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A COMPACTIFICATION OF OUTER SPACE WHICH IS AN ABSOLUTE RETRACT

by Mladen BESTVINA & Camille HORBEZ (*)

ABSTRACT. — We define a new compactification of outer space CV_N (the *Pacman compactification*) which is an absolute retract, for which the boundary is a Z -set. The classical compactification \overline{CV}_N made of very small F_N -actions on \mathbb{R} -trees, however, fails to be locally 4-connected as soon as $N \geq 4$. The Pacman compactification is a blow-up of \overline{CV}_N , obtained by assigning an orientation to every arc with nontrivial stabilizer in the trees.

RÉSUMÉ. — Nous définissons une nouvelle compactification de l'outre espace CV_N (la *compactification de Pacman*) qui est un rétract absolu et dont le bord est un Z -ensemble. À l'inverse, pour tout $N \geq 4$, la compactification classique \overline{CV}_N , qui consiste en les actions très petites de F_N sur des arbres réels, n'est pas localement 4-connexe. La compactification de Pacman est un éclatement de \overline{CV}_N , obtenu en attribuant une orientation à tout arc à stabilisateur non trivial dans ces arbres réels.

1. Introduction

The study of the group $\text{Out}(F_N)$ of outer automorphisms of a finitely generated free group has greatly benefited from the study of its action on Culler–Vogtmann's outer space CV_N . It is therefore reasonable to look for compactifications of CV_N that have “nice” topological properties. The goal of the present paper is to construct a compactification of CV_N which is a compact, contractible, finite-dimensional absolute neighborhood retract (ANR), for which the boundary is a Z -set.

One motivation for finding “nice” actions of a group G on absolute retracts comes from the problem of solving the Farrell–Jones conjecture for

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G , see [3, Section 1]. For instance, it was proved in [6] that the union of the Rips complex of a hyperbolic group together with the Gromov boundary is a compact, contractible ANR, and this turned out to be a crucial ingredient in the proof by Bartels–Lück–Reich of the Farrell–Jones conjecture for hyperbolic groups [3]. A similar approach was recently used by Bartels to extend these results to the context of relatively hyperbolic groups [1], and by Bartels–Bestvina to the context of mapping class groups [2].

We review some terminology. A compact metrizable space X is said to be an *absolute (neighborhood) retract* (AR or ANR) if for every compact metrizable space Y that contains X as a closed subset, the space X is a (neighborhood) retract of Y . Given $x \in X$, we say that X is *locally contractible (LC)* at x if for every open neighborhood \mathcal{U} of x , there exists an open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of x such that the inclusion map $\mathcal{V} \hookrightarrow \mathcal{U}$ is nullhomotopic. More generally, X is *locally n -connected (LC^n)* at x if for every open neighborhood \mathcal{U} of x , there exists an open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of x such that for every $0 \leq i \leq n$, every continuous map $f : S^i \rightarrow \mathcal{V}$ from the i -sphere is nullhomotopic in \mathcal{U} . We then say that X is *LC* (or *LC^n*) if it is *LC* (or *LC^n*) at every point $x \in X$.

A nowhere dense closed subset Z of a compact metrizable space X is a *Z-set* if X can be instantaneously homotoped off of Z , i.e. if there exists a homotopy $H : X \times [0, 1] \rightarrow X$ so that $H(x, 0) = x$ and $H(X \times (0, 1]) \subseteq X \setminus Z$. Given $z \in Z$, we say that Z is *locally complementarily contractible (LCC)* at z , resp. *locally complementarily n -connected (LCC^n)* at z , if for every open neighborhood \mathcal{U} of z in X , there exists a smaller open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of z in X such that the inclusion $\mathcal{V} \setminus Z \hookrightarrow \mathcal{U} \setminus Z$ is nullhomotopic in X , resp. trivial in π_i for all $0 \leq i \leq n$. We then say that Z is *LCC* (resp. *LCC^n*) in X if it is *LCC* (resp. *LCC^n*) at every point $z \in Z$.

Every ANR space is locally contractible. Further, if X is an ANR, and $Z \subseteq X$ is a Z-set, then Z is *LCC* in X . Conversely, it is a classical fact that every finite-dimensional, compact, metrizable, locally contractible space X is an ANR, and further, an n -dimensional *LC^n* compact metrizable space is an ANR, see [23, Theorem V.7.1]. If X is further assumed to be contractible then X is an AR. In Appendix B of the present paper, we will establish (by similar methods) a slight generalization of this fact, showing that if X is an n -dimensional compact metrizable space, and $Z \subseteq X$ is a nowhere dense closed subset which is *LCC^n* in X , and such that $X \setminus Z$ is *LC^n* , then X is an ANR and Z is a Z-set in X .

Culler–Vogtmann’s *outer space* CV_N can be defined as the space of all F_N -equivariant homothety classes of free, minimal, simplicial, isometric

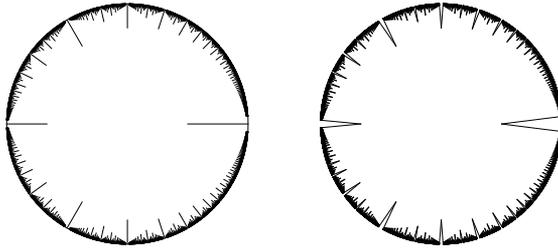


Figure 1.1. The reduced parts of the classical compactification \overline{CV}_2 (on the left), and of the Pacman compactification \overline{CV}_2 (on the right).

F_N -actions on simplicial metric trees (with no valence 2 vertices). Culler–Morgan’s compactification of outer space [9] can be described by taking the closure in the space of all F_N -equivariant homothety classes of minimal, nontrivial F_N -actions on \mathbb{R} -trees, equipped with the equivariant Gromov–Hausdorff topology. The closure \overline{CV}_N identifies with the space of homothety classes of minimal, very *small* F_N -trees [4, 8, 22], i.e. those trees whose arc stabilizers are cyclic and root-closed (possibly trivial), and whose tripod stabilizers are trivial. Outer space CV_N is contractible and locally contractible [10].

When $N = 2$, the closure \overline{CV}_2 was completely described by Culler–Vogtmann in [11]. The closure of reduced outer space (where one does not allow for separating edges in the quotient graphs) is represented on the left side of Figure 1.1: points in the circle at infinity represent actions dual to measured foliations on a once-punctured torus, and there are “spikes” coming out corresponding to simplicial actions where some edges have non-trivial stabilizer. These spikes prevent the boundary $\overline{CV}_2 \setminus CV_2$ from being a Z -set in \overline{CV}_2 : these are locally separating subspaces in \overline{CV}_2 , and therefore \overline{CV}_2 is not LCC (it is not even LCC^0) at points on these spikes. More surprisingly, while \overline{CV}_2 is an absolute retract (and we believe that so is \overline{CV}_3), this property fails as soon as $N \geq 4$.

THEOREM 1.1. — *For all $N \geq 4$, the space \overline{CV}_N is not locally 4-connected, hence it is not an AR.*

There are however many trees in \overline{CV}_N at which \overline{CV}_N is locally contractible: for example, we prove in Section 2.2 that \overline{CV}_N is locally contractible at any tree with trivial arc stabilizers. The reason why local 4-connectedness fails in general is the following. When $N \geq 4$, there are trees in \overline{CV}_N that contain both a subtree dual to an arational measured foliation

on a nonorientable surface Σ of genus 3 with a single boundary curve c , and a simplicial edge with nontrivial stabilizer c . We construct such a tree T_0 (see Section 2 for its precise definition), at which \overline{CV}_N fails to be locally 4-connected, due to the combination of the following two phenomena.

- The space $X(c)$ of trees where c fixes a nondegenerate arc locally separates \overline{CV}_N at T_0 .
- The subspace of $\mathcal{PMF}(\Sigma)$ made of foliations which are dual to very small F_N -trees contains arbitrarily small embedded 3-spheres which are not nullhomologous: these arise as $\mathcal{PMF}(\Sigma')$ for some orientable subsurface $\Sigma' \subseteq \Sigma$ which is the complement of a Möbius band in Σ . Notice here that a tree dual to a geodesic curve on Σ may fail to be very small, in the case where the curve is one-sided in Σ .

We will find an open neighborhood \mathcal{U} of T_0 in \overline{CV}_N such that for any smaller neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of T_0 , we can find a 3-sphere S^3 in $X(c) \cap \mathcal{V}$ (provided by the second point above) which is not nullhomologous in $X(c) \cap \mathcal{U}$, but which can be capped off by balls B_{\pm}^4 in each of the two complementary components of $X(c) \cap \mathcal{V}$ in \mathcal{V} . By gluing these two balls along their common boundary S^3 , we obtain a 4-sphere in \mathcal{V} , which is shown not to be nullhomotopic within \mathcal{U} by appealing to a Čech homology argument, presented in Appendix A of the paper.

It would be of interest to have a better understanding of the topology of the space $\mathcal{PMF}(\Sigma)$ in order to have a precise understanding of the failure of local connectivity of \overline{CV}_N .

QUESTION. — *Is \overline{CV}_N locally 3-connected for every N ? Is \overline{CV}_3 locally contractible?*

However, we also build a new compactification \widehat{CV}_N of CV_N (a blow-up of \overline{CV}_N) which is an absolute retract, for which the boundary is a Z -set. The remedy to the bad phenomena described above is to prescribe orientations in an F_N -equivariant way to all arcs with nontrivial stabilizers in trees in \overline{CV}_N , which has the effect in particular to “open up” \overline{CV}_N at the problematic spaces $X(c)$. In other words, the characteristic set of any element $g \in F_N \setminus \{e\}$ in a tree T is given an orientation as soon as it is not reduced to a point: when g acts as a hyperbolic isometry of T , its axis comes with a natural orientation, and we also decide to orient the edges with nontrivial stabilizers. Precise definitions of \widehat{CV}_N and its topology are given in Section 3 of the present paper. In rank 2, this operation has the

effect of “cutting” along the spikes (see Figure 1.1), which leads us to call this new compactification the *Pacman compactification* of outer space.

THEOREM 1.2. — *The space \widehat{CV}_N is an absolute retract of dimension $3N - 4$, and $\widehat{CV}_N \setminus CV_N$ is a Z -set.*

The space \widehat{CV}_N is again compact, metrizable and finite-dimensional: this is established in Section 3 of the present paper from the analogous results for \overline{CV}_N . Also, we show in Section 4 that every point in \widehat{CV}_N is a limit of points in CV_N . The crucial point for proving Theorem 1.2 is to show that the boundary $\widehat{CV}_N \setminus CV_N$ is locally complementarily contractible.

The proof of this last fact is by induction on the rank N , and the strategy is the following. Given a tree $T \in \widehat{CV}_N \setminus CV_N$, one can first approximate T by trees that split as graphs of actions over free splittings of F_N , and admit 1-Lipschitz F_N -equivariant maps to T . Using our induction hypothesis (and working in the outer space of each of the factors that are elliptic in the splitting), we prove that subspaces in \widehat{CV}_N made of trees that split as graphs of actions over a given free splitting are locally complementarily contractible at every point in the boundary. We also find a continuous way of deforming a neighborhood of T into one of these subspaces, so that T is sent to a nearby tree. This enables us to prove that $\widehat{CV}_N \setminus CV_N$ is locally complementarily contractible at T .

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2. The space \widehat{CV}_N is not an AR when $N \geq 4$.

