrow{\sim} V$  as a composition:

$$f_{\theta_0} := (r_{\mathcal{L}}^{\theta_0})^{-1} \circ \Gamma(I_{\theta_0}, \eta_{\theta_0}^{-1}) \circ r_{\text{gr}\mathcal{L}}^{\theta_0}.$$

Similarly, we define  $f_{\theta_0}^* : V \xrightarrow{\sim} \bigoplus_{c \in C} V_c$  as a composition:

$$f_{\theta_0}^* := (r_{\text{gr}\mathcal{L}}^{\theta_0-\pi})^{-1} \circ \Gamma(I_{\theta_0-\pi}, \eta_{\theta_0-\pi}) \circ r_{\mathcal{L}}^{\theta_0-\pi}.$$

Then,  $f_{\theta_0}^! := \bigoplus_c T_c^{-1} \circ f_{\theta_0}^* \circ T$  is described as a composition:

$$f_{\theta_0}^! := (r_{\text{gr}\mathcal{L}}^{\theta_0+\pi})^{-1} \circ \Gamma(I_{\theta_0+\pi}, \eta_{\theta_0+\pi}) \circ r_{\mathcal{L}}^{\theta_0+\pi}.$$

PROPOSITION 2.5. — *The data  $((V_c, T_c)_{c \in C}, f_{\theta_0}, f_{\theta_0}^*)$  are Stokes data on  $(V_{\mathcal{L}}, T_{\mathcal{L}})$  of type  $(C, \tau_{\theta_0})$ . Moreover, the correspondence*

$$(\mathcal{L}, \mathcal{L}_{\bullet}) \mapsto ((V_{\mathcal{L}}, T_{\mathcal{L}}), (V_c, T_c)_{c \in C}, f_{\theta_0}, f_{\theta_0}^*),$$

and  $\lambda \mapsto \Gamma(\mathbb{R}, p^{-1}\lambda)$  (where  $\lambda$  is a morphism in  $\text{St}_{\mathbf{k}}^C$ ) gives a fully faithful functor  $\mathfrak{A}_{\theta_0} : \text{St}_{\mathbf{k}}^C \rightarrow \text{Std}_{\mathbf{k}}(C, \tau_{\theta_0})$ .

*Proof.* — Let  $r^{\theta, \theta'} : V_{\text{gr}\mathcal{L}} \rightarrow \Gamma(I_{\theta} \cap I_{\theta'}, \text{gr}\mathcal{L})$  be the restriction map for  $\theta, \theta' \in \mathbb{R}$ . The composition  $f_{\theta_0}^* \circ f_{\theta_0}$  can be described as the composition:

$$(r^{\theta_0, \theta_0-\pi})^{-1} \circ \Gamma(I_{\theta_0} \cap I_{\theta_0-\pi}, \eta_{\theta_0-\pi}) \circ \Gamma(I_{\theta_0} \cap I_{\theta_0-\pi}, \eta_{\theta_0}^{-1}) \circ r^{\theta_0, \theta_0-\pi}.$$

The fact that  $\eta_{\theta_\circ - \pi}$  and  $\eta_{\theta_\circ}$  are splitting implies  $f_{\theta_\circ, c}^* \circ f_{\theta_\circ, c} = \text{id}_{V_c}$ . Since  $\eta_{\theta_\circ - \pi}$  and  $\eta_{\theta_\circ}$  are compatible with filtration, we have  $f_{\theta_\circ, c'}^* \circ f_{\theta_\circ, c} = 0$  if  $c <_{\theta_\circ - \pi/2} c'$ . This proves the condition (1) of the Definition 2.3 because  $c <_{\theta_\circ - \pi/2} c'$  is equivalent to  $\tau_{\theta_\circ}(c') < \tau_{\theta_\circ}(c)$ . Similarly, the composition  $f_{\theta_\circ}^! \circ f_{\theta_\circ}$  is described as the following composition:

$$(r^{\theta_\circ, \theta_\circ + \pi})^{-1} \circ \Gamma(I_{\theta_\circ} \cap I_{\theta_\circ + \pi}, \eta_{\theta_\circ + \pi}) \circ \Gamma(I_{\theta_\circ} \cap I_{\theta_\circ + \pi}, \eta_{\theta_\circ}^{-1}) \circ r^{\theta_\circ, \theta_\circ + \pi}.$$

Since  $c <_{\theta_\circ + \pi/2} c'$  is equivalent to  $\tau_{\theta_\circ}(c) < \tau_{\theta_\circ}(c')$ , we have the condition (2) of Definition 2.3.

Let  $(\mathcal{L}, \mathcal{L}_\bullet)$  and  $(\mathcal{L}', \mathcal{L}'_\bullet)$  be objects in  $\text{St}_k^C$ . Let  $\lambda : \mathcal{L} \rightarrow \mathcal{L}'$  be a morphism of local systems. It induces a morphism

$$g := \Gamma(\mathbb{R}, p^{-1}\lambda) : (V_{\mathcal{L}}, T_{\mathcal{L}}) \rightarrow (V_{\mathcal{L}'}, T_{\mathcal{L}'}).$$

Proposition 2.2(2) implies that  $\lambda$  is compatible with Stokes filtration if and only if  $g$  is compatible with Stokes data constructed above. This shows that the correspondence gives a fully faithful functor

$$\mathfrak{A}_{\theta_\circ} : \text{St}_k^C \rightarrow \text{Std}_k(C, \tau_{\theta_\circ}). \quad \square$$

We shall prove the essential surjectivity of  $\mathfrak{A}_{\theta_\circ}$ . We recall a classification result of Stokes filtered local systems with a fixed graded Stokes structure. For a Stokes filtered local system  $(\mathcal{L}, \mathcal{L}_\bullet)$ ,  $\mathcal{E}nd(\mathcal{L})$  is equipped with a natural Stokes filtration. We put

$$\text{Aut}^{<0}(\mathcal{L}) := \text{id}_{\mathcal{L}} + \mathcal{E}nd(\mathcal{L})_{<0} \subset \mathcal{E}nd(\mathcal{L})_{\leq 0}.$$

A local section of  $\text{Aut}^{<0}(\mathcal{L})$  is a local endomorphism  $\lambda : \mathcal{L} \rightarrow \mathcal{L}$  of the local system which is compatible with the Stokes filtration such that the graded morphism  $\text{gr}(\lambda)$  is identity. For a set of local systems  $(\mathcal{G}_c)_{c \in C}$  indexed by  $C$ , put  $\mathcal{G} := \bigoplus_{c \in C} \mathcal{G}_c$  and define a Stokes filtration on  $\mathcal{G}$  by  $\mathcal{G}_{\leq c, \theta} := \bigoplus_{c' \leq_{\theta} c} \mathcal{G}_{c'}$ .

LEMMA 2.6 ([51, Proposition 1.42]). — *The set of isomorphism classes of Stokes filtered local systems  $(\mathcal{L}, \mathcal{L}_\bullet)$  with an isomorphism from the graded part  $\text{gr}\mathcal{L}$  to  $\mathcal{G}$  is identified with  $H^1(S^1; \text{Aut}^{<0}(\mathcal{G}))$ .*

*Proof.* — The proof is standard and shown in more general contexts. Here, we only give the construction of the Stokes filtered local system from a class  $\alpha \in H^1(S^1; \text{Aut}^{<0}(\mathcal{G}))$ . Let  $(I_k)_{k=1}^\ell$  be a covering on  $S^1$  by open intervals such that  $I_k \cap I_{k'} = \emptyset$  if  $|k - k'| \neq 1, \ell - 1$ . Then  $\alpha$  is represented by a Čech cocycle  $(\alpha_k)_{k=1}^\ell$  where  $\alpha_k \in H^0(I_k \cap I_{k+1}; \text{Aut}^{<0}(\mathcal{G}))$  for  $k = 1, 2, \dots, \ell - 1$  and  $\alpha_\ell \in H^0(I_\ell \cap I_1; \text{Aut}^{<0}(\mathcal{G}))$ . Then, by gluing, there exists a unique (up to isomorphisms) local system  $\mathcal{L}$  with isomorphisms  $\beta_k : \mathcal{L}|_{I_k} \xrightarrow{\sim} \mathcal{G}|_{I_k}$  such that  $\alpha_k = \beta_{k+1}(\beta_k)^{-1}$  for  $k = 1, 2, \dots, \ell - 1$  and

$\alpha_\ell = \beta_1 \circ (\beta_\ell)^{-1}$ . The Stokes filtration on  $\mathcal{L}$  is defined via  $\beta_k$  and it is well defined and its graded part  $\text{gr}\mathcal{L}$  is isomorphic to  $\mathcal{G}$  since  $\alpha_k$  is a local section of  $\text{Aut}^{<0}(\mathcal{G})$ . □

**THEOREM 2.7.** — *The functor  $\mathfrak{A}_{\theta_\circ} : \text{St}_{\mathbf{k}}^C \rightarrow \text{Std}_{\mathbf{k}}(C, \tau_{\theta_\circ})$  constructed in Proposition 2.5 is essentially surjective, and hence gives an equivalence of categories.*

*Proof.* — Let  $((V, T), (V_c, T_c)_{c \in C}, f, f^*)$  be an object in  $\text{Std}_{\mathbf{k}}(C, \tau_{\theta_\circ})$ . Let  $\mathcal{G}_c$  be the local system corresponding to  $(V_c, T_c)$ . Set  $\mathcal{G} := \bigoplus_{c \in C} \mathcal{G}_c$ . Define a local endomorphism  $\lambda^* : \mathcal{G}|_{p(I_{\theta_\circ - \pi} \cap I_{\theta_\circ})} \rightarrow \mathcal{G}|_{p(I_{\theta_\circ - \pi} \cap I_{\theta_\circ})}$  by the following composition:

$$\Gamma(I_{\theta_\circ - \pi} \cap I_{\theta_\circ}, p^{-1}\mathcal{G}) \xleftarrow{\sim} \bigoplus_c V_c \xrightarrow{f^* \circ f} \bigoplus_c V_c \xrightarrow{\sim} \Gamma(I_{\theta_\circ - \pi} \cap I_{\theta_\circ}, p^{-1}\mathcal{G})$$

where the first and the third isomorphisms are the restriction maps regarding  $V_c$  as  $\Gamma(\mathbb{R}, p^{-1}\mathcal{G}_c)$ . By Definition 2.3(1),  $\lambda^*$  is a local section of  $\text{Aut}^{<0}(\mathcal{G})$  on  $p(I_{\theta_\circ - \pi} \cap I_\theta)$ . Similarly, we can define a local endomorphism  $\lambda^! : \mathcal{G}|_{p(I_{\theta_\circ - \pi} \cap I_\theta)} \rightarrow \mathcal{G}|_{p(I_{\theta_\circ - \pi} \cap I_\theta)}$  by using  $f^! \circ f$  and it defines a local section of  $\text{Aut}^{<0}(\mathcal{G})$  on  $p(I_{\theta_\circ - \pi} \cap I_\theta)$ . The pair  $(\lambda^*, \lambda^!)$  defines a class in  $H^1(S^1; \text{Aut}^{<0}(\mathcal{G}))$  as Čech cocycle for the covering  $S^1 = p(I_{\theta_\circ}) \cup p(I_{\theta_\circ + \pi})$ . By Lemma 2.6, it defines a Stokes filtered local system  $(\mathcal{L}, \mathcal{L}_\bullet)$ . The fact that  $\mathfrak{A}_{\theta_\circ}(\mathcal{L}, \mathcal{L}_\bullet) \simeq ((V, T), (V_c, T_c)_{c \in C}, f, f^*)$  is obvious by the construction of  $\mathfrak{A}_{\theta_\circ}$  and the proof of Lemma 2.6. □

*Remark 2.8.* — By this theorem, the notion of Stokes data introduced here is equivalent to the notion of Stokes data introduced in [23].

### 2.2. Mutations on Stokes data

**DEFINITION 2.9.** — *Let  $((V, T), (V_c, T_c)_{c \in C}, f, f^*)$  be an object in the category  $\text{Std}_{\mathbf{k}}(C, \tau)$ . For each  $i \in \{1, 2, \dots, m\}$ , we define the endomorphisms  $R_i, R_i^*, L_i, L_i^!$  on  $V$  as follows:*

$$\begin{aligned} R_i &:= \text{id}_V - f_{\tau^{-1}(i)} \circ f_{\tau^{-1}(i)}^*, \\ R_i^* &:= \text{id}_V - T \circ f_{\tau^{-1}(i)} \circ T_{\tau^{-1}(i)}^{-1} \circ f_{\tau^{-1}(i)}^*, \\ L_i &:= \text{id}_V - f_{\tau^{-1}(i)} \circ f_{\tau^{-1}(i)}^!, \\ L_i^! &:= \text{id}_V - T^{-1} \circ f_{\tau^{-1}(i)} \circ T_{\tau^{-1}(i)} \circ f_{\tau^{-1}(i)}^!. \end{aligned}$$

We remark that  $T^{-1} \circ R_i^* \circ T = L_i$  and  $T \circ L_i^! \circ T^{-1} = R_i$ . We also have  $R_i^* \circ R_i = R_i \circ R_i = R_i$ , and  $L_i^! \circ L_i = L_i \circ L_i = L_i$  by easy computation.

PROPOSITION 2.10. — *Let  $((V, T), (V_c, T_c)_{c \in C}, f, f^*)$  be an object in  $\text{Std}_{\mathbf{k}}(C, \tau)$ . For  $i \in \{2, \dots, m\}$ , we define*

$$\begin{aligned}
 (\mathbb{R}_i f)_c &:= \begin{cases} R_i \circ f_c & (\tau(c) = i - 1) \\ f_c & \text{otherwise,} \end{cases} \\
 (\mathbb{R}_i f^*)_c &:= \begin{cases} f_c^* \circ R_i^* & (\tau(c) = i - 1) \\ f_c^* & \text{otherwise.} \end{cases}
 \end{aligned}$$

*Then the tuple  $((V, T), (V_c, T_c)_{c \in C}, \mathbb{R}_i f, \mathbb{R}_i f^*)$  is an object of the category  $\text{Std}_{\mathbf{k}}(C, (i, i - 1) \circ \tau)$ . Similarly, for  $i \in \{1, \dots, m - 1\}$ , we define*

$$\begin{aligned}
 (\mathbb{L}_i f)_c &:= \begin{cases} L_i \circ f_c & (\tau(c) = i + 1) \\ f_c & \text{otherwise,} \end{cases} \\
 (\mathbb{L}_i f^*)_c &:= \begin{cases} f_c^* \circ T \circ L_i^! \circ T^{-1} & (\tau(c) = i + 1) \\ f_c^* & \text{otherwise.} \end{cases}
 \end{aligned}$$

*Then the tuple  $((V, T), (V_c, T_c)_{c \in C}, \mathbb{L}_i f, \mathbb{L}_i f^*)$  is an object of the category  $\text{Std}_{\mathbf{k}}(C, (i, i + 1) \circ \tau)$ .*

*Proof.* — We show the first half of the proposition. The second half is shown similarly. Let  $c, c'$  be two distinct elements of  $C$ . We only consider the case where  $\tau(c) = i - 1$  and  $\tau(c') = i$  since the discussion is easier in the other cases.

We first show that  $(\mathbb{R}_i f)_c = R_i \circ f_c$  is injective. Let  $u$  be any vector in  $V_c$  and assume  $R_i f_c u = 0$ . It implies  $f_c u = f_{c'} \circ f_{c'}^* \circ f_c u$ . Next, we show that the intersection of the image  $(\mathbb{R}_i f)_c(V_c)$  and  $f_{c'} V_{c'}$  is  $(0)$ . Let  $v$  be a vector in  $V_c$  such that  $\mathbb{R}_i f v \in V_{c'}$ . It implies that  $f_c v - f_{c'} f_{c'}^* f v \in V_{c'}$ . Hence  $f_c v \in V_{c'}$ , and it implies  $v = 0$ . The first and the second assertions show that  $\mathbb{R}_i f$  defines an isomorphism of vector spaces.

Finally, we show that  $(\mathbb{R}_i f, \mathbb{R}_i f^*)$  satisfies the conditions (1) and (2) in Definition 2.3. We only need to consider the case where  $\tau(c) = i - 1$  and  $\tau(c') = i$ . We have

$$(2.6) \quad (\mathbb{R}_i f^*)_c \circ (\mathbb{R}_i f)_c = (f_c^* - f_c^* T f_{c'} T_{c'}^{-1} f_{c'}^*)(f_c - f_{c'} f_{c'}^* f_c)$$

$$(2.7) \quad = f_c^* f_c - f_c^* T f_{c'} T_{c'}^{-1} f_{c'}^* f_c + f_c^* T f_{c'} T_{c'}^{-1} f_{c'}^* f_c$$

$$(2.8) \quad = \text{id}_{V_c} .$$

Here, we used  $f_c^* f_{c'} = 0$  from (2.6) to (2.7) and  $f_c^* f_c = \text{id}$  ( $f_{c'}^* f_{c'} = \text{id}$ ) from (2.7) to (2.8). We also have

$$\begin{aligned} (\mathbb{R}_i f^*)_{c'} \circ (\mathbb{R}_i f)_c &= f_{c'}^* \circ (f_c - f_{c'} f_{c'}^* f_c) \\ &= 0. \end{aligned}$$

by using  $f_{c'}^* f_{c'} = \text{id}$ . They show the condition (1). The condition (2) is shown similarly, which proves the proposition (see Remark 2.4).  $\square$

We put  $\sigma_i : \text{Std}_{\mathbf{k}}(C, \tau) \rightarrow \text{Std}_{\mathbf{k}}(C, (i \ i + 1) \circ \tau)$  to be the functor  $((V, T), (V_c, T_c)_{c \in C}, f, f^*) \mapsto ((V, T), (V_c, T_c)_{c \in C}, \mathbb{R}_{i+1} f, \mathbb{R}_{i+1} f^*)$  and put  $\sigma_i^{-1} : \text{Std}_{\mathbf{k}}(C, \tau) \rightarrow \text{Std}_{\mathbf{k}}(C, (i \ i + 1) \circ \tau)$  to be the functor

$$((V, T), (V_c, T_c)_{c \in C}, f, f^*) \mapsto ((V, T), (V_c, T_c)_{c \in C}, \mathbb{L}_i f, \mathbb{L}_i f^*).$$

These functors act trivially on the morphisms. It can easily be checked that  $\sigma_i, \sigma_i^{-1}$  actually define functors.

PROPOSITION 2.11. — *The functors  $\{\sigma_i, \sigma_i^{-1}\}_{i=1}^{m-1}$  satisfy the following braid relations.*

- (1)  $\sigma_i \circ \sigma_i^{-1} = \text{id}, \sigma_i \circ \sigma_i^{-1} = \text{id},$
- (2)  $\sigma_i \circ \sigma_{i+1} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1},$  and
- (3)  $\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i \quad (|i - j| \geq 2).$

In particular, by the relation (1),  $\sigma_i$  and  $\sigma_i^{-1}$  are equivalences of categories.

*Proof.* — The component  $(\mathbb{L}_i \mathbb{R}_{i+1} f)_c$  is the composition  $L_{i+1} R_{i+1} f_c$  if  $\tau(c) = i$ , otherwise  $f_c$ . Easy calculation shows that  $L_{i+1} R_{i+1} f_c = f_c$ . Similarly, we have  $\mathbb{L}_i \mathbb{R}_{i+1} f^* = f^*$ . This implies  $\sigma_i^{-1} \circ \sigma_i = \text{id}$ . The relation  $\sigma_i \circ \sigma_i^{-1} = \text{id}$  is shown similarly. Hence we get (1).

We shall prove (2). Let

$$((V, T), (V_c, T_c)_{c \in C}, f, f^*)$$

be an object of  $\text{Std}_{\mathbf{k}}(C, \tau)$ . Then we have the following:

$$\begin{aligned} (\mathbb{R}_{i+1} \mathbb{R}_{i+2} \mathbb{R}_{i+1} f)_c &= \begin{cases} R_{i+2} R_{i+1} f_c & \tau(c) = i \\ R_{i+1} f_c & \tau(c) = i + 1 \\ f_c & \text{otherwise,} \end{cases} \\ (\mathbb{R}_{i+2} \mathbb{R}_{i+1} \mathbb{R}_{i+2} f)_c &= \begin{cases} R'_{i+2} R_{i+2} f_c & \tau(c) = i \\ R_{i+1} f_c & \tau(c) = i + 1 \\ f_c & \text{otherwise,} \end{cases} \end{aligned}$$

where  $R'_{i+2} := \text{id}_V - R_{i+2} f_{\tau^{-1}(i+1)} f_{\tau^{-1}(i+1)}^* R_{i+2}^*$ . Put  $R'_{i+1} := R_{i+2}$ . Then we have  $R'_{i+2} \circ R'_{i+1} = R_{i+2} \circ R_{i+1}$ , which implies  $\mathbb{R}_{i+1} \mathbb{R}_{i+2} \mathbb{R}_{i+1} f = \mathbb{R}_{i+2} \mathbb{R}_{i+1} \mathbb{R}_{i+2} f$ . Indeed, we have

$$\begin{aligned} R'_{i+2} \circ R'_{i+1} &= (\text{id} - R_{i+2} \circ (\text{id} - R_{i+1}) \circ R_{i+2}^*) \circ R_{i+2} \\ &= R_{i+2} - R_{i+2} \circ (\text{id} - R_{i+1}) \circ R_{i+2} \\ &= R_{i+2} \circ R_{i+1} \circ R_{i+2}, \end{aligned}$$

and

$$\begin{aligned} R_{i+2} \circ R_{i+1} \circ (\text{id} - R_{i+2}) &= R_{i+2} \circ (\text{id} - f_{\tau^{-1}(i+1)} \circ f_{\tau^{-1}(i+1)}^*) \circ (f_{\tau^{-1}(i+2)} \circ f_{\tau^{-1}(i+2)}^*) \\ &= R_{i+2} \circ (f_{\tau^{-1}(i+2)} \circ f_{\tau^{-1}(i+2)}^*) \\ &= R_{i+2} \circ (\text{id} - R_{i+2}) = 0. \end{aligned}$$

The relation  $\mathbb{R}_{i+2} \mathbb{R}_{i+1} \mathbb{R}_{i+2} f^* = \mathbb{R}_{i+1} \mathbb{R}_{i+2} \mathbb{R}_{i+1} f^*$  is shown similarly. The relation (3) is obvious by the definition. □

Let  $\text{Br}_m$  denote the braid group of  $m$ -strands with the standard generator  $\sigma_1, \dots, \sigma_{m-1}$ . For an element  $\sigma \in \text{Br}_m$ , let  $\bar{\sigma}$  denote the image of  $\sigma$  via the quotient map  $\text{Br}_m \rightarrow S_m$  to the symmetric group of degree  $m$ . In particular,  $s_i = \bar{\sigma}_i$  are the permutations  $s_i = (i \ i + 1)$ , ( $i = 1, 2, \dots, m - 1$ ).

DEFINITION 2.12. — For  $\sigma \in \text{Br}_m$ , define a functor

$$\mathbb{M}_\sigma : \text{Std}_k(C, \tau) \xrightarrow{\sim} \text{Std}_k(C, \bar{\sigma} \circ \tau)$$

as the composition of mutations defined above.

### 2.3. Stokes factors and Mutations

Fix a finite subset  $C \subset \mathbb{C}$  and a bijection  $\tau : C \xrightarrow{\sim} \{1, \dots, m\}$ . Let  $((V, T), (V_c, T_c)_{c \in C}, f, f^*)$  be an object of  $\text{Std}(C, \tau)$ .

DEFINITION 2.13. — We define a map  $(\cdot)_R : S_m \rightarrow \text{Br}_m$  by  $(s)_R := \sigma_{i_1} \cdots \sigma_{i_k}$  for a reduced expression  $s = s_{i_1} \cdots s_{i_k}$ . This map is well defined (see, for example, [5, Chapter 4, §1.5, Proposition 5], [41, Theorem 2]).

For  $s \in S_m$ , set  $I_i(s) := \{j \in \{1, \dots, m\} \mid i < j, s(i) > s(j)\}$ , and  $I(s) := \{(i, j) \mid j \in I_i(s)\}$ .

*Notation 2.14.* — Let  $A = \{a_1, a_2, \dots, a_k\}$  be a finite ordered set with  $a_1 < a_2 < \dots < a_k$ . Let  $(h_a)_{a \in A}$  be a sequence of endomorphisms of  $V$  indexed by  $A$ . Then, we use the following notation:

$$\prod_{a \in A}^{\leftarrow} h_a := h_{a_k} \circ h_{a_{k-1}} \circ \dots \circ h_{a_1}, \quad \prod_{a \in A}^{\rightarrow} h_a := h_{a_1} \circ h_{a_2} \circ \dots \circ h_{a_k}.$$

Recall that for  $\sigma \in \text{Br}_m$ , we have  $\mathbb{M}_\sigma : \text{Std}_{\mathbf{k}}(C, \tau) \xrightarrow{\sim} \text{Std}_{\mathbf{k}}(C, \bar{\sigma} \circ \tau)$  (Definition 2.12). Define  $\sigma f$  and  $\sigma f^*$  by  $\mathbb{M}_\sigma((V, T), (V_c, T_c)_{c \in C}, f, f^*) = ((V, T), (V_c, T_c)_{c \in C}, \sigma f, \sigma f^*)$ .

