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ASYMPTOTIC GEOMETRY OF DISCRETE INTERLACED PATTERNS: PART II

by Erik DUSE & Anthony METCALFE (*)

Abstract. — We study the asymptotic boundary of the *liquid region* of large random lozenge tiling models defined by uniformly random interlacing particle systems. In particular, we study a *non-phase separating* part of the boundary, i.e., a part of the boundary that does not border to a frozen phase. This is called the singular part of the boundary. We prove that isolated components of this boundary are lines and classify four different cases. Moreover, we show that the singular part of the boundary can have infinite one-dimensional Hausdorff measure. This has implications to the study of the *free boundary problem* arising in the variational problem studied by Kenyon and Okounkov and in a related work by De Silva and Savin.

Résumé. — Nous étudions la limite asymptotique de la région liquide de grands modèles aléatoires de tuiles rhombiques définies par des systèmes de particules uniformément et aléatoirement entrelacés. En particulier nous étudions une partie séparatrice, non phasée, de la limite, i.e. la partie de la limite qui ne touche pas une phase gelée. Cela s'appelle la partie singulière de la limite. Nous prouvons que les composantes isolées de cette limite sont des droites qui se classent en quatre cas différents. De plus, nous montrons que la partie singulière de la limite peut avoir une mesure de Hausdorff infinie et unidimensionnelle. Cela a des implications pour l'étude du problème de la limite libre découlant du problème variationnel étudié par Kenyon et Okounkov et un travail relié de De Silva et Savin.

1. Introduction

1.1. Uniformly Random Lozenge Tiling Models

Consider a tiling of a regular hexagon. An example of such is shown in Figure 1.1. In Figure 1.2 we also depict the "frozen regions" and the "frozen boundary" separating the frozen region from the "liquid region". Also, the

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asymptotic shape of the frozen boundary is shown. For a precise definition of asymptotic limit shape and frozen boundary, see [7, Definition 1.3 and 1.5].

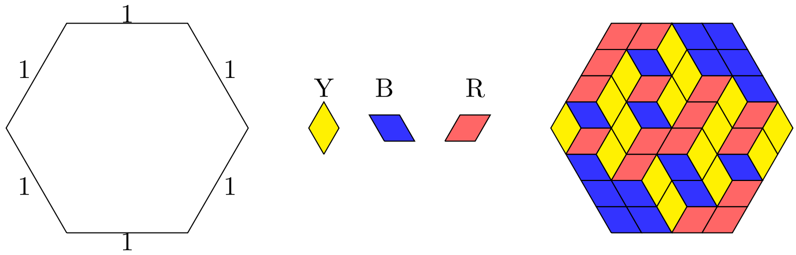

Figure 1.1. Left: A regular hexagon with sides of length 1*. Middle: The three different types of lozenges with sides of length* $1/n$*. Right: An example tiling when* $n = 4$*.*

Figure 1.2. Left: The frozen boundary of the example tiling of Figure 1.1. Right: The asymptotic shape of the frozen boundary of a "typical random tiling" as $n \to \infty$*. See* [14] *and* [7]

We will now restrict our attention to the configuration of yellow tiles. We see that we may encode the configuration of yellow tiles as interlacing systems of particles. In Figure 1.3 this is done in two different ways. In the figure to the left in Figure 1.3, we encode the configuration of yellow tiles as two interlacing systems. One interlacing system between row 1 and row 4, and one interlacing system between row 7 and row 4, being glued together along their common row, row 4. Moreover, since we pick tessellations of the regular hexagon uniformly at random, the configuration of tiles at row 4 is a random configuration. On the other hand, in the figure to the right in Figure 1.3 we encode the configuration of yellow tiles as one interlacing system by adding virtual particles on the side of the polygon. Now, the configuration of yellow tiles at row 8, the “top line”, is deterministic as opposed to the configuration of tiles on row 4.

We may now note that the configuration of yellow tiles entirely fixes the configurations of the other tiles, the red and the blue tiles. This can be seen as follows. In between every row of yellow tiles we may we have a row of red and blue tiles. Firstly, we see that the position of the yellow tile on the first row fixes the positions of the tiles on first row of red and blue tiles. Secondly, we see that the position of yellow tiles on the first and the second row fixes the position of red and blue tiles on the second row of blue and red tiles. The process may be continued until we reach the “top row” of the first interlacing system, row 4, and the position of the tiles on row 3 and row 4 have determined the position of the red and blue tiles on the fourth row of the rows of red and blue tiles. We have now fixed the position of all the tiles in the lower interlacing system. We may now repeat the process in the upper interlacing system, using that the position of the yellow tile on row 7 fixes position of red and blue tiles on row 8 and so on. This completes the tessellation of the regular hexagon, given the position of the yellow tiles. In fact we notice that the choice of considering yellow tiles was arbitrary, we may in fact equally well have chosen to consider the configurations of red or the blue tiles instead.

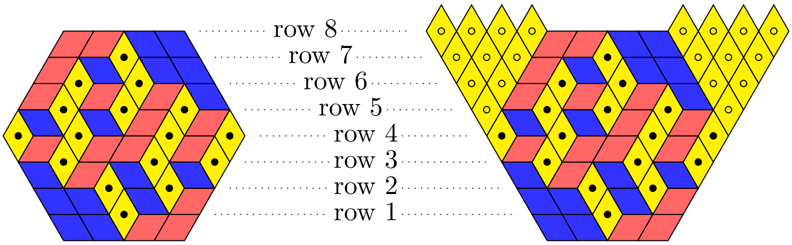

Figure 1.3. Left: Equivalent interlaced particle configuration of the example tiling of Figure 1.1. Right: Equivalent interlaced particle configuration with added deterministic lozenges/particles. The unfilled circles represent the deterministic particles.

We will now be more precise on how we encode the positions of the yellow tiles as an interlaced particle system. Let $y_i^{(r)}$ denote the position of the i:th particle on the r:th row and let $x_i^{(n)} := y_i^{(n)}$ denote the position of the particles on the *top line*, indicated by unfilled circles inside the tiles. Then the particles on row $r+1$ will interlace with the particles on row r

according to

$$y_1^{(r+1)} > y_1^{(r)} > y_2^{(r+1)} > y_2^{(r)} \cdots > y_r^{(r)} > y_{r+1}^{(r+1)},$$

for every $r = 1, \ldots, n-1$. In particular, the positions of the yellow tiles $\{x_i^{(n)}\}_{i=1}^n$ occur in densely packed blocks. Therefore, due to the interlacing constraint, they form deterministic wedge like regions as shown in the figure to the right in Figure 1.3. Due to this, interlacing systems of yellow tiles are in a bijective correspondence with lozenge tilings of certain types of polygons, i.e., the tiling of the regular hexagon in the figure to the left in Figure 1.3. Uniformly random lozenge tilings of such classes of polygons where studied in [14]. In particular, it is shown that as $n \to \infty$, a typical tiling will display frozen regions of only one type of tile in the corners of the polygon. Theses regions will be separated by an algebraic curve, whose interior region is a disordered region for the tiles, i.e., one expects to see all different species of tiles as $n \to \infty$. This region is called the *liquid region*, and its boundary *frozen boundary*. In particular, the algebraic curve is tangent to (not necessarily all) the sides of the polygon and possesses only cusp singularities. It is shown in [14] that at regular points of the frozen boundary one sees universal scaling limits.

Now if we associate to a top line configuration of yellow tiles the empirical measure

$$\mu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i^{(n)}/n},$$

then, in the situation studied by Petrov in [14], then μ_n converges weakly to a measure μ, whose density is the sum of characteristic functions of a finite union of disjoint intervals. One may now try to relax this assumption and study what happens asymptotically with the boundary of the disordered region if we instead assume that μ_n converges weakly to a more general measure μ. It will be convenient when considering interlacing system to make a coordinate transformation according to Figure 1.4. For more details see [7, Section 1.4] and [14, Section 2.1].

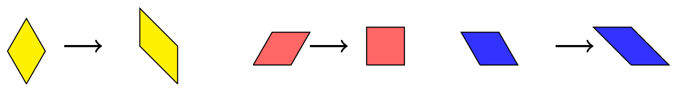

Figure 1.4. Coordinate transformation of lozenge tiles.

After the coordinate transformation, Figure 1.1 becomes Figure 1.5. Furthermore the interlacing condition between row $r+1$ and row r has changed into

$$y_1^{(r+1)} \geqslant y_1^{(r)} > y_2^{(r+1)} \geqslant y_2^{(r)} \cdots \geqslant y_r^{(r)} > y_{r+1}^{(r+1)}.$$

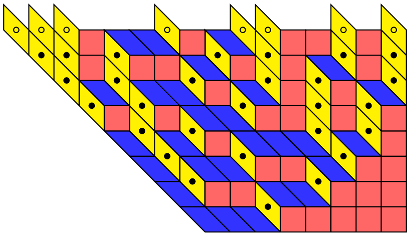

Figure 1.5. *An example tiling and its equivalent interlaced particle configuration after the coordinate transformation. The unfilled circles represent the deterministic lozenges/particles.*

Figure 1.5 is an example of a more general type of interlacing system in the yellow tiles where we no longer assume that the density of the limiting measures are sums of characteristic functions of a finite union of disjoint intervals. It turns out that with these more general boundary conditions on the top line novel phenomena occurs. In particular, the boundary of the liquid region is partitioned into two sets, one of which we call *the edge* $\mathcal{E}$, which corresponds to the frozen boundary in the models in [14], where one expects to see universal scaling limits, and one set where we conjecture that one no longer has this property. Moreover, the "non-universal boundary" can have a very complicated geometry. In particular, parts of this set can be pieces of straight lines going to the liquid region. An example of the geometry of the boundary of the liquid region is shown in Figure 1.6.

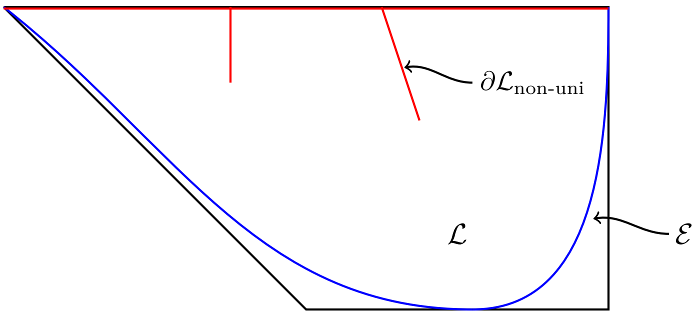

Figure 1.6. *The boundary of the liquid region $\partial\mathcal{L} = \mathcal{E} \cup \partial\mathcal{L}_{\text{non-uni}}$ is the union of two parts. On $\mathcal{E}$ we expect universal edge behaviour and on $\partial\mathcal{L}_{\text{non-uni}}$ we do not expect to see universal edge behaviour.*

1.2. Variational Problem for the Height Function

In the limit $n \to \infty$, when the size of tiles go to zero, we denote ρ_Y, ρ_R and ρ_B the local densities of the respective tile (which we assume exist). In particular, $\rho_Y(x) + \rho_R(x) + \rho_B(x) = 1$ for all x in our asymptotic tiling domain. To the asymptotic distribution of tiles we can associate a height function h, determined up to a constant, by letting

$$\nabla h(x) := (\rho_Y(x), \rho_R(x)). \tag{1.1}$$

In particular we note that $\nabla h(x) \in K \subset \mathbb{R}^2$, where K is the compact convex set given by the convex hull of the points $\{(0,0),(1,0),(0,1)\}$. However, already on the discrete level, we may associate a *random height* function to each random configuration of tiles. For details, see [12]. In [5] it was shown that $\lim_{n\to\infty} \mathbb{E}[h_n] = h$ almost surely. Moreover, let D denote the asymptotic tiling domain and let

$$\mathscr{C}_K(D,g) := \{u \in C^{0,1}(\overline{D}) : u|_{\partial D} = g,\, \nabla u(x) \in K, \text{ for a.e. } x \in D\}. \tag{1.2}$$

denote the space of possible height functions for a fixed asymptotic boundary value g. Then, also in [5], it was shown that the asymptotic height function h is the unique minimizer of the following convex functional,

$$\mathcal{E}[u] := \int_D \sigma(\nabla u(x))\mathrm{d}x, \quad u \in \mathscr{C}_K(D,g), \tag{1.3}$$

where $\sigma : K \to \mathbb{R}$ is a strictly convex function called the *surface tension* determined by the spectral curve of the lattice, see [12]. The set

$$\mathcal{L} := \{x \in D : \nabla h(x) \in K^\circ\}, \tag{1.4}$$

where K° denotes the interior of the set K, is called the *liquid region* and can be thought as the set where the gradient constraint $\nabla h(x) \in K$ is *not* active. From direct methods of the calculus of variations one can only conclude that h is a Lipschitz function, the gradient $\nabla h(x)$ need a priori only be defined almost everywhere by Rademacher's theorem. Therefore, the set $\mathcal{L}$ could have a very complicated structure. In particular it does not directly follow that $\mathcal{L}$ is an open set. If however $\mathcal{L}$ is open, then one can show that h must solve the Euler–Lagrange equation

$$\nabla \cdot \nabla\sigma(\nabla h(x)) = 0 \tag{1.5}$$

on $\mathcal{L}$. At this point it is useful to compare this variational problem to another more studied one with a convex gradient constraint. Consider the minimization of the Dirichlet energy

$$\mathcal{I}[u] := \int_D \nabla u(x) \cdot \nabla u(x)\mathrm{d}x, \quad u \in \mathscr{C}_K(D,g). \tag{1.6}$$

This minimization problem is called the *elastic-plastic torsion problem*, where $K = \overline{\mathbb{D}}$ is the closed disc, and has been studied in for example [1] and [4]. Let

$$\mathfrak{M}(x) := \sup_{u \in \mathscr{C}_K(D,g)} \{u(x)\}, \tag{1.7}$$

$$\mathfrak{m}(x) := \inf_{u \in \mathscr{C}_K(D,g)} \{u(x)\}. \tag{1.8}$$

Using a method due to Brezis and Sibony in [3], one can show that the gradient constraint $\nabla u \in \overline{\mathbb{D}}$ is equivalent to the constraint $\mathfrak{m} \leqslant u \leqslant \mathfrak{M}$, and in particular

$$\{x \in D : \nabla h(x) \in K^\circ\} = \{x \in D : \mathfrak{m}(x) < h < \mathfrak{M}(x)\}$$

and since both $\mathfrak{m}, \mathfrak{M}$ and the minimizer h of (1.6) are Lipschitz continuous, we can conclude that $\mathcal{L}$ is open. Unfortunately, this method relies on the fact that the surface tension, in the Dirichlet case $\sigma(x) = |x|^2$, has a continuous convex extension to a larger open set containing $\overline{\mathbb{D}}$. However, for the variational problem (1.3), the only convex extension of σ is to let

$$\sigma_{ex}(x) := \begin{cases} \sigma(x) & \text{if } x \in K \\ +\infty & \text{if } x \in K^c. \end{cases}$$

Therefore, one cannot conclude that the two constraints are equivalent. Due to this fact one can view the constraint $\nabla u(x) \in K$ as a *natural gradient constraint*. Therefore, to conclude that $\mathcal{L}$ is open is already a very non-trivial problem. For this case, and a large class of similar convex functionals, this was proven in [17]. Since $\mathcal{L}$ is not given, determining $\mathcal{L}$ is of great interest, since the minimizer h solves the Euler–Lagrange equation on that set. In particular one is interested in determining the regularity of the set

$$\Gamma := \partial\mathcal{L} \cap D^\circ, \tag{1.9}$$

called the *free boundary*. However, due to the natural gradient constraint, many standard methods in PDE and free boundary problems fail. In [11], this problem was studied when D was in a particular class of polygonal domains, and where the boundary value g was determined by ∂D up to a constant. The main conclusion in [11] is that $\overline{\Gamma}$ is in fact an *algebraic curve*. In this paper we study the case when the domain D is a four-sided polygon $\mathcal{P}$, but where we on one side allow arbitrary boundary values. However, we do not study Γ using the variational problem (1.3). Instead we use an alternative approach, using the special properties of the domain $\mathcal{P}$, so that one can study the discrete problem directly, using methods of algebraic combinatorics and asymptotic analysis. This builds upon the work in [7]. In particular we will use methods of harmonic analysis. One of the new results of this paper (Lemma 4.1) is that one can have $\mathcal{H}^1(\Gamma) = +\infty$ even for smooth boundary values. This is a result not easily proven using other methods. Moreover, the decomposition of $\partial\mathcal{L}$, and in particular Γ into the sets $\mathcal{E}$ and $\partial\mathcal{L}_{\text{non-uni}}$ also reflect the regularity properties of the minimizer h. In particular the set $\mathcal{S}_{nt}^{\text{sing},I}(\mu)$ and $\mathcal{S}_{nt}^{\text{sing,II}}(\mu)$ in Definition 1.6 correspond to sets where h is not C^1, i.e. $h \in C^1(\mathcal{P}^\circ \backslash (\mathcal{S}_{nt}^{\text{sing},I}(\mu) \cup \mathcal{S}_{nt}^{\text{sing,II}}(\mu)))$. This is consistent with Theorem 1.3 in [17].

1.3. Discrete Interlacing Sequences

We recall the underlying probabilistic model described in [7]. A *discrete Gelfand–Tsetlin pattern* of depth n is an n-tuple, denoted

$$(y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in \mathbb{Z} \times \mathbb{Z}^2 \times \cdots \times \mathbb{Z}^n,$$

which satisfies the interlacing constraint

$$y_1^{(r+1)} \geqslant y_1^{(r)} \geqslant y_2^{(r+1)} \geqslant y_2^{(r)} > \cdots \geqslant y_r^{(r)} \geqslant y_{r+1}^{(r+1)},$$

for all $r \in \{1, \dots, n-1\}$, denoted $y^{(r+1)} \succ y^{(r)}$. For each $n \geqslant 1$, fix $x^{(n)} \in \mathbb{Z}^n$ with $x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)}$, and consider the following probability measure on the set of patterns of depth n:

$$\nu_n[(y^{(1)}, \dots, y^{(n)})] := \frac{1}{Z_n} \cdot \begin{cases} 1; & \text{when } x^{(n)} = y^{(n)} \succ y^{(n-1)} \succ \cdots \succ y^{(1)}, \\ 0; & \text{otherwise,} \end{cases}$$

where $Z_n > 0$ is a normalisation constant. This can equivalently be considered as a measure on configurations of interlaced particles in $\mathbb{Z} \times \{1, \dots, n\}$ by placing a particle at position $(u, r) \in \mathbb{Z} \times \{1, \dots, n\}$ whenever u is an element of $y^{(r)}$. ν_n is then the uniform probability measure on the set of all such interlaced configurations with the particles on the top row in the deterministic positions defined by $x^{(n)}$. This measure also arises naturally from certain tiling models (see [7] and [14] for further details). In [7] and [14] it was independently shown that this process is determinantal. The correlation kernel, $K_n : (\mathbb{Z} \times \{1, \dots, n\})^2 \to \mathbb{C}$, acts on pairs of particle positions. Note that the deterministic top row and the interlacing constraint implies that it is sufficient to restrict to those positions, $(u, r), (v, s) \in \mathbb{Z} \times \{1, \dots, n-1\}$, with $u \geqslant x_n^{(n)} + n - r$ and $v \geqslant x_n^{(n)} + n - s$. For all such (u, r) and (v, s),

$$K_n((u, r), (v, s)) = \widetilde{K}_n((u, r), (v, s)) - \phi_{r,s}(u, v), \tag{1.10}$$

where

$$\widetilde{K}_n((u,r),(v,s))$$
$$:= \frac{1}{(2\pi\,\mathrm{i})^2}\frac{(n-s)!}{(n-r-1)!}\oint_{\gamma_n}\mathrm{d}w\oint_{\Gamma_n}\mathrm{d}z\,\frac{\prod_{k=u+r-n+1}^{u-1}(z-k)}{\prod_{k=v+s-n}^{v}(w-k)}\frac{1}{w-z}\prod_{i=1}^{n}\left(\frac{w-x_i^{(n)}}{z-x_i^{(n)}}\right),$$

and

$$\phi_{r,s}(u,v)$$
$$:= 1_{(v\geqslant u)}\cdot\begin{cases}0; & \text{when } s\leqslant r,\\ 1; & \text{when } s=r+1,\\ \frac{1}{(s-r-1)!}\prod_{j=1}^{s-r-1}(v-u+s-r-j); & \text{when } s>r+1.\end{cases}$$

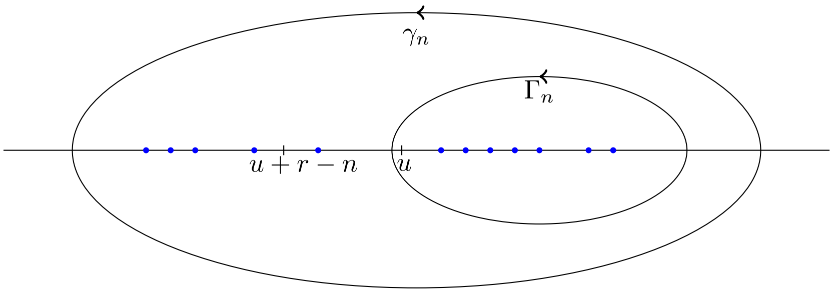

Figure 1.7. Γ_n *contains* $\{x_j : x_j \geqslant u\}$ *and none of* $\{x_j : x_j \leqslant u+r-n\}$. γ_n *contains* Γ_n *and* $\{v, v-1, \ldots, v+s-n)\}$. *Both contours are oriented counter-clockwise. Here the blue dots indicate a possible distribution of the set* $\{x_j\}_j$.

1.4. Asymptotic Assumptions and Geometric Behaviour of the Liquid Region

It is natural to consider the asymptotic behaviour of the determinantal system introduced in the previous section as $n \to \infty$, under the assumption that the (rescaled) empirical distribution of the deterministic particles on the top row converges weakly to a measure with compact support. More exactly, assume that

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i^{(n)}/n} \to \mu$$

as $n \to \infty$, in the sense of weak convergence of measures, where μ is a probability measure with compact support, $\mathrm{supp}(\mu)$. We additionally assume that the convex hull of $\mathrm{supp}(\mu)$ is of length strictly greater than 1. This condition ensures that the density of μ is not equal to 1 almost everywhere. This in turn implies that $\mathcal{L} \neq \varnothing$.