2.1. Review: Outer space and Culler–Morgan’s compactification

Outer space and its closure. Let $N \geq 2$. *Outer space* CV_N (resp. *unprojectivized outer space* cv_N) is the space of F_N -equivariant homothety (resp. isometry) classes of simplicial, free, minimal, isometric F_N -actions on simplicial metric trees, with no valence 2 vertices. Here we recall that an F_N -tree is *minimal* if it does not contain any proper F_N -invariant subtree. Unprojectivized outer space can be embedded into the space of all F_N -equivariant isometry classes of minimal F_N -actions on \mathbb{R} -trees, which is equipped with the *equivariant Gromov–Hausdorff topology* introduced in [29, 30]. This is the topology for which a basis of open neighborhoods of a tree T is given by the sets $\mathcal{N}_T(K, X, \varepsilon)$ (where $K \subseteq T$ is a finite set of

points, $X \subseteq F_N$ is a finite subset, and $\varepsilon > 0$), defined in the following way: an F_N -tree T' belongs to $\mathcal{N}_T(K, X, \varepsilon)$ if there exists a finite set $K' \subseteq T'$ and a bijection $K \rightarrow K'$ such that for all $x, y \in K$ and all $g \in X$, one has $|d_{T'}(x', gy') - d_T(x, gy)| < \varepsilon$ (where x', y' are the images of x, y under the bijection). The closure $\overline{cv_N}$ was identified in [4, 8, 22] with the space of F_N -equivariant isometry classes of minimal, very small actions of F_N on \mathbb{R} -trees (an action is called *very small* if arc stabilizers are cyclic and root-closed [possibly trivial], and tripod stabilizers are trivial). Note that we allow for the trivial action of F_N on a point in $\overline{cv_N}$. The compactification $\overline{CV_N}$ is the space of homothety classes of nontrivial actions in $\overline{cv_N}$. In the present paper, for carrying induction arguments, we will need to allow for the case where $N = 1$, in which case cv_1 is the collection of all possible isometry classes of \mathbb{Z} -actions on the real line (these are just parameterized by the translation length of the generator), and $\overline{cv_1}$ is obtained by adding the trivial action on a point.

Structure of the trees in $\overline{cv_N}$: the Levitt decomposition. A *splitting* of F_N is a minimal, simplicial F_N -tree. A tree $T \in \overline{cv_N}$ is said to *split* as a *graph of actions* over a splitting S if there exist

- for each vertex u of S with stabilizer G_u , a G_u -tree T_u such that if e is an edge of S incident on u , then G_e is elliptic in T_u ,
- for each edge $e = uv$ of S , points $x_{e,u} \in T_u$ and $x_{e,v} \in T_v$, both stabilized by G_e ,
- for each edge $e = uv$ of S , a segment $I_e = [y_u, y_v]$ (possibly of length 0),

where all these data are F_N -equivariant, such that T is obtained from the disjoint union of the trees T_u and the segments I_e by attaching every vertex $x_{e,u} \in T_u$ to the extremity y_u of I_e . See e.g. [27], although in the present paper, we allow some of the segments I_e to have length 0.

By a result of Levitt [27], every tree $T \in \overline{cv_N}$ splits uniquely as a graph of actions in such a way that vertices of the splitting correspond to connected components of the closure of the set of branch points in T , and edges correspond to maximal arcs whose interior contains no branch point of T . This splitting will be referred to as the *Levitt decomposition* of T . All vertex actions $G_v \curvearrowright T_v$ of this decomposition have dense G_v -orbits (the group G_v might be trivial, and the tree T_v might be reduced to a point). The underlying simplicial tree S of the splitting is very small, and all its edges e yield segments I_e of positive length in T . Every very small F_N -tree with dense orbits has trivial arc stabilizers, see e.g. [4, Remark 1.9] or [15,

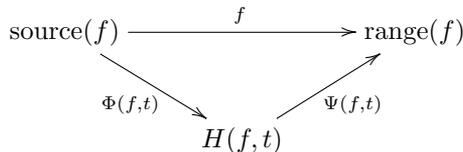
Proposition I.10]. Therefore, every tree $T \in \overline{CV_N}$ has only finitely many orbits of maximal arcs with nontrivial stabilizer. Moreover, if $1 \neq Z < F_N$ is cyclic the fixed point set $\text{Fix}(Z) \subset T$ is empty, or a point, or an arc, and if in addition $1 \neq Z' < Z$ then $\text{Fix}(Z') = \text{Fix}(Z)$.

Characteristic sets of elements in a very small F_N -tree. The characteristic set $\text{Char}_T(g)$ of an element $g \in F_N$ in an F_N -tree T is its axis if g is hyperbolic and its fixed point set if g is elliptic. When T is very small, the characteristic set of a nontrivial elliptic element is a closed interval (possibly a point). An important observation for us is that if characteristic sets of g and h intersect in more than a point in T , then the same is true in a neighborhood of (the homothety class of) T in $\overline{CV_N}$.

Morphisms between F_N -trees and a semi-flow on $\overline{CV_N}$. A morphism between two F_N -trees T and T' is an F_N -equivariant map $f : T \rightarrow T'$, such that every segment $I \subseteq T$ can be subdivided into finitely many subsegments I_1, \dots, I_k , so that for all $i \in \{1, \dots, k\}$, the map f is an isometry when restricted to I_i . Notice in particular that every morphism is 1-Lipschitz. A morphism $f : T \rightarrow T'$ is optimal if in addition, for every $x \in T$, there is an open arc $I \subseteq T$ containing x in its interior on which f is one-to-one. We denote by \mathcal{A} the space of isometry classes of all F_N -trees, and by $\text{Opt}(\mathcal{A})$ the space of optimal morphisms between trees in \mathcal{A} , which is equipped with the equivariant Gromov–Hausdorff topology, see [18, Section 3.2]. The following statement can be found in [18, Section 3], it is based on work of Skora [34] inspired by an idea of Steiner.

PROPOSITION 2.1 (Skora [34], Guirardel–Levitt [18]). — *There exist continuous maps $H : \text{Opt}(\mathcal{A}) \times [0, 1] \rightarrow \mathcal{A}$ and $\Phi, \Psi : \text{Opt}(\mathcal{A}) \times [0, 1] \rightarrow \text{Opt}(\mathcal{A})$ such that*

- for all $f \in \text{Opt}(\mathcal{A})$ and all $t \in [0, 1]$, the tree $H(f, t)$ is the range of the morphism $\Phi(f, t)$ and the source of the morphism $\Psi(f, t)$,
- for all $f \in \text{Opt}(\mathcal{A})$, we have $\Phi(f, 0) = \text{id}$ and $\Psi(f, 0) = f$,
- for all $f \in \text{Opt}(\mathcal{A})$, we have $\Phi(f, 1) = f$ and $\Psi(f, 1) = \text{id}$, and
- for all $f \in \text{Opt}(\mathcal{A})$ and all $t \in [0, 1]$, we have $\Psi(f, t) \circ \Phi(f, t) = f$.



The proof of Proposition 2.1 goes as follows: given a morphism $f : T_0 \rightarrow T_1$, one first defines for all $t \in [0, 1]$ a minimal F_N -tree T_t , as the quotient space T_0/\sim_t , where $a \sim_t b$ whenever $f(a) = f(b)$ and $\tau(a, b) := \sup_{x \in [a, b]} d_{T_1}(f(a), f(x)) \leq t$. The morphism f factors through optimal morphisms $\phi_t : T_0 \rightarrow T_t$ and $\psi_t : T_t \rightarrow T_1$ which vary continuously with f . We then let $H(f, t) := T_t$, $\Phi(f, t) := \phi_t$ and $\Psi(f, t) := \psi_t$. The path $(H(f, t))_{t \in [0, 1]}$ will be called the *canonical folding path* directed by f .

We will now make a few observations about the above construction. We recall that the *bounded backtracking (BBT) constant* of a morphism $f : T_0 \rightarrow T_1$, denoted by $\text{BBT}(f)$, is defined as the maximal real number such that for all $x, y \in T_0$ and all $z \in [x, y]$, we have $d_{T_1}(f(z), [f(x), f(y)]) \leq C$. We make the following observation.

LEMMA 2.2. — *For all $t \in [0, 1]$, we have $\text{BBT}(\psi_t) \leq \text{BBT}(f)$.*

Proof. — Let $x, y \in T_t$, and let $z \in [x, y]$. Let x_0, y_0 be ϕ_t -preimages of x, y in T_0 . Then $\phi_t([x_0, y_0])$ contains $[x, y]$, so we can find a ϕ_t -preimage z_0 of z in the segment $[x_0, y_0]$. We then have $d_{T_1}(\psi_t(z), [\psi_t(x), \psi_t(y)]) = d_{T_1}(f(z_0), [f(x_0), f(y_0)]) \leq \text{BBT}(f)$, which shows that $\text{BBT}(\psi_t) \leq \text{BBT}(f)$. □

LEMMA 2.3. — *Assume that f is isometric when restricted to any arc of T_0 with nontrivial stabilizer. Then for every $t \in [0, 1]$, the map $\psi_t : T_t \rightarrow T_1$ is isometric when restricted to any arc of T_t with nontrivial stabilizer.*

Proof. — Let $[a_t, b_t] \subseteq T_t$ be a nondegenerate arc with nontrivial stabilizer $\langle g \rangle$. We aim to show that $\psi_t(a_t) \neq \psi_t(b_t)$, which is enough to conclude since ψ_t is a morphism. By definition of T_t , there exist $a, b \in T_0$ satisfying $\tau(a, ga) \leq t$ and $\tau(b, gb) \leq t$, with $f(ga) = f(a)$ and $f(gb) = f(b)$, such that $a_t = \phi_t(a)$ and $b_t = \phi_t(b)$.

Assume towards a contradiction that $\psi_t(a_t) = \psi_t(b_t)$. Then $f(a) = f(b)$. If g does not fix any nondegenerate arc in T_0 , then the segment $[a, b]$ is contained in the union of all g^k -translates of $[a, ga]$ and $[b, gb]$, with k varying over \mathbb{Z} . It follows that $\tau(a, b) \leq \max\{\tau(a, ga), \tau(b, gb)\} \leq t$, and hence $a_t = b_t$, a contradiction. Assume now that g fixes a nondegenerate arc $[a', b'] \subseteq T_0$, and let a'' (resp. b'') be the projection of a (resp. b) to $[a', b']$. Using the fact that $f(a) = f(b)$ and that f is isometric when restricted to $[a', b']$, we have $f([a'', b'']) \subseteq f([a, a'']) \cup f([b, b''])$, and therefore we get that $\tau(a, b) = \max\{\sup_{x \in [a, a'']} d_{T_1}(f(a), f(x)), \sup_{y \in [b, b'']} d_{T_1}(f(b), f(y))\}$. Since $[a, a''] \subseteq [a, ga]$ and $[b, b''] \subseteq [b, gb]$, we then obtain as above that $\tau(a, b) \leq t$, so again $a_t = b_t$, a contradiction. □

Remark 2.4. — Together with [20, Proposition 4.4], which says that arc stabilizers in the intermediate trees are root-closed if arc stabilizers are root-closed in T_0 and T_1 , Lemma 2.3 implies that if $T_0, T_1 \in \overline{cv_N}$, and if f is isometric when restricted to arcs with nontrivial stabilizer, then all intermediate trees belong to $\overline{cv_N}$. It is also known [18, Proposition 3.6] that if $T_0, T_1 \in cv_N$, then all intermediate trees belong to cv_N .

2.2. Local contractibility at trees with trivial arc stabilizers

The goal of the present section is to prove that $\overline{cv_N}$ is locally contractible at every tree with all arc stabilizers trivial (Proposition 2.9). The following lemma provides nice approximations of trees in $\overline{cv_N}$ with trivial arc stabilizers.

