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A GEOMETRIC APPROACH TO CATLIN’S BOUNDARY SYSTEMS

by Dmitri ZAITSEV

ABSTRACT. — For a point p in a smooth real hypersurface $M \subset \mathbb{C}^n$, where the Levi form has the nontrivial kernel $K_p^{10} \neq 0$, we introduce an invariant cubic tensor

$$\tau_p^3 : \mathbb{C}T_p \times K_p^{10} \times \overline{K_p^{10}} \rightarrow \mathbb{C} \otimes (T_p/H_p),$$

which together with Ebenfelt’s 3rd order tensor, constitutes the full set of the 3rd order invariants of M at p .

Next, in addition, assume $M \subset \mathbb{C}^n$ to be (*weakly*) pseudoconvex. Then τ_p^3 must identically vanish. In this case we further define an invariant quartic tensor

$$\tau_p^4 : \mathbb{C}T_p \times \mathbb{C}T_p \times K_p^{10} \times \overline{K_p^{10}} \rightarrow \mathbb{C} \otimes (T_p/H_p),$$

and for every $q = 0, \dots, n-1$, an invariant submodule sheaf $\mathcal{S}^{10}(q)$ of $(1,0)$ vector fields in terms of the Levi form, and an invariant ideal sheaf $\mathcal{I}(q)$ of complex functions generated by certain components and derivatives of the Levi form, such that the set of points of Levi rank q is locally contained in real smooth submanifolds defined by real parts of the functions in $\mathcal{I}(q)$, whose tangent spaces have explicit algebraic description in terms of the quartic tensor τ^4 .

Most recently, the constructions of τ^3 and τ^4 inspired Raich and Harrington [35] to discover new related invariants in the non-pseudoconvex case and connect them with the closed range property for the $\bar{\partial}$ operator.

Finally, we relate the introduced invariants with D’Angelo’s finite type, Catlin’s multitype and Catlin’s boundary systems.

RÉSUMÉ. — Soit p un point d’une hypersurface $M \subset \mathbb{C}^n$, réelle et lisse, en lequel la forme de Levi possède un noyau non trivial, noté $K_p^{10} \neq 0$. On introduit alors un tenseur d’ordre 3, invariant

$$\tau_p^3 : \mathbb{C}T_p \times K_p^{10} \times \overline{K_p^{10}} \rightarrow \mathbb{C} \otimes (T_p/H_p),$$

qui, avec le tenseur d’ordre 3 d’Ebenfelt, fournit l’ensemble de tous les tenseurs d’ordre 3 invariants de M au point p .

Si, de plus, on suppose que M est pseudo-convexe, alors τ_p^3 est identiquement nul. On définit alors un tenseur d’ordre 4, invariant

$$\tau_p^4 : \mathbb{C}T_p \times \mathbb{C}T_p \times K_p^{10} \times \overline{K_p^{10}} \rightarrow \mathbb{C} \otimes (T_p/H_p).$$

Keywords: Catlin multitype, subelliptic estimates, boundary systems, Levi form, pseudoconvexity, real hypersurfaces, invariant tensors, ideal sheaves.

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On introduit également pour $q = 0, \dots, n-1$, un faisceau de sous-modules invariant, noté $\mathcal{S}^{1,0}(q)$, constitué de champs de vecteurs de type $(1, 0)$, construits à partir de la forme de Levi, ainsi qu'un faisceau d'idéaux invariant, noté $\mathcal{I}(q)$, constitué de fonctions complexes construites à partir de certaines composantes et dérivées de la forme de Levi. L'ensemble des points dont la forme de Levi est de rang q est alors contenu localement dans des sous-variétés réelles et lisses définies par des parties réelles de fonctions dans $\mathcal{I}(q)$, et dont les espaces tangents ont des descriptions algébriques explicites en termes du tenseur τ^4 .

Tout récemment, la construction de τ^3 et τ^4 a inspiré Raich et Harrington [35] qui ont découvert de nouveaux invariants associés à ces derniers dans le cas où M n'est pas pseudo-convexe et les ont reliés à la « closed range property » pour l'opérateur $\bar{\partial}$.

Enfin, on fait le lien entre ces invariants définis plus haut et le type fini au sens de D'Angelo, le multitype au sens de Catlin et les « boundary systems » de Catlin.

1. Introduction

1.1. Overview for broader audience

In this brief overview we put this paper's material in somewhat broader context. The methods and tools introduced here may be of interest for general systems of Partial Differential Equations (PDE), beyond the context of the $\bar{\partial}$ -equations. An evidence for this is the recent breakthrough paper by Siu [59] extending to general PDE systems the celebrated multiplier ideal technique of Kohn [46].

In the study of PDE systems and their solutions, an important general approach is that of the *a priori estimates* $\|u\|_1 \leq \|u\|_2$, where $\|\cdot\|_1, \|\cdot\|_2$ is a pair of semi-norms and u is a test function. Finding a priori estimates is typically a difficult problem whose solution in most cases relies on the specific nature of the system, with very few general approaches known. The multiplier ideal technique by Kohn [46] and the potential-theoretic approach by Catlin [8, 9], are ones of the few known general approaches for the $\bar{\partial}$ -Neumann problem, the boundary value problem for the $\bar{\partial}$ -equation with Neumann boundary conditions, see recent expositions in [11, 51, 58, 59, 62].

In both approaches, understanding singularities of the Cauchy–Riemann structure induced on the boundary by the ambient complex structure, is of utmost importance. Our goal here is to develop new geometric invariants to tackle this problem. A particular advantage of geometric invariants comes from the freedom of using them in arbitrary coordinates, as well as providing certain adapted coordinates, where the computations can be significantly simplified.

The invariants introduced are *tensors*, *ideal sheaves of functions* and *submodule sheaves of vector fields*. The tensors arise from derivatives of the Levi form, a fundamental invariant of the induced Cauchy–Riemann structure. However, only derivatives along certain special vector fields lead to invariant tensors, as illustrated by examples in this paper. This observation leads to the study of suitable submodule sheaves, where the vector fields must belong, in order to define tensors.

While being a powerful tool for computations in arbitrary coordinates, higher order tensors, as opposed to the Levi form, have the fundamental limitation of not well-behaving across singularities, since they are defined on kernels of varying dimension. In order to achieve some better behaviour and control, more flexible objects are needed, such as ideal and submodule sheaves mentioned above. In this paper we introduce invariant ideal sheaves generalizing functions in Catlin's boundary systems [8], whereas the invariant tensors control the transversality and nondegeneracy property of those functions.

1.2. Specialized overview

In more special terms, the goal of this paper is to introduce new geometric invariants giving insight into some techniques developed by Catlin in his celebrated papers [8, 9], following previous foundational work by Kohn [43, 44, 45, 46] on the $\bar{\partial}$ -Neumann problem. In particular, we introduce new invariant ideal sheaves containing functions arising in Catlin's boundary systems, and new invariant tensors permitting to simplify the iterative construction of the boundary systems by more direct computations of the tensors' kernels. The obtained geometric approach may lead to sharper subelliptic estimates, as well as to advance our understanding of the Kohn's multiplier ideal sheaves [46], in view of their relation with Catlin's technique, as indicated by Siu [57, 58, 59] (see also Nicoara [53]). Furthermore, in his recent fundamental work [59], Siu proposed new techniques for generating multipliers for general systems of partial differential equations, also including a new procedures even for the case of the $\bar{\partial}$ -Neumann problem.

1.3. Conditions of property (P) type

The importance of having a better understanding is further underlined by the role played by the Catlin's potential-theoretic “Property (P)” type

conditions (see e.g. [3, 30, 51, 61, 62] for recent surveys) that found vast applications in multiple research areas such as:

- (1) Compactness of the Kohn's $\bar{\partial}$ -Neumann solution operator by Henkin and Iordan [36], McNeal [50], Raich and Straube [55], Harrington [32, 34], Çelik and Şahutoğlu [12]. The compactness was even proved to be equivalent to Property (P) for Hartogs domains in \mathbb{C}^2 by Christ and Fu [20].
- (2) Subelliptic estimates by Fornæss and Sibony [27], Straube [60], Herbig [37], Harrington [32].
- (3) Invariant metric estimates due to Catlin [10], Cho [17, 18, 19], Boas–Straube–Yu [6], McNeal [49] and Herbort [38] and via subelliptic estimates by McNeal [48].
- (4) Stein neighborhood bases by Harrington [33] and Şahutoğlu [56].
- (5) Estimates and comparison of the Bergman and Szegő kernels by Boas [5], Nagel, Rosay, Stein and Wainger and [52], Boas–Straube–Yu [6], Charpentier and Dupain [13, 14, 15] Chen and Fu [16], Khanh and Raich [41].
- (6) Holomorphic Morse inequalities and eigenvalue asymptotics for $\bar{\partial}$ -Neumann Laplacian by Fu and Jacobowitz [29].
- (7) Tangential $\bar{\partial}_b$ and complex Green operator by Raich–Straube [55], Raich [54], Straube [63], Khanh, Pinton and Zampieri [40].
- (8) Construction of peak and bumping functions by Diederich and Herbort [24], Fornæss and McNeal [26], by Yu [65], Bharali and Stensønes [2], as well as some generalisations of Property (P) by Khanh–Zampieri [42].
- (9) Division problems for holomorphic functions with C^∞ boundary values by Bierstone and Milman [4], as an application of global regularity, whose proof for smooth finite type boundaries relies on Catlin's method.
- (10) Regularity of solutions to the complex Monge–Ampère equation by Ha and Khanh [31] and Baracco, Khanh, Pinton and Zampieri [1].

For bounded pseudoconvex domains with *real-analytic* boundaries of finite type in \mathbb{C}^n , Kohn's [46] celebrated theory of subelliptic multipliers provides an alternative approach to Catlin's in establishing subelliptic estimates. The same approach also yields subelliptic estimates for smoothly bounded domains of finite type in \mathbb{C}^2 , that were already treated by Kohn in his earlier paper [45]. However, for general *smoothly bounded domains of finite type*, it remains open at the time of writing, whether the multiplier

approach yields subelliptic estimates, with Catlin's method being currently the only available.

1.4. Submanifolds containing multitype level sets

A key geometric aspect of Catlin's subelliptic estimates proof consists of showing the existence of the so-called *weight functions* satisfying certain boundedness and positivity estimates for their complex Hessians, that are known as “Property (P)” type conditions (see e.g. [51, 3] for recent surveys). A major difficulty when constructing such weight functions under geometric conditions (such as finite type), is to keep the uniform nature of the estimates across points of varying “*degeneracy*” for the underlying geometry. A simple example of a degeneracy measure is the *rank of the Levi form* of the boundary $M := \partial D$ (where D is a domain in \mathbb{C}^n). A more refined measure is the Catlin multitype [8], see also Section 6. To deal with points of varying multitype, Catlin developed his machinery of boundary systems [8]. The main idea to gain control of the multitype level sets is by including them locally into certain “*containing submanifolds*”. A result of this type is the content of [8, Main Theorem, Part (2)], where a containing submanifold is constructed by a collection of inductively chosen *boundary system functions* that arise as certain carefully selected (vector field) derivatives of the Levi form.

In this paper we focus on geometric invariants behind the containing submanifold construction, with the goal to extend and simplify the boundary system approach. Our main discovery is that at the 4th order level, the boundary systems, as well as the type and the multitype, can be described in terms of the new invariant objects, such as tensors, submodule and ideal sheaves.

At the 4th order level, the multitype level sets boil down to simpler *level sets of the Levi (form) rank* (see Proposition 6.3 for details). Recall that Catlin's boundary system functions [8] are constructed inductively with every new equation depending on chosen solutions for previous ones. In comparison, we here collect defining functions for the Levi rank level sets into *invariant ideal sheaves* $\mathcal{I}(q)$ on M , for each Levi rank q . The sheaf $\mathcal{I}(q)$ is generated by certain 1st order Levi form derivatives as described in Theorem 2.1 below. In particular, arbitrary defining functions from $\mathcal{I}(q)$ can be combined without any additional relations. Furthermore, additional derivatives of the Levi form along arbitrary complex vector fields L^3 (in Theorem 2.1(5)), including transversal ones, are allowed for functions in

$\mathcal{I}(q)$. In comparison, for a related boundary system function given by the same formula, the outside vector field L^3 would have to be in a special subbundle inside the holomorphic tangent bundle. As a result, we obtain richer classes of functions allowing for more flexibility and control for the submanifolds containing Levi rank level sets (see Example 2.2), that may potentially lead to sharper a priori estimates.

In parallel to the ideal sheaf $\mathcal{I}(q)$ construction, we introduce *invariant quartic tensors* τ^4 , giving a precise control over differentials of the functions in $\mathcal{I}(q)$. This is expressed in Theorem 2.1(2), where the tangent space of the containing manifold S equals the real kernel of the tensor. Importantly, the full tangent space of S (rather than only the tangential part) is controlled here via the kernel of τ^4 , which means that vector fields transversal to the complex tangent must also be allowed among tensor arguments. The tensors are constructed in Lemma 4.14 as certain 2nd order Levi form derivatives, where the Levi form is first evaluated along certain special vector fields and then differentiated along all possible vector fields, including ones transversal to the complex tangent. In comparison, only derivatives with respect to $(1,0)$ and $(0,1)$ vector fields can appear in the boundary systems.

For reader's convenience, we summarize the main results and constructions in Theorem 2.1, leaving more detailed and general statements with their proofs in the chapters following.