Definition 1.1. — *For clarity we explicitly state the class of measures in which μ lies: $\mu \in \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ is the set of Borel measures on $\mathbb{R}$. Moreover, $\mu \leqslant \lambda$ where λ is Lebesgue measure (recall $x^{(n)} \in \mathbb{Z}^n$), $\|\mu\| = 1$, μ has compact support. We will denote this set of measures by $\mu \in \mathcal{M}^{\lambda}_{c,1}(\mathbb{R})$. Additionally we note that μ admits a density w.r.t. λ, which is uniquely defined up to a set of zero Lebesgue measure. Denoting the density by f, and $[a,b]$ the convex hull of $\mathrm{supp}(\mu)$, ($b-a>1$), it satisfies $f \in L^\infty(\mathbb{R})$, $f(x) = 0$ for all $x \in \mathbb{R} \setminus [a,b]$, $\int_{\mathbb{R}} f(x)\mathrm{d}x = 1$, and $0 \leqslant f(x) \leqslant 1$ for all $x \in [a,b]$. We write $f \in \rho^{\lambda}_{c,1}(\mathbb{R})$. Note that $\mathbb{R} \setminus \mathrm{supp}(\mu)$ is the largest open set on which $f = 0$ almost everywhere, and $\mathbb{R} \setminus \mathrm{supp}(\lambda - \mu)$ is the largest open set on which $f = 1$ almost everywhere. Finally we note that the set $\mathcal{M}^{\lambda}_{c,1}(\mathbb{R})$ is convex, i.e., if $\sigma, \nu \in \mathcal{M}^{\lambda}_{c,1}(\mathbb{R})$, then for all $t \in [0,1]$, $t\sigma + (1-t)\nu \in \mathcal{M}^{\lambda}_{c,1}(\mathbb{R})$.*

Note, rescaling the vertical and horizontal positions of the particles of the Gelfand–Tsetlin patterns by $\frac{1}{n}$, that the weak convergence and the interlacing constraint imply that the rescaled particles almost surely lie asymptotically in the following set:

$$\mathcal{P} = \{(\chi,\eta) \in \mathbb{R}^2 : a \leqslant \chi + \eta - 1 \leqslant \chi \leqslant b, 0 \leqslant \eta \leqslant 1\}$$

Fixing $(\chi,\eta) \in \mathcal{P}$, the local asymptotic behaviour of particles near (χ,η) can be examined by considering the asymptotic behaviour of $K_n((u_n,r_n),(v_n,s_n))$ as $n \to \infty$, where $\{(u_n,r_n)\}_{n\geqslant 1} \subset \mathbb{Z}^2$ and $\{(v_n,s_n)\}_{n\geqslant 1} \subset \mathbb{Z}^2$ satisfy

$$\frac{1}{n}(u_n,r_n) \to (\chi,\eta), \quad \frac{1}{n}(v_n,s_n) \to (\chi,\eta)$$

as $n \to \infty$. Assume this additional asymptotic behaviour, substitute (u_n,r_n) and (v_n,s_n) into (1.10), and rescale the contours by $\frac{1}{n}$ to get,

$$\widetilde{K}_n((u_n,r_n),(v_n,s_n)) = \frac{A_n}{(2\pi\,\mathrm{i})^2} \oint_{\gamma_n} \mathrm{d}w \oint_{\Gamma_n} \mathrm{d}z \, \frac{\exp(nf_n(w) - n\tilde{f}_n(z))}{w-z}, \tag{1.11}$$

for all $n \in \mathbb{N}$. Now Γ_n contains $\{\frac{1}{n}x_i^{(n)} : x_i^{(n)} \geqslant u_n\}$ and none of $\{\frac{1}{n}x_i^{(n)} \leqslant u_n + r_n - n\}$, and γ_n contains Γ_n and $\{\frac{1}{n}(v_n + s_n - n), \dots, \frac{1}{n}v_n\}$. Also

$A_n := \frac{(n-s_n)!}{(n-r_n-1)!}\, n^{s_n-r_n-1}$,

$$f_n(w) := \frac{1}{n}\sum_{i=1}^{n}\log\left(w-\frac{x_i^{(n)}}{n}\right) - \frac{1}{n}\sum_{j=v_n+s_n-n}^{v_n}\log\left(w-\frac{j}{n}\right),$$

$$\tilde{f}_n(z) := \frac{1}{n}\sum_{i=1}^{n}\log\left(z-\frac{x_i^{(n)}}{n}\right) - \frac{1}{n}\sum_{j=u_n+r_n-n+1}^{u_n-1}\log\left(z-\frac{j}{n}\right).$$

Finally, inspired by the asymptotic assumptions and the forms of f_n and $\tilde{f}_n$, we define

$$f_{(\chi,\eta)}(w) := \int_{\mathbb{R}}\log(w-t)\mathrm{d}\mu(t) - \int_{\chi+\eta-1}^{\chi}\log(w-t)\mathrm{d}t, \tag{1.12}$$

for all $w \in \mathbb{C}\setminus\mathbb{R}$.

Remark 1.2. — Do not confuse the asymptotic function $f_{(\chi,\eta)}(w)$ with the density f of the measure μ. The authors apologize for this unfortunate notation and hope that it will not cause any confusion. Furthermore, the asymptotic function will only be mentioned in the introduction, and in all other sections of this paper, f will always denote the density of the measure.

Steepest descent analysis and equations (1.10) and (1.11) suggest that, as $n \to \infty$, the asymptotic behaviour of $K_n((u_n, r_n),(v_n, s_n))$ depends on the behaviour of the roots of $f'_{(\chi,\eta)}$:

$$f'_{(\chi,\eta)}(w) = \int_{\mathbb{R}}\frac{\mathrm{d}\mu(t)}{w-t} - \int_{\chi+\eta-1}^{\chi}\frac{\mathrm{d}t}{w-t}, \tag{1.13}$$

for all $w \in \mathbb{C}\setminus\mathbb{R}$. In [7], we define the *liquid region*, $\mathcal{L}$, as the set of all $(\chi,\eta) \in \mathcal{P}$ for which $f'_{(\chi,\eta)}$ has a unique root in the upper-half plane, $\mathbb{H} := \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$. Whenever $(\chi,\eta) \in \mathcal{L}$, one expects universal bulk asymptotic behaviour, i.e., that the local asymptotic behaviour of the particles near (χ,η) are governed by the extended discrete *Sine kernel* as $n \to +\infty$. Also, one expects that the particles are not asymptotically densely packed. Moreover, when considering the corresponding tiling model and its associated height function, one would expect to see the Gaussian Free Field asymptotically. See for example [14], [15] for a special case.

Let $W_{\mathcal{L}} : \mathcal{L} \to \mathbb{H}$ map $(\chi,\eta) \in \mathcal{L}$ to the corresponding unique root of $f'_{(\chi,\eta)}$ in $\mathbb{H}$. In [7], we show that $W_{\mathcal{L}}$ is a homeomorphism with inverse

$W_{\mathcal{L}}^{-1}(w) = (\chi_{\mathcal{L}}(w), \eta_{\mathcal{L}}(w))$ for all $w \in \mathbb{H}$, where

$$\chi_{\mathcal{L}}(w) := w + \frac{(w-\bar{w})(\mathrm{e}^{C(\bar{w})} - 1)}{\mathrm{e}^{C(w)} - \mathrm{e}^{C(\bar{w})}}, \tag{1.14}$$

$$\eta_{\mathcal{L}}(w) := 1 + \frac{(w-\bar{w})(\mathrm{e}^{C(w)} - 1)(\mathrm{e}^{C(\bar{w})} - 1)}{\mathrm{e}^{C(w)} - \mathrm{e}^{C(\bar{w})}}, \tag{1.15}$$

and $C : \mathbb{C} \setminus \operatorname{supp}(\mu) \to \mathbb{C}$ is the *Cauchy* transform of μ:

$$C(w) := \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{w-t}. \tag{1.16}$$

Thus $\mathcal{L}$ is a non-empty, open (with respect to $\mathbb{R}^2$), simply connected subset of $\mathcal{P}$.

Define the complex slope $\Omega = \Omega(\chi, \eta) \in \mathbb{C}$ by

$$\Omega(\chi, \eta) = \frac{W_{\mathcal{L}}(\chi, \eta) - \chi}{W_{\mathcal{L}}(\chi, \eta) - \chi - \eta + 1}. \tag{1.17}$$

The equation $f'(\chi, \eta)(w)\big|_{w = W_{\mathcal{L}}(\chi,\eta)} = 0$ implies that the *complex slope* Ω satisfies the equation

$$\frac{1}{\Omega} = \exp \int_{\mathbb{R}} \left(\chi + \frac{(1-\eta)\Omega}{1-\Omega} - t \right)^{-1} \mathrm{d}\mu(t). \tag{1.18}$$

Note that since

$$\Omega = \exp \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{t - W_{\mathcal{L}}(\chi, \eta)} \tag{1.19}$$

and $W_{\mathcal{L}}(\chi, \eta) \in \mathbb{H}$, it follows that $\operatorname{Im}[\Omega] > 0$ for all $(\chi, \eta) \in \mathcal{L}$. Moreover, by differentiating (1.18) with respect to χ and η respectively, one see that Ω satisfies the *complex Burgers equation*

$$\Omega \frac{\partial \Omega}{\partial \chi} = -(1-\Omega) \frac{\partial \Omega}{\partial \eta}. \tag{1.20}$$

For a connection to lozenge tiling problems see [11].

Using the complex slope Ω one define the Beta kernel $\mathcal{B}_{\Omega} : \mathbb{Z}^2 \to \mathbb{C}$, according to:

$$\mathcal{B}_{\Omega}(m, l) = \frac{1}{2\pi \mathrm{i}} \int_{\bar{\Omega}}^{\Omega} (1-z)^m z^{-l-1} \mathrm{d}z, \tag{1.21}$$

where the integration contours are such that they cross $(0,1) \subset \mathbb{R}$ when $m \geqslant 0$, and $(-\infty, 0) \subset \mathbb{R}$ when $m < 0$. It was shown in [14], that if one let $\mu = \lambda\big|_{\cup_{k=1}^m I_k}$, where $I_k = [a_k, b_k]$, and $\cup_{k=1}^m I_k$ is a disjoint union of intervals, then if one assumes that

$$\lim_{n\to\infty} \frac{1}{n}(x_i^{(n)}, y_i^{(n)}) = (\chi, \eta) \in \mathcal{L}, \quad \text{for } i = 1, 2, \ldots, r$$

and,

$$x_i^{(n)} - x_j^{(n)} = l_{ij} \in \mathbb{Z} \quad \text{and} \quad y_i^{(n)} - y_j^{(n)} = m_{ij} \in \mathbb{Z}$$

are fixed whenever n is sufficiently large, then

$$\lim_{n\to\infty} \rho_r((x_1^{(n)}, y_1^{(n)}), (x_2^{(n)}, y_2^{(n)}), \dots, (x_r^{(n)}, y_r^{(n)}) = \det[\mathcal{B}_\Omega(m_{ij}, l_{ij})]_{i,j=1}^r$$

Though it is not done in this paper, this result can be easily extended to the case when $\mu \in \mathcal{M}_{c,1}^\lambda(\mathbb{R})$. In particular note that this implies that the macroscopic density of particles are given by

$$\rho(\chi, \eta) = \frac{1}{2\pi\,\mathrm{i}} \int_{\overline{\Omega}}^{\Omega} \frac{\mathrm{d}z}{z} = \frac{1}{\pi} \arg \Omega(\chi, \eta).$$

In [7], we also study $\partial\mathcal{L}$. Our motivation for doing this is that edge-type behavior is expected at $\partial\mathcal{L}$ for appropriate scaling limits. It is therefore necessary to understand the geometry of $\partial\mathcal{L}$. We study $\partial\mathcal{L}$ using the above homeomorphism: $\partial\mathcal{L}$ is the set of all $(\chi, \eta) \in \mathcal{P}$ for which there exists a sequence, $\{w_n\}_{n\geqslant 1} \subset \mathbb{H}$, with $W_{\mathcal{L}}^{-1}(w_n) = (\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \to (\chi, \eta)$ as $n \to \infty$, and either $|w_n| \to \infty$ or $w_n \to x \in \mathbb{R} = \partial\mathbb{H}$ as $n \to \infty$.

The situation when $|w_n| \to \infty$ is trivial: $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \to (\frac{1}{2} + \int t\mathrm{d}\mu(t), 0)$ as $n \to \infty$. In order to consider the situation when $w_n \to x \in \mathbb{R} = \partial\mathbb{H}$, recall that $\mu \leqslant \lambda$. In [7], we consider the case where $w_n \to x \in R$, where $R \subset \mathbb{R}$ is the open set,

$$R := R_\mu \cup R_{\lambda-\mu} \cup R_0 \cup R_1 \cup R_2, \tag{1.22}$$

and

- $R_\mu := \mathbb{R} \setminus \operatorname{supp}(\mu) \cap \{t \in \mathbb{R} : C(t) \neq 0\}$.
- $R_{\lambda-\mu} := \mathbb{R} \setminus \operatorname{supp}(\lambda - \mu)$.
- $R_0 := \mathbb{R} \setminus \operatorname{supp}(\mu) \cap \{t \in \mathbb{R} : C(t) = 0\}$
- R_1 is the set of all $t \in \mathbb{R}$ for which there exists an $\eta > 0$ with $(t, t+\eta) \subset \mathbb{R}\setminus\operatorname{supp}(\mu)$ and $(t-\eta, t) \subset \mathbb{R}\setminus\operatorname{supp}(\lambda-\mu)$. In particular, the density ρ of μ restricted to the interval $(t-\eta, t+\eta)$ equals $\rho|_{(t-\eta,t+\eta)}(x) = \chi_{(t-\eta,t)}(x)$.
- R_2 is the set of all $t \in \mathbb{R}$ for which there exists an $\eta > 0$ with $(t, t+\eta) \subset \mathbb{R}\setminus\operatorname{supp}(\lambda-\mu)$ and $(t-\eta, t) \subset \mathbb{R}\setminus\operatorname{supp}(\mu)$. In particular, the density ρ of μ restricted to the interval $(t-\eta, t+\eta)$ equals $\rho|_{(t-\eta,t+\eta)}(x) = \chi_{(t,t+\eta)}(x)$.

We show that $(\chi_{\mathcal{L}}(w_n),\eta_{\mathcal{L}}(w_n)) \to (\chi_{\mathcal{E}}(t),\eta_{\mathcal{E}}(t))$ as $n\to\infty$, where $\chi_{\mathcal{E}},\eta_{\mathcal{E}}: R\to\mathbb{R}$ are the real-analytic functions defined by,

$$(1.23)\quad (\chi_{\mathcal{E}}(t),\eta_{\mathcal{E}}(t)) = \begin{cases} \left(t+\frac{1-\mathrm{e}^{-C(t)}}{C'(t)}, 1+\frac{\mathrm{e}^{C(t)}+\mathrm{e}^{-C(t)}-2}{C'(t)}\right) & \text{if } t\in R_\mu\cup R_0 \\ \left(t+\frac{1-(\frac{t-t_1}{t-t_2})\,\mathrm{e}^{-C(t)}-1}{C_I'(t)+\frac{1}{t-t_2}-\frac{1}{t-t_1}}, 1+\frac{(\frac{t-t_2}{t-t_1})\,\mathrm{e}^{C_I(t)}+(\frac{t-t_1}{t-t_2})\,\mathrm{e}^{-C_I(t)}-2}{C_I'(t)+\frac{1}{t-t_2}-\frac{1}{t-t_1}}\right) & \text{if } t\in R_{\lambda-\mu} \\ (t, 1-\mathrm{e}^{C_I(t)}(t-t_2)) & \text{if } t\in R_1 \\ (t-\mathrm{e}^{-C_I(t)}(t-t_1), 1+\mathrm{e}^{-C_I(t)}(t-t_1)) & \text{if } t\in R_2 \end{cases}$$

Above $I := (t_2,t_1)$ is any interval which satisfies $t\in I\subset\mathbb{R}\setminus\operatorname{supp}(\lambda-\mu)$ whenever $t\in\mathbb{R}\setminus\operatorname{supp}(\lambda-\mu)$, and the requirements of (1.22) whenever $x\in R_1\cup R_2$. Also, C is the Cauchy transform of (1.16), and $C_I(t) := \int_{\mathbb{R}\setminus I}\frac{\mathrm{d}\mu(x)}{t-x}$ for all $t\in I$. It follows from above that $(\chi_{\mathcal{E}}(\,\cdot\,),\eta_{\mathcal{E}}(\,\cdot\,)) : R\to\partial\mathcal{L}$ is the unique continuous extension, to R, of $(\chi_{\mathcal{L}}(\,\cdot\,),\eta_{\mathcal{L}}(\,\cdot\,)) : \mathbb{H}\to\mathcal{L}$. In [7] we show that the extension is injective, and we define the *edge*, $\mathcal{E}\subset\partial\mathcal{L}$, as the image space of the extension. We argue that $\mathcal{E}$ is a natural subset of $\partial\mathcal{L}$ on which to expect edge asymptotic behaviour. This will be examined in the upcoming papers, [8] and [6]. In these papers we will show, for example, as $n\to\infty$ and choosing the parameters (u_n,r_n) and (v_n,s_n) appropriately, that $K_n((u_n,r_n),(v_n,s_n))$ converges to the *Airy* or *Cusp-Airy* kernel when $x\in\mathbb{R}\setminus\operatorname{supp}(\mu)$ or $\in\mathbb{R}\setminus\operatorname{supp}(\lambda-\mu)$ and $(\chi,\eta) = (\chi_{\mathcal{E}}(t),\eta_{\mathcal{E}}(t))$. Similarly when $t\in\mathbb{R}\setminus\operatorname{supp}(\lambda-\mu)$, except now the asymptotic behaviour of the correlation kernel of the "holes" is examined. Thus $\mathcal{E}$ is a subset of $\partial\mathcal{L}$ where we expect standard, universal type edge behavior. Furthermore, in [7], we defined the sets $\mathcal{E}_\mu = W_{\mathcal{E}}^{-1}(R_\mu)$, $\mathcal{E}_{\lambda-\mu} = W_{\mathcal{E}}^{-1}(R_\mu)$, $\mathcal{E}_0 = W_{\mathcal{E}}^{-1}(R_0)$, $\mathcal{E}_1 = W_{\mathcal{E}}^{-1}(R_1)$, and $\mathcal{E}_2 = W_{\mathcal{E}}^{-1}(R_2)$. One can show that for any sequence $\{(\chi_n,\eta_n)\}_n\subset\mathcal{L}$, such that $\lim_{n\to\infty}(\chi_n,\eta_n) = (\chi_{\mathcal{E}},\eta_{\mathcal{E}})\in\mathcal{E}$, the boundary value of the complex slope Ω exists and equals

$$(1.24)\qquad \lim_{n\to\infty}\Omega(\chi_n,\eta_n) = \begin{cases} \mathrm{e}^{-C(t)}\in\mathbb{R} & \text{if } (\chi_{\mathcal{E}},\eta_{\mathcal{E}})\in\mathcal{E}_\mu \\ \frac{t-t_2}{t-t_1}\,\mathrm{e}^{-C_I(t)}\in\mathbb{R} & \text{if } (\chi_{\mathcal{E}},\eta_{\mathcal{E}})\in\mathcal{E}_{\lambda-\mu} \\ 1 & \text{if } (\chi_{\mathcal{E}},\eta_{\mathcal{E}})\in\mathcal{E}_0 \\ 0 & \text{if } (\chi_{\mathcal{E}},\eta_{\mathcal{E}})\in\mathcal{E}_1 \\ \infty & \text{if } (\chi_{\mathcal{E}},\eta_{\mathcal{E}})\in\mathcal{E}_2 \end{cases}$$

where $t = W_{\mathcal{E}}(\chi_{\mathcal{E}},\eta_{\mathcal{E}})$, and where $\lim_{n\to\infty}\Omega(\chi_n,\eta_n) = \infty$ is viewed as a limit on the Riemann sphere $\mathbb{C}\cup\{\infty\}$. Hence, we may view $\mathcal{E}$ as a *shock* of the complex Burgers equation (1.20).

Remark 1.3. — In principle the convergence of $K_n((u_n,r_n),(v_n,s_n))$ could depend on how the empirical measure μ_n converges to μ. However, such questions will be considered in an upcoming paper [8].

Remark 1.4. — Note that $R_1 \cap R_2 = \emptyset$. Also $R_1 \cup R_2 = \partial(\mathbb{R} \setminus \operatorname{supp}(\mu)) \cap \partial(\mathbb{R} \setminus \operatorname{supp}(\lambda-\mu))$, the set of all common boundary points of the disjoint open sets $\mathbb{R} \setminus \operatorname{supp}(\mu)$ and $\mathbb{R} \setminus \operatorname{supp}(\lambda-\mu)$. Therefore we can alternatively write, $R = (\ \overline{(\mathbb{R} \setminus \operatorname{supp}(\mu)) \cup (\mathbb{R} \setminus \operatorname{supp}(\lambda-\mu))}\ )^\circ$.

Note that $R = \mathbb{R} = \partial\mathbb{H}$ in the special case when μ is Lebesgue measure restricted to a finite number of disjoint intervals. In this case $\mathcal{E} = \partial\mathcal{L}$, and was examined by Petrov, [14]. For general μ, however, $\mathbb{R} \setminus R$ is non-empty. It therefore remains to consider sequences, $\{w_n\}_{n\geqslant 1} \subset \mathbb{H}$, with $w_n \to x \in \mathbb{R} \setminus R$ as $n \to \infty$. In [7], letting f denotes the density of μ (see Definition 1.1), we show that:

Lemma 1.5. — *$(x,1) \in \partial\mathcal{L}$ for $x \in \mathbb{R} \setminus R = (\operatorname{supp}(\mu) \cap \operatorname{supp}(\lambda-\mu)) \setminus (R_1 \cup R_2)$ whenever there exists an $\epsilon > 0$ for which one of the following cases is satisfied:*

(1) $\sup_{t\in(x-\epsilon,x+\epsilon)} f(t) < 1$ *and* $\inf_{t\in(x-\epsilon,x+\epsilon)} f(t) > 0$.
(2) $\sup_{t\in(x-\epsilon,x)} f(t) < 1$, $\inf_{t\in(x-\epsilon,x)} f(t) > 0$ *and* $f(t) = 0$ *for all* $t \in (x, x+\epsilon)$.
(3) $\sup_{t\in(x-\epsilon,x)} f(t) < 1$, $\inf_{t\in(x-\epsilon,x)} f(t) > 0$ *and* $f(t) = 1$ *for all* $t \in (x, x+\epsilon)$.
(4) $\sup_{t\in(x,x+\epsilon)} f(t) < 1$, $\inf_{t\in(x,x+\epsilon)} f(t) > 0$ *and* $f(t) = 0$ *for all* $t \in (x-\epsilon, x)$.
(5) $\sup_{t\in(x,x+\epsilon)} f(t) < 1$, $\inf_{t\in(x,x+\epsilon)} f(t) > 0$ *and* $f(t) = 1$ *for all* $t \in (x-\epsilon, x)$.

Moreover $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \to (x,1)$ as $n \to \infty$ for all $\{w_n\}_{n\geqslant 1} \subset \mathbb{H}$ with $w_n \to x$.

Recall that for a general $f \in \rho^{\lambda}_{c,1}(\mathbb{R})$ the assumptions of Lemma 1.5 need not be satisfied, and so the above lemma gives an incomplete picture. The main goal of this paper is to extend this result. In particular, we will examine the novel and subtle geometric behaviour of $\partial\mathcal{L}$ when the conditions of the above lemma are violated. This analysis is surprisingly difficult, and naturally leads to questions in harmonic analysis.

Finally, it is natural to consider the asymptotic behaviour of the correlation kernel in (1.11) when $(\chi,\eta) \in \partial\mathcal{L} \setminus \mathcal{E}$. Though we do not consider such questions in this paper, we conjecture that this behaviour is "non-universal". Indeed, we conjecture that the local behaviour of the deterministic particles, $x^{(n)}$, strongly influence the asymptotics in this case. Some intuition about why this may so can be obtained from steepest descent considerations. In paper [8], for example, we consider a natural subset of $\mathcal{E}$ whereby each (χ, η) in this subset is characterised by a unique real-valued root of $f'_{(\chi,\eta)}$ (see (1.13)) of multiplicity 2. Using this root, we then perform a standard, though quite technical, steepest descent analysis of the contour integral expression in (1.12), and prove that the asymptotics of the correlation kernel are governed by the *Airy kernel*. Thus we confirm universal edge asymptotic behaviour for each (χ, η) in this subset. Of course, the existence and uniqueness of the real-valued root of multiplicity 2 is central to the analysis, and through this we can show that the local behaviour of $x^{(n)}$ have no influence on the asymptotics. Points $(\chi, \eta) \in \partial\mathcal{L} \setminus \mathcal{E}$, however, have no analogous characterisation.

1.5. Introduction to The Geometry of $\partial\mathcal{L}\backslash\mathcal{E}$ and the Non-Trivial Support of μ

As explained in the previous section, fixing $\mu \in \mathcal{M}^{\lambda}_{c,1}(\mathbb{R})$ (see Remark 1.1) and defining $\chi_{\mathcal{L}}$ and $\eta_{\mathcal{L}}$ as in (1.14) and (1.15), we wish to examine the boundary behaviour of the homeomorphism $(\chi_{\mathcal{L}}(\,\cdot\,), \eta_{\mathcal{L}}(\,\cdot\,)) : \mathbb{H} \to \mathcal{L}$ in the neighbourhood of the following set:

Definition 1.6. — *Given* $\mu \in \mathcal{M}^{\lambda}_{c,1}(\mathbb{R})$*, the* non-trivial support *of* μ*, denoted* $\mathcal{S}_{nt}(\mu) \subset \mathbb{R}$*, is the complement of the open set defined in* (1.22)*. More exactly,*

$$\mathcal{S}_{nt}(\mu) := \mathrm{supp}(\mu) \cap \mathrm{supp}(\lambda - \mu) \setminus (R_1 \cup R_2),$$

where λ *is Lebesgue measure and*

$$R_1 \cup R_2 = \partial(\mathbb{R} \setminus \mathrm{supp}(\mu)) \cap \partial(\mathbb{R} \setminus \mathrm{supp}(\lambda - \mu))$$

(see Remark 1.4).

Throughout the remainder of this paper we therefore make the following assumptions:

$$\text{Fix } \mu \in \mathcal{M}^{\lambda}_{c,1}(\mathbb{R}) \text{ for which } \mathcal{S}_{nt}(\mu)^{\circ} \text{ is non-empty.} \tag{1.25}$$

Remark 1.7. — Assumption 1.25 excludes densities of the form $f(t) = \phi(t)\chi_{\mathfrak{K}}(t)$, where $\phi \in \rho^{\lambda}_{c,1}(\mathbb{R})$, and $\mathfrak{K}$ is a measurable closed set such that $\mathfrak{K}^{\circ} = \varnothing$. Then $\mathcal{S}_{nt}(\mu) \subset \mathfrak{K}$. In particular, we will not consider examples of the form $f(t) = \chi_{\mathfrak{C}}(t)$, where $\mathfrak{C}$ is a fat Cantor set, that is a nowhere dense set such that $\lambda(\mathfrak{C}) > 0$.