LEMMA 2.5 ([22, Theorem 5.3]). — *Given a tree $T \in \overline{cv_N}$ with all arc stabilizers trivial, and any open neighborhood \mathcal{U} of T in $\overline{cv_N}$, there exists a tree $U \in \mathcal{U} \cap cv_N$ that admits an optimal morphism onto T .*

Proof. — It follows from [21, Theorem 3.6] that we can always find a simplicial tree $U' \in \mathcal{U} \cap \overline{cv_N}$ that admits an optimal morphism onto T (optimality is not stated in [21], however it follows from the construction which essentially relies on the approximation techniques from [16], see also Proposition 4.7 for an argument).

However, the F_N -action on U' may *a priori* not be free (but U' has trivial arc stabilizers because T has trivial arc stabilizers). One way to replace U' by a tree in cv_N is to replace any vertex v with nontrivial stabilizer G_v by a free G_v -tree wedged at the point v , and perform this operation equivariantly and at all nontrivial fixed vertices; however, the natural map $U' \rightarrow U$ will collapse these trees and will not be a morphism. Instead, we define a tree U'_ε in the following way (see Figure 2.1).

Let e_1, \dots, e_k be a choice of representatives of the orbits of edges incident on v in U' , made such that for all $i \neq j$, the turn $\{e_i, e_j\}$ is f -legal, i.e. no initial segment of e_i is identified with an initial segment of e_j by f : this is possible because the tree T has trivial arc stabilizers, so two edges in the same G_v -orbit never get identified by f . For all $i \in \{1, \dots, k\}$, we denote by $\ell(e_i)$ the length of the edge e_i . Let $\{a_1, \dots, a_l\}$ be a free basis of G_v .

Let U'_ε be the tree obtained from U' by giving length $\ell(e_1) - \varepsilon$ to e_1 , and blowing up the vertex v to a free G_v -tree, as depicted in Figure 2.1 (where we have represented the quotient graph U'_ε/F_N). Then there is an optimal

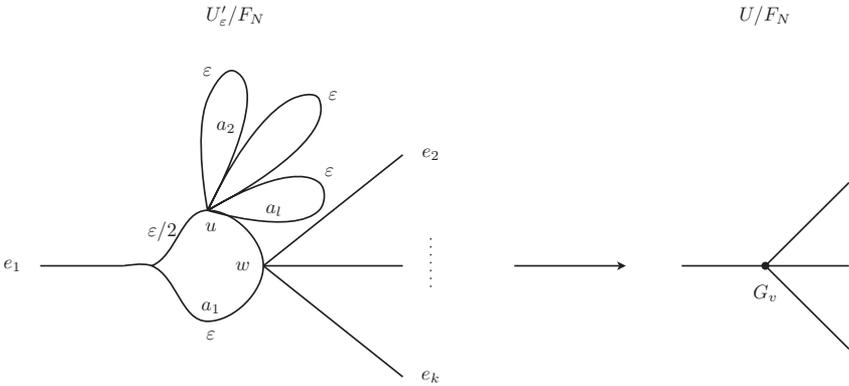


Figure 2.1. The situation in the proof of Lemma 2.5.

morphism $g : U'_\epsilon \rightarrow U'$, which is an isometry in restriction to each complementary component of the blown-up edges. We need to show that the composition $g \circ f$ is again optimal. Optimality at any point x distinct from w (with the notation from the picture) follows from the optimality of g . Optimality at w follows from the fact that a_1 does not fix any nondegenerate arc in T . □

LEMMA 2.6. — *Let $f : S \rightarrow T$ be an optimal morphism from a tree $S \in cv_N$ to a tree $T \in \overline{cv_N}$. Then there is a neighborhood \mathcal{W} of T and a continuous map*

$$\Psi_S : \mathcal{W} \rightarrow \text{Opt}(\overline{cv_N})$$

such that for all $W \in \mathcal{W}$, the source of $\Psi_S(W)$ is a tree $S' \in cv_N$, in the same (cone on a) simplex as S and varying continuously, the range of $\Psi_S(W)$ is the tree W , and $\Psi_S(W)$ is an optimal morphism. In addition, $\Psi_S(T) = f$.

Proof. — Let v be a vertex of S . Since f is optimal, the point v belongs to a line $l \subseteq S$ such that the restriction $f|_l$ is an isometry (notice however that we may not assume in general that l is the axis of an element of F_N). Denote by v_1 and v_2 the two vertices of S which are adjacent to v on the line l . We can then find two hyperbolic elements $\gamma_{1,v}, \gamma_{2,v} \in F_N$ whose axes in T both intersect $f(l)$ but do not intersect each other, and such that the segment joining $Ax_T(\gamma_{1,v})$ to $Ax_T(\gamma_{2,v})$ contains $f(v_1), f(v)$ and $f(v_2)$ in its interior. Let $d \in \mathbb{R}$ denote the distance from $Ax_T(\gamma_{1,v})$ to $f(v)$.

If W is sufficiently close to T , the elements $\gamma_{1,v}$ and $\gamma_{2,v}$ are hyperbolic in W and their axes are disjoint and lie at distance at least d from each other.

We denote by x_W^v the point at distance d from $Ax_W(\gamma_{1,v})$ on the segment from $Ax_W(\gamma_{1,v})$ to $Ax_W(\gamma_{2,v})$. Given a choice v_1, \dots, v_k of representatives of the orbits of the vertices of S , there is a unique choice of a (new) metric on S , giving a tree S_W , so that the linear extension of g_W (defined on vertices by sending v_i to $x_W^{v_i}$, and extending equivariantly) is a morphism (up to restricting to a smaller neighborhood of T , we can assume that no edge gets length 0). Using the fact that f_l is an isometry, we get that this morphism is also optimal: indeed, the segment joining $Ax_T(\gamma_{1,v})$ to $Ax_T(\gamma_{2,v})$ contains $[f(v_1), f(v_2)]$. It follows that when W is close to T the segment joining $Ax_W(\gamma_{1,v})$ to $Ax_W(\gamma_{2,v})$ overlaps $[g_W(v_1), g_W(v_2)]$ in a segment that contains $g_W(v)$ in its interior. In particular, g_W sends the two directions at v determined by l to distinct directions. Thus we set $\Psi_S(W) = g_W$. It is standard that Ψ_S is continuous, see [17] for example. \square

Given $T_0 \in \overline{cv_N}$, the following corollary enables us to choose basepoints continuously in all trees in a neighborhood of T_0 , with a prescribed choice on T_0 . In the statement, we fix a Cayley tree R of F_N with respect to a free basis of F_N , and a vertex $* \in R$, and we denote by $\text{Map}(F_N, \overline{cv_N})$ the collection of all F_N -equivariant maps from a tree obtained from R by possibly varying edge lengths, to trees in $\overline{cv_N}$.

COROLLARY 2.7.

- (1) *Let $c, c' \in F_N$ be two nontrivial elements that do not belong to the same cyclic subgroup. Then the function*

$$b : \overline{cv_N} \rightarrow \text{Map}(F_N, \overline{cv_N})$$

that sends $T \in \overline{cv_N}$ to the morphism $R \rightarrow T$ that sends the basepoint in R to the projection of $\text{Char}_T(c)$ to $\text{Char}_T(c')$ when the two are disjoint, and to the midpoint of the overlap when they intersect, is continuous.

- (2) *Let $N \geq 2$, let $T_0 \in \widehat{cv_N}$, let $A \subseteq F_N$ be a free factor of F_N , and let $x_0 \in T_0$ be a point which is contained in the union of all characteristic sets of elements of A . There exists a continuous map $b : \overline{cv_N} \rightarrow \text{Map}(F_N, \overline{cv_N})$ such that for all $T \in \overline{cv_N}$, the range of $b(T)$ is T , and $b(T_0)(*) = x_0$, and $b(T)(*)$ is contained in the union of all characteristic sets of elements of A .*

Proof. — The first part of the corollary was proved in [17, p. 166]. We prove (2). From (1), we get a global choice of basepoints b_1 contained in the union of all characteristic sets of elements of A . From Lemma 2.6, we have a choice of basepoints b'_2 defined in a neighborhood \mathcal{W} of T_0 , with $b'_2(T_0)(*) = x_0$. Let $a \in A$ be an element whose characteristic set contains

x_0 . By projecting b'_2 to the characteristic set of a , we get a continuous choice of basepoints b_2 in \mathcal{W} , all contained in the union of all characteristic sets of elements of A . Choose a continuous function $\phi : \overline{cv_N} \rightarrow [0, 1]$ which is 1 at T_0 and is 0 outside a compact subset of \mathcal{W} . Then define

$$b(T)(*) = (1 - \phi(T)) b_1(T)(*) + \phi(T) b_2(T)(*). \quad \square$$

LEMMA 2.8. — *Let $T \in \overline{cv_N}$, let \mathcal{U} be an open neighborhood of T in $\overline{cv_N}$. Then there exist $\varepsilon > 0$ and an open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of T in $\overline{cv_N}$ such that if $U \in \mathcal{V}$ is a tree that admits a $(1 + \varepsilon)$ -Lipschitz F_N -equivariant map f onto T , and if $U' \in \overline{cv_N}$ is a tree such that f factors through $(1 + \varepsilon)$ -Lipschitz F_N -equivariant maps from U to U' and from U' to T , then $U' \in \mathcal{U}$.*

Proof. — By definition of the equivariant Gromov–Hausdorff topology, there exist $\delta \in (0, 1)$ and a finite set $\{g_1, \dots, g_k\}$ of elements of F_N such that \mathcal{U} contains

$$\mathcal{U}' := \{T' \in \overline{cv_N} \mid (1 - \delta)\|g_i\|_T \leq \|g_i\|_{T'} \leq (1 + \delta)\|g_i\|_T \text{ for all } i \in \{1, \dots, k\}\}.$$

Let $\varepsilon > 0$ be such that $\varepsilon < \min\{\delta, \frac{1}{1 - \delta} - 1\}$, and let $\delta' > 0$ be such that $\delta' < \frac{1 + \delta}{1 + \varepsilon} - 1$ (this exists because $\varepsilon < \delta$). Notice in particular that we have

$$(2.1) \quad 1 - \delta < \frac{1}{1 + \varepsilon}$$

and

$$(2.2) \quad (1 + \varepsilon)(1 + \delta') < 1 + \delta.$$

Let \mathcal{V} be an open neighborhood of T in $\overline{cv_N}$ contained in

$$\{U \in \overline{cv_N} \mid (1 - \delta')\|g_i\|_T \leq \|g_i\|_U \leq (1 + \delta')\|g_i\|_T \text{ for all } i \in \{1, \dots, k\}\}.$$

If $U \in \mathcal{V}$ and $U' \in \overline{cv_N}$ are trees such that f factors through $(1 + \varepsilon)$ -Lipschitz F_N -equivariant maps from U to U' and from U' to T , then for all $i \in \{1, \dots, k\}$, we have

$$\|g_i\|_T \leq (1 + \varepsilon)\|g_i\|_{U'} \leq (1 + \varepsilon)^2\|g_i\|_U \leq (1 + \varepsilon)^2(1 + \delta')\|g_i\|_T.$$

Using Equations (2.1) and (2.2), this implies that

$$(1 - \delta)\|g_i\|_T \leq \|g_i\|_{U'} \leq (1 + \delta)\|g_i\|_T,$$

so $U' \in \mathcal{U}$. □

PROPOSITION 2.9. — *The space $\overline{cv_N}$ is locally contractible at every tree with all arc stabilizers trivial.*

Proof. — Let $T \in \overline{cv_N}$ be a tree with all arc stabilizers trivial, and let \mathcal{U} be an open neighborhood of T in $\overline{cv_N}$. Let $\varepsilon > 0$ and $\mathcal{V} \subseteq \mathcal{U}$ be a smaller neighborhood of T , as provided by Lemma 2.8. Let $S \in \mathcal{V} \cap cv_N$ be such that there exists an optimal morphism from S to T (this exists by Lemma 2.5). Then there exists a smaller neighborhood $\mathcal{W} \subseteq \mathcal{V}$ of T such that for all $T' \in \mathcal{W}$, there is a $(1 + \varepsilon)$ -Lipschitz F_N -equivariant map from S to the source S' of the morphism $\Psi_S(T')$ given by Lemma 2.6. Since morphisms are 1-Lipschitz, in view of Lemma 2.8, this implies that all trees that belong to either the straight path from S to S' , or to the canonical folding path directed by $\Psi_S(T')$, belong to \mathcal{U} . As $\Psi_S(T')$ varies continuously with T' , this gives a homotopy of \mathcal{W} onto S that stays within \mathcal{U} . \square

A variant of the above argument shows the following statement, which will be useful in our proof of the fact that $\overline{CV_N}$ is not an AR.