1.5. More details on invariant tensors and ideal sheaves

Our first step in defining invariant tensors is a byproduct result giving a complete set of *cubic invariants* for a general real hypersurface M , without pseudoconvexity assumption. This is achieved by constructing an invariant cubic tensor τ^3 obtained by differentiating the Levi form along vector fields with values in the Levi kernel, see Lemma 3.4. Remarkably, to obtain tensoriality of the Levi form derivatives, it is of crucial importance to require *both vector fields inside the Levi form to take values in the Levi kernel* as explained in Example 3.1. This stands further, in remarkable contrast with the cubic tensor defined by Ebenfelt [25] (by means of the Lie derivatives of the contact form), where only one of the tensorial arguments needs to be in the Levi kernel. On the other hand, Ebenfelt's cubic tensor ψ_3 does not allow for transversal directions, in contrast to τ^3 . It turns out that the pair (ψ_3, τ^3) yields a complete set of cubic invariants, as demonstrated by the normal form (of order 3) in Proposition 3.6 eliminating all other terms that are not part of either of the tensors.

We also investigate the construction based on double Lie brackets (also considered by Webster [64]). This approach, however, in order to yield a tensor, has to require all vector fields to be in the complexified holomorphic tangent bundle, leading only to a restriction of the cubic tensor τ^3 . Again, the double Lie bracket construction is only tensorial when both vector fields inside the inner bracket take their values in the Levi kernel (see Example 3.1).

As mentioned earlier, the cubic tensor τ^3 is constructed without any pseudoconvexity assumption. On the other hand, in presence of pseudoconvexity, the whole tensor τ^3 must vanish identically (Lemma 3.14). The only cubic terms that may survive are of the form (3.14) which can never appear in the lowest weight terms, and hence never play a role in Catlin's multitype and boundary system theory.

Motivated by the above, we next look for quartic tensors. It turns out (Example 4.1) that this time, neither second order Levi form derivatives nor quartic Lie brackets provide tensorial invariants even when all vector field arguments take their values in the Levi kernel. To overcome this problem, we restrict the choice of the vector fields involved by requiring a certain kind of condition of "Levi kernel inclusion up to higher order" (Definition 4.2). In Lemma 4.6 we show that this additional condition always holds for any vector field that is Levi-orthogonal to a maximal Levi-nondegenerate subbundle, which, in particular, arises in Catlin's boundary system construction. However, the mentioned Levi-orthogonality lacks some invariance as it depends on the choice of the subbundle. In contrast, the Levi kernel inclusion up to order 1 is an invariant condition that only depends on the 1-jet of the vector field at the reference point.

With that restriction on the vector fields, the invariant quartic tensor τ^4 can now be defined in a similar fashion. In particular, the this tensor's restriction τ^{40} appears in the lowest weight normal form with weights $\geq 1/4$, see Proposition 4.17. It turns out that the restriction τ^{40} provides exactly the missing information at the lowest weight level for hypersurfaces of finite type 4 (where the finite type 3 is known to never occur for pseudoconvex points, see e.g. Corollary 3.15). For example, both D'Angelo finite type 4 and Catlin's multitype up to entry 4 can be completely characterized in terms of τ^{40} . Specifically, having the finite type 4 is equivalent to the nonvanishing of τ^{40} on complex lines (Proposition 5.1), whereas having a multitype up to entry 4 is equivalent to τ^{40} having trivial kernel (Propositions 6.3).

In Section 7 we use the quartic tensor τ^4 to characterize differentials of the functions in the ideal sheaf $\mathcal{I}(q)$ as well as minimal tangent spaces of “containing manifolds” defined by a transversal set of functions in $\mathcal{I}(q)$.

Finally, in Section 8 we obtain a characterization for a Catlin’s boundary system, where the most difficult part of obtaining vector field directions of nonvanishing Levi form derivatives at the lowest weight is replaced by the nonvanishing of the tensor τ^{40} on the vector fields’ values at the reference point, a purely algebraic condition.

In a forthcoming paper will shall extend the present geometric approach towards its *approximate versions* with necessary control to perform the induction step in the subelliptic estimate proof.

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2. Notation and main results

We shall work in the smooth (C^∞) category unless stated otherwise. Let $M \subset \mathbb{C}^n$ be a (smooth) real hypersurface. We write $T := TM$ for its tangent bundle, $H = HM \subset T$ for the complex (or holomorphic) tangent bundle, $Q := T/H$ for the normal bundle, as well as

$$\mathbb{C}T := \mathbb{C} \otimes T, \quad \mathbb{C}H := \mathbb{C} \otimes H, \quad \mathbb{C}Q := \mathbb{C} \otimes Q,$$

for their respective complexifications. Further, H^{10} and $H^{01} = \overline{H^{10}}$ denote $(1,0)$ and $(0,1)$ bundles respectively, such that $\mathbb{C}H = H^{10} \oplus \overline{H^{10}}$. By a small abuse of notation, we write $L \in V$ whenever a vector field L is a local section in V , where V can be a bundle or a sheaf. Similarly we write $V_1 \subset V_2$ when V_1 is a local subbundle or subsheaf of V_2 , where it will be convenient to allow V_1 to be defined over smaller open sets than those where V_2 is defined.

A subscript p always denotes evaluation at a point $p \in M$, i.e. L_p for the value of a vector field L at p , or V_p for the space of all values of elements in V , which is the fiber when V is a vector bundle.

On the dual side, $\Omega = \Omega M$ stands for the bundle of all real 1-forms on M , $\mathbb{C}\Omega$ for all complex 1-forms, Ω^0 for all *contact forms*, i.e. forms Ω that are vanishing on H and real-valued on T , and $\mathbb{C}\Omega^0$ for the corresponding complexification.

Recall that a *(local) defining function* of M is any real-valued function ρ in a neighborhood in \mathbb{C}^n of a point in M , with $d\rho \neq 0$ such that M given

by $\rho = 0$ in the domain of ρ . For any defining function ρ , the one-form $\theta := i\partial\rho$ spans (over \mathbb{R}) the bundle Ω^0 of all contact forms.

We shall consider the standard \mathbb{C} -bilinear pairing $\langle \theta, L \rangle := \theta(L)$ for $\theta \in \mathbb{C}\Omega$, $L \in \mathbb{C}T$. By a slight abuse, we keep the same notation also for the induced pairing

$$\langle \cdot, \cdot \rangle: \mathbb{C}\Omega^0 \times \mathbb{C}Q \rightarrow \mathbb{C}$$

between the (complex) contact forms and the normal bundle. With this notation, we regard the *Levi form tensor* at a point $p \in M$ as the \mathbb{C} -bilinear map

$$\tau_p^2: H_p^{10} \times \overline{H_p^{10}} \rightarrow \mathbb{C}Q_p,$$

which is uniquely determined by the identity

$$(2.1) \quad \langle \theta_p, \tau_p^2(L_p^2, L_p^1) \rangle = i\langle \theta, [L^2, L^1] \rangle_p, \quad L^2 \in H^{10}, L^1 \in \overline{H^{10}},$$

where, as mentioned before, the membership notation for L^2, L^1 (such as $L^2 \in H^{10}$) means being local sections of the corresponding bundles. The normalization of τ^2 used here is chosen such that for the quadric

$$\rho = -2 \operatorname{Re} w + q(z, \bar{z}) = 0, \quad (w, z) \in \mathbb{C} \times \mathbb{C}^{n-1},$$

with $(1, 0)$ vector fields

$$L_j := \partial_{z_j} + q_{z_j} \partial_w, \quad j = 1, 2,$$

and the contact form $\theta = i\partial\rho = i(-dw + \partial q)$, we have

$$\langle \theta_0, \tau_0^2(\partial_{z_j}, \partial_{\bar{z}_k}) \rangle = i\langle \theta, [L_j, \overline{L_k}] \rangle_0 = -\langle dw, q_{z_j \bar{z}_k}(\partial_{\bar{w}} - \partial_w) \rangle_0 = \partial_{z_j} \partial_{\bar{z}_k} q,$$

or more generally

$$(2.2) \quad \langle \theta_0, \tau_0^2(v^2, v^1) \rangle = \partial_{v^2} \partial_{v^1} q, \quad v^2 \in H_0^{10}, v^1 \in \overline{H_0^{10}}.$$

Here we use the subscript notation q_{z_j} for the partial derivative, and ∂_v denotes the directional derivative along the constant vector field identified with the vector $v \in \{0\} \times \mathbb{C}^{n-1}$ in some local holomorphic coordinates.

The tensor τ_p^2 has the unique \mathbb{C} -bilinear symmetric extension

$$\tau_p^2: \mathbb{C}H_p \times \mathbb{C}H_p \rightarrow \mathbb{C}Q_p,$$

for which we still write τ_p^2 by a slight abuse, where we extend by symmetry to $\overline{H_p^{10}} \times H_p^{10}$ and by zero to $H_p^{10} \times H_p^{10}$ and $\overline{H_p^{10}} \times \overline{H_p^{10}}$. For the above quadric example, (2.2) still holds for the symmetric extension. The choice of the *symmetric \mathbb{C} -bilinear tensor* rather than hermitian, as common for the Levi form, will help us to keep the notation lighter for the subsequent Levi form derivatives, as that way we shall never need to remember, which arguments are \mathbb{C} -linear and which are \mathbb{C} -antilinear.

Recall that M is pseudoconvex at p if and only if there exists a nonzero covector $\theta_0 \in \Omega_p^0$ with

$$\langle \theta_p, \tau_p^2(v, \bar{v}) \rangle \geq 0, \quad v \in H_p^{10}.$$

We shall always assume this choice of θ , whenever M is pseudoconvex.

We say that a point $p \in M$ is of Levi rank q , if the Levi form τ_p^2 at p has rank q . A subbundle $V \subset H^{10}$ is called Levi-nondegenerate, if the Levi form is nondegenerate on $V \times \bar{V}$. For every such subbundle V , we write

$$V^\perp \subset H^{10}, \quad V^\perp = \cup_x V_x^\perp, \quad V_x^\perp = \{v \in H_x^{10} : \tau_x^2(v, \bar{v}^1) = 0 \text{ for all } v^1 \in V\},$$

for the orthogonal complement with respect to the Levi form, which is necessarily a subbundle.

Finally, we write $K_p^{10} \subset H_p^{10}$ and $K_p^{01} = \overline{K_p^{10}} \subset H_p^{01}$ for the Levi kernel components at p , $\mathbb{C}K_p = K_p^{10} \otimes \overline{K_p^{10}}$ for the complexification and $K_p = \mathbb{C}K_p \cap T_p$ for the corresponding real part.

The following is an overview of some of the main results (further results below don't assume pseudoconvexity):

THEOREM 2.1. — Let $M \subset \mathbb{C}^n$ be a pseudoconvex real hypersurface. Then for every $q \in \{0, \dots, n-1\}$, there exist an invariant submodule sheaf $\mathcal{S}^{10}(q)$ of $(1,0)$ vector fields, an invariant ideal sheaf $\mathcal{I}(q)$ of complex functions, and for every $p \in M$ of Levi rank q , an invariant quartic tensor

$$\tau_p^4: \mathbb{C}T_p \times \mathbb{C}T_p \times K_p^{10} \times \overline{K_p^{10}} \rightarrow \mathbb{C}Q_p,$$

and a real submanifold $S \subset M$ through p , such that the following hold:

- (1) S contains the set of all points $x \in M$ of Levi rank q in a neighborhood of p .
- (2) The tangent space of S at p equals the real part of the kernel of τ_p^4 :

$$T_p S = \operatorname{Re} \ker \tau_p^4 = \{v \in T_p : \tau_p^4(v, v^3, v^2, v^1) = 0 \text{ for all } v^3, v^2, v^1\}.$$

- (3) In suitable holomorphic coordinates vanishing at p , M admits the form

$$2 \operatorname{Re} w = \sum_{j=1}^q |z_{2j}|^2 + \varphi^4(z_4, \bar{z}_4) + o_w(1), \quad (w, z_2, z_4) \in \mathbb{C} \times \mathbb{C}^q \times \mathbb{C}^{n-q-1},$$

where $o_w(1)$ indicates terms of weight greater than 1, with weights 1, 1/2 and 1/4 assigned to the components of w , z_2 and z_4 respectively, and where φ^4 is a plurisubharmonic homogenous polynomial of degree 4 representing the quartic tensor τ_p^4 in the sense that

$$\tau_p^4(v^4, v^3, v^2, v^1) = \partial_{v^4} \partial_{v^3} \partial_{v^2} \partial_{v^1} \varphi^4$$

holds for $v^4, v^3 \in \mathbb{C}K_0$ and $v^2, \bar{v}^1 \in K_0^{10}$.

- (4) S is given by

$$S = \{f^1 = \cdots = f^m = 0\}, \quad df^1 \wedge \cdots \wedge df^m \neq 0, \quad f^j \in \text{Re } \mathcal{I}(q).$$

In fact, any $f \in \text{Re } \mathcal{I}(q)$ vanishes on the set of points of Levi rank q .

- (5) The ideal sheaf $\mathcal{I}(q)$ is generated by all functions g, f of the form

$$g = \langle \theta, [L^2, L^1] \rangle, \quad f = L^3 \langle \theta, [L^2, L^1] \rangle,$$

where $\theta \in \Omega^0$ is a contact form, $L^3 \in \mathbb{C}T$ is an arbitrary complex vector field, and $L^2, \bar{L}^1 \in \mathcal{S}^{10}(q)$ are arbitrary sections of the submodule sheaf.

- (6) The submodule sheaf $\mathcal{S}^{10}(q)$ contains all germs of $(1,0)$ vector fields L satisfying $L \in V_L^\perp$, with $V_L \subset H^{10}$ being some Levi-nondegenerate subbundle of rank q in a neighborhood of p (that may depend on L). In particular, $\mathcal{S}^{10}(q)$ generates the Levi kernel at each point of Levi rank q .

