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HOMOTOPY GROUPS OF GENERIC LEAVES OF LOGARITHMIC FOLIATIONS

by Diego RODRÍGUEZ-GUZMÁN (*)

ABSTRACT. — We study the homotopy groups of generic leaves of logarithmic foliations on complex projective manifolds. We exhibit a relation between the homotopy groups of a generic leaf and of the complement of the polar divisor of the logarithmic foliation.

RÉSUMÉ. — Nous étudions les groupes d'homotopie des feuilles génériques des feuilletages logarithmiques sur les variétés projectives complexes. Nous montrons une relation entre les groupes d'homotopie d'une feuille générique et ceux du complément du diviseur des pôles du feuilletage logarithmique.

1. Introduction

A logarithmic foliation \mathcal{F} on a complex projective manifold X is defined by a closed logarithmic 1-form ω with polar locus $D = \sum_j D_j$, with D_j irreducible hypersurface of X . Here \mathcal{L} denotes a non singular leaf of \mathcal{F} , which is an immersed complex manifold of codimension one in X . In general, \mathcal{L} is a transcendental leaf, that is, it is not contained in any projective hypersurface of X .

We will consider the following topological properties of complex projective manifolds:

- (i) If X is a smooth hypersurface of the projective space \mathbb{P}^{n+1} with $n > 1$, then X is simply connected.

Also, if a complex projective manifold X , with dimension n , lies in \mathbb{P}^m and $H \subset \mathbb{P}^m$ is a general hyperplane, the Lefschetz hyperplane section theorem, implies that the following assertions hold:

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- (ii) If $n > 1$ then the hyperplane section $X \cap H$ is connected.
- (iii) If $n > 2$ then the fundamental groups of X and $X \cap H$ are isomorphic.

Based on the statements above, Dominique Cerveau in [1, Section 2.10] proposes to study which conditions over a logarithmic foliation \mathcal{F} on a complex projective manifold are sufficient for a generic leaf \mathcal{L} of \mathcal{F} to satisfy these topological properties.

In this article, we shall study this issue using Homotopy Theory. Our first result exhibits a relation between the homotopy groups of a generic leaf \mathcal{L} and of the complement $X - D$ of the polar divisor D of the closed logarithmic 1-form ω defining the foliation \mathcal{F} on a complex projective manifold X .

THEOREM 1.1. — *Let $\mathcal{F}, \omega, D, X$ satisfy the above assumptions, with $\dim_{\mathbb{C}} X = n + 1$ and $n > 1$. If D is a simple normal crossing ample divisor, then the fundamental group of \mathcal{L} is isomorphic to the group*

$$G := \left\{ [\gamma] \in \pi_1(X - D) \mid \int_{\gamma} \omega = 0 \right\},$$

where γ is a closed curve in $X - D$. Furthermore, the morphisms of homotopy groups

$$i_* : \pi_l(\mathcal{L}) \rightarrow \pi_l(X - D),$$

induced by the inclusion $i : \mathcal{L} \hookrightarrow X - D$ are isomorphisms if $1 < l < n$ and epimorphisms if $l = n$.

If X is the projective space \mathbb{P}^{n+1} , we prove the following Lefschetz hyperplane section type theorem.

THEOREM 1.2. — *Let \mathcal{F} be a logarithmic foliation defined by a logarithmic 1-form ω on \mathbb{P}^{n+1} , $n \geq 1$, with a simple normal crossing polar divisor D . Let $H \subset \mathbb{P}^{n+1}$ be a hyperplane such that $H \cap D$ is a reduced divisor with simple normal crossings in H . Suppose the leaves $\mathcal{L}, \mathcal{L} \cap H$ are generic leaves of $\mathcal{F}, \mathcal{F}|_H$ respectively. Then the morphism between homotopy groups*

$$(i)_* : \pi_l(\mathcal{L} \cap H) \rightarrow \pi_l(\mathcal{L}),$$

induced by the inclusion $i : \mathcal{L} \cap H \hookrightarrow \mathcal{L}$ is

- (1) an isomorphism if $l < n - 1$,
- (2) an epimorphism if $l = n - 1$.

This result shows that the claims (ii) and (iii) are true for generic leaves of logarithmic foliations on \mathbb{P}^{n+1} . For generic logarithmic foliations on \mathbb{P}^{n+1} ,

Theorem 1.1 implies that this foliations have simply connected generic leaves.

In order to prove these results we shall adapt a result of Carlos Simpson [9, Corollary 21], which concerns the topology of integral varieties of a closed holomorphic 1-form on a projective variety.