(1.26) Let $X := \{t : 0 < f(t) < 1, \mathrm{d}\mu(t) = f(t)\mathrm{d}t\}$.

Assume that for any open interval $I \subset \mathcal{S}_{nt}(\mu)^\circ$, $\lambda(X \cap I) > 0$.

Remark 1.8. — This assumption is non-trivial. In [16], it is shown that there exists a Borel set $A \subset [0,1]$ such that for any interval $I \subset [0,1]$ one has

$$0 < \lambda(A \cap I) < \lambda(I). \tag{1.27}$$

Taking $f\big|_{[0,1]}(t) = \chi_A(t)$, (1.27) shows that $[0,1] \subset \mathcal{S}_{nt}(\mu)$. However, $\lambda(\{t : 0 < f(t) < 1\} \cap [0,1]) = 0$.

Fix $x \in \mathcal{S}_{nt}(\mu)$ and a sequence $\{w_n\}_{n\geqslant 1} \subset \mathbb{H}$ with $w_n \to x$ as $n \to \infty$. Assuming these hypothesises, we wish to examine the behaviour of $\{(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n))\}_{n\geqslant 1}$ as $n \to \infty$ for the various possibilities of the point $x \in \mathcal{S}_{nt}(\mu)$ and the sequence $\{w_n\}_{n\geqslant 1} \subset \mathbb{H}$. More precisely, we introduce the following equivalence relation:

Definition 1.9. — *The sequences $\omega_x = \{w_n\}_{n=1}$ and $\omega'_x = \{w'_m\}_{m=1}$ are said to be equivalent if the following holds:*

- $\lim_{n\to\infty} w_n = \lim_{m\to\infty} w'_m = x$.
- *There exist $N > 0$ and $M > 0$, depending on ω_x and ω'_x such that $w_{N+n} = w'_{M+n}$ whenever $n > 0$.*

This is easily seen to be an equivalence relation. We denote this by $\omega_x \sim \omega'_x$ and denote $[\omega]$ by its equivalence class. Furthermore, for each $x \in \mathbb{R}$, let S_x denote the set of equivalence classes of sequences converging to x.

Now let

$$\begin{aligned}\partial\mathcal{L}_\omega(x) &:= \overline{\{(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) : n \geqslant 1\}} \backslash \{(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) : n \geqslant 1\} \\ &= \{(\chi', \eta') \in \mathcal{P} : \{w_{n_k}\}_k \subset \{w_n\}_{n=1}, \lim_{k\to\infty} (\chi_{\mathcal{L}}(w_{n_k}), \eta_{\mathcal{L}}(w_{n_k}) = (\chi', \eta')\}.\end{aligned} \tag{1.28}$$

Then clearly $\partial\mathcal{L}_\omega(x) = \partial\mathcal{L}_{\omega'}(x) = \partial\mathcal{L}_{[\omega]}(x)$ whenever $\omega \sim \omega'$. Finally let

$$\partial\mathcal{L}(x) = \bigcup_{[\omega]\in S_x} \partial\mathcal{L}_{[\omega]}(x). \tag{1.29}$$

We now note that by Lemma A.1 in the appendix,

$$\partial\mathcal{L} = \partial\mathcal{L}(\infty) \cup \left(\bigcup_{x\in\mathbb{R}} \partial\mathcal{L}(x) \right).$$

In Lemma 4.1, we show for every $x \in \mathcal{S}_{nt}(\mu)^\circ$ that we can always choose $\{w_n\}_{n\geqslant 1}$ such that $(\chi_\mathcal{L}(w_n), \eta_\mathcal{L}(w_n)) \to (x,1)$. In other words, $\mathcal{S}_{nt}(\mu)^\circ \times \{1\} \subset \partial\mathcal{L}$. We define the generic case as that in which this limit is observed for arbitrary sequences:

Definition 1.10. — *$x \in \mathcal{S}_{nt}(\mu)$ is said to be* generic *whenever $\partial\mathcal{L}(x) = \{(x,1)\}$. In particular, this is equivalent to $(\chi_\mathcal{L}(w_n), \eta_\mathcal{L}(w_n)) \to (x,1)$ as $n \to \infty$ for arbitrary sequences $\{w_n\}_{n\geqslant 1} \subset \mathbb{H}$ converging to x. The set of generic points will be denoted by $\mathcal{S}_{nt}^{\mathrm{ren}}(\mu)$.*

The homeomorphism, $(\chi_\mathcal{L}(\,\cdot\,), \eta_\mathcal{L}(\,\cdot\,)) : \mathbb{H} \to \mathcal{L}$, therefore has a unique continuous extension to $x \in \mathcal{S}_\mu$ whenever x is generic. In Theorem 3.6, we show that for a typical set $G \subset \mathcal{S}_{nt}(\mu)^\circ$, where G is defined in Proposition 3.1, $G \subset \mathcal{S}_{nt}^{\mathrm{ren}}(\mu)$ is dense in $\mathcal{S}_{nt}(\mu)^\circ$.

We are particularly interested in those parts of $\partial\mathcal{L}$ that arise from non-generic points. Recall in the previous section, we defined the edge, $\mathcal{E} \subset \partial\mathcal{L}$, by extending $(\chi_\mathcal{L}(\,\cdot\,), \eta_\mathcal{L}(\,\cdot\,))$ uniquely and continuously to $\mathbb{R} \setminus \mathcal{S}_{nt}(\mu)$. In particular $\mathcal{E} = \bigcup_{x\in R} \partial\mathcal{L}(x)$. Also the point $\partial\mathcal{L}(\infty) = (\frac{1}{2} + \int t\mathrm{d}\mu(t), 0)$ is obtained by extending the homeomorphism uniquely and continuously to "infinity". Finally, as observed above, $\mathcal{S}_{nt}(\mu)^\circ \times \{1\} \subset \partial\mathcal{L}$. We therefore define the *singular part of* $\partial\mathcal{L}$, denoted $\partial\mathcal{L}_{\mathrm{sing}} \subset \partial\mathcal{L}$, as:

$$\partial\mathcal{L}_{\mathrm{sing}} := \partial\mathcal{L} \setminus \left(\mathcal{E} \cup \left\{\left(\frac{1}{2} + \int t\mathrm{d}\mu(t), 0\right)\right\} \cup (\mathcal{S}_{nt}^{\mathrm{ren}}(\mu) \times \{1\})\right). \tag{1.30}$$

In view of Lemma A.1, this leads to the natural decomposition of the boundary $\partial\mathcal{L}$ according to

$$\partial\mathcal{L} = \left\{\left(\frac{1}{2} + \int t\mathrm{d}\mu(t), 0\right)\right\} \cup \mathcal{E} \cup (\mathcal{S}_{nt}^{\mathrm{ren}}(\mu) \times \{1\}) \cup \partial\mathcal{L}_{\mathrm{sing}}. \tag{1.31}$$

In particular we have

$$\partial\mathcal{L}_{\mathrm{sing}} = \bigcup_{x\in\mathbb{R}\setminus(R\cup\mathcal{S}_{nt}^{\mathrm{ren}}(\mu))} \partial\mathcal{L}(x). \tag{1.32}$$

We begin our analysis by expressing $((\chi_\mathcal{L}(w), \eta_\mathcal{L}(w)) = ((\chi_\mathcal{L}(u,v), \eta_\mathcal{L}(u,v))$ in real and imaginary parts of $C(w)$, where $w = u + iv$. Using that

$$\mathrm{Re}(C(w)) = \int_\mathbb{R} \frac{(u-t)f(t)\mathrm{d}t}{(u-t)^2+v^2} := \pi H_v f(u) \tag{1.33}$$

$$-\mathrm{Im}(C(w)) = \int_\mathbb{R} \frac{vf(t)\mathrm{d}t}{(u-t)^2+v^2} = \pi P_v f(u), \tag{1.34}$$

equations (1.14) and (1.15) then become

$$\chi_{\mathcal{L}}(u,v) = u + v\frac{\mathrm{e}^{-\pi H_v f(u)} - \cos(\pi P_v f(u))}{\sin(\pi P_v f(u))}, \tag{1.35}$$

$$\eta_{\mathcal{L}}(u,v) = 1 - v\frac{\mathrm{e}^{\pi H_v f(u)} + \mathrm{e}^{-\pi H_v f(u)} - 2\cos(\pi P_v f(u)))}{\sin(\pi P_v f(u)))}. \tag{1.36}$$

Remark 1.11. — Recall that $P_v f(u)$ is the Poisson kernel of f and $H_v f(u)$ is the harmonic conjugate of $P_v f(u)$. Also note that by Lemma 2.7, $0 < \pi P_v f(u) < \pi$ for all $(u,v) \in \mathbb{H}$. It is a well-known fact from harmonic analysis that

$$\lim_{v\to 0^+} P_v f(u) = f(u) \qquad \text{for a.e } u \tag{1.37}$$

$$\lim_{v\to 0^+} H_v f(u) = \mathcal{H} f(u) \qquad \text{for a.e } u, \tag{1.38}$$

where $\mathcal{H}f$ denotes the Hilbert transform of f. In fact, the limits exist for every u in the Lebesgue set of f and the Lebesgue set of $\mathcal{H}f$ respectively, where the Lebesgue set $\mathscr{L}_f$ of an $L^1_{\mathrm{loc}}(\mathbb{R})$ function f is the set of all $x \in \mathbb{R}$ such that

$$\lim_{h\to 0^+} \frac{1}{2h}\int_{x-h}^{x+h} |f(t) - f(x)|\mathrm{d}t = 0.$$

We now distinguish between different types of sequences that will be of use:

Definition 1.12. — $\{w_n\}_{n\geqslant 1} = \{u_n + \mathrm{i}\, v_n\}_{n\geqslant 1}$ *is said to converge* non-tangentially *to x whenever there exists a constant $k > 0$ for which $|\frac{u_n - x}{v_n}| < k$ for all n sufficiently large and such that $\lim_{n\to\infty} w_n = x$. $\{w_n\}_{n\geqslant 1}$ is said to converge* tangentially *to x whenever $|\frac{u_n - x}{v_n}| \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} w_n = x$.*

Note, we can alternatively define the above sequences by considering the following *truncated cones*: For all $k > 0$ and $h > 0$, define $\Gamma_k^h(x) \subset \Gamma_k(x) \subset \mathbb{H}$ by,

$$\Gamma_k^h(x) := \{(u,v) \in \mathbb{H} : 0 < v < h \text{ and } |u - x| < kv\},$$
$$\Gamma_k(x) := \{(u,v) \in \mathbb{H} : v > 0 \text{ and } |u - x| < kv\}.$$

These are shown in Figure (1.8). Note that $\{w_n\}_{n\geqslant 1}$ converges non-tangentially to x iff $w_n \to x$ and there exists a $k > 0$ for which $w_n \in \Gamma_k(x)$ for all n sufficiently large. Also, $\{w_n\}_{n\geqslant 1}$ converges tangentially to x iff $w_n \to x$ and there exists an $n(k)$ for which $w_n \notin \Gamma_k(x)$ for all $n > n(k)$.

Of course, arbitrary sequences $\{w_n\}_{n\geqslant 1} \subset \mathbb{H}$ such that $\lim_{n\to\infty} w_n = x$ are not-necessarily tangential nor non-tangential. However by considering

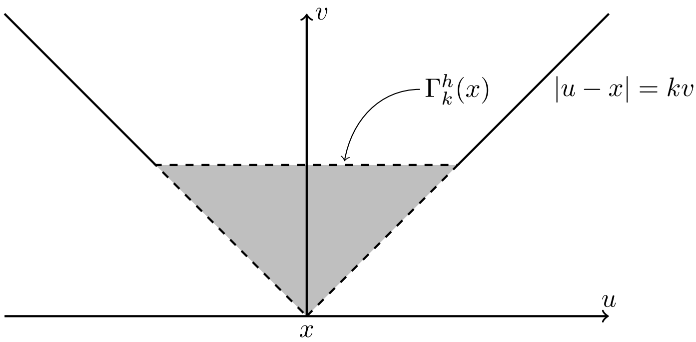

Figure 1.8. Truncated Cone

sub-sequences, one can assume that the sequence is either tangential or non-tangential.

Definition 1.13. — *$x \in \mathcal{S}_{nt}(\mu)$ is said to be* regular *if and only if $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \to (x,1)$ as $n \to \infty$ whenever $\{w_n\}_{n\geqslant 1}$ converges non-tangentially to x. The set of regular points is denoted by $\mathcal{S}_{nt}^{\mathrm{reg}}(\mu)$.*

In Section 6 we consider non-generic situations:

Definition 1.14. — *$x \in \mathcal{S}_{nt}(\mu)$ is said to be* singular *if it is not regular, and the set of all singular points will be denoted by $\mathcal{S}_{nt}^{\mathrm{sing}}(\mu)$. We identify four classes singular points:*

- *$x \in \mathcal{S}_{nt}^{\mathrm{sing},I}(\mu)$ if and only if there exists a $\delta > 0$ and a function $\varphi : \mathbb{R} \to \mathbb{R}$ for which x is in the Lebesgue set of φ, $\varphi(x) = 0$, $|\mathcal{H}\varphi(x)| < +\infty$, and $f(t) = \chi_{[x-\delta,x]}(t) + \varphi(t)$ for almost all t.*
- *$x \in \mathcal{S}_{nt}^{\mathrm{sing,II}}(\mu)$ if and only if there exists a $\delta > 0$ and a function $\varphi : \mathbb{R} \to \mathbb{R}$ for which x is in the Lebesgue set of φ, $\varphi(x) = 0$, $|\mathcal{H}\varphi(x)| < +\infty$, and $f(t) = \chi_{[x,x+\delta]}(t) + \varphi(t)$ for almost all t.*
- *$x \in \mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)$ if and only if $\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x-t)^2} < +\infty$ and $\mathcal{H}f(x) \neq 0$.*
- *$x \in \mathcal{S}_{nt}^{\mathrm{sing,IV}}(\mu)$ if and only if $\int_{\mathbb{R}} \frac{1-f(t)\mathrm{d}t}{(x-t)^2} < +\infty$ and $\mathcal{H}(1-f)(x) \neq 0$.*

We will only consider the case $x \in \mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)$ in this paper, as the similar ideas applies to the other cases as well. The fact that $x \in \mathcal{S}_{nt}(\mu)$ is singular whenever $x \in \mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)$ is shown in Propositions 4.2. Indeed we show, whenever $x \in \mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)$ and $\{w_n\}_{n\geqslant 1}$ is non-tangential, that $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n))$ converges to a point which is different from $(x,1)$.

In particular the set $\mathcal{S}_{nt}^{\mathrm{sing}}(\mu)$ can be seen as an obstruction to extending the map $W_{\mathcal{L}}^{-1}(\mu) : \mathbb{H} \to \mathcal{L}$ to a homeomorphism of the boundary. In particular, when $\mathcal{S}_{nt}^{\mathrm{sing}}(\mu) \neq \varnothing$, then $\partial\mathcal{L}$ is not homeomorphic to S^1.

When considering the boundary behavior of the map $W_{\mathcal{L}}^{-1}$ for sequence $\{w_n\}_n \in \mathbb{H}$ such that $\lim_{n\to\infty} w_n = x \in \mathcal{S}_{nt}^{\mathrm{sing}}(\mu)$ we will almost exclusively consider the case of isolated singular points. Furthermore, it will be shown that to study boundary behavior at such points one will be forced to consider particular classes of tangential sequences converging to x. More precisely, under an additional technical assumption on the density f, we prove in Proposition 4.7 and Theorem 4.12 that: If $x \in \mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)$ and there exists an $\varepsilon > 0$ such that $\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(y-t)^2} < \infty$ for all $y \in (x-\varepsilon, x) \cup (x, x+\varepsilon)$, then

$$\partial\mathcal{L}(x) = \overline{\left\{\left(x + \frac{1-\mathrm{e}^{-\pi\mathcal{H}f(x)}}{\xi - \pi(\mathcal{H}f)'(x)}, 1 - \frac{\mathrm{e}^{\pi\mathcal{H}f(x)} + \mathrm{e}^{-\pi\mathcal{H}f(x)} - 2}{\xi - \pi(\mathcal{H}f)'(x)}\right) : \xi \in (0, +\infty)\right\}}.$$

Note that the geometry of $\partial\mathcal{L}(x)$ in these cases is entirely characterized by either $\mathcal{H}\varphi(x)$ or the numbers $\mathcal{H}f(x)$ and $(\mathcal{H}f)'(x)$. An additional reason why we choose to only consider those singular points which satisfied some additional criteria for isolatedness, is that we do not believe that the same type of simple characterization of $\partial\mathcal{L}(x)$ is possible in the case of dense singular points, or in the case when the assumptions of Proposition 4.7 are violated. Finally, if one applies Definition 1.14 to points $x \in R$ one would find that every x in $R_\mu \cup R_{\lambda-\mu} \cup R_1 \cup R_2$ were singular points. Therefore the case of considering boundary behavior of non-isolated boundary points of $\partial\mathcal{S}_{nt}(\mu)$ is similar to the case of non-isolated singular points. We will therefore restrict ourselves to consider only isolated points of $\partial\mathcal{S}_{nt}(\mu)$.

We now consider the boundary behaviour of the complex slope Ω for sequences $\{(\chi_n, \eta_n)\}_n \subset \mathcal{L}$ such that $\lim_{n\to\infty}(\chi_n, \eta_n) = (\chi, \eta) \in \partial\mathcal{L}\backslash\mathbb{R}$. First consider the case when $(\chi, \eta) \in \{(x, 1) : x \in \mathcal{S}_{nt}^{\mathrm{ren}}(\mu)\}$. One can show that almost all *non-tangential* limits exists and

$$\lim_{n\to\infty} \Omega(\chi_n, \eta_n) = \mathrm{e}^{-\pi\mathcal{H}f(x) + \mathrm{i}\,\pi f(x)} \in \mathbb{C}.$$

Thus, such limit is thus not in general real, which should be contrasted to the case when $(\chi, \eta) \in \mathcal{E}$. On the other hand, if we assume that $x \in \mathcal{S}_{nt}^{\mathrm{sing}}(\mu)$, and that in addition x is an isolated singular point, then for *all* sequences

$\{(\chi_n,\eta_n)\}_n \subset \mathcal{L}$ such that $\lim_{n\to\infty}(\chi_n,\eta_n) = (\chi,\eta) \in \partial\mathcal{L}(x)$ we get

$$\lim_{n\to\infty} \Omega(\chi_n,\eta_n) = \begin{cases} \mathrm{e}^{-\pi\mathcal{H}f(x)} \in \mathbb{R} & \text{if } x \in \mathcal{S}_{nt}^{\text{sing,III}}(\mu) \\ -\,\mathrm{e}^{\pi\mathcal{H}(1-f)(x)} \in \mathbb{R} & \text{if } x \in \mathcal{S}_{nt}^{\text{sing,IV}}(\mu) \\ 0 & \text{if } x \in \mathcal{S}_{nt}^{\text{sing},I}(\mu) \\ \infty & \text{if } x \in \mathcal{S}_{nt}^{\text{sing,II}}(\mu) \end{cases} \tag{1.39}$$

This shows that at least a subset of $\partial\mathcal{L}_{\text{sing}}$ are shocks of the complex Burgers equation in the same way as $\mathcal{E}$.

We will conclude this introduction by discussing open problems not solved in this paper.

Conjecture 1.15. — $\mathcal{S}_{nt}^{\text{reg}}(\mu) = \mathcal{S}_{nt}^{\text{ren}}(\mu)$.

Conjecture 1.16. — *$\mathcal{S}_{nt}^{\text{sing}}(\mu)$ is meagre set in $\mathbb{R}$.*

However, note that $\mathcal{S}_{nt}^{\text{sing}}(\mu)$ is not necessarily negligible from a measure theoretic point of view. This is proven in Lemma 4.4, where we show that there exists a $\mu \in \mathcal{M}_{c,1}^{\lambda}(\mathbb{R})$ such that $\lambda(\mathcal{S}_{nt}^{\text{sing}}(\mu)) > 0$ and $\mathcal{H}^1(\partial\mathcal{L}) = +\infty$, where $\mathcal{H}^1$ denotes the one dimensional Hausdorff measure. Moreover, in Lemma 4.6, we show that the set $\mathcal{S}_{nt}^{\text{sing}}(\mu)$ may be dense in $\mathcal{S}_{nt}(\mu)^\circ$.

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2. Preliminaries

2.1. Integral Means and the Boundary Behavior of $\mathrm{e}^{H_{v_n}f(u_n)}$ and $P_{v_n}f(u_n)$

When studying the asymptotic behaviour of $(\chi_{\mathcal{L}}(w_n),\eta_{\mathcal{L}}(w_n))$ for non-tangential sequences $\{w_n\}_n$ such that $\lim_{n\to\infty} w_n = x$, it is natural to first try to estimate $\mathrm{e}^{H_{v_n}f(u_n)}$ and $P_{v_n}f(u_n)$ separately. We will not attempt to classify all possible situations for which a point is regular, but contend ourselves with providing sufficient conditions which cover many interesting cases. In particular, we provide sufficient conditions for $v_n\,\mathrm{e}^{\pi|H_{v_n}f(u_n)|} \to 0$ for a non-tangential sequence $u_n + \mathrm{i}\,v_n \to x \in \mathcal{S}_{nt}(\mu)$ as $n \to +\infty$. To achieve this it will be natural to consider certain means of the function f.

We recall again that the Lebesgue set $\mathscr{L}_f$ of an $L^1_{\text{loc}}(\mathbb{R})$ function f is the set of all $x \in \mathbb{R}$ such that

$$\lim_{h\to 0^+} \frac{1}{2h} \int_{x-h}^{x+h} |f(t) - f(x)| \mathrm{d}t = 0. \tag{2.1}$$

It is a well known result that the set of points which fails to be Lebesgue points has Lebesgue measure zero, see [21]. If x does not belong to the Lebesgue set of f one may try to redefine the value of $f(x)$ at x such that (2.1) holds. If this is not possible then x does not belong to the Lebesgue set of f for any $f \in [f] \in L^1(\mathbb{R})$, where $[f]$ denotes the equivalence class of f in $L^1(\mathbb{R})$. In particular we note that if $f \in \rho^\lambda_{c,1}(\mathbb{R})$ and (2.1) holds then

$$\begin{aligned}\lim_{h\to 0^+} \frac{1}{2h} \int_{x-h}^{x+h} f(t)\mathrm{d}t &= \lim_{h\to 0^+} \frac{1}{h} \int_{x}^{x+h} f(t)\mathrm{d}t \\ &= \lim_{h\to 0^+} \frac{1}{h} \int_{x-h}^{x} f(t)\mathrm{d}t = f(x).\end{aligned}$$

Of course the converse of this is not true in general. However, if $f(x) = 0$ or $f(x) = 1$ then in the first case we have

$$\lim_{h\to 0^+} \frac{1}{2h} \int_{x-h}^{x+h} |f(t) - f(x)| \mathrm{d}t = \lim_{h\to 0^+} \frac{1}{2h} \int_{x-h}^{x+h} f(t)\mathrm{d}t = 0$$

or in the second case

$$\begin{aligned}\lim_{h\to 0^+} \frac{1}{2h} \int_{x-h}^{x+h} |f(t) - f(x)| \mathrm{d}t &= \lim_{h\to 0^+} \frac{1}{2h} \int_{x-h}^{x+h} \mathrm{d}t - \frac{1}{2h} \int_{x-h}^{x+h} f(t)\mathrm{d}t \\ &= 0.\end{aligned}$$

Therefore let

$$f(x) = F'(x) := \lim_{h\to 0^+} \frac{F(x+h) - F(x-h)}{2h}, \tag{2.2}$$

where

$$F(x) = \int_{-\infty}^{x} \mathrm{d}\mu(t),$$

and note that the limit (2.2) exists for almost every x, in particular for every x in the Lebesgue set of f. Functions $f \in L^1(\mathbb{R})$ defined through (2.2) are said to be *strictly defined*, (see [19, p. 192]). We will therefore always assume that the density f in the equivalence class of densities of the measure μ is defined by (2.2), and the Lebesgue set of f will always be with respect to this density. Moreover, it will be important to study not only the properties of the density f but also of its Hilbert transform $\mathcal{H}f$, where

$$\mathcal{H}f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_{|x-t|>\varepsilon} \frac{f(t)\mathrm{d}t}{x-t},$$

and where this limit exists for almost every x. It is a well-known fact in the theory of singular integrals that the Hilbert transform is a bounded operator on $L^p(\mathbb{R})$ for every $1 < p < \infty$, see for example Theorem 4.1.7 in [9]. Since $f \in \rho^{\lambda}_{c,1}(\mathbb{R})$ it follows that $f \in L^p(\mathbb{R})$ for every $1 \leqslant p \leqslant \infty$, and hence that $\mathcal{H}f \in L^p(\mathbb{R})$ for every $1 < p < \infty$.