- (7) The tensor τ_p^4 has the positivity property

$$\tau_p^4(v^2, v^2, v^1, \bar{v}^1) \geqslant 0, \quad v^2 \in T_p, \quad v^1 \in K_p^{10}.$$

In addition, when M is of finite type at most 4 at p , the following also holds:

- (i) The intersection $T_p S \cap K_p$ with the Levi kernel K_p is totally real.
(ii) For every $v \in K_p^{10}$, the tensor τ_p^4 does not identically vanish on

$$(\mathbb{C}v + \mathbb{C}\bar{v}) \times (\mathbb{C}v + \mathbb{C}\bar{v}) \times \mathbb{C}v \times \mathbb{C}\bar{v}.$$

In particular, the regular type at p equals to the D'Angelo type and is either 2 or 4. The type is 4 whenever $K_p^{10} \neq 0$.

- (iii) The (Catlin's) multitype at p equals

$$(1, \underbrace{2, \dots, 2}_q, \underbrace{4, \dots, 4}_{n-q-1}),$$

where the number of 2's equals the Levi rank q at p . In particular, the multitype is determined by the Levi rank.

Note that we prefer the reversed order of the vector fields L^3, L^2, L^1 , e.g. $L^3 \langle \theta, [L^2, L^1] \rangle$ that better reflects the logical order of the operations: first form the Lie bracket $[L^2, L^1]$ inside, then differentiate by L^3 from outside (after pairing with θ).

For the proofs and more detailed and general statements, see the respective sections below. The submodule sheaves $\mathcal{S}^{10}(q)$ from (6) are defined in Section 4.2, and the ideal sheaves $\mathcal{I}(q)$ from (5) in Section 7. In particular,

local sections in $\mathcal{I}(q)$ vanish at points of Levi rank q by Corollary 7.5. The quartic tensor τ^4 is constructed in Section 4.3. In view of Proposition 7.9, the intersection of real kernels of differentials df for $f \in \text{Re } \mathcal{I}(q)$ coincides with $\text{Re } \ker \tau_p^4$. Hence we can choose functions f^j satisfying (4) and (2). The normal form in (3) follows from Proposition 4.17.

When M is of finite type 4, Proposition 5.1 implies that τ^4 has no holomorphic kernel, and therefore its (real) kernel as in (2) is totally real, as stated in (i). Statement (ii) is also part of Proposition 5.1. Finally, statement (iii) about the multitype is contained in Section 6.

The following simple example illustrates one of the differences between functions in the ideal sheaf $\mathcal{I}(q)$ and the boundary systems (as defined in [8, Section 2], see also Section 8 below).

Example 2.2. — Consider the hypersurface $M \subset \mathbb{C}_{w,z}^2$ given by

$$2 \operatorname{Re} w = \varphi(z, \bar{z}, \operatorname{Im} w), \quad \varphi(z, \bar{z}, u) := |z|^4 + u^2|z|^2,$$

which is pseudoconvex and of finite type 4. Then a boundary system $\{L_2; r_2\}$ defines the 2-dimensional submanifold $S := \{r_2 = 0\} \subset M$, which contains all points of Levi rank 0. However, since r_2 is of the form

$$(2.3) \quad r_2 = \operatorname{Re} L^3 \langle \theta, [L^2, L^1] \rangle$$

(cf. the notation of Theorem 2.1(5)), its differential at 0 is given by

$$dr_2(v) = \operatorname{Re} \tau_0^4(v, L_0^3, L_0^2, L_0^1),$$

which vanishes on the transversal space $\{dz = 0\}$. Consequently, any S defined by a boundary system function r_2 must be tangent to the real line $\{dz = 0\}$.

On the other hand, in the ideal sheaf $\mathcal{I}(0)$ we can choose a function given by (2.3) with *transversal* L^3 , i.e. $L_p^3 \notin \mathbb{C}H_p$. That allows us to reduce the submanifold S in Theorem 2.1 down to only the origin $z = w = 0$, which, in fact, is the set of points of Levi rank 0.

In particular, the ideal sheaf $\mathcal{I}(0)$ captures as its zero set with linearly independent differentials the 0-dimensional singular stratum of all Levi-degenerate points, which cannot be achieved with boundary systems.

3. Invariant cubic tensors

We begin by investigating the 3rd order invariants without the pseudoconvexity assumption.

3.1. Double Lie brackets

In presence of a nontrivial Levi kernel K_p^{10} at a point $p \in M$, it is natural to look for cubic tensors arising from double Lie brackets with one of the vector fields having its value inside the Levi kernel at the reference point:

$$(3.1) \quad \langle \theta, [L^3, [L^2, L^1]] \rangle, \quad \theta \in \Omega^0, \quad L^3, L^2 \in H^{10}, \quad L^1 \in \overline{H^{10}}, \quad L_p^1 \in \overline{K_p^{10}}.$$

However, the following simple example shows that (3.1) does not define a tensor in general:

Example 3.1. — Let $M \subset \mathbb{C}_{z_1, z_2, w}^3$ be the degenerate quadric

$$(3.2) \quad \rho = -(w + \bar{w}) + z_1 \bar{z}_1 = 0,$$

and consider the $(1, 0)$ vector fields

$$L^3 := \partial_{z_2}, \quad L^2 := \partial_{z_2} + cz_2 L^1, \quad L^1 := \partial_{z_1} + \bar{z}_1 \partial_w.$$

The main idea here is to “twist” the vector field L^2 with $L_0^2 \in K_0^{10}$ by adding a multiple of the other vector field L^1 , along which the Levi form is nonzero.

Then

$$[L^3, [L^2, \overline{L^1}]] = c[L^1, \overline{L^1}] = c(\partial_{\bar{w}} - \partial_w),$$

and hence for any fixed contact form $\theta \in \Omega^0$, the value

$$\langle \theta, [L^3, [L^2, \overline{L^1}]] \rangle_0$$

depends on c , even though all values L_0^j are independent of c . Note that both L_0^2 and L_0^3 (but not $\overline{L_0^1}$) are inside the Levi kernel K_0^{10} . Hence the double Lie bracket does not define any tensor $K_p^{10} \times K_p^{10} \times \overline{H_p^{10}} \rightarrow \mathbb{C}Q_p$ with $p = 0$.

Similarly, taking

$$L^3 := \partial_{z_2}, \quad L^2 := \partial_{z_1} + \bar{z}_1 \partial_w, \quad L^1 := \partial_{z_2} + c\bar{z}_2 L^1,$$

we conclude that

$$\langle \theta, [L^3, [L^2, \overline{L^1}]] \rangle_0$$

depends on c , thus also not defining any tensor $K_p^{10} \times H_p^{10} \times \overline{K_p^{10}} \rightarrow \mathbb{C}Q_p$.

The same example also shows that the Levi form derivative $L^3 \langle \theta, [L^2, \overline{L^1}] \rangle$ considered below does not behave tensorially on the same spaces.

In contrast, we do get an invariant tensor when *both vector fields inside the inner Lie bracket have their values in the Levi kernel at the reference point*. We write τ^{31} for the corresponding tensor, emphasizing the fact that it will become a restriction of the full tensor τ^3 below.

LEMMA 3.2. — *The double Lie bracket $[L^3, [L^2, L^1]]$ defines an invariant tensor*

$$(3.3) \quad \tau_p^{31} : \mathbb{C}K_p \times K_p^{10} \times \overline{K_p^{10}} \rightarrow \mathbb{C}Q_p,$$

i.e. there exists an unique τ_p^{31} as above satisfying

$$\begin{aligned} \tau_p^{31}(L_p^3, L_p^2, L_p^1) &= i[L^3, [L^2, L^1]]_p \mod \mathbb{C}H_p, \\ L^3, L^2, \overline{L^1} &\in H^{10}, \quad L_p^3, L_p^2, \overline{L_p^1} \in K_p^{10}. \end{aligned}$$

Furthermore, τ_p^{31} is symmetric on $K_p^{10} \times K_p^{10} \times \overline{K_p^{10}}$ in its K^{10} -arguments, and on $\overline{K_p^{10}} \times K_p^{10} \times \overline{K_p^{10}}$ in its $\overline{K^{10}}$ -arguments, and satisfies the reality condition

$$(3.4) \quad \overline{\tau_p^{31}(v^3, v^2, v^1)} = \tau_p^{31}(\bar{v}^3, \bar{v}^1, \bar{v}^2).$$

Proof. — It suffices to show that

$$(3.5) \quad [L^3, [L^2, L^1]]_p \in \mathbb{C}H_p$$

holds whenever any of the values L_p^j is 0. Since any such L^j can be written as linear combination $\sum a_k L_k$ with $a_k(p) = 0$ and L_k being in the same bundle (either H^{10} or $\overline{H^{10}}$), it suffices to assume $L^j = a\tilde{L}^j$ with $a(p) = 0$. Then, any term giving a nonzero value at p in (3.5), must involve differentiation of the function a , either by one of the vector fields L^j , with the bracket of the other two as factor:

$$(L^{j_3}a)[L^{j_2}, L^{j_1}],$$

or by two of the vector fields L^j , with the third one as factor:

$$(L^{j_3}L^{j_2}a)L^{j_1}.$$

In the second case (3.5) is clear. In the first case, we obtain a bracket of two L^j , one of which has value at p contained in the kernel $K_p^{10} \oplus \overline{K_p^{10}}$, implying (3.5).

The reality condition is straightforward and the symmetries follow from the Jacobi identity. \square

Remark 3.3. — A closely related construction is the one of the cubic form $c = c(L^3, L^2, L^1)$ by Webster [64], defined for triples of vector fields in a neighborhood of a reference point $p \in M$, whose value c_p at p depends on the 1-jets of the vector fields L^3, L^2, L^1 , but in general, is not uniquely determined by their values at p , as demonstrated by Example 3.1, unless all three vector fields are valued in kernel at p as in Lemma 3.2.

3.2. The Levi form derivative

As alternative to the double Lie bracket tensor, one can differentiate the Levi form after pairing with a contact form, which is similar to the approach employed by Catlin in his boundary system construction:

$$(3.6) \quad L^3 \langle \theta, [L^2, L^1] \rangle.$$

Again Example 3.1 shows that (3.6) does not define a tensor if either of the vector fields L^2, L^3 inside the Lie bracket is not in the Levi kernel at p . On the other hand, if both vector fields L^1, L^2 inside the Lie bracket have their value at p contained in the Levi kernel, we do obtain a tensor even when the outside vector field L^3 is not necessarily contained in $\mathbb{C}H$:

LEMMA 3.4. — There exists unique cubic tensor

$$\tau_p^3 : \mathbb{C}T_p \times K_p^{10} \times \overline{K_p^{10}} \rightarrow \mathbb{C}Q_p,$$

satisfying

$$(3.7) \quad \begin{aligned} \langle \theta_p, \tau_p^3(L_p^3, L_p^2, L_p^1) \rangle &= i(L^3 \langle \theta, [L^2, L^1] \rangle)_p, \\ \theta \in \Omega^0, \quad L^3 \in \mathbb{C}T, \quad L^2, \overline{L^1} &\in H^{10}, \quad L_p^2, \overline{L_p^1} \in K_p^{10}. \end{aligned}$$

Furthermore, τ_p^3 satisfies the reality condition

$$(3.8) \quad \overline{\tau_p^3(v^3, v^2, v^1)} = \tau_p^3(\overline{v^3}, \overline{v^1}, \overline{v^2}),$$

note the switch of the last two arguments.

Proof. — The proof is similar to that of Lemma 3.2, and the symmetry follows directly from the definition. \square

Remark 3.5. — Note that the tensor τ^3 in Lemma 3.4 is defined when the first argument is any complex vector field on M , not necessarily in the subbundle $\mathbb{C}H$, in contrast to the tensor τ^{31} in Lemma 3.2. This shows that taking derivatives of the Levi form provides more information than taking iterated Lie brackets.

3.3. A normal form of order 3 and complete set of cubic invariants of a real hypersurface

To compare tensors τ^{31} and τ^3 , it is convenient to use a partial normal form for the cubic terms. In the following we write $\varphi^{j_1 \dots j_m}(x_{j_1}, \dots, x_{j_m})$ for a polynomial of the multi-degree (j_1, \dots, j_m) in its corresponding variables. We also write $z_k = (z_{k1}, \dots, z_{km}) \in \mathbb{C}^m$ for the coordinate vectors and their components.

PROPOSITION 3.6. — *For every real hypersurface M in \mathbb{C}^n and point $p \in M$ of Levi rank q , there exist local holomorphic coordinates*

$$(w, z) = (w, z_2, z_3) \in \mathbb{C} \times \mathbb{C}^q \times \mathbb{C}^{n-q-1},$$

vanishing at p , where M takes the form

$$w + \bar{w} = \varphi(z, \bar{z}, i(w - \bar{w})), \quad \varphi(z, \bar{z}, u) = \varphi^2(z, \bar{z}, u) + \varphi^3(z, \bar{z}, u) + O(4),$$

where

$$\varphi^2(z, \bar{z}, u) = \varphi^{11}(z_2, \bar{z}_2) = \sum \pm |z_{2j}|^2,$$

and

$$\varphi^3(z, \bar{z}, u) = 2 \operatorname{Re} \varphi^{21}(z, \bar{z}_3) + \varphi^{111}(z_3, \bar{z}_3, u),$$

with

$$\begin{aligned} \varphi^{21} &= \sum \varphi_{j_1 j_2 j_3}^{21,1} z_{2j_1} z_{2j_2} \bar{z}_{3j_3} + \sum \varphi_{j_1 j_2 j_3}^{21,2} z_{2j_1} z_{3j_2} \bar{z}_{3j_3} \\ &\quad + \sum \varphi_{j_1 j_2 j_3}^{21,3} z_{3j_1} z_{3j_2} \bar{z}_{3j_3}, \\ \varphi^{111} &= \sum \varepsilon_j |z_{3j}|^2 u, \quad \varepsilon_j \in \{-1, 0, 1\}, \end{aligned}$$

and $O(4)$ stands for all terms of total order at least 4.