2. Generic leaves of logarithmic foliations

Let ω be a closed logarithmic 1-form on a complex projective manifold X with polar divisor $D = \sum D_i$. Note that for any point $p \in X$ there is a neighborhood U of p in X such that $\omega|_U$ can be written as

$$(2.1) \quad \omega_0 + \sum_{j=1}^r \lambda_j \frac{df_j}{f_j},$$

where ω_0 is a closed holomorphic 1-form on U , $\lambda_j \in \mathbb{C}^*$ and $f_j \in \mathcal{O}(U)$, and $\{f_j = 0\}$, $j = 1, \dots, r$, are the reduced equations of the irreducible components of $D \cap U$.

If the polar divisor $D \subset X$ is simple normal crossing, then each irreducible component of D is smooth and locally near of each point D can be represented in a chart $(x_i) : U \rightarrow M$ as the locus $\{x_0 \cdots x_k = 0\}$ with $k + 1 \leq \dim(X)$. Moreover, there is a coordinate chart $(U, (y_j))$ for each point $q \in X$ such that ω can be written as

$$(2.2) \quad ((y_j)^{-1})^* \omega = \sum_{j=0}^k \lambda_j \frac{dy_j}{y_j}.$$

Consider a singular point p of ω not in $\text{Sing}(D)$. Since D is simple normal crossing, (2.2) shows that the connected component S_p of $\{x \in X \mid \omega(x) = 0\}$, which contains p , has empty intersection with D (see [2, Theorem 3] for more details). When D is an ample divisor in X , then any complex subvariety of dimension greater than zero has nonempty intersection with D . In particular, we have the following result.

LEMMA 2.1. — *Let ω, p, S_p be as above. If D is a simple normal crossing ample polar divisor of ω , then S_p is the isolated point p .*

A closed logarithmic 1-form ω on X with polar divisor D defines the following group morphism

$$(2.3) \quad \begin{aligned} \phi : \pi_1(X - D) &\rightarrow (\mathbb{C}, +) \\ [\gamma] &\mapsto \int_\gamma \omega, \end{aligned}$$

where γ is a closed curve in $X - D$. Consider a normal subgroup G of $\pi_1(X - D)$, which is contained in $\ker \phi$. The group G defines a regular covering space $\rho : Y \rightarrow X - D$ with the property that $\rho_*(\pi_1(Y)) = G$. Taking a closed curve $\eta : I \rightarrow Y$ we see that $[\rho \circ \eta] \in \ker \phi$. We thus get

$$\int_{\eta} \rho^* \omega = \int_{\rho \circ \eta} \omega = 0.$$

Consequently, for a fixed point $y_0 \in Y$ the function

$$(2.4) \quad g(y) = \int_{y_0}^y \rho^* \omega$$

is well defined for $y \in Y$. In this way, we obtain what will be referred to as a ω -exact covering space of $X - D$.

If the kernel of ϕ is a non trivial group, then there are at least two ω -exact covering spaces of $X - D$: the universal cover and the regular cover $\rho : Y \rightarrow X - D$ such that $\rho_*(\pi_1(Y)) = \ker \phi$.

The following theorem is an adaptation of [9, Corollary 21], which allows us to exhibit relations between the homotopy groups of a cover of an integral manifold of ω and the regular cover of the complement of the polar divisor of ω .

THEOREM 2.2 (Lefschetz–Simpson Theorem). — *Let ω be a closed logarithmic 1-form on a projective manifold X of dimension $n + 1$, $n \geq 1$. Assume that the polar divisor D of ω is simple normal crossing ample divisor. Then for a ω -exact covering space $\rho : Y \rightarrow X - D$ and a function g (2.4), the pair $(Y, g^{-1}(c))$ is n -connected for any $c \in \mathbb{C}$.*

The n -connectedness of the pair $(Y, g^{-1}(c))$ is equivalent to the morphisms of homotopy groups

$$\pi_i(g^{-1}(c)) \rightarrow \pi_i(Y),$$

induced by the inclusion $g^{-1}(c) \hookrightarrow Y$ being isomorphisms if $i < n$ and epimorphism for $i = n$.

The statement above will be proved in the next section. We will use this theorem and standard techniques of Homotopy Theory to prove the Theorems 1.1 and 1.2.

Now, we study the homotopy groups of generic leaves \mathcal{L} of \mathcal{F} . The following relation between homotopy groups of a path connected topological space Y and its regular covering space $\rho : \tilde{Y} \rightarrow Y$ will be very useful.

The covering space projection ρ induces isomorphisms

$$\rho_* : \pi_i(\tilde{Y}, \tilde{y}_0) \rightarrow \pi_i(Y, y_0),$$

with $\tilde{y}_0 \in \rho^{-1}(y_0)$, between homotopy groups of dimension greater than 1.