As was remarked before, we will be interested in considering non-tangential limits. That is, if $u : \mathbb{H} \to \mathbb{R}$, we will say that u has a non-tangential limit l at $x_0 \in \mathbb{R} = \partial\mathbb{H}$, if for each $\alpha > 0$,

$$\lim_{\substack{(x,y)\to(x_0,0)\\(x,y)\in\Gamma_\alpha(x_0)}} u(x,y) = l.$$

Similarly, we will say that a function $u : \mathbb{H} \to \mathbb{R}$ is *non-tangentially bounded* at x_0 if for every $\alpha > 0$ we have that

$$\sup_{(x,y)\in\Gamma^1_\alpha(x_0)} |u(x,y)| < \infty.$$

For many estimates it will prove useful to introduce the Hardy–Littlewood maximal function m_f, defined at $x \in \mathbb{R}$ for $f \in L^p(\mathbb{R})$ for $1 \leqslant p \leqslant \infty$ by

$$m_f(x) := \sup_{h>0} \frac{1}{2h}\int_{x-h}^{x+h} |f(t)|\mathrm{d}t.$$

Recall that, $P_v f(u)$ is the Poisson integral of the density f. However, by Lemma 1.5 in [19, Chapter VI], $H_v f(u) = P_v(\mathcal{H}f)(u)$, that is $H_v f(u)$ is the Poisson integral of the Hilbert transform of f. Now Theorem 3.16 in [19, Chapter II] implies that $\lim_{n\to\infty} P_{v_n} f(u_n) = x$ for non-tangential limits at each $x \in \mathscr{L}_f$, thus in particular almost everywhere. Similarly, $H_v f(u)$ has the non-tangential limit $\mathcal{H}f(x)$ at every $x \in \mathscr{L}_{\mathcal{H}f}$, thus in particular, almost everywhere. Finally, Theorem 1.4 in [19, Chapter VI] shows that $m_{\mathcal{H}f}$ dominates $H_v f$ in the following sense:

$$\sup_{(u,v)\in\Gamma^1_\alpha(x)} |H_v f(u)| \leqslant d_\alpha m_{\mathcal{H}f}(x), \tag{2.3}$$

where the constant d_α does not depend on x. Moreover, Lemma 1.2 in [19, Chapter VI] states that

$$\lim_{v\to 0^+}\left\{H_v f(x) - \int_{0<v\leqslant|t|} \frac{f(x-t)\mathrm{d}t}{t}\right\} = 0 \tag{2.4}$$

at each point x in $\mathscr{L}_f$.

Remark 2.1. — Note that in Lemma 1.2 in [19], $\mathscr{L}_f$ is the Lebesgue set of f and not of $\mathcal{H}f$. It should be noted that (2.4) does not apply for arbitrary non-tangential limits, as can be seen by considering the function $f(t) = (\log(|t|^{-1}))^{-1}\chi_{[-a,a]}(t)$ at 0, for some $a > 0$. However, if f satisfies the following Dini-type condition:

$$\int_{x-1}^{x+1} \frac{|f(x) - f(t)|\mathrm{d}t}{|x-t|} < +\infty, \tag{2.5}$$

then for all non-tangential limits $\{u_n + \mathrm{i}\, v_n\}_n$ that converge to x,

$$\lim_{n\to\infty} H_{v_n} f(u_n) = \mathcal{H}f(x).$$

For a proof of this fact see Proposition A.2 in the appendix.

So far we have not used the fact that $f \in L^\infty(\mathbb{R})$ and its consequences for its Hilbert transform $\mathcal{H}f$. However, the fact that $f \in L^\infty(\mathbb{R})$ implies that $\mathcal{H}f \in \mathrm{BMO}$, where BMO, denotes the class of functions of bounded mean oscillation. A function $f \in \mathrm{BMO}$ if

$$\sup_{\substack{x\in\mathbb{R}\\ h>0}} \frac{1}{2h} \int_{x-h}^{x+h} |f(t) - Mf(x,h)|\mathrm{d}t < +\infty, \tag{2.6}$$

where

$$Mf(x,h) := \frac{1}{2h} \int_{x-h}^{x+h} f(t)\mathrm{d}t.$$

The left hand side of (2.6) is the BMO norm of f and is denoted by $\|f\|_{\mathrm{BMO}}$. In particular it follows that if $f \in \mathrm{BMO}$, then $m_f \in \mathrm{BMO}$, see [2, Theorem 4.2 (b)]. Moreover, functions of bounded mean oscillation are $L^p_{\mathrm{loc}}(\mathbb{R})$ for every $0 < p < \infty$. Finally, functions $f \in \mathrm{BMO}$ satisfies the John–Nirenberg inequality:

$$\begin{aligned} |\{t \in [x-h, x+h] : |f(t) - Mf(x,h)| > \alpha\}| \\ \leqslant c_1 \exp\left(-c_2 \frac{\alpha}{\|f\|_{\mathrm{BMO}}}\right) 2h \end{aligned} \tag{2.7}$$

for some positive constants c_1, c_2 independent of x. For more details see for example [10] or [20].

If $f \in \mathsf{C}^{\lambda,\alpha}_{c,1}(\mathbb{R})$, then for every $x, y \in \overline{I}_k$, and every k, there exists a constant C, such that $|f(x) - f(y)| \leqslant C|x-y|^\alpha$. This implies that for every $x, y \in I_k$, there exists a constant c, that depends on x, such that $|\mathcal{H}f(x) - \mathcal{H}f(y)| \leqslant c|x-y|^\alpha$. Thus, in particular $\mathcal{H}f \in C(I_k)$, for every k. Note however that $\mathcal{H}f$ need not be continuous on the set $\bigcup_k \partial I_k$.

For $f \in L^1(\mathbb{R})$ and $y > 0$ let

$$\begin{aligned} M_R f(x,y) &:= \frac{1}{y}\int_x^{x+y} f(t)\mathrm{d}t \\ M_L f(x,y) &:= \frac{1}{y}\int_{x-y}^{x} f(t)\mathrm{d}t \\ \Delta M f(x,y) &:= \frac{1}{y}\int_x^{x+y} f(t)\mathrm{d}t - \frac{1}{y}\int_{x-y}^{x} f(t)\mathrm{d}t \\ \Delta m_f(x) &:= \sup_{y>0} |\Delta M f(x,y)| \\ m_f^\delta(x) &:= \sup_{0<y<\delta} |M f(x,y)| \end{aligned}$$

for $x \in \mathbb{R}$ and $y \in \mathbb{R}^+$. It follows from the fact that $0 \leqslant f(t) \leqslant 1$, that $0 \leqslant M_R f(x,y) \leqslant 1$, $0 \leqslant M_L f(x,y) \leqslant 1$ and $-1 \leqslant \Delta M f(x,y) \leqslant 1$ for all $(x,y) \in \mathbb{H}$. In particular $\Delta m_f(x)$ is a maximal function for the cancellation of the right sided and left sided means, i.e., it measure the maximal difference between the right and left sided means of f, and $m_f^\delta(x)$ is a truncated maximal function. As will be shown in Lemma 2.4, it is the size of $\Delta M f(u_n, v_n)$ that controls the growth rate of the function $\pi|H_{v_n} f(u_n)|$ for non-tangential sequences $u_n + \mathrm{i}\, v_n \in \mathbb{H}$ as $u_n + \mathrm{i}\, v_n \to x \in \mathbb{R}$ as $n \to +\infty$. In particular, we have the following important Lemma:

Lemma 2.2. — *Assume that $x \in \mathcal{S}_{nt}(\mu)^\circ$ and that $f \in \rho_{c,1}^\lambda(\mathbb{R})$. Then*

$$|\Delta M f(x,y)| < 1, \tag{2.8}$$

and

$$|\Delta M f(x,y)| \leqslant \frac{1}{y} \tag{2.9}$$

for all $y > 0$.

Proof. — Assume the contrary. Then there exists a $y^* > 0$ such that $|\Delta M f(x,y^*)| = 1$. It is clear from the definition of $M_R f$ and $M_L f$ that either $M_R f(x,y^*) = 1$ and $M_L f(x,y^*) = 0$ or that $M_L f(x,y^*) = 1$ and $M_R f(x,y^*) = 0$. In the first case this implies that $f(t) = 1$ for a.e. $t \in [x, x+y]$ and that $f(t) = 0$ for a.e. $t \in [x-y, x]$, This implies that $(x, x+y) \subset \mathbb{R} \setminus \operatorname{supp}(\lambda - \mu)$ and $(x-y, x) \subset \mathbb{R} \setminus \operatorname{supp}(\mu)$. Thus, $x \in R_2$. This however contradicts the assumption that $x \in \mathcal{S}_{nt}(\mu)^\circ$. The other case is analogous. To prove (2.9), we note that

$$|\Delta M f(x,y)| \leqslant \frac{1}{y}\int_{x-y}^{x+y} f(t)\mathrm{d}t \leqslant \frac{1}{y}\int_{\mathbb{R}} f(t)\mathrm{d}t = \frac{1}{y},$$

since $f \geqslant 0$ and $f \in \rho_{c,1}^\lambda(\mathbb{R})$. □

In what follows it will be useful to define:

Definition 2.3.

$$f_R^+(x) := \limsup_{h\to 0^+} M_R f(x,h) = \limsup_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t \tag{2.10}$$

$$f_R^-(x) := \liminf_{h\to 0^+} M_R f(x,h) = \liminf_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t \tag{2.11}$$

$$f_L^+(x) := \limsup_{h\to 0^+} M_L f(x,h) = \limsup_{h\to 0^+} \frac{1}{h}\int_{x-h}^{x} f(t)\mathrm{d}t \tag{2.12}$$

$$f_L^-(x) := \liminf_{h\to 0^+} M_L f(x,h) = \liminf_{h\to 0^+} \frac{1}{h}\int_{x-h}^{x} f(t)\mathrm{d}t. \tag{2.13}$$

Lemma 2.4. — *Fix $x \in \mathcal{S}_{nt}(\mu)$. Let*

$$c(x) := \max\{|f_R^+(x) - f_L^-(x)|, |f_L^+(x) - f_R^-(x)|\}$$

Then for every $\varepsilon > 0$ and every non-tangentially convergent sequence $\{u_n + \mathrm{i}\, v_n\}_n$ to x, such that $\{u_n + \mathrm{i}\, v_n\}_n \subset \Gamma_k(x)$, there exists an $N > 0$ and a constant $C = C(\varepsilon, x, k)$, such that

$$v_n\, \mathrm{e}^{|H_{v_n} f(u_n)|} \leqslant C v_n^{1-c(x)-\varepsilon}. \tag{2.14}$$

If in particular $x \in \mathscr{L}_f$, then

$$v_n\, \mathrm{e}^{|H_{v_n} f(u_n)|} \leqslant C v_n^{1-\varepsilon}. \tag{2.15}$$

Finally, we have the identity

$$\pi H_v f(u) = \int_0^{+\infty} t \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{t^2+v^2}\right) \Delta M f(u,t)\mathrm{d}t. \tag{2.16}$$

Proof. — Assume that $\{u_n + \mathrm{i}\, v_n\}_n$ is non-tangentially convergent to x. Then $\{u_n + \mathrm{i}\, v_n\}_n \subset \Gamma_k(x)$ for some $k > 0$. An integration by parts gives

$$\begin{aligned}\pi H_{v_n} f(u_n) &= \int_{\mathbb{R}} \frac{t f(u_n - t)\mathrm{d}t}{t^2+v_n^2} = \int_0^{+\infty} \frac{t}{t^2+v_n^2}\big[f(u_n-t) - f(u_n+t)\big]\mathrm{d}t \\ &= -\int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{t^2+v_n^2}\right)\left[\int_0^t \big[f(u_n-s) - f(u_n+s)\big]\mathrm{d}s\right]\mathrm{d}t \\ &= \int_0^{+\infty} t\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{t^2+v_n^2}\right)\Delta M f(u_n,t)\mathrm{d}t.\end{aligned}$$

Choose $d = \max\{1, k\}$. Write,

$$\pi H_{v_n} f(u_n) = \int_0^{dv_n} t \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{t^2+v_n^2}\right) \Delta M f(u_n, t)\mathrm{d}t + \int_{dv_n}^{+\infty} t \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{t^2+v_n^2}\right) \Delta M f(u_n, t)\mathrm{d}t \tag{2.17}$$

$$= I_1^{(n)} + I_2^{(n)}. \tag{2.18}$$

By Lemma 2.2

$$|I_1^{(n)}| \leqslant \int_0^{dv_n} t \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{t^2+v_n^2}\right)\right| \mathrm{d}t = \int_0^{dv_n} t \left|\frac{v_n^2 - t^2}{(t^2+v_n^2)^2}\right| \mathrm{d}t \tag{2.19}$$

$$\leqslant \int_0^{dv_n} \frac{d}{v_n} \mathrm{d}t = d^2. \tag{2.20}$$

Now consider $I_2^{(n)}$ so that $t \geqslant dv_n \geqslant kv_n > |u_n - x|$. Then,

$$\begin{aligned}
\Delta M f(u_n, t) &= \frac{1}{t}\left[\int_{u_n}^{u_n+t} f(y)\mathrm{d}y - \int_{u_n-t}^{u_n} f(y)\mathrm{d}y\right] \\
&= \frac{1}{t}\left[\int_{x+(u_n-x)}^{x+(u_n-x)+t} f(y)\mathrm{d}y - \int_{x+(u_n-x)-t}^{x+(u_n-x)} f(y)\mathrm{d}y\right] \\
&= \frac{1}{t}\left[\int_{x}^{x+(u_n-x)+t} f(y)\mathrm{d}y - \int_{x}^{x+(u_n-x)} f(y)\mathrm{d}y \right. \\
&\qquad \left. - \int_{x+(u_n-x)-t}^{x} f(y)\mathrm{d}y - \int_{x}^{x+(u_n-x)} f(y)\mathrm{d}y\right] \\
&= \frac{1}{t}\left[\frac{(u_n-x)+t}{(u_n-x)+t} \int_{x}^{x+(u_n-x)+t} f(y)\mathrm{d}y \right. \\
&\qquad - 2\frac{(u_n-x)}{(u_n-x)} \int_{x}^{x+(u_n-x)} f(y)\mathrm{d}y \\
&\qquad\qquad \left. - \frac{t-(u_n-x)}{t-(u_n-x)} \int_{x+(u_n-x)-t}^{x} f(y)\mathrm{d}y\right] \\
&= \frac{(u_n-x)+t}{t} M_R f(x, (u_n-x)+t) \\
&\qquad - 2\frac{(u_n-x)}{t} M_R f(x, (u_n-x)) \\
&\qquad\qquad - \frac{t-(u_n-x)}{t} M_L f(x, t-(u_n-x)).
\end{aligned}$$

If $u_n - x < 0$, then similarly,

$$\begin{aligned}\Delta Mf(u_n,t) = \frac{t-(x-u_n)}{t} M_R f(x, t-(x-u_n)) \\ + 2\frac{(x-u_n)}{t} M_L f(x, x-u_n) \\ - \frac{t+(x-u_n)}{t} M_L f(x, t+(x-u_n)).\end{aligned}$$

Let $0 < \varepsilon < 1$. By definition, there exists an $N = N(\varepsilon)$ and an $h = h(\varepsilon, x) < 1$, such that

$$\begin{aligned} f_R^-(x) - \varepsilon \leqslant M_R f(x,(u_n-x)+t) \leqslant f_R^+(x)+\varepsilon \\ f_R^-(x) - \varepsilon \leqslant M_R f(x,(u_n-x)) \leqslant f_R^+(x)+\varepsilon \\ f_L^-(x) - \varepsilon \leqslant M_L f(x,(u_n-x)-t) \leqslant f_L^+(x)+\varepsilon \end{aligned}$$

whenever $n > N$ and $dv_n < t < h$ and $u_n - x \geqslant 0$, and

$$\begin{aligned} f_R^-(x) - \varepsilon \leqslant M_R f(x,t-(x-u_n)) \leqslant f_R^+(x)+\varepsilon \\ f_L^-(x) - \varepsilon \leqslant M_L f(x,(x-u_n)) \leqslant f_L^+(x)+\varepsilon \\ f_L^-(x) - \varepsilon \leqslant M_L f(x,(x-u_n)+t) \leqslant f_L^+(x)+\varepsilon \end{aligned}$$

whenever $n > N$ and $dv_n < t < h$ and $u_n - x < 0$. Thus, when $u_n - x \geqslant 0$ and $n > N$ and $dv_n < t < h$,

$$\begin{aligned}\Delta Mf(u_n,t) &\leqslant \frac{(u_n-x)+t}{t}(f_R^+(x)+\varepsilon) - 2\frac{(u_n-x)}{t}(f_R^-(x)-\varepsilon) \\ &\qquad - \frac{t-(u_n-x)}{t}(f_L^-(x)-\varepsilon) \\ &\leqslant f_R^+(x) - f_L^-(x) + 2\varepsilon + \frac{(u_n-x)}{t}(f_R^+(x)+\varepsilon - 2(f_R^-(x)-\varepsilon) \\ &\qquad + (f_L^-(x)-\varepsilon)) \\ &= f_R^+(x) - f_L^-(x) + 2\varepsilon \\ &\qquad + \frac{(u_n-x)}{t}(f_R^+(x) - 2f_R^-(x) + f_L^-(x) + 2\varepsilon) \\ &\leqslant f_R^+(x) - f_L^-(x) + 2\varepsilon + \frac{6(u_n-x)}{t},\end{aligned}$$

and

$$\begin{aligned}\Delta Mf(u_n,t) \geqslant \frac{(u_n-x)+t}{t}(f_R^-(x)-\varepsilon) - 2\frac{(u_n-x)}{t}(f_R^+(x)+\varepsilon)\\ - \frac{t-(u_n-x)}{t}(f_L^+(x)+\varepsilon)\\ \geqslant f_R^-(x)-f_L^+(x)-2\varepsilon+\frac{(u_n-x)}{t}(f_R^-(x)-\varepsilon-2(f_R^+(x)+\varepsilon)\\ +(f_L^+(x)+\varepsilon))\\ = f_R^-(x)-f_L^+(x)-2\varepsilon\\ +\frac{(u_n-x)}{t}(f_R^-(x)-2f_R^+(x)+f_L^+(x)-2\varepsilon)\\ \geqslant f_R^-(x)-f_L^+(x)-2\varepsilon-\frac{6(u_n-x)}{t}.\end{aligned}$$

If instead $u_n - x < 0$, then whenever $n > N$ and $dv_n < t < h$ and ε sufficiently small, then similarly

$$\Delta Mf(u_n,t) \leqslant f_R^+(x)-f_L^-(x)+2\varepsilon+\frac{6(x-u_n)}{t}$$

and

$$\Delta Mf(u_n,t) \geqslant f_R^-(x)-f_L^+(x)-2\varepsilon-\frac{6(x-u_n)}{t}$$

Hence, changing ε to $\varepsilon/2$ in the calculation above we have for $dv_n \leqslant t < h(\varepsilon, x)$

$$|\Delta Mf(u_n,t)| \leqslant c(x)+\varepsilon+\frac{6|u_n-x|}{t}. \tag{2.21}$$

Note that $\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{t^2+v_n^2}\right)\right| = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{t^2+v_n^2}\right)$ when $t \geqslant v_n$. With $dv_n \leqslant t \leqslant h$, we can write

$$I_2^{(n)} = \left(\int_{dv_n}^{h}+\int_h^1+\int_1^{+\infty}\right)\left(-t\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{t^2+v_n^2}\right)\Delta Mf(u_n,t)\right)\mathrm{d}t.$$

Use the estimate (2.21) for $dv_n \leqslant t \leqslant h$, (2.8) for $h \leqslant t \leqslant 1$ and (2.9) for $t > 1$. This gives

$$\begin{aligned}|I_2^{(n)}| = \int_{dv_n}^{h}\left(-t\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{t^2+v_n^2}\right)\right)\left(c(x)+\varepsilon+\frac{6|u_n-x|}{t}\right)\\ +\int_h^1\left(-t\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{t^2+v_n^2}\right)\right)\mathrm{d}t+\int_1^{+\infty}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{t^2+v_n^2}\right)\right).\end{aligned}$$

We can now evaluate the integrals and use $|u_n - x| \leqslant kv_n$. Some straightforward estimates give

$$|I_2^{(n)}| \leqslant (c(x)+\varepsilon)\log\sqrt{\frac{h^2+v_n^2}{(d^2+1)v_n^2}} + 2 + \frac{6k}{d} + \log\sqrt{\frac{R^2+v_n^2}{h^2+v_n^2}}.$$

Together with (2.17) and (2.19) this proves (2.14) together with an appropriate constant $C = C(\varepsilon, x, h)$. Finally, inequality (2.15) follows from the fact that $c(x) = 0$ whenever $x \in \mathscr{L}_f$. □

Remark 2.5. — We note that inequality (2.14) is trivial whenever $c(x) = 1$ since $v\,\mathrm{e}^{\pi|H_v f(u)|} \leqslant \tilde{c}$, for some positive constant $\tilde{c}$, whenever $f \in \rho^{\lambda}_{c,1}(\mathbb{R})$.

The following Lemma is a similar to Lemma 2.4, but we only consider orthogonal limits. The estimate we derive will not depend on some $\varepsilon > 0$. This will be needed in the proof of Theorem 3.6.

Lemma 2.6. — *We have the estimate*

$$v\,\mathrm{e}^{\pi|H_v f(x)|} \leqslant c v^{1-\Delta m_f(x)}, \tag{2.22}$$

for $v > 0$, where c is a positive constant that does not depend on x.

Proof. — Using (2.16), we get the estimate

$$\begin{aligned}\pi|H_v f(x)| \leqslant \sup_{0\leqslant t\leqslant v} |\Delta M f(x,t)| \int_0^v t\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{t^2+v^2}\right)\mathrm{d}t \\ + \sup_{v\leqslant t\leqslant 1} |\Delta M f(x,t)| \int_v^1 -t\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{t^2+v^2}\right)\mathrm{d}t \\ + \int_1^\infty -t\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{t^2+v^2}\right)|\Delta M f(x,t)|\mathrm{d}t.\end{aligned}$$

Now using (2.8) in the first term, the definition of $\Delta m_f(x)$ in the second expression and (2.9) in the last expression to see that

$$\pi|H_v f(x)| \leqslant C + \Delta m_f(x)\log(v^{-1}),$$

where C is a numerical constant. □

We will now consider the denominator $\sin(\pi P_v f(u))$ in (1.35) and (1.36). As will be shown in Lemma 2.8, the size of $\sin(\pi P_v f(u))$ can be estimated from below by $\frac{2d}{1+d^2} M f(u, dv)$ for some arbitrary $d > 0$, rather than the quantity $\Delta M f(u,v)$ as in Lemma 2.4.