LEMMA 2.10. — *Let $T_0 \in \overline{cv_N}$ be a tree with trivial arc stabilizers that is dual to a measured foliation on a surface Σ with a single boundary component c . Let $Z \subseteq \overline{cv_N}$ be the set of all trees dual to a measured foliation on Σ . Then for every open neighborhood \mathcal{V} of T_0 in $\overline{cv_N}$, there exist a smaller neighborhood $\mathcal{W} \subseteq \mathcal{V}$ of T_0 in $\overline{cv_N}$, a tree $S \in \mathcal{W} \cap cv_N$, and a continuous map $H : (Z \cap \mathcal{W}) \times [0, 1] \rightarrow \mathcal{V}$ such that $H(z, 0) = z$ and $H(z, 1) = S$ for all $z \in Z \cap \mathcal{W}$, and c is hyperbolic in $H(z, t)$ for all $z \in Z$ and all $t > 0$.*

Proof. — The proof of Lemma 2.10 is the same as the proof of Proposition 2.9, except that we have to show in addition that c remains hyperbolic until it reaches z along the canonical folding path from S' to z determined by the morphism $\Psi_S(z)$. Notice that c is not contained in any proper free factor of F_N , so whenever c becomes elliptic along a canonical folding path, the tree T reached by the path contains no simplicial edge with trivial stabilizer. In view of Lemma 2.3, no two simplicial edges with nontrivial stabilizer can get identified by the folding process. In addition, an arc in a subtree with dense orbits from the Levitt decomposition of T as a graph of actions cannot get identified with a simplicial edge with nontrivial stabilizer, and two such arcs cannot either get identified together [21, Lemmas 1.9 and 1.10]. This implies that the canonical folding path $(H(\Psi_S(z), t))_{t \in [0, 1]}$ becomes constant once it reaches T . To conclude the proof of Lemma 2.10, it remains to reparametrize this canonical folding path to ensure that it does not reach T before $t = 1$. Notice that $\|c\|_{H(\Psi_S(z), t)}$ decreases strictly as t increases, until it becomes equal to 0, and in addition the tree $H(\Psi_S(z), t)$ is the same for all $t \in [0, 1]$ such that $\|c\|_{H(\Psi_S(z), t)} = 0$. We can therefore reparametrize the canonical folding path by the translation length of c : for

all $l \leq \|c\|_{H(\Psi_S(z),0)}$, we let $H'(\Psi_S(z), \|c\|_{H(\Psi_S(z),0)} - l)$ be the unique tree T on the folding path for which $\|c\|_T = l$. To get a continuous map from $(Z \cap \mathcal{W}) \times [0, 1]$ to \mathcal{V} , we then renormalize the parameter l by dividing it by $\|c\|_{H(\Psi_S(z),0)}$. \square

2.3. The space \overline{CV}_N is not locally 4-connected when $N \geq 4$.

We will now prove that Culler–Morgan’s compactification \overline{CV}_N of outer space is not an AR as soon as $N \geq 4$.

THEOREM 2.11. — *For all $N \geq 4$, the space \overline{CV}_N is not locally 4-connected, hence it is not an AR.*

Remark 2.12. — It can actually be shown however that the closure \overline{CV}_2 is an absolute retract, and we also believe that \overline{CV}_3 is an absolute retract, though establishing this fact certainly requires a bit more work than the arguments from the present paper.

An embedded 3-sphere in $\mathcal{PMF}^{vs}(\Sigma)$. When Σ is a compact surface of negative Euler characteristic $\chi(\Sigma) < 0$ we denote by $\mathcal{PML}(\Sigma)$ the space of projectivized measured geodesic laminations on Σ so that every boundary component has measure 0. The space $\mathcal{PML}(\Sigma)$ is homeomorphic to the sphere $S^{-3\chi(\Sigma)-b-1}$ where b is the number of boundary components (see [37, Theorem 3] or [19, Proposition 1.5]).

We now specialize and let Σ be a nonorientable surface of genus 3 with one boundary component (so that its Euler characteristic is -2). Thus $\mathcal{PML}(\Sigma) = S^4$. We mention that Σ admits pseudo-Anosov homeomorphisms [31, 37]. We denote by $\mathcal{PML}^{vs}(\Sigma)$ the subspace of $\mathcal{PML}(\Sigma)$ made of all laminations that are dual to very small trees (equivalently, all laminations that do not contain any 1-sided compact leaf, see [35]). Let γ be a simple closed curve on Σ that separates Σ into a Möbius band and an orientable surface Σ_0 (which is a compact surface of genus 1 with two boundary components). Denote by γ_0 the core of the Möbius band, which is 1-sided and geodesic. Then the space $\mathcal{PML}(\Sigma_0) = S^3$ is a subset of $\mathcal{PML}^{vs}(\Sigma)$, however $\gamma_0 \notin \mathcal{PML}^{vs}(\Sigma)$. The key observation for constructing 4-spheres in \overline{CV}_N showing that \overline{CV}_N is not locally 4-connected will be the following.

LEMMA 2.13. — *The subset $\mathcal{PML}(\Sigma_0) \subseteq \mathcal{PML}^{vs}(\Sigma)$ is a 3-sphere which is a retract of $\mathcal{PML}^{vs}(\Sigma)$.*

Proof. — The space $\mathcal{PML}(\Sigma_0)$ is a topologically embedded 3-dimensional sphere in the 4-dimensional sphere $\mathcal{PML}(\Sigma)$, so it separates $\mathcal{PML}(\Sigma)$. We prove below that both sides of $\mathcal{PML}(\Sigma) \setminus \mathcal{PML}(\Sigma_0)$ contain curves arising as the core of a Möbius band, for which the dual tree is not very small. This implies that $\mathcal{PML}(\Sigma_0)$ is a retract of $\mathcal{PML}^{vs}(\Sigma)$.

To prove the assertion first notice that one of the two complementary components is a cone of $\mathcal{PML}(\Sigma_0)$ over γ_0 . If ϕ is a pseudo-Anosov homeomorphism of Σ , then $\phi(\gamma_0)$ is the core of a Möbius band, and it belongs to the other complementary component. □

Definition of the tree T_0 . We will now define a tree $T_0 \in \overline{CV}_N$ at which \overline{CV}_N will fail to be locally 4-connected.

Let $N \geq 4$. Let $T_0 \in \overline{CV}_N$ be the tree defined in the following way (see Figure 2.2). Let \mathcal{F}_0 be an arational measured foliation on Σ , obtained as the attracting foliation of a pseudo-Anosov diffeomorphism f of Σ . Let A be the fundamental group of Σ , which is free of rank 3, let T_A be the very small A -tree dual to \mathcal{F}_0 , and let $x_A \in T_A$ be the unique point fixed by a nontrivial element c_A corresponding to the boundary curve of Σ .

Let $B = F_{N-2}$, which we write as a free product $B = B' * \langle c_B \rangle$ for some element $c_B \in B$. Let $T_B \in \overline{cv}_{N-2}$ be a tree that splits as a graph of actions over this free splitting of B , with vertex actions a free and simplicial action $T_{B'} \in cv_{N-3}$, and the trivial action of $\langle c_B \rangle$ on a point, where the edge with trivial stabilizer from the splitting is given length 0. Let x_B be the point fixed by c_B in T_B . Notice that x_B belongs to the B' -minimal subtree $T_{B'}$ of T_B .

Write $F_N = A *_{c_A=c_B} B$, and let T_0 be the very small F_N -tree obtained as a graph of actions over this amalgamated free product, with vertex actions T_A and T_B and attaching points x_A and x_B , where the simplicial edges with nontrivial stabilizers coming from the splitting are assigned length 1. We let $c := c_A = c_B \in F_N$.

Finding embedded 4-spheres in a neighborhood of T_0 . Choose an element $g \in F_N$ which is hyperbolic in T_0 and whose axis crosses the arc fixed by c . Let \mathcal{U} be an open neighborhood of T_0 in \overline{CV}_N consisting of trees where g is hyperbolic, the characteristic sets of g and c overlap, and the surface group A is not elliptic. The crucial property satisfied by such a neighborhood \mathcal{U} of T_0 is that we have the *oriented translation length* of c , namely the continuous function

$$\theta : \mathcal{U} \rightarrow \mathbb{R}$$

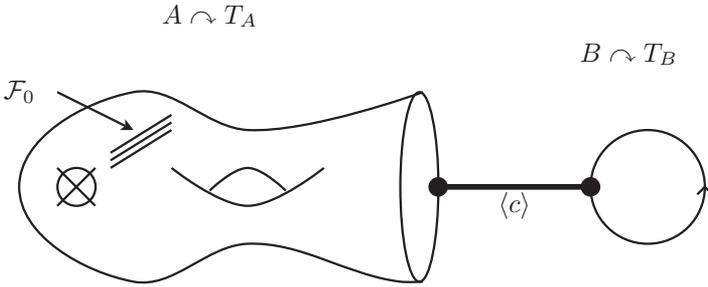


Figure 2.2. The tree T_0 at which \overline{CV}_N fails to be locally 4-connected.

defined by

$$\theta(T) = \varepsilon_T \frac{\|c\|_T}{\|g\|_T}$$

where $\varepsilon_T = 1$ (resp. -1) if c is hyperbolic in T and the axes of c and g give the same (resp. opposite) orientation to the overlap, and $\varepsilon_T = 0$ if c is elliptic. We will denote by \mathcal{U}_+ (resp. \mathcal{U}_-) the subset of \mathcal{U} made of trees such that $\theta(T) \geq 0$ (resp. $\theta(T) \leq 0$). Similarly, given any smaller neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of T_0 , we will let $\mathcal{V}_+ := \mathcal{V} \cap \mathcal{U}_+$ and $\mathcal{V}_- := \mathcal{V} \cap \mathcal{U}_-$.