DEFINITION 2.3. — Let \mathcal{F} be a logarithmic foliation on X defined by a closed logarithmic 1-form ω with polar divisor D . Let $\rho : Y \rightarrow X - D$ and $g : Y \rightarrow \mathbb{C}$ be as above. We will say that a leaf \mathcal{L} of \mathcal{F} is generic if for some component C of $\rho^{-1}(\mathcal{L})$ the value of g on C is a regular value of g .

PROPOSITION 2.4. — Under the conditions stated above, we assume that the polar divisor D of ω is a simple normal crossing ample divisor. Then there exists a ω -exact covering space $\rho : Y \rightarrow X - D$ such that for a primitive g of $\rho^*\omega$ defined by (2.4) the inverse image $g^{-1}(c)$ of a regular value $c \in \mathbb{C}$ is biholomorphic to a generic leaf \mathcal{L} of \mathcal{F} .

Proof. — Consider the ω -exact covering space $\rho : Y \rightarrow X - D$ satisfying $\rho_*(\pi_1(Y)) = \ker \phi$, with ϕ defined by (2.3). Write $G = \ker \phi$.

We will show that for a regular value $c \in \mathbb{C}$ of g the restriction

$$\rho|_{g^{-1}(c)} : g^{-1}(c) \rightarrow \mathcal{L}$$

is a biholomorphism.

Suppose not. Then there exist distinct points $y_0, y_1 \in \rho^{-1}(x_0)$, with $x_0 \in \mathcal{L}$, such that $y_0, y_1 \in g^{-1}(c)$.

Take $\tilde{\gamma} : I \rightarrow Y$ with $\tilde{\gamma}(0) = y_0$ y $\tilde{\gamma}(1) = y_1$. Since $n \geq 1$, Theorem 2.2 implies that the pair $(Y, g^{-1}(c))$ is 1-connected. This implies that there exists γ' contained in $g^{-1}(c)$ homotopic to $\tilde{\gamma}$ with fixed endpoints. Therefore $\gamma = \rho \circ \gamma'$ is a curve in \mathcal{L} which is not homotopically trivial in $X - D$. But since it is contained in a leaf of the foliation we have that

$$\int_{\gamma} \omega = 0.$$

Hence γ is homotopic to an element of G a contradiction. Thus $\rho|_{g^{-1}(c)}$ is a biholomorphism. This is the desired conclusion. □

Proof of Theorem 1.1. — Let $\rho : Y \rightarrow X - D$ be the ω -exact cover given by Proposition 2.4. By Theorem 2.2, the morphisms

$$((\rho|_{g^{-1}(c)})^{-1} \circ i)_* : \pi_l(\mathcal{L}) \rightarrow \pi_l(Y)$$

are isomorphisms if $l < n$ and epimorphisms if $l = n$, where the map $\rho|_{g^{-1}(c)}$ is the biholomorphism between $g^{-1}(c)$ and \mathcal{L} , and $i : \mathcal{L} \hookrightarrow X - D$ is the inclusion map. As n is greater than 1 we have that $\pi_1(\mathcal{L})$ is isomorphic to

$$\pi_1(Y) \cong \left\{ [\gamma] \in \pi_1(X - D) \mid \int_{\gamma} \omega = 0 \right\}.$$

Since Y is a covering space of $X - D$, it follows that the morphisms $(\rho)_*$ between the groups $\pi_l(Y)$ and $\pi_l(X - D)$ are isomorphisms if $l > 1$. Therefore the morphisms

$$i_* : \pi_l(\mathcal{L}) \rightarrow \pi_l(X - D)$$

are isomorphisms if $1 < l < n$ and epimorphisms if $l = n$. □

We will use the statements in [4, 5, 7] about the topology of the complement of a divisor in a projective manifold to establish the results below.

Let H be an abelian free group generated by the components D_i of a divisor $D \subset X$. If X is simply connected and each irreducible component of the simple normal crossing divisor $D = \sum_{i=0}^k D_i$ is ample, Corollary 2.2 of [7] implies that the fundamental group of the complement $X - D$ is isomorphic to the cokernel of the morphism

$$\begin{aligned} h : H_2(X, \mathbb{Z}) &\rightarrow H \\ a &\mapsto \sum_{i=0}^k (a, D_i) D_i, \end{aligned}$$

where (a, D_i) is the Kronecker pairing, here we associate D_i with its Chern class in $H^2(X, \mathbb{Z})$ (see [3, p. 15] for more details). If X is the projective space \mathbb{P}^{n+1} , then the image of the morphism h is generated by (d_0, \dots, d_k) , where d_i is the degree of D_i . In particular, we have the following result