Lemma 2.7. — *For any $u \in \mathbb{R}$ and $v > 0$,*

$$0 < P_v f(u) < 1.$$

Proof. — Clearly, $P_v f(u) > 0$ since $\|f\|_1 = 1$ and $f(t) \geqslant 0$. Similarly, using that f has compact support so that $\operatorname{supp}(f) \subset [-R, R]$ for $R > 0$ sufficiently large, we get

$$P_v f(u) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{v f(t) \mathrm{d}t}{(u-t)^2 + v^2} \leqslant \frac{1}{\pi} \int_{-R}^{R} \frac{v \mathrm{d}t}{(u-t)^2 + v^2} < 1$$

since $f(t) \leqslant 1$. □

Lemma 2.8. — *For any fixed $d > 0$,*

$$\sin(\pi P_v f(u)) \geqslant \frac{d}{1+d^2} \min\{Mf(u, dv), M(1-f)(u, dv)\}.$$

Proof. — Using the inequality

$$\sin t \geqslant \frac{\pi}{4} - \frac{1}{4}|2t - \pi|$$

valid for $t \in [0, \pi]$, we get using Lemma 2.7,

$$\sin(\pi P_v f(u)) \geqslant \frac{\pi}{4} - \frac{1}{4}|2\pi P_v f(u) - \pi|.$$

We now use the inequality

$$\frac{v}{(u-t)^2 + v^2} \geqslant \frac{1}{1+d^2} \frac{1}{v}$$

valid for $t \in [u - dv, u + dv]$ and any fixed $d > 0$, to get

$$\begin{aligned} \pi P_v f(u) &\geqslant \int_{u-dv}^{u+dv} \frac{v f(t) \mathrm{d}t}{(u-t)^2 + v^2} \\ &\geqslant \frac{1}{1+d^2} \frac{1}{v} \int_{u-dv}^{u+dv} f(t) \mathrm{d}t = \frac{2d}{1+d^2} Mf(u, dv), \end{aligned}$$

and similarly,

$$\pi P_v f(u) = \pi - \pi P_v (1-f)(u) \leqslant \pi - \frac{2d}{1+d^2} M(1-f)(u, dv).$$

Since

$$|2\pi P_v f(u)) - \pi| = \begin{cases} 2\pi P_v f(u) - \pi & \text{if } P_v f(u) \geqslant \frac{1}{2} \\ \pi - 2\pi P_v f(u) & \text{if } P_v f(u) < \frac{1}{2}, \end{cases}$$

$$\begin{aligned} \frac{\pi}{4} - \frac{1}{4}|2\pi P_v f(u) - \pi| &= \begin{cases} \frac{\pi}{2} - \frac{\pi}{2} P_v f(u) & \text{if } P_v f(u) \geqslant \frac{1}{2} \\ \frac{\pi}{2} P_v f(u)) & \text{if } P_v f(u) < \frac{1}{2} \end{cases} \\ &\geqslant \begin{cases} \frac{d}{1+d^2} M(1-f)(u, dv) & \text{if } P_v f(u) \geqslant \frac{1}{2} \\ \frac{d}{1+d^2} Mf(u, dv) & \text{if } P_v f(u) < \frac{1}{2}. \end{cases} \\ &\geqslant \frac{d}{1+d^2} \min\{Mf(u, dv), M(1-f)(u, dv)\} \end{aligned}$$

□

LEMMA 2.9. — *Fix $x \in \mathcal{S}_{nt}(\mu)$. Let*

$$b(x) := \frac{1}{4}\min\{2 - f_R^+(x) - f_L^+(x), f_R^-(x) + f_L^-(x)\} \tag{2.23}$$

Consider a non-tangentially convergent sequence such that $\{u_n + \mathrm{i}\, v_n\}_n \subset \Gamma_k(x)$ and fix $\varepsilon > 0$. Then,

$$\sin(\pi P_{v_n} f(u_n)) \geqslant \frac{2k}{1+4k^2}(b(x) - \varepsilon)$$

for n sufficiently large.

Proof. — By definition $|u_n - x| \leqslant kv_n$ for all n. Choose $d = 2k$ in Lemma 2.8. Assume $u_n - x > 0$. Then,

$$\begin{aligned} Mf(u_n, 2kv_n) &= \frac{1}{4kv_n}\int_{u_n-2kv_n}^{u_n+2kv_n} f(t)\mathrm{d}t = \frac{1}{4kv_n}\int_{x+(u_n-x)-2kv_n}^{x+(u_n-x)+2kv_n} f(t)\mathrm{d}t \\ &\geqslant \frac{1}{4kv_n}\int_{x}^{x+kv_n} f(t)\mathrm{d}t + \frac{1}{4kv_n}\int_{x-kv_n}^{x} f(t)\mathrm{d}t \\ &\geqslant \frac{1}{4}(M_R f(x, kv_n) + M_L f(x, kv_n)) \end{aligned}$$

and the same estimate holds if $u_n - x < 0$. Thus,

$$\begin{aligned} &\min\{Mf(u_n, kv_n), M(1-f)(u_n, kv_n)\} \\ &\geqslant \frac{1}{4}\min\{M_R f(x, kv_n) + M_L f(x, kv_n), M_R(1-f)(x, kv_n) + M_L(1-f)(x, kv_n)\} \\ &\geqslant b(x) - \varepsilon \end{aligned}$$

whenever $n > N = N(\varepsilon)$ say. Then Lemma 2.8 implies that

$$\sin(\pi P_{v_n} f(u_n)) \geqslant \frac{2k}{1+4k^2}(b(x) - \varepsilon)$$

whenever $n > N$. □

We now give a version of Lemma 2.9 for orthogonal limits that will be need in Theorem 3.6.

LEMMA 2.10. — *Fix $x \in \mathcal{S}_{nt}(\mu)$. Then, for any fixed $\delta > 0$*

$$\sin(\pi P_v f(x)) \geqslant \frac{1}{2}\min\{1 - m_f^\delta(x), 1 - m_{1-f}^\delta(x)\}$$

for all $0 < v < \delta$.

Proof. — Since

$$\inf_{0<v<\delta} Mf(x,v) = 1 - \sup_{0<v<\delta} M(1-f)(x,v) \geqslant 1 - m_{1-f}^\delta(x),$$

and

$$\inf_{0<v<\delta} 1 - Mf(x,v) = 1 - \sup_{0<v<\delta} Mf(x,v) \geqslant 1 - m_f^\delta(x),$$

the result follows immediately from Lemma 2.8 with $d = 1$. □

Lemma 2.11. — *Fix $x \in \mathcal{S}_{nt}(\mu)$. Then for every sequence $\{u_n + \mathrm{i}\, v_n\}_n \subset \Gamma_k(x)$ which converges non-tangentially to x, we have for every $\varepsilon > 0$ sufficiently small*

$$|(\chi_{\mathcal{L}}(u_n, v_n) - u_n, \eta_{\mathcal{L}}(u_n, v_n) - 1)| \leqslant \frac{1 + 4k^2}{2k} \frac{\sqrt{20} C v_n^{1-c(x)-\varepsilon}}{b(x) - \varepsilon}, \tag{2.24}$$

where C is the same constant as in Lemma 2.4.

Proof. — From (1.35) and (1.36) we see that

$$\begin{aligned} |\sin[\pi P_{v_n} f(u_n)]|^2 &|(\chi_{\mathcal{L}}(u_n, v_n) - u_n), \eta(u_n, v_n)_{\mathcal{L}} - 1)|^2 \\ &\leqslant (\mathrm{e}^{-\pi H_{v_n} f(u_n)} - \cos(\pi P_{v_n} f(u))^2 \\ &\qquad + (\mathrm{e}^{\pi H_{v_n} f(u_n)} + \mathrm{e}^{-\pi H_{v_n} f(u_n)} - 2\cos(\pi P_{v_n} f(u_n))^2 v_n^2 \\ &\leqslant (\mathrm{e}^{\pi |H_{v_n} f(u_n)|} + 1)^2 + (2\, \mathrm{e}^{\pi |H_{v_n} f(u_n)|} + 2)^2 v_n^2 \\ &\leqslant 5(1 + 3\, \mathrm{e}^{2\pi |H_{v_n} f(u_n)|}) v_n^2 \\ &\leqslant 20 C^2 v_n^{2(1-c(x)-\varepsilon)} \end{aligned}$$

by Lemma 2.4, whenever n is sufficiently large. Hence, by Lemma 2.9,

$$|(\chi_{\mathcal{L}}(u_n, v_n) - u_n, \eta_{\mathcal{L}}(u_n, v_n) - 1)| \leqslant \frac{1 + 4k^2}{2k} \frac{\sqrt{20} C v_n^{1-c(x)-\varepsilon}}{b(x) - \varepsilon},$$

if $\varepsilon < b(x)$. □

We conclude this section with a similar estimate as in Lemma 2.11, for orthogonal sequences, but where the constant is independent of x.

Lemma 2.12. — *For every $x \in \mathcal{S}_{nt}(\mu)$ and $\delta > 0$, there exists a constant $c > 0$ independent of x and δ, such that*

$$|(\chi_{\mathcal{L}}(x, v) - x, \eta_{\mathcal{L}}(x, v) - 1)| \leqslant \frac{2\sqrt{20} c v^{1-\Delta m_f(x)}}{\min\{1 - m_f^\delta(x), 1 - m_{1-f}^\delta(x)\}}, \tag{2.25}$$

whenever $v < \delta$.

Proof. — Combining Lemma 2.6 and Lemma 2.9, a similar computation as in the proof of Lemma 2.11 gives (2.25). □

3. Generic Points

3.1. Generic Points Are Dense

Proposition 3.1. — *Define the set* $G \subset \mathcal{S}_{nt}(\mu)^\circ = (\operatorname{supp}(\mu) \cap \operatorname{supp}(\lambda - \mu))^\circ$ *according to*

$$G := \mathcal{S}_{nt}(\mu)^\circ \cap \mathscr{L}_f \cap \mathscr{L}_{m_{\mathcal{H}f}} \cap \{t \in \mathbb{R} : 0 < f(t) < 1\} \tag{3.1}$$

Then every $x \in G$ *is regular, and* G *is dense in* $\mathcal{S}_{nt}(\mu)^\circ$*. Furthermore, for every interval* $I \subset \mathcal{S}_{nt}(\mu)^\circ$, $|G \cap I| = \lambda(G \cap I) > 0$*. If in addition the inequality* $0 < f(t) < 1$ *holds almost everywhere in* $\mathcal{S}_{nt}(\mu)^\circ$*, then almost every* $x \in \mathcal{S}_{nt}(\mu)^\circ$ *belongs to* G*.*

Proof. — Since x belongs to the Lebesgue set of f and $1 < f(x) < 1$ by assumption, we have by Lemma 2.4, that $v_n \,\mathrm{e}^{\pi |H_{v_n} f(u_n)|} \to 0$ for a non-tangential sequence $u_n + \mathrm{i}\, v_n \in \mathbb{H}$ such that $\lim_{n\to+\infty} u_n + \mathrm{i}\, v_n = x$. Moreover, (see page 11)

$$\lim_{n\to+\infty} P_{v_n} f(u_n) = f(x)$$

also holds for every such sequence. Hence

$$\lim_{n\to\infty} \frac{v_n \,\mathrm{e}^{-\pi H_{v_n} f(u_n)} - v_n \cos(\pi P_{v_n} f(u_n))}{\sin(\pi P_{v_n} f(u_n))} = 0$$

and

$$\lim_{n\to\infty} \frac{v_n \,\mathrm{e}^{\pi H_{v_n} f(u_n)} + v_n \,\mathrm{e}^{-\pi H_{v_n} f(u_n)} - 2 v_n \cos(\pi P_{v_n} f(u_n))}{\sin(\pi P_{v_n} f(u_n))} = 0$$

hold, which implies the claim by (1.35) and (1.36). Hence, every $x \in G$ is regular. Since G is the finite intersection of measurable sets, G is measurable. Since $f, m_{\mathcal{H}f} \in L^1_{\mathrm{loc}}(\mathbb{R})$ it follows that almost every $x \in \mathcal{S}_{nt}(\mu)^\circ$ belongs to $\mathscr{L}_f \cap \mathscr{L}_{m_{\mathcal{H}f}}$. Let $X = \mathcal{S}_{nt}(\mu)^\circ \cap \{t \in \mathbb{R} : 0 < f(t) < 1\}$. By Hypothesis 1.26, $\lambda(G \cap I) > 0$ for every interval $I \subset \mathcal{S}_{nt}(\mu)^\circ$, thus in particular $I \cap G \neq \varnothing$. This proves that G is dense in $\mathcal{S}_{nt}(\mu)^\circ$. Finally, if the inequality $0 < f(t) < 1$ holds almost everywhere in $\mathcal{S}_{nt}(\mu)^\circ$, then $\lambda(X \cap \mathscr{L}_f \cap \mathscr{L}_{m_{\mathcal{H}f}} \cap \mathcal{S}_{nt}(\mu)^\circ) = \lambda(\mathcal{S}_{nt}(\mu)^\circ)$. □

Remark 3.2. — We believe that Hypothesis 1.26 is not necessary. That is, we believe that if Hypothesis 1.26 is not true, then Proposition 3.1 remains true if the set G is changed to

$$\begin{aligned} G = &\mathcal{S}_{nt}(\mu)^\circ \cap \mathscr{L}_f \cap \mathscr{L}_{m_{\mathcal{H}f}} \\ &\cap \left(\mathcal{S}_{nt}(\mu)^\circ \setminus \left(\left\{x : \int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x-t)^2} < +\infty\right\} \cup \left\{x : \int_{\mathbb{R}} \frac{1-f(t)\mathrm{d}t}{(x-t)^2} < +\infty\right\}\right)\right). \end{aligned}$$

This change of typical set however, would require a substantial change of Lemma 3.3.

We now give a lemma which will be useful for proving that $\partial\mathcal{L}(x) = \{(x,1)\}$ for a typical point x in $\mathcal{S}_{nt}(\mu)^{\circ}$.

Lemma 3.3. — *Assume that $x \in G$. Then there exists sequences $\{r_n\}_n \subset G$ and $\{l_n\}_n \subset G$, such that $x < r_{n+1} < r_n$ and $l_{n+1} > l_n > x$ for all n and $\lim_{n\to\infty} r_n = \lim_{n\to\infty} l_n = x$. Moreover*

$$\max\{\sup_n m_{\mathcal{H}f}(r_n), \sup_n m_{\mathcal{H}f}(l_n)\} < +\infty \tag{3.2}$$

and

$$\min\{\inf_n f(r_n), \inf_n(1-f(r_n)), \inf_n f(l_n), \inf_n(1-f(l_n))\} > 0. \tag{3.3}$$

Proof. — Since $0 < f(x) < 1$ we can take $\varepsilon > 0$ so that $0 < f(x) - \varepsilon < f(x) + \varepsilon < 1$, and since $x \in \mathscr{L}_f \cap \mathscr{L}_{m_{\mathcal{H}f}}$, there exists an $\delta = \delta(\varepsilon, x)$ such that

$$\frac{1}{h}\int_x^{x+h} |f(t) - f(x)|\mathrm{d}t < \frac{\varepsilon}{2} \tag{3.4}$$

$$\frac{1}{h}\int_x^{x+h} |m_{\mathcal{H}f}(t) - m_{\mathcal{H}f}(x)|\mathrm{d}t < \frac{\varepsilon}{2} \tag{3.5}$$

whenever $h < \delta$. Now, assume that

$$\liminf_{h\to 0^+} \frac{|\{t \in [x, x+h] : |f(t) - f(x)| < \varepsilon\}|}{h} = 0.$$

Then there exists a sequence $\{h_k\}_k$ such that $\lim_{k\to\infty} h_k = 0$ and

$$\frac{|\{t \in [x, x+h_k] : |f(t) - f(x)| < \varepsilon\}|}{h_k} < \varepsilon/2.$$

Consequently,

$$|\{t \in [x, x+h_k] : |f(t) - f(x)| \geqslant \varepsilon\}| \geqslant h_k(1 - \varepsilon/2),$$

and so

$$\frac{1}{h_k}\int_x^{x+h_k} |f(t) - f(x)|\mathrm{d}t \geqslant \varepsilon(1-\varepsilon/2) > \frac{\varepsilon}{2}$$

for all k since $\varepsilon < 1/2$. However, this contradicts (3.4). Therefore, let

$$\inf_{0<h<\delta} h^{-1}|\{t \in [x, x+h] : |f(t) - f(x)| < \varepsilon\}| = d > 0.$$

Recall the John–Nirenberg inequality (2.7) and choose an N so large that

$$c_1 \exp\left(-c_2 \frac{N}{\|m_{\mathcal{H}f}\|_{\mathrm{BMO}}}\right) < \frac{d}{4}.$$

Then

$$\frac{|\{t\in[x-h,x+h]:|m_{\mathcal{H}f}(t)-m_{\mathcal{H}f}(x)|>N+\varepsilon\}|}{2h}\leqslant\frac{d}{4}$$

so that

$$|\{t\in[x,x+h]:|m_{\mathcal{H}f}(t)-m_{\mathcal{H}f}(x)|\leqslant N-\varepsilon\}|\geqslant h-\frac{dh}{2},$$

where we have used that if $|m_{\mathcal{H}f}(t)-Mm_{\mathcal{H}f}(x,h)|>N$ then $|m_{\mathcal{H}f}(t)-m_{\mathcal{H}f}(x)|>N+|m_{\mathcal{H}f}(x)-Mm_{\mathcal{H}f}(x,h)|>N+\varepsilon$ by (3.5). Therefore, by the inclusion-exclusion principle

$$|\{t\in[x,x+h]:|m_{\mathcal{H}f}(t)-m_{\mathcal{H}f}(x)|\leqslant N-\varepsilon\}\\ \cap\{t\in[x,x+h]:|f(t)-f(x)|<\varepsilon\}|\geqslant\frac{dh}{2}$$

for every $0<h<\delta$. Fix an $0<h_0<\delta$ and choose an r_0 in $\{t\in(x,x+h_0):|m_{\mathcal{H}f}(t)-m_{\mathcal{H}f}(x)|\leqslant N\}\cap|\{t\in(x,x+h_0):|f(t)-f(x)|<\varepsilon\}|$. Now take $h_1<\min\{r_0,h_0/2\}$, and choose r_1 in $\{t\in(x,x+h_1):|m_{\mathcal{H}f}(t)-m_{\mathcal{H}f}(x)|\leqslant N\}\cap\{t\in(x,x+h_1):|f(t)-f(x)|<\varepsilon\}$. Iteration of this process gives a sequence $\{r_n\}_n$ with the desired properties since N is fixed, $f(r_n)>f(x)-\varepsilon$ and $1-f(r_n)>1-(f(x)+\varepsilon)>0$. A similar argument as above also yields the sequence $\{l_n\}_n$. □

We now want to consider the question of whether we can determine $\partial\mathcal{L}(x)$ whenever $x\in\mathcal{S}_{nt}^{\mathrm{reg}}(\mu)$. Recall Lemma 2.5 in [7], where we showed that if $x\in\mathcal{S}_{nt}(\mu)^\circ$ and there exists a neighborhood N_x of x such that

$$\sup_{t\in N_x}\{f(t),1-f(t)\}<1, \tag{3.6}$$

then $\partial\mathcal{L}(x)=\{(x,1)\}$. If condition (3.6) is not satisfied for $x\in\mathcal{S}_{nt}^{\mathrm{reg}}(\mu)^\circ$, then it is considerably harder to prove that $\partial\mathcal{L}(x)=\{(x,1)\}$. The reason is the following: Even though for some point x, one knows that for every point x' in a neighborhood N_x of x one has that $\lim_{n\to+\infty}(\chi_{\mathcal{L}}(w_n),\eta_{\mathcal{L}}(w_n))=(x',1)$ whenever $\{w_n\}_n$ is a non-tangential sequence such that $\lim_{n\to+\infty}w_n=x'$, this does not necessarily imply that $\partial\mathcal{L}(x)=\{(x,1)\}$. The difficulty comes from the fact that tangential limits also have to be considered. The next example illustrates the difficulty.

Example 3.4. — Let

$$\varphi(x,y)=\mathrm{e}^{16/x^8}\exp\left\{-\frac{1}{(y-x^2)(y-2x^2)}\right\}\chi_{x^2<y<2x^2}(x,y).$$

Then $\varphi\in C^\infty(\mathbb{H})$, and $\mathrm{supp}(\varphi)=\{(x,y)\in\overline{\mathbb{H}}:x^2\leqslant y\leqslant 2x^2\}$. Moreover, for every non-tangential limit $\{w_n\}_n\in\mathbb{H}$, such that $\lim_{n\to\infty}w_n=x$, we have $\lim_{n\to\infty}\varphi(w_n) = 0$. However, $\lim_{x\to 0^+}\varphi(x, 3x^2/2) = 1$. Hence, $\varphi \notin C(\overline{\mathbb{H}})$.

The argument that is missing in order to conclude that for a regular point $x \in \mathcal{S}_{nt}^{\mathrm{reg}}(\mu)$, one has $\partial\mathcal{L}(x) = \{(x,1)\}$, is that if one for example specialize to orthogonal limits, then one need that $\lim_{v\to 0^+}(\chi_{\mathcal{L}}(y,v), \eta_{\mathcal{L}}(y,v)) = (y,1)$ uniformly, for every y in a compact neighborhood of x. We now prove that this is sufficient.

Lemma 3.5. — *Assume that $x \in \mathcal{S}_{nt}(\mu)^\circ = (\mathrm{supp}(\mu) \cap \mathrm{supp}(\lambda - \mu))^\circ$ is regular. Furthermore, assume that there exists sequences of regular points $\{r_n\}_n$ and $\{l_m\}_m$ such that $r_n > x$ and $r_m < x$ for all n, m, and such that $\lim_{n\to\infty} r_n = \lim_{m\to\infty} l_m = x$, and such that $\lim_{v\to 0^+}(\chi(r_n,v), \eta(r_n,v)) = (r_n,1)$ and $\lim_{v\to 0^+}(\chi_{\mathcal{L}}(l_m,v), \eta_{\mathcal{L}}(l_m,v)) = (l_m,1)$ uniformly for all n,m. Then x is generic. More exactly, for any sequence $u_n + \mathrm{i}\, v_n \in \mathbb{H}$ such that $\lim_{n\to+\infty} u_n + \mathrm{i}\, v_n = x$,*

$$\lim_{n\to+\infty}(\chi_{\mathcal{L}}(u_n,v_n), \eta_{\mathcal{L}}(u_n,v_n)) = (x,1).$$

Proof. — By possibly passing to a subsequence, we may assume that the sequence $\{w_l\}_l$ is tangential and that $u_l > x$ for all l. According to the assumptions, there exists a sequence of regular points $\{r_n\}$ such that $r_n > x$ and $\lim_{n\to\infty} r_n = x$, and $\lim_{v\to 0^+}(\chi_{\mathcal{L}}(r_n,v), \eta_{\mathcal{L}}(r_n,v)) = (r_n,1)$ uniformly for all n. In particular, for every $\varepsilon > 0$ sufficiently small, there exists a $\delta = \delta(\varepsilon)$, such that

$$|(\chi_{\mathcal{L}}(r_n,v) - r_n, \eta_{\mathcal{L}}(r_n,v) - 1)| < \varepsilon$$

whenever $v < \delta$ for all n. Choose $k < \varepsilon/(r_1 - x)$ and consider the non-tangential line $\{t + ikt : t \in (0,+\infty]\}$. Since x is regular, it follows that $\lim_{t\to 0^+}(\chi_{\mathcal{L}}(x+t,kt), \eta_{\mathcal{L}}(x+t,kt)) = (x,1)$. Consider the sequence of open sets $X_n^{(k)}$ defined according to

$$X_n^{(k)} := \{(u,v) \in \mathbb{H} : x < u < r_n, 0 < v < k(u-x)\}.$$

Then $w_l \in X_n^{(k)}$ whenever $l > L$, for some $L = L(n)$. Moreover,

$$(\chi_{\mathcal{L}}(w_l), \eta_{\mathcal{L}}(w_l)) \in \overline{W_{\mathcal{L}}^{-1}(X_n^{(k)})}$$

since the map $W_{\mathcal{L}}$ is a homeomorphism. Then,

$$|(\chi_{\mathcal{L}}(w_l) - x, \eta_{\mathcal{L}}(w_l) - 1)| \leqslant d((x,1), \overline{W_{\mathcal{L}}^{-1}(X_n^{(k)})}) = d((x,1), \partial W_{\mathcal{L}}^{-1}(X_n^{(k)}))$$

whenever $l > L(n)$. Note that d denotes the Hausdorff distance between sets, that is, if $X, Y \subset \mathbb{R}^2$, then

$$d(X,Y) = \max\{\sup_{x\in X}\inf_{y\in Y}|x-y|, \sup_{y\in Y}\inf_{x\in X}|x-y|\}.$$

Let T_n be the closed region, whose boundary ∂T_n, can be decomposed into three components according to

$$\begin{aligned}\partial T_n^1 &= \{(t,1) : x \leqslant t \leqslant r_n\}\\ \partial T_n^2 &= \{(\chi_{\mathcal{L}}(x+t,kt),\eta_{\mathcal{L}}(x+t,kt)) : t \in (0,r_n-x)\}\\ \partial T_n^3 &= \{(\chi_{\mathcal{L}}(r_n,t),\eta_{\mathcal{L}}(r_n,t)) : t \in (0,k(r_n-x)]\}\end{aligned}$$

See Figure 3.1 and 3.2. Since all $\{r_n\}_n$ and x are regular points and $W_{\mathcal{L}}^{-1}$ is a homeomorphism it follows that $\partial T_n^2 \cup \partial T_n^3 \subset \partial W_{\mathcal{L}}^{-1}$. Again, since $W_{\mathcal{L}}^{-1}$ is a homeomorphism and T_n is a closed set it follows that $\overline{W_{\mathcal{L}}^{-1}(X_n^{(k)})} \subset T_n$. First note that by assumption on the sequence $\{r_n\}_n$, there exists an $N_1 = N_1(\varepsilon)$ such that $r_n - x < \varepsilon$, whenever $n > N_1$. Hence,

$$d((x,1),\partial T_n^1) < \varepsilon$$

whenever $n > N_1$. By the assumption that x was regular, there exists an $N_2 = N_2(\varepsilon)$, such that

$$d((x,1),\partial T_n^2) < \varepsilon$$

whenever $n > N_2$. Finally, as discussed above by the assumption of uniform convergence,

$$d((x,1),\partial T_n^3) < \varepsilon$$

for all n. Take $N = \max\{N_1,N_2\}$, and take $n > N$ and $l > L(N)$, then

$$|(\chi_{\mathcal{L}}(w_l)-x,\eta_{\mathcal{L}}(w_l)-1)| \leqslant d((x,1),\partial T_n) < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, this implies that $\lim_{l\to\infty}|(\chi_{\mathcal{L}}(w_l)-x,$ $\eta_{\mathcal{L}}(w_l)-1)| = 0$, and the proof is complete. □

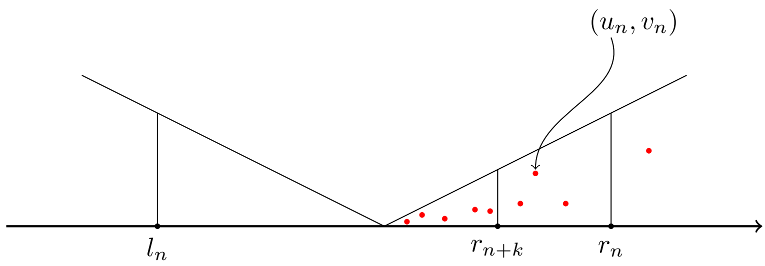

Figure 3.1. The region $X_n^{(k)}$ is depicted above. The red dots represents the positions of the sequence $\{u_n + \mathrm{i}\, v_n\}_{n=1}^{+\infty}$

We now show that all points in the set G defined in Proposition 3.1 are generic.