Proof. — It is well-known that the quadratic term φ^2 can be transformed into $\varphi^{11}(z_2, \bar{z}_2)$ representing the (nondegenerate part of) the Levi form. Furthermore, as customary, we may assume that the cubic term φ^3 has no harmonic terms.

Next, by suitable polynomial transformations

$$(z, w) \mapsto \left(z + \sum_{j=1}^r z_{2j} h_j(z, w), w \right),$$

we can eliminate all cubic monomials of the form $\bar{z}_{2j} h(z, u)$ and their conjugates, where $h(z, w)$ is any holomorphic quadratic monomial. The proof is completed by inspecting the remaining cubic monomials and diagonalizing the quadratic form in the component φ^{111} . \square

Next we use the convenient $(1, 0)$ vector fields with obvious notation:

LEMMA 3.7. — *For a real hypersurface $M \subset \mathbb{C}^n$ given by*

$$(3.9) \quad w + \bar{w} = \varphi(z, \bar{z}, i(w - \bar{w})), \quad (w, z) \in \mathbb{C} \times \mathbb{C}^{n-1},$$

the subbundle H^{10} of $(1, 0)$ vector fields is spanned by

$$L_j := \partial_{z_j} + \frac{\varphi_{z_j}}{1 - i\varphi_u} \partial_w, \quad j = 1, \dots, n-1.$$

More generally, H^{10} is spanned by all vector fields of the form

$$(3.10) \quad L_v := \partial_v + \frac{\varphi_v}{1 - i\varphi_u} \partial_w, \quad v \in \{0\} \times \mathbb{C}^{n-1},$$

where the subscript v denotes the differentiation in the direction of v .

Calculating with the special vector fields from Lemma 3.7, we obtain:

COROLLARY 3.8. — Let M be in the normal form given by Proposition 3.6. Then tensors τ_p^{31} and τ_p^3 defined in Lemmas 3.2 and 3.4 respectively satisfy

$$(3.11) \quad \langle \theta_0, \tau_0^{31}(v^3, v^2, v^1) \rangle = \langle \theta_0, \tau_0^3(v^3, v^2, v^1) \rangle = \partial_{v^3} \partial_{v^2} \partial_{v^1} \varphi^3,$$

where

$$v^3, v^2, \overline{v^1} \in K_0^{10} \cong \{0\} \times \mathbb{C}^{n-q-1}, \quad \theta = i\partial\rho, \quad \rho = -2 \operatorname{Re} w + \varphi.$$

Furthermore, the second identity in (3.11) still holds for $v^3 \in \mathbb{C}H_0$.

In particular, τ^{31} is a restriction of τ^3 to $\mathbb{C}T_p \times K_p^{10} \times \overline{K_p^{10}}$, explaining the notation.

Remark 3.9. — The term φ^{21} in Proposition 3.6 represents, up to a nonzero constant multiple, the cubic invariant tensor introduced by Ebenfelt [25], defined by means of the Lie derivative \mathcal{T} :

$$(3.12) \quad \psi_p^3 : H_p^{10} \times H_p^{10} \times \overline{K_p^{10}} \rightarrow \mathbb{C}Q_p, \quad \langle \theta_p, \psi^3(L_p^3, L_p^2, L_p^1) \rangle = \langle \mathcal{T}_{L^3} \mathcal{T}_{L^2} \theta, L^1 \rangle_p,$$

where

$$\mathcal{T}_L := d \circ \iota_L + \iota_L \circ d,$$

and ι is the contraction. Here the phenomenon illustrated by Example 3.1 of the lack of tensoriality in the last argument does not occur as the right-hand side obviously depends only on the value L_p^1 . In fact, it follows from the transformation law of the Lie derivative,

$$\mathcal{T}_L(f\theta) = f\mathcal{T}_L\theta + (Lf)\theta, \quad \mathcal{T}_{fL}\theta = f\mathcal{T}_L\theta + \theta(L)df,$$

that the same right-hand side in (3.12) defines a tensor even on the larger spaces

$$(3.13) \quad \tau_p^{21} : \mathbb{C}H_p \times \mathbb{C}H_p \times \mathbb{C}K_p \rightarrow \mathbb{C}Q_p.$$

On the other hand, the same expression *does not define any tensor when the first argument varies arbitrarily in $\mathbb{C}T$* , even for a Levi-flat hypersurface $M \subset \mathbb{C}^2$. Indeed, taking

$$M = \{\rho = 0\}, \quad \rho = -w - \bar{w},$$

with

$$\theta = -i(1+z+\bar{z})dw, \quad L^3 = i(\partial_w - \partial_{\bar{w}}), \quad L^2 = \partial_z, \quad L^1 = \partial_{\bar{z}},$$

we compute

$$\mathcal{T}_{fL^3}\mathcal{T}_{L^2}\theta = \mathcal{T}_{fL^3}(\iota_{L^2}d\theta) = f\omega + d\theta(L^2, L^3)df = f\omega + df,$$

where f is any smooth complex function and ω is some 1-form. Then choosing f with $f(p) = 0$ and $df_p = cd\bar{z}$, we conclude that $\langle \mathcal{T}_{fL^3}\mathcal{T}_{L^2}\theta, L^1 \rangle_p$ depends on c and hence is not tensorial.

Finally, as consequence from Proposition 3.6, we obtain:

COROLLARY 3.10. — *The tensors τ^3 and τ^{21} , given by respectively Lemma 3.4 and (3.13), coincide up to a constant on their common set of definition, and constitute together the full set of cubic invariants of M at p .*

3.4. Symmetric extensions

As consequence of Corollary 3.8, τ^3 is symmetric in K^{10} - or in $\overline{K^{10}}$ -vectors whenever two of them occur in any two arguments. This property leads to a natural symmetric extension:

LEMMA 3.11. — *The restriction*

$$\tau_p^{30}: \mathbb{C}K_p \times K_p^{10} \times \overline{K^{10}} \rightarrow \mathbb{C}Q_p$$

of the cubic tensor τ_p^3 admits an unique symmetric extension

$$\tilde{\tau}_p^{30}: \mathbb{C}K_p \times \mathbb{C}K_p \times \mathbb{C}K_p \rightarrow \mathbb{C}Q_p,$$

satisfying

$$\langle \theta_0, \tilde{\tau}_0^{30}(v^3, v^2, v^1) \rangle = \partial_{v^3}\partial_{v^2}\partial_{v^1}\varphi^3,$$

whenever M is in a normal form $\rho = -2\operatorname{Re} w + \varphi = 0$ as in Proposition 3.6 and $\theta = i\partial\rho$.

Remark 3.12. — Note that since φ^3 has no harmonic terms in a normal form, the extension tensor $\tilde{\tau}^{30}$ always vanishes whenever its arguments are either all in K^{10} or all in $\overline{K^{10}}$.

Example 3.13. — In contrast to τ^{30} , the full cubic tensor τ^3 does not in general have any invariant extension to $\mathbb{C}T \times \mathbb{C}K \times \mathbb{C}K$. Indeed, consider the cubic $M \subset \mathbb{C}^2$ given by

$$\rho := -2\operatorname{Re} w + \varphi^3 = 0, \quad \varphi^3 = 2\operatorname{Re}(z^2\bar{z}).$$

Then $\partial_{\bar{w}}\partial_z\partial_z\varphi^3 = 0$. Now consider a change of coordinates with linear part $(w, z) \mapsto (w, z + iw)$ transforming $\partial_{\bar{w}}$ into $\partial_{\bar{w}} - i\partial_{\bar{z}}$. Then, after removing harmonic terms, the new cubic term takes form

$$\varphi^3 = 2\operatorname{Re}(z^2\bar{z}) - 4\operatorname{Im}wz\bar{z}.$$

But then $(\partial_{\bar{w}} - i\partial_{\bar{z}})\partial_z\partial_z\varphi^3 \neq 0$, i.e. the 3rd derivatives of φ^3 do not transform as tensor when passing to another normal form.

3.5. Cubic tensors vanish for pseudoconvex hypersurfaces

If M is pseudoconvex, the Levi form $\langle \theta, [L, \bar{L}] \rangle$ does not change sign. Then the right-hand side in (3.7) is, up to a constant, a derivative of a nonnegative function at its zero, hence vanishes. Therefore the whole cubic tensor τ^3 must vanish identically. We obtain:

LEMMA 3.14. — *Let M be a pseudoconvex hypersurface and $p \in M$. Then the cubic tensor τ_p^3 (and therefore its restriction τ_p^{31}) vanishes identically. Equivalently, the cubic normal form in Proposition 3.6 satisfies*

$$(3.14) \quad \varphi^{21}(z, \bar{z}_3) = \sum_{jkl} c_{jkl} z_{2j} z_{2k} \bar{z}_{3l}, \quad \varphi^{111}(z_3, \bar{z}_3, u) = 0.$$

The remaining cubic terms in (3.14) can be absorbed into higher weight terms as follows. We write $o_w(m)$ for terms of weights higher than m .

COROLLARY 3.15. — *A pseudoconvex hypersurface M in suitable holomorphic coordinates is given by*

$$(3.15) \quad w + \bar{w} = \varphi(z, \bar{z}, i(w - \bar{w})), \quad \varphi = \sum_j |z_{2j}|^2 + o_w(1),$$

where o_w is calculated for (w, z_2, z_3) , and their conjugates, having weights $1, \frac{1}{2}, \frac{1}{3}$ respectively. In particular, M cannot be of (D'Angelo) type 3.

The last statement follows directly from (3.15), since the contact orders with lines in the directions of z^3 are at least 4.

3.6. Freeman's modules and uniformly Levi-degenerate hypersurfaces

Freeman [28] introduced for any smooth real hypersurface $M \subset \mathbb{C}^n$, a decreasing sequence of invariantly defined submodules of the module of all

smooth $(1, 0)$ vector fields on M . In particular, Freeman's second submodule N'_2 is defined by the Lie bracket relation

$$N'_2 := \{L \in H^{10} : [L, \overline{L^1}] \in \mathbb{C}H \text{ for all } L^1 \in H^{10}\},$$

or equivalently, by the inclusion relation in the Levi kernels K_p^{10} :

$$N'_2 := \{L \in H^{10} : L_p \in K_p^{10} \text{ for all } p\}.$$

For a fixed $p_0 \in M$, it is easy to see that vector fields in N'_2 span the Levi kernel $K_{p_0}^{10}$ if and only if $\dim K_p^{10}$ is constant for $p \in M$ near p_0 , i.e. when M is *uniformly Levi-degenerate* in a neighborhood of p_0 . Indeed, since $\dim K_p^{10}$ is upper semi-continuous, whereas the dimension of the span at p of all values of N'_2 is lower semi-continuous, the only way both dimensions can match at p_0 is when they are both constant in its neighborhood, i.e. when M is uniformly Levi-degenerate. In the latter case, the tensor τ^{21} in (3.12) can be computed by means of double Lie brackets of vector fields in N'_2 , as shown in [39, Appendix].

On the other hand, if M is of finite type, one necessarily has $K_p = 0$ on a dense set of $p \in M$ (otherwise nontrivial integral surfaces of K^{10} would be complex-analytic subsets of M). Hence in this case, the module N'_2 is always trivial, whereas the tensor τ^3 may not be so. And even when the module N'_2 is not trivial but $\dim K_p$ is not constant in any neighborhood of a point p_0 , it is easy to see that the set of values L_p for $L \in N'_2$ can never span the full Levi kernel K_p .

Remark 3.16. — In the *uniformly Levi-degenerate case*, i.e. when the Levi kernel dimension $\dim K_p$ is constant, alternatively to the Lie derivative approach in (3.12), both double Lie brackets (as in Lemma 3.2) and Levi form derivative approaches (as in Lemma 3.4) can be used to define τ^{21} by *imposing additional restrictions* on the vector fields to be contained in the Levi kernel subbundle everywhere, rather than only at the reference point as in (3.12).

On the other hand, without the uniformity assumption on $\dim K_p$, only the Lie derivative approach leads to an invariant definition of τ^{21} , whereas only the Levi form derivative approach is suitable to define the full cubic tensor τ^3 as in Lemma 3.4. It is quite remarkable that no single approach seems to work to define the complete system of cubic invariants, consisting of the pair (τ^{21}, τ^3) .

4. Invariant quartic tensors

If the cubic tensor τ^3 vanishes, it is natural to look for higher order invariants by taking iterated Lie brackets or higher order derivatives of Levi form. However, in contrast to the statements of Lemmas 3.2 and 3.4, we don't obtain any tensor in this way even when all vector field values are in the Levi kernel, as demonstrated by our next counter-example:

Example 4.1. — Let $M \subset \mathbb{C}_{z_1, z_2, w}^3$ be again the degenerate quadric from Example 3.1, and set

$$L := \partial_{z_2} + cz_2(\partial_{z_1} + \bar{z}_1\partial_w).$$

Then

$$[L, \bar{L}] = |cz_2|^2(\partial_{\bar{w}} - \partial_w),$$

and both $\langle \theta, [L, [\bar{L}, [L, \bar{L}]]] \rangle_0$ and $(L\bar{L}\langle \theta, [L, \bar{L}] \rangle)_0$ depends on c (and hence on the 1-jet of L), even though the value L_0 is contained in the Levi kernel K_0^{10} , and the cubic tensor τ_0^3 identically vanishes.