COROLLARY 2.5. — *Under the hypotheses of Theorem 1.1, if $X = \mathbb{P}^{n+1}$ then the fundamental group of a generic leaf \mathcal{L} is isomorphic to the following group*

$$\left\{ (m_0, \dots, m_k) \in \mathbb{Z}^{k+1} \mid \sum_{i=0}^k \lambda_i m_i = 0 \right\} / \mathbb{Z}(d_0, \dots, d_k),$$

where $\lambda_i = \text{Res}(D_i, \omega)$ are the residues of ω around D_i .

Example 2.6. — Let us consider the case where the polar divisor of the logarithmic 1-form ω has only two irreducible components, say D_0 and D_1 . If the degrees d_0, d_1 are equal then the leaves \mathcal{L} of the foliation \mathcal{F} are contained in elements of the pencil

$$\{aF_0 + bF_1 \mid (a : b) \in \mathbb{P}^1\}.$$

In particular the generic leaf \mathcal{L} is of the form $\{aF_0 + bF_1 = 0\} - D$ for $(a : b)$ generic. The Corollary 2.5 implies that

$$\pi_1(\mathcal{L}) = \frac{\mathbb{Z}}{d\mathbb{Z}},$$

which is the torsion subgroup of $\pi_1(\mathbb{P}^3 - D)$.

COROLLARY 2.7. — *Let X^{n+1} be a complete intersection in \mathbb{P}^N . Let D be an arrangement of hyperplanes in \mathbb{P}^N such that $D \cap X$ is simple normal crossing in X . If a logarithmic foliation \mathcal{F} on X has polar divisor $D \cap X$ then any generic leaf of \mathcal{F} satisfies $\pi_l(\mathcal{L}) = 0$ for $1 < l < n$.*

Proof. — From Theorem 1.1 the homotopy groups of dimension l , with $1 < l < n$, of a generic leaf are isomorphic to the respective homotopy groups of $X - D$. By [7, Theorem 2.4] the homotopy groups of $X - D$ of dimension $1 < l < n$ are trivial, and the corollary follows. \square

Let ω be a closed logarithmic 1-form as in Corollary 2.5. The residues $\{\lambda_j\}_{j=0}^k$ of ω are non resonant if all integer solutions $(m_0, \dots, m_k) \in \mathbb{Z}^{k+1}$ of the equation $\sum_{j=0}^k m_j \lambda_j = 0$ are contained in $\mathbb{Z}(d'_0, \dots, d'_k)$, where $d'_j \cdot \gcd(d_0, \dots, d_k) = d_j$.

If the residues $\{\lambda_j\}_{j=0}^k$ are non resonant and $\gcd(d_0, \dots, d_k) = 1$, the Corollary 2.5 implies that the generic leaf \mathcal{L} of the foliation \mathcal{F} defined by ω has trivial fundamental group. Therefore we deduce from Corollary 2.7,

COROLLARY 2.8. — *Under the assumptions of Corollary 2.5, if moreover $d_j = 1$ and the residues λ_i are non resonant. Then the generic leaf \mathcal{L} of the foliation \mathcal{F} is $(n - 1)$ -connected.*

Next result gives a relation between the homotopy groups of a generic leaf of a logarithmic foliation on \mathbb{P}^{n+1} and its general hyperplane sections.

Proof of Theorem 1.2. — Let $D(H) = H \cap D$. The inclusion i from $H - D(H)$ to $\mathbb{P}^{n+1} - D$ induces the morphisms

$$(2.5) \quad i_* : \pi_l(H - D(H)) \rightarrow \pi_l(\mathbb{P}^{n+1} - D)$$

in homotopy. From the Lefschetz–Zariski type [5, Theorem 0.2.1] we have that i_* is an isomorphism for $l < n$ and an epimorphism for $l = n$.