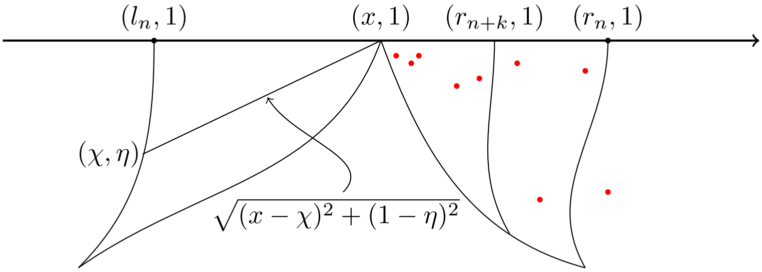

Figure 3.2. Depiction of the set T_n. The dots represent the images of the tangential sequence under the homeomorphism $W_{\mathcal{L}}^{-1}$.

Theorem 3.6. — *Assume that $x \in G$. Then $\partial\mathcal{L}(x) = \{(x, 1)\}$.*

Proof. — Let $x \in G$. Take sequences $\{r_n\}_n \subset G$ and $\{l_n\}_n \subset G$ as in Lemma 3.3 and $N, \varepsilon > 0$ such that

$$\sup_{n,k}\{\sup_{v>0} |H_v f(r_n)|, \sup_{v>0} |H_v f(l_k)|\} < N \tag{3.7}$$

and

$$\min\{\inf_n \min\{f(r_n), 1 - f(r_n)\}, \inf_k \min\{f(l_k), 1 - f(l_k)\}\} > \varepsilon. \tag{3.8}$$

Assume that $u \in X := \bigcup_n \{r_n\} \cup \{x\} \bigcup_n \{l_n\}$. Then an integration by parts gives

$$\begin{aligned} v^{-1}\pi P_v f(u) &= 2\int_{-\infty}^{+\infty} \frac{(t-u)}{((t-u)^2+v^2))^2} \int_u^t f(t')\mathrm{d}t'\mathrm{d}t \\ &\geqslant \int_u^{+\infty} \frac{(t-u)^2}{((t-u)^2+v^2))^2}(M_R f(u, t-u)\chi_{t>u})\mathrm{d}t. \end{aligned}$$

Since $u \in X \subset G$, and thus in particular $x \in \mathscr{L}_f \cap \{t \in \mathcal{S}_{nt}(\mu) : 0 < f(t) < 1\}$, there exists a $\delta = \delta(u)$ such that $\min\{M_R f(u, t-u), M_R(1-f)(u, t-u)\} \geqslant \varepsilon/2$ whenever $|t-u| < \delta$. This implies that

$$\min\{v^{-1}P_v f(u), v^{-1}P_v(1-f)(u)\} \geqslant \frac{\varepsilon}{2\pi} \int_u^\delta \frac{(t-u)^2}{((t-u)^2+v^2))^2}\mathrm{d}t \to +\infty \tag{3.9}$$

as $v \to 0^+$. We get from (1.35) and (1.36)

$$\begin{aligned} &|\sin[\pi P_v f(u)])|^2 |(\chi_{\mathcal{L}}(u, v) - u, \eta_{\mathcal{L}}(u, v) - 1)|^2 \\ &= v^2 \left(\mathrm{e}^{-H_v f(u)} - \cos(\pi P_v f(u)\right)^2 + v^2 \left(\mathrm{e}^{H_v f(u)} + \mathrm{e}^{-H_v f(u)} - 2\cos(\pi P_v f(u)\right)^2 \\ &\leqslant v^2(1 + \mathrm{e}^{|H_v f(u)|})^2 + v^2(2\,\mathrm{e}^{|H_v f(u)|} + 2)^2 \leqslant 20 v^2\,\mathrm{e}^{2N} v^2, \end{aligned}$$

by (3.7). Since $\sin(\pi P_v f(u)) \geqslant \frac{\pi}{2}\min\{P_v f(u), P_v(1-f)(u)\}$,

$$\begin{aligned}(3.10)\quad |(\chi_{\mathcal{L}}(u,v)-u, \eta_{\mathcal{L}}(u,v)-1)| \\ &\leqslant \frac{\sqrt{20}v\,\mathrm{e}^N}{v\min\{\pi P_v f(u), \pi P_v(1-f)(u)\}} \\ &= \frac{\sqrt{20}\,\mathrm{e}^N}{\min\{v^{-1}\pi P_v f(u), v^{-1}\pi P_v(1-f)(u)\}} := g_v(u).\end{aligned}$$

We now show that $g_v(u)$ is an increasing function in v for each u. We must show that

$$\min\{v'^{-1}\pi P_{v'} f(u), v'^{-1}\pi P_{v'}(1-f)(u)\} < \min\{v^{-1}\pi P_v f(u), v^{-1}\pi P_v(1-f)(u)\}$$

for $v' < v$. Since both $v^{-1}\pi P_v f(u)$ and $v^{-1}\pi P_v(1-f)(u)$ are decreasing functions in v it follows that $g_v(u)$ is an increasing function in v for all $x \in X$.

Note that by (3.10), $g_v(u) \to 0$ as $v \to 0^+$ for all $u \in X$. We now show that $g_v(u)$ is continuous function for all fixed v. It is sufficient to show that for some sequence $\{r_{m_n}\}_n$ such that $\lim_{n\to\infty} r_{m_n} = x$ we have $\lim_{n\to\infty} g_v(r_{m_n}) = g_v(x)$. However, this follows immediately from the fact that $P_v(u)$ is a continuous function on $\mathbb{H}$. Since X is compact in the subspace topology from $\mathbb{R}$ and 0 is a continuous function on X, it follows by Dini's theorem that $g_v(u) \to 0$ as $v \to 0^+$ uniformly on X. The estimate (3.10) then shows that $(\chi_{\mathcal{L}}(u,v), \eta_{\mathcal{L}}(u,v)) \to (u,1)$ uniformly on X. Hence, by Lemma 3.5, it follows that $\partial\mathcal{L}(x) = \{(x,1)\}$. □

Remark 3.7. — Recall that the typical set G satisfies $\lambda(I \cap G) > 0$ for every interval I. Note however that this does not imply that G is a comeagre set. To construct a counter-example one may consider a nowhere dense set $\mathfrak{S}$, such that $\lambda(\mathfrak{S}) > 0$ and consider a countable union of rational translations of $\mathfrak{S}$. Also note that by Theorem 1.6 in [13], $\mathcal{S}_{nt}(\mu)^\circ$ can be written as a disjoint union of a meagre set and a null set. Since we conjecture that $\mathcal{S}_{nt}^{\mathrm{sing}}(\mu)$ is a meagre set, we could in principle have that $\lambda(\mathcal{S}_{nt}^{\mathrm{ren}}(\mu)) = \lambda(\mathcal{S}_{nt}(\mu)\backslash\mathcal{S}_{nt}^{\mathrm{sing}}(\mu)) = 0$, in which case one could question the definition of $\mathcal{S}_{nt}^{\mathrm{ren}}(\mu)$. However, Theorem 3.6 and Proposition 3.1 show that this is not the case.

4. Singular Points

4.1. Sufficient Conditions for the Existence of Singular Points of the Non-Trivial Support

For $x \in \mathcal{S}_{nt}(\mu)$, we recall that x is a singular point of the support of μ if there exists a non-tangential sequence $\{w_n\}_{n=1}^{+\infty} \subset \mathbb{H}$ such that $\lim_{n\to+\infty} w_n = x$ but $\lim_{n\to\infty}(\chi_{\mathcal{L}}(u_n,v_n),\eta_{\mathcal{L}}(u_n,v_n)) \neq (x,1)$. Consequently this means that $\partial\mathcal{L}(x)$ contains more points than $(x,1)$. For the types of singular points considered in this article, the limit $\lim_{n\to+\infty}(\chi_{\mathcal{L}}(w_n),\eta_{\mathcal{L}}(w_n))$ will exist for every sequence $\{w_n\}_n$ that converges non-tangentially to x and be independent of the non-tangential sequence chosen. Let the limit be $(\chi_\Gamma(x),\eta_\Gamma(x))$.

Lemma 4.1. — *For every $x \in \mathcal{S}_{nt}(\mu)^\circ$ there exists a sequence $\{w_n\}_{n=1}^{\infty}$ such that*

$$\lim_{n\to\infty}(\chi_{\mathcal{L}}(u_n,v_n),\eta_{\mathcal{L}}(u_n,v_n)) = (x,1).$$

In particular this implies that $\{(x,1)\} \subset \partial\mathcal{L}(x)$ for all $x \in \mathcal{S}_{nt}(\mu)^\circ = (\mathrm{supp}(\mu)\cap\mathrm{supp}(\lambda-\mu))^\circ$.

Proof. — According to Proposition 3.1 the set of regular points is dense in $x \in \mathcal{S}_{nt}(\mu)^\circ$. Therefore, for every x there exists a sequence $\{u_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} u_n = x$ and such that u_n is regular for every n. Hence, for every $\varepsilon > 0$ sufficiently small and each n there exists a v_n such that

$$|(\chi_{\mathcal{L}}(u_n,v_n),\eta_{\mathcal{L}}(u_n,v_n)) - (u_n,1)| < \frac{\varepsilon}{n}$$

This implies that

$$\lim_{n\to\infty}(\chi_{\mathcal{L}}(u_n,v_n),\eta_{\mathcal{L}}(u_n,v_n)) = (x,1)$$

which concludes the proof. □

From this and the fact that $\mathcal{L}$ is simply connected and $W_{\mathcal{L}}$ is a homeomorphism, $(\chi_\Gamma(x),\eta_\Gamma(x))$ has to be connected to the point $(x,1)$. But since for all non-tangentially convergent sequences $\{w_n\}_n$ to x,

$$\lim_{n\to+\infty}(\chi_{\mathcal{L}}(w_n),\eta_{\mathcal{L}}(w_n)) = (\chi_\Gamma(x),\eta_\Gamma(x)),$$

we will have to consider tangentially convergent sequences to x in order to determine the whole of $\partial\mathcal{L}(x)$. We will not attempt to determine $\partial\mathcal{L}(x)$ in full generality, nor will we attempt a complete classification of all singular points, but contend ourselves with some more restrictive assumptions on the density f.

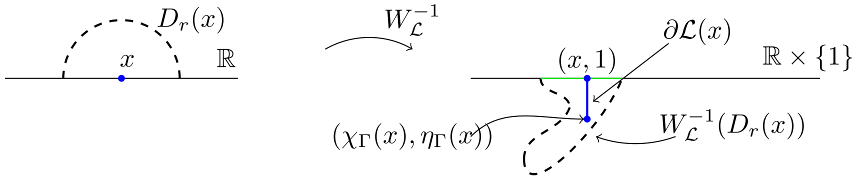

Figure 4.1. Depiction of a singular point.

Proposition 4.2. — *Assume that $x \in \mathcal{S}_{nt}^{\text{sing,III}}(\mu)$. Then $x \in \mathscr{L}_f$ and $|\mathcal{H}f(x)| < +\infty$. Furthermore, x is a singular point. Finally, for every non-tangential sequence $\{u_n + \mathrm{i}\, v_n\}_{n=1}^{\infty} \in \mathbb{H}$ that converges to x,*

$$(4.1)\qquad \lim_{n\to+\infty} (\chi_{\mathcal{L}}(u_n,v_n),\eta_{\mathcal{L}}(u_n,v_n))$$
$$= \Big(x-(\pi(\mathcal{H}f)'(x))^{-1}(\mathrm{e}^{-\pi\mathcal{H}f(x)}-1), 1+(\pi(\mathcal{H}f)'(x))^{-1}(\mathrm{e}^{\pi\mathcal{H}f(x)}+\mathrm{e}^{-\pi\mathcal{H}f(x)}-2)\Big)$$
$$:= (\chi_\Gamma^{\text{III}}(x),\eta_\Gamma^{\text{III}}(x)),$$

where

$$(\mathcal{H}f)'(x) := -\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x-t)^2}.$$

Proof. — Consider the case when $x \in \mathcal{S}_{nt}^{\text{sing,III}}(\mu)$. We first show that x belongs to the Lebesgue set of f. Recall that

$$\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x-t)^2} < +\infty.$$

Hence,

$$\lim_{h\to 0^+}\int_{x-h}^{x+h} \frac{f(t)\mathrm{d}t}{(x-t)^2} = 0.$$

But for small h,

$$\int_{x-h}^{x+h} \frac{f(t)\mathrm{d}t}{(x-t)^2} > h^{-2}\int_{x-h}^{x+h} f(t)\mathrm{d}t > h^{-1}\int_{x-h}^{x+h} f(t)\mathrm{d}t.$$

This implies that

$$\lim_{h\to 0^+}\frac{1}{2h}\int_{x-h}^{x+h} f(t)\mathrm{d}t = 0,$$

so $f(x) = 0$ and $x \in \mathscr{L}_f$ by the discussion in the beginning of Section 2.1. Take $\delta > 0$ small, then

$$\int_{x-\delta}^{x+\delta} \frac{f(t)\mathrm{d}t}{|x-t|} < \int_{x-\delta}^{x+\delta} \frac{f(t)\mathrm{d}t}{(x-t)^2} < +\infty,$$

by assumption, so $|\mathcal{H}f(x)| < +\infty$. Moreover, by Proposition A.2 this also implies that for every $k > 0$

$$\lim_{\substack{(u,v)\to(x,0)\\(u,v)\in\Gamma_k(x)}} H_v f(u) = \mathcal{H}f(x).$$

By definition

$$\lim_{\substack{(u,v)\to(x,0)\\(u,v)\in\Gamma_k(x)}} \frac{\sin(P_v f(u))}{v} = \lim_{\substack{(u,v)\to(x,0)\\(u,v)\in\Gamma_k(x)}} \int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(u-t)^2+v^2}.$$

Note that $(x-t)^2 = 2((x-u)+(u-t))^2 \leqslant (2k^2+2)(v^2+(u-t)^2)$ for all $(u,v) \in \Gamma_k(x)$, $t \in \mathbb{R}$. Hence, Lebesgue's dominated convergence theorem implies that

$$\lim_{\substack{(u,v)\to(x,0)\\(u,v)\in\Gamma_k(x)}} \int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(u-t)^2+v^2} = \int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(t-x)^2}.$$

In addition, since $x \in \mathscr{L}_f$, it follows that

$$\lim_{\substack{(u,v)\to(x,0)\\(u,v)\in\Gamma_k(x)}} P_v f(u) = 0.$$

Consequently,

$$\lim_{\substack{(u,v)\to(x,0)\\(u,v)\in\Gamma_k(x)}} \frac{v(e^{-}\pi H_v f(u) - \cos(\pi P_v f(u))}{\sin(\pi P_v f(u))} = \frac{\mathrm{e}^{-\pi\mathcal{H}f(x)}-1}{\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x-t)^2}} \quad (\neq 0 \text{ since } \mathcal{H}f(x) \neq 0)$$

and

$$\lim_{\substack{(u,v)\to(x,0)\\(u,v)\in\Gamma_k(x)}} \frac{v(\mathrm{e}^{\pi H_v f(u)} + \mathrm{e}^{-\pi H_v f(u)} - 2\cos(\pi P_v f(u))}{\sin(\pi P_v f(u))} = \frac{\mathrm{e}^{\pi\mathcal{H}f(x)} + \mathrm{e}^{-\pi\mathcal{H}f(x)} - 2}{\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x-t)^2}}.$$

Hence $\lim_{n\to\infty}(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \neq (x,1)$, and so x is singular. Again we notice that the limit is independent of k. □

Remark 4.3. — It is interesting to compare the results of Proposition 4.2 with the parametrization of the *edge* $\mathcal{E}$. One sees that the set $\mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)$ is analogous to the parametrization set R_μ, i.e., $(\chi_{\mathcal{E}}(x), \eta_{\mathcal{E}}(x)) = (\chi_\Gamma^{\mathrm{III}}(x), \eta_\Gamma^{\mathrm{III}}(x))$ whenever $x \in \mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)$.

We now conclude this section by two lemmas that show that the set $\mathcal{S}_{nt}^{\text{sing}}(\mu)$ can have $\lambda(\mathcal{S}_{nt}^{\text{sing}}(\mu)) > 0$ and that it may be dense in $\mathcal{S}_{nt}(\mu)^{\circ}$. In particular, this shows that the set $\mathcal{S}_{nt}^{\text{sing}}(\mu)$ can be at least a meagre set. Moreover, it also shows that the boundary $\partial\mathcal{L}$ can be very complicated.

Lemma 4.4. — *There exists a function $f \in \rho_{1,c}^{\lambda}(\mathbb{R})$ such that*

$$\lambda(\mathcal{S}_{nt}^{\text{sing,III}}(\mu)) > 0.$$

Proof. — Let $I = (a,b)$ be an open interval. Associate to it the function

$$\phi_I(t) = \exp\left(-\frac{1}{(t-a)^2(t-b)^2}\right)\chi_{(a,b)}(t). \tag{4.2}$$

Then $\phi_I \in C^{\infty}(\mathbb{R})$ and $\text{supp}(\phi_I) = [a,b]$. Consider the interval $[0,1]$. Remove from it the middle $1/4$ interval $(3/8, 5/8)$. Let $I_1^1 = (3/8, 5/8)$ and $\varphi_1(t) = \phi_{I_1^1}(t)$. Now remove the middle $1/16$ interval from the middle of the remaining intervals. That is, let $I_2^1 = (5/32, 7/32)$ and $I_2^2 = (25/32, 27/32)$, and let $\varphi_2(t) = \phi_{I_1^1}(t) + \phi_{I_2^1}(t) + \phi_{I_2^2}(t)$. Then $[0,1] - (I_1^1 \cup I_2^1 \cup I_2^2) = [0, 5/32] \cup [7/32, 3/8] \cup [5/8, 25/32] \cup [27/32, 1]$. Continue this process, by at step n, remove the middle $1/2^{2n}$:th interval from the remaining 2^{n-1} intervals. Let I_n^k be the k:th of the 2^{n-1} open intervals that are deleted at each step, and let $\varphi_n(t) = \sum_{m=1}^{n}\sum_{k=1}^{2^{m-1}} \phi_{I_m^k}(t)$. Let $\mathfrak{C} = [0,1] \backslash (\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^{n-1}} I_n^k)$. The set $\mathfrak{C}$ is called the Smith–Volterra–Cantor set. It can be shown that $\mathfrak{C}$ is a compact nowhere dense set such that $\lambda(\mathfrak{C}) = 1/2$. In particular $\mathfrak{C}^{\circ} = \varnothing$. Let $\varphi(t) = \lim_{n\to\infty} \varphi_n(t)$. Since all the intervals I_n^k are disjoint and $\varphi \in C^{\infty}(\overline{I_n^k})$ for all n and k it follows that $\varphi \in C^{\infty}(\mathbb{R})$ and $\text{supp}(\varphi) = [0,1] \backslash (\mathfrak{C} \backslash \partial\mathfrak{C})$. Moreover, $0 \leqslant \varphi(t) < 1$ for all t. However, if we choose a measure μ such that $\mu\big|_{[0,1]} = \varphi(t)\mathrm{d}t$, then the measure-theoretic support of μ restricted to $[0,1]$ equals $[0,1]$. This is because, for every $x \in [0,1]$, we have for every neighbourhood N_x of x that $\lambda(N_x \cap \text{supp}(\varphi)) > 0$. We now note that $0 \leqslant \varphi_n(t) \leqslant \varphi_{n+1}(t) \leqslant \varphi(t)$ for all n. Hence, by Lebesgue's monotone convergence theorem we have for every $x \in \mathfrak{C}$

$$\lim_{n\to\infty}\int_{\mathbb{R}} \frac{\varphi_n(t)\mathrm{d}t}{(x-t)^2} = \int_{\mathbb{R}} \frac{\varphi(t)\mathrm{d}t}{(x-t)^2}.$$

We now observe that for all $0 \leqslant a < b \leqslant 1$ and $t \in \mathbb{R}$ and $x \in \mathbb{R}\backslash(a,b)$ we have

$$\phi_I(t) \leqslant \mathrm{e}^{-1/(t-x)^2}.$$

Therefore,

$$\int_{\mathbb{R}} \frac{\varphi_n(t)\mathrm{d}t}{(x-t)^2} = \sum_{m=1}^{n} \sum_{k=1}^{2^{m-1}} \int_{I_m^k} \frac{\phi_{I_m^k}(t)\mathrm{d}t}{(x-t)^2} \leqslant \sum_{m=1}^{n} \sum_{k=1}^{2^{m-1}} \int_{I_m^k} \frac{\mathrm{e}^{-1/(x-t)^2}\,\mathrm{d}t}{(x-t)^2}$$
$$\leqslant \int_{\mathbb{R}} \frac{\mathrm{e}^{-1/(x-t)^2}\,\mathrm{d}t}{(x-t)^2} < +\infty,$$

for all n. Hence $\int_{\mathbb{R}} \frac{\varphi(t)\mathrm{d}t}{(x-t)^2} < +\infty$. Now assume that $\lambda(\{x \in \mathfrak{C} : \mathcal{H}\varphi(x) \neq 0\}) = 0$. Take the density f to be

$$f(t) = \varphi(t) + \chi_{[-1,-\|\varphi\|_1]}(t)$$

Then $\mathcal{H}f = \mathcal{H}\varphi + \mathcal{H}\chi_{[-1,-\|\varphi\|_1]}$. Then for every $x \in [0,1]$, $\mathcal{H}\chi_{[-1,-\|\varphi\|_1]}(x) > 0$. Thus $\lambda(\{x \in \mathfrak{C} : \mathcal{H}f(x) \neq 0\}) = \lambda(\mathfrak{C})$. This implies that $\lambda(\mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)) = \lambda(\mathfrak{C}) > 0$. If on the other hand $\lambda(\{x \in \mathfrak{C} : \mathcal{H}\varphi(x) \neq 0\}) > 0$, then let $f(t) = \varphi(\delta t)$, where $\delta = \|\varphi\|_1$. Then $\int_{\mathbb{R}} f(t)\mathrm{d}t = \int_{\mathbb{R}} \varphi(t\delta)\mathrm{d}t = \delta^{-1}\int_{\mathbb{R}} \varphi(x)\mathrm{d}x = 1$ and $0 \leqslant f(t) < 1$. Moreover, since $\mathcal{H}f(x) = \mathcal{H}\varphi(\delta t)(x) = \delta\mathcal{H}\varphi(\delta x)$, it follows that $\lambda(\{x \in \mathfrak{C} : \mathcal{H}f(x) \neq 0\}) > 0$. Hence $\lambda(\{x \in \delta^{-1}\mathfrak{C} : \mathcal{H}f(x) \neq 0\}) > 0$, and consequently $\lambda(\mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)) > 0$. □

Remark 4.5. — Note that the set $\mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)$ in Lemma 4.4 is uncountable. Moreover, by Proposition 4.7 and Remark 4.8 it follows that

$$\mathcal{H}^1(\partial\mathcal{L}(x)) > 1 - \eta_\Gamma^{\mathrm{III}}(x) > 0 \text{ for every } x \in \mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu).$$

Therefore, $\mathcal{H}^1(\partial\mathcal{L}) > \sum_{x\in\mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)}(1-\eta_\Gamma^{\mathrm{III}}(x))$. However, every uncountable sum of positive real numbers is always infinite. Therefore $\mathcal{H}^1(\partial\mathcal{L}(x)) = +\infty$ for the measure in Lemma 4.4.