4.1. Vector fields that are in the Levi kernel up to order 1

In view of Example 4.1, in order to obtain a tensor, we need to restrict the choice of the vector fields. This motivates the following definition:

DEFINITION 4.2. — *Let L be a $(1, 0)$ vector field. We say that L is in the Levi kernel up to order 1 at p if, for any vector fields $L^1 \in H^{10}$, $L^2 \in CT$, and any contact form θ , the following holds:*

$$(4.1) \quad \langle \theta, [L^1, \bar{L}] \rangle_p = (L^2 \langle \theta, [L^1, \bar{L}] \rangle)_p = 0.$$

More generally, we have the following “microlocal” version of this definition as follows. For a fixed tangent vector $v \in CT_p$, we say that L is in the Levi kernel up to v -order 1 at p if (4.1) holds whenever $L_p^2 = v$, i.e. we differentiate the Levi form only in the fixed given direction. (The latter property obviously depends only on the value v rather than its vector field extension L^2 .) If the above property holds for all v in a vector subspace $V \subset CT_p$, we also say that L is in the Levi kernel up to V -order 1 at p .

If L is a $(0, 1)$ vector field, we say that L is in the Levi kernel up to order 1 at p whenever \bar{L} is. Similarly, we extend all the other terminology in this definition to $(0, 1)$ vector fields.

It is straightforward to see that:

LEMMA 4.3. — For any $(0, 1)$ vector field \bar{L} with $\bar{L}_p \in \overline{K_p^{10}}$, the expression

$$(L^2\langle\theta, [L^1, \bar{L}]\rangle)_p$$

only depends on the values L_p^2, L_p^1 and θ_p , as well as on the 1-jet of L at p . In particular, L being in the Levi kernel up to order 1, is a linear condition on the 1-jet of L at p .

Example 4.4. — In the setting of Example 4.1, choosing $L^1 := \partial_{z_1} + \bar{z}_1 \partial_w$, we compute

$$(L\langle\theta, [L^1, \bar{L}]\rangle)_0 \neq 0,$$

which shows that here L is not in the Levi kernel of order 1, even though its value at 0 is contained in the Levi kernel.

Remark 4.5. — Using any normal form as in Proposition 3.6 and calculating with vector fields (3.10), we can obtain a condition equivalent to (4.1) with L^2 in $\mathbb{C}H$ (rather than in $\mathbb{C}T$), which can be stated in terms of the double Lie brackets instead of the Levi form derivatives:

$$(4.2) \quad \langle\theta_p, [L^1, \bar{L}]_p\rangle = \langle\theta_p, [L^2, [L^1, \bar{L}]]_p\rangle = \langle\theta_p, [\bar{L}^2, [L^1, \bar{L}]]_p\rangle = 0.$$

A priori, it is not at all clear that vector fields as in Definition 4.2 exist. The following lemma provides an easy way of constructing them.

LEMMA 4.6. — Let M have the Levi rank q at $p \in M$, with Levi kernel K_p^{10} . Assume that $L \in H^{10}$ is in the Levi kernel up to order 1 at p , as per Definition 4.2. Then $L_p \in K_p^{10}$ and

$$(4.3) \quad \tau_p^3(L_p^2, L_p^1, \bar{L}_p) = 0, \quad L^2 \in \mathbb{C}T, \quad L^1 \in H^{10}, \quad L_p^1 \in K_p^{10}.$$

must hold for all L^2, L^1 . (Equivalently, \bar{L}_p is contained in the kernel of τ_p^3 in the last argument).

Vice versa, assume that $L_p \in K_p^{10}$ and (4.3) holds. Let

$$(\tilde{L}^1, \dots, \tilde{L}^q)$$

be a Levi-nondegenerate system of $(1, 0)$ vector fields at p (i.e. the matrix $\tau_p^2(\tilde{L}_p^j, \tilde{L}_p^k)$ is nondegenerate), such that L is Levi-orthogonal to each \tilde{L}^j , $j = 1, \dots, q$, in a neighborhood of p . Then L is in the Levi kernel up to order 1 at p .

Proof. — The first part follows directly from the definitions.

Vice versa, since the Levi form has rank r in p , and L_p is Levi-orthogonal to each \tilde{L}^j , it follows that L_p is in the Levi kernel, i.e. the first expression in (4.1) must vanish.

Next, (4.3) implies that the second expression in (4.1) vanishes whenever $L_p^1 \in K_p^{10}$. Similarly, in view of the symmetry (3.8), also the third expression vanishes under the same assumption.

Finally, to a general L^1 with $L_p^1 \notin K_p^{10}$, we can always add a linear combination of \tilde{L}^j to achieve the inclusion of the value at p in the Levi kernel. Since L is Levi-orthogonal to each \tilde{L}^j identically in a neighborhood of p , this does not change (4.1), completing the proof. \square

In particular, in view of Lemma 3.14 we obtain:

COROLLARY 4.7. — *Let M be pseudoconvex. Then every $v \in K_p^{10}$ extends to a $(1, 0)$ vector field, which is in the Levi kernel up to order 1 at p .*

Remark 4.8. — More generally, a similar result can be obtained without pseudoconvexity for a $v \in K_p^{10}$ whose conjugate \bar{v} is in the kernel of τ^3 (in the last argument), i.e. satisfying

$$\tau_p^3(L_p^3, L_p^2, \bar{v}) = 0$$

for all L^3, L^2 . Then there exists a $(0, 1)$ vector field \bar{L} extension of \bar{v} , which is in the Levi kernel up to order 1 at p .

4.2. Invariant submodule sheaves of vector fields

The notion of the *Levi kernel inclusion up to order 1* has been defined pointwise in Definition 4.2. In order to have a uniform control for Levi kernels in nearby points, we shall need to define corresponding sheaves of submodules of vector fields as follows.

DEFINITION 4.9. — *Let $M \subset \mathbb{C}^n$ be a real hypersurface. Denote by \mathcal{T}^{10} the sheaf of all $(1, 0)$ vector fields on M . For every $q \leq n - 1$, define $\mathcal{S}^{10}(q) \subset \mathcal{T}^{10}$ to be the submodule sheaf consisting of all germs of vector fields on M which are contained in the Levi kernel up to order 1 at every point of Levi rank $\leq q$.*

Clearly we have the inclusions

$$\mathcal{S}^{10}(0) \supset \cdots \supset \mathcal{S}^{10}(n - 1).$$

As a direct consequence of Lemma 4.6, we obtain the following strengthening of Corollary 4.7:

COROLLARY 4.10. — *Let M be a pseudoconvex hypersurface. Then for every q , local sections of $\mathcal{S}^{10}(q)$ span the Levi kernel K_x^{10} at every point $x \in M$ of Levi rank q .*

Note that to guarantee the existence of sufficiently many sections as in Corollary 4.10, it is important to restrict the property underlying Definition 4.9 only to points of Levi rank $\leq q$. Without that restriction, the sheaf would become trivial e.g. for any manifold M that is generically Levi-nondegenerate (which is the case for any M of finite type).

Definition 4.9 requires to check the condition at every point of Levi rank $\leq q$, which can be difficult to deal with in practice, when the set of such points is not “nice”. However, Lemma 4.6 implies:

COROLLARY 4.11. — *Suppose that M is pseudoconvex. Let $V \subset H^{10}$ be a Levi-nondegenerate subbundle of rank q in a neighborhood of p . Then any section in the Levi-orthogonal complement V^\perp is contained in the sheaf $\mathcal{S}^{10}(q)$. In particular, local sections of $\mathcal{S}^{10}(q)$ span the Levi kernel K_x^{10} at every point $x \in M$ of Levi rank $\leq q$.*

Recall that in Section 2, we call a subbundle $V \subset H^{10}$ Levi-nondegenerate whenever the Levi form restriction to $V \times \overline{V}$ is nondegenerate.

Example 4.12. — If $p \in M$ is a point of Levi rank 0, where the cubic tensor τ_p^3 vanishes, any $(1, 0)$ vector field is automatically contained in the Levi kernel up to order 1 at p in view of Corollary 4.7. In particular, if M is pseudoconvex, the sheaf $\mathcal{S}^{10}(0)$ consists of all germs of $(1, 0)$ vector fields. When M is not pseudoconvex, the condition $f \in \mathcal{S}^{10}(0)$ is more delicate, requiring f to belong to the kernel of the cubic tensor τ_p^3 at every point of Levi rank 0.

On the opposite end, for $q = n - 1$, the sheaf $\mathcal{S}^{10}(n - 1)$ consists of all germs of $(1, 0)$ vector fields that are contained in the Levi kernel at every point. Indeed, any point is of Levi rank $\leq n - 1$, hence any germ in $\mathcal{S}^{10}(n - 1)$ is automatically contained in the Levi kernel at every point. Vice versa, for any such germ L , the first term in (4.2) vanishes identically and hence also the second vanishes by differentiation.

The sheaves $\mathcal{S}^{10}(q)$ for $1 < q < n - 1$ are more interesting:

Example 4.13. — Let $M \subset \mathbb{C}_{z_1, z_2, w}^3$ be given by

$$2 \operatorname{Re} w = |z_1|^4 + |z_2|^4.$$

The condition $L \in \mathcal{S}^{10}(1)$ only involves points of Levi rank ≤ 1 , i.e. the subset $M^1 := \{z_1 z_2 = 0\}$. Hence, at points outside M^1 , the sheaf $\mathcal{S}^{10}(1)$ contains all germs of $(1, 0)$ vector fields.

Next, for $p = (z_1, 0, w) \in M^1$ with $z_1 \neq 0$, a $(1, 0)$ vector field L satisfying

$$(4.4) \quad L = a^1 \partial_{z_1} + a^2 \partial_{z_2} \mod (\partial_{\bar{w}} - \partial_w)$$

for some functions a^1, a^2 , is in the Levi kernel up to order 1 at p if and only if the coefficient a^1 vanishes up to order 1 at p . A similar property holds for $p = (0, z_2, w) \in M^1$. Finally, for $p = (0, 0, w) \in M^1$, any L is in the Levi kernel up to order 1 at p . However, any $L \in \mathcal{S}^{10}(1)$ must have both a^1, a^2 vanish at 0 up to order 1 by continuity. Summarizing, a germ $L \in \mathcal{S}^{10}(1)$ if and only if a^j vanishes up to order 1 at every point of M^1 with $z_{3-j} = 0$ for $j = 1, 2$.

4.3. Construction of the quartic tensor

Equipped with special vector fields as in Definition 4.2, we can now define an invariant quartic tensor by means of the second order derivatives of the Levi form:

LEMMA 4.14. — *Let M be such that the cubic tensor τ_p^3 vanishes for some $p \in M$. Then there exists an unique tensor*

$$\tau_p^4: \mathbb{C}T_p \times \mathbb{C}T_p \times K_p^{10} \times \overline{K_p^{10}} \rightarrow \mathbb{C}Q_p,$$

such that for any $(1, 0)$ vector fields $\bar{L}^1, L^2 \in H^{10}$ that are in the Levi kernel up to order 1 at p , any vector fields $L^3, L^4 \in \mathbb{C}T$, and any contact form $\theta \in \Omega^0$,

$$(4.5) \quad \langle \theta_p, \tau_p^4(L_p^4, L_p^3, L_p^2, L_p^1) \rangle = i(L^4 L^3 \langle \theta, [L^2, L^1] \rangle)_p.$$

More generally, (4.5) still holds whenever both L^1 and L^2 are in the Levi kernel up to L_p^j -order 1 at p , for $j = 3, 4$.

Proof. — Similar to the proof of Lemma 3.2, it suffices to prove that the right-hand side of (4.5) vanishes whenever either $L^k = a\tilde{L}^k$ for some $k = 1, 2, 3, 4$, or $\theta = a\tilde{\theta}$, where a is a smooth function vanishing at p . In the following \tilde{a} will denote either a or the conjugate \bar{a} and we assume (without loss of generality) that each of L^3, L^4 is contained in either H^{10} or H^{01} .

Now the vanishing of the right-hand side in (4.5) is obvious for $k = 4$. For $k = 3$, it takes the form

$$(L^4 \tilde{a})_p (\tilde{L}^3 \langle \theta, [L^2, L^1] \rangle)_p,$$

which must vanish in view of Definition 4.2. For $k = 1$, we obtain

$$(L^4\bar{a})_p(L^3\langle\theta, [L^2, \tilde{L}^1]\rangle)_p + (L^3\bar{a})_p(L^4\langle\theta, [L^2, \tilde{L}^1]\rangle)_p \\ + (L^4L^3\bar{a})_p(\langle\theta, [L^2, \tilde{L}^1]\rangle)_p,$$

which again vanishes in view of Definition 4.2. For $k = 2$, the proof follows from the case $k = 1$ by exchanging L^2 and L^1 and conjugating. Finally, for $\theta = a\tilde{\theta}$, we obtain

$$(L^4a)_p(L^3\langle\tilde{\theta}, [L^2, L^1]\rangle)_p + (L^3a)_p(L^4\langle\tilde{\theta}, [L^2, L^1]\rangle)_p \\ + (L^4L^3a)_p(\langle\tilde{\theta}, [L^2, L^1]\rangle)_p,$$

which vanishes by the same argument. \square

Remark 4.15. — In higher generality, when the cubic tensor τ_p^3 may not vanish completely, a quartic tensor τ_p^4 can still be constructed via (4.5) along certain kernels of τ_p^3 . We will not pursue this direction as our focus here is on the pseudoconvex case when τ_p^3 always vanishes identically.