We denote by $\overline{cv}(A)$ the space of all very small A -actions on \mathbb{R} -trees (which is also the closure of the outer space associated to A). Let \mathcal{U}_A be an open neighborhood of T_A in $\overline{cv}(A)$ such that there exists an element $a \in A$ such that the characteristic sets of c_A and a have empty intersection in all trees in \mathcal{U}_A . For all $U_A \in \mathcal{U}_A$, we then denote by $b(U_A)$ the projection of $\text{Char}_{U_A}(a)$ to $\text{Char}_{U_A}(c_A)$: this gives us a continuous choice of a basepoint in every tree $U_A \in \mathcal{U}_A$. Given a tree $U_A \in \overline{cv}(A)$ in which c_A is hyperbolic, and $t \in \mathbb{R}$, we let $b(U_A, t)$ be the unique point on the axis of c_A at distance $|t|$ from $b(U_A)$, and such that $[b(U_A), b(U_A, t)]$ is oriented in the same direction as the axis of c_A if $t \geq 0$, and in the opposite direction if $t \leq 0$. We then let $\mathcal{T}(U_A, t) \in \overline{cv}(A)$ be the tree which splits as a graph of actions over the free product $F_N = A * B'$, with vertex actions U_A and $T_{B'}$, and attaching points $b(U_A, t)$ and x_B (recall from the above that $x_B \in T_{B'}$), where the simplicial edge from the splitting is assigned length 0. The following lemma follows from the argument in the proof of [22, Lemma 5.6].

LEMMA 2.14. — *For every open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of T_0 in \overline{CV}_N , there exists an open neighborhood $\mathcal{V}_A \subseteq \mathcal{U}_A$ of T_A in $\overline{cv}(A)$ such that for all $U_A \in \mathcal{V}_A$ such that c_A is hyperbolic in U_A , we have $\mathcal{T}(U_A, 1) \in \mathcal{V}_+$, and $\mathcal{T}(U_A, -1) \in \mathcal{V}_-$.*

Denote by $X(c)$ the subspace of \overline{CV}_N made of all trees where c is elliptic, and by $X(c)^*$ the subset of $X(c)$ consisting of trees where the surface group A is not elliptic. Notice that our choice of neighborhood \mathcal{U} of T_0 ensures that $\mathcal{U} \cap X(c) \subseteq X(c)^*$. For any tree T in $X(c)^*$, the minimal A -invariant subtree is dual to some measured foliation on Σ [35], so we have a map $\psi : X(c)^* \rightarrow \mathcal{PML}^{vs}(\Sigma)$.

PROPOSITION 2.15. — *For every open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of T_0 , there exists a topologically embedded 3-sphere S^3 in $X(c) \cap \mathcal{V}$ which is a retract of $X(c) \cap \mathcal{U}$, and such that the inclusion map $\iota : S^3 \hookrightarrow X(c) \cap \mathcal{V}$ extends to continuous maps $\iota_{\pm} : B_{\pm}^4 \rightarrow \mathcal{V}_{\pm}$ (where B_{\pm}^4 are 4-balls with boundary S^3), with $\iota_{\pm}(B_{\pm}^4 \setminus S^3) \subseteq \overline{CV}_N \setminus X(c)$.*

Proof. — We identify $\mathcal{PML}(\Sigma)$ with a continuous lift in $\mathcal{ML}(\Sigma)$. This gives a continuous injective map $\phi : \mathcal{PML}^{vs}(\Sigma) \rightarrow X(c)^*$, mapping every measured foliation \mathcal{F} to the tree in \overline{CV}_N that splits as a graph of actions over $F_N = A *_{\langle c \rangle} B$, with vertex actions the tree dual to \mathcal{F} , and the fixed action $B \curvearrowright T_B$, where the attaching points are the unique points fixed by c , and the simplicial edge from the splitting is assigned length 1. The composition $\psi\phi$ is the identity. Let $\Sigma_0 \subset \Sigma$ be an oriented subsurface as in Lemma 2.13. We can assume that the 3-sphere $\phi(\mathcal{PML}(\Sigma_0))$ is contained in \mathcal{V} : to achieve this, use uniform north-south dynamics of the pseudo-Anosov homeomorphism f on $\mathcal{PML}(\Sigma)$ to find $k \in \mathbb{Z}$ so that $\phi(\mathcal{PML}(f^k(\Sigma_0))) \subseteq X(c) \cap \mathcal{V}$ (see [24, Theorem 3.5], which is stated there in the case of an orientable surface, but the similar statement for a nonorientable surface follows by considering the orientable double cover of Σ). Using Lemma 2.13, we get that the 3-sphere $\phi(\mathcal{PML}(\Sigma_0))$ is a retract of $X(c) \cap \mathcal{U}$ (notice also that its image in $\overline{cv}(A)$ is again an embedded 3-sphere which we denote by S_A^3).

Let now \mathcal{V}_A be an open neighborhood of T_A in $\overline{cv}(A)$ provided by Lemma 2.14, and let $\mathcal{W}_A \subseteq \mathcal{V}_A$ be a smaller neighborhood of T_A provided by Lemma 2.10. We can assume that the sphere constructed in the above paragraph is such that $S_A^3 \subseteq \mathcal{W}_A$. By Lemma 2.10, there exists a tree $V_A \in \mathcal{V}_A \cap \overline{cv}(A)$ and a homotopy $H_A : S_A^3 \times [0, 1] \rightarrow \mathcal{V}_A$ such that

- for all $z \in S_A^3$, we have $H_A(z, 0) = z$, and $H_A(z, 1) = V_A$, and
- for all $z \in S_A^3$ and all $t > 0$, the element c is hyperbolic in $H_A(z, t)$.

We now define $H_{\pm} : S^3 \times [0, 1] \rightarrow \mathcal{V}_{\pm}$ by letting $H_{\pm}(z, 0) = z$ for all $z \in S^3$, and $H_{\pm}(z, t) = \mathcal{T}(H_A(z, t), \pm 1)$ for all $z \in S^3$ and all $t > 0$. Continuity follows from the argument from [22, Lemma 5.6]. This enables

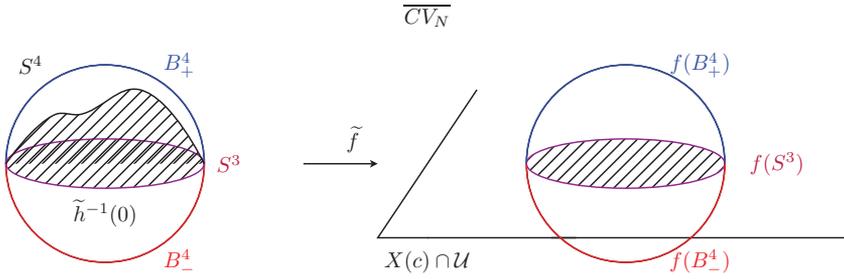


Figure 2.3. Constructing the sphere S^4 showing failure of local 4-connectivity of \overline{CV}_N .

us to construct the maps ι_{\pm} , where the segment joining a point $z \in S^3$ to the center of B_{\pm}^4 , is mapped to $H_{\pm}(z, [0, 1])$. □

Our proof of Theorem 2.11 uses a Čech homology argument, given in Appendix A of the paper.

Proof of Theorem 2.11. — We will show that for every neighborhood $\mathcal{V} \subset \mathcal{U}$ of T_0 there is a map $S^4 \rightarrow \mathcal{V}$ which is not nullhomotopic in \mathcal{U} . This is illustrated in Figure 2.3. Let S^3 be a topologically embedded 3-sphere in $X(c) \cap \mathcal{V}$ provided by Proposition 2.15. In particular, the inclusion map $\iota : S^3 \hookrightarrow X(c) \cap \mathcal{U}$ is nontrivial in homology (either singular homology, or Čech homology \check{H}_3 with $\mathbb{Z}/2$ coefficients). In addition, we can extend ι to maps $\iota_{\pm} : B_{\pm}^4 \rightarrow \mathcal{V}_{\pm}$, with $\iota_{\pm}(B_{\pm}^4 \setminus S^3) \subseteq \overline{CV}_N \setminus X(c)$. Gluing these two along the boundary produces a map $f : S^4 \rightarrow \mathcal{V}$, which restricts to ι on S^3 , and such that the composition $h = \theta f : S^4 \rightarrow \mathbb{R}$ (where we recall that θ is the oriented translation length of c) is standard (i.e. $h^{-1}(0) = S^3$, $h^{-1}([0, +\infty)) = B_+^4$ and $h^{-1}((-\infty, 0]) = B_-^4$).

Now suppose that f extends to $\tilde{f} : B^5 \rightarrow \mathcal{U}$, and let $\tilde{h} = \theta \tilde{f} : B^5 \rightarrow \mathbb{R}$. If $\tilde{h}^{-1}(0)$ were a manifold with boundary S^3 , then we would immediately deduce that the inclusion

$$S^3 \hookrightarrow \tilde{h}^{-1}(0)$$

is trivial in singular homology H_3 . This may fail to be true, but we can still apply Lemma A.1 to deduce that the above inclusion is trivial in Čech homology \check{H}_3 with $\mathbb{Z}/2$ coefficients. Applying \tilde{f} and observing that $\tilde{f}(\tilde{h}^{-1}(0)) \subseteq X(c) \cap \mathcal{U}$ we deduce that

$$f|_{S^3} = \iota : S^3 \rightarrow X(c) \cap \mathcal{U}$$

is trivial in \check{H}_3 , a contradiction. □

3. The Pacman compactification of outer space

In the present paper, we will be interested in another compactification \widehat{CV}_N of outer space, which we call the *Pacman compactification*. We also define \widehat{cv}_N as the unprojectivized version of \widehat{CV}_N . We will now define \widehat{CV}_N and establish some basic topological properties.

A point in \widehat{CV}_N is given by a (homothety class of a) tree $T \in \widehat{CV}_N$, together with an F_N -equivariant choice of orientation of the characteristic set of every nontrivial element of F_N that fixes a nondegenerate arc in T (notice that every element $g \in F_N$ which is hyperbolic in T determines a natural orientation on its axis, which is isometric to a real line, and the element g^{-1} determines the opposite orientation on this line). Precisely, given any nontrivial element $c \in F_N$ which is not a proper power, and whose characteristic set in T is a nondegenerate arc $\text{Char}_T(c) := [x, y]$ fixed by c , we prescribe an orientation of $\text{Char}_T(c)$, and the orientation of $\text{Char}_T(c^{-1})$ is required to be opposite to the orientation of $\text{Char}_T(c)$. The F_N -translates of $[x, y]$ get the induced orientation as required by F_N -equivariance. Notice that there is an F_N -equivariant surjective map $\pi : \widehat{CV}_N \rightarrow \widehat{CV}_N$, which consists in forgetting the orientations of the edges with nontrivial stabilizer.

Given two oriented (possibly finite or infinite) geodesics l and l' in an \mathbb{R} -tree T with nondegenerate intersection, we define the *relative orientation* of l and l' as being equal to $+1$ if the orientations of l and l' agree on their intersection, and -1 otherwise. Given $T \in \widehat{CV}_N$, and two elements $\alpha, \beta \in F_N$ whose characteristic sets have nondegenerate intersection in T , we define the *relative orientation* of the pair (α, β) in T as being equal to the relative orientation of their characteristic sets.

We now define a topology on \widehat{CV}_N . Given an open set $U \subseteq \widehat{CV}_N$, and a finite (possibly empty) collection of pairs $(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)$ of elements of F_N , such that for all $i \in \{1, \dots, k\}$, β_i is hyperbolic and the characteristic sets of α_i and β_i have nondegenerate intersection in all trees in U , we let

$$U((\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k))$$

be the set of all $T \in \widehat{CV}_N$ such that $\pi(T) \in U$, and the relative orientation of (α_i, β_i) in T is equal to $+1$ for all $i \in \{1, \dots, k\}$. Notice that given two open sets $U, U' \subseteq \widehat{CV}_N$, and finite collections of elements $\alpha_i, \beta_i, \alpha'_j, \beta'_j \in F_N$, the intersection $U((\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)) \cap U'((\alpha'_1, \beta'_1), \dots, (\alpha'_l, \beta'_l))$ is equal to $(U \cap U')((\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k), (\alpha'_1, \beta'_1), \dots, (\alpha'_l, \beta'_l))$. This shows the following lemma.