LEMMA 2.15. — For  $s \in S_m$ , we have the following formulas:

$$(2.9) \quad ((s)_R f)_c = \left( \prod_{i \in I_{\tau(c)}(s)}^{\leftarrow} R_i \right) \circ f_c,$$

$$(2.10) \quad ((s)_R f^*)_c = f_c^* \circ \left( \prod_{i \in I_{\tau(c)}(s)}^{\rightarrow} R_i^* \right).$$

*Proof.* — For  $s \in S_m$ , let  $\ell(s)$  be the length of  $s$ . We shall show the lemma by induction on  $\ell(s)$ . If  $\ell(s) = 0$ ,  $s$  is identity and the lemma is obvious. If  $\ell(s) > 0$ , then there exists some  $s_j$  such that  $s = s' \cdot s_j$  with  $\ell(s) = \ell(s') + 1$ . Set

$$f'_c := \begin{cases} f_c & (\tau(c) \neq j) \\ R_{j+1} \circ f_c & (\tau(c) = j), \end{cases}$$

$$R'_i := \begin{cases} R_i & (i \neq j, j + 1) \\ R_{j+1} & (i = j) \\ \text{id} - R_{j+1} \circ f_{c_j} \circ f_{c_j}^* \circ R_{j+1}^* & (i = j + 1), \end{cases}$$

where  $c_j = \tau^{-1}(j)$ . By induction hypothesis, we have

$$((s)_R f)_c = \left( \prod_{i \in I_{s_j \circ \tau(c)}(s')}^{\leftarrow} R'_i \right) \circ f'_c.$$

Since  $\ell(s) = \ell(s') + 1$ , we have  $j + 1 \notin I_j(s')$ , and

$$I_k(s) = \begin{cases} s_j(I_{s_j(k)}(s')) & k \neq j \\ s_j(I_{s_j(k)}(s')) \cup \{j + 1\} & k = j. \end{cases}$$

If  $j + 1 \notin I_{s_j \circ \tau(c)}(s')$ , then

$$\prod_{i \in I_{s_j \circ \tau(c)}(s')}^{\leftarrow} R'_i = \prod_{i \in s_j(I_{s_j \circ \tau(c)}(s'))}^{\leftarrow} R_i,$$

hence we have (2.9). If  $j + 1 \in I_{s_j \circ \tau(c)}(s')$ , then  $j \in I_{s_j \circ \tau(c)}(s')$  since  $s'(j) < s'(j + 1)$  by  $j + 1 \notin I_j(s')$ . By the proof of Proposition 2.11, we have  $R'_{j+1} \circ R'_j = R_{j+1} \circ R_j$ . This implies (2.9). The equation (2.10) is shown similarly.  $\square$

As a special case, we have the following.

PROPOSITION 2.16. — Set  $\Delta := (w_0)_R \in \text{Br}_m$  where  $w_0 \in S_m$  is the longest element, i.e.,

$$w_0 = \begin{pmatrix} 1 & 2 & \cdots & m \\ m & m-1 & \cdots & 1 \end{pmatrix}.$$

Then we have  $\Delta f = (f^*)^{-1}$ , and  $\Delta f^* = (\bigoplus_c T_c) \circ f^{-1} \circ T^{-1}$ .

*Proof.* — Set  $\Delta f^! := (\bigoplus_c T_c)^{-1} \circ \Delta f^* \circ T$ . We show  $\Delta f \circ f^* \circ f = f$ , and  $f^! \circ f \circ \Delta f^! = f^!$ . We use the notation  $c_i = \tau^{-1}(i)$  ( $i = 1, \dots, m$ ). Then the restriction  $\Delta f \circ f^* \circ f|_{V_{c_i}}$  is

$$\begin{aligned} & \sum_{i \leq j} R_m \circ \cdots \circ R_{j+1} \circ f_{c_j} \circ f_{c_j}^* \circ f_{c_i} \\ &= R_m \circ \cdots \circ R_{i+1} \circ f_{c_i} + \sum_{i < j} R_m \circ \cdots \circ R_{j+1} \circ f_{c_j} \circ f_{c_j}^* \circ f_{c_i} \\ &= R_m \circ \cdots \circ R_{i+1} \circ f_{c_i} + \sum_{i < j \leq m} R_m \circ \cdots \circ R_{j+1} \circ f_{c_i} \\ &\quad - \sum_{i < j \leq m} R_m \circ \cdots \circ R_j \circ f_{c_i} \\ &= f_{c_i}. \end{aligned}$$

We also see that the composition  $f_{c_i}^! \circ f \circ \Delta f^!$  is

$$\begin{aligned} & \sum_{i \leq j} f_{c_i}^! \circ f_{c_j} \circ f_{c_j}^! \circ L_{j+1} \circ \cdots \circ L_m \\ &= f_{c_i}^! \circ L_{i+1} \circ \cdots \circ L_m + \sum_{i < j \leq m} f_{c_i}^! \circ L_{j+1} \circ \cdots \circ L_m \\ &\quad - \sum_{i < j \leq m} f_{c_i}^! \circ L_j \circ \cdots \circ L_m \\ &= f_{c_i}^!. \end{aligned}$$

This implies the proposition. □

For an endomorphism  $g \in \text{End}(\bigoplus_c V_c)$  and  $(c, c') \in C \times C$ , let  $g_{cc'}$  denote the component  $V_c \rightarrow V_{c'}$ . Using this notation, we define

$$(2.11) \quad \text{End}^\diamond \left( \bigoplus_{c \in C} V_c \right) := \left\{ g \in \text{End} \left( \bigoplus_{c \in C} V_c \right) \mid g_{cc} = \text{id}_{V_c} \text{ for all } c \in C \right\},$$

and

$$(2.12) \quad \text{Sf}_\tau(s) := \left\{ g \in \text{End}^\diamond \left( \bigoplus_{c \in C} V_c \right) \mid \begin{array}{l} g_{cc'} = 0 \text{ for } (\tau(c), \tau(c')) \notin I(s) \\ \text{with } c \neq c' \end{array} \right\},$$

where  $s$  is a element of  $S_m$ , and  $\tau$  is an isomorphism  $\tau : C \xrightarrow{\sim} \{1, \dots, m\}$ .

*Remark 2.17.* — If  $(i, j) \in I(s)$ , then  $i < j$ . If  $(i, j), (j, k) \in I(s)$ , then  $(i, k) \in I(s)$ . These properties imply  $\text{Sf}_\tau(s)$  is a group.

**LEMMA 2.18.** — *Take  $s \in S_m$  and put  $\sigma := (s)_R$ , then  $(\sigma f)^{-1} \circ f \in \text{Sf}_\tau(s)$ .*

*Proof.* — Since  $\text{Sf}_\tau(s)$  is a group, it is enough to show  $f^{-1} \circ \sigma f \in \text{Sf}_\tau(s)$ . We have

$$\begin{aligned} \sigma f|_{V_c} &= \left( \prod_{i \in I_\tau(c)(s)}^{\leftarrow} R_i \right) \circ f_c \\ &= \left( \prod_{(\tau(c), \tau(c')) \in I(s)}^{\leftarrow} (\text{id}_V - f_{c'} \circ f_{c'}^*) \right) \circ f_c. \end{aligned}$$

This implies  $f^{-1} \circ \sigma f \in \text{Sf}_\tau(s)$ . □

### 2.4. Stokes factors and mutations for Stokes filtered local systems

Let  $C$  be a set of  $m$ -points in  $\mathbb{C}$ . Let  $\theta_\circ$  be a  $C$ -generic real number. Recall that we have the bijection  $\tau_{\theta_\circ} : C \xrightarrow{\sim} \{1, 2, \dots, m\}$  such that

$$\tau_{\theta_\circ}(c) < \tau_{\theta_\circ}(c') \Leftrightarrow c <_{\theta_\circ + \pi/2} c'.$$

We put  $c_i := \tau_{\theta_\circ}^{-1}(i)$ . We also have an equivalence of categories

$$\mathfrak{A}_{\theta_\circ} : \text{St}_k^C \rightarrow \text{Std}_k(C, \tau_{\theta_\circ}).$$

We also recall that we have a functor

$$\mathbb{M}_\sigma : \text{Std}_k(C, \tau_{\theta_\circ}) \xrightarrow{\sim} \text{Std}_k(C, \bar{\sigma} \circ \tau_{\theta_\circ})$$

for  $\sigma \in \text{Br}_m$ . The purpose of Section 2.4 is to show the following:

**THEOREM 2.19.** — *Let  $\theta_\circ, \theta'_\circ$  be  $C$ -generic real numbers. Then there exists an element  $\sigma \in \text{Br}_m$  such that  $\mathbb{M}_\sigma \circ \mathfrak{A}_{\theta_\circ} \simeq \mathfrak{A}_{\theta'_\circ}$ . Moreover, if we have  $\theta_\circ \geq \theta'_\circ \geq \theta_\circ - \pi$ , then we can take  $\sigma = (\tau_{\theta'_\circ} \circ \tau_{\theta_\circ}^{-1})_R$ .*

Take an object  $(\mathcal{L}, \mathcal{L}_\bullet) \in \text{St}_k^C$  and put  $((V, T), (V_c, T_c)_{c \in C}, f_\theta, f_\theta^*) := \mathfrak{A}_\theta(\mathcal{L}, \mathcal{L}_\bullet)$  for a  $C$ -generic real number  $\theta \in \mathbb{R}$ .

**DEFINITION 2.20.** — *For a real number  $\theta \in \mathbb{R}$ , we define the following:*

$$\begin{aligned} \text{R}(\theta) &:= \{(c, c') \in C \times C \mid \Im(e^{-i\theta}(c - c')) = 0, \text{ and } \Re(e^{-i\theta}(c - c')) > 0\}, \\ \text{Sf}(\theta) &:= \left\{ g \in \text{End}^\diamond \left( \bigoplus_{c \in C} V_c \right) \mid g_{cc'} = 0 \text{ for } (c, c') \notin \text{R}(\theta) \text{ with } c \neq c' \right\}, \\ \text{Sm}(\theta) &:= \left\{ g \in \text{End}^\diamond \left( \bigoplus_{c \in C} V_c \right) \mid g_{cc'} = 0 \text{ for } c' <_{\theta+\pi/2} c \right\}, \end{aligned}$$

where  $\text{End}^\diamond(\bigoplus_c V_c)$  is defined in (2.11). For a fixed  $c \in C$ , we also define  $\text{R}_c(\theta)$  as the subset of elements  $c'$  of  $C$  such that  $(c, c')$  is in  $\text{R}(\theta)$ .

For  $C$ -generic  $\theta, \theta' \in \mathbb{R}$ , set  $s_{\theta, \theta'} := \tau_{\theta'} \circ \tau_\theta^{-1} \in S_m$ . For  $\theta \in \mathbb{R}$ , set  $s_\theta := s_{\theta+\varepsilon, \theta-\varepsilon}$  for  $0 < \varepsilon \ll 1$ .

**LEMMA 2.21.** — *If  $\theta \geq \theta' \geq \theta - \pi$ , then we have*

$$(2.13) \quad I_{\tau_\theta(c)}(s_{\theta, \theta'}) = \tau_\theta \left( \bigsqcup_{\theta > \theta'' > \theta'} \text{R}_c(\theta'') \right).$$

*Proof.* — The condition  $c' \in \tau_\theta^{-1}(I_{\tau_\theta(c)}(s_{\theta, \theta'}))$  is equivalent to  $c <_{\theta+\pi/2} c', c' <_{\theta'+\pi/2} c$ . Consider the following function  $\text{Im} : \varphi \mapsto \Im(e^{-i\varphi}(c - c'))$ ,  $\varphi \in [\theta', \theta]$ . Then  $c' \in \tau_\theta^{-1}(I_{\tau_\theta(c)}(s_{\theta, \theta'}))$  is equivalent to  $\text{Im}(\theta) < 0 < \text{Im}(\theta')$ . Since  $\theta \geq \theta' \geq \theta - \pi$ , this is equivalent to the existence and uniqueness of  $\theta'' \in ]\theta', \theta[$  such that  $\text{Im}(\theta'') = 0$ , and  $\text{Im}(\theta'' + \varepsilon') < 0 < \text{Im}(\theta'' - \varepsilon')$  for a sufficiently small positive number  $\varepsilon'$ . This is equivalent to  $c' \in \text{R}_c(\theta'')$ . □

**COROLLARY 2.22.** — *We have  $\text{Sf}(\theta) = \text{Sf}_{\tau_{\theta+\varepsilon}}(s_\theta)$  for  $(0 < \varepsilon \ll 1)$ . If  $\theta$  is  $C$ -generic, then we have  $\text{Sm}(\theta) = \text{Sf}_{\tau_\theta}(w_0)$ .*

Fix  $C$ -generic  $\theta_\circ \in \mathbb{R}$ . Take  $\theta_i$  ( $i = 1, \dots, \ell$ ) so that  $\theta_\circ > \theta_1 > \theta_2 > \dots > \theta_\ell > \theta_\circ - \pi$ ,  $\{\theta_1, \dots, \theta_\ell\} = \{\theta \in \mathbb{R} \mid \theta_\circ - \pi < \theta < \theta_\circ \text{ and } \text{R}(\theta) \neq \emptyset\}$ . Then we have

$$\{(c, c') \in C \times C \mid c <_{\theta_\circ+\pi/2} c'\} = \bigsqcup_i \text{R}(\theta_i).$$

LEMMA 2.23 ([1, Lemma 2]). — For all  $g \in \text{Sm}(\theta_o)$ , there exists a unique element  $(g_i)_i \in \prod_{1 \leq i \leq \ell} \text{Sf}(\theta_i)$  such that

$$(2.14) \quad g = g_\ell \circ g_{\ell-1} \circ \cdots \circ g_1,$$

i.e.,  $\text{Sf}(\theta_\ell) \times \cdots \times \text{Sf}(\theta_1) \xrightarrow{\sim} \text{Sm}(\theta_o)$ .

*Proof.* — For a pair  $(i, j)$ ,  $1 \leq i < j \leq m$ , let  $k(i, j)$  be the number such that  $(c_i, c_j) \in \text{R}(\theta_{k(i,j)})$ . Let  $g_{ij}$  be the  $(c_i, c_j)$ -component of  $g_{k(i,j)}$ . Then the  $(c_i, c_j)$ -component of the right hand side of (2.14) is

$$(2.15) \quad \sum_{i=i_0 < i_1 < \cdots < i_a = j} g_{i_{a-1}i_a} \cdots g_{i_0i_1}$$

where  $i = i_0 < i_1 < \cdots < i_a = j$  runs so that

$$\theta_{k(i_0, i_1)} < \theta_{k(i_1, i_2)} < \cdots < \theta_{k(i_{a-1}, i_a)}.$$

Since (2.15) is the sum of  $g_{ij}$  and products of  $g_{i'j'}$  with  $j' - i' < j - i$ , we can uniquely determine  $g_{ij}$  by induction on  $(j - i)$  for given  $g$ . □

Set  $\Delta_{\theta_i} := (s_{\theta_i})_R \in \text{Br}_m$ ,  $(f_o, f_o^*) := (f_{\theta_o}, f_{\theta_o}^*)$ , and

$$(f_i, f_i^*) = (\Delta_{\theta_i} f_{i-1}, \Delta_{\theta_i} f_{i-1}^*).$$

We remark that  $f_i^{-1} \circ f_{i-1} = (\Delta_{\theta_i} f_{i-1})^{-1} f_{i-1} \in \text{Sf}_{\tau_{\theta_i+\varepsilon}}(s_{\theta_i}) = \text{Sf}(\theta_i)$ .

LEMMA 2.24. —  $\Delta_{\theta_\ell} \cdot \Delta_{\theta_{\ell-1}} \cdots \Delta_{\theta_1} = \Delta$ .

*Proof.* — We have  $s_{\theta_\ell} \cdots s_{\theta_1} = \tau_{\theta_o-\pi} \circ \tau_{\theta_o}^{-1} = w_0$ , and  $\sum_{i=1}^\ell \ell(s_{\theta_i}) = \sum_{i=1}^\ell \# \text{R}(\theta_i) = \ell(w_0)$ . This implies the lemma. □

LEMMA 2.25. — Set  $\text{Sf}_{\theta_i} := f_{\theta_i-\varepsilon}^{-1} \circ f_{\theta_i+\varepsilon}$ . Then  $\text{Sf}_{\theta_i}$  is an element of  $\text{Sf}(\theta_i)$ .

*Proof.* — Since  $f_{\theta_i-\varepsilon}$  and  $f_{\theta_i+\varepsilon}$  preserve the filtration on  $I_{\theta_i-\varepsilon} \cap I_{\theta_i+\varepsilon}$ ,  $(\text{Sf}_{\theta_i})_{cc'} = 0$  if there exists a  $\varphi \in I_{\theta_i-\varepsilon} \cap I_{\theta_i+\varepsilon}$  such that  $c <_\varphi c'$ . Therefore, if  $(\text{Sf}_{\theta_i})_{cc'} \neq 0$  and  $c \neq c'$ , then  $\theta_i - \varepsilon < \arg(c - c') < \theta_i + \varepsilon$  which implies  $(c, c') \in \text{R}(\theta_i)$ . Since  $f_{\theta_i+\varepsilon}$  and  $f_{\theta_i-\varepsilon}$  are splitting,  $(\text{Sf}_{\theta_i})_{cc} = \text{id}$ , which implies the lemma. □

LEMMA 2.26. — We have  $\Delta f_{\theta_o} = f_{\theta_o-\pi}$  and  $\Delta f_{\theta_o}^* = f_{\theta_o-\pi}^*$ .

*Proof.* — By definition, we have  $(f_{\theta_o}^*)^{-1} = f_{\theta_o-\pi}$ . Using Proposition 2.16, we have the following:

$$\begin{aligned} (\Delta f_{\theta_o}, \Delta f_{\theta_o}^*) &= \left( (f_{\theta_o}^*)^{-1}, \left( \bigoplus_c T_c \right) \circ f_{\theta_o}^{-1} \circ T^{-1} \right) \\ &= (f_{\theta_o-\pi}, f_{\theta_o-2\pi}^{-1}) \\ &= (f_{\theta_o-\pi}, f_{\theta_o-\pi}^*). \end{aligned}$$

□

LEMMA 2.27. — We have  $f_i = f_{\theta_i - \varepsilon}$  and  $f_i^* = f_{\theta_i - \varepsilon}^*$ .

*Proof.* — For  $\sigma \in \text{Br}_m$ , set  $\widetilde{\sigma f} := (\sigma f)^{-1} \circ f$ . Then we have

$$\begin{aligned} \text{Sf}_{\theta_\ell} \cdots \text{Sf}_{\theta_1} &= f_{\theta_\circ - \pi}^{-1} \circ f_{\theta_\circ} \\ &= \widetilde{\Delta f_{\theta_\circ}} \\ &= \widetilde{\Delta_{\theta_\ell} \cdots \Delta_{\theta_1} f_0} \\ &= \widetilde{\Delta_{\theta_\ell} f_{\ell-1}} \circ \cdots \circ \widetilde{\Delta_{\theta_1} f_0}. \end{aligned}$$

This implies  $\text{Sf}_{\theta_i} = \widetilde{\Delta_{\theta_i} f_{i-1}}$ . Hence we have  $f_i = f_{\theta_i - \varepsilon}$ . Set  $\theta'_\circ := \theta_\circ - \pi$ ,  $\theta'_i := \theta_i - \pi$ ,  $(f'_\circ, f'^*_\circ) := (f_{\theta'_\circ}, f^*_{\theta'_\circ})$ , and  $(f'_i, f'^*_i) := (\Delta_{\theta'_i} f'_{i-1}, \Delta_{\theta'_i} f'^*_{i-1})$ . By the first part of this lemma, we have  $f'_i = f_{\theta'_i - \varepsilon}$ . By Lemma 2.26, we see that

$$(\Delta_{\theta_\ell} \cdots \Delta_{\theta_{i+1}} f_i, \Delta_{\theta_\ell} \cdots \Delta_{\theta_{i+1}} f_i^*) = (f_{\theta'_\circ}, f^*_{\theta'_\circ}),$$

which implies  $f'_i = \Delta_{\theta'_i} \cdots \Delta_{\theta'_1} \cdot \Delta_{\theta_\ell} \cdots \Delta_{\theta_{i+1}} f_i = \Delta f_i$ . Combined with Proposition 2.16, we have  $f_{\theta'_i - \varepsilon} = (f_i^*)^{-1}$ , which proves the lemma. □

COROLLARY 2.28. — If  $\theta, \theta' \in \mathbb{R}$  are  $C$ -generic and  $\theta \geq \theta' \geq \theta - \pi$ , then

$$f_{\theta',c} = \left( \prod_{i \in I_{\tau_\theta(c)}(s_{\theta,\theta'})} \overleftarrow{R}_i \right) \circ f_{\theta,c}, \quad f^*_{\theta',c} = f^*_{\theta,c} \circ \prod_{i \in I_{\tau_\theta(c)}(s_{\theta,\theta'})} \overrightarrow{R}_i^*.$$

*Proof of Theorem 2.19.* — In the case  $\theta \geq \theta' \geq \theta - \pi$ , Theorem 2.19 is a direct consequence of Corollary 2.28. The general case can be reduced to this case. □

### 2.5. Pairings on Stokes filtered local systems and Mutation systems

For  $(V, T) \in \text{Rep}_{\mathbb{Z}}(\mathbf{k})$ , a pairing  $[\cdot, \cdot] : V \otimes V \rightarrow \mathbf{k}$  is called compatible with  $T$  if

$$(2.16) \quad [v, w] = [Tw, v] \text{ for all } v, w \in V.$$

If the map  $[\cdot, \cdot]$  is non-degenerate, then the monodromy  $T$  is determined by the compatibility condition. Here, non-degenerate means that the induced map  $v \mapsto (w \mapsto [v, w])$  is an isomorphism. The condition is equivalent to that the map  $v \mapsto (w \mapsto [w, v])$  is an isomorphism. A pair  $(V, [\cdot, \cdot])$  of a

vector space  $V$  and a non-degenerate pairing  $[\cdot, \cdot]$  is called polarized vector space.

We often assume that the pairing is non-degenerate without a mention. For two polarized vector spaces  $(V, [\cdot, \cdot])$  and  $(V', [\cdot, \cdot]')$ , a linear map  $f : V \rightarrow V'$  is called a morphism of polarized vector spaces if it is compatible with the pairings:  $[v, w] = [fv, fw]'$ .

DEFINITION 2.29. — Let  $((V, T), (V_c, T_c)_{c \in C}, f, f^*)$  be an object of the category  $\text{Std}(C, \tau)$ . A pairing  $[\cdot, \cdot]$  on  $V$  compatible with  $T$  is called compatible with the Stokes data if the following conditions hold:

- The induced map  $[v, w]_c := [f_c v, f_c w]$  ( $v, w \in V_c$ ) is non-degenerate on  $V_c$  and compatible with  $T_c$  for all  $c \in C$ .
- For every  $c \in C$ , the map  $f_c^* : V \rightarrow V_c$  is left adjoint to  $f_c$  in the sense that  $[v, f_c v_c] = [f_c^* v, v_c]_c$  for all  $v \in V, v_c \in V_c$ .

A representation  $(V, T)$  equipped with Stokes data and a compatible pairing is equivalent to the following structure, which we call “mutation systems”.

DEFINITION 2.30. — A mutation system is a tuple

$$((V, [\cdot, \cdot]), (V_c, [\cdot, \cdot]_c)_{c \in C}, \tau, f)$$

consisting of

- (1) a polarized vector space  $(V, [\cdot, \cdot])$ ,
- (2) a family of polarized vector spaces  $(V_c, [\cdot, \cdot]_c)_{c \in C}$  indexed by a finite set  $C$ ,
- (3) a bijection  $\tau : C \xrightarrow{\sim} \{1, 2, \dots, m\}$  (the pair  $(C, \tau)$  is called type of the mutation system), and
- (4) an isomorphism  $f : \bigoplus_{c \in C} V_c \xrightarrow{\sim} V$  of vector spaces

such that  $(\tau, f)$  gives a semiorthogonal decomposition of  $(V, [\cdot, \cdot])$  with respect to  $(V_c, [\cdot, \cdot]_c)_{c \in C}$  in the sense that

- (a) for every  $c \in C$ , the restriction  $f_c := f|_{V_c}$  is a morphism of polarized vector spaces, and
- (b) if  $v \in V_c, w \in V_{c'}$  and  $\tau(c) > \tau(c')$ , then  $[f_c v, f_{c'} w] = 0$ .

We call the underlying pair  $(\tau, f)$  the splitting data of the mutation system. The category of mutation systems with fixed type  $(C, \tau)$  (whose morphisms are the morphisms of underlying Stokes data compatible with the pairings) is denoted by  $\text{Mut}_{\mathbf{k}}(C, \tau)$ .

We remark that we can reconstruct the maps  $f_c^*$  (resp.  $f_c^!$ ) for  $c \in C$  by the condition  $[v, f_c w] = [f_c^* v, w]_c$  (resp.  $[f_c w, v] = [w, f_c^! v]_c$ ) for all

$v \in V, w \in V_c$ . We also have  $[v, R_i w] = [R_i^* v, w]$ , and  $[L_i v, w] = [v, L_i^! w]$ . The functor  $\mathbb{M}_\sigma : \mathbf{Std}_k(C, \tau) \xrightarrow{\sim} \mathbf{Std}_k(C, \bar{\sigma} \circ \tau)$  defined in Definition 2.12 for  $\sigma \in \text{Br}_m$  can be extended to the functor  $\mathbb{M}_\sigma : \mathbf{Mut}_k(C, \tau) \xrightarrow{\sim} \mathbf{Mut}_k(C, \bar{\sigma} \circ \tau)$ .