Lemma 4.6. — *There exists a function $f \in \rho_{1,c}^{\lambda}(\mathbb{R})$ and an interval $I \subset \mathrm{supp}(f)$, such that $\mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)$ is dense in I.*

Proof. — Consider the dyadic set

$$S = \{x_k^r = k2^{-r} : r \geqslant 1, 1 \leqslant k < 2^r, k \text{ odd}\}.$$

The set S is dense in the interval $[0,1]$. Let $I_k^r = (x_k^r - 2^{-2r-1}, x_k^r + 2^{-2r-1})$. Then $U = \bigcup_{r,k} I_k^r$ is an open cover of the dense set S such that

$$|U| \leqslant \sum_{r=1}^{\infty} \sum_{\substack{k=1\\ k \text{ odd}}}^{2^r} |I_k^r| \leqslant \sum_{r=1}^{\infty}\sum_{k=1}^{2^r} \frac{1}{2^{2r}} = \sum_{r=1}^{\infty} \frac{1}{2^r} = \frac{1}{2}.$$

Define the function φ according to

$$\varphi(t) := \frac{1}{2}\chi_{[0,1]\setminus U}(t) + \inf_{r,k}\{(x_k^r - t)^2 \chi_{I_k^r}(t)\}.$$

We will show that $\inf_{r,k}\{(x_k^r - t)^2\chi_{I_k^r}(t)\}$ is not identically 0. Consider the set $E = \{t \in U\backslash S : \inf_{r,k}\{(x_k^r - t)^2\chi_{I_k^r}(t)\} = 0\}$. We now estimate the measure of E. It is clear that if $t \in U\backslash S$ is such that there exists an r_0, such that

$$t \notin \bigcup_{r=r'}^{\infty} \bigcup_{\substack{k=1\\ k \text{ odd}}}^{2^r} I_k^r$$

whenever $r' \geqslant r_0$, then $\inf_{r,k}\{(x_k^r - t)^2\chi_{I_k^r}(t)\} > 0$. Thus $E \subset \bigcup_{r=r_0}^{\infty} \bigcup_{\substack{k=1\\ k \text{ odd}}}^{2^r} I_k^r$ for every $r_0 \geqslant 1$. Hence for every $\varepsilon > 0$

$$|E| \leqslant \sum_{r=r_0}^{\infty} \sum_{\substack{k=1\\ k \text{ odd}}}^{2^r} |I_k^r| \leqslant \sum_{r=r_0}^{\infty} \sum_{k=1}^{2^r} \frac{1}{2^{2r}} = \sum_{r=r_0}^{\infty} \frac{1}{2^r} < \varepsilon,$$

whenever r_0 is sufficiently large. Thus $|E| = 0$. Moreover, we clearly have that for any $x_k^r \in S$

$$\begin{aligned}\int_{\mathbb{R}} \frac{\varphi(t)\mathrm{d}t}{(x_k^r - t)^2} &\leqslant \frac{1}{2}\int_{[0,1]\backslash I_k^r} \frac{\mathrm{d}t}{(x_k^r - t)^2} + \frac{1}{2}\int_{I_k^r} \frac{\inf_{m,l}\{(x_l^m - t)^2\chi_{I_l^m}(t)\}\mathrm{d}t}{(x_k^r - t)^2} \\ &\leqslant \frac{1}{2}\int_{[0,1]\backslash I_k^r} \frac{\mathrm{d}t}{(x_k^r - t)^2} + \frac{1}{2}\int_{I_k^r} \frac{(x_k^r - t)^2\chi_{I_k^r}(t)\mathrm{d}t}{(x_k^r - t)^2} < +\infty.\end{aligned}$$

Now assume that for any interval $J \subset [0,1]$, the set $\{x \in S : \mathcal{H}\varphi(x) \neq 0\}$ is not dense in J. Then the set $\{x \in S : \mathcal{H}\varphi(x) = 0\}$ is dense in $[0,1]$. Let

$$f(t) = \varphi(t) + \chi_{[-1,-\|\varphi\|_1]}(t).$$

Then $\mathcal{H}\chi_{[-1,-\|\varphi\|_1]}(x) > 0$ for all $x \in [0,1]$. Hence the set $\{x \in S : \mathcal{H}f(x) \neq 0\}$ is dense in $[0,1]$. Thus $\mathcal{S}_{nt}^{\text{sing,III}}(\mu)$ is dense in [0,1]. If on the other hand the set $\{x \in S : \mathcal{H}\varphi(x) \neq 0\}$ is dense in some interval $J \subset [0,1]$, let $f(t) = \varphi(\delta t)$, where $\delta = \|\varphi\|_1$. Then $\{x \in \delta^{-1}S : \mathcal{H}f(x) \neq 0\}$ is dense in $\delta^{-1}J$. Thus $\mathcal{S}_{nt}^{\text{sing,III}}(\mu)$ is dense in $\delta^{-1}J$. □

4.2. Geometry of $\partial\mathcal{L}(x)$ when $x \in \mathcal{S}_{nt}^{\text{sing}}(\mu)$

We have seen that if x is singular then $\partial\mathcal{L}(x) \neq \{(x,1)\}$. In this section we will determine subsets of $\partial\mathcal{L}(x)$, and under additional assumptions on the density f, we will be able to determine the entire set $\partial\mathcal{L}(x)$.

Proposition 4.7. — *Let $x \in (\mathcal{S}_{nt}^{\text{sing,III}}(\mu) \cap \mathcal{S}_{nt}(\mu)^\circ$. Assume that there exists a $\delta > 0$ such that for all $y \in (x-\delta, x) \cup (x, x+\delta)$*

$$\int_{x-\delta}^{x+\delta} \frac{f(t)\mathrm{d}t}{(y-t)^2} = +\infty. \tag{4.3}$$

Then the parametrized curve

$$\begin{aligned}&\{(\chi_{\mathrm{III}}(\xi),\eta_{\mathrm{III}}(\xi)):\xi\in(0,+\infty)\}\\&:=\left\{\left(x+\frac{1-\mathrm{e}^{-\pi\mathcal{H}f(x)}}{\xi-\pi(\mathcal{H}f)'(x)},1-\frac{\mathrm{e}^{\pi\mathcal{H}f(x)}+\mathrm{e}^{-\pi\mathcal{H}f(x)}-2}{\xi-\pi(\mathcal{H}f)'(x)}\right):\xi\in(0,+\infty)\right\}\\&\qquad\qquad\subset\{(\chi,\eta)\in\mathbb{R}^2:\eta-1=(1-\mathrm{e}^{\pi\mathcal{H}f(x)})(\chi-x)\}\end{aligned}$$

is a subset of $\partial\mathcal{L}(x)$. In particular, there exist two one-parameter families of tangential continuous curves $\{s+iv_\xi^+(s):s\in(0,+\infty),\xi\in(0,+\infty)\}$ and $\{s+iv_\xi^-(s):s\in(-\infty,0),\xi\in(0,+\infty)\}$ where $v_\xi^\pm(s)$ is a continuous function of s for each $\xi\in(0,+\infty)$, such that

$$\lim_{s\to0^\pm}\chi_{\mathcal{L}}(x+s+iv_\xi^\pm(s)):=\chi_{\mathrm{III}}(\xi)$$
$$\lim_{s\to0^\pm}\eta_{\mathcal{L}}(x+s+iv_\xi^\pm(s)):=\eta_{\mathrm{III}}(\xi).$$

The curves $\{s+iv_\xi^+(s):s\in(0,+\infty),\xi\in(0,+\infty)\}$ and $\{s+iv_\xi^-(s):s\in(-\infty,0),\xi\in(0,+\infty)\}$ satisfy the equation

$$G_2(s,v_\xi^\pm(s);x)=\xi$$

where

$$G_2(s,v;x)=1_{s>0}\int_x^{x+2s}\frac{f(t)\mathrm{d}t}{(x+s-t)^2+v^2}+1_{s<0}\int_{x+2s}^{x}\frac{f(t)\mathrm{d}t}{(x+s-t)^2+v^2}.$$

Remark 4.8. — It is worth mentioning that the assumption that $x\in(\mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)\cap\mathcal{S}_{nt}(\mu)^\circ$ is never strictly used. In particular, the assumption that there exists an $\varepsilon>0$ such that for all $y\in(x-\varepsilon,x)\cup(x,x+\varepsilon)$

$$\int_{-\delta}^{\delta}\frac{f(t)\mathrm{d}t}{(y-t)^2}=+\infty.$$

implies that $((x-\varepsilon,x)\cup(x,x+\varepsilon))\cap(\mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)\cup R_\mu)=\varnothing$.

Proof. — We begin by studying the limit of the integral

$$\int_{\mathbb{R}}\frac{f(t)\mathrm{d}t}{(u(s)-t)^2+v_\xi(s)^2}\tag{4.4}$$

under a one parameter family of curves $w_\xi(s)=u(s)+iv_\xi(s)\in\mathbb{H}$ such that $\lim_{s\to0^+}w_\xi(s)=x$. In particular we may take $u(s)=x+s$ and assume that $s>0$. The analysis of the case when $s<0$ is completely analogous to the case when $s>0$. The idea is to split (4.4) into two parts, one of which dominates the integrand, and apply Lebesgues dominated convergence theorem. Write

$$\begin{aligned}(4.5)\quad \int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x+s-t)^2+v^2} &= \int_{|x+s-t|\geqslant s} \frac{f(t)\mathrm{d}t}{(x+s-t)^2+v^2} + \int_x^{x+2s} \frac{f(t)\mathrm{d}t}{(x+s-t)^2+v^2} \\ &= I(s,v;x) + G_2(s,v;x)\end{aligned}$$

We now fix $s > 0$. Then clearly, the function $G_2(s,v;x)$ is monotonically decreasing in v. By Fatou's lemma and (4.3)

$$\liminf_{v\to 0^+} G_2(s,v;x) = \liminf_{v\to 0^+} \int_x^{x+2s} \frac{f(t)\mathrm{d}t}{(x+s-t)^2+v^2} = +\infty$$

for s sufficiently small. By Lebesgue's dominated convergence theorem we also have

$$\lim_{v\to+\infty} G_2(s,v;x) = 0.$$

Therefore, since $G_2(s,v;x)$ is monotonically decreasing in v, the equation

$$G_2(s,v;x) = \xi \tag{4.6}$$

has a unique solution $v_\xi(s)$ for all $\xi \in (0,+\infty)$. Differentiation under the integral sign gives

$$\begin{aligned}\frac{\partial G_2(s,v;x)}{\partial v} &= \int_x^{x+2s} \frac{\partial}{\partial v} \frac{1}{(x+s-t)^2+v^2} f(t)\mathrm{d}t \\ &= -2\int_x^{x+2s} \frac{v}{((x+s-t)^2+v^2)^2} f(t)\mathrm{d}t < 0\end{aligned}$$

for all $v > 0$. Hence the implicit function theorem implies that there exists a continuous path $(x+s, v_\xi(s))$ for $s \in (0,\delta)$ such that $G_2(s, v_\xi(s); x) = \xi$. Now, assume that $(x+s, v_\xi(s))$ contains a non-tangential subsequence $\{(x+s_j, v_\xi(s_j))\}_{j=0}^{+\infty}$, i.e., sequence such that there exists a $k>0$ such that $v_j = v_\xi(s_j) > ks_j$ for all j. This implies that

$$\begin{aligned}G_2(s_j, v_j; x) &= \int_x^{x+2s_j} \frac{f(t)\mathrm{d}t}{(x+s_j-t)^2+v_j^2} \\ &\leqslant \int_x^{x+2s_j} \frac{f(t)\mathrm{d}t}{k^2 s_j^2} = \frac{1}{k^2 s_j^2} \int_x^{x+2s_j} f(t)\mathrm{d}t.\end{aligned}$$

However, since $x \in \mathcal{S}_{nt}^{\text{sing,III}}(\mu)$ by assumption, for every $\varepsilon > 0$ there exists a J such that whenever $j > J$

$$\int_x^{x+2s_j} \frac{f(t)}{(x-t)^2}\mathrm{d}t < \varepsilon.$$

Since,

$$\int_x^{x+2s_j} \frac{f(t)}{(x-t)^2}\mathrm{d}t \geqslant \frac{1}{4s_j^2}\int_x^{x+2s_j} f(t)\mathrm{d}t$$

we find that

$$G_2(s_j, v_j; x) \leqslant \frac{4\varepsilon s_j^2}{k^2 s_j^2} = \frac{4\varepsilon}{k}.$$

As ε was arbitrary this implies that $\lim_{j\to+\infty} G_2(s_j, v_j; x) = 0$, a contradiction. Thus the path $(x+s, v_\xi(s))$ becomes tangential to the real axis, i.e. $\lim_{s\to 0^+} v_\xi(s)/s = 0$. We now consider $I(s, v_\xi(s); x)$. Since $|x-t| < 2|x+s-t|$ whenever $|x+s-t| \geqslant s$ we have

$$\frac{f(t)\chi_{\mathbb{R}\setminus[x,x+2s]}(t)}{(x+s-t)^2 + v_\xi(s)^2} \leqslant \frac{4f(t)}{(x-t)^2},$$

and since $\lim_{s\to 0^+} v_\xi(s) = 0$, Lebesgue's dominated convergence theorem implies that

$$\lim_{s\to 0^+}\int_{\mathbb{R}} \frac{f(t)\chi_{\mathbb{R}\setminus[x,x+2s](t)}}{(x-s-t)^2 + v_\xi(s)^2} = \int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x-t)^2}.$$

It now remains to study $\pi H_{v_\xi(s)} f(x+s)$ as $s \to 0^+$. We have

$$\begin{aligned}\pi H_{v_\xi(s)} f(x+s) &= \int_{\mathbb{R}} \frac{(x+s-t)f(t)\mathrm{d}t}{(x+s-t)^2 + v_\xi(s)^2}\\ &= \int_{|x+s-t|\geqslant s} \frac{(x+s-t)f(t)\mathrm{d}t}{(x+s-t)^2 + v_\xi(s)^2}\\ &\qquad + \int_x^{x+2s} \frac{(x+s-t)f(t)\mathrm{d}t}{(x+s-t)^2 + v_\xi(s)^2}\\ &= J_1(s) + J_2(s).\end{aligned}$$

Note that for $t \in \mathbb{R}\setminus[x, x+2s]$ $|x-t| < 2|x+s-t|$, which implies that

$$\frac{|x+s-t|f(t)}{(x+s-t)^2 + v_\xi(s)^2} \leqslant \frac{2f(t)}{|x-t|}.$$

Again by Lebesgue's dominated convergence theorem,

$$\lim_{s\to 0^+} J_1(s) = \lim_{s\to 0^+}\int_{\mathbb{R}} \frac{(x+s-t)f(t)\chi_{\mathbb{R}\setminus[x,x+2s](t)}}{(x-s-t)^2 + v_\xi(s)} = \int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{x-t}.$$

Finally,

$$|J_2(s)| \leqslant \int_x^{x+2s} \frac{|x+s-t| f(t)\mathrm{d}t}{(x+s-t)^2 + v_\xi(s)^2} \leqslant 2s \int_x^{x+2s} \frac{f(t)\mathrm{d}t}{(x+s-t)^2 + v_\xi(s)^2} = 2s\xi,$$

by (4.6). Hence,

$$\lim_{s\to 0^+} J_2(s) = 0.$$

Altogether, this implies that

$$\lim_{s\to 0^+} \frac{v_\xi(s)\left\{ \mathrm{e}^{\pi H_{v_\xi(s)} f(x+s)} - \cos(\pi P_{v_\xi(s)} f(x+s))\right\}}{\sin(\pi P_{v_\xi(s)} f(x+s))} = \lim_{s\to 0^+} \frac{\mathrm{e}^{\pi H_{v_\xi(s)} f(x+s)} - \cos(\pi P_{v_\xi(s)} f(x+s))}{\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x+s-t)^2 + v_\xi(s)^2}} = \frac{\mathrm{e}^{\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{x-t}} - 1}{\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x-t)^2} + \xi}$$

and

$$\lim_{s\to 0^+} \frac{v_\xi(s)\left\{1 - \mathrm{e}^{-\pi H_{v_\xi(s)} f(x+s)} \cos(\pi P_{v_\xi(s)} f(x+s))\right\}}{\sin(\pi P_{v_\xi(s)} f(x+s))} = \frac{1 - \mathrm{e}^{-\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{x-t}}}{\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x-t)^2} + \xi}.$$

Recall that the distributional derivative of the Cauchy principal value integral p. v.$\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{x-t}$ equals

$$\frac{\mathrm{d}}{\mathrm{d}x} \text{p.v.} \int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{x-t} = -\text{f.p.} \int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x-t)^2},$$

where f.p. $\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x-t)^2}$ denotes Hadamard's finite part integral. However, as the integrals $\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{x-t}$ and $\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x-t)^2}$ exists in the ordinary sense we have that

$$-\frac{\mathrm{d}}{\mathrm{d}x} \text{p.v.} \int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{x-t} = -\pi(\mathcal{H}f)'(x) = \text{f.p.} \int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x-t)^2} = \int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x-t)^2}$$

Using this we find that

$$\lim_{s\to 0^+} \chi_{\mathcal{L}}(w_\xi(s)) = \chi_{\mathrm{III}}(\xi) = x + \frac{1 - \mathrm{e}^{-\pi\mathcal{H}f(x)}}{\xi - \pi(\mathcal{H}f)'(x)}$$

and

$$\lim_{s\to 0^+} \eta_{\mathcal{L}}(w_\xi(s)) = \eta_{\mathrm{III}}(\xi) = 1 - \frac{\mathrm{e}^{\pi\mathcal{H}f(x)} + \mathrm{e}^{-\pi\mathcal{H}f(x)} - 2}{\xi - \pi(\mathcal{H}f)'(x)}$$

for each fixed $\xi \in (0,+\infty)$. In particular we note that this is a parametrization of a part of line given by the equation

$$\frac{\eta-1}{\chi-x} = -\frac{2-\mathrm{e}^{\pi\mathcal{H}f(x)}-\mathrm{e}^{-\pi\mathcal{H}f(x)}}{1-\mathrm{e}^{-\pi\mathcal{H}f(x)}} = 1-\mathrm{e}^{\pi\mathcal{H}f(x)}\,. \qquad \square$$

Corollary 4.9. — *Assume that $x \in \mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)$. Then for every fixed $\xi \in (-\infty,+\infty)$ there exists a sequence $\{x+s_j+iv_\xi(s_j)\}_{j=1}^{+\infty} \in \mathbb{H}$, such that $\lim_{j\to+\infty} x+s_j+iv_\xi(s_j) = x$ and*

$$\lim_{j\to+\infty} \chi_{\mathcal{L}}(x+s_j+iv_\xi(s_j)) = \chi_{\mathrm{III}}(\xi)$$
$$\lim_{j\to+\infty} \eta_{\mathcal{L}}(x+s_j+iv_\xi(s_j)) = \eta_{\mathrm{III}}(\xi).$$

Proof. — We may repeat the proof of Proposition 4.7 replacing a continuous path everywhere with a sequence $\{(x+s_j+v(s_j)\}_j$. The only difference is that since we are not assuming (4.3) we may not conclude that there exists a solution to the equation $G_2(s,v;x)=\xi$ for every s sufficiently small. However, since we are considering sequences instead of paths we can always find a sequence $s_j\to 0$ as $j\to+\infty$ such that $G_2(s_j,v;x)=\xi$. The rest of the proof remains the same. $\square$

In general the equation $G_2(s,v;x)=\xi$ in Proposition 4.7 can of course not be solved explicitly. However there exists an important special case when $f(t)$ or $1-f(t)$ is convex in a neighborhood of the point x, when one can solve the equation $G_2(s,v;x)=\xi$ approximately.

Proposition 4.10. — *Let $x \in \mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)\cap\mathcal{S}_{nt}(\mu)^\circ$ and let $G_2(s,v;x)$ be the function defined in Proposition 4.7. Assume that there is an $\varepsilon>0$ such that $f(t)$ is convex in $[x-\varepsilon,x+\varepsilon]$ and $f(x+2s)/f(x+s)$ is uniformly bounded for $|s|\leqslant\varepsilon$. Fix $\xi>0$ and define*

$$v(s) = \frac{\pi}{\xi} f(x+s) \tag{4.7}$$

for $|s|\leqslant\varepsilon$. Then,

$$\lim_{s\to 0} G_2(s,v(s);x) = \xi. \tag{4.8}$$

Thus,

$$\begin{aligned}\lim_{s\to 0}\chi_{\mathcal{L}}(x+s+iv(s)) &= \chi_{\mathrm{III}}(\xi)\\ \lim_{s\to 0}\eta_{\mathcal{L}}(x+s+iv(s)) &= \eta_{\mathrm{III}}(\xi).\end{aligned} \tag{4.9}$$

Proof. — Consider the case $s > 0$. Let

$$I_1(s) = \int_x^{x+2s} \frac{f(x+s)\mathrm{d}t}{(x+s-t)^2 + v(s)^2},$$
$$I_2(s) = \int_{x+s}^{x+2s} \frac{(f(t) - f(x+s))\mathrm{d}t}{(x+s-t)^2 + v(s)^2} - \int_x^{x+s} \frac{(f(x+s) - f(t))\mathrm{d}t}{(x+s-t)^2 + v(s)^2}$$

so that

$$G_2(x, v(s); x) = I_1(s) + I_2(s).$$

Let $F(x)$ be a convex function on an interval I and let $x, y, w \in I$ with $x < y < w$. Then

$$\frac{F(y) - F(x)}{y - x} \leqslant \frac{F(w) - F(x)}{w - x} \tag{4.10}$$

(see [18, Proposition 1.25]). From this and $f(x) = 0$ since $x \in \mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)$ we see that $f(x+s)/s$ is an increasing function in $(0, \varepsilon)$ and hence

$$a = \lim_{s \to 0^+} \frac{f(x+s)}{s}$$

exists and is $\geqslant 0$. We must have $a = 0$, since if $a > 0$ then

$$\infty = \int_0^\varepsilon \frac{a}{s}\mathrm{d}s \leqslant \int_0^\varepsilon \frac{f(x+s)}{s^2}\mathrm{d}s \leqslant \int_{\mathbb{R}} \frac{f(t)}{(x-t)^2}\mathrm{d}t,$$

which contradicts $x \in \mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)$. Thus,

$$\frac{v(s)}{s} = \frac{\pi}{\xi} \frac{f(x+s)}{s} \to 0 \tag{4.11}$$

as $s \to 0^+$. It follows that

$$I_1(s) = \frac{\xi}{\pi} \int_x^{x+2s} \frac{v(s)\mathrm{d}t}{(x+s-t)^2 + v(s)^2} = \frac{2\xi}{\pi} \arctan \frac{s}{v(s)} \to \xi$$

as $s \to 0^+$. Hence, to prove (4.8) it remains to show that $I_2(s) \to 0^+$ as $s \to 0^+$. Notice that we can write

$$I_2(s) = \int_0^s (f(x+s+t) + f(x+s-t) - 2f(x+s)) \frac{\mathrm{d}t}{t^2 + v(s)^2}. \tag{4.12}$$

Since f is convex in $[x - \varepsilon, x + \varepsilon]$ we see that for $0 \leqslant s \leqslant \varepsilon/2$,

$$\frac{1}{2}(f(x+s+t) + f(x+s-t)) \geqslant f\left(\frac{x+s+t+x+s-t}{2}\right) = f(x+s)$$

and consequently $I_2(s) \geqslant 0$. It follows from (4.10) that

$$\frac{f(x+s) - f(x)}{s} \geqslant \frac{f(x+s-t) - f(x)}{s-t}$$

for $t \in [0,s)$ and since $f(x)=0$ we see that

$$f(x+s) \geqslant \frac{s}{s-t} f(x+s-t) \geqslant f(x+s-t).$$

From (4.10) we also see that

$$\frac{f(x+s+t)-f(x+s)}{t} \leqslant \frac{f(x+2s)-f(x+s)}{s} \leqslant \frac{f(x+2s)}{s}$$

for $t \in [0,s]$. Thus,

$$\begin{aligned} (4.13)\quad & f(x+s+t)+f(x+s-t)-2f(x+s) \\ &= t\left(\frac{f(x+s+t)-f(x+s)}{t}\right) - (f(x+s)-f(x+s-t)) \\ &\leqslant t\frac{f(x+2s)}{s} \leqslant Ct\frac{f(x+s)}{s}, \end{aligned}$$

for some constant C. In the last estimate we used our assumption that $f(x+2s)/f(x+s)$ is uniformly bounded for $s \in [0,\varepsilon]$. If we use (4.11) together with the estimate (4.13) in (4.12), we see that we have proved that

$$\begin{aligned} 0 \leqslant I_2(s) \leqslant \frac{Cf(x+s)}{s}\int_0^s \frac{t\mathrm{d}t}{t^2+v(s)^2} &= \frac{C\xi v(s)}{\pi s}\int_0^s \frac{t\mathrm{d}t}{t^2+v(s)^2} \\ &= \frac{C\xi v(s)}{2\pi s}\log\left(1+\frac{s^2}{v(s)^2}\right) \to 0 \end{aligned}$$

as $s \to 0^+$ by (4.11). This proves (4.8) and (4.9) follows as in the proof of Proposition 4.7. □

Remark 4.11. — In particular we note that the assumption that $\frac{f(x+2s)}{f(x+s)}$ is uniformly bounded in s holds if $f(x+s) \sim g(s)|s|^\alpha$, for some positive and bounded function $g(s)$ and some $\alpha > 0$ such that $g(s)|s|^\alpha$ is convex in a neighborhood of 0.