LEMMA 3.1. — *The sets $U((\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k))$ form a basis of open sets for a topology on \widehat{CV}_N .*

From now on, we will equip \widehat{CV}_N with the topology generated by these sets. Since trees in CV_N have trivial arc stabilizers, there is an inclusion map $\iota : CV_N \hookrightarrow \widehat{CV}_N$.

LEMMA 3.2. — *The map $\iota : CV_N \hookrightarrow \widehat{CV}_N$ is a topological embedding.*

Proof. — It follows from the definition of the Gromov–Hausdorff topology that if $T \in CV_N$, and if $(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)$ is a finite set of pairs of elements of F_N such that for all $i \in \{1, \dots, k\}$, the axes of α_i and β_i in T have nondegenerate intersection, and their orientations agree on their intersection, then the same remains true in a neighborhood of T . Therefore, the topology on \widehat{CV}_N restricts to the Gromov–Hausdorff topology on CV_N , which shows that ι is a topological embedding. \square

For every tree $T \in \widehat{CV}_N$, there are only finitely many conjugacy classes of elements of F_N that fix a nondegenerate arc in T . Therefore, the map $\pi : \widehat{CV}_N \rightarrow \overline{CV}_N$ has finite fibers. The map π is continuous for the above topology on \widehat{CV}_N because the π -preimage of any open set $U \subseteq \overline{CV}_N$ is the open set $U(\emptyset)$ in \widehat{CV}_N .

PROPOSITION 3.3. — *The space \widehat{CV}_N is second countable.*

Proof. — Since \overline{CV}_N is second countable, we can choose a countable basis $(U_i)_{i \in \mathbb{N}}$ of open sets of \overline{CV}_N . Then the countable collection of all sets of the form $U_i((\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k))$ (where for all $j \in \{1, \dots, k\}$, the element β_j is hyperbolic and the characteristic sets of α_j and β_j have nondegenerate intersection in all trees in U_i) is a basis of open sets of \widehat{CV}_N . \square

PROPOSITION 3.4. — *The space \widehat{CV}_N is Hausdorff.*

Proof. — Let $T \neq T' \in \widehat{CV}_N$. If $\pi(T) \neq \pi(T')$, then since \overline{CV}_N is Hausdorff, we can find disjoint open neighborhoods U of $\pi(T)$ and U' of $\pi(T')$ in \overline{CV}_N , and these yield disjoint open neighborhoods of T and T' in \widehat{CV}_N . So we can assume that $\pi(T) = \pi(T')$, and there is an arc e with nontrivial stabilizer $\langle c \rangle$ in $\pi(T)$ whose orientation is not the same in T and T' . Let $h \in F_N$ be an element which is hyperbolic in $\pi(T)$, whose axis contains the edge e . One can then find an open neighborhood U of $\pi(T)$ in \overline{CV}_N such that for all trees $Y \in U$, the element h is hyperbolic in Y , and the characteristic sets of h and c have nondegenerate intersection in Y . Then exactly one of the points $T, T' \in \widehat{CV}_N$ belongs to $U(c, h)$, the other belongs to $U(c, h^{-1})$, and $U(c, h) \cap U(c, h^{-1}) = \emptyset$. This shows that \widehat{CV}_N is Hausdorff. \square

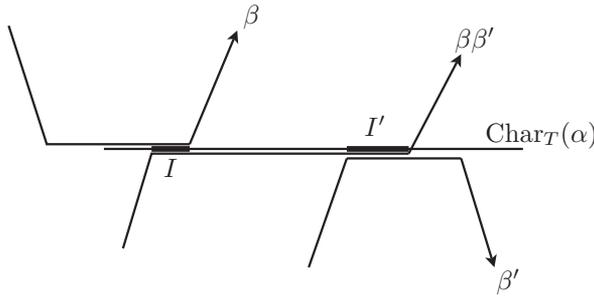


Figure 3.1. Typical position of the characteristic set of α and the axes of β, β' and $\beta\beta'$ in the proof of Lemma 3.5.

The following lemma gives a useful criterion for checking that a sequence converges in \widehat{CV}_N .

LEMMA 3.5. — Let $(T_n)_{n \in \mathbb{N}} \in \widehat{CV}_N^{\mathbb{N}}$, and let $T \in \widehat{CV}_N$. Assume that $(\pi(T_n))_{n \in \mathbb{N}}$ converges to $\pi(T)$ in \widehat{CV}_N , and that for every element $\alpha \in F_N$ fixing a nondegenerate arc in T , there exists a hyperbolic element β whose axis in T has nondegenerate intersection with $\text{Char}_T(\alpha)$, such that the relative orientations of (α, β) eventually agree in T_n and in T . Then $(T_n)_{n \in \mathbb{N}}$ converges to T .

Proof. — As $(\pi(T_n))_{n \in \mathbb{N}}$ converges to $\pi(T)$, it is enough to prove that given any two elements $\alpha, \beta' \in F_N$ whose characteristic sets in T have nondegenerate intersection, with β' hyperbolic in T , the relative orientations of (α, β') eventually agree in T_n and in T .

This is clearly true if α is also hyperbolic in T , so we assume that α fixes a nondegenerate arc in T . By hypothesis, there exists $\beta \in F_N$ whose axis in T has nondegenerate intersection with $\text{Char}_T(\alpha)$, such that the relative orientation of (α, β) is equal to 1 both in T and in T_n for n sufficiently large.

Since β and β' are hyperbolic in T , they are also both hyperbolic in all trees T_n with $n \in \mathbb{N}$ large enough. A typical situation is depicted in Figure 3.1. Up to replacing β and β' by their inverses, we can assume that $\beta\beta'$ is hyperbolic in T , and that the characteristic sets of α, β and $\beta\beta'$ in T contain a common nondegenerate segment I , and likewise the characteristic sets of α, β' and $\beta\beta'$ contain a common nondegenerate segment I' . This also holds in T_n for all sufficiently large $n \in \mathbb{N}$ (in particular $\beta\beta'$ is hyperbolic in T_n for all sufficiently large $n \in \mathbb{N}$). Since β and $\beta\beta'$ are both hyperbolic, their relative orientation eventually agrees in T_n and in T . By hypothesis,

so does the relative orientation of (α, β) . The existence of the segment I thus ensures that the relative orientation of $(\alpha, \beta\beta')$ eventually agrees in T_n and in T . Since β' and $\beta\beta'$ are both hyperbolic, their relative orientation eventually agrees in T_n and in T . The existence of the segment I' thus ensures that the relative orientation of (α, β') eventually agrees in T_n and in T , as required. \square

PROPOSITION 3.6. — *The space \widehat{CV}_N is compact.*

Proof. — Since \widehat{CV}_N is second countable, it is enough to prove sequential compactness. Let $(T_n)_{n \in \mathbb{N}} \in \widehat{CV}_N^{\mathbb{N}}$. Since \overline{CV}_N is compact [9, Theorem 4.5], up to passing to a subsequence, we can assume that $(\pi(T_n))_{n \in \mathbb{N}}$ converges to a tree $T \in \overline{CV}_N$.

Let $\alpha \in F_N$ be an element that fixes a nondegenerate arc in T , and let $\beta \in F_N$ be a hyperbolic element in T , whose axis has nondegenerate intersection with $\text{Char}_T(\alpha)$. Then up to passing to a subsequence, we can assume that the relative orientation of (α, β) in T_n is eventually constant, and assign the corresponding orientation to $\text{Char}_T(\alpha)$. If we do this equivariantly for each of the finitely many orbits of maximal arcs with nontrivial stabilizer in T , Lemma 3.5 ensures that we have found a subsequence of $(T_n)_{n \in \mathbb{N}}$ that converges to T . \square

Being compact, Hausdorff, and second countable, the space \widehat{CV}_N is metrizable.

COROLLARY 3.7. — *The space \widehat{CV}_N is metrizable.*

PROPOSITION 3.8. — *The space \widehat{CV}_N has finite topological dimension equal to $3N - 4$.*

Proof. — The map $\pi : \widehat{CV}_N \rightarrow \overline{CV}_N$ is a continuous map between compact metrizable spaces, with finite point preimages. It follows from Hurewicz's theorem (see [14, Theorem 4.3.6]) that $\dim(\widehat{CV}_N) \leq \dim(\overline{CV}_N)$. In addition, the space \widehat{CV}_N contains CV_N as a topologically embedded subspace (Lemma 3.2). As $\dim(\overline{CV}_N) = \dim(CV_N) = 3N - 4$ (see [4, 15]), the result follows. \square

Corollary 4.3 below will show in addition that CV_N is dense in \widehat{CV}_N , so \widehat{CV}_N is a compactification of CV_N . The $\text{Out}(F_N)$ -action clearly extends to a continuous action on this compactification.

4. The space \widehat{CV}_N is an AR, and the boundary is a Z -set.

The goal of the present section is to prove Theorem 1.2 from the introduction.

4.1. General strategy

We first explain the general strategy of our proof. Let X be a topological space, let $Z \subset X$ be a closed subspace, and let $z \in Z$. We say that (X, Z) is *locally complementarily contractible (LCC)* at z if for every neighborhood \mathcal{U} of z in X there is a smaller neighborhood \mathcal{V} of z in X such that the inclusion $\mathcal{V} \setminus Z \hookrightarrow \mathcal{U} \setminus Z$ is nullhomotopic. If this is true for every $z \in Z$ we say that (X, Z) is *LCC*.

Remark 4.1. — Notice that if (X, Z) is LCC, then $X \setminus Z$ is dense in X , as otherwise we would be able to find a point $z \in Z$ and an open neighborhood \mathcal{V} of z in X such that $\mathcal{V} \setminus Z = \emptyset$, and the map $\emptyset \rightarrow \emptyset$ is not nullhomotopic by convention.

To detect that \widehat{CV}_N is an AR we will use the criterion in the appendix, Theorem B.1. We will apply it with $X = \widehat{CV}_N$ and $Z = X \setminus CV_N$. We are thus reduced to proving that (X, Z) is LCC at every $T_0 \in Z$. We will actually work in the unprojectivized version \widehat{cv}_N and prove the following statement.

THEOREM 4.2. — *For all $N \geq 1$, the pair $(\widehat{cv}_N, \widehat{cv}_N \setminus cv_N)$ is LCC.*

In view of Remark 4.1, we deduce the following result.

COROLLARY 4.3. — *For all $N \geq 1$, every point in \widehat{cv}_N is a limit of points in cv_N .*

Proof of Theorem 1.2 from Theorem 4.2. — Since CV_N is contractible and locally contractible, and since \widehat{CV}_N is compact, metrizable and finite-dimensional, in view of Theorem B.1, it is enough to show that $(\widehat{CV}_N, \widehat{CV}_N \setminus CV_N)$ is LCC.

Denote by $*$ the trivial tree in \widehat{cv}_N . We claim that there exists a continuous lift $\phi : \widehat{CV}_N \rightarrow \widehat{cv}_N$ of the projection map $\psi : \widehat{cv}_N \setminus \{*\} \rightarrow \widehat{CV}_N$. Indeed, by [32, I.6.5, Corollaire 2], there exists a finite set $\{g_1, \dots, g_k\}$ of elements of F_N such that for every nontrivial tree $T \in \widehat{cv}_N$, there exists $i \in \{1, \dots, k\}$ with $\|g_i\|_T > 0$. The lift ϕ is then defined by sending a point $[T] \in \widehat{CV}_N$ to the unique representative for which $\sum_{i=1}^k \|g_i\|_T = 1$.