Let  $\mathcal{L}$  be a local system on  $S^1$ . A sesquilinear pairing on  $\mathcal{L}$  is a morphism  $h : \iota^{-1} \mathcal{L} \otimes \mathcal{L} \rightarrow \mathbf{k}_{S^1}$  of local systems. It induces two morphisms

$$\ell_h, \mathbb{D}\ell_h : \mathcal{L} \rightarrow \mathbb{D}\mathcal{L}.$$

Here,  $\ell_h$  is defined by  $\ell_h t(s) := h(s, t)$  where  $t \in \mathcal{L}, s \in \iota^{-1} \mathcal{L}$ , and  $\mathbb{D}\ell_h$  is its dual. It is called non-degenerate if  $\ell_h$  is an isomorphism. It is called symmetric if  $\iota^{-1} h \circ \text{ex} = h$  where  $\text{ex} : \iota^{-1} \mathcal{L} \otimes \mathcal{L} \xrightarrow{\sim} \mathcal{L} \otimes \iota^{-1} \mathcal{L}$  is the exchanging operator. This is equivalent to the condition  $\mathbb{D}\ell_h = \ell_h$ .

LEMMA 2.31. — *Let  $\mathcal{L}$  be a local system on  $S^1$  and  $(V, T) = (V_{\mathcal{L}}, T_{\mathcal{L}})$  be the corresponding object in  $\mathbf{Rep}(S^1)$ . Then there is a natural one-to-one correspondence between the set of non-degenerate symmetric sesquilinear pairings on  $\mathcal{L}$  and the set of pairings on  $V$  compatible with  $T$ .*

*Proof.* — Let  $h$  be a non-degenerate symmetric sesquilinear pairing on  $\mathcal{L}$ . Then the pairing  $[\cdot, \cdot]_h$  on  $V$  is given by the formula

$$(2.17) \quad [s, t]_h := p^{-1} h(\tau_{\mathcal{L}} s, t) \quad (s, t \in V_{\mathcal{L}} = \Gamma(\mathbb{R}, p^{-1} \mathcal{L})).$$

Here,  $\tau_{\mathcal{L}} : V_{\mathcal{L}} \xrightarrow{\sim} V_{\iota^{-1} \mathcal{L}}$  is given by  $\tau_{\mathcal{L}} s(\theta) = s(\theta - \pi)$  for  $\theta \in \mathbb{R}$ . Symmetry of  $h$  implies compatibility of  $[\cdot, \cdot]_h$  with  $T$  and non-degeneracy of  $h$  implies that of  $[\cdot, \cdot]_h$ . □

DEFINITION 2.32. — *Let  $(\mathcal{L}, \mathcal{L}_\bullet)$  be a Stokes filtered local system. We call a sesquilinear pairing  $h : \iota^{-1} \mathcal{L} \otimes \mathcal{L} \rightarrow \mathbf{k}_{S^1}$  compatible with the Stokes filtration  $\mathcal{L}_\bullet$  if the induced morphism  $\ell_h$  is a morphism of Stokes filtered local systems. The category of Stokes filtered local systems with exponents  $C$  with symmetric non-degenerate sesquilinear pairings compatible with the Stokes filtration is denoted by  $\mathbf{StP}_k^C$ . The morphism in this category is the morphism in  $\mathbf{St}_k^C$  such that the pairing is preserved.*

Remark that  $h$  is compatible with the Stokes filtration if and only if  $\ell_h : \mathcal{L} \rightarrow \mathbb{D}\mathcal{L}$  is a morphism of Stokes filtered local systems.

LEMMA 2.33. — *We have a functorial isomorphism of Stokes data*

$$\Phi_{\theta_0}(\mathbb{D}\mathcal{L}, \mathbb{D}\mathcal{L}_\bullet) \cong \left( (V^\vee, (T^\vee)^{-1}), (V_c^\vee, (T_c^\vee)^{-1})_{c \in C}, (f_{\theta_0}^*)^\vee, \left( \bigoplus_{c \in C} T_c^\vee \right)^{-1} f_{\theta_0}^\vee T^\vee \right).$$

Via this isomorphism,  $\Phi_{\theta_0}(\ell_h)$  is identified with the map

$$V \xrightarrow{\sim} V^\vee; v \mapsto [\bullet, v]_h.$$

*Proof.* — The morphism  $\tau_{\mathcal{L}}$  (in the proof of Lemma 2.31) gives an isomorphism  $\tau_{\mathcal{L}}^\vee : V_{\mathbb{D}\mathcal{L}} \xrightarrow{\sim} V_{\mathcal{L}}^\vee$ . The morphism  $\Phi_{\theta_0}(\ell_h) : V_{\mathcal{L}} \rightarrow V_{\mathbb{D}\mathcal{L}}$  is by definition identified with  $V \xrightarrow{\sim} V^\vee; v \mapsto [\bullet, v]_h$ . We also have the isomorphism  $\tau_{\text{gr}_c \mathcal{L}}^\vee : V_{\text{gr}_c \mathbb{D}\mathcal{L}} \xrightarrow{\sim} V_c^\vee$ . Via these isomorphisms, the pair  $((f_{\theta_0-\pi}^\vee)^{-1}, (f_{\theta_0-\pi}^*)^{-1})$  underlies the Stokes data  $\Phi_{\theta_0}(\mathbb{D}\mathcal{L}, \mathbb{D}\mathcal{L}_\bullet)$ . Since we have  $f_{\theta_0-\pi} = (f_{\theta_0}^*)^{-1}$ , and  $f_{\theta_0-\pi}^* = T^{-1}f_{\theta_0}^{-1}(\bigoplus_c T_c)$ , we have the conclusion.  $\square$

LEMMA 2.34. — Let  $((V, T), (V_c, T_c)_{c \in C}, f, f^*) \in \text{Std}(C, \tau)$  be an object of  $\text{Std}(C, \tau)$ . Then the tuple

$$(2.18) \quad \left( (V^\vee, (T^\vee)^{-1}), (V_c^\vee, (T_c^\vee)^{-1})_{c \in C}, (f^*)^\vee, \left( \bigoplus_{c \in C} T_c^\vee \right)^{-1} f^\vee T^\vee \right)$$

is also an object in  $\text{Std}(C, \tau)$ . A pairing  $[\cdot, \cdot]$  on  $V$  compatible with  $T$  is compatible with the Stokes data if and only if the induced map

$$V \xrightarrow{\sim} V^\vee; v \mapsto [\bullet, v]$$

gives an isomorphism of Stokes data between  $((V, T), (V_c, T_c)_{c \in C}, f, f^*)$  and (2.18).

*Proof.* — Put  $g := (f^*)^\vee$ , and  $g^* := (\bigoplus_c T_c^\vee)^{-1} f^\vee T^\vee$ . We also put  $g^! := (\bigoplus_c T_c^\vee) g^* (T^\vee)^{-1}$ . Then we have  $g^* g = ((\bigoplus_c T_c) f^! f (\bigoplus_c T_c)^{-1})^\vee$ , and  $g^! g = (f^* f)^\vee$ . This implies the first part of the lemma. Let  $\ell : V \rightarrow V^\vee$  be the map  $v \mapsto [\bullet, v]$ . We also define  $\text{gr}_c(\ell) : V_c \rightarrow V_c^\vee$  by  $v_c \mapsto [f_c \bullet, f_c v_c]$ . The induced morphism  $\bigoplus_c V_c \rightarrow \bigoplus_c V_c^\vee$  is denoted by  $\text{gr}(\ell)$ . The composition  $g^{-1} \ell f$  is given by  $w \mapsto [(f^*)^{-1} \bullet, f w]$ . This map is graded if and only if it is equal to  $\text{gr}(\ell)$ . This condition is equivalent to the compatibility condition of the pairing with the Stokes data. By (2.16), we have  $T^\vee \ell = \ell^\vee$ . Hence the composition  $g^* \ell (f^*)^{-1}$  is given by  $(g^{-1} \ell f (\bigoplus_c T_c^{-1}))^\vee$ . Therefore,  $g^* \ell (f^*)^{-1}$  is graded if and only if  $g^{-1} \ell f$  is graded. This completes the proof.  $\square$

By these lemmas, the equivalence  $\mathfrak{A}_{\theta_0} : \text{St}_{\mathbf{k}}^C \xrightarrow{\sim} \text{Std}_{\mathbf{k}}(C, \tau_{\theta_0})$  gives an equivalence  $\text{StP}_{\mathbf{k}}^C \xrightarrow{\sim} \text{Mut}_{\mathbf{k}}(C, \tau_{\theta_0})$ , which is also denoted by  $\mathfrak{A}_{\theta_0}$ . We remark that the pairing  $[\cdot, \cdot]_c$  ( $c \in C$ ) underlying  $\mathfrak{A}_{\theta_0}(\mathcal{L}, \mathcal{L}_\bullet, h)$  is canonically identified with  $[\cdot, \cdot]_{\text{gr}_c(h)}$ . In particular, it does not depend on the choice of  $\theta_0$ .

*Remark 2.35.* — We also consider a  $\mathbb{Z}/2\mathbb{Z}$ -graded version of these structures. If  $V$  is a  $\mathbb{Z}/2\mathbb{Z}$ -graded vector space, the compatibility condition of pairing  $[\cdot, \cdot]$  with  $T$  is defined by

$$[Tv, w] = (-1)^{\deg v}[w, v],$$

where  $v, w$  are homogeneous elements. The symmetry of pairings on local systems is replaced by the graded-symmetry. For  $\mathbb{Z}/2\mathbb{Z}$ -graded local system  $\mathcal{L} = \mathcal{L}^0 \oplus \mathcal{L}^1$ ,  $h$  is graded symmetric if the following equalities hold:  $h(\mathcal{L}^i, \mathcal{L}^j) = 0$  for  $i \neq j$ , and  $\iota^{-1}h \circ \text{ex} = (-1)^i h$  on  $\iota^{-1}\mathcal{L}^i \otimes \mathcal{L}^i$  for  $i = 0, 1$ . The functor  $\mathfrak{A}_{\theta_\circ} : \text{St}_k^C \xrightarrow{\sim} \text{Mut}_k(C, \tau_{\theta_\circ})$  enhanced to these categories is also denoted by the same notation.

### 2.6. A generalization of the construction of Stokes data

We generalize the construction of Stokes data from Stokes filtered local systems. This construction is only used in the proof of Lemma 6.14 and Theorem 7.9.

Let  $(\mathcal{L}, \mathcal{L}_\bullet) \in \text{St}_k^C$  be a Stokes filtered local system with exponents  $C \subset \mathbb{C}$ . Fix  $C$ -generic  $\theta_\circ$ . A tuple of real numbers  $\theta_\bullet = \{\theta_c\}_{c \in C}$  is called  $C$ -generic if  $\theta_c$  are  $C$ -generic for all  $c \in C$ . Set

$$L_{\theta_c}^- := \{c - re^{i\theta_c} \mid 0 \leq r\} \subset \mathbb{C}.$$

**DEFINITION 2.36.** — *Let  $\theta_\bullet$  be a  $C$ -generic tuple of real numbers with  $\theta_\circ \geq \theta_c > \theta_\circ - 2\pi$  for all  $c \in C$ . We define a binary relation  $<_{\theta_\bullet + \pi/2}$  as follows:*

$$c <_{\theta_\bullet + \pi/2} c' \Leftrightarrow \begin{cases} \theta_c > \theta_{c'} \text{ and } L_{\theta_c}^- \cap L_{\theta_{c'}}^- = \emptyset \text{ or} \\ \theta_c = \theta_{c'} \text{ and } c <_{\theta_c + \pi/2} c'. \end{cases}$$

The next lemma gives another description of  $<_{\theta_\bullet + \pi/2}$ .

**LEMMA 2.37.**

$$c <_{\theta_\bullet + \pi/2} c' \Leftrightarrow \begin{cases} c <_{\theta_c + \pi/2} c' \text{ or } c <_{\theta_{c'} + \pi/2} c' & (\theta_c \geq \theta_{c'} > \theta_c - \pi), \\ c' <_{\theta_c + \pi/2} c \text{ or } c' <_{\theta_{c'} + \pi/2} c & (\theta_c - \pi \geq \theta_{c'}). \end{cases}$$

*Proof.* — Set

$$L_{\theta_c} := \{c - re^{i\theta_c} \mid r \in \mathbb{R}\}, \quad L_{\theta_c}^+ := L_{\theta_c} \setminus L_{\theta_c}^-.$$

If  $\theta_c = \theta_{c'}$  or  $\theta_c = \theta_{c'} \pm \pi$ , then the lemma easily follows. Hence we consider the following two cases:

- (1)  $\theta_{c'} \in ]\theta_c - \pi, \theta_c[ \pmod{2\pi}$ .
- (2)  $\theta_{c'} \in ]\theta_c, \theta_c + \pi[ \pmod{2\pi}$ .

We first consider the case (1). In this case, we see that  $c <_{\theta_c + \pi/2} c'$  if and only if  $L_{\theta_c} \cap L_{\theta_{c'}} \subset L_{\theta_{c'}}^+$  and  $c <_{\theta_{c'} + \pi/2} c'$  if and only if  $L_{\theta_c} \cap L_{\theta_{c'}} \subset L_{\theta_c}^+$ . Therefore  $c <_{\theta_c + \pi/2} c'$  or  $c <_{\theta_{c'} + \pi/2} c'$  if and only if  $L_{\theta_c}^- \cap L_{\theta_{c'}}^- = \emptyset$ . The lemma easily follows from this. The proof of the case (2) is similar.  $\square$

In general, the binary relation  $<_{\theta_\bullet + \pi/2}$  is not a partial ordering.

DEFINITION 2.38. — *A tuple of real numbers  $\theta_\bullet$  is said to be ordered if  $\theta_\bullet$  is  $C$ -generic,  $\theta_\circ \geq \theta_c > \theta_\circ - 2\pi$  for all  $c \in C$ , and  $<_{\theta_\bullet + \pi/2}$  gives a total ordering.*

For ordered  $\theta_\bullet$ , we have the isomorphism of ordered sets

$$C \xrightarrow{\sim} \{1, 2, \dots, m\},$$

where the order of  $C$  is given by  $<_{\theta_\bullet + \pi/2}$ . This isomorphism is denoted by  $\tau_{\theta_\bullet}$ . Set

$$f_{\theta_\bullet} := \coprod_{c \in C} f_{\theta_c, c}, \quad f_{\theta_\bullet}^* := \coprod_{c \in C} f_{\theta_c, c}^*,$$

where the symbol  $\coprod$  means the direct sum.

PROPOSITION 2.39. — *If  $\theta_\bullet$  is ordered, then  $((V_c, T_c)_{c \in C}, f_{\theta_\bullet}, f_{\theta_\bullet}^*)$  are Stokes data on  $(V, T) = (V_{\mathcal{L}}, T_{\mathcal{L}})$  of type  $(C, \tau_{\theta_\bullet})$ .*

*Proof.* — It is sufficient to show that  $f_{\theta_c, c}^* \circ f_{\theta_{c'}, c'} = 0$  and  $f_{\theta_{c'}, c'}^! \circ f_{\theta_c, c} = 0$  for  $c <_{\theta_\bullet + \pi/2} c'$ . We first consider the case  $\theta_c \geq \theta_{c'} > \theta_c - \pi$ . In this case, we see that

$$\theta_c - \pi/2, \theta_{c'} - \pi/2 \in I_{\theta_c - \pi} \cap I_{\theta_{c'}}.$$

Combined with Lemma 2.37, we have  $f_{\theta_c, c}^* \circ f_{\theta_{c'}, c'} = 0$  (see also the proof of Proposition 2.5). By rotating  $\pi$ , we also see that

$$\theta_c + \pi/2, \theta_{c'} + \pi/2 \in I_{\theta_c} \cap I_{\theta_{c'} + \pi},$$

which implies  $f_{\theta_{c'}, c'}^! \circ f_{\theta_c, c} = 0$ . The case  $\theta_c - \pi \geq \theta_{c'}$  is similar. Note that in this case we have

$$\theta_c - 3\pi/2, \theta_{c'} + \pi/2 \in I_{\theta_c - \pi} \cap I_{\theta_{c'}}. \quad \square$$

Similar to Proposition 2.5, this correspondence gives a fully faithful functor

$$\mathfrak{A}_{\theta_\bullet} : \text{St}_k^C \rightarrow \text{Std}_k(C, \tau_{\theta_\bullet}).$$

The following is a direct consequence of the above proposition.

COROLLARY 2.40. — *If  $(\mathcal{L}, \mathcal{L}_\bullet, h) \in \text{StP}_k^C$ , then the corresponding pairing  $[\cdot, \cdot]$  on  $(V_{\mathcal{L}}, T_{\mathcal{L}})$  is compatible with the Stokes data  $\mathfrak{A}_{\theta_\bullet}(\mathcal{L}, \mathcal{L}_\bullet)$ .*

By the above corollary, we have a functor from  $\text{StP}_k^C$  to  $\text{Mut}_k(C, \tau_{\theta_\bullet})$ . To simplify notation, this functor is also denoted by  $\mathfrak{A}_{\theta_\bullet}$ .

*Remark 2.41.* — We also consider a  $\mathbb{Z}/2\mathbb{Z}$ -graded version (see Remark 2.35). This generalized functor is also denoted by  $\mathfrak{A}_{\theta_\bullet}$ .

### 2.7. A reminder for Riemann–Hilbert correspondence

We recall some fundamental results on Riemann–Hilbert correspondence for meromorphic connections on the germ  $(\mathbb{C}, 0)$  of a complex plane at zero. A comprehensive reference is [51]. Let  $\mathcal{M}$  be a finite dimensional  $\mathcal{O}_{\mathbb{C},0}(*\{0\})$ -vector space together with a  $\mathbb{C}$ -linear map  $\nabla : \mathcal{M} \rightarrow \mathcal{M} \otimes \Omega_{\mathbb{C},0}^1$  satisfying the Leibniz rule:  $\nabla(as) = a\nabla s + s \otimes da$ , ( $a \in \mathcal{O}_{\mathbb{C},0}(*\{0\})$ ,  $s \in \mathcal{M}$ ). We call such a pair  $(\mathcal{M}, \nabla)$  a meromorphic connection on  $(\mathbb{C}, 0)$ . We often abbreviate  $\nabla$ .

For  $c \in \mathbb{C}$ , we define a meromorphic connection  $\mathcal{E}^{c/z} := (\mathcal{O}_{\mathbb{C},0}(*\{0\}), d + d(c/z))$ .

DEFINITION 2.42. — A meromorphic connection  $\mathcal{M}$  is called of exponential type with exponents  $C \subset \mathbb{C}$  if there is an isomorphism of formal meromorphic connections:

$$(2.19) \quad \mathcal{M} \otimes \mathbb{C}((z)) \simeq \left( \bigoplus_{c \in C} \mathcal{E}^{-c/z} \otimes \mathcal{R}_c \right) \otimes \mathbb{C}((z)),$$

where  $\mathcal{R}_c$  is a regular singular meromorphic connection for each  $c$ . The category of meromorphic connections on  $(\mathbb{C}, 0)$  of exponential type is denoted by  $\text{Me}$ . For a finite subset  $C \subset \mathbb{C}$ ,  $\text{Me}_C$  denotes the full subcategory such that the exponents of objects are contained in  $C$ .

Let  $\varpi : \text{Bl}_0^{\mathbb{R}}(\mathbb{C}) \rightarrow \mathbb{C}$  be the real blowing up of the complex plane  $\mathbb{C}$  at zero. We have natural inclusions  $j_{\mathbb{C}^*} : \mathbb{C}^* \hookrightarrow \text{Bl}_0^{\mathbb{R}}(\mathbb{C})$  and

$$i_{\partial} : S^1 \simeq \partial \text{Bl}_0^{\mathbb{R}}(\mathbb{C}) \hookrightarrow \text{Bl}_0^{\mathbb{R}}(\mathbb{C})$$

where  $\partial \text{Bl}_0^{\mathbb{R}}(\mathbb{C})$  is the boundary. We put  $\tilde{\mathcal{O}} := (j_{\mathbb{C}^*})_* \mathcal{O}_{\mathbb{C}^*}$  where  $\mathcal{O}_{\mathbb{C}^*}$  is the sheaf of holomorphic functions on  $\mathbb{C}^*$ . We define subsheaves  $\mathcal{A}^{\text{mod}}$  and  $\mathcal{A}^{\text{rd}}$  of  $\tilde{\mathcal{O}}$  as follows: For an open subset  $U$  in  $\text{Bl}_0^{\mathbb{R}}(\mathbb{C})$ , put  $U^* := U \cap \mathbb{C}^*$ . A section  $f$  of  $\tilde{\mathcal{O}}(U) = \mathcal{O}(U^*)$  is a section of  $\mathcal{A}^{\text{mod}}(U)$  if and only if for any compact subset  $K \subset U$ , there exist a constant  $C_K$  and a non-negative integer  $N_K$  such that  $|f(z)| \leq C_K |z|^{-N_K}$  for all  $z \in U^* \cap K$ . A section  $g \in \mathcal{O}(U^*)$  is a section of  $\mathcal{A}^{\text{rd}}(U)$  if and only if for any compact subset  $K \subset U$ , and for any non-negative integer  $N$ , there is a constant  $C_{K,N}$  such

that  $|g(z)| \leq C_{K,N}|z|^N$  for all  $z \in U^* \cap K$ . For  $c \in \mathbb{C}$ , we also have the subsheaves  $e^{c/z} \mathcal{A}^{\text{mod}}$ , and  $e^{c/z} \mathcal{A}^{\text{rd}}$  where  $e^{c/z}$  is considered as a section of  $\tilde{\mathcal{O}}$ .

Using these sheaves as coefficients, we define various de Rham complexes as follows:

- $\widetilde{\text{DR}}(\mathcal{M}) := \{\tilde{\mathcal{O}} \otimes \varpi^{-1} \mathcal{M} \rightarrow \tilde{\mathcal{O}} \otimes \varpi^{-1}(\Omega^1 \otimes \mathcal{M})\},$
- $\text{DR}_{\leq c}(\mathcal{M}) := \{e^{c/z} \mathcal{A}^{\text{mod}} \otimes \varpi^{-1} \mathcal{M} \rightarrow e^{c/z} \mathcal{A}^{\text{mod}} \otimes \varpi^{-1}(\Omega^1 \otimes \mathcal{M})\},$
- $\text{DR}_{< c}(\mathcal{M}) := \{e^{c/z} \mathcal{A}^{\text{rd}} \otimes \varpi^{-1} \mathcal{M} \rightarrow e^{c/z} \mathcal{A}^{\text{rd}} \otimes \varpi^{-1}(\Omega^1 \otimes \mathcal{M})\},$

where  $\widetilde{\text{DR}}(\mathcal{M})$  has cohomology in degree 0 at most.

Then the pair

$$\text{RH}(\mathcal{M}) := (\mathcal{H}^0 \widetilde{\text{DR}}(\mathcal{M}), \mathcal{H}^0 \text{DR}_{\leq \bullet}(\mathcal{M}))$$

is considered as a Stokes filtered local system on  $S^1$  via  $i_{\partial}^{-1}$ , where  $\mathcal{H}^0$  is the cohomology of degree zero.

**THEOREM 2.43** ([38], [51, Theorem 5.7]). — *The Riemann–Hilbert functor  $\text{RH} : \text{Me}_C \rightarrow \text{St}_C^C$  is an equivalence of categories.*

*Remark 2.44.* — Let  $(\mathcal{L}, \mathcal{L}_{\bullet})$  be the Stokes filtered local system corresponding to  $(\mathcal{M}, \nabla) \in \text{Me}_C$  via RH. Then the rank of  $\text{gr}_{c_i} \mathcal{L}$  is equal to the rank of  $\mathcal{R}_i$  in (2.19). It is also equal to the dimension of  $V_{c_i}$  of the corresponding Stokes data.

Let  $\iota : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$  be the involution  $z \mapsto -z$ . Set

$$\mathcal{M}^{\vee} := \mathcal{H}om(\mathcal{M}, \mathcal{O}(*\{0\})).$$

Then we define  $\mathbb{D}(\mathcal{M}) := \iota^{-1} \mathcal{M}^{\vee}$ .

**PROPOSITION 2.45** ([51, Proposition 5.15]). — *The Riemann–Hilbert functor is compatible with duality, i.e.  $\text{RH}(\mathbb{D}\mathcal{M}) \simeq \mathbb{D} \text{RH}(\mathcal{M})$ .*

*Proof.* — The compatibility with dual  $\mathcal{M} \mapsto \mathcal{M}^{\vee}$  is shown in [51, Proposition 5.15]. The compatibility with the involution  $\iota^{-1}$  is shown similarly. □

**DEFINITION 2.46.** — *We define the category  $\text{MeP}$  of meromorphic connections of exponential type with pairings as follows:*

- (1) *An object in  $\text{MeP}$  is a pair  $(\mathcal{M}, \mathcal{Q})$  of a meromorphic connection  $\mathcal{M} \in \text{Me}$  and an isomorphism*

$$(2.20) \quad \mathcal{Q} : \mathcal{M} \xrightarrow{\sim} \mathbb{D}\mathcal{M}$$

*of meromorphic connections such that  $\mathbb{D}\mathcal{Q} = \mathcal{Q}$ .*

(2) Let  $(\mathcal{M}, \mathcal{Q})$  and  $(\mathcal{M}', \mathcal{Q}')$  be objects in  $\text{MeP}$ . A morphism from  $(\mathcal{M}, \mathcal{Q})$  to  $(\mathcal{M}', \mathcal{Q}')$  is a morphism  $\lambda \in \text{Hom}_{\text{Me}}(\mathcal{M}, \mathcal{M}')$  such that  $\mathbb{D}\lambda \circ \mathcal{Q}' \circ \lambda = \mathcal{Q}$ .

For a finite set  $C \subset \mathbb{C}$ , we also define  $\text{MeP}_C$  as a full subcategory of  $\text{MeP}$  whose exponents are contained in  $C$ .