Consider the regular cover $\rho : Y \rightarrow \mathbb{P}^{n+1} - D$ given by Proposition 2.4. Let g be a primitive of $\rho^*\omega$. Let $Y(H) = \rho^{-1}(H - D(H))$. Notice that $Y(H)$ is a connected regular covering space of $H - D(H)$. Let g_H be the restriction of g to $Y(H)$. Let $c \in \mathbb{C}$ be a regular value of g and g_H . Since $l \leq n - 1$, Theorem 2.2 implies that the morphisms

$$i_* : \pi_l(g^{-1}(c)) \rightarrow \pi_l(Y) \quad \text{and} \quad i_* : \pi_l(g_H^{-1}(c)) \rightarrow \pi_l(Y(H))$$

induced by the inclusion $Y(H) \hookrightarrow Y$, are isomorphisms if $l < n - 1$ and epimorphisms if $l = n - 1$. Considering the long exact sequence of homotopy

groups we obtain the following commutative diagram for $l > 0$:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & \pi_{l+1}(Y(H), g_H^{-1}(c)) & \longrightarrow & \pi_l(g_H^{-1}(c)) & \longrightarrow & \pi_l(Y(H)) \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & \pi_{l+1}(Y, g^{-1}(c)) & \longrightarrow & \pi_l(g^{-1}(c)) & \longrightarrow & \pi_l(Y) \longrightarrow \cdots
 \end{array}$$

Since $Y, Y(H)$ are covers of $\mathbb{P}^{n+1} - D, H - D(H)$ respectively, the morphism (2.5) shows that the morphisms

$$\pi_l(Y(H)) \rightarrow \pi_l(Y)$$

are isomorphisms for $l < n$ and epimorphisms for $l = n$. Analogously, Theorem 2.2 implies that the morphisms $\pi_l(Y(H), g_H^{-1}(c)) \rightarrow \pi_l(Y, g^{-1}(c))$ are isomorphisms for $l < n$ and epimorphisms for $l = n$. Applying the five Lemma, we have that the morphisms $\pi_l(g_H^{-1}(c)) \rightarrow \pi_l(g^{-1}(c))$ are isomorphisms for $l < n - 1$ and epimorphisms for $l = n - 1$. Hence the theorem follows from the biholomorphism given by Proposition 2.4. \square

This verifies the claim (iii) for generic leaves of logarithmic foliations on projective spaces.

3. Lefschetz–Simpson Theorem

In order to prove the Theorem 1.2 we adapt the proof of [9, Theorem 1] to the case of logarithmic closed 1-forms with a simple normal crossing ample polar divisor. One of the key steps in the proof of Theorem 2.2 consists in establishing an Ehresmann type result for the function g outside an open neighborhood of the singular locus of $\rho^*\omega$. Before proving Theorem 2.2 we will need to introduce some notation and to establish some preliminary results. Also, we follow the notation used in [9].

We will use some properties in [6, 9] about Homotopy Theory to establish the statements bellow.

Singular Theory

Under the hypotheses of Theorem 2.2, the Lemma 2.1 implies that the singular locus of ω in $X - D$ is a finite union of isolated points. Let $\{p_i\}$ be the finite set of isolated singularities of ω in $X - D$. Fix a metric μ on

X . Since X is compact, μ is complete. We can choose $\varepsilon_1 > 0$ sufficiently small such that the closed balls

$$B_\mu(p_i, \varepsilon_1) = M_i$$

are pairwise disjoint and the restriction of ω to an open neighborhood of M_i is exact. We define primitives $g_i(x) = \int_{p_i}^x \omega$ for $x \in M_i$. Since the points p_i are isolated singularities, it follows from [8, Theorems 4.8, 5.10] the existence of $\varepsilon_2 > 0$ sufficiently small such that

- (1) $0 \in B(0, \varepsilon_2) \subset \mathbb{C}$ is the unique critical value for the primitive g_i ;
- (2) the intersections $g_i^{-1}(0) \cap \partial M_i$ and $g_i^{-1}(B(0, \varepsilon_2)) \cap \partial M_i = T_i$ are smooth, and the restriction of ω to T_i is a 1-form on T_i which never vanishes.

LEMMA 3.1. — *Let $F_i = g_i^{-1}(0)$ and $E_i = g_i^{-1}(c)$ with $c \in B(0, \varepsilon_2) - \{0\}$ be fibers of g_i restricted to*

$$N_i = M_i \cap g_i^{-1}(B(0, \varepsilon_2)).$$

For small ε_2 the pair (N_i, F_i) is l -connected for every $l \in \mathbb{N}$ and the pair (N_i, E_i) is n -connected.

Proof. — For ε_2 sufficiently small [8, Theorem 5.2] implies that F_i is a deformation retract of N_i . Therefore the pair (N_i, F_i) is l -connected for any $l \in \mathbb{N}$.