In Propositions 4.7 we determined a subset of $\partial\mathcal{L}(x)$ when $x \in \mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)$. We now want to show that under some additional assumptions on the density f, this set is in fact all of $\partial\mathcal{L}(x)$.

Theorem 4.12. — *Assume that $x \in \mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)$ and that the assumptions of Proposition 4.7 are satisfied. Furthermore, assume that there exists sequences $\{r_n\}_n \subset G$ and $\{l_n\}_n \subset G$ of regular points such that $r_n > x$ and $l_n < x$ for all n and such that $\lim_{n\to\infty} r_n = \lim_{n\to\infty} l_n = x$. Finally assume that*

$$\max\{\sup_n |m_{\mathcal{H}f}(r_n)|, \sup_n |m_{\mathcal{H}f}(l_n)|\} < +\infty.$$

Then,

$$\partial\mathcal{L}(x) = \overline{\{(\chi_{\mathrm{III}}(\xi), \eta_{\mathrm{III}}(\xi)) : \xi \in (0, +\infty)\}},$$

where the functions $\chi_{\mathrm{III}}(\xi)$ and $\eta_{\mathrm{III}}(\xi)$ are defined in Proposition 4.7.

Remark 4.13. — In particular the assumptions of Theorem 4.12 hold if $x \in \mathscr{L}_{m_{\mathcal{H}f}}$ by a modification of the proof of Lemma 3.3.

Proof. — Let $x \in \mathcal{S}_{nt}^{\mathrm{sing,III}}(\mu)$. We know from Proposition 4.7 that $\ell = \overline{\{(\chi_{\mathrm{III}}(\xi), \eta_{\mathrm{III}}(\xi) : 0 < \xi < \infty\}} \subset \partial\mathcal{L}(x)$ and we want to prove that equality holds. Let $w_n = u_n + \mathrm{i}\, v_n \in \mathbb{H}$, $n \geqslant 0$, be any sequence such that $w_n \to x$ as $n \to \infty$. We want to show that all limit points of $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n))$ belong to A. By taking subsequences we can assume that $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n))$ converges. Set

$$\xi_n = \int_x^{x+2(u_n-x)} \frac{f(t)\mathrm{d}t}{(u_n-t)^2+v_n^2}.$$

By taking a further subsequence we can assume that $\xi_n \to \xi \in [0, \infty]$ as $n \to \infty$ $u_n > x$ for all n. If $\xi \in [0, \infty)$, a repetition of the arguments of the proof of Proposition 4.7 gives

$$\lim_{n\to\infty} (\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) = \left(x + \frac{1-\mathrm{e}^{-\pi\mathcal{H}f(x)}}{\xi - \pi(\mathcal{H}f)'(x)}, 1 - \frac{\mathrm{e}^{\pi\mathcal{H}f(x)} + \mathrm{e}^{-\pi\mathcal{H}f(x)} - 2}{\xi - \pi(\mathcal{H}f)'(x)}\right).$$

It remains to consider the case $\xi = \infty$. We want to show that in this case $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \to (x, 1)$ as $n \to \infty$. Since $\xi_n \to \infty$, for every $\xi > 0$ there is an $N(\xi)$ such that

$$\int_x^{x+2(u_n-x)} \frac{f(t)\mathrm{d}t}{(u_n-t)^2+v_n^2} > \xi \tag{4.14}$$

whenever $n > N(\xi)$. Let $v_\xi(s)$ be the continuous function defined in Proposition 4.7. Then the inequality (4.14) above implies that

$$G_2(u_n - x, v_n; x) > G_2(u_n - x, v_\xi(u_n - x); x).$$

Since the function $G_2(s, v; x)$ is monotonically decreasing in v this implies that $v_n < v_\xi(u_n - x, v_n; x)$ for all $n > N(\xi)$. This implies that the sequence $\{w_n\}_n$ is trapped inside the set

$$\{(u, v) \in \mathbb{H} : x \leqslant u < u_{N(\xi)}, 0 < v < v_\xi(u - x)\}$$

whenever $n > N(\xi)$. In particular for every n there exists an $r_{k_n} \in G$ such that $r_{k_n} > u_n$ and $\lim_{n\to\infty} r_{k_n} = x$. Let $X_n^{(\xi)}$ be the open set

$$X_n^{(\xi)} = \{(u, v) \in \mathbb{R}^2 : x < u < r_n, 0 < v < v_\xi(u - x)\}.$$

In particular w_n belongs to $X_n^{(\xi)}$ for every $n > N(\xi)$. Let T_n be the closed set, whose boundary equals

$$\begin{aligned}\partial T_n = &\{(t,1) : x \leqslant t \leqslant r_{k_n}\} \cup \{W_{\mathcal{L}}^{-1}(r_{k_n} + \mathrm{i}\, t) : 0 < t \leqslant v_\xi(r_{k_n} - x)\} \\ &\cup \{W_{\mathcal{L}}^{-1}(t + \mathrm{i}\, v_\xi(t)) : 0 < t \leqslant r_{k_n} - x\} \\ &\cup \left\{\left(x + \frac{1 - \mathrm{e}^{-\pi\mathcal{H}f(x)}}{\xi' - \pi(\mathcal{H}f)'(x)}, 1 - \frac{\mathrm{e}^{\pi\mathcal{H}f(x)} + \mathrm{e}^{-\pi\mathcal{H}f(x)} - 2}{\xi' - \pi(\mathcal{H}f)'(x)}\right) : \xi' \geqslant \xi\right\} \\ := &\,\partial T_n^1 \cup \partial T_n^2 \cup \partial T_n^3 \cup \partial T_n^4.\end{aligned}$$

We now show that $\overline{W_{\mathcal{L}}^{-1}(X_n^{(\xi)})} \subset T_n$. Since $x \in \mathcal{S}_{nt}(\mu)^\circ$, it follows from Lemma 4.1 that all points of $\partial T_n^1 \subset \partial W_{\mathcal{L}}^{-1}(X_n^{(\xi)})$. Since $r_{k_n} \in G$, it follows that $\lim_{v\to 0^+}(\chi_{\mathcal{L}}(r_{k_n} + \mathrm{i}\, v), \eta_{\mathcal{L}}(r_{k_n} + \mathrm{i}\, v)) = (r_{k_n}, 1)$. Hence $\partial T_n^2 \subset \partial W_{\mathcal{L}}^{-1}(X_n^{(\xi)})$. By Proposition 4.7, $(\partial T_n^3 \cup \partial T_n^4) \subset \partial W_{\mathcal{L}}^{-1}(X_n^{(\xi)})$. On the other hand, by Lemma 4.1 and the fact that $W_{\mathcal{L}}^{-1}$ is a homeomorphism, we have that $W_{\mathcal{L}}^{-1}(X_n^{(\xi)}) \subset T_n^\circ$. Thus $\overline{W_{\mathcal{L}}^{-1}(X_n^{(\xi)})} \subset T_n$. This fact follows from Lemma 4.1 and Proposition 4.7 and the assumption that r_{k_n} is a regular point in $\mathcal{S}_{nt}(\mu)^\circ$. In particular, $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \in T_n$ for every $n > N(\xi)$. The trapping regions T_n are illustrated in Figure 4.2. We will now show that $\limsup_{n\to+\infty} d((x,1), \partial T_n) \leqslant C/\xi$ for some positive constant C independent of ξ, which implies that $\lim_{n\to+\infty}(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) = (x,1)$. Recall that

$$d((x,1), \partial T_n) = \sup_{(x',y')\in\partial T_n} d((x,1),(x',y')).$$

Clearly,

$$d((x,1), \partial T_n^1) = r_{k_n} - x \to 0$$

as $n \to +\infty$. Similarly, from the proof of Proposition 4.7 it follows that

$$\lim_{n\to\infty} d((x,1), \partial T_n^3) = \left|\left(\frac{1 - \mathrm{e}^{-\pi\mathcal{H}f(x)}}{\xi - \pi(\mathcal{H}f)'(x)}, \frac{\mathrm{e}^{\pi\mathcal{H}f(x)} + \mathrm{e}^{-\pi\mathcal{H}f(x)} - 2}{\xi - \pi(\mathcal{H}f)'(x)}\right)\right|$$

and

$$d((x,1), \partial T_n^4) = \left|\left(\frac{1 - \mathrm{e}^{-\pi\mathcal{H}f(x)}}{\xi - \pi(\mathcal{H}f)'(x)}, \frac{\mathrm{e}^{\pi\mathcal{H}f(x)} + \mathrm{e}^{-\pi\mathcal{H}f(x)} - 2}{\xi - \pi(\mathcal{H}f)'(x)}\right)\right|.$$

We now estimate $d((x,1), \partial T_n^2)$. This is the most subtle part of the proof, and here the choice of the sequence $\{r_{k_n}\}_n$ is critical. By assumption the sequence $\{m_{\mathcal{H}f}(r_{k_n})\}_n$ is bounded and hence by estimate (2.3) $|H_v f(r_{k_n})|$, is uniformly bounded. Thus, there is a constant C' independent of ξ such that for all n and $0 < v < v_\xi(r_{k_n} - x)$,

$$v\left|\frac{1 - \mathrm{e}^{-\pi H_v f(r_{k_n})}\cos(\pi P_v f(r_{k_n}))}{\sin(\pi P_v f(r_{k_n}))}\right| \leqslant v\frac{C'}{|\sin(\pi P_v f(r_{k_n}))|}$$

and

$$v\left|\frac{\mathrm{e}^{\pi H_v f(r_{k_n})} + \mathrm{e}^{-\pi H_v f(r_{k_n})} - 2\cos(\pi P_v(r_{k_n}))}{\sin(\pi P_v f(r_{k_n}))}\right| \leqslant v\frac{C'}{|\sin(\pi P_v f(r_{k_n}))|}$$

for all n. In addition,

$$\begin{aligned}
&\frac{v}{|\sin(\pi P_v f(r_{k_n}))|}\\
&\leqslant \frac{v}{\min\{P_v f(r_{k_n}), P_v(1-f)(r_{k_n}))\}}\\
&\leqslant \frac{1}{\min\{v^{-1}P_v f(r_{k_n}), v^{-1}P_v(1-f)(r_{k_n}))\}}\\
&\leqslant \frac{1}{\min\{v_\xi(r_{k_n}-x)^{-1}P_{v_\xi(r_{k_n}-x)}f(r_{k_n}), v_\xi(r_{k_n}-x)^{-1}P_{v_\xi(r_{k_n}-x)}(1-f)(r_{k_n}))\}}
\end{aligned}$$

for all $0 < v < v_\xi(r_{k_n} - x)$, by the monotonicity of the function $v^{-1}\pi P_v f(r_{k_n})$. By the same argument that was used to control (4.5) in the proof of Proposition 4.7 we see that

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(r_{k_n} - t)^2 + v_\xi(r_{k_n} - x)^2}\\
&= \int_{|r_{k_n} - t| > r_{k_n} - x} \frac{f(t)\mathrm{d}t}{(r_{k_n} - t)^2 + v_\xi(r_{k_n} - x)^2} + G_2(r_{k_n} - x, v_\xi(r_{k_n} - x); x)\\
&\to \int_{\mathbb{R}} \frac{f(t)\mathrm{d}t}{(x-t)^2} + \xi,
\end{aligned}$$

as $n \to \infty$. Furthermore,

$$\begin{aligned}
\int_{\mathbb{R}} \frac{(1 - f(t))\mathrm{d}t}{(r_n - t)^2 + v_\xi(r_n - x)^2} &= \int_{\mathbb{R}} \frac{\mathrm{d}t}{(r_n - t)^2 + v_\xi(r_n - x)^2} - \xi + O(1)\\
&= \frac{\pi}{v_\xi(r_n - x)} - \xi + O(1) \geqslant \xi
\end{aligned}$$

whenever n is sufficiently large. Hence,

$$\limsup_{n\to\infty} d((x,1), \partial T_n^2) \leqslant \frac{C}{\xi}.$$

Combining our estimates we have proved that there is a constant C such that

$$\limsup_{n\to\infty} d\Big((x,1), \partial T_n)\Big) \leqslant \frac{C}{\xi}.$$

Since $\xi \in [0, \infty)$ was arbitrary, the result follows. □

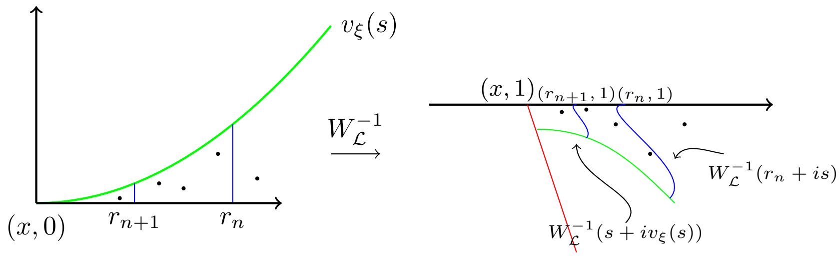

Figure 4.2. This figure illustrates the trapping regions T_n. The black dots represent the positions of the sequence $\{u_n + \mathrm{i}\, v_n\}_{n=1}^{+\infty}$

Appendix A. Additional Results

Lemma A.1.

$$\partial\mathcal{L} = \partial\mathcal{L}(\infty) \cup \left(\bigcup_{x\in\mathbb{R}} \partial\mathcal{L}(x) \right). \tag{A.1}$$

Proof. — Let $\omega_x = \{w_n\}_n \subset \mathbb{H}$ be a sequence such that $\lim_{n\to\infty} w_n = x$. Then $\{W_{\mathcal{L}}^{-1}(w_n)\} \subset \mathcal{L}$. Since $\mathcal{L} \subset \mathcal{P}$, $\overline{\mathcal{L}}$ is compact. By Heine–Borel theorem, it follows that $\partial\mathcal{L}_{[\omega]}(x) \neq \varnothing$. Assume that there exists a point $(\chi', \eta') \in \mathcal{L} \cap \partial\mathcal{L}_{[\omega]}(x)$. Then there exists a subsequence $\{w_{n_k}\}_k$ such that $(\chi_{\mathcal{L}}(w_{n_k}), \eta_{\mathcal{L}}(w_{n_k})) \to (\chi', \eta')$. However, since $W_{\mathcal{L}}$ is a homeomorphism, it follows that $\lim_{k\to\infty} w_{n_k} = w' = W_{\mathcal{L}}((\chi', \eta'))$, a contradiction. Hence $\partial\mathcal{L}_{[\omega]}(x) \subset \partial\mathcal{L}$. Since this holds for every such sequence $\omega = \omega_x$, it follows that

$$\partial\mathcal{L}(x) = \bigcup_{[\omega]\in S_x} \partial\mathcal{L}_{[\omega]}(x) \subset \partial\mathcal{L}.$$

In particular this holds for every $x \in \mathbb{R}$. Thus,

$$\bigcup_{x\in\mathbb{R}} \partial\mathcal{L}(x) \subset \partial\mathcal{L}.$$

Finally, Lemma 2.1 in [7] proves that for any sequence $\{w_n\}_n \subset \mathbb{H}$, such that $\lim_{n\to\infty} |w_n| = \infty$, $\lim_{n\to\infty}(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) = (\frac{1}{2} + \int_{\mathbb{R}} x \mathrm{d}\mu(x), 0) \in \partial\mathcal{L}$. This shows that

$$\partial\mathcal{L}(\infty) \cup \left(\bigcup_{x\in\mathbb{R}} \partial\mathcal{L}(x) \right) \subset \partial\mathcal{L}.$$

We now show the reverse inclusion. Let $(\chi', \eta') \in \partial\mathcal{L}$. Then there exists a sequence $\{(\chi_n, \eta_n)\}_n \subset \mathcal{L}$ such that $\lim_{n\to\infty}(\chi_n, \eta_n) = (\chi', \eta')$. Let $w_n = W_{\mathcal{L}}((\chi_n, \eta_n))$. Assume that the sequence $\{w_n\}_n$ is unbounded.

Then it contains a subsequence $\{w_{n_k}\}_k$ such that $\lim_{k\to\infty}|w_{n_k}|=\infty$. Then Lemma 2.1 in [7] shows that $(\chi',\eta')=(\frac{1}{2}+\int_{\mathbb{R}}x\mathrm{d}\mu(x),0)$. However, this implies that $\lim_{n\to\infty}|w_n|=\infty$. Thus, we may assume that the sequence $\{w_n\}_n$ is bounded in $\mathbb{H}$. Consider the set of limit points of $\{w_n\}_n$, that is $\overline{\{w_n\}_n}\backslash\{w_n\}_n$. Assume that $w'\in\overline{\{w_n\}_n}\backslash\{w_n\}_n\cap\mathbb{H}$. Then there exists a subsequence $\{w_{n_k}\}_k$ such that $\lim_{k\to\infty}w_{n_k}=w'$. However since $W_{\mathcal{L}}$ is a homeomorphism, this implies that $\lim_{k\to\infty}W_{\mathcal{L}}^{-1}(w_{n_k})=W_{\mathcal{L}}^{-1}(w')\neq(\chi',\eta')$, a contradiction. Thus $\overline{\{w_n\}_n}\backslash\{w_n\}_n\subset\mathbb{R}$. This shows that

$$\partial\mathcal{L}\subset\partial\mathcal{L}(\infty)\cup\left(\bigcup_{x\in\mathbb{R}}\partial\mathcal{L}(x)\right).\qquad\square$$

Proposition A.2. — *Let $f\in L^p(\mathbb{R})$ where $p>1$. Assume that*

$$\int_{x-1}^{x+1}\frac{|f(x)-f(t)|\mathrm{d}t}{|x-t|}<+\infty.\tag{A.2}$$

Then for every non-tangential convergent sequence $\{u_n+\mathrm{i}\,v_n\}_n$ to x,

$$\lim_{n\to\infty}H_{v_n}f(u_n)=\mathcal{H}f(x).\tag{A.3}$$

Moreover, $x\in\mathscr{L}_f$.

Proof. — We first note that (A.2) implies that

$$\lim_{h\to 0^+}\int_{x-h}^{x+h}\frac{|f(x)-f(t)|\mathrm{d}t}{|x-t|}\geqslant\lim_{h\to 0^+}\int_{x-h}^{x+h}\frac{|f(x)-f(t)|\mathrm{d}t}{2h}=0.$$

Thus, $x\in\mathscr{L}_f$. We now show that $\mathcal{H}f(x)$ exists. We have for every $R>0$ sufficiently large

$$\begin{aligned}\int_{|x-t|>\varepsilon}\frac{f(t)\mathrm{d}t}{x-t}=\int_{\varepsilon<|x-t|<R}&\frac{(f(t)-f(x))\mathrm{d}t}{x-t}\\&+f(x)\underbrace{\int_{\varepsilon<|x-t|<R}\frac{\mathrm{d}t}{x-t}}_{=0}+\int_{|x-t|>R}\frac{f(t)\mathrm{d}t}{x-t}\\=J_1+J_2.&\end{aligned}$$

We first estimate I_2. Since $f\in L^p(\mathbb{R})$ we have by Hölder's inequality

$$|J_2|\leqslant\|f\|_p\left(\int_{|x-t|>R}\frac{\mathrm{d}t}{|x-t|^q}\right)^{1/q}=\frac{\|f\|_p2^{1/q}}{(q-1)^{1/q}R^{(q-1)/q}},$$

where $q=\frac{p}{p-1}>1$. Moreover, since

$$\lim_{\varepsilon\to 0^+}\frac{(f(x)-f(t))\chi_{|t|>\varepsilon}}{x-t}=\frac{f(x)-f(t)}{x-t}$$

for all $t \neq x$, and

$$\frac{|f(t)-f(x)|\chi_{|t|>\varepsilon}}{|x-t|} \leqslant \frac{|f(t)-f(x)|}{|x-t|},$$

it follows by (A.2) and Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon\to 0^+}\int_{\varepsilon<|x-t|<R}\frac{f(t)-f(x)}{x-t}\mathrm{d}t = \int_{|x-t|<R}\frac{f(t)-f(x)}{x-t}\mathrm{d}t.$$

Since $R>0$ was arbitrary and $f \in L^p(\mathbb{R})$ it follows that

$$\pi\mathcal{H}f(x) = \lim_{R\to\infty}\int_{|x-t|<R}\frac{f(x)-f(t)}{x-t}\mathrm{d}t = \int_{\mathbb{R}}\frac{f(x)-f(t)}{x-t}\mathrm{d}t$$

exists. Now consider a non-tangentially convergent sequence $\{u_n+\mathrm{i}\,v_n\}_n$ to x. Then $\{u_n+\mathrm{i}\,v_n\}_n \subset \Gamma_k(x)$ for some $k>0$. We may assume that $u_n - x \geqslant 0$. Then

$$\begin{aligned}-\pi H_{v_n}f(u_n) &= \int_{\mathbb{R}}\frac{-(u_n-t)f(t)\mathrm{d}t}{(u_n-t)^2+v_n^2} = \int_{\mathbb{R}}\frac{(u_n-t)(f(x)-f(t))\mathrm{d}t}{(u_n-t)^2+v_n^2}\\ &= \int_x^{x+2(u_n-x)}\frac{(u_n-t)(f(x)-f(t))\mathrm{d}t}{(u_n-t)^2+v_n^2}\\ &\qquad + \int_{\mathbb{R}\setminus[x,x+2(u_n-x)]}\frac{(u_n-t)(f(x)-f(t))\mathrm{d}t}{(u_n-t)^2+v_n^2}\\ &:= I_1^{(n)}+I_2^{(n)},\end{aligned}$$

We first consider $I_2^{(n)}$. Since

$$\frac{|u_n-t|\chi_{\mathbb{R}\setminus[x,x+2(u_n-x)]}(t)}{(u_n-t)^2+v_n^2} \leqslant \frac{2}{|x-t|}$$

for all t, and

$$\lim_{n\to\infty}\frac{(u_n-t)(f(x)-f(t))\chi_{\mathbb{R}\setminus[x,x+2(u_n-x)]}(t)}{(u_n-t)^2+v_n^2} = \frac{f(x)-f(t)}{x-t}$$

for all $t \neq x$, we get form (A.2) and Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty}\int_{\mathbb{R}\setminus[x,x+2(u_n-x)]}\frac{(u_n-t)(f(x)-f(t))\mathrm{d}t}{(u_n-t)^2+v_n^2} = -\pi\mathcal{H}f(x).$$

We now consider $I_1^{(n)}$. Since $\{u_n + \mathrm{i}\, v_n\}_n \subset \Gamma_k(x)$,

$$\begin{aligned}|I_1^{(n)}| &\leqslant \int_x^{u_n+(u_n-x)} \frac{|(u_n-t)(f(x)-f(t))|\mathrm{d}t}{(u_n-t)^2+v_n^2}\\ &\leqslant \frac{1}{v_n}\int_x^{u_n+(u_n-x)} |f(x)-f(t)|\mathrm{d}t\\ &\leqslant \frac{(u_n-x)}{v_n}\frac{1}{(u_n-x)}\int_{x-2(u_n-x)}^{x+2(u_n-x)} |f(x)-f(t)|\mathrm{d}t\\ &\leqslant k\frac{1}{(u_n-x)}\int_{x-2(u_n-x)}^{x+2(u_n-x)} |f(x)-f(t)|\mathrm{d}t \to 0\end{aligned}$$

as $n \to \infty$, since $x \in \mathscr{L}_f$. □

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