**COROLLARY 2.47.** — *The Riemann–Hilbert functor  $\text{RH} : \text{MeP}_C \rightarrow \text{StP}_C^C$  is well defined and gives an equivalence of categories.*

*Remark 2.48.* — As in Remark 2.35, we consider  $\mathbb{Z}/2\mathbb{Z}$ -graded meromorphic connections. The connections are assumed to be grade-preserving. The only difference with non-graded case is the pairing. The pairing is defined to be graded-symmetric so that the equivalence  $\text{RH} : \text{MeP}_C \rightarrow \text{StP}_C^C$  is generalized to the  $\mathbb{Z}/2\mathbb{Z}$ -graded case. The generalized functor is also denoted by  $\text{RH}$ .

### 3. A-mutation systems

#### 3.1. Quantum connections of exponential type

Let  $X$  be a Fano manifold, that is,  $X$  is a smooth projective variety whose anti-canonical bundle  $\omega_X^{-1} := \det TX$  is ample. Let  $H^\bullet(X)$  denote the Betti cohomology group of  $X$  over  $\mathbb{C}$ . For  $\alpha_1, \alpha_2, \dots, \alpha_n \in H^\bullet(X)$ , let  $\langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle_{0,n,d}^X$  denote the genus-zero  $n$ -points Gromov–Witten invariant of degree  $d \in H_2(X, \mathbb{Z})$ . The quantum cup product  $\alpha_1 *_\tau \alpha_2$  of two classes  $\alpha_1, \alpha_2 \in H^\bullet(X)$  with parameter  $\tau \in H^\bullet(X)$  is given by

$$(3.1) \quad (\alpha_1 *_\tau \alpha_2, \alpha_3)_X = \sum_{d \in \text{Eff}(X)} \sum_{n=0}^{\infty} \langle \alpha_1, \alpha_2, \alpha_3, \tau, \dots, \tau \rangle_{0,3+n,d}^X$$

where  $(\alpha, \beta)_X := \int_X \alpha \cup \beta$  is the Poincaré pairing and  $\text{Eff}(X) \subset H_2(X; \mathbb{Z})$  is the set of effective curve classes. It is not known if the quantum products  $*_\tau$  converge in general, however the quantum cup product for  $\tau \in H^2(X)$  makes sense since  $X$  is a Fano manifold.

**DEFINITION 3.1** ([14]). — *Consider the trivial vector bundle  $\mathcal{H}_X := H^\bullet(X) \otimes \mathcal{O}_{\mathbb{C}}$  over  $\mathbb{C}_z$ . Define the meromorphic flat connection*

$$\nabla : \mathcal{H}_X \rightarrow \mathcal{H}_X \otimes \Omega_{\mathbb{C}}^1(\log\{0\}) \otimes \mathcal{O}_{\mathbb{C}}(\{0\})$$

*called a quantum connection by*

$$(3.2) \quad \nabla := d - \left( \frac{1}{z} (c_1(X) *_{0}) - \mu \right) \frac{dz}{z},$$

where  $z$  denotes the coordinate on  $\mathbb{C}$ ,  $\mu \in \text{End}(H^\bullet(X))$  is the grading operator defined by  $\mu|_{H^p(X)} := (p - \dim_{\mathbb{C}} X)/2 \cdot \text{id}_{H^p(X)}$ , and  $c_1(X) \in H^2(X; \mathbb{Z})$  is the first Chern class of  $X$ . We also set the meromorphic connection  $\mathcal{M}_X := \mathcal{H}_X \otimes \mathcal{O}_{\mathbb{C},0}$ . We define a  $\mathbb{Z}/2\mathbb{Z}$ -graded symmetric sesquilinear pairing  $Q_X : \iota^* \mathcal{H}_X \otimes \mathcal{H}_X \rightarrow \mathcal{O}_{\mathbb{C}}$  by  $Q_X(s, t)(z) := (s(-z), t(z))_X$  where  $s, t \in \mathcal{H}_X$ . The induced pairing on  $\mathcal{M}_X$  is denoted by  $\mathcal{Q}_X$ .

We consider the following assumption [30, Conjecture 3.4]:

(3.3) The meromorphic connection  $\mathcal{M}_X$  is of exponential type.

Under this assumption, we have the following:

COROLLARY 3.2 (Corollary of assumption (3.3)). — Let  $C_X$  be the set of eigenvalues of  $c_1(X)_{*0}$ . The set of exponents of  $\mathcal{M}_X$  coincide with  $-C_X$ . In other words, we have an isomorphism

$$\mathcal{M}_X \otimes \mathbb{C}((z)) \simeq \left( \bigoplus_{c \in C_X} \mathcal{E}^{c/z} \otimes \mathcal{R}_c \right) \otimes \mathbb{C}((z))$$

where  $\mathcal{R}_c$  is regular singular. The rank of the regular singular part  $\mathcal{R}_c$  is the dimension of the eigenspace of  $c_1(X)_{*0}$  associated with  $c$ .

Proof. — For the lattice  $\mathcal{H}_X$ , we have  $(z^2 \nabla_{\partial_z})|_{z=0} = -c_1(X)_{*0}$  identifying the fiber of  $\mathcal{H}_X$  at  $z = 0$  with  $H^\bullet(X)$ . By Exercise 5.9 in [49, II], we have a decomposition  $\mathcal{M}_X \otimes \mathbb{C}((z)) \simeq \bigoplus_{c \in C_X} \mathcal{M}_c$  compatible with the lattice and the generalized eigenvalue decomposition of  $c_1(X)_{*0}$ . On the other hand, by assumption (3.3), we have an isomorphism

$$\mathcal{M}_X \otimes \mathbb{C}((z)) \xrightarrow{\sim} \left( \bigoplus_{i=1}^m \mathcal{E}^{c_i/z} \otimes \mathcal{R}_i \right) \otimes \mathbb{C}((z))$$

for some distinct complex numbers  $c_1, \dots, c_m$ . Consider the induced morphism  $\phi : \mathcal{M}_c \rightarrow (\mathcal{E}^{c_i/z} \otimes \mathcal{R}_i) \otimes \mathbb{C}((z))$  for some  $c \in C_X$  and  $i = 1, \dots, m$ . We claim that  $\phi = 0$  if  $c \neq c_i$  (This claim implies the corollary). Take a frame  $\mathbf{v} = (v_k)_k$  of  $\mathcal{M}_c$  so that  $v_k \in \mathcal{H}_X \otimes \mathbb{C}[[z]]$ . Then we have

$$\nabla_{z\partial_z} \mathbf{v} = \mathbf{v}(-z^{-1}(c \cdot \text{Id} + N) + A(z))$$

where  $\text{Id}$  is the identity matrix,  $N$  is a nilpotent constant matrix, and  $A(z)$  is a matrix with entries in  $\mathbb{C}[[z]]$ . We also take a frame  $\mathbf{w}$  in  $(\mathcal{E}^{c_i/z} \otimes \mathcal{R}_i) \otimes \mathbb{C}((z))$  so that  $\nabla_{z\partial_z} \mathbf{w} = \mathbf{w}(-z^{-1}c_i \cdot \text{Id} + A'(z))$  where  $A'(z)$  is a matrix with entries in  $\mathbb{C}[[z]]$ . If we take a matrix  $B = B(z)$  with entries in  $\mathbb{C}((z))$  so that  $\phi(\mathbf{v}) = \mathbf{w}B$ , the flatness condition of  $\phi$  is written as follows:

(3.4) 
$$z\partial_z B + (A'B - BA) = z^{-1}B((c_i - c) \cdot \text{Id} + N).$$

Since we assume that  $c \neq c'$  and  $N$  is nilpotent,  $((c_i - c) \cdot \text{Id} + N)$  is an invertible constant matrix. By comparing orders of entries between both sides of (3.4), we conclude that  $B = 0$ . □

### 3.2. Fundamental solutions, pairings, and A-mutation systems

Although the following proposition is proved for even degrees of the cohomology, the same proof can be applied for the  $\mathbb{Z}/2\mathbb{Z}$ -graded case.

PROPOSITION 3.3 ([16, Lemma 2.4, and Lemma 2.5], see also [17, Proposition 2.3.1]). — *There exists a unique holomorphic function*

$$S : \mathbb{P}^1 \setminus \{0\} \rightarrow \text{End}(H^\bullet(X))$$

with  $S(\infty) = \text{id}_{H^\bullet(X)}$  such that

$$\nabla(S(z)z^{-\mu}z^{\rho_X}\alpha) = 0 \text{ for all } \alpha \in H^\bullet(X),$$

$$T(z) := z^\mu S(z)z^{-\mu} \text{ is regular at } z = \infty \text{ and } T(\infty) = \text{id}_{H^\bullet(X)},$$

where  $\rho_X = (c_1(X) \cup) \in \text{End}(H^\bullet(X))$  and we define  $z^{-\mu} := \exp(-\mu \log z)$ ,  $z^{\rho_X} := \exp(\rho_X \log z)$ .

Moreover, we have

$$(S(-z)\alpha, S(z)\beta)_X = (\alpha, \beta)_X \text{ for all } \alpha, \beta \in H^\bullet(X).$$

Let  $\theta_\circ$  be a real number generic with respect to  $-C_X$ . Under assumption (3.3), we have the mutation system  $\mathfrak{A}_{\theta_\circ} \text{RH}(\mathcal{M}_X, \mathcal{Q}_X)$ . Denote the underlying polarized vector space by  $(V_{\mathcal{M}_X}, [\cdot, \cdot]_{\mathcal{Q}_X})$ , which is independent of a choice of  $\theta_\circ$ .

Let  $\text{Bl}_0^{\mathbb{R}}(\mathbb{C})$  be the real blowing up of  $\mathbb{C}$  at 0. Let  $\widetilde{\text{Bl}}_0^{\mathbb{R}}(\mathbb{C})$  be its universal covering. The universal covering  $\widetilde{\mathbb{C}}_z^*$  of  $\mathbb{C}_z^*$  can be considered as an open subset of  $\widetilde{\text{Bl}}_0^{\mathbb{R}}(\mathbb{C})$ . Its complement is identified with  $\mathbb{R}$ , which is universal covering of  $S^1$ . By construction, the vector space  $V_{\mathcal{M}_X}$  can be identified with the space of flat section of  $\widetilde{p}^* \mathcal{H}_X$  where  $\widetilde{p} : \widetilde{\mathbb{C}}_z^* \rightarrow \mathbb{C}_z$  denotes the composition of the universal covering and the inclusion. Namely, we can canonically identify  $V_{\mathcal{M}_X}$  with

$$(3.5) \quad \left\{ s : \widetilde{\mathbb{C}}_z^* \rightarrow H^\bullet(X) \mid \nabla s = 0 \right\}.$$

The first half of Proposition 3.3 gives an isomorphism  $H^\bullet(X) \simeq V_{\mathcal{M}_X}$  by

$$\Phi(\alpha) := (2\pi)^{-\dim X/2} S(z)z^{-\mu}z^{\rho_X}\alpha,$$

where  $V_{\mathcal{M}_X}$  is identified with (3.5). By the second half of Proposition 3.3, if we put

$$[\alpha, \beta]_X := \frac{1}{(2\pi)^{\dim X}} \int_X e^{\pi i \mu} e^{-\pi i \rho_X} \alpha \cup \beta$$

for  $\alpha, \beta \in H^\bullet(X)$ , the isomorphism  $\Phi$  is compatible with the pairings  $[\cdot, \cdot]_X$  and  $[\cdot, \cdot]_{\mathcal{Q}_X}$ .

DEFINITION 3.4. — *The mutation system on  $(H^\bullet(X), [\cdot, \cdot]_X)$  defined via the isomorphism*

$$\Phi : (H^\bullet(X), [\cdot, \cdot]_X) \xrightarrow{\sim} (V_{\mathcal{M}_X}, [\cdot, \cdot]_{\mathcal{Q}_X})$$

described above is called an  $A$ -mutation system of  $X$ , which is also denoted by  $\mathfrak{A}_{\theta_0} \text{RH}(\mathcal{M}_X, \mathcal{Q}_X)$ . The pair  $(\tau_{\theta_0}, {}^A f_{\theta_0, c})$  denotes the underlying splitting data.

We can describe this mutation system more concretely as follows:

LEMMA 3.5. — *Fix a hermitian metric  $\|\cdot\|$  on  $H^\bullet(X)$ . Let  $c$  be a complex number in  $-C_X$ . A class  $\alpha \in H^\bullet(X)$  is in  $\text{Im } {}^A f_{\theta_0, c}$  if and only if there exists a non-negative integer  $N$  such that*

$$(3.6) \quad \|e^{-c/z} \Phi(\alpha)(z)\| \leq O(|z|^{-N})$$

for  $\Im \log z \in I_{\theta_0}$ .

*Proof.* — In general, for  $(\mathcal{L}, \mathcal{L}_\bullet) \in \text{St}^C$  and  $C$ -generic  $\theta_0 \in \mathbb{R}$ , we have

$$\text{Im } f_{\theta_0, c} = \{s \in V_{\mathcal{L}} \mid s_\theta \in (p^{-1} \mathcal{L}_{\leq c})_\theta \text{ for } \theta \in I_{\theta_0}\}.$$

In the case  $(\mathcal{L}, \mathcal{L}_\bullet) = \text{RH}(\mathcal{M}_X)$ , the germ  $(p^{-1} \mathcal{L}_{\leq c})_\theta$  can be identified with the space of  $\alpha \in H^\bullet(X)$  with the following properties: There exists an open neighborhood  $U$  of  $\theta$  in  $\text{Bl}_0^{\mathbb{R}}(\mathbb{C})$  (remark that  $\theta \in \mathbb{R} \subset \text{Bl}_0^{\mathbb{R}}(\mathbb{C})$ ) such that for any compact subset  $K$  in  $U$  there exists a positive integer  $N$  with  $\|e^{-c/z} \Phi(\alpha)(z)\| \leq O(|z|^{-N})$  for  $\log z \in K \cap \widetilde{C}^*$ . Hence the condition (3.6) implies  $\alpha \in \text{Im } {}^A f_{\theta_0, c}$ . On the other hand, by replacing  $I_{\theta_0}$  with a bigger  $(-C_X)$ -good open interval, we see that  $\alpha \in \text{Im } {}^A f_{\theta_0, c}$  implies the condition (3.6). □

Remark 3.6. — By Remark 2.44 and Corollary 3.2, the dimension of  $\text{Im } f_{\theta_0, c}$  is equal to the dimension of the eigenspace of  $-c_1(X)_*0$  associated with  $c$ .

### 4. B-mutation systems

#### 4.1. Hochschild homology for smooth projective varieties

We recall some definitions and properties of Hochschild homology of smooth projective varieties. We mainly follow the formulation of [25, §5] (see also [37]).

Let  $X, Y, Z$  be smooth projective varieties defined over  $\mathbb{C}$ . The dimension of  $X, Y, Z$  are denoted by  $d_X, d_Y, d_Z$  respectively. We denote by  $D^b(X)$  the triangulated category of bounded complexes of coherent sheaves on  $X$ . The shift functor is denoted by [1]. For a morphism  $f : X \rightarrow Y$ , we denote by  $f_*$  the right derived direct image functor and  $f^*$  the left derived inverse image functor. Moreover the left derived tensor product is denoted by  $\otimes$ .

Let  $\mathcal{E} \in D^b(X \times Y)$  and  $\mathcal{F} \in D^b(Y \times Z)$  be bounded complexes. We define the exact functor

$$\Phi_{\mathcal{E}} : D^b(X) \rightarrow D^b(Y)$$

by

$$\Phi_{\mathcal{E}}(-) := (\pi_2)_*(\pi_1^*(-) \otimes \mathcal{E}),$$

where  $\pi_i$  is the projection from  $X \times Y$  to the  $i$ -th factor. For the diagonal sheaf

$$\mathcal{O}_{\Delta} := \Delta_* \mathcal{O}_X \in D^b(X \times X),$$

we have  $\Phi_{\mathcal{O}_{\Delta}} \cong \text{id}$ , where  $\Delta : X \rightarrow X \times X$  is the diagonal embedding. We define the composition of kernels  $\mathcal{E}$  and  $\mathcal{F}$  by

$$\mathcal{F} \circ \mathcal{E} := (\pi_{1,3})_*(\pi_{1,2}^* \mathcal{E} \otimes \pi_{2,3}^* \mathcal{F}) \in D^b(X \times Z),$$

where  $\pi_{i,j}$  is the projection from  $X \times Y \times Z$  to the  $i^{\text{th}} \times j^{\text{th}}$  factor. Then we have  $\Phi_{\mathcal{F} \circ \mathcal{E}} \cong \Phi_{\mathcal{F}} \circ \Phi_{\mathcal{E}}$ .

We define the Hochschild homology of  $X$  as follows:

$$\text{HH}_k(X) := \bigoplus_{p-q=k} H^q(X, \Omega_X^p), \quad \text{HH}_{\bullet}(X) := \bigoplus_{k \in \mathbb{Z}} \text{HH}_k(X)$$

This is a  $\mathbb{Z}$ -graded vector space.

*Remark 4.1* (see, e.g., [6], [24], [53], [57]). — More precisely, the definition of the Hochschild homology is

$$\text{HH}_k(X) := \text{Hom}_{D^b(X)}(\mathcal{O}_X[k], \Delta^* \mathcal{O}_{\Delta})$$

and we have the Hochschild–Kostant–Rosenberg isomorphism

$$\text{I}_{\text{HKR}} : \bigoplus_{k \in \mathbb{Z}} \text{HH}_k(X) \xrightarrow{\sim} \bigoplus_{k \in \mathbb{Z}} \bigoplus_{p-q=k} H^q(X, \Omega_X^p).$$

We use the right hand side as a definition of the Hochschild homology.

For a homogeneous element  $\alpha \in \text{HH}_\bullet(X)$ , the degree of  $\alpha$  is denoted by  $\text{deg } \alpha$ . We identify  $\text{HH}_\bullet(X)$  with  $H^\bullet(X)$  via the Hodge decomposition (as  $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces). For a morphism  $f : X \rightarrow Y$ , we denote by  $f_*$  the Gysin map.

*Remark 4.2.* — By definition,  $f_*$  satisfies the following:

$$\frac{1}{(2\pi i)^{d_Y}} \int_Y f_* \alpha \cup \beta = \frac{1}{(2\pi i)^{d_X}} \int_X \alpha \cup f^* \beta.$$

Here  $\alpha \in \text{HH}_\bullet(X)$  and  $\beta \in \text{HH}_\bullet(Y)$ .

For  $\mathcal{E} \in D^b(X \times Y)$ , a morphism  $\phi_{\mathcal{E}} : \text{HH}_\bullet(X) \rightarrow \text{HH}_\bullet(Y)$  is defined as follows:

$$\phi_{\mathcal{E}}(-) := (\pi_2)_*(\pi_1^*(-) \cup v(\mathcal{E})),$$

where

$$v(\mathcal{E}) := \text{Ch}(\mathcal{E}) \sqrt{\text{Td}_{X \times Y}}$$

is the Mukai vector. Then we have

$$\phi_{\mathcal{O}_\Delta} = \text{id}, \quad \phi_{\mathcal{F}} \circ \phi_{\mathcal{E}} = \phi_{\mathcal{F} \circ \mathcal{E}}.$$

We note that  $\phi_{\mathcal{E}}$  preserves the  $\mathbb{Z}$ -grading.

*Remark 4.3.* — By definition, Chern characters  $\text{Ch}$  and Todd classes  $\text{Td}$  of bounded complexes of coherent sheaves on  $X$  are elements of

$$\text{Im} \left( \bigoplus_{k=0}^{d_X} H^{2k}(X; \mathbb{Z}(k)) \rightarrow H^\bullet(X) \right),$$

where  $\mathbb{Z}(k)$  is the  $k$ -th Tate twist of  $\mathbb{Z}$  (see [17, §3.4]).

An exact functor  $F : D^b(X) \rightarrow D^b(Y)$  is called a Fourier–Mukai functor if there exists  $\mathcal{E} \in D^b(X \times Y)$  such that  $F \cong \Phi_{\mathcal{E}}$ . The complex  $\mathcal{E}$  is called a Fourier–Mukai kernel of  $F$ . For a Fourier–Mukai functor  $F \cong \Phi_{\mathcal{E}}$ , we define  $\phi_F$  by  $\phi_{\mathcal{E}}$ . By the next lemma, we see  $\phi_F$  is independent of a choice of a Fourier–Mukai kernel  $\mathcal{E}$ .

LEMMA 4.4 ([8, Corollary 4.4]). — *Let  $\mathcal{E}_1, \mathcal{E}_2 \in D^b(X \times Y)$ . If we have an equivalence  $\Phi_{\mathcal{E}_1} \cong \Phi_{\mathcal{E}_2}$ , then  $\phi_{\mathcal{E}_1} = \phi_{\mathcal{E}_2}$ .*

*Proof.* — By [8, Corollary 4.4], we have  $[\mathcal{E}_1] = [\mathcal{E}_2]$  in the  $K$ -group, which implies the lemma. □

Let  $\mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow \mathcal{E}_1[1]$  be an exact triangle in  $D^b(X \times Y)$ . Since Chern characters are additive, we have

$$\phi_{\mathcal{E}_2} = \phi_{\mathcal{E}_1} + \phi_{\mathcal{E}_3}.$$

By considering the case  $\mathcal{E}_2 \cong 0$ , we have  $\phi_{\mathcal{E}[1]} = -\phi_{\mathcal{E}}$ .

We define the left and right adjoint kernels  $\mathcal{E}^*, \mathcal{E}^! \in D^b(Y \times X)$  by

$$\mathcal{E}^* := (\sigma_{XY})^*(\mathcal{E}^\vee \otimes \pi_2^* \omega_Y[d_Y]), \quad \mathcal{E}^! := (\sigma_{XY})^*(\mathcal{E}^\vee \otimes \pi_1^* \omega_X[d_X]),$$

where  $\sigma_{XY} : Y \times X \rightarrow X \times Y$  is the natural isomorphism,  $\mathcal{E}^\vee$  is the dual  $R\mathcal{H}om(\mathcal{E}, \mathcal{O}_{X \times Y})$ , and  $\omega_X, \omega_Y$  are the canonical bundles. Then we see that  $\Phi_{\mathcal{E}^*}$  is the left adjoint of  $\Phi_{\mathcal{E}}$  and  $\Phi_{\mathcal{E}^!}$  is the right adjoint of  $\Phi_{\mathcal{E}}$ . Since the operations  $*$  and  $!$  preserve exact triangles, it follows that

$$(4.1) \quad \phi_{\mathcal{E}_2^*} = \phi_{\mathcal{E}_1^*} + \phi_{\mathcal{E}_3^*}, \quad \phi_{\mathcal{E}_2^!} = \phi_{\mathcal{E}_1^!} + \phi_{\mathcal{E}_3^!}$$

for an exact triangle  $\mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow \mathcal{E}_1[1]$ .

### 4.2. Hochschild homology for admissible subcategories

We define Hochschild homology for admissible subcategories and construct objects of  $\text{Rep}_{\mathbb{Z}}(\mathbb{C})$ . Similar construction has already been considered by many people (e.g., [7], [32], [45], [52]).

For a functor  $F$ , we denote by  $F^*$  (resp.  $F^!$ ) the left (resp. right) adjoint functor. Let  $\mathcal{A}$  be a full triangulated subcategory of  $D^b(X)$  and we denote by  $i_{\mathcal{A}} : \mathcal{A} \hookrightarrow D^b(X)$  the inclusion functor.  $\mathcal{A}$  is called admissible if  $i_{\mathcal{A}}$  has left and right adjoint functors. Note that an admissible subcategory  $\mathcal{A}$  is saturated, and hence all fully faithful functors to triangulated categories of finite type are also admissible (see [4, §2]). For a functor  $F$  to  $\mathcal{A}$ , the functor  $i_{\mathcal{A}} \circ F$  is denoted by  $\tilde{F}$ . For an admissible subcategory  $\mathcal{A}$ , we define left and right orthogonal to  $\mathcal{A}$  by

$$\begin{aligned} {}^\perp\mathcal{A} &:= \{A' \in D^b(X) \mid \forall A \in \mathcal{A} \text{ Hom}(A', A) = 0\}, \\ \mathcal{A}^\perp &:= \{A' \in D^b(X) \mid \forall A \in \mathcal{A} \text{ Hom}(A, A') = 0\}. \end{aligned}$$

Then  ${}^\perp\mathcal{A}$  and  $\mathcal{A}^\perp$  are also admissible (see [4, Proposition 3.6]). Moreover we have semiorthogonal decompositions

$$D^b(X) \cong \langle \mathcal{A}, {}^\perp\mathcal{A} \rangle \cong \langle \mathcal{A}^\perp, \mathcal{A} \rangle.$$

See Definition 4.12 for the definition of semiorthogonal decompositions. We denote by  $L_{\mathcal{A}}$  (resp.  $R_{\mathcal{A}}$ ) the projection functor from  $D^b(X)$  to  $\mathcal{A}$  with respect to the semiorthogonal decomposition  $\langle \mathcal{A}, {}^\perp\mathcal{A} \rangle$  (resp.  $\langle \mathcal{A}^\perp, \mathcal{A} \rangle$ ).

LEMMA 4.5. —  $i_{\mathcal{A}}^* \cong L_{\mathcal{A}}$  and  $i_{\mathcal{A}}^! \cong R_{\mathcal{A}}$ .