4.4. Positivity of the quartic tensor

As direct consequence of Lemma 4.14 we obtain:

COROLLARY 4.16. — *Let M be pseudoconvex. Then the quartic tensor τ_p^4 satisfies the following positivity property:*

$$\tau_p^4(v^2, v^2, v^1, \overline{v^1}) \geq 0, \quad v^2 \in T_p, \quad v^1 \in K_p^{10}.$$

Proof. — Since the Levi form $\tau_p^2(v^1, \overline{v^1})$ vanishes for all $v^1 \in K_p^{10}$, the function $x \mapsto i\langle\theta, [L^2, L^1]\rangle_x$ in (4.5) for fixed L^2 and $L^1 = \overline{L^2}$ achieves its local minimum at p . Hence its differential also vanishes at p and the real hessian is positive semidefinite. Then the desired conclusion follows from (4.5). \square

4.5. A normal form up to weight 1/4

Since the cubic normal form for pseudoconvex hypersurfaces (3.15) is in some sense “lacking nondegenerate terms”, we extend it by lowering the weight of z_3 from 1/3 to 1/4 (and renaming z_3 to z_4 to reflect the weight change):

PROPOSITION 4.17. — *For every pseudoconvex real hypersurface M in \mathbb{C}^n and point $p \in M$ of Levi rank q , there exist local holomorphic coordinates*

$$(w, z) = (w, z_2, z_4) \in \mathbb{C} \times \mathbb{C}^q \times \mathbb{C}^{n-q-1},$$

vanishing at p , such that M takes the form

$$(4.6) \quad \rho = 0, \quad \rho = -2 \operatorname{Re} w + \varphi(z, \bar{z}, i(w - \bar{w})), \quad \varphi = \varphi^2 + \varphi^4 + o_w(1),$$

where

$$\varphi^2(z, \bar{z}, u) = \sum_{j=1}^q |z_{2j}|^2, \quad \varphi^4(z, \bar{z}, u) = 2 \operatorname{Re} \varphi^{31}(z_4, \bar{z}_4) + \varphi^{22}(z_4, \bar{z}_4),$$

with

$$\varphi^{31} = \sum \varphi_{j_1 \dots j_4}^{31} z_{4j_1} z_{4j_2} z_{4j_3} \bar{z}_{4j_4} \quad \varphi^{22} = \sum \varphi_{j_1 \dots j_4}^{22} z_{4j_1} z_{4j_2} \bar{z}_{4j_3} \bar{z}_{4j_4},$$

where the weight estimate o_w is calculated for u, z_j^2, z_k^4 , and their conjugates, being assigned the weights $1, \frac{1}{2}, \frac{1}{4}$ respectively. Each polynomial φ^{jk} here is bihomogeneous of bidegree (j, k) in (z^4, \bar{z}^4) .

Furthermore, the following hold:

- (1) For every $v \in K_0^{10} \cong \{0\} \times \mathbb{C}^{n-q-1}$, the vector field L_v given by (3.10) is in the Levi kernel up to v^0 -order 1 at 0 for any $v^0 \in \mathbb{C}K_0$.
- (2) For $v^4, v^3 \in \mathbb{C}K_0$ and $v^2, \bar{v}^1 \in K_0^{10}$, we have

$$(4.7) \quad \tau_p^4(v^4, v^3, v^2, v^1) = \partial_{v^4} \partial_{v^3} \partial_{v^2} \partial_{v^1} \varphi^4.$$

In particular, the restriction

$$(4.8) \quad \tau_p^{40}: \mathbb{C}K_p \times \mathbb{C}K_p \times K_p^{10} \times \overline{K_p^{10}} \rightarrow \mathbb{C}Q_p$$

of τ_p^4 is symmetric in whatever arguments can be exchanged and satisfies the reality condition

$$\overline{\tau_p^4(v^4, v^3, v^2, v^1)} = \tau_p^4(\overline{v^4}, \overline{v^3}, \overline{v^2}, \overline{v^1}),$$

note the switch of the last two arguments.

Proof. — The existence of the desired normal form is a direct consequence of Lemma 3.14. A direct calculation shows the special vector fields in (3.10) with $v \in \{0\} \times \mathbb{C}^{n-q-1}$ are in the Levi kernel up to tangential order 1 as claimed. The remaining properties are straightforward. \square

Similarly to Corollary 3.8, one can also show the following quartic Lie bracket representation:

LEMMA 4.18. — *The restriction τ^{40} of τ^4 satisfies*

$$\langle \theta_p, \tau_p^{41}(L_p^4, L_p^3, L_p^2, L_p^1) \rangle = i \langle \theta_p, [L^4, [L^3, [L^2, L^1]]]_p \rangle$$

whenever $L^2, \bar{L}^1 \in H^{10}$ are in the Levi kernel up to $\mathbb{C}K$ -order 1 at p , $L^3, L^4 \in \mathbb{C}H$, and $\theta \in \Omega^0$ is any contact form.

Here we use the “microlocal” variant of the containment condition in the Levi kernel from Definition 4.2.

Remark 4.19. — It is easy to see that pseudoconvexity of M implies that the quartic polynomial φ^4 in (4.6) is plurisubharmonic. Conversely, every plurisubharmonic φ^4 appears in a normal form of some pseudoconvex hypersurface, e.g. the model hypersurface

$$w + \bar{w} = \sum_{j=1}^q |z_{2j}|^2 + \varphi^4(z_4, \bar{z}_4).$$

Furthermore, taking averages along circles, it is easy to see that plurisubharmonicity of φ^4 implies that of its bidegree $(2, 2)$ component φ^{22} .

4.6. Normal form for vector fields in the Levi kernel up to order 1

In any normal form as in (4.6), our special vector fields that are in the Levi kernel up to order 1, have particularly simple weighted expansion. In fact, we obtain this conclusion under a slightly more general assumption that only requires to differentiate the Levi form in the directions of the Levi kernel. As customary, we shall assign weight $-a$ to coordinate vector field ∂_{z_j} whenever the weight of the coordinate z_j is a .

PROPOSITION 4.20. — *Let M be of the form (4.6) and $L \in H^{10}$ be any vector field such that*

$$(4.9) \quad \langle \theta, [L^1, \bar{L}] \rangle_0 = (L^2 \langle \theta, [L^1, \bar{L}] \rangle)_0 = 0$$

holds for any $\theta \in \Omega^0$, $L^1 \in H^{10}$ and $L^2 \in \mathbb{C}H$ with $L_0^2 \in \mathbb{C}K_0$. Then in the given coordinates and weights as in Proposition 4.17, the vector field L must have weight at least $-1/4$ and there exists vector $v \in \{0\} \times \mathbb{C}^{n-q-1}$ such that L has a weighted expansion

$$(4.10) \quad L = \sum_j a_j (\partial_{z_{4j}} + \varphi_{z_{4j}}^4 (\partial_{\bar{w}} - \partial_w)) + O_w(0),$$

$$a_j \in \mathbb{C}, \quad j = 1, \dots, n-q-1.$$

In particular, L cannot have any other terms of weight $-1/4$ such as

$$(4.11) \quad z_{4j}\partial_{z_{2k}}, \quad \bar{z}_{4j}\partial_{\bar{z}_{2k}}.$$

Vice versa, any vector field (4.10) satisfies (4.9).

Proof. — Since L is in the Levi kernel at 0, its expansion cannot have any vector fields $\partial_{z_{2k}}$, hence L must have weight $\geq -1/4$.

To show (4.10), it suffices to prove that none of the terms (4.11) can occur in the expansion of L . But the latter fact is a direct consequence of (4.9) with

$$L^1 = \partial_{z_{2k}} \quad L^2 = \partial_{z_{4j}} \text{ or } L^2 = \partial_{\bar{z}_{4j}} \quad \text{mod } \partial_w.$$

Finally, assume L has expansion (4.10). In particular, $L_0 \in K_0^{10}$ must hold, hence the first expression in (4.9) vanishes. To show that also the second expression vanishes, in view of Remark 4.8, it suffices to assume that L^1 is either in the Levi kernel up to order 1 or is of the form

$$(4.12) \quad L^1 = \partial_{z_{2j}} + \bar{z}_{2j}\partial_w.$$

In the first case, L^1 also has a weighted expansion similar to (4.10). Since $\theta = i\partial\rho$ has weight ≥ 1 , a direct calculation shows that $[L^1, \bar{L}]$ has weight $\geq -1/4$, and hence $L^2\langle\theta, [L^1, \bar{L}]\rangle$ has weight $\geq -1/4 + 1 - 1/4 = 1/2$, and therefore must vanish at 0.

In the second case, when L^1 is given by (4.12), a direct calculation shows that $[L^1, \bar{L}]$ has weight $\geq -1/2$ and hence $L^2\langle\theta, [L^1, \bar{L}]\rangle$ has weight $\geq -1/4 + 1 - 1/2 = 1/4$, and therefore again must vanish at 0. \square

As a byproduct, we obtain the following consequence:

COROLLARY 4.21. — Let L and L' be two vector fields satisfying the assumptions of Proposition 4.20. Then

$$[L, L']_0, [\bar{L}, L']_0 \in \mathbb{C}K_0.$$

Proof. — In view of Proposition 4.20, L, L' and their conjugates commute in their components of weight $-1/4$. Hence their brackets must have weight $\geq -1/4$ and the statement follows. \square

Recall that in Definition 4.9 we introduced invariant submodule sheaves $\mathcal{S}^{10}(q) \subset H^{10}$ by requiring the condition to be in the Levi kernel up to order 1 to hold at every point of Levi rank $\leq q$. Then as direct application of Corollary 4.21, we obtain:

COROLLARY 4.22. — The invariant submodule sheaf $\mathcal{S}(q) = \mathcal{S}^{10}(q) \oplus \overline{\mathcal{S}^{10}(q)}$ satisfies the following formal integrability condition at all points q

of Levi rank $\leq q$:

$$[\mathcal{S}(q), \mathcal{S}(q)]_p \in \mathbb{C}K_p.$$

Proof. — The statement follows by applying Corollary 4.21 at each point of Levi rank $\leq q$. \square

4.7. Symmetric extension

Similarly to Lemma 3.11, we obtain a symmetric extension for the Levi kernel restriction of τ^4 :

LEMMA 4.23. — *The restriction*

$$\tau_p^{40}: \mathbb{C}K_p \times \mathbb{C}K_p \times K_p^{10} \times \overline{K_p^{10}} \rightarrow \mathbb{C}Q_p$$

of the quartic tensor τ_p^4 admits an unique symmetric extension

$$\tilde{\tau}_p^{40}: \mathbb{C}K_p \times \mathbb{C}K_p \times \mathbb{C}K_p \times \mathbb{C}K_p \rightarrow \mathbb{C}Q_p,$$

satisfying

$$(4.13) \quad \langle \theta_0, \tilde{\tau}_0^{40}(v^4, v^3, v^2, v^1) \rangle = \partial_{v^4} \partial_{v^3} \partial_{v^2} \partial_{v^1} \varphi^4,$$

whenever M is in a normal form $\rho = -2\operatorname{Re} w + \varphi = 0$ as in Proposition 4.17 and $\theta = i\partial\rho$. In fact, (4.13) holds whenever φ satisfies $d\varphi_0 = 0$ and $\partial_{v^j} \partial_{v^2} \partial_{v^1} \varphi^3 = 0$ for $j = 3, 4$.

5. Applications and properties of the quartic tensor

5.1. Relation with the D'Angelo finite type

The quartic tensor τ^4 can be used to completely characterize the finite type up to 4 in the sense of D'Angelo [22] (see also “Property P” in [22, Definition 5.1]):

PROPOSITION 5.1. — *Let M be a pseudoconvex hypersurface with non-trivial Levi kernel at p . Then M is of D'Angelo type 4 at p if and only if for every nonzero vector $v \in K_p^{10}$, the tensor τ_p^4 does not vanish when restricted to*

$$(5.1) \quad (\mathbb{C}v + \mathbb{C}\bar{v}) \times (\mathbb{C}v + \mathbb{C}\bar{v}) \times \mathbb{C}v \times \mathbb{C}\bar{v}.$$

In fact, the latter property implies the following stronger nonvanishing condition:

$$\tau_p^4(v, \bar{v}, v, \bar{v}) \neq 0.$$

Proof. — We may assume M is put into its normal form as in Proposition 4.17.

If the restriction of τ_p^4 vanishes on (5.1) for some $v \neq 0$, we may assume $v = \partial_{z_{31}}$, where $z_3 = (z_{31}, \dots, z_{3,n-r})$. Then it follows from the normal form that the line $\mathbb{C}v$ has order of contact with M higher than 4, hence the D'Angelo type at p is also higher than 4.

On the other hand, suppose the restriction of τ_p^4 to (5.1) does not vanish for any $v \neq 0$. Assume by contradiction, there exists a nontrivial holomorphic curve

$$\gamma: (\mathbb{C}, 0) \rightarrow (\mathbb{C}^{n+1}, 0), \quad \gamma(t) = \sum_{k \geq k_0} a_k t^k, \quad a_{k_0} \neq 0,$$

whose contact order with M at 0 is higher than 4. Recall that the contact order is given by

$$\frac{\nu(\rho \circ \gamma)}{\nu(\gamma)},$$

where ρ is any defining function of M and ν is the vanishing order at 0, in particular, $\nu(\gamma) = k_0 \geq 1$. Taking $\rho := -2 \operatorname{Re} w + \varphi$, we must have $a_{k_0} \in \{0\} \times \mathbb{C}^n$, otherwise the contact order would be 1. Similarly, expanding $\rho \circ \gamma$, it follows by induction that

$$a_l \in \{0\} \times \mathbb{C}^n, \quad l < 4k_0,$$

and

$$a_k \in \{0\} \times \{0\} \times \mathbb{C}^{n-r}, \quad k < 2k_0,$$

for otherwise the contact order would be less than 4. Finally collecting terms of order $4k_0$ and using our assumption that the contact order is greater than 4, we obtain

$$(5.2) \quad \varphi^2(a_{2k_0} t^{2k_0}, \bar{a}_{2k_0} \bar{t}^{2k_0}) + \varphi^4(a_{k_0} t^{k_0}, \bar{a}_{k_0} \bar{t}^{k_0}) = 0.$$

In particular, it follows that

$$\varphi^4(a_{k_0} \xi, \bar{a}_{k_0} \bar{\xi}) = c \xi^2 \bar{\xi}^2.$$

Since φ^4 is plurisubharmonic, we must have $c \geq 0$. Hence both terms in (5.2) are nonnegative, and therefore must vanish. In particular, $c a_{k_0}^2 \bar{a}_{k_0}^2 = 0$, implying

$$\varphi^4(a_{k_0} \xi, \bar{a}_{k_0} \bar{\xi}) = 0,$$

which is in contradiction with our nonvanishing assumption on τ_p^4 . Hence the D'Angelo type is 4 completing the proof of the converse direction.