We know from [8, Theorems 5.11, 6.5] that E_i has the homotopy type of a bouquet of spheres $\mathbb{S}^n \vee \dots \vee \mathbb{S}^n$ for ε_2 sufficiently small. Thus the fiber E_i is $(n - 1)$ -connected. Since the neighborhood N_i can be contracted to p_i the long exact sequence of Homotopy Theory implies that the pair (N_i, E_i) is n -connected. □

Ehresmann type result

Let $\rho : Y \rightarrow X - D$ be a ω -exact covering space and the function g a primitive of $\rho^*\omega$. We will use $j \in J_i$ as an index set for the points \tilde{p}_j of the discrete set $\rho^{-1}(p_i)$ and we will denote the union $\cup J_i$ by J . Fix $\varepsilon_1, \varepsilon_2 > 0$ sufficiently small such that

- in each connected component \widetilde{M}_j of $\rho^{-1}(M_i)$ containing the point \tilde{p}_j , the restriction of ρ in \widetilde{M}_j is a biholomorphism; and
- the subsets N_i, T_i, F_i, E_i and the function g_i satisfy the properties above for every i .

We define a primitive $\tilde{g}_j = g_i \circ \rho$ for the restriction of $\rho^*\omega$ to \tilde{M}_j such that $g|_{\tilde{M}_j} = \tilde{g}_j + a_j$ for some $a_j \in \mathbb{C}$, with $j \in J_i$. The subsets $\tilde{N}_j, \tilde{T}_j, \tilde{F}_j, \tilde{E}_j$ of \tilde{M}_j are the analogues of the subsets N_i, T_i, F_i, E_i of M_i .

We choose δ such that $0 < 5\delta < \varepsilon_2$. For each $b \in \mathbb{C}$, we define the subset $J(b)$ of J formed by the indexes j such that $|b - a_j| < 3\delta$. Let $U_b = B(b, \delta) \subset \mathbb{C}$ and define the open subset of the covering space Y

$$W(b) = g^{-1}(U_b) \cap \left(\bigcup_{j \in J(b)} \tilde{N}_j^\circ \right),$$

where \tilde{N}_j° denotes the interior of \tilde{N}_j , which satisfies $(g^{-1}(U_b) - W(b)) \cap \overline{W(b)}$ is contained in $\bigcup_{j \in J(b)} \tilde{T}_j$.

We can now formulate our Ehresmann type result.

PROPOSITION 3.2. — *There exists a trivialization of $g^{-1}(U_b) - W(b)$ with trivializing diffeomorphism*

$$\Phi : U_b \times (g^{-1}(b) - W(b)) \rightarrow g^{-1}(U_b) - W(b),$$

such that the restriction to the boundary satisfies

$$\Phi(U_b \times (g^{-1}(b) - W(b)) \cap \overline{W(b)}) = (g^{-1}(U_b) - W(b)) \cap \overline{W(b)}.$$

Proof. — For each point q in the polar divisor $D \subset X$ of the logarithmic 1-form ω , we have a coordinate chart $(V(q), \psi)$ such that

$$(3.1) \quad \omega = \psi^* \left(\sum_{j=1}^{r(q)} \lambda_j \frac{dy_j}{y_j} \right).$$

We can take a finite number of points $q_\beta \in D$ with coordinate charts (V_β, ψ_β) satisfying (3.1) and such that the union $\bigcup_\beta V_\beta$ covers D .

Let $U_i \subset N_i$ be open balls containing the singular points of ω in $X - D$ such that the diameter of $g_i(U_i)$ is smaller than $\delta/10$. Using partition of unity we construct two C^∞ complete real vector fields u, v on X such that

- (1) their restrictions in V_β satisfy that

$$D\psi_\beta(u) = \sum_{j=1}^{r(q_\beta)} \frac{y_j}{\lambda_j} \frac{\partial}{\partial y_j}, \quad D\psi_\beta(v) = \sqrt{-1} \sum_{j=1}^{r(q_\beta)} \frac{y_j}{\lambda_j} \frac{\partial}{\partial y_j},$$

- (2) at any point p in $X - (\bigcup_\beta V_\beta \cup \bigcup_i U_i)$ they satisfy $\omega(u_p) = 1$, $\omega(v_p) = \sqrt{-1}$ and if p belongs to T_i then these vector fields are tangent to T_i ;
- (3) their restriction to $\overline{U_i}$ vanishes in p_i .

The vector fields u, v leave the divisor D invariant. It follows that the restriction of u, v to $X - D$ are still complete vector fields and they satisfy $\omega(u) = 1, \omega(v) = \sqrt{-1}$ outside of $D \cup (\cup_i U_i)$.