*Proof.* — For  $E \in D^b(X)$ , we have an exact triangle

$$\widetilde{R}_{\perp\mathcal{A}}E \rightarrow E \rightarrow \widetilde{L}_{\mathcal{A}}E \rightarrow \widetilde{R}_{\perp\mathcal{A}}E[1].$$

Since  $\text{Hom}_{D^b(X)}(\widetilde{R}_{\perp\mathcal{A}}E, A) = 0$  for  $A \in \mathcal{A}$ , by applying  $\text{Hom}_{D^b(X)}(-, A)$  to the above exact triangle, we have a functorial isomorphism

$$\text{Hom}_{\mathcal{A}}(L_{\mathcal{A}}E, A) \cong \text{Hom}_{D^b(X)}(E, A).$$

This implies  $i_{\mathcal{A}}^* \cong L_{\mathcal{A}}$ . The proof of  $i_{\mathcal{A}}^! \cong R_{\mathcal{A}}$  is similar. □

To define  $\phi_{\widetilde{L}_{\mathcal{A}}}$  and  $\phi_{\widetilde{R}_{\mathcal{A}}}$ , we need the following lemma:

LEMMA 4.6 ([32]). —  $\widetilde{L}_{\mathcal{A}}$  and  $\widetilde{R}_{\mathcal{A}}$  are Fourier–Mukai functors.

*Proof.* — Apply [32, Theorem 3.7] for semiorthogonal decompositions  $\langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle$  and  $\langle \mathcal{A}^{\perp}, \mathcal{A} \rangle$ . □

Using the above lemma, we define the Hochschild homology of  $\mathcal{A}$  as follows:

DEFINITION 4.7. — We define the Hochschild homology  $\text{HH}_{\bullet}(\mathcal{A})$  of  $\mathcal{A}$  by  $\text{Im } \phi_{\widetilde{R}_{\mathcal{A}}} \subset \text{HH}_{\bullet}(X)$ .

Let  $\mathcal{B} \subset D^b(Y)$  be an admissible subcategory. A functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is called a Fourier–Mukai functor if there exists  $\mathcal{E} \in D^b(X \times Y)$  such that  $\Phi_{\mathcal{E}}|_{\mathcal{A}} \cong \widetilde{F}$ . The complex  $\mathcal{E}$  is called a Fourier–Mukai kernel of  $F$ . To define  $\phi_F$  for a Fourier–Mukai functor  $F$ , we need some lemmas.

LEMMA 4.8. — Let  $\mathcal{B} \subset D^b(Y)$  be an admissible subcategory of  $D^b(Y)$  and  $\mathcal{E} \in D^b(X \times Y)$  be a bounded complex. If  $\Phi_{\mathcal{E}}(\mathcal{A}) \subset \mathcal{B}$ , then we have  $\phi_{\mathcal{E}}(\text{HH}_{\bullet}(\mathcal{A})) \subset \text{HH}_{\bullet}(\mathcal{B})$ .

*Proof.* — By assumption, we have

$$\Phi_{\mathcal{E}} \circ \widetilde{R}_{\mathcal{A}} \cong \widetilde{R}_{\mathcal{B}} \circ \Phi_{\mathcal{E}} \circ \widetilde{R}_{\mathcal{A}}.$$

Hence we have

$$\text{Im}(\phi_{\mathcal{E}} \circ \phi_{\widetilde{R}_{\mathcal{A}}}) \subset \text{Im } \phi_{\widetilde{R}_{\mathcal{B}}},$$

which implies the lemma. □

LEMMA 4.9. — If  $\Phi_{\mathcal{E}_1}|_{\mathcal{A}} \cong \Phi_{\mathcal{E}_2}|_{\mathcal{A}}$ , then  $\phi_{\mathcal{E}_1}|_{\text{HH}_{\bullet}(\mathcal{A})} = \phi_{\mathcal{E}_2}|_{\text{HH}_{\bullet}(\mathcal{A})}$ .

*Proof.* — By assumption, we have  $\Phi_{\mathcal{E}_1} \circ \widetilde{R}_{\mathcal{A}} \cong \Phi_{\mathcal{E}_2} \circ \widetilde{R}_{\mathcal{A}}$ . This implies the lemma. □

For a Fourier–Mukai functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  with a Fourier–Mukai kernel  $\mathcal{E}$ , we define  $\phi_F$  by  $\phi_{\mathcal{E}}|_{\mathrm{HH}_{\bullet}(\mathcal{A})}$ . By Lemma 4.8, we can consider  $\phi_F$  as a morphism from  $\mathrm{HH}_{\bullet}(\mathcal{A})$  to  $\mathrm{HH}_{\bullet}(\mathcal{B})$ . By Lemma 4.9, this is independent of a choice of a Fourier–Mukai kernel. Note that  $\mathrm{id}_{\mathcal{A}}$  and  $i_{\mathcal{A}}$  are Fourier–Mukai functors with a Fourier–Mukai kernel  $\mathcal{O}_{\Delta}$ . Hence we have  $\phi_{\mathrm{id}_{\mathcal{A}}} = \mathrm{id}_{\mathrm{HH}_{\bullet}(\mathcal{A})}$  and  $\phi_{i_{\mathcal{A}}}$  is the natural embedding of  $\mathrm{HH}_{\bullet}(\mathcal{A})$ . To define  $\mathrm{HH}_{\bullet}(\mathcal{A})$ , we used the specific projection  $\tilde{R}_{\mathcal{A}}$ . The next lemma implies that the definition of  $\mathrm{HH}_{\bullet}(\mathcal{A})$  is independent of a choice of a projection.

LEMMA 4.10. — *Let  $P : D^b(X) \rightarrow \mathcal{A}$  be a Fourier–Mukai functor which satisfies  $P|_{\mathcal{A}} \cong \mathrm{id}_{\mathcal{A}}$ . Then  $\mathrm{Im} \phi_P = \mathrm{HH}_{\bullet}(\mathcal{A})$ .*

*Proof.* — By assumption, we have  $P \circ \tilde{R}_{\mathcal{A}} \cong R_{\mathcal{A}}$  and  $R_{\mathcal{A}} \circ \tilde{P} \cong P$ . Hence we have  $\mathrm{Im} \phi_{R_{\mathcal{A}}} \subset \mathrm{Im} \phi_P$  and  $\mathrm{Im} \phi_P \subset \mathrm{Im} \phi_{R_{\mathcal{A}}}$ . □

We construct an object of  $\mathrm{Rep}_{\mathbb{Z}}(\mathbb{C})$  from  $\mathcal{A}$  and the Serre functor. We recall that for  $E \in D^b(X)$ , the functor  $-\otimes E$  is a Fourier–Mukai functor with a Fourier–Mukai kernel  $\Delta_*E$ . Hence the Serre functor  $S_X \cong - \otimes \omega_X[d_X]$  of  $D^b(X)$  is a Fourier–Mukai functor.

LEMMA 4.11. —  *$R_{\mathcal{A}} \circ S_X \circ i_{\mathcal{A}}$  is the Serre functor  $S_{\mathcal{A}}$  of  $\mathcal{A}$ . Especially,  $S_{\mathcal{A}}$  is a Fourier–Mukai functor.*

*Proof.* — Let  $A_1, A_2 \in \mathcal{A}$ . This lemma is proved by the following functorial isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(A_1, A_2) &\cong \mathrm{Hom}_{D^b(X)}(A_2, S_X(A_1))^{\vee} \\ &\cong \mathrm{Hom}_{\mathcal{A}}(A_2, R_{\mathcal{A}} \circ S_X(A_1))^{\vee}. \end{aligned} \quad \square$$

Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a Fourier–Mukai functor with a Fourier–Mukai kernel  $\mathcal{E}$ . Then we have an equivalence  $F \cong R_{\mathcal{B}} \circ \Phi_{\mathcal{E}} \circ i_{\mathcal{A}}$ , which implies the left adjoint  $F^* \cong L_{\mathcal{A}} \circ \Phi_{\mathcal{E}^*} \circ i_{\mathcal{B}}$  is a Fourier–Mukai functor. Hence, if  $F$  is an isomorphism, the quasi-inverse  $F^{-1} \cong F^*$  is a Fourier–Mukai functor and  $\phi_F$  is an isomorphism with the inverse  $\phi_{F^{-1}}$ . Set

$$T_{\mathcal{A}} := (-1)^{\mathrm{deg}} \phi_{S_{\mathcal{A}}^{-1}},$$

where  $(-1)^{\mathrm{deg}}$  is the sign operator defined by  $(-1)^k \cdot \mathrm{id}$  on  $\mathrm{HH}_k(X)$ . The above argument shows that  $(\mathrm{HH}_{\bullet}(\mathcal{A}), T_{\mathcal{A}})$  is an element of  $\mathrm{Rep}_{\mathbb{Z}}(\mathbb{C})$ . Moreover, for an isomorphism  $F$ , we have  $F \circ S_{\mathcal{A}} \cong S_{\mathcal{B}} \circ F$  (see, e.g., [25, Lemma 1.30]), which implies

$$\phi_F \circ T_{\mathcal{A}} = T_{\mathcal{B}} \circ \phi_F.$$

Thus  $\phi_F$  is a morphism in  $\mathrm{Rep}_{\mathbb{Z}}(\mathbb{C})$ . For simplicity of notation, we write  $T_X$  instead of  $T_{D^b(X)}$ . Then we have  $(\mathrm{HH}_{\bullet}(X), T_X) \in \mathrm{Rep}_{\mathbb{Z}}(\mathbb{C})$ .

### 4.3. Stokes data from semiorthogonal decompositions

In this section we construct Stokes data from (framed) semiorthogonal decompositions. We first recall the definition of semiorthogonal decompositions.

DEFINITION 4.12. — Let  $\mathcal{T}$  be a triangulated category. A sequence of full triangulated subcategories  $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$  is called a semiorthogonal decomposition of  $\mathcal{T}$  if  $\mathcal{A}_i \subset \mathcal{A}_j^\perp$  for  $i < j$  and for every  $T \in \mathcal{T}$  there exists the following sequence of exact triangles:

$$\begin{array}{ccccccccccc}
 0 = T_m & \longrightarrow & T_{m-1} & \longrightarrow & T_{m-2} & \longrightarrow & \cdots & \longrightarrow & T_1 & \longrightarrow & T_0 = T. \\
 & & \swarrow & & \swarrow & & & & \swarrow & & \swarrow \\
 & & \mathcal{A}_m & & \mathcal{A}_{m-1} & & & & \mathcal{A}_1 & & 
 \end{array}$$

Here  $A_i$  are objects of  $\mathcal{A}_i$ .

Let  $C$  be a finite set and  $\tau : C \xrightarrow{\sim} \{1, 2, \dots, m\}$  be a bijection. Let  $\{\text{gr}_c D^b(X)\}_{c \in C}$  be a family of admissible subcategories in  $D^b(X)$ .

DEFINITION 4.13. — A pair  $(\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in C})$  is called a framed semiorthogonal decomposition of type  $(C, \tau)$  with the frame  $\{\text{gr}_c D^b(X)\}_{c \in C}$  if  $\{\mathcal{A}_i\}_{1 \leq i \leq m}$  is a semiorthogonal decomposition  $D^b(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m \rangle$  and  $\{F_c\}_{c \in C}$  is a tuple of Fourier–Mukai isomorphisms

$$F_c : \text{gr}_c D^b(X) \xrightarrow{\sim} \mathcal{A}_{\tau(c)}.$$

We denote by  $P_i$  the projection functor from  $D^b(X)$  to  $\mathcal{A}_i$  with respect to the semiorthogonal decomposition  $\{\mathcal{A}_i\}_{1 \leq i \leq m}$ . Then we have

$$P_i \circ \tilde{P}_j \cong \begin{cases} P_i & (i = j) \\ 0 & (i \neq j). \end{cases}$$

The next theorem of Kuznetsov is essential for our construction.

THEOREM 4.14 ([32, Theorem 3.7]). — Every  $P_i$  is a Fourier–Mukai functor. Moreover, we have the following sequence of exact triangles in  $D^b(X \times X)$ :

$$\begin{array}{ccccccccccc}
 0 = \mathcal{D}_m & \longrightarrow & \mathcal{D}_{m-1} & \longrightarrow & \mathcal{D}_{m-2} & \longrightarrow & \cdots & \longrightarrow & \mathcal{D}_1 & \longrightarrow & \mathcal{D}_0 = \mathcal{O}_\Delta \\
 & & \swarrow & & \swarrow & & & & \swarrow & & \swarrow \\
 & & \mathcal{P}_m & & \mathcal{P}_{m-1} & & & & \mathcal{P}_1 & & 
 \end{array}$$

where  $\mathcal{P}_i$  is a Fourier–Mukai kernel of  $P_i$ .

Since  $P_i$  is a projection to  $\mathcal{A}_i$ , using Lemma 4.10, we have

$$\text{Im } \phi_{P_i} = \text{HH}_\bullet(\mathcal{A}_i).$$

Hence, as a corollary of Theorem 4.14, we have the following:

COROLLARY 4.15. —  $\text{HH}_\bullet(X) = \bigoplus_{i=1}^m \text{HH}_\bullet(\mathcal{A}_i)$ .

Let  $(\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in C})$  be a framed semiorthogonal decomposition of type  $(C, \tau)$  with a frame  $\{\text{gr}_c D^b(X)\}_{c \in C}$ . Then

$$\phi_{F_c} : (\text{HH}_\bullet(\text{gr}_c D^b(X)), T_{\text{gr}_c D^b(X)}) \xrightarrow{\sim} (\text{HH}_\bullet(\mathcal{A}_{\tau(c)}), T_{\mathcal{A}_{\tau(c)}})$$

is an isomorphism in  $\text{Rep}_{\mathbb{Z}}(\mathbb{C})$ . Set

$$\begin{aligned} Bf_c &:= \phi_{i_{\mathcal{A}_{\tau(c)}}} \circ \phi_{F_c} : \text{HH}_\bullet(\text{gr}_c D^b(X)) \hookrightarrow \text{HH}_\bullet(X), & Bf &:= \prod_{c \in C} Bf_c, \\ Bf_c^* &:= \phi_{F_c}^{-1} \circ \phi_{L_{\mathcal{A}_{\tau(c)}}} : \text{HH}_\bullet(X) \rightarrow \text{HH}_\bullet(\text{gr}_c D^b(X)), & Bf^* &:= \prod_{c \in C} Bf_c^*, \\ Bf_c^! &:= \phi_{F_c}^{-1} \circ \phi_{R_{\mathcal{A}_{\tau(c)}}} : \text{HH}_\bullet(X) \rightarrow \text{HH}_\bullet(\text{gr}_c D^b(X)), & Bf^! &:= \prod_{c \in C} Bf_c^!. \end{aligned}$$

Since  $R_{\mathcal{A}_i} \cong i_{\mathcal{A}_i}^! \cong S_{\mathcal{A}_i} \circ i_{\mathcal{A}_i}^* \circ S_X^{-1}$ , we have  $Bf_c^! = T_c^{-1} \circ Bf_c^* \circ T_X$ .

THEOREM 4.16. —  $((\text{HH}_\bullet(\text{gr}_c D^b(X)), T_{\text{gr}_c D^b(X)})_{c \in C}, Bf, Bf^*)$  defines Stokes data on  $(\text{HH}_\bullet(X), T_X)$  of type  $(C, \tau)$ .

*Proof.* — By Corollary 4.15,  $Bf$  is an isomorphism. By the definition of semiorthogonal decomposition, we have  $\mathcal{A}_j \subset {}^\perp \mathcal{A}_i$  for  $i < j$ . Hence we have

$$L_{\mathcal{A}_i} \circ i_{\mathcal{A}_j} \cong \begin{cases} \text{id}_{\mathcal{A}_j} & (i = j) \\ 0 & (i < j). \end{cases}$$

This implies

$$Bf_c^* \circ Bf_{c'} = \begin{cases} \text{id}_{V_{c'}} & (c = c') \\ 0 & (\tau(c) < \tau(c')). \end{cases}$$

Similarly, we have

$$Bf_c^! \circ Bf_{c'} = \begin{cases} \text{id}_{V_{c'}} & (c = c') \\ 0 & (\tau(c) > \tau(c')), \end{cases}$$

which proves the theorem. □

DEFINITION 4.17. — Let  $(\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in C})$  be a framed semiorthogonal decomposition of type  $(C, \tau)$  with a frame  $\{\text{gr}_c D^b(X)\}_{c \in C}$ . We define Stokes data

$$\mathfrak{B}((\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in C}))$$

on  $(\mathrm{HH}_\bullet(X), T_X)$  of type  $(C, \tau)$  by the tuple

$$\left( (\mathrm{HH}_\bullet(\mathrm{gr}_c D^b(X)), T_{\mathrm{gr}_c D^b(X)})_{c \in C}, {}^{Bf}, {}^{Bf*} \right).$$

#### 4.4. Mutations of framed semiorthogonal decompositions

In this section, we define mutations of framed semiorthogonal decompositions. Essentially, this construction is due to [4] (see also [32]).

Let  $(\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in C})$  be a framed semiorthogonal decomposition of type  $(C, \tau)$  with a frame  $\{\mathrm{gr}_c D^b(X)\}_{c \in C}$ .

DEFINITION 4.18 (cf. [32, §2.4]). — For each  $i \in \{1, 2, \dots, m - 1\}$ , we define

$$\begin{aligned} (\mathbb{R}_{i+1} \mathcal{A}_\bullet)_j &:= \begin{cases} \mathcal{A}_j & (j \neq i, i + 1) \\ \mathcal{A}_{i+1} & (j = i) \\ {}^\perp \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathcal{A}_{i+1} \rangle \cap \langle \mathcal{A}_{i+2}, \dots, \mathcal{A}_m \rangle^\perp & (j = i + 1), \end{cases} \\ (\mathbb{R}_{i+1} F_\bullet)_c &:= \begin{cases} F_c & (\tau(c) \neq i) \\ R_{\perp \mathcal{A}_{i+1}} \circ F_c & (\tau(c) = i). \end{cases} \end{aligned}$$

The pair  $(\{(\mathbb{R}_{i+1} \mathcal{A}_\bullet)_j\}_{1 \leq j \leq m}, \{(\mathbb{R}_{i+1} F_\bullet)_c\}_{c \in C})$  is called the right mutation of  $(\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in C})$ . Similarly, we define the left mutation

$$(\{(\mathbb{L}_i \mathcal{A}_\bullet)_j\}_{1 \leq j \leq m}, \{(\mathbb{L}_i F_\bullet)_c\}_{c \in C})$$

as follows:

$$\begin{aligned} (\mathbb{L}_i \mathcal{A}_\bullet)_j &:= \begin{cases} \mathcal{A}_j & (j \neq i, i + 1) \\ \mathcal{A}_i & (j = i + 1) \\ {}^\perp \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1} \rangle \cap \langle \mathcal{A}_i, \mathcal{A}_{i+2}, \dots, \mathcal{A}_m \rangle^\perp & (j = i), \end{cases} \\ (\mathbb{L}_i F_\bullet)_c &:= \begin{cases} F_c & (\tau(c) \neq i + 1) \\ L_{\mathcal{A}_i^\perp} \circ F_c & (\tau(c) = i + 1). \end{cases} \end{aligned}$$

To prove the propositions below (Propositions 4.20 and 4.21), we need the following:

LEMMA 4.19 ([4]). —  $R_{\perp \mathcal{A}_{i+1}}|_{\mathcal{A}_{i+1}^\perp} : \mathcal{A}_{i+1}^\perp \rightarrow {}^\perp \mathcal{A}_{i+1}$  is an isomorphism and the quasi-inverse is given by  $L_{\mathcal{A}_{i+1}^\perp}|_{{}^\perp \mathcal{A}_{i+1}}$ .

Proof. — See the proof of [4, Lemma 1.9]. □

PROPOSITION 4.20. — *The pairs  $(\{(\mathbb{R}_{i+1}\mathcal{A}_\bullet)_j\}_{1 \leq j \leq m}, \{(\mathbb{R}_{i+1}F_\bullet)_c\}_{c \in C})$  and  $(\{(\mathbb{L}_i\mathcal{A}_\bullet)_j\}_{1 \leq j \leq m}, \{(\mathbb{L}_iF_\bullet)_c\}_{c \in C})$  are framed semiorthogonal decompositions of type  $(C, s_i \circ \tau)$  with the frame  $\{\text{gr}_c D^b(X)\}_{c \in C}$ .*

*Proof.* — We only prove that  $(\{(\mathbb{R}_{i+1}\mathcal{A}_\bullet)_j\}_{1 \leq j \leq m}, \{(\mathbb{R}_{i+1}F_\bullet)_c\}_{c \in C})$  is a framed semiorthogonal decomposition. The proof for the left mutation is similar. By definition,  $\{(\mathbb{R}_{i+1}\mathcal{A}_\bullet)_j\}_{1 \leq j \leq m}$  is a semiorthogonal decomposition of  $D^b(X)$ . Thus it is sufficient to show that  $(\mathbb{R}_{i+1}F_\bullet)_c$  is an isomorphism between  $\text{gr}_c D^b(X)$  and  $(\mathbb{R}_{i+1}\mathcal{A}_\bullet)_{s_i \circ \tau(c)}$ . We only consider the case  $c = \tau^{-1}(i)$  since the case  $c \neq \tau^{-1}(i)$  is obvious. For  $A_i \in \mathcal{A}_i$ , we have the following exact triangle:

$$R_{\perp \mathcal{A}_{i+1}}(A_i) \rightarrow A_i \rightarrow L_{\mathcal{A}_{i+1}}(A_i) \rightarrow R_{\perp \mathcal{A}_{i+1}}(A_i)[1].$$

This implies

$$R_{\perp \mathcal{A}_{i+1}}(A_i) \in (\langle \mathcal{A}_i, \mathcal{A}_{i+1} \rangle \cap {}^\perp \mathcal{A}_{i+1}) = (\mathbb{R}_{i+1}\mathcal{A}_\bullet)_{i+1}.$$

From Lemma 4.19 and  $\mathcal{A}_i \subset \mathcal{A}_{i+1}^\perp$ , it follows that  $R_{\perp \mathcal{A}_{i+1}}|_{\mathcal{A}_i}$  is fully faithful.

It remains to show that  $R_{\perp \mathcal{A}_{i+1}}|_{\mathcal{A}_i} : \mathcal{A}_i \rightarrow (\mathbb{R}_{i+1}\mathcal{A}_\bullet)_{i+1}$  is essentially surjective. Since  $\mathcal{A}_i$  is saturated,  $\text{Im } R_{\perp \mathcal{A}_{i+1}}|_{\mathcal{A}_i} \subset (\mathbb{R}_{i+1}\mathcal{A}_\bullet)_{i+1}$  is admissible. Hence it is sufficient to show that

$$(\mathbb{R}_{i+1}\mathcal{A}_\bullet)_{i+1} \cap {}^\perp(\text{Im } R_{\perp \mathcal{A}_{i+1}}|_{\mathcal{A}_i}) \cong 0.$$

Choose  $A \in (\mathbb{R}_{i+1}\mathcal{A}_\bullet)_{i+1} \cap {}^\perp(\text{Im } R_{\perp \mathcal{A}_{i+1}}|_{\mathcal{A}_i})$ . Then, for all  $B \in \mathcal{A}_i$ , we have

$$\text{Hom}_{D^b(X)}(A, B) = \text{Hom}_{\perp \mathcal{A}_{i+1}}(A, R_{\perp \mathcal{A}_{i+1}}B) = 0.$$

Thus we have  $A \in {}^\perp \mathcal{A}_i \cap (\mathbb{R}_{i+1}\mathcal{A}_\bullet)_{i+1} \cong 0$ , which completes the proof.  $\square$

We construct an action of the braid group  $\text{Br}_m$  on the set of framed semiorthogonal decompositions. Set

$$\begin{aligned} \sigma_i(\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in C}) &:= (\{(\mathbb{R}_{i+1}\mathcal{A}_\bullet)_j\}_{1 \leq j \leq m}, \{(\mathbb{R}_{i+1}F_\bullet)_c\}_{c \in C}), \\ \sigma_i^{-1}(\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in C}) &:= (\{(\mathbb{L}_i\mathcal{A}_\bullet)_j\}_{1 \leq j \leq m}, \{(\mathbb{L}_iF_\bullet)_c\}_{c \in C}). \end{aligned}$$

PROPOSITION 4.21.

- (1)  $\sigma_i^{-1}\sigma_i = \sigma_i\sigma_i^{-1} = \text{id}$ .
- (2)  $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ .
- (3)  $\sigma_i\sigma_j = \sigma_j\sigma_i \quad (|i - j| \geq 2)$ .

*In other words,  $\sigma_i, \sigma_i^{-1}$  satisfy the braid relations.*

*Proof.*

(1). — By definition, we have

$$(\mathbb{L}_i(\mathbb{R}_{i+1}F_\bullet)_\bullet)_c = \begin{cases} F_c & (\tau(c) \neq i) \\ L_{\mathcal{A}_{i+1}^\perp} \circ R_{\perp\mathcal{A}_{i+1}} \circ F_c & (\tau(c) = i). \end{cases}$$

From Lemma 4.19 and  $\mathcal{A}_i \subset \mathcal{A}_{i+1}^\perp$ , we see that

$$L_{\mathcal{A}_{i+1}^\perp} \circ R_{\perp\mathcal{A}_{i+1}} \circ F_{\tau^{-1}(i)} \cong F_{\tau^{-1}(i)}.$$

This implies  $\sigma_i^{-1}\sigma_i = \text{id}$ . Similarly, we can prove  $\sigma_i\sigma_i^{-1} = \text{id}$ .