Let $[T] \in \widehat{CV}_N$, and let T be a lift to \widehat{cv}_N . Let $\mathcal{U}_{[T]}$ be a neighborhood of $[T]$ in \widehat{CV}_N , and let \mathcal{U}_T be the full preimage of $\mathcal{U}_{[T]}$ in \widehat{cv}_N , which is a neighborhood of T in \widehat{cv}_N . Since $(\widehat{cv}_N, \widehat{cv}_N \setminus cv_N)$ is LCC at T , there exists a neighborhood $\mathcal{V}_T \subseteq \mathcal{U}_T$ of T in \widehat{cv}_N such that the inclusion $\mathcal{V}_T \cap cv_N \subseteq \mathcal{U}_T \cap cv_N$ is nullhomotopic; we denote by $H : (\mathcal{V}_T \cap cv_N) \times [0, 1] \rightarrow \mathcal{U}_T \cap cv_N$ a homotopy from $\mathcal{V}_T \cap cv_N$ to a point. Let $\mathcal{V}_{[T]}$ be a neighborhood of $[T]$ in \widehat{CV}_N such that $\phi(\mathcal{V}_{[T]}) \subseteq \mathcal{V}_T$. Then $\psi \circ H \circ \phi$ is a homotopy from $\mathcal{V}_{[T]} \cap CV_N$ to a point that stays inside $\mathcal{U}_{[T]} \cap CV_N$. This shows that $(\widehat{CV}_N, \widehat{CV}_N \setminus CV_N)$ is LCC, as claimed. \square

We now explain our proof of Theorem 4.2; Theorem 4.4 below summarizes the strategy. In order to state it, we extend the notion of morphisms to \widehat{cv}_N as follows: a *morphism* between two trees $T, T' \in \widehat{cv}_N$ is a morphism between $\pi(T)$ and $\pi(T')$ which is further assumed to be isometric and orientation-preserving when restricted to every edge with nontrivial stabilizer.

THEOREM 4.4. — *Let $T_0 \in \widehat{cv}_N$ be a nontrivial tree and let \mathcal{U} be a neighborhood of T_0 . Then there is a tree $U \in \mathcal{U}$ and an optimal morphism $f : U \rightarrow T_0$ such that*

- (1) U splits as a graph of actions over a 1-edge free splitting S of F_N .

Further, there is a subset $X_{S,U} \subseteq \widehat{cv}_N$ that contains U , a continuous map $\rho : \widehat{cv}_N \rightarrow X_{S,U}$, and a homotopy $H : \widehat{cv}_N \times [0, 1] \rightarrow \widehat{cv}_N$ between the identity and ρ such that

- (2) $\rho(T_0) = U$,
- (3) $H(\{T_0\} \times [0, 1]) \subseteq \mathcal{U}$,
- (4) $H(cv_N \times [0, 1]) \subseteq cv_N$, and
- (5) Assume that the pair $(\widehat{cv}_k, \widehat{cv}_k \setminus cv_k)$ is LCC for all $k < N$. If $U \not\subseteq cv_N$ then $(X_{S,U}, X_{S,U} \setminus cv_N)$ is LCC at U .

The fact that every tree in \widehat{cv}_N can be approximated by trees that split as graphs of actions over free splittings (Property (1)) will be explained in Section 4.3; it relies on classical approximation arguments, using the Rips machine. The construction of $X_{S,U}$ and of the maps ρ and H will be carried out in Section 4.4. Roughly speaking, the set $X_{S,U}$ will consist of trees that split as a graph of actions over the one-edge free splitting S , but the details of the definition will depend on U . To prove Property (5), we will take advantage of this splitting, which will allow for an inductive argument.

Proof of Theorem 4.2 assuming Theorem 4.4. — The proof is by induction on N . The statement is obvious when $N = 1$, so we assume that $N \geq 2$.

First, we observe that the pair $(\widehat{cv}_N, \widehat{cv}_N \setminus cv_N)$ is LCC at the trivial tree $*$. Indeed, by [32, I.6.5, Corollary 2], we can find a finite set $\{\gamma_1, \dots, \gamma_k\}$ of elements of F_N such that for every nontrivial tree $T \in \widehat{cv}_N$, there exists $i \in \{1, \dots, k\}$ such that $\|\gamma_i\|_T > 0$. Then any open set \mathcal{U} that contains $*$ contains an open set of the form

$$\mathcal{V} = \left\{ T \in \widehat{cv}_N \mid \sum_{i=1}^k \|\gamma_i\|_T < \varepsilon \right\}$$

for some $\varepsilon > 0$. By scaling, $\mathcal{V} \cap cv_N$ deformation retracts to the slice

$$\left\{ T \in cv_N \mid \sum_{i=1}^k \|\gamma_i\|_T = \varepsilon/2 \right\}$$

which is homeomorphic to CV_N and hence contractible. This shows that $\mathcal{V} \cap cv_N \hookrightarrow \mathcal{U} \cap cv_N$ is nullhomotopic.

Let now $T_0 \in \widehat{cv}_N$ be a nontrivial tree and let \mathcal{U} be a neighborhood of T_0 . Let $U, X_{S,U}, \rho, H$ be as in Theorem 4.4. There are now two cases, depending whether U belongs to cv_N or to the boundary.

If $U \in cv_N$, choose a contractible neighborhood \mathcal{W} of U in $cv_N \cap \mathcal{U}$. By the continuity of ρ and H and Properties (2) and (3), there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ of T_0 such that $H(\mathcal{V} \times [0, 1]) \subseteq \mathcal{U}$ and $\rho(\mathcal{V}) \subseteq \mathcal{W}$. By Property (4), we have $H((\mathcal{V} \cap cv_N) \times [0, 1]) \subseteq \mathcal{U} \cap cv_N$, so H homotopes $\mathcal{V} \cap cv_N$ into \mathcal{W} , staying within $\mathcal{U} \cap cv_N$. As \mathcal{W} is contractible, this shows that the inclusion $\mathcal{V} \cap cv_N \hookrightarrow \mathcal{U} \cap cv_N$ is nullhomotopic.

Now suppose $U \notin cv_N$. By induction we may assume that the pair $(\widehat{cv}_k, \widehat{cv}_k \setminus cv_k)$ is LCC for all $k < N$. Therefore by Property (5), there is a neighborhood $\mathcal{W} \subset \mathcal{U} \cap X_{S,U}$ of U in $X_{S,U}$ such that $\mathcal{W} \cap cv_N \hookrightarrow \mathcal{U} \cap cv_N$ is nullhomotopic. By continuity of ρ and H , there is a neighborhood \mathcal{V} of T_0 such that $H(\mathcal{V} \times [0, 1]) \subseteq \mathcal{U}$ and $\rho(\mathcal{V}) \subseteq \mathcal{W}$, and by Property (4) we have $H((\mathcal{V} \cap cv_N) \times [0, 1]) \subseteq \mathcal{U} \cap cv_N$. Therefore H homotopes $\mathcal{V} \cap cv_N$ into $\mathcal{W} \cap cv_N$, staying within $\mathcal{U} \cap cv_N$. As $\mathcal{W} \cap cv_N \hookrightarrow \mathcal{U} \cap cv_N$ is nullhomotopic, we deduce that the inclusion $\mathcal{V} \cap cv_N \hookrightarrow \mathcal{U} \cap cv_N$ is nullhomotopic. \square

4.2. Morphisms between trees in $\widehat{cv_N}$ and extension of the semi-flow

We recall from the previous section that a *morphism* between two trees $T, T' \in \widehat{cv_N}$ is a morphism between $\pi(T)$ and $\pi(T')$ which is further assumed to be isometric and orientation-preserving when restricted to every edge with nontrivial stabilizer. A morphism between two trees $T, T' \in \widehat{cv_N}$ is *optimal* if the corresponding morphism from $\pi(T)$ to $\pi(T')$ is optimal. The space $\text{Mor}(\widehat{cv_N})$ of all morphisms between trees in $\widehat{cv_N}$ is topologized by saying that two morphisms are close whenever the corresponding morphisms between the projections of the trees in $\widehat{cv_N}$ are close, and in addition the sources and ranges of the morphisms are close in $\widehat{cv_N}$. As before, this space is second-countable. We note that the canonical map $\text{Mor}(\widehat{cv_N}) \rightarrow \text{Mor}(\overline{cv_N})$ is bounded-to-one, and it is injective when restricted to morphisms between two fixed elements of $\widehat{cv_N}$. We denote by $\text{Opt}(\widehat{cv_N})$ the space of all optimal morphisms between trees in $\widehat{cv_N}$. We now extend Proposition 2.1 to morphisms between trees in $\widehat{cv_N}$.

PROPOSITION 4.5. — *There exist continuous maps $\widehat{H} : \text{Opt}(\widehat{cv_N}) \times [0, 1] \rightarrow \widehat{cv_N}$ and $\widehat{\Phi}, \widehat{\Psi} : \text{Opt}(\widehat{cv_N}) \times [0, 1] \rightarrow \text{Opt}(\widehat{cv_N})$ such that*

- for all $f \in \text{Opt}(\widehat{cv_N})$ and all $t \in [0, 1]$, the tree $\widehat{H}(f, t)$ is the range of the morphism $\widehat{\Phi}(f, t)$ and the source of the morphism $\widehat{\Psi}(f, t)$,
- for all $f \in \text{Opt}(\widehat{cv_N})$, we have $\widehat{\Phi}(f, 0) = \text{id}$ and $\widehat{\Psi}(f, 0) = f$,
- for all $f \in \text{Opt}(\widehat{cv_N})$, we have $\widehat{\Phi}(f, 1) = f$ and $\widehat{\Psi}(f, 1) = \text{id}$, and
- for all $f \in \text{Opt}(\widehat{cv_N})$ and all $t \in [0, 1]$, we have $\widehat{\Psi}(f, t) \circ \widehat{\Phi}(f, t) = f$.

Proof. — Let $f \in \text{Opt}(\widehat{cv_N})$ be a morphism with source S and range T . By Proposition 2.1, the corresponding morphism from $\pi(S)$ to $\pi(T)$ factors through trees $H(f, t)$. Notice that any morphism satisfies the hypothesis from Lemma 2.3, i.e. f is isometric when restricted to any arc of $\pi(T)$ with trivial stabilizer. By Lemma 2.3, the induced morphism from $H(f, t)$ to $\pi(T)$ is isometric when restricted to any arc of $H(f, t)$ with nontrivial stabilizer. This enables us to define $\widehat{H}(f, t)$ for all $(f, t) \in \text{Opt}(\widehat{cv_N}) \times [0, 1]$, by pulling back the orientations on the edges of T in all trees $H(f, t)$. We also get morphisms $\widehat{\Phi}(f, t)$ and $\widehat{\Psi}(f, t)$. We will check that the map \widehat{H} defined in this way is continuous, from which it follows that $\widehat{\Phi}$ and $\widehat{\Psi}$ are also continuous. Since $\text{Opt}(\widehat{cv_N})$ is second-countable, it is enough to show sequential continuity.