Let \tilde{u}, \tilde{v} be the liftings of u, v with respect to Y . Notice that the vector fields \tilde{u}, \tilde{v} are complete vector fields on Y , which restricted to $g^{-1}(U_b) - W(b)$ satisfy $\rho^*\omega(\tilde{u}) = 1, \rho^*\omega(\tilde{v}) = \sqrt{-1}$. It implies the existence of the diffeomorphism

$$\begin{aligned} \Phi : U_b \times (g^{-1}(b) - W(b)) &\rightarrow g^{-1}(U_b) - W(b) \\ (t_1 + b, t_2 + b) \times \{q\} &\mapsto \Phi_1(t_1, \Phi_2(t_2, q)), \end{aligned}$$

where Φ_1, Φ_2 are flows of \tilde{u}, \tilde{v} , respectively. The vector fields \tilde{u}, \tilde{v} are tangent to \tilde{T}_j for every $j \in J$. In particular, they are tangent to $\cup_{j \in J(b)} \tilde{T}_j$. It follows that

$$\Phi(U_b \times (g^{-1}(b) - W(b)) \cap \overline{W(b)}) = (g^{-1}(U_b) - W(b)) - (g^{-1}(\partial U_b) \cap \overline{W(b)})$$

as we wanted. □

Example 3.3. — Let ω be a closed logarithmic 1-form on \mathbb{P}^{n+1} , with a simple normal crossing polar divisor $D = H_0 + \dots + H_k$ with $1 \leq k \leq n + 1$. Let H_j be hyperplanes defined by $H_j = \{z_j = 0\}$ where $[z_0 : \dots : z_{n+1}]$ are homogeneous coordinates for \mathbb{P}^{n+1} . Take the universal covering

$$\begin{aligned} \rho : \mathbb{C}^{n+1} &\rightarrow \mathbb{P}^{n+1} - D \\ [1 : x_1 : \dots : x_{n+1}] &\mapsto [1 : e^{2\pi\sqrt{-1}x_1} : \dots : e^{2\pi\sqrt{-1}x_k} : x_{k+1} : \dots : x_{n+1}]. \end{aligned}$$

If we denote the residues by $\text{Res}(\omega, H_j) = \lambda_j$, then the pull-back $\rho^*\omega$ admits the following expression

$$2\pi\sqrt{-1} \sum_{j=0}^k \lambda_j dx_j,$$

which is a linear 1-form on \mathbb{C}^{n+1} . In this case, there are no singularities outside the divisor and the primitive g of $\rho^*\omega$ is a fibration of \mathbb{C}^{n+1} with fiber \mathbb{C}^n . In particular, the pair $(\mathbb{C}^{n+1}, g^{-1}(c))$ is l -connected for every l .

Example 3.4. — Consider the closed rational 1-form

$$\omega = d \left(\frac{x^2 + y^2 + z^2}{xy} \right)$$

in homogeneous coordinates $[x : y : z]$ of \mathbb{P}^2 . The polar divisor D of ω has only two irreducible components $D_0 = \{x = 0\}, D_1 = \{y = 0\}$, with $D = 2D_0 + 2D_1$. The singularities of ω outside of D are the points $p_1 = [1 : 1 : 0], p_2 = [-1 : 1 : 0]$.

The 1-form ω is exact in $\mathbb{P}^2 - D$. The leaves of the foliation \mathcal{F} defined by ω in $\mathbb{P}^2 - D$ coincide with

$$\{x^2 + y^2 + z^2 - \alpha xy = 0\} - D \quad \text{with } \alpha \in \mathbb{C}.$$

If we assume that Proposition 3.2 is true in this situation, we would have for $\delta > 0$ sufficiently small a diffeomorphism

$$\Phi : g^{-1}(B(2, \delta)) - W(2) \cong B(2, \delta) \times (g^{-1}(2) - W(2)).$$

But this is impossible since the set $g^{-1}(2)$ consists of two lines and the set $g^{-1}(2) - W(2)$ is not connected and the set $g^{-1}(B(2, \delta)) - W(2)$ is connected.

The construction of the vector field used to prove Proposition 3.2 fails in this case, since at the singular points $q_1 = [1 : 0 : 1]$, $q_2 = [-1 : 0 : 1]$, $q_3 = [0 : 1 : 1]$, $q_4 = [0 : -1 : 1]$ the vector field

$$u = \frac{x^2 y}{x^2 - y^2 - z^2} \frac{\partial}{\partial x} + \frac{y^2 x}{y^2 - x^2 - z^2} \frac{\partial}{\partial y} + \frac{2z}{xy} \frac{\partial}{\partial z}$$

cannot be extended.

The proof of Theorem 2.2

Define the following sets

$$\begin{aligned} P(b, V) &= g^{-1}(V) \cup W(b), & R(b) &= g^{-1}(U_b) - W(b), \\ P^R(b, V) &= P(b, V) - W(b), \end{aligned}$$

where V is contained in U_b .