(2). — Set  $\mathcal{A}'_{i+2} := \text{Im } R_{\perp\mathcal{A}_{i+2}}|_{\mathcal{A}_{i+1}}$ . By definition, we have

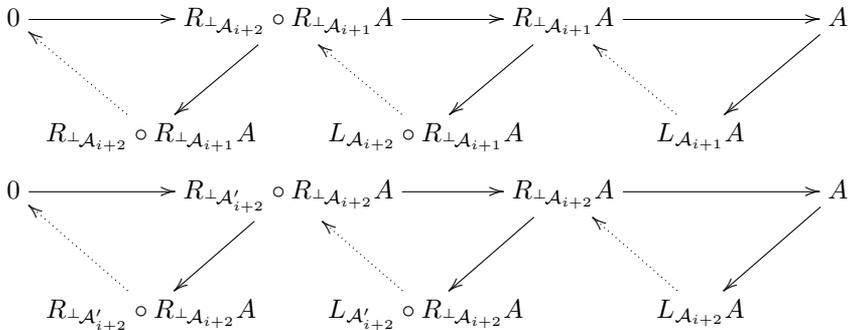
$$(\mathbb{R}_{i+1}(\mathbb{R}_{i+2}(\mathbb{R}_{i+1}F_\bullet)_\bullet)_\bullet)_c = \begin{cases} F_c & (\tau(c) \neq i, i+1) \\ R_{\perp\mathcal{A}_{i+2}} \circ F_c & (\tau(c) = i+1) \\ R_{\perp\mathcal{A}_{i+2}} \circ R_{\perp\mathcal{A}_{i+1}} \circ F_c & (\tau(c) = i), \end{cases}$$

$$(\mathbb{R}_{i+2}(\mathbb{R}_{i+1}(\mathbb{R}_{i+2}F_\bullet)_\bullet)_\bullet)_c = \begin{cases} F_c & (\tau(c) \neq i, i+1) \\ R_{\perp\mathcal{A}_{i+2}} \circ F_c & (\tau(c) = i+1) \\ R_{\perp\mathcal{A}'_{i+2}} \circ R_{\perp\mathcal{A}_{i+2}} \circ F_c & (\tau(c) = i). \end{cases}$$

Hence it is sufficient to show that

$$R_{\perp\mathcal{A}_{i+2}} \circ R_{\perp\mathcal{A}_{i+1}} \cong R_{\perp\mathcal{A}'_{i+2}} \circ R_{\perp\mathcal{A}_{i+2}}.$$

Choose  $A \in D^b(X)$ . Then we obtain the following sequences of exact triangles:



From these sequences of exact triangles, we see that  $R_{\perp\mathcal{A}_{i+2}} \circ R_{\perp\mathcal{A}_{i+1}}$  is the projection to  ${}^\perp\langle \mathcal{A}_{i+1}, \mathcal{A}_{i+2} \rangle$  and  $R_{\perp\mathcal{A}'_{i+2}} \circ R_{\perp\mathcal{A}_{i+2}}$  is the projection to  ${}^\perp\langle \mathcal{A}_{i+2}, \mathcal{A}'_{i+2} \rangle$ . Since

$$\langle \mathcal{A}_{i+1}, \mathcal{A}_{i+2} \rangle = \langle \mathcal{A}_{i+2}, \mathcal{A}'_{i+2} \rangle$$

as subcategories of  $D^b(X)$ , it follows that

$$R_{\perp \mathcal{A}_{i+2}} \circ R_{\perp \mathcal{A}_{i+1}} \cong R_{\perp \mathcal{A}'_{i+2}} \circ R_{\perp \mathcal{A}_{i+2}}.$$

(3). — Obvious from definition. □

*Remark 4.22.* — More precisely, the braid group  $\text{Br}_m$  acts on the set of equivalence classes of framed semiorthogonal decompositions with the frame  $\{\text{gr}_c D^b(X)\}_{c \in C}$ . Here two framed semiorthogonal decompositions

$$(\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in C}) \text{ and } (\{\mathcal{A}'_i\}_{1 \leq i \leq m}, \{F'_c\}_{c \in C})$$

with the frame  $\{\text{gr}_c D^b(X)\}_{c \in C}$  are equivalent if and only if  $\tilde{F}_c \cong \tilde{F}'_c$  for all  $c \in C$ .

For  $\sigma \in \text{Br}_m$ , we can define the framed semiorthogonal decomposition  $\sigma(\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in C})$  of type  $(C, \bar{\sigma} \circ \tau)$  with the frame  $\{\text{gr}_c D^b(X)\}_{c \in C}$  as the composition of mutations.

**THEOREM 4.23.** — *We have*

$$\mathfrak{B}(\sigma(\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in C})) = \mathbb{M}_\sigma(\mathfrak{B}(\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in C})).$$

*Proof.* — By the braid relations, it is sufficient to show the case  $\sigma = \sigma_i$  for some  $i$ . We only show that

$$\begin{aligned} \phi_{(\mathbb{R}_{i+1} F_\bullet)_c} &= R_{i+1} \circ f_c, \\ \phi_{(\mathbb{R}_{i+1} F_\bullet)_c}^* &= f_c^* \circ R_{i+1}^*. \end{aligned}$$

for  $c = \tau^{-1}(i)$  since the rest of the proof is evident. The first equality is shown by

$$R_{i+1} = \text{id}_{V_X} - \phi_{\tilde{L}_{\mathcal{A}_{i+1}}} = \phi_{\tilde{R}_{\perp \mathcal{A}_{i+1}}}.$$

We will show the second equality. Note that, for an admissible subcategory  $\mathcal{A}$ , we have

$$(\tilde{R}_{\mathcal{A}})^* = (i_{\mathcal{A}} \circ R_{\mathcal{A}})^* \cong \tilde{L}_{\mathcal{A}}.$$

Since  $(\mathbb{R}_{i+1} F_\bullet)_c \cong \tilde{R}_{\perp \mathcal{A}_{i+1}} \circ i_{\mathcal{A}_i} \circ F_c$ , we have

$$(\mathbb{R}_{i+1} F_\bullet)_c^* \cong F_c^{-1} \circ L_{\mathcal{A}_i} \circ \tilde{L}_{\perp \mathcal{A}_{i+1}}.$$

Hence it follows that

$$\phi_{(\mathbb{R}_{i+1} F_\bullet)_c}^* = f_c^* \circ \phi_{\tilde{L}_{\perp \mathcal{A}_{i+1}}}.$$

Since  $\tilde{L}_{\perp \mathcal{A}_{i+1}} \cong (\tilde{R}_{\perp \mathcal{A}_{i+1}})^* \cong S_X^{-1} \circ \tilde{R}_{\perp \mathcal{A}_{i+1}}^1 \circ S_X$ , we have

$$\phi_{\tilde{L}_{\perp \mathcal{A}_{i+1}}} = T_X \circ \phi_{\tilde{R}_{\perp \mathcal{A}_{i+1}}^1} \circ T_X^{-1}.$$

By simple computation, it follows

$$\begin{aligned} \phi_{\widetilde{R}^1_{\perp \mathcal{A}_{i+1}}} &= \text{id}_{V_X} - \phi_{\widetilde{L}^1_{\mathcal{A}_{i+1}}} \\ &= \text{id}_{V_X} - \phi_{\widetilde{R}_{\mathcal{A}_{i+1}}} \\ &= L_{i+1}, \end{aligned}$$

where in the first line we use (4.1). Thus we have

$$\phi_{\widetilde{L}^{\perp \mathcal{A}_{i+1}}} = T_X \circ L_{i+1} \circ T_X^{-1} = R_{i+1}^*,$$

which proves the claim. □

### 4.5. Pairings on Hochschild homology and B-mutation systems

In this section, we introduce a slightly modified version of the generalized Mukai pairing and show that this pairing is compatible with the monodromy  $T_X$  and the Stokes data constructed from a framed semiorthogonal decomposition.

DEFINITION 4.24 (see also [7], [44], [46], [47], [52]). — We define a Pairing on  $\text{HH}_\bullet(X)$  by

$$[\alpha, \beta]_X := \frac{1}{(2\pi i)^{d_X}} \int_X e^{\pi i \rho_X} W(\alpha) \cup \beta,$$

where  $W(\alpha) := i^{p+q} \alpha$  for  $\alpha \in H^q(X, \Omega_X^p)$ .

Since

$$e^{\pi i \mu} e^{-\pi i \rho_X} = e^{\pi i \rho_X} e^{\pi i \mu} = i^{-d_X} e^{\pi i \rho_X} W,$$

we see that  $[\cdot, \cdot]_X = [\cdot, \cdot]_X$ . To compute  $T_X$ , we show the following lemma.

LEMMA 4.25. — For  $E \in D^b(X)$ , we have  $\phi_{-\otimes E} = - \cup \text{Ch}(E)$ .

Proof. — Let  $\pi_1$  (resp.  $\pi_2$ ) be the projection from  $X \times X$  to the first (resp. second) factor.

We recall that  $-\otimes E$  is a Fourier–Mukai functor with a Fourier–Mukai kernel  $\Delta_* E$ . We compute  $\phi_{-\otimes E}(\alpha)$ . By definition, we have

$$\phi_{-\otimes E}(\alpha) = \pi_{2*}(\pi_1^*(\alpha) \cup \text{Ch}(\Delta_* E) \sqrt{\text{Td}_{X \times X}}).$$

Using the Riemann–Roch theorem and the projection formula, we have

$$\begin{aligned} \text{Ch}(\Delta_* E) \sqrt{\text{Td}_{X \times X}} &= \Delta_*(\text{Ch}(E) \text{Td}_X) \cup \sqrt{\text{Td}_{X \times X}}^{-1} \\ &= \Delta_*(\text{Ch}(\mathcal{E}) \Delta^*(\sqrt{\text{Td}_{X \times X}})) \cup \sqrt{\text{Td}_{X \times X}}^{-1} \\ &= \Delta_*(\text{Ch}(E)). \end{aligned}$$

Thus we have

$$\begin{aligned} \phi_{-\otimes E}(\alpha) &= \pi_{2*}(\pi_1^*(\alpha) \cup \Delta_*(\text{Ch}(E))) \\ &= \pi_{2*}\Delta_*(\Delta^*\pi_1^*(\alpha) \cup \text{Ch}(E)) \\ &= \alpha \cup \text{Ch}(E). \end{aligned} \quad \square$$

As a special case of  $E = \omega_X^{-1}[-d_X]$ , we have  $T_X = (-1)^{d_X}(-1)^{\text{deg}}e^{2\pi i\rho_X}$ .

To show some statements about  $[\cdot, \cdot]_X$ , we use the following properties of the morphism  $W$ :

- $W$  is a ring homomorphism.
- For  $\alpha, \beta \in \text{HH}_\bullet(X)$ , we have  $\int_X (-1)^{d_X} W(\alpha) \cup \beta = \int_X W(\beta) \cup \alpha$ .
- $W(c_1(X)) = -c_1(X)$ ,  $W(\text{Ch}(E^\vee)) = \text{Ch}(E)$ ,  
 $W(\sqrt{\text{Td}_X}) = \sqrt{\text{Td}_X} \cup e^{-\pi i c_1(X)}$ .
- For a map  $f : X \rightarrow Y$ , we have

$$W \circ f^* = f^* \circ W, \quad W \circ f_* = (-1)^{d_Y - d_X} f_* \circ W.$$

PROPOSITION 4.26. —  $[\cdot, \cdot]_X$  is compatible with  $T_X$ .

*Proof.* — We need to show  $[T_X \alpha, \beta]_X = (-1)^{\text{deg } \alpha} [\beta, \alpha]_X$ , i.e.,

$$\frac{1}{(2\pi i)^{d_X}} \int_X e^{\pi i \rho_X} W((-1)^{\text{deg } \alpha - d_X} e^{2\pi i \rho_X} \alpha) \cup \beta = \frac{(-1)^{\text{deg } \alpha}}{(2\pi i)^{d_X}} \int_X e^{\pi i \rho_X} W(\beta) \cup \alpha$$

for homogeneous elements  $\alpha, \beta \in \text{HH}_\bullet(X)$ . Using the above properties of  $W$ , this follows from simple computation. □

The next proposition gives a characterization of  $\phi_{\mathcal{E}^*}, \phi_{\mathcal{E}^\dagger}$  in terms of the pairing  $[\cdot, \cdot]_X$ .

PROPOSITION 4.27 (cf. [7, Theorem 8]). — For  $\mathcal{E} \in D^b(X \times Y), \alpha \in \text{HH}_\bullet(Y), \beta \in \text{HH}_\bullet(X)$ , we have

- (1)  $[\alpha, \phi_{\mathcal{E}}(\beta)]_Y = [\phi_{\mathcal{E}^*}(\alpha), \beta]_X$ ,
- (2)  $[\phi_{\mathcal{E}}(\beta), \alpha]_Y = [\beta, \phi_{\mathcal{E}^\dagger}(\alpha)]_X$ .

*Proof.* — Set  $\tilde{\pi}_1 := \pi_2 \circ \sigma_{XY}$  and  $\tilde{\pi}_2 := \pi_1 \circ \sigma_{XY}$ . We first show

$$(4.2) \quad (-1)^{d_Y} W(\nu(\mathcal{E}^*)) \cup \tilde{\pi}_2^* e^{\pi i c_1(X)} = \sigma_{XY}^*(\nu(\mathcal{E}) \cup \pi_2^* e^{\pi i c_1(Y)}).$$

By definition, we have  $\text{Ch}(\mathcal{E}^*) = \sigma_{XY}^*(\text{Ch}(\mathcal{E}^\vee) \cup (-1)^{d_Y} \pi_2^* e^{-2\pi i c_1(Y)})$ , which implies

$$W(\text{Ch}(\mathcal{E}^*)) = \sigma_{XY}^*(\text{Ch}(\mathcal{E}) \cup (-1)^{d_Y} \pi_2^* e^{2\pi i c_1(Y)}).$$

Hence we have

$$\begin{aligned} W(\nu(\mathcal{E}^*)) &= W(\text{Ch}(\mathcal{E}^*)) \cup W(\sqrt{\text{Td}_{Y \times X}}) \\ &= \sigma_{XY}^*(\text{Ch}(\mathcal{E}) \cup (-1)^{d_Y} \pi_2^* e^{2\pi i c_1(Y)}) \cup W(\sqrt{\text{Td}_{X \times Y}}), \end{aligned}$$

where we use  $\sigma_{XY}^* \sqrt{\text{Td}_{X \times Y}} = \sqrt{\text{Td}_{Y \times X}}$ . Using

$$W(\sqrt{\text{Td}_{X \times Y}}) = \sqrt{\text{Td}_{X \times Y}} \cup e^{-\pi i c_1(X \times Y)},$$

it follows that

$$\begin{aligned} W(\nu(\mathcal{E}^*)) &= \sigma_{XY}^*(\text{Ch}(\mathcal{E}) \cup (-1)^{d_Y} \pi_2^* e^{\pi i c_1(Y)} \cup \sqrt{\text{Td}_{X \times Y}} \cup \pi_1^* e^{-\pi i c_1(X)}) \\ &= \sigma_{XY}^*(\text{Ch}(\mathcal{E}) \cup \pi_2^* e^{\pi i c_1(Y)} \cup \sqrt{\text{Td}_{X \times Y}}) \cup (-1)^{d_Y} \tilde{\pi}_2^* e^{-\pi i c_1(X)}, \end{aligned}$$

which implies the equality (4.2).

We next show (1). By definition, we have

$$\begin{aligned} [\phi_{\mathcal{E}^*}(\alpha), \beta]_X &= \frac{1}{(2\pi i)^{d_X}} \int_X e^{\pi i \rho_X} W(\tilde{\pi}_{2*}(\tilde{\pi}_1^* \alpha \cup \nu(\mathcal{E}^*))) \cup \beta \\ &= \frac{(-1)^{d_Y}}{(2\pi i)^{d_X}} \int_X \tilde{\pi}_{2*}(\tilde{\pi}_1^* W(\alpha) \cup W(\nu(\mathcal{E}^*))) \cup e^{\pi i \rho_X} \beta \\ &= \frac{(-1)^{d_Y}}{(2\pi i)^{d_{X \times Y}}} \int_{Y \times X} \tilde{\pi}_1^* W(\alpha) \cup W(\nu(\mathcal{E}^*)) \cup \tilde{\pi}_2^*(e^{\pi i \rho_X} \beta), \end{aligned}$$

where in the third line we use the projection formula. Using the equality (4.2) and the projection formula, we have

$$\begin{aligned} [\phi_{\mathcal{E}^*}(\alpha), \beta]_X &= \frac{1}{(2\pi i)^{d_{X \times Y}}} \int_{Y \times X} \sigma_{XY}^*(\pi_2^* W(\alpha) \cup \nu(\mathcal{E}) \cup \pi_2^* e^{\pi i c_1(Y)} \cup \pi_1^* \beta) \\ &= \frac{1}{(2\pi i)^{d_{X \times Y}}} \int_{X \times Y} \pi_2^*(e^{\pi i \rho_Y} W(\alpha)) \cup \pi_1^* \beta \cup \nu(\mathcal{E}) \\ &= \frac{1}{(2\pi i)^{d_Y}} \int_Y e^{\pi i \rho_Y} W(\alpha) \cup \pi_{2*}(\pi_1^* \beta \cup \nu(\mathcal{E})). \end{aligned}$$

This proves (1).

Finally, we show (2). Since  $\Phi_{\mathcal{E}!} \cong S_X \circ \Phi_{\mathcal{E}^*} \circ S_Y^{-1}$ , we see that

$$\phi_{\mathcal{E}!} = T_X^{-1} \circ \phi_{\mathcal{E}^*} \circ T_Y.$$

Hence the statement follows from (1) and Proposition 4.26. □

Let  $\mathcal{A}$  be an admissible subcategory of  $D^b(X)$ . We define a pairing  $[\cdot, \cdot]_{\mathcal{A}}$  on  $\text{HH}_{\bullet}(\mathcal{A})$  by the restriction of  $[\cdot, \cdot]_X$  to  $\text{HH}_{\bullet}(\mathcal{A})$ .

LEMMA 4.28. — *The pairing  $[\cdot, \cdot]_{\mathcal{A}}$  is compatible with  $T_{\mathcal{A}}$ .*

*Proof.* — By Lemma 4.11, we have  $S_{\mathcal{A}} \cong R_{\mathcal{A}} \circ S_X \circ i_{\mathcal{A}}$ . Hence we have  $S_{\mathcal{A}}^{-1} \cong S_{\mathcal{A}}^* \cong L_{\mathcal{A}} \circ S_X^{-1} \circ i_{\mathcal{A}}$  and  $T_{\mathcal{A}} = \phi_{L_{\mathcal{A}}} \circ T_X|_{\text{HH}_{\bullet}(\mathcal{A})}$ . Since  $L_{\mathcal{A}}^1 \cong i_{\mathcal{A}}$ , the statement follows from Proposition 4.26 and (2) of Proposition 4.27. □

LEMMA 4.29. — *Let  $\mathcal{A}$  and  $\mathcal{B}$  be admissible subcategories of  $D^b(X)$ . Suppose that  $\mathcal{B} \subseteq \mathcal{A}^{\perp}$ . Then  $[\alpha, \beta]_X = 0$  for  $\alpha \in \text{HH}_{\bullet}(\mathcal{A})$  and  $\beta \in \text{HH}_{\bullet}(\mathcal{B})$ .*

*Proof.* — Note that  $\tilde{L}_{\mathcal{A}}^! = (i_{\mathcal{A}} \circ L_{\mathcal{A}})^! \cong \tilde{R}_{\mathcal{A}}$ . Hence we have

$$[\alpha, \beta]_X = [\phi_{\tilde{L}_{\mathcal{A}}}(\alpha), \beta]_X = [\alpha, \phi_{\tilde{R}_{\mathcal{A}}}(\beta)]_X.$$

Since  $\mathcal{B} \subset \mathcal{A}^\perp$  we see  $\tilde{R}_{\mathcal{A}}|_{\mathcal{B}} \cong 0$ , which implies the lemma. □

Applying this lemma to  $\mathcal{A}$  and  $\mathcal{A}^\perp$ , we see that  $[\cdot, \cdot]_{\mathcal{A}}$  is non-degenerate. In general, for a Fourier–Mukai functor  $F$ , the morphism  $\phi_F$  does not preserve the pairings.

LEMMA 4.30. — *Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a Fourier–Mukai functor. Suppose that  $F$  is fully faithful. Then  $\phi_F$  preserves the pairings.*

*Proof.* — Note that  $F^*$  is a Fourier–Mukai functor. Since  $F$  is fully faithful, it follows  $F^* \circ F \cong \text{id}_{\mathcal{A}}$  (e.g., [25, Corollary 1.22]). Thus we have

$$[\phi_F(\alpha), \phi_F(\beta)]_{\mathcal{B}} = [\phi_{F^* \circ F}(\alpha), \beta]_{\mathcal{A}} = [\alpha, \beta]_{\mathcal{A}}. \quad \square$$

Let  $(\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in C})$  be a framed semiorthogonal decomposition of type  $(C, \tau)$  with a frame  $\{\text{gr}_c D^b(X)\}_{c \in C}$  of  $D^b(X)$ . Combining Lemmas 4.28, 4.29, and 4.30, we conclude

THEOREM 4.31. — *The pairing  $[\cdot, \cdot]_X$  is compatible with the Stokes data  $\mathfrak{B}((\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in C}))$ , and hence gives a mutation system.*

DEFINITION 4.32. — *This mutation system is called a B-mutation system.*

To simplify notation, the B-mutation system is also denoted by

$$\mathfrak{B}((\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in C})).$$

### 5. Dubrovin type conjectures

Let  $X$  be a Fano manifold. Fix  $-C_X$ -generic  $\theta_o \in \mathbb{R}$ . In Section 3, under assumption (3.3), we construct the A-mutation system  $\mathfrak{A}_{\theta_o}(\text{RH}(\mathcal{M}_X, \mathcal{Q}_X))$  with the splitting data  $(\tau_{\theta_o}, A f_{\theta_o})$ . On the other hand, in Section 4, we construct the B-mutation system  $\mathfrak{B}((\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in -C_X}))$  with the splitting data  $(\tau_{\theta_o}, B f)$  from a framed semiorthogonal decomposition of type  $(-C_X, \tau_{\theta_o})$  with a frame  $\{\text{gr}_c D^b(X)\}_{c \in -C_X}$ .

Let

$$\hat{\Gamma}_X = \prod_{i=1}^{d_X} \Gamma(1 + \delta_i) \in H^\bullet(X)$$

be the Gamma class of  $X$ , where  $\delta_1, \delta_2, \dots, \delta_{d_X} \in H^\bullet(X; \mathbb{Z})$  are the Chern roots of  $TX$  and  $\Gamma(z)$  is the Gamma function. We define an isomorphism

$$\Gamma : \text{HH}_\bullet(X) \rightarrow H^\bullet(X)$$

by  $\Gamma(\alpha) := (\alpha \cup \hat{\Gamma}_X) / \sqrt{\text{Td}_X}$ , where we identify  $H^\bullet(X)$  with  $\text{HH}_\bullet(X)$  via the Hodge decomposition.

LEMMA 5.1. — For  $\alpha, \beta \in \text{HH}_\bullet(X)$ , we have  $[\Gamma(\alpha), \Gamma(\beta)]_X = [\alpha, \beta]_X$ .

Proof. — Note that  $[\alpha, \beta]_X = [\alpha, \beta]_X$ . Hence it is sufficient to show that  $[\Gamma(\alpha), \Gamma(\beta)]_X = [\alpha, \beta]_X$ . By the identity

$$e^{\pi iz} \Gamma(1 - z) \Gamma(1 + z) = \frac{2\pi iz}{1 - e^{-2\pi iz}},$$

we see that

$$e^{\pi i \rho_X} W(\hat{\Gamma}_X) \cup \hat{\Gamma}_X = \text{Td}_X.$$

From this and  $W(\sqrt{\text{Td}_X}) = e^{-\pi i \rho_X} \sqrt{\text{Td}_X}$ , we obtain

$$e^{\pi i \rho_X} W(\Gamma(\alpha)) \cup \Gamma(\beta) = e^{\pi i \rho_X} W(\alpha) \cup \beta.$$

Hence we have  $[\Gamma(\alpha), \Gamma(\beta)]_X = [\alpha, \beta]_X$ . □

DEFINITION 5.2. — We say that  $X$  satisfies Dubrovin type conjecture if the following conditions hold:

- The meromorphic connection  $\mathcal{M}_X$  is of exponential type (3.3).
- There exists a framed semiorthogonal decomposition of type  $(-C_X, \tau_{\theta_o})$  with a frame  $\{\text{gr}_c D^b(X)\}_{c \in -C_X}$  such that  $\Gamma$  gives an isomorphism of mutation systems, i.e.,

$$\text{Im } Af_{\theta_o, c} = \text{Im}(\Gamma \circ Bf_c)$$

for all  $c \in -C_X$ .

Remark 5.3. — For another  $-C_X$ -generic  $\theta'_o$ , we can construct another A-mutation system. By Theorem 2.19, these mutation systems are related by the braid group action. From this and Theorem 4.23, we see that  $X$  satisfies Dubrovin type conjecture for  $\theta_o$  if and only if  $X$  satisfies Dubrovin type conjecture for  $\theta'_o$ .

Remark 5.4.