Finally, the last statement follows from the plurisubharmonicity of φ^4 in any normal form. \square

The last conclusion suggests a connection with the so-called regular type. Recall that the *regular type* of M at p is the maximum (possibly infinite) of the vanishing order $\nu(\rho \circ \gamma)$, where ρ is a defining function of M in a neighborhood of p and $\gamma: (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, p)$ a germ of a regular complex-analytic curve, i.e. satisfying $\gamma'(0) \neq 0$. As a consequence of Proposition 5.1, we obtain the following characterization:

COROLLARY 5.2. — *Let M be a pseudoconvex hypersurface with non-trivial Levi kernel at p . Then the following are equivalent:*

- (1) *M is of D'Angelo type 4 at p ;*
- (2) *M is of regular type 4 at p ;*
- (3) *the quartic tensor satisfies $\tau_p^4(v, \bar{v}, v, \bar{v}) \neq 0$ whenever v is a nonzero vector in the Levi kernel at p .*

Proof. — The equivalence of (1) and (3) is contained in Proposition 5.1. The equivalence of (2) and (3) is obtained by repeating the proof of the proposition for a regular curve. \square

The pseudoconvexity assumption in Proposition 5.1 cannot be dropped:

Example 5.3. — Let $M \subset \mathbb{C}_{w, z_1, z_2}^3$ be given by

$$2 \operatorname{Re} w = |z_1|^2 - |z_2|^4.$$

Then M contains the image of the curve $t \mapsto (0, t^2, t)$ and is hence of infinite type at 0. On the other hand, M is in the normal form (4.6) and hence $\tau_0^4(v, \bar{v}, v, \bar{v}) \neq 0$ for any $v \neq 0 \in K_0^{10}$.

5.2. Uniformity of the quartic tensor

The sheaves $\mathcal{S}^{10}(q)$ introduced in Definition 4.9 can be used to obtain a uniform behavior of τ_p^4 as p varies over the set of nearby points of bounded Levi rank. In fact, as direct consequence from the definition and Corollary 4.11, we obtain that τ_p^4 can be calculated using local sections of $\mathcal{S}^{10}(q)$:

COROLLARY 5.4. — *For every vector fields $L^4, L^3 \in \mathbb{C}T$ and $L^2, \bar{L}^1 \in \mathcal{S}^{10}(q)$ defined in an open set $U \subset M$, the identity (4.5) holds simultaneously for all points $p \in U$ of Levi rank q .*

Remark 5.5. — In the context of Corollary 5.4, it is essential to require the vector fields L^2, L^1 to be contained in Levi kernels up order 1 (rather than merely contained in Levi kernels). In fact, for a higher order perturbation of Examples 4.1 where 0 is the only Levi-degenerate point, choosing

vector fields L^j as higher order perturbations of the vector field L in the example or its conjugate would violate (4.5).

It is important to note that the conclusion of Corollary 5.4 may not hold for points $p \in U$ of Levi rank $> q$ when $L_p^2, \bar{L}_p^1 \in K_p^{10}$. In fact, τ_p^4 may not even be continuous e.g. may vanish for p of higher Levi rank even when $\tau_{p_0}^4$ does not vanish on any line for p_0 of Levi rank q . This is illustrated by D'Angelo's celebrated example where the finite type is not upper-semicontinuous [21, 22], see also Example 4 and its continuation on p. 135–136 in [23]:

Example 5.6 (D'Angelo). — Let $M \subset \mathbb{C}_{w,z_1,z_2}^3$ be given by

$$2 \operatorname{Re} w = |z_1^2 - wz_2|^2 + |z_2|^4.$$

Then M is of Levi rank 0 and finite type 4 at 0 and hence τ_0^4 does not vanish on the lines products (5.1) in view of Proposition 5.1. In fact, M is in its normal form as in Proposition 4.17 with $\varphi^4 = |z_1|^4 + |z_2|^4$, and hence

$$\tau_0^4(v, \bar{v}, v, \bar{v}) = 4(|v_1|^4 + |v_2|^4), \quad v \in K_0^{10} \cong \{0\} \times C_{z_1, z_2}^2.$$

On the other hand, at every $p = (it, 0, 0)$ on the imaginary axis with $t \neq 0$, the Levi rank is 1, and M can be locally transformed into a normal form (4.6) with vanishing φ^4 implying $\tau_p^4(v, \bar{v}, v, \bar{v}) = 0$ for any $v \in K_p^{10}$. Thus $\tau_p^4(v, \bar{v}, v, \bar{v})$ cannot be continuous for any $v = v(p)$ converging to any $v(0) \neq 0$ as $p \rightarrow 0$.

Of course, this phenomenon is closely related to the lack of upper-semicontinuity of the type as demonstrated by D'Angelo. The additional importance of this example and its generalisation in [22, Example 5.16] and [23, Example 4] is the occurrence of the “worst possible lack of semicontinuity” of the type for pseudoconvex hypersurfaces, demonstrating sharpness of the D'Angelo's bound controlling the type in a neighborhood of a point $p \in M$ in terms of the type at p , see [22, Theorem 5.5].

5.3. Kernels of quartic tensors

For any homogenous polynomial, consider the following notion of holomorphic kernel:

DEFINITION 5.7. — *The holomorphic kernel of a homogeneous polynomial $P(z, \bar{z})$, $z \in \mathbb{C}^n$, is defined to be the subspace of all $(1, 0)$ vectors v such that*

$$(5.3) \quad \partial_v P(z, \bar{z}) \equiv \partial_{\bar{v}} P(z, \bar{z}) \equiv 0.$$

Equivalently, the holomorphic kernel is the space of all v such that both v and \bar{v} belong to the kernel of the polarization of p .

It is straightforward to see the following simple characterization of the kernel:

LEMMA 5.8. — The holomorphic kernel of p is the maximal subspace V such that, there exists a linear change of coordinates such that

$$V = \bigoplus_{j=1}^l (\mathbb{C}\partial_{z_j} \oplus \mathbb{C}\partial_{\bar{z}_j})$$

and $P(z, \bar{z})$ is independent of the variables z_1, \dots, z_l and their conjugates.

DEFINITION 5.9. — The rank of P is $n - d$, where d is the dimension of the holomorphic kernel.

Also separating bihomogeneous components in (5.3), we obtain:

LEMMA 5.10. — Let

$$P(z, \bar{z}) = \sum P_{kl}(z, \bar{z})$$

be a decomposition into components P_{kl} of bidegree (k, l) in (z, \bar{z}) . Then the holomorphic kernel of P equals the intersection of kernels of P_{kl} for all k, l .

Next we compare the holomorphic kernel of the polynomial φ^4 in the normal form given by Proposition 4.17 and the restriction

$$\tau_p^{40} : \mathbb{C}K_p \times \mathbb{C}K_p \times K_p^{10} \times \overline{K_p^{10}} \rightarrow \mathbb{C}Q_p.$$

of the quartic tensor τ_p^4 to the Levi kernel in each component.

DEFINITION 5.11. — The holomorphic kernel of τ_p^{40} is $V \cap \overline{V}$, where

$$V = \ker \tau_p^{40} = \{v \in \mathbb{C}K_p : \tau_p^{40}(v, v^3, v^2, v^1) = 0 \text{ for all } v^3, v^2, v^1\}.$$

First of all, remark that without pseudoconvexity assumption, the holomorphic kernel of τ_p^{40} may get larger than that of φ^4 :

Example 5.12. — Let $M \subset \mathbb{C}_{w, z_1, z_2}^3$ be given by

$$2 \operatorname{Re} w = \varphi^4(z, \bar{z}) := 2 \operatorname{Re}(z_1^3 \bar{z}_2).$$

Then the arguments in the proof of Proposition 4.17 can be used to show that (4.7) still holds, implying that ∂_{z_2} and $\partial_{\bar{z}_2}$ are in the kernel of τ_0^{40} in the 1st and 2nd arguments but not in the 3rd one.

On the other hand, in presence of pseudoconvexity, both kernels must coincide as the following lemma shows. As a matter of convention, for a multilinear function $f(v^1, \dots, v^m)$, we call its kernel in the k th argument the space of all v^k such that $f(v^1, \dots, v^m) = 0$ holds for all v^j with $j \neq k$.

LEMMA 5.13. — *Let M be in its normal form given by Proposition 4.17, and assume that M is pseudoconvex. Then both holomorphic kernels of τ_p^{40} in the 1st and 2nd arguments coincide with holomorphic kernel V of φ^4 . Furthermore, the kernels of τ_p^{40} in the 3rd and 4th arguments coincide respectively with V and \bar{V} .*

Proof. — As direct consequence of (4.7) we obtain that the holomorphic kernel of φ^4 is contained in the kernel of τ_p^{40} in each argument.

Vice versa, let v be $(1, 0)$ vector in the holomorphic kernel of τ_p^{40} (in the 1st argument). We write $\xi = z_4$ for brevity. After a linear change of coordinates we may assume $v = \partial_{\xi_1}$, where ξ_1 is the first component of ξ in the notation of Proposition 4.17. Then it follows from (4.7) that $\partial_{\xi_1} \varphi^4$ is harmonic. Since φ^4 has no harmonic terms, it must have the form

$$\varphi^4 = 2 \operatorname{Re}(\bar{\xi}_1 h(\xi)) + R,$$

where h is holomorphic and R is independent of ξ_1 . Now since M is pseudoconvex, φ^4 is plurisubharmonic, in particular,

$$(5.4) \quad (\partial_{\xi_1} + t\partial_{\bar{\xi}_j})(\partial_{\bar{\xi}_1} + t\partial_{\bar{\xi}_j})\varphi^4 \geq 0$$

holds for all $t \in \mathbb{R}$. Then for $t = 0$, we obtain $\partial_{\xi_1} h \geq 0$. Since h is holomorphic, we must have $\partial_{\xi_1} h \equiv 0$. Hence the linear part of (5.4) must be ≥ 0 and therefore equal to 0, since t is any real number. But this means $h \equiv 0$, and hence $v = \partial_{\xi_1}$ is in the holomorphic kernel of φ^4 as claimed.

The claimed statements for kernels in other arguments of τ_p^{40} are obtained by repeating the same proof. \square

In view of Lemma 5.13, we simply refer to the *holomorphic kernel of τ_p^{40}* for its kernel in the 1st (and, equivalently, in the 2nd) argument. Also the *rank of τ_p^{40}* is $\dim K_p^{10} - d$, where d is the dimension of its holomorphic kernel, which coincides with the rank of φ^4 in the sense of Definition 5.9.

6. Relation with Catlin's multitype

We first recall the definition of the multitype due to Catlin [7]. Consider arbitrary ordered weights

$$(6.1) \quad \Lambda = (\lambda_1, \dots, \lambda_n), \quad 1 \leq \lambda_1 \leq \dots \leq \lambda_n \leq \infty,$$

and for a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$, define its weight by

$$\|\alpha\|_\Lambda := \lambda_1^{-1}\alpha_1 + \dots + \lambda_n^{-1}\alpha_n.$$

DEFINITION 6.1. — A weight Λ is called admissible if for each $k = 1, \dots, n$, either $\lambda_k = +\infty$ or there exists a k -tuple of nonnegative integers $a = (a_1, \dots, a_k)$ satisfying $a_k > 0$ and $\|a\|_\Lambda = 1$.

Next, for a smooth function $\rho(z, \bar{z})$ defined in a neighborhood of 0, write

$$(6.2) \quad \rho = O_\Lambda(1)$$

whenever all nonzero monomials $\rho_{z^\alpha \bar{z}^\beta} z^\alpha \bar{z}^\beta$ in the Taylor expansion of ρ at 0 satisfy $\|\alpha + \beta\|_\Lambda \geq 1$, i.e. are of weight ≥ 1 , or, equivalently, whenever the estimate

$$|\rho(z, \bar{z})| \leq C(|z_1|^{\lambda_1} + \dots + |z_n|^{\lambda_n})$$

holds in a neighborhood of 0 for some $C > 0$. Finally, we regard (μ_1, \dots, μ_n) as lexicographically smaller than $(\lambda_1, \dots, \lambda_n)$ whenever $\mu_j < \lambda_j$ holds for the smallest j such that $\mu_j \neq \lambda_j$.

DEFINITION 6.2 ([7]). — The multitype of a smooth real hypersurface $M \subset \mathbb{C}^n$ at a point $p \in M$ is the lexicographic supremum of the set of all admissible weights Λ such that there exist local holomorphic coordinates in a neighborhood of p and vanishing at p , where M is given by a defining function ρ satisfying $\rho = O_\Lambda(1)$.

Note that since a defining function ρ is unique up to a nonvanishing smooth factor, the property $\rho = O_\Lambda(1)$ is independent of the choice of ρ (but, of course, in general, it does depend on the coordinates).

Catlin's multitype is an essential ingredient in his proof of the Property (P) [8], based on [7], required for the proof of global regularity of the $\bar{\partial}$ -Neumann problem, as well as for its quantitative analogue in [9] required for the proof of subelliptic estimates. However, in practice, the multitype is difficult to compute, due to the nature of its definition requiring taking lexicographic supremum over arbitrary holomorphic coordinate charts.

In [47], Kolar gave a remarkable general algorithm for computing the multitype of a given hypersurface, which involves taking certain consecutive coordinate normalisations. However, the number of steps involved there equals the dimension, and each step depends on the previous coordinate choice.