LEMMA 3.5. — *Let $V \subset U_b$ be a contractible subset. If there exists a continuous map $\xi : U_b \times [0, 1] \rightarrow U_b$ such that $\xi(y, 0) = y$, and the sets $\xi(V \times [0, 1])$, $\xi(U_b \times \{1\})$ lie in V . Then the pair $(g^{-1}(U_b), g^{-1}(V))$ is n -connected.*

Proof. — For each \tilde{T}_j with $j \in J(b)$, we can choose a vector field ν_j tangent to the level sets of g and pointing to the interior of $W(b)$. The vector field ν on $\partial W(b)$ defined by $\nu|_{\tilde{T}_j} + \nu_j$ allows us to construct a deformation $h : W(b) \times [0, 1] \rightarrow W(b)$ such that $h(y, 0) = y$, and the image of $h(W(b) \times \{1\}) = W'(b)$ has empty intersection with $R(b)$.

The map $h(y, 1 - t)$ gives us a deformation of the pair $(g^{-1}(U_b) - W'(b), P(b, V) - W'(b))$ to the pair $(R(b), P^R(b, V))$. By [9, 5.5 Excision II], the pairs $(g^{-1}(U_b), P(b, V))$ and $(R(b), P^R(b, V))$ have the same l -connectivity.

Therefore, Proposition 3.2. implies that the pair $(g^{-1}(U_b), P(b, V))$ is l -connected for every l .

Now, consider the pair $(\tilde{N}_j^\circ \cap g^{-1}(U_b), \tilde{N}_j^\circ \cap g^{-1}(V)) = (U_{b,j}, V_j)$ with $j \in J(b)$. Since V is contractible and the restriction of g to $\tilde{N}_j - \tilde{F}_j$ is a trivial fibration, Lemma 3.1 and Property [9, 5.3 Deformation] imply that the pair $(U_{b,j}, V_j)$ is l -connected for every l if $\tilde{F}_j \subset V_j$, and n -connected if \tilde{F}_j is not contained in V_j . Therefore the pair $(W(b), \cup_{j \in J(b)} V_j)$ is at least n -connected.

The vector field $-\nu$ points toward the interior of $R(b)$. Analogously, we define a deformation $h' : R(b) \times [0, 1] \rightarrow R(b)$ such that the closure of the image $h'(R(b) \times \{1\}) = R'(b)$ has empty intersection with $\overline{W(b)}$. Considering the set $R'(b) \cap g^{-1}(V)$ and the tangency of ν to the level sets of g , [9, 5.5 Excision II] implies that the pair $(P(b, V), g^{-1}(V))$ is n -connected. Since $(g^{-1}(U_b), P(b, V))$ is l -connected for every l , by [9, 5.2 Transitivity] the pair $(g^{-1}(U_b), g^{-1}(V))$ is n -connected. \square

Proof of Theorem 2.2. — Take a triangulation Δ of \mathbb{C} by equilateral triangles with sides of length δ , such that one of the vertices in V_Δ is $c \in \mathbb{C}$. Let H_l be the family of concentric hexagons with center c and vertices in V_Δ . Label by c_i the vertices V_Δ such that between c_i and c_{i+1} there always exists an edge $e_i \in E_\Delta$ of the triangulation Δ , and $c_0 = c$. Also, the vertices c_i with $6(l-1)l/2 < i \leq 6l(l+1)/2$ are in the hexagon H_l .

Consider the open sets $U_i = B_{\mathbb{C}}(c_i, \delta)$ and $W_i = \cup_{j \leq i} U_j$. Since the intersection $U_i \cap W_{i-1} = V_i$ is contractible in U_i , Lemma 3.5 gives that the pair $(g^{-1}(U_i), g^{-1}(V_i))$ is n -connected.

The W_i -closures of the sets $(W_i - W_{i-1})$ and $(W_{i-1} - U_i)$ are disjoint, thus the previous paragraph combined with [9, 5.4 Excision] imply that the pair $(g^{-1}(W_i), g^{-1}(W_{i-1}))$ is n -connected for every i . From [9, 5.2 Transitivity] we deduce that the pair $(g^{-1}(W_i), g^{-1}(W_0))$ is n -connected for every i . Taking $V = c$ in Lemma 3.5, we see that $(g^{-1}(W_0), g^{-1}(c))$ is n -connected. Hence $(g^{-1}(W_i), g^{-1}(c))$ is n -connected for all i . As any representative element of a class in $\pi_l(Y, g^{-1}(c))$ is contained in some pair $(g^{-1}(W_i), g^{-1}(c))$ we conclude that the pair $(Y, g^{-1}(c))$ is n -connected. \square

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