- If (even part of) the quantum cohomology of  $X$  is semisimple and  $D^b(X)$  has a full exceptional collection, then Dubrovin type conjecture defined above follows from Gamma conjecture II ([17, Conjecture 4.6.1]).
- Gamma conjecture II is closely related to Dubrovin's (original) conjecture [15] (see [17, §4.6] for a more detailed explanation)

## 6. Properties of quantum connections of Fano manifolds

### 6.1. Symmetry of fundamental solutions of quantum connections

Let  $X$  be a Fano manifold. We denote by  $r_X$  the Fano index of  $X$ , that is,

$$r_X := \max \left\{ r \in \mathbb{Z}_{>0} \mid \frac{c_1(X)}{r} \in H^2(X; \mathbb{Z}) \right\}.$$

Recall that  $\text{Pic}(X) \cong H^2(X; \mathbb{Z})$  since  $X$  is Fano. Let  $\mathcal{O}(k)$  be the line bundle which satisfies  $c_1(\mathcal{O}(k)) = kc_1(X)/r_X$ . Set  $H_X := c_1(\mathcal{O}(1))$ . Let  $\mathbb{C}_q$  be the complex plane with the coordinate  $q$ . Set  $c_1(X)*_q := c_1(X)*_{c_1(X) \log q}$ . Note that  $c_1(X)*_q \in \text{End}(H^\bullet(X)) \otimes \mathbb{C}[q^{r_X}]$  by the divisor axiom. Recall that the set of eigenvalues of the operator  $c_1(X)*_0$  is denoted by  $C_X$ . We denote by  $E(c)$  the generalized eigenspace of  $c_1(X)*_0$  associated with the eigenvalue  $c \in C_X$ . Set  $\log \omega_k := -(2\pi i k)/r_X$  and  $\omega_k := e^{\log \omega_k}$  for  $k \in \mathbb{Z}$ . We recall the following equation (See, e.g., [17, §2.2]):

$$(6.1) \quad q^\mu (c_1(X)*_q) q^{-\mu} = q(c_1(X)*_0).$$

By substituting  $\omega_k$  for  $q$ , we have the following lemma:

LEMMA 6.1. — *If  $c \in C_X$ , then we have  $\omega_k c \in C_X$ . Moreover, we have  $E(c) \cong E(\omega_k c)$  and this isomorphism is given by  $\alpha \rightarrow \omega_k^{-\mu} \alpha$ .*

*Proof.* — Since  $c_1(X)*_q \in \text{End}(H^\bullet(X)) \otimes \mathbb{C}[q^{r_X}]$ , it follows that

$$c_1(X)*_{\omega_k} = c_1(X)*_0.$$

Hence the lemma follows from the equation (6.1). □

We introduce a one parameter deformation of the quantum connection  $(\mathcal{H}_X, \nabla)$  of  $X$ , which is the restriction of the usual Dubrovin connection ([14], [15]) to the  $c_1(X)$  direction (see also [12, Chapter 8], [31, §3.2.1] and references therein). We consider the trivial bundle

$$\tilde{\mathcal{H}}_X := (H^\bullet(X) \times \mathbb{C}_z \times \mathbb{C}_q \rightarrow \mathbb{C}_z \times \mathbb{C}_q)$$

with the meromorphic connection

$$\tilde{\nabla} := d - \left( \frac{1}{z} (c_1(X)*_q) - \mu \right) \frac{dz}{z} + \left( \frac{c_1(X)*_q}{z} \right) \frac{dq}{q}.$$

By definition, the restriction of  $(\tilde{\mathcal{H}}_X, \tilde{\nabla})$  to  $q = 1$  is the quantum connection  $(\mathcal{H}_X, \nabla)$ .

Set

$$\tilde{S}(z, q) := q^{-\mu} S\left(\frac{z}{q}\right) \left(\frac{z}{q}\right)^{-\mu} \left(\frac{z}{q}\right)^{\rho_X}.$$

This is a holomorphic map defined on the universal cover  $\tilde{\mathbb{C}}_z^* \times \tilde{\mathbb{C}}_q^*$ .

LEMMA 6.2. —  $\tilde{S}$  gives a fundamental solution of the meromorphic connection  $(\tilde{\mathcal{H}}_X, \tilde{\nabla})$ , that is,  $\tilde{\nabla}\tilde{S} = 0$ .

*Proof.* — Set  $S'(z) := S(z)z^{-\mu}z^{\rho_X}$ . This lemma is proved by the following computation:

$$\begin{aligned} \tilde{\nabla}_{\frac{\partial}{\partial z}} \tilde{S} &= q^{-\mu} \frac{1}{q} \left(\frac{dS'}{dz}\right) \left(\frac{z}{q}\right) + q^{-\mu} \frac{1}{q} \left(\frac{q}{z}\right) \mu S' \left(\frac{z}{q}\right) \\ &\quad - q^{-\mu} \frac{1}{q} \left(\frac{q}{z}\right)^2 (c_1(X)*_0) S' \left(\frac{z}{q}\right) \\ &= q^{-\mu} \frac{1}{q} (\nabla_{\frac{d}{dz}} S') \left(\frac{z}{q}\right) \\ &= 0, \end{aligned}$$

$$\begin{aligned} \tilde{\nabla}_{\frac{\partial}{\partial q}} \tilde{S} &= -\mu q^{-\mu} \frac{1}{q} S' \left(\frac{z}{q}\right) - q^{-\mu} \left(\frac{z}{q^2}\right) \left(\frac{dS'}{dz}\right) \left(\frac{z}{q}\right) + q^{-\mu} \frac{1}{z} (c_1(X)*_0) S' \left(\frac{z}{q}\right) \\ &= -q^{-\mu} \left(\frac{z}{q^2}\right) (\nabla_{\frac{d}{dz}} S') \left(\frac{z}{q}\right) \\ &= 0, \end{aligned}$$

where we use the equation (6.1). □

Recall that  $T(z) = z^\mu S(z)z^{-\mu}$ .

PROPOSITION 6.3. —  $T(z/\omega_k) = T(z)$ .

*Proof.* — Since  $\tilde{\nabla}\tilde{S} = 0$  and  $c_1(X)*_{\omega_k} = c_1(X)*_0$ , we see that

$$\tilde{S}(z, \omega_k) = z^{-\mu} T(z/\omega_k) z^{\rho_X} \omega_k^{-\rho_X}$$

is a fundamental solution of  $(\mathcal{H}_X, \nabla)$ . Hence  $z^{-\mu} T(z/\omega_k) z^{\rho_X}$  is also a fundamental solution. It is obvious that  $T(z/\omega_k) = \text{id}$  at  $z = \infty$ . This implies  $T(z/\omega_k) = T(z)$  by uniqueness (Proposition 3.3). □

### 6.2. Gamma conjecture I

In this section, we recall Gamma conjecture I and derive some assertions. First, we recall Property  $\mathcal{O}$ . Set

$$T_X := \max \{|c| \mid c \in C_X\}.$$

DEFINITION 6.4 ([17, Definition 3.1.1]). — We say that  $X$  satisfies Property  $\mathcal{O}$  if the following conditions hold:

- $T_X \in C_X$ .
- If  $c \in C_X$  and  $|c| = T_X$ , then  $c = \omega_k T_X$  for some  $k$ .
- The multiplicity of  $T_X$  is one, i.e.,  $\dim E(T_X) = 1$ .

Remark 6.5. — Galkin–Golyshev–Iritani [17] only considered the even part cohomology  $H^{\text{ev}}(X) := \bigoplus_{k \in 2\mathbb{Z}} H^k(X)$ . But the next lemma implies our definition of Property  $\mathcal{O}$  is equivalent to the original one. Note that the set of eigenvalues of  $(c_1(X) * 0)|_{H^{\text{ev}}(X)}$  is equal to  $C_X$  (see, e.g., the proof of [18, Proposition 7.1]). Hence we easily see that our Property  $\mathcal{O}$  implies the original one.

LEMMA 6.6. — Assume that  $H^{\text{ev}}(X)$  satisfies the same conditions as Definition 6.4. Then  $X$  satisfies Property  $\mathcal{O}$ .

Proof. — The proof is the same as [22, Proposition 1.2]. Set

$$H^{\text{odd}}(X) := \bigoplus_{k-1 \in 2\mathbb{Z}} H^k(X).$$

Let  $\alpha$  be an element of  $H^{\text{odd}}(X) \cap E(c) \setminus \{0\}$ . Take  $\beta \in H^{\text{odd}}(X)$  such that  $(\alpha, \beta)_X = 1$ . Then we have  $\alpha * 0 \beta \in H^{\text{ev}}(X) \cap E(c) \setminus \{0\}$ , and  $(\alpha * 0 \beta)^2 = 0$ . Thus we have  $\dim(H^{\text{ev}}(X) \cap E(c)) \geq 2$ , which implies the lemma.  $\square$

We define the subspace

$$\mathcal{A}_{(c,\theta)} \subset \left\{ s : \tilde{\mathbb{C}}_z^* \rightarrow H^\bullet(X) \mid \nabla s = 0 \right\}$$

of flat sections as follows (cf. [17, §3.3]). A section  $s$  is an element of  $\mathcal{A}_{(c,\theta)}$  if and only if there exists a constant  $C$  and a non-negative integer  $N$  such that  $\|e^{c/z} s(z)\| \leq C|z|^{-N}$  for  $\log z \in \tilde{\mathbb{C}}_z^*$  with  $\Im \log z = \theta, |z| < 1$ . Here the norm of  $e^{c/z} s$  is given by a fixed hermitian metric on  $H^\bullet(X)$ .  $\mathcal{A}_{(\omega_k T_X, -2\pi k/r_X)}$  is denoted briefly by  $\mathcal{A}_k$ . Recall that the isomorphism

$$\Phi : H^\bullet(X) \rightarrow \left\{ s : \tilde{\mathbb{C}}_z^* \rightarrow H^\bullet(X) \mid \nabla s = 0 \right\}$$

is defined by  $\Phi(\alpha)(z) = (2\pi)^{-d_X/2} S(z) z^{-\mu} z^{\rho_X} \alpha$ .

DEFINITION 6.7 ([17, Conjecture 3.4.3]). — We say that  $X$  satisfies Gamma conjecture I if  $X$  satisfies Property  $\mathcal{O}$  and  $\mathcal{A}_0 = \mathbb{C}\Phi(\hat{\Gamma}_X)$ .

We have the following lemma:

LEMMA 6.8. —  $\mathcal{A}_{(c,\theta)} \cong \mathcal{A}_{(\omega_k c, \theta - 2\pi k/r_X)}$ . This isomorphism is given by  $\alpha \rightarrow \text{Ch}(\mathcal{O}_X(k)) \cup \alpha$  via the isomorphism  $\Phi$ .

*Proof.* — For  $s \in \mathcal{A}_{(c,\theta)}$ , choose  $\alpha \in H^\bullet(X)$  which satisfies  $s = \Phi(\alpha)$ . By Lemma 6.2, we have  $\omega_k^{-\mu} s(z/\omega_k) \in \mathcal{A}_{(\omega_k c, \theta - 2\pi k/r_X)}$ . By Proposition 6.3, we have  $\omega_k^{-\mu} s(z/\omega_k) = \Phi(\omega_k^{-\rho_X} \cup \alpha)$ . The lemma is proved by the equation  $\omega_k^{-\rho_X} = \text{Ch}(\mathcal{O}_X(k))$ . □

From this lemma together with Gamma conjecture I, we can calculate  $\mathcal{A}_k$ .

PROPOSITION 6.9. — *If  $X$  satisfies Gamma conjecture I, then we have*

$$\mathcal{A}_k = \mathbb{C}\Phi(\hat{\Gamma}_X \text{Ch}(\mathcal{O}_X(k))).$$

*Proof.* — Obvious from Lemma 6.8. □

### 6.3. Vanishing cycles

In this section, we fix  $(-C_X)$ -generic  $\theta_o$  and assume  $\mathcal{M}_X$  is of exponential type (3.3). Let  $\mathfrak{A}_{\theta_o}(\text{RH}(\mathcal{M}_X, \mathcal{Q}_X))$  be the A-mutation system with the splitting data  $(\tau_{\theta_o}, {}^A f_{\theta_o})$ .

DEFINITION 6.10. — *Let  $\mathcal{A} \subset D^b(X)$  be an admissible subcategory. We say that  $\mathcal{A}$  is a vanishing cycle at  $(c, \theta_o)$  if  $\text{Im } {}^A f_{\theta_o, -c} = \Gamma(\text{HH}_\bullet(\mathcal{A}))$ .*

Recall that  $E \in D^b(X)$  is called an exceptional object if

$$\text{Hom}(E, E[k]) = \begin{cases} \mathbb{C} & (k = 0) \\ 0 & (k \neq 0). \end{cases}$$

Let  $\langle E \rangle$  be the smallest full strict triangulated subcategory of  $D^b(X)$  which contains  $E$ . We note that  $\langle E \rangle$  is an admissible subcategory of  $D^b(X)$  if  $E$  is an exceptional object. (see, e.g., [25, Lemma 1.58]).

LEMMA 6.11. — *Let  $E \in D^b(X)$  be an exceptional object. Then*

$$\Gamma(\text{HH}_\bullet(\langle E \rangle)) = \mathbb{C}\hat{\Gamma}_X \text{Ch}(E).$$

*Proof.* — We consider  $E$  as an object of  $D^b(\{\text{pt}\} \times X)$ . Let

$$\Phi_E : D^b(\{\text{pt}\}) \rightarrow D^b(X)$$

be the Fourier–Mukai functor with the Fourier–Mukai kernel  $E$ . Then we have  $\text{Im } \Phi_E \subset \langle E \rangle$  and this inclusion induces an isomorphism. Hence we see that  $\phi_E$  induces an isomorphism  $\text{HH}_\bullet(D^b(\{\text{pt}\})) \cong \text{HH}_\bullet(\langle E \rangle)$ . By definition, we have  $\text{HH}_\bullet(\{\text{pt}\}) \cong \mathbb{C}$ , which implies  $\text{Im } \phi_E = \mathbb{C}\nu(E)$ , where  $\nu(E)$  is the Mukai vector. The statement follows from  $\Gamma(\nu(E)) = \hat{\Gamma}_X \text{Ch}(E)$ . □

We note that  $\mathcal{O}(k)$  is an exceptional object since  $X$  is Fano. Moreover, we easily see that the collection of objects  $(\mathcal{O}_X(0), \mathcal{O}_X(1), \dots, \mathcal{O}_X(r_X - 1))$  is exceptional, i.e.,  $\langle \mathcal{O}_X(k) \rangle \subset \langle \mathcal{O}_X(l) \rangle^\perp$  for  $k < l$ .

PROPOSITION 6.12. — *Suppose that  $-\pi/2 \leq \theta_o \leq \pi/2$ . If  $X$  satisfies Gamma conjecture I, then the subcategory  $\langle \mathcal{O}_X(k) \rangle$  is a vanishing cycle at  $(\omega_k \Gamma_X, \theta_o - 2\pi k/r_X)$ .*

*Proof.* — By Lemma 3.5, we have

$$\text{Im } {}^A f_{\theta_o - 2\pi k/r_X, -\omega_k \Gamma_X} \subset \mathcal{A}_k = \mathbb{C}\Phi(\hat{\Gamma}_X \text{Ch}(\mathcal{O}_X(k))).$$

By Remark 3.6 and Lemma 6.1, we have  $\dim \text{Im } {}^A f_{\theta_o - 2\pi k/r_X, -\omega_k \Gamma_X} = 1$ . Thus we have

$$\text{Im } {}^A f_{\theta_o - 2\pi k/r_X, -\omega_k \Gamma_X} = \mathbb{C}\Phi(\hat{\Gamma}_X \text{Ch}(\mathcal{O}_X(k))).$$

Combined with Lemma 6.11, we have the statement. □

In the proof of Lemma 6.14, which is used in the proof of Theorem 7.9, we use the following notation:

Notation 6.13. — Let  $(V, [\cdot, \cdot])$  be a finite dimensional vector space  $V$  over a field  $k$  with a pairing  $[\cdot, \cdot]$  and  $W$  be a subspace of  $V$ . Set

$$\begin{aligned} {}^\perp W &:= \{v \in V \mid [v, w] = 0 \text{ for all } w \in W\}, \\ W^\perp &:= \{v \in V \mid [w, v] = 0 \text{ for all } w \in W\}. \end{aligned}$$

LEMMA 6.14. — *Suppose that  $\theta_\bullet = \{\theta_c\}_{c \in -C_X}$  is ordered. Take  $c_o \in -C_X$ . Let  $\{\mathcal{A}_{(-c, \theta_c)}\}_{c \in -C_X \setminus \{c_o\}}$  be a family of vanishing cycles. Assume that  $\mathcal{A}_{(-c, \theta_c)} \subset \mathcal{A}_{(-c', \theta_{c'})}^\perp$  for  $c < \theta_\bullet + \pi/2 c'$ . Set*

$$\mathcal{A}_{(-c_o, \theta_{c_o})} := \left( \bigcap_{c < \theta_\bullet + \pi/2 c_o} {}^\perp \mathcal{A}_{(-c, \theta_c)} \right) \cap \left( \bigcap_{c_o < \theta_\bullet + \pi/2 c} \mathcal{A}_{(-c, \theta_c)}^\perp \right).$$

Then  $\mathcal{A}_{(-c_o, \theta_{c_o})}$  is a vanishing cycle at  $(-c_o, \theta_{c_o})$ .

*Proof.* — By construction,  $\{\mathcal{A}_{(-c, \theta_c)}\}_{c \in -C_X}$  gives a framed semiorthogonal decomposition

$$(\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in -C_X})$$

of type  $(-C_X, \tau_{\theta_\bullet})$  with a frame  $\{\mathcal{A}_{(-c, \theta_c)}\}_{c \in -C_X}$  by requiring  $F_c = \text{id}$  for all  $c$ . Hence we can construct a B-mutation system

$$\mathfrak{B}((\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in -C_X}))$$

with the splitting data  $(\tau_{\theta_\bullet}, Bf)$ . By semiorthogonality of the pairing  $[\cdot, \cdot]_X$ , we have

$$(6.2) \quad \text{Im}^{Bf_{c_0}} = \left( \bigcap_{c < \theta_\bullet + \pi/2c_0} {}^\perp(\text{Im}^{Bf_c}) \right) \cap \left( \bigcap_{c_0 < \theta_\bullet + \pi/2c} (\text{Im}^{Bf_c})^\perp \right).$$

On the other hand,  $\text{Im}^{Af_{\theta_\bullet, c_0}}$  of the splitting data  $(\tau_{\theta_\bullet}, Af_{\theta_\bullet})$  of the mutation system  $\mathfrak{A}_{\theta_\bullet}(\text{RH}(\mathcal{M}_X, \mathcal{Q}_X))$  has a similar expression as the equation (6.2), i.e.,

$$(6.3) \quad \text{Im}^{Af_{\theta_\bullet, c_0}} = \left( \bigcap_{c < \theta_\bullet + \pi/2c_0} {}^\perp(\text{Im}^{Af_{\theta_\bullet, c}}) \right) \cap \left( \bigcap_{c_0 < \theta_\bullet + \pi/2c} (\text{Im}^{Af_{\theta_\bullet, c}})^\perp \right).$$

By the definition of vanishing cycles, we have  $\text{Im}(\Gamma \circ Bf_c) = \text{Im}^{Af_{\theta_\bullet, c}}$  for  $c \neq c_0$ . Thus we have  $\text{Im}(\Gamma \circ Bf_{c_0}) = \text{Im}^{Af_{\theta_\bullet, c_0}}$ , which proves the lemma. □

### 7. Examples

In this section, we prove that complete intersection Fano manifolds in projective spaces with Fano index greater than one satisfy Dubrovin type conjectures. The proof is similar to the proof of Gamma conjectures for projective spaces ([17, §5]).

#### 7.1. Calculations of cohomology

Let  $F$  be an ample vector bundle on a Fano manifold  $Y$ , that is, the tautological line bundle of the projective space bundle  $\mathbb{P}(F)$  is ample. By [42, Proposition 1.8],  $F$  is generated by global sections. Hence  $F$  is convex, that is,  $H^1(\mathbb{P}^1, f^*F) = 0$  for all non-constant holomorphic maps  $f : \mathbb{P}^1 \rightarrow Y$ . We assume  $1 \leq \text{rk } F \leq d_Y$ . Let  $s$  be a global section of  $F$  and set  $X := s^{-1}(0)$ . By a Bertini type theorem ([42, Theorem1.10]), we can choose general  $s$  so that  $X$  is reduced and smooth. We assume  $X$  is a Fano manifold. We denote by  $i : X \rightarrow Y$  the inclusion and set  $H_{\text{amb}}^\bullet(X) := \text{Im } i^*$ . Since  $F$  is convex,  $H_{\text{amb}}^\bullet(X)$  is closed under the quantum cup product  $\ast_\tau$  for  $\tau \in H_{\text{amb}}^\bullet(X)$  (see, e.g., [11], [27, §2.4], [28, §3.4]). The next lemma is well known (see, e.g., [39, §2.2]) and we only use the case  $Y = \mathbb{P}^{d_Y}$ , but we give a detailed proof for completeness.

LEMMA 7.1.

- $H^\bullet(X) = H_{\text{amb}}^\bullet(X) \oplus H_{\text{amb}}^\bullet(X)^\perp$ .
- $H_{\text{amb}}^\bullet(X)^\perp \subset H^{d_X}(X)$ .

Here  $H_{\text{amb}}^\bullet(X)^\perp$  is the orthogonal subspace of  $H_{\text{amb}}^\bullet(X)$  with respect to the Poincaré pairing.

*Proof.* — By definition, we have

$$H_{\text{amb}}^\bullet(X) \cap H_{\text{amb}}^\bullet(X)^\perp = \left\{ i^* \alpha \mid \int_X i^* \alpha \cup i^* \beta = 0 \text{ for all } \beta \in H^\bullet(Y) \right\}.$$

Using the projection formula and  $i_* i^* = (2\pi i)^{d_Y - d_X} e(F) \cup$ , we obtain

$$\int_X i^* \alpha \cup i^* \beta = \int_Y e(F) \cup \alpha \cup \beta,$$

where  $e(F) \in H^\bullet(Y; \mathbb{Z})$  is the Euler class of  $F$ . Thus we have

$$H_{\text{amb}}^\bullet(X) \cap H_{\text{amb}}^\bullet(X)^\perp = i^* \ker(e(F) \cup -).$$

By Sommese’s theorem (see, e.g., [35, Theorem 7.1.1]), it follows that  $i^* : H^k(Y) \rightarrow H^k(X)$  is an isomorphism for  $k < d_X$ . Choose an ample line bundle  $L$  on  $Y$ . Then

$$(i^* c_1(L) \cup -)^{d_X - k} : H^k(X) \rightarrow H^{2d_X - k}(X)$$

is an isomorphism for  $k < d_X$  by the hard Lefschetz theorem. Consider the following commutative diagram:

$$\begin{array}{ccc} H^k(Y) & \xrightarrow{i^*} & H^k(X) \\ (c_1(L) \cup -)^{d_X - k} \downarrow & & \downarrow (i^* c_1(L) \cup -)^{d_X - k} \\ H^{2d_X - k}(Y) & \xrightarrow{i^*} & H^{2d_X - k}(X) \end{array}$$

Here the top and right arrows are isomorphisms for  $k < d_X$ . This diagram implies that

$$i^* : H^k(Y) \rightarrow H^k(X)$$

is surjective for  $k \neq d_X$ . Hence we obtain  $H_{\text{amb}}^\bullet(X)^\perp \subset H^{d_X}(X)$ . By the hard Lefschetz theorem for ample bundles (e.g., [35, Theorem 7.1.10]), it follows that

$$e(F) \cup - : H^{d_X}(Y) \rightarrow H^{2d_Y - d_X}(Y)$$

is injective. Thus  $H_{\text{amb}}^{d_X}(X) \cap H_{\text{amb}}^\bullet(X)^\perp = \{0\}$ , which proves the lemma. □

We consider the quantum connection  $\mathcal{M}_X$  as a  $\mathcal{O}_{\mathbb{C},0}(*\{0\})\langle \partial_z \rangle$ -module.

LEMMA 7.2. — *The decomposition  $H^\bullet(X) = H^\bullet_{\text{amb}}(X) \oplus H^\bullet_{\text{amb}}(X)^\perp$  induces the decomposition*

$$H^\bullet(X) \otimes \mathcal{O}_{\mathbb{C},0}(*\{0\}) = H^\bullet_{\text{amb}}(X) \otimes \mathcal{O}_{\mathbb{C},0}(*\{0\}) \oplus H^\bullet_{\text{amb}}(X)^\perp \otimes \mathcal{O}_{\mathbb{C},0}(*\{0\})$$

as a  $\mathcal{O}_{\mathbb{C},0}(*\{0\})\langle\partial_z\rangle$ -module. Moreover  $H^\bullet_{\text{amb}}(X)^\perp \otimes \mathcal{O}_{\mathbb{C},0}(*\{0\})$  is of exponential type.