In our case, we can use the quartic tensor τ_4 to have an algebraically invariant way of calculating part of the multitype, that works regardless of the dimension n and independent of coordinates. In case τ_4 has trivial kernel, this approach gives the complete multitype.

PROPOSITION 6.3. — *Let $M \subset \mathbb{C}^n$ be a pseudoconvex hypersurface, and $p \in M$ a point with Levi form of rank q_2 and the restricted quartic tensor τ^{40} of rank q_4 . Then the multitype $\Lambda = (\lambda_1, \dots, \lambda_n)$ of M at p satisfies*

$$(6.3) \quad \lambda_1 = 1, \quad \lambda_2 = \dots \lambda_{q_2+1} = 2, \quad \lambda_{q_2+2} = \dots = \lambda_{q_2+q_4+1} = 4,$$

and

$$(6.4) \quad \lambda_k > 4, \quad k > q_2 + q_4 + 1.$$

In particular, if τ^{40} has only trivial kernel, the multitype is $(1, 2, \dots, 2, 4, \dots, 4)$, where the number of 2's equals the Levi rank.

Proof. — By Lemma 5.8, in addition to the normal form in Proposition 4.17, we can make φ^4 independent of the last d coordinates, where d is the dimension of the $(1, 0)$ kernel of τ_4 . This shows that it is possible to achieve (6.2) with weights satisfying both (6.3) and (6.4).

The actual multitype may only be lexicographically higher, in particular, (6.4) is already satisfied. Assume by contradiction that we have another choice of coordinates with higher weights failing one of the equalities in (6.3). However, we must obviously have $\lambda_1 = 1$ and the Levi form invariance forces the next q_2 weights to be equal 2. Therefore we must have some $\lambda_k > 4$ for $k \leq q_2 + q_4 + 1$. In those coordinates, we would have the same normal form as in Proposition 4.17 with φ^4 being independent of z_j at least for $j \geq q_2 + q_4 + 1$. That, however, would mean that the rank of τ_4 is less than q_4 , which is a contradiction. Hence the multitype must satisfy all of (6.3) as claimed, where the admissibility condition from Definition 6.1 is clearly satisfied. \square

7. Ideal sheaves for the Levi rank level sets

We use the vector field submodule sheaves $\mathcal{S}^{10}(q)$ in Definition 4.9 to define invariant ideal sheaves of smooth functions $\mathcal{I}(q)$ (as in Theorem 2.1(5)):

DEFINITION 7.1. — *Let $M \subset \mathbb{C}^n$ be a pseudoconvex hypersurface. For every q , define $\mathcal{I}(q)$ to be the ideal sheaf generated by all (smooth complex) functions g, f of the form*

$$g = \langle \theta, [L^2, L^1] \rangle, \quad f = L^3 \langle \theta, [L^2, L^1] \rangle,$$

where $\theta \in \Omega^0$ is a contact form, $L^3 \in \mathbb{C}T$ arbitrary complex vector field, and $L^2, \overline{L^1} \in \mathcal{S}^{10}(q)$ arbitrary sections.

Remark 7.2. — The same ideal sheaf $\mathcal{I}(q)$ is generated by the functions g, f as in Definition 7.1, where the vector fields L^j can be chosen from a fixed finite set of generators of $\mathbb{C}T$ and $\mathcal{S}^{10}(q)$. That is due to the linearity of g with respect to smooth functions, and the linearity of f modulo the ideal generated by all functions g .

Example 7.3. — Example 4.12 shows that for $q = 0$, the sheaf $\mathcal{S}^{10}(0)$ contains all germs of all $(1, 0)$ vector fields. Then the ideal sheaf $\mathcal{I}(0)$ is generated by all Levi form entries and their first order derivatives.

On the opposite end, Example 4.12 shows that for $q = n - 1$, the sheaf $\mathcal{S}^{10}(n - 1)$ consists of all germs of $(1, 0)$ vector fields that are everywhere contained in the Levi kernel. Such sheaf is always trivial when the hypersurface M is generically Levi-nondegenerate, which is the case e.g. whenever M is of finite type. In the latter case, the ideal sheaf $\mathcal{I}(n - 1)$ is also trivial (identically zero), which corresponds to the simple fact that the set of points of the maximal Levi rank $n - 1$ is never contained in a proper submanifold.

Example 7.4. — Let M be as in Example 4.13. Then away from the subset $M^1 = \{z_1 z_2 = 0\} \subset M$ of the points of Levi rank ≤ 1 , the ideal sheaf $\mathcal{I}(1)$ is generated by the Levi form entries $|z_1|^2, |z_2|^2$, that generate all smooth germs of functions there. This is expected as that set consists of the Levi-nondegenerate points.

On the other hand, along the Levi-degeneracy set M^1 , the Levi form and its derivatives in Definition 7.1 need to be computed along vector fields $L^2, \overline{L^1}$ from the submodule $\mathcal{S}^{10}(1)$. In view of Example 4.13, the Levi form entries $g = \langle \theta, [L^2, L^1] \rangle$ always vanish of order at least 2 along M^1 , whereas their derivatives $f = L^3 \langle \theta, [L^2, L^1] \rangle$ generate the maximal ideal of M^1 away from $z_1 = z_2 = 0$.

As a direct consequence of Corollary 4.11 and Lemma 3.14, we obtain a general way of constructing submanifolds containing level sets of the Levi rank:

COROLLARY 7.5. — *Let M be a pseudoconvex hypersurface. Then every local section in $\mathcal{I}(q)$ vanishes at all points of Levi rank q . In particular, for any collection f^1, \dots, f^m of real functions from the real part $\text{Re } \mathcal{I}(q)$ defined in an open set $U \subset M$ satisfying*

$$(7.1) \quad df^1 \wedge \cdots \wedge df^m \neq 0,$$

the submanifold

$$S = \{f^1 = \cdots = f^m = 0\}$$

contains the set of all points of Levi rank q in U . In fact, the set S still has the same property without assuming (7.1).

Remark 7.6. — Note that due to our definition of $\mathcal{I}(q)$, any complex multiple of a local section is again a local section. Consequently, it suffices to take only sections in $\text{Re}\mathcal{I}(q)$ to define the same set.

We next apply the quartic tensor to describe the differentials of sections in $\mathcal{I}(q)$.

DEFINITION 7.7. — For an ideal sheaf \mathcal{I} define its kernel at p

$$\ker_p \mathcal{I} \subset \mathbb{C}T_p,$$

to be the intersection of kernels of all differentials df_p , where f is any local section of \mathcal{I} in a neighborhood of p .

Then Corollary 7.5 implies:

LEMMA 7.8. — Let $p \in M$ be a point of Levi rank q . Then the kernel of the q th sublevel ideal $\mathcal{I}(q)$ at p coincides with the kernel of the quartic tensor τ_p^4 .

We can now summarize this paragraph's results as follows:

PROPOSITION 7.9. — Let $M \subset \mathbb{C}^n$ be a pseudoconvex real hypersurface, and $p \in M$ a point of Levi rank q . Then in a neighborhood of p , the set of all points of the same Levi rank q is contained in a real submanifold $S \subset M$ through p such that

$$T_p S = \ker \tau_p^4,$$

and S is given by the vanishing of local sections

$$f^1, \dots, f^m \in \mathcal{I}(q), \quad df^1 \wedge \cdots \wedge df^m \neq 0.$$

In particular, when M is of finite type 4 at p , the intersection of $T_p S$ with the Levi kernel at p is totally real.

8. Relation with Catlin's boundary systems

8.1. Maximal Levi-nondegenerate subbundles

Recall that Catlin's boundary system construction for a hypersurface M at a point $p \in M$ begins with a maximal collection of $(1, 0)$ vector fields L_2, \dots, L_{q+1} tangent to M such that the Levi form matrix

$$(\langle \theta, [L_j, \bar{L}_k] \rangle)_{2 \leq j, k \leq q}$$

is nonsingular. In particular, q must be equal to the Levi rank at p .

Invarantly, consider any *maximal Levi-nondegenerate subbundle* through p , i.e. any smooth subbundle $V^{10} \subset H^{10}$ where the restriction of the Levi form is nondegenerate. Then obviously any such V^{10} appears as the span of the first q vector fields in Catlin's boundary system, and vice versa, every such span is a maximal Levi-nondegenerate subbundle.

Next Catlin considers the Levi-orthogonal subbundle

$$S^{10} := (V^{10})^\perp \subset H^{10}$$

(T_{q+2}^{10} in Catlin's notation). In particular, the subbundle S^{10} contains all Levi kernels K_x^{10} at all points $x \in M$ near p , even when $\dim K_x^{10}$ depends on x . That makes the fiber $(V_x^{10})^\perp$ unique whenever the Levi rank at x is the same as p , even when V_x^{10} itself may not be unique. On the other hand, at points x of higher Levi rank, $(V_x^{10})^\perp$ clearly depends on the choice of V_x^{10} .

The rest of Catlin's boundary system construction only depends on the subbundle S^{10} rather than on V^{10} and its chosen basis.

8.2. Levi kernel inclusion of higher order

As mentioned before, S^{10} contains the Levi kernel at every point. On the other hand, if M is pseudoconvex, we have shown in Lemma 4.6 that S^{10} is itself contained in the Levi kernel up to order 1 at p as defined in Definition 4.2. That permits to use arbitrary sections of S^{10} in the calculation of the quartic tensor τ^4 :

COROLLARY 8.1. — *Let M be a pseudoconvex hypersurface, $V^{10} \subset H^{10}$ a maximal Levi-nondegenerate subbundle at $p \in M$, and S^{10} the Levi-orthogonal complement of V^{10} . Then the quartic tensor τ_p^4 defined by Lemma 4.14 satisfies*

$$\langle \theta_p, \tau_p^4(L_p^4, L_p^3, L_p^2, L_p^1) \rangle = i(L^4 L^3 \langle \theta, [L^2, L^1] \rangle)_p$$

for any $L^4, L^3 \in \mathbb{C}T$, $L^2, \overline{L^1} \in S^{10}$ and $\theta \in \Omega^0$.

8.3. Relation with the rest of Catlin's boundary system construction

The remaining part of Catlin's construction is based on the higher order Levi form derivatives

$$(8.1) \quad \mathcal{L}\theta := L^m \dots L^3 \langle \theta, [L^2, L^1] \rangle, \quad \mathcal{L} = (L^m, \dots, L^1),$$

where $\theta = \partial r$ and r is a defining function of M . Then a boundary system

$$(8.2) \quad \mathcal{B} = \{r_1, r_{q_2+2}, \dots, r_\nu; L_2, \dots, L_\nu\}, \quad q_2 + 2 \leq \nu \leq n,$$

is constructed together with associated weights

$$\alpha_1 = 1 < \alpha_2 = \dots = \alpha_q = 2 < \alpha_{q+1} \leq \dots \leq \alpha_\nu \leq \infty,$$

where $r_1 = r$ is the given defining function, L_j and r_j are respectively smooth $(1, 0)$ vector fields and smooth real functions in a neighborhood of p . The construction proceeds by induction as follows. Assuming a boundary system is constructed for given ν , define the next subbundle

$$T_{\nu+1}^{10} := \{L \in T_{q_2+2}^{10} : \partial r_{q_2+2}(L) = \dots = \partial r_\nu(L) = 0\}.$$

Then count all previous L_j and their conjugates with weight α_j , and consider a new vector field $L_{\nu+1} \in T_{\nu+1}^{10}$ and its conjugate, whose weight $\alpha = \alpha_{\nu+1}$ is to be determined. Now look for all lists $\mathcal{L} = (L^m, \dots, L^1)$ with each $L^k \in \{L_{q_2+2}, \dots, L_{\nu+1}\}$, which are of total weight 1 and ordered, i.e. L_j, \bar{L}_j precede L_k, \bar{L}_k whenever $j > k$, such that

$$(8.3) \quad (\mathcal{L}\partial\rho)_p \neq 0.$$

The list must contain the new vector field $L_{\nu+1}$ or its conjugate, and the new weight $\alpha_{\nu+1}$ is chosen to be minimal possible with that property. Finally set either

$$r_{\nu+1} := \operatorname{Re} L^{m-1} \dots L^3 \langle \theta, [L^2, L^1] \rangle \text{ or } r_{\nu+1} := \operatorname{Im} L^{m-1} \dots L^3 \langle \theta, [L^2, L^1] \rangle$$

such that

$$(L_{\nu+1} r_{\nu+1})_p \neq 0,$$

which is always possible in view of (8.3), since the first vector field in the list, L^m is either $L_{\nu+1}$ or its conjugate. Restating Lemma 4.6 and Corollary 8.1, we have:

COROLLARY 8.2. — Let M be a pseudoconvex hypersurface with Levi form of rank q at p . Fix a (Catlin's) boundary system $\{L_2, \dots, L_{q+1}\}$ at p . Then $S^{10} = T_{q+2}^{10} = V^\perp$ for

$$V := \operatorname{span}\{L_2, \dots, L_{q+1}\}.$$

Furthermore, for any vector fields $L^4, L^3 \in S^{10} + \overline{S^{10}}$, $L^2 \in S^{10}$, $L^1 \in \overline{S^{10}}$, we have

$$\begin{aligned} L^3 \langle \theta, [L^2, L^1] \rangle_p &= 0, \\ L^4 L^3 \langle \theta, [L^2, L^1] \rangle_p &= \tau_p^{40}(L_p^4, L_p^3, L_p^2, L_p^1). \end{aligned}$$

In other words, for lists \mathcal{L} of length 3, the derivative $(\mathcal{L}\theta)_p$ vanishes, whereas for lists of length 4, it only depends on the vector field values at p and is

given by the restricted quartic tensor τ_4 (regardless of the choice of the boundary system).

Thus via the quartic tensor restriction τ_p^{40} , the nonvanishing condition in (8.3) is reduced to a purely algebraic property only depending on the vector fields' values at p .

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