*Proof.* — Recall that  $H^\bullet_{\text{amb}}(X)$  is closed under the quantum product  $*_0$ . For  $\alpha \in H^\bullet_{\text{amb}}(X)^\perp$  and  $\beta \in H^\bullet_{\text{amb}}(X)$ , we have

$$(c_1(X) *_0 \alpha, \beta)_X = (\alpha, c_1(X) *_0 \beta)_X = 0,$$

where we use  $c_1(X) = i^*(c_1(Y) - c_1(F)) \in H^\bullet_{\text{amb}}(X)$ . This implies that  $c_1(X) *_0 \alpha \in H^\bullet_{\text{amb}}(X)^\perp$ . Hence  $c_1(X) *_0$  and  $\mu$  are compatible with the decomposition  $H^\bullet(X) = H^\bullet_{\text{amb}}(X) \oplus H^\bullet_{\text{amb}}(X)^\perp$ , which implies the first statement. We will show the second statement. Since  $\mu = 0$  on the subspace  $H^\bullet_{\text{amb}}(X)^\perp \subset H^{d_X}(X)$ , the differential  $\partial_z$  acts on  $H^\bullet_{\text{amb}}(X)^\perp \subset H^{d_X}(X)$  as  $d/dz - (c_1(X) *_0)/z^2$ . Let  $c_1(X) *_0 = S + N$  be the Jordan decomposition, where  $S$  is the semisimple part and  $N$  is the nilpotent part. Without loss of generality, we can assume  $N$  has only one Jordan block. Take a basis  $e_1, e_2, \dots, e_k$  of  $H^\bullet_{\text{amb}}(X)^\perp$  such that  $N(e_i) = e_{i+1}$  for  $1 \leq i \leq k - 1$  and  $N(e_k) = 0$ . We define a grading operator  $\text{Gr}$  by  $\text{Gr}(e_i) := i \cdot e_i$ . Then we easily see that

$$z^{\text{Gr}} \left( \frac{d}{dz} - \frac{c_1(X) *_0}{z^2} \right) z^{-\text{Gr}} = \frac{d}{dz} - \frac{\text{Gr}}{z} - \frac{N}{z} - \frac{S}{z^2}$$

and the right hand side is obviously of exponential type. □

If the Fano index of  $X$  is greater than one, we can calculate the action of  $c_1(X) *_0$  on  $H^\bullet_{\text{amb}}(X)^\perp$  as follows:

LEMMA 7.3 ([18, §7]). — *If  $r_X \geq 2$ , then  $c_1(X) *_0 \alpha = 0$  for  $\alpha \in H^\bullet_{\text{amb}}(X)^\perp$ .*

*Proof.* — Since  $\alpha \in H^\bullet_{\text{amb}}(X)^\perp \subset H^{d_X}(X)$  and  $r_X \geq 2$ , we have

$$c_1(X) *_0 \alpha \in H^{d_X+2}(X) \oplus \bigoplus_{k \leq d_X-2} H^k(X).$$

But  $c_1(X) *_0 \alpha \in H^\bullet_{\text{amb}}(X)^\perp \subset H^{d_X}(X)$ . Hence we have  $c_1(X) *_0 \alpha = 0$ . □

**7.2. Dubrovin type conjectures for complete intersections in projective spaces**

In this section, we assume that  $Y$  is the projective space  $\mathbb{P}^{d_Y}$  of dimension  $d_Y$  and  $F = \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_k)$ . Moreover, we assume

$$d_i \geq 2, d_X = d_Y - k \geq 3, r_X = d_Y + 1 - d_1 - d_2 - \cdots - d_k \geq 2.$$

PROPOSITION 7.4. — *The quantum connection  $\mathcal{M}_X$  is of exponential type.*

*Proof.* — By Lemma 7.2, it is sufficient to show that the meromorphic connection  $(H_{\text{amb}}^\bullet(X) \otimes \mathcal{O}_{\mathbb{C},0}(*\{0\}), \nabla^{\text{amb}})$  is of exponential type, where  $\nabla^{\text{amb}}$  is the restriction of  $\nabla$  to  $H_{\text{amb}}^\bullet(X) \otimes \mathcal{O}_{\mathbb{C},0}(*\{0\})$ . Let  $\log q \in \mathbb{C}$ . Set  $q := e^{\log q}, w := qz$ , and  $\tau := c_1(X) \log q$ . Note that  $c_1(X) *_\tau$  defines an endomorphism of  $H_{\text{amb}}^\bullet(X)$  since  $\tau \in H_{\text{amb}}^\bullet(X)$ . Then we can easily check that

$$q^{-\mu} \nabla_{\frac{d}{dz}}^{\text{amb}} q^\mu = \left( w \frac{d}{dw} + \mu - \frac{1}{w} c_1(X) *_\tau \right) \Big|_{H_{\text{amb}}^\bullet(X) \otimes \mathcal{O}_{\mathbb{C},0}(*\{0\})}.$$

By [48, Corollary 6.14], we see that the right hand side is equipped with a non-commutative Hodge structure for  $|q| \ll 1$  (see [50] for the definition), and hence of exponential type. □

As a corollary of Givental’s mirror theorem, we calculate the ring structure of  $H_{\text{amb}}^\bullet(X)$  with respect to the quantum product.

LEMMA 7.5. — *Set  $D_X = d_1^{d_1} d_2^{d_2} \cdots d_k^{d_k}$ . Then*

$$H_{\text{amb}}^\bullet(X) \cong \mathbb{C}[x] / \langle x^{d_X+1-r_X} (x^{r_X} - D_X) \rangle$$

as a ring.

*Proof.* — By [20, Corollary 9.3],  $H_X \in H^2(X)$  satisfies the relation  $H_X^{d_X+1} - D_X H_X^{d_1+d_2+\cdots+d_k-k} = 0$ . Hence we have the ring morphism

$$\mathbb{C}[x] / \langle x^{d_X+1-r_X} (x^{r_X} - D_X) \rangle \rightarrow H_{\text{amb}}^\bullet(X)$$

which sends  $x$  to  $H_X$ . We easily see that these rings have the same dimension. Since

$$H_X^l = \overbrace{H_X \cup H_X \cup \cdots \cup H_X}^l + (\text{lower degree term}),$$

the ambient cohomology  $H_{\text{amb}}^\bullet(X)$  is generated by  $H_X$ . Hence the above morphism is an isomorphism. □

*Remark 7.6.* — Similarly, by using [20, Corollary 10.9], we have

$$H_{\text{amb}}^\bullet(X) \cong \mathbb{C}[x]/\langle (x + D'_X)^{d_X} (x^{r_X} - D_X + D'_X) \rangle$$

for the case  $r_X = 1$ , where  $D'_X := d_1!d_2! \cdots d_k!$ .

**COROLLARY 7.7.** —  $T_X = r_X D_X^{1/r_X}$ ,  $C_X = \{T_X \omega_k \mid k \in \mathbb{Z}\} \cup \{0\}$ . Moreover,  $X$  satisfies Property  $\mathcal{O}$ .

*Proof.* — This statement easily follows from Lemmas 7.3 and 7.5. □

**PROPOSITION 7.8.** —  $X$  satisfies Gamma conjecture.

*Proof.* — By [17, Theorem 5.0.1], projective spaces satisfy Gamma conjectures I. By [18, Corollary 3.9, Theorem 8.3], it follows that  $X$  satisfies Gamma conjecture I by induction on  $k$ . □

**THEOREM 7.9.** —  $X$  satisfies Dubrovin type conjecture.

*Proof.* — Choose a sufficiently small positive real number  $\theta_\circ \in \mathbb{R}$ . Set  $c_\circ := 0$  and  $c_k := -T_X \omega_k$  ( $k = 0, 1, \dots, r_X - 1$ ). We define a ordered tuple of real numbers  $\theta_\bullet$  by

$$\theta_c := \begin{cases} \theta_\circ & (c = c_\circ) \\ -2\pi k/r_X + \theta_\circ & (c = c_k). \end{cases}$$

Then the order  $<_{\theta_\bullet + \pi/2}$  on  $-C_X$  is given by

$$c_\circ <_{\theta_\bullet + \pi/2} c_0 <_{\theta_\bullet + \pi/2} c_1 <_{\theta_\bullet + \pi/2} \cdots <_{\theta_\bullet + \pi/2} c_{r_X - 1}.$$

Note that  $\theta_\circ$  and  $\theta_\bullet$  are  $(-C_X)$ -generic. By Corollary 2.40, we can construct a mutation system  $\mathfrak{A}_{\theta_\bullet}(\text{RH}(\mathcal{M}_X, \mathcal{Q}_X))$  with the splitting data  $(\tau_{\theta_\bullet}, {}^A f_{\theta_\bullet})$ .

Since  $X$  satisfies Gamma conjecture I, using Proposition 6.12, we see that  $\mathcal{A}_{-c_k} := \langle \mathcal{O}(k) \rangle$  is a vanishing cycle at  $(-c_k, \theta_{c_k})$ . It is easy to see that  $\{\mathcal{A}_{-c_k}\}_{c_k \in -C_X \setminus \{c_\circ\}}$  satisfies the assumption of Lemma 6.14, which implies

$$\mathcal{A}_{-c_\circ} := \langle \mathcal{O}(0), \mathcal{O}(1), \dots, \mathcal{O}(r_X - 1) \rangle^\perp$$

is a vanishing cycle at  $(-c_\circ, \theta_{c_\circ})$ . We define a framed semiorthogonal decomposition

$$(\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in -C_X})$$

of type  $(-C_X, \tau_{\theta_\bullet})$  with the frame  $\{\text{gr}_c D^b(X)\}_{c \in -C_X} := \{\mathcal{A}_{-c}\}_{c \in C_X}$  by requiring  $F_c = \text{id}_{\mathcal{A}_{-c}}$ . We consider the corresponding B-mutation system

$$\mathfrak{B}((\{\mathcal{A}_i\}_{1 \leq i \leq m}, \{F_c\}_{c \in -C_X}))$$

with the splitting data  $(\tau_{\theta_\bullet}, {}^B f)$ . Then, by the definition of vanishing cycle categories, we have  $\text{Im}(\Gamma \circ {}^B f_c) = \text{Im} {}^A f_{\theta_\bullet, c}$  for all  $c \in -C_X$ .

Let  $\mathfrak{A}_{\theta_o}(\text{RH}(\mathcal{M}_X, \mathcal{Q}_X))$  be the A-mutation system with the splitting data  $(\tau_{\theta_o}, {}^A f_{\theta_o})$ . By Theorem 4.23, it is sufficient to show that

$$(\tau_{\theta_\bullet}, {}^A f_{\theta_\bullet}) = (\bar{\sigma} \circ \tau_{\theta_o}, \sigma {}^A f_{\theta_o})$$

for some  $\sigma \in \text{Br}_{r_X+1}$ . To show this statement, we introduce the following ordered tuple of real numbers  $\{\phi_\bullet\}_{c \in -C_X}$ :

$$\phi_c := \begin{cases} \theta_c & (\theta_c > \theta_o - \pi) \\ \theta_o - \pi & (\theta_c \leq \theta_o - \pi). \end{cases}$$

We can easily see that  $\phi_\bullet$  is  $-C_X$ -generic. Let  $\mathfrak{A}_{\phi_\bullet}(\text{RH}(\mathcal{M}_X, \mathcal{Q}_X))$  be the corresponding mutation system with the splitting data  $(\tau_{\phi_\bullet}, {}^A f_{\phi_\bullet})$ . Note that  $V_{\mathcal{M}_X}$  is identified with  $H^\bullet(X)$  via the isomorphism  $\Phi$  (see Definition 3.4). We define elements of the symmetric group  $s, s'$  by  $\tau_{\phi_\bullet} \circ \tau_{\theta_o}^{-1}, \tau_{\theta_\bullet} \circ \tau_{\phi_\bullet}^{-1}$  respectively. We claim that

$$(\tau_{\phi_\bullet}, {}^A f_{\phi_\bullet}) = (s \circ \tau_{\theta_o}, (s)_R {}^A f_{\theta_o}), \quad (\tau_{\theta_\bullet}, {}^A f_{\theta_\bullet}) = (s' \circ \tau_{\phi_\bullet}, (s')_R {}^A f_{\phi_\bullet}).$$

We first show  $(\tau_{\phi_\bullet}, {}^A f_{\phi_\bullet}) = (s \circ \tau_{\theta_o}, (s)_R {}^A f_{\theta_o})$ . By Lemma 2.15, we have

$$((s)_R {}^A f_{\theta_o})_c = \left( \prod_{i \in I_{\tau_{\theta_o}(c)}(s)}^{\leftarrow} R_i \right) \circ {}^A f_{\theta_o, c}.$$

On the other hand, by Corollary 2.28, we have

$${}^A f_{\phi_c, c} = \left( \prod_{i \in I_{\tau_{\theta_o}(c)}(s_{\theta_o, \phi_c})}^{\leftarrow} R_i \right) \circ {}^A f_{\theta_o, c}.$$

Hence it is sufficient to show that

$$\left\{ c' \in -C_X \mid \begin{matrix} c < \theta_o + \pi/2 & c' \\ c' < \phi_\bullet + \pi/2 & c \end{matrix} \right\} = \left\{ c' \in -C_X \mid \begin{matrix} c < \theta_o + \pi/2 & c' \\ c' < \phi_c + \pi/2 & c \end{matrix} \right\}$$

for all  $c \in -C_X$ . By simple consideration, we see that both sets are equal to

$$\begin{cases} \emptyset & (c = c_o) \\ \{c_l \mid c < \theta_o + \pi/2, c_l, l < k\} & (c = c_k, \theta_{c_k} > \theta_o - \pi) \\ \{c_l \mid c < \theta_o + \pi/2, c_l\} \cup \{c_o\} & (c = c_k, \theta_{c_k} \leq \theta_o - \pi). \end{cases}$$

We next show  $(\tau_{\theta_\bullet}, {}^A f_{\theta_\bullet}) = (s' \circ \tau_{\phi_\bullet}, (s')_R {}^A f_{\phi_\bullet})$ . It is sufficient to show that

$$\left\{ c' \in -C_X \mid \begin{matrix} c < \phi_\bullet + \pi/2 & c' \\ c' < \theta_\bullet + \pi/2 & c \end{matrix} \right\} = \left\{ c' \in -C_X \mid \begin{matrix} c < \phi_c + \pi/2 & c' \\ c' < \theta_c + \pi/2 & c \end{matrix} \right\}$$

for all  $c \in -C_X$  as ordered sets, where order of the left hand side (resp. right hand side) is defined by  $\langle \phi_{\bullet} + \pi/2$  (resp.  $\langle \phi_c + \pi/2$ ). We see that both sets are equal to

$$\begin{cases} \emptyset & (c = c_o) \\ \emptyset & (c = c_k, \theta_{c_k} > \theta_o - \pi) \\ \{c_l \mid c \langle_{\theta_o - \pi/2} c_l, l < k\} & (c = c_k, \theta_{c_k} \leq \theta_o - \pi). \end{cases}$$

Since  $\phi_{c_k} = \theta_o - \pi$  for  $c_k$  with  $\theta_{c_k} < \theta_o - \pi$ , we see that  $\langle_{\phi_{\bullet} + \pi/2}$  and  $\langle_{\phi_c + \pi/2}$  define the same order on this set, which completes the proof.  $\square$

*Remark 7.10.* — Similarly, we can prove that  $X$  satisfies Dubrovin type conjecture if  $r_X = 1$  and  $d_X$  is odd (see also Lemma 6.6 and Remark 7.6).

### Acknowledgement

Both of the authors would like to thank Hiroshi Iritani for discussions and encouragement in many occasions, Masahiro Futaki and Yuuki Shiraishi for informal seminar. The first author would like to thank Tatsuki Kuwagaki and Kohei Yahiro for discussions. The second author would like to express his gratitude for his supervisor Takuro Mochizuki. He is also grateful Claus Hertling and Claude Sabbah for their advise, encouragement, and kindness.

The first author was supported by JSPS Grant-in-Aid for Scientific Research numbers 23224002 and 15H02054. The second author was supported by Grant-in-Aid for JSPS Research Fellow number 16J02453 and the Kyoto Top Global University Project (KTGU).

### BIBLIOGRAPHY

- [1] W. BALSER, W. B. JURKAT & D. A. LUTZ, “Birkhoff invariants and Stokes’ multipliers for meromorphic linear differential equations”, *J. Math. Anal. Appl.* **71** (1979), no. 1, p. 48-94.
- [2] A. BAYER & Y. I. MANIN, “(Semi)simple exercises in quantum cohomology”, in *The Fano Conference (Torino, 2002)*, Università di Torino, 2004, p. 143-173.
- [3] A. I. BONDAL, “A symplectic groupoid of triangular bilinear forms and the braid group”, *Izv. Ross. Akad. Nauk, Ser. Mat.* **68** (2004), no. 4, p. 19-74.
- [4] A. I. BONDAL & M. M. KAPRANOV, “Representable functors, Serre functors, and reconstructions”, *Izv. Akad. Nauk SSSR, Ser. Mat.* **53** (1989), no. 6, p. 1183-1205, 1337.
- [5] N. BOURBAKI, *Lie groups and Lie algebras. Chapters 4–6*, Springer, 2002, Translated from the 1968 French original by Andrew Pressley, xii+300 pages.
- [6] A. CĂLDĂRARU, “The Mukai pairing. II. The Hochschild–Kostant–Rosenberg isomorphism”, *Adv. Math.* **194** (2005), no. 1, p. 34-66.

- [7] A. CĂLDĂRARU & S. WILLERTON, “The Mukai pairing. I. A categorical approach”, *New York J. Math.* **16** (2010), p. 61-98.
- [8] A. CANONACO & P. STELLARI, “Non-uniqueness of Fourier–Mukai kernels”, *Math. Z.* **272** (2012), no. 1-2, p. 577-588.
- [9] P.-E. CHAPUT, L. MANIVEL & N. PERRIN, “Quantum cohomology of minuscule homogeneous spaces III. Semi-simplicity and consequences”, *Can. J. Math.* **62** (2010), no. 6, p. 1246-1263.
- [10] G. CIOLLI, “On the quantum cohomology of some Fano threefolds and a conjecture of Dubrovin”, *Int. J. Math.* **16** (2005), no. 8, p. 823-839.
- [11] T. COATES, “The Quantum Lefschetz Principle for Vector Bundles as a Map Between Givental Cones”, <https://arxiv.org/abs/1405.2893>, 2014.
- [12] D. A. COX & S. KATZ, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs, vol. 68, American Mathematical Society, 1999, xxii+469 pages.
- [13] J. A. CRUZ MORALES & M. VAN DER PUT, “Stokes matrices for the quantum differential equations of some Fano varieties”, *Eur. J. Math.* **1** (2015), no. 1, p. 138-153.
- [14] B. DUBROVIN, “Geometry of 2D topological field theories”, in *Integrable systems and quantum groups (Montecatini Terme, 1993)*, Lecture Notes in Mathematics, vol. 1620, Springer, 1996, p. 120-348.
- [15] ———, “Geometry and analytic theory of Frobenius manifolds”, in *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, no. Extra Vol. II, 1998, p. 315-326.
- [16] ———, “Painlevé transcendents in two-dimensional topological field theory”, in *The Painlevé property*, CRM Series in Mathematical Physics, Springer, 1999, p. 287-412.
- [17] S. GALKIN, V. GOLYSHEV & H. IRITANI, “Gamma classes and quantum cohomology of Fano manifolds: gamma conjectures”, *Duke Math. J.* **165** (2016), no. 11, p. 2005-2077.
- [18] S. GALKIN & H. IRITANI, “Gamma conjecture via mirror symmetry”, <https://arxiv.org/abs/1508.00719>, 2015.
- [19] S. GALKIN, A. MELLIT & M. SMIRNOV, “Dubrovin’s conjecture for  $IG(2,6)$ ”, *Int. Math. Res. Not.* (2015), no. 18, p. 8847-8859.
- [20] A. GIVENTAL, “Equivariant Gromov–Witten invariants”, *Int. Math. Res. Not.* (1996), no. 13, p. 613-663.
- [21] D. GUZZETTI, “Stokes matrices and monodromy of the quantum cohomology of projective spaces”, *Commun. Math. Phys.* **207** (1999), no. 2, p. 341-383.
- [22] C. HERTLING, Y. I. MANIN & C. TELEMAN, “An update on semisimple quantum cohomology and  $F$ -manifolds”, *Tr. Mat. Inst. Steklova* **264** (2009), p. 69-76.
- [23] C. HERTLING & C. SABBAH, “Examples of non-commutative Hodge structures”, *J. Inst. Math. Jussieu* **10** (2011), no. 3, p. 635-674.
- [24] G. HOCHSCHILD, B. KOSTANT & A. ROSENBERG, “Differential forms on regular affine algebras”, *Trans. Am. Math. Soc.* **102** (1962), p. 383-408.
- [25] D. HUYBRECHTS, *Fourier–Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, Clarendon Press, 2006, viii+307 pages.
- [26] H. IRITANI, “An integral structure in quantum cohomology and mirror symmetry for toric orbifolds”, *Adv. Math.* **222** (2009), no. 3, p. 1016-1079.
- [27] ———, “Quantum cohomology and periods”, *Ann. Inst. Fourier* **61** (2011), no. 7, p. 2909-2958.
- [28] H. IRITANI, E. MANN & T. MIGNON, “Quantum Serre theorem as a duality between quantum  $D$ -modules”, *Int. Math. Res. Not.* (2016), no. 9, p. 2828-2888.

- [29] K. IWAKI & A. TAKAHASHI, “Stokes matrices for the quantum cohomologies of a class of orbifold projective lines”, *J. Math. Phys.* **54** (2013), no. 10, article ID 101701 (18 pages).
- [30] L. KATZARKOV, M. KONTSEVICH & T. PANTEV, “Hodge theoretic aspects of mirror symmetry”, in *From Hodge theory to integrability and TQFT tt\*-geometry*, Proceedings of Symposia in Pure Mathematics, vol. 78, American Mathematical Society, 2008, p. 87-174.
- [31] ———, “Bogomolov–Tian–Todorov theorems for Landau–Ginzburg models”, *J. Differ. Geom.* **105** (2017), no. 1, p. 55-117.
- [32] A. KUZNETSOV, “Hochschild homology and semiorthogonal decompositions”, <https://arxiv.org/abs/0904.4330>, 2009.
- [33] ———, “Base change for semiorthogonal decompositions”, *Compos. Math.* **147** (2011), no. 3, p. 852-876.
- [34] ———, “Exceptional collections in surface-like categories”, *Mat. Sb.* **208** (2017), no. 9, p. 116-147.
- [35] R. LAZARSFELD, *Positivity in algebraic geometry. II. Positivity for Vector Bundles, and Multiplier Ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 49, Springer, 2004, xviii+385 pages.
- [36] A. LIBGOBER, “Chern classes and the periods of mirrors”, *Math. Res. Lett.* **6** (1999), no. 2, p. 141-149.
- [37] V. A. LUNTS, “Lefschetz fixed point theorems for Fourier-Mukai functors and DG algebras”, *J. Algebra* **356** (2012), p. 230-256.
- [38] B. MALGRANGE, “La classification des connexions irrégulières à une variable”, in *Mathematics and physics (Paris, 1979/1982)*, Progress in Mathematics, vol. 37, Birkhäuser, 1983, p. 381-399.
- [39] E. MANN & T. MIGNON, “Quantum  $\mathcal{D}$ -modules for toric nef complete intersections”, *Int. J. Math.* **28** (2017), no. 6, article ID 1750047 (49 pages).
- [40] M. MARCOLLI & G. TABUADA, “From exceptional collections to motivic decompositions via noncommutative motives”, *J. Reine Angew. Math.* **701** (2015), p. 153-167.
- [41] H. MATSUMOTO, “Générateurs et relations des groupes de Weyl généralisés”, *C. R. Math. Acad. Sci. Paris* **258** (1964), p. 3419-3422.
- [42] S. MUKAI, “Polarized  $K3$  surfaces of genus 18 and 20”, in *Complex projective geometry (Trieste, 1989/Bergen, 1989)*, London Mathematical Society Lecture Note Series, vol. 179, Cambridge University Press, 1992, p. 264-276.
- [43] D. Y. NOGIN, “Helices on some Fano threefolds: constructivity of semiorthogonal bases of  $K_0$ ”, *Ann. Sci. Éc. Norm. Supér.* **27** (1994), no. 2, p. 129-172.
- [44] A. POLISHCHUK, “Lefschetz type formulas for dg-categories”, *Sel. Math., New Ser.* **20** (2014), no. 3, p. 885-928.
- [45] A. POLISHCHUK & A. VAINTROB, “Chern characters and Hirzebruch–Riemann–Roch formula for matrix factorizations”, *Duke Math. J.* **161** (2012), no. 10, p. 1863-1926.
- [46] A. C. RAMADOSS, “The relative Riemann-Roch theorem from Hochschild homology”, *New York J. Math.* **14** (2008), p. 643-717.
- [47] ———, “The Mukai pairing and integral transforms in Hochschild homology”, *Mosc. Math. J.* **10** (2010), no. 3, p. 629-645.
- [48] T. REICHELDT & C. SEVENHECK, “Non-affine Landau–Ginzburg models and intersection cohomology”, *Ann. Sci. Éc. Norm. Supér.* **50** (2017), no. 3, p. 665-753.
- [49] C. SABBAH, *Isomonodromic deformations and Frobenius manifolds. An introduction*, Universitext, Springer; EDP Sciences, 2007, xiv+279 pages.
- [50] ———, “Non-commutative Hodge structures”, *Ann. Inst. Fourier* **61** (2011), no. 7, p. 2681-2717.

- [51] ———, *Introduction to Stokes structures*, Lecture Notes in Mathematics, vol. 2060, Springer, 2013, xiv+249 pages.
- [52] D. SHKLYAROV, “Hirzebruch–Riemann–Roch-type formula for DG algebras”, *Proc. Lond. Math. Soc.* **106** (2013), no. 1, p. 1-32.
- [53] R. G. SWAN, “Hochschild cohomology of quasiprojective schemes”, *J. Pure Appl. Algebra* **110** (1996), no. 1, p. 57-80.
- [54] K. UEDA, “Stokes matrices for the quantum cohomologies of Grassmannians”, *Int. Math. Res. Not.* (2005), no. 34, p. 2075-2086.
- [55] ———, “Stokes Matrix for the Quantum Cohomology of Cubic Surfaces”, <https://arxiv.org/abs/math/0505350>, 2005.
- [56] L. DE THANHOFFER DE VOLCSEY & M. VAN DEN BERGH, “On an analogue of the Markov equation for exceptional collections of length 4”, <https://arxiv.org/abs/1607.04246>, 2016.
- [57] A. YEKUTIELI, “The continuous Hochschild cochain complex of a scheme”, *Can. J. Math.* **54** (2002), no. 6, p. 1319-1337.

Manuscrit reçu le 11 septembre 2017,  
révisé le 15 novembre 2018,  
accepté le 17 janvier 2019.

Fumihiko SANDA  
Graduate school of Mathematics, Nagoya University,  
Furo-cho  
Chikusa-ku, Nagoya 464-8602 (Japan)  
[fumhiko.sanda@gmail.com](mailto:fumhiko.sanda@gmail.com)

Yota SHAMOTO  
Kavli Institute for the Physics and Mathematics of  
the Universe, University of Tokyo  
5-1-5 Kashiwanoha  
Kashiwa, Chiba 277-8583 (Japan)  
[yota.shamoto@ipmu.jp](mailto:yota.shamoto@ipmu.jp)