Let $(f, t) \in \text{Opt}(\widehat{cv_N}) \times [0, 1]$, and let $((f_n, t_n))_{n \in \mathbb{N}} \in (\text{Opt}(\widehat{cv_N}) \times [0, 1])^{\mathbb{N}}$ be a sequence that converges to (f, t) . By Proposition 2.1, the sequence

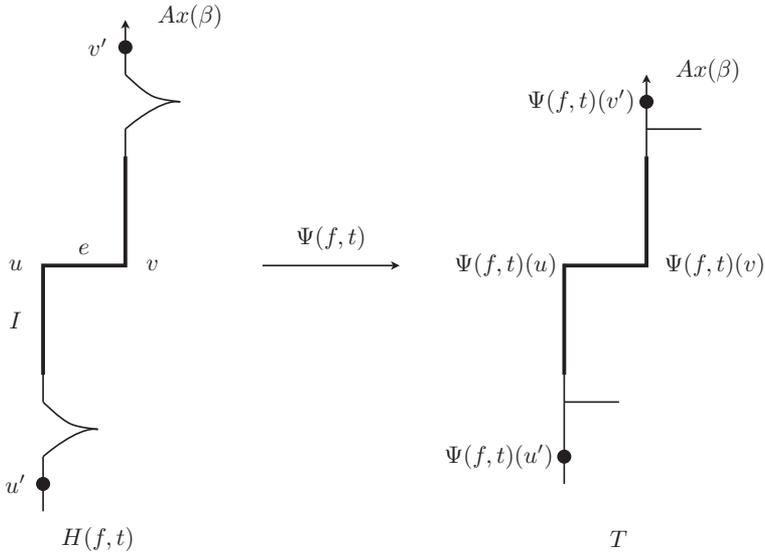


Figure 4.1. The situation in the proof of Proposition 4.5.

$(\pi(\widehat{H}(f_n, t_n)))_{n \in \mathbb{N}}$ converges to $\pi(\widehat{H}(f, t))$. Let $\alpha \in F_N$ be an element that fixes a nondegenerate arc $e = [u, v]$ in $\widehat{H}(f, t)$. We denote by $l(e)$ the length of e . Then α also fixes a nondegenerate arc in the range T of f . Let $C := \text{BBT}(f)$ be the bounded backtracking constant of f (whose definition is recalled in the paragraph preceding Lemma 2.2). Recall from Lemma 2.2 that $\text{BBT}(\Psi(f, t)) \leq \text{BBT}(f)$ for all $t \in [0, 1]$.

We observe that there exists a legal segment I of length larger than $2\text{BBT}(f) + l(e)$, centered at the midpoint of e . Indeed, every segment contained in one of the subtrees with dense orbits of the Levitt decomposition is legal (see e.g. [5, Lemma 2.2]), and since $\Psi(f, t)$ is optimal, every legal segment ending at a vertex of $\pi(\widehat{H}(f, t))$ can be enlarged to a longer legal segment. We can then find an element $\beta \in F_N$ which is hyperbolic in $\widehat{H}(f, t)$ and whose axis contains I (this is obtained by multiplying two hyperbolic elements whose axes are disjoint, and such that the bridge between their axes contains I). This is illustrated in Figure 4.1.

We claim that β is hyperbolic in T , that its axis crosses the image of e , and that the relative orientations of (α, β) are the same in $\widehat{H}(f, t)$ and in T . To prove this, we first observe that there is no point u' on the axis of β in $H(f, t)$ which has the same image as u in T : otherwise, our choice of I would imply that the image of $[u, u']$ in T contains a point at distance larger

than $\text{BBT}(f)$ from $\Psi(f, t)(u) = \Psi(f, t)(u')$, contradicting the definition of the BBT. Similarly, there is no point v' on the axis of β in $H(f, t)$ which has the same image as v in T . In particular, if u' and v' are two points lying on the axis of β in $H(f, t)$, sufficiently far from u and v , and such that u', u, v, v' are aligned in this order in $H(f, t)$, then their images in T are also aligned in this order. Using this observation, we can thus find three points of the form $u, \beta u$ and $\beta^2 u$ lying on the axis of β in $H(f, t)$, whose images in T are still aligned in this order. This implies that β is hyperbolic in T , and that the axis of β in T crosses the image of e , and the relative orientations of (α, β) in $\widehat{H}(f, t)$ and in T are the same, as claimed.

For all $n \in \mathbb{N}$, let T_n be the range of the morphism f_n . Since $(T_n)_{n \in \mathbb{N}}$ converges to T in \widehat{cv}_N , the relative orientations of (α, β) in T_n and in T eventually agree. Since the characteristic sets of α and β have nondegenerate overlap in all trees $\widehat{H}(f, t')$ with $t' \geq t$, a compactness argument shows that for $n \in \mathbb{N}$ large enough, the characteristic sets of α and β have nondegenerate overlap in all trees $\widehat{H}(f_n, t')$ with $t' \geq t_n$. Therefore, the relative orientation of (α, β) cannot change along the path from $\widehat{H}(f_n, t_n)$ to T_n , so it is the same in $\widehat{H}(f_n, t_n)$ and in T_n . This implies that the relative orientations of (α, β) eventually agree in $\widehat{H}(f_n, t_n)$ and in $\widehat{H}(f, t)$. Lemma 3.5 then shows that $\widehat{H}(f_n, t_n)$ converges to $\widehat{H}(f, t)$. \square

4.3. Approximations by trees that split over free splittings

The following lemma is a version of Lemma 2.8 for \widehat{cv}_N , which easily follows from the version in \widehat{cv}_N .

LEMMA 4.6. — *Let $T \in \widehat{cv}_N$, and let \mathcal{U} be an open neighborhood of T in \widehat{cv}_N . Then there exists an open neighborhood $\mathcal{W} \subseteq \mathcal{U}$ of T in \widehat{cv}_N such that if $U \in \mathcal{W}$ is a tree that admits a morphism f onto T , and if $U' \in \widehat{cv}_N$ is a tree such that f factors through morphisms from U to U' and from U' to T , then $U' \in \mathcal{U}$.*

A one-edge free splitting of F_N is the Bass–Serre tree of a graph of groups decomposition of F_N , either as a free product $F_N = A * B$, or as an HNN extension $F_N = A *$. If $T \in \widehat{cv}_N$ is a tree that splits as a graph of actions over a splitting S of F_N , we say that an attaching point of a vertex action T_v is *admissible* if either it belongs to the G_v -minimal subtree of T_v , or else it is an endpoint of an arc with nontrivial stabilizer contained in G_v . The following proposition extends the analogous result for \widehat{cv}_N (see [21, Theorems 3.6 and 3.11]).

PROPOSITION 4.7. — *Let $T \in \widehat{cv}_N$, and let \mathcal{U} be an open neighborhood of T . Then there exists a tree $U \in \mathcal{U}$ that splits as a graph of actions over a one-edge free splitting S of F_N with admissible attaching points, coming with an optimal morphism $f : U \rightarrow T$ such that*

- every arc with nontrivial stabilizer in T is the f -image of an arc with nontrivial stabilizer in U ,
- the f -preimage of any arc with nontrivial stabilizer in T consists of a single arc with nontrivial stabilizer in U ,
- f is an isometry when restricted to the minimal subtree of any vertex action.

Proof. — The proof of Proposition 4.7 builds on classical approximation techniques of very small F_N -trees, we present a sketch of the argument. Let X be the underlying graph of groups of the Levitt decomposition of T . We can assume that all edge stabilizers of X are nontrivial, otherwise the result is obvious by choosing $U = T$ and f to be the identity. We may do the same if T is simplicial (with all edge groups nontrivial) as such trees are graphs of actions with skeleton a 1-edge free splitting (see [33, 36] or [4, Lemma 4.1]). We now assume T is not simplicial.

If some vertex action T_v of the Levitt decomposition is nongeometric (i.e. not dual to a measured foliation on a 2-complex [28]), then one can approximate T_v by a geometric tree $U'_v \in \mathcal{U}$ that contains a simplicial edge e with trivial stabilizer and admits an optimal morphism $f_v : U'_v \rightarrow T_v$, keeping point stabilizers the same; these induce an optimal morphism $f' : U' \rightarrow T$. Let U' be the tree defined as a graph of actions over X , where T_v is replaced by U'_v (the attaching points are prescribed by the stabilizers because all edge stabilizers in X are nontrivial). The edge e is dual to a one-edge free splitting S of F_N , and U splits as a graph of actions over S . Up to slightly folding e (and slightly increasing its length), we can assume that attaching points are admissible. By folding within every vertex action $A \curvearrowright U'_A$ of U' , the morphism f' factors through a tree U in such a way that the induced morphism from U to T is an isometry when restricted to the A -minimal subtree of U . In view of Lemma 4.6, we can also ensure that $U \in \mathcal{U}$.

We now assume that $T \in \widehat{cv}_N$ is geometric. Every geometric F_N -tree splits as a graph of actions with indecomposable vertex actions, which are either dual to arational measured foliations on surfaces, or of Levitt type.

If T contains a Levitt component, we approximate it by free and simplicial actions by first running the Rips machine, and then cutting along a little arc transverse to the foliation in a naked band, see [4, 16]: this

remains true in \widehat{cv}_N because each of these indecomposable trees has trivial arc stabilizers. This gives an approximation of T by a tree U that splits as a graph of actions over a one-edge free splitting of F_N , and comes equipped with an optimal morphism onto T . We then get the required conditions on f as in the nongeometric case.

Finally, one is left with the case where T is geometric, and all its indecomposable subtrees are dual to measured foliations on surfaces. In this situation, it follows from [22, Proposition 5.10] that either T splits as a graph of actions over a one-edge free splitting of F_N , or else some indecomposable subtree from the geometric decomposition has an *unused* boundary component (this means that in the skeleton of the decomposition of T into its indecomposable geometric components, the boundary curve is not attached to any incident edge). In the latter situation, we can again cut along a little arc with extremity on the unused boundary component, and transverse to the foliation, to approximate the indecomposable component. The required condition on f is obtained as above. \square

4.4. Definition of $X_{S,U}$, construction of the deformation, and proof of Theorem 4.4

In this section, given $T_0 \in \widehat{cv}_N$ and a neighborhood \mathcal{U} we will define a deformation H of \widehat{cv}_N onto a particularly nice set of trees (the set $X_{S,U}$ in Theorem 4.4). Here, S will be a 1-edge free splitting of F_N and all trees in $X_{S,U}$ will split as graphs of actions over the splitting S . The tree U , which comes from Proposition 4.7, will be near T_0 and will be the image of T_0 under the deformation. The construction of $X_{S,U}$ and the deformation (without Property (5) from Theorem 4.4) work equally well in \overline{cv}_N .

From now on, we fix a tree $T_0 \in \widehat{cv}_N$, and an open neighborhood \mathcal{U} of T_0 in \widehat{cv}_N . Let $\mathcal{W} \subseteq \mathcal{U}$ be a smaller neighborhood of T_0 provided by Lemma 4.6, i.e. such that whenever a morphism from $U \in \mathcal{W}$ to T_0 factors through a tree U' , then $U' \in \mathcal{U}$. We choose a one-edge free splitting S of F_N , and a tree $U \in \mathcal{W}$ that splits as a graph of actions over S with admissible attaching points, provided by Proposition 4.7. In particular, there is an optimal morphism $f : U \rightarrow T_0$ such that for every vertex group A of S with vertex action U_A , the restriction of f to the A -minimal subtree $U_A^{\min} \subseteq U_A$ is an isometry.

The description of the deformation depends on whether the splitting S is $F_N = A*B$ or $F_N = A*_1$. We will concentrate on the case of a free product, and only briefly explain how to adapt to the case of an HNN extension, as the details differ only in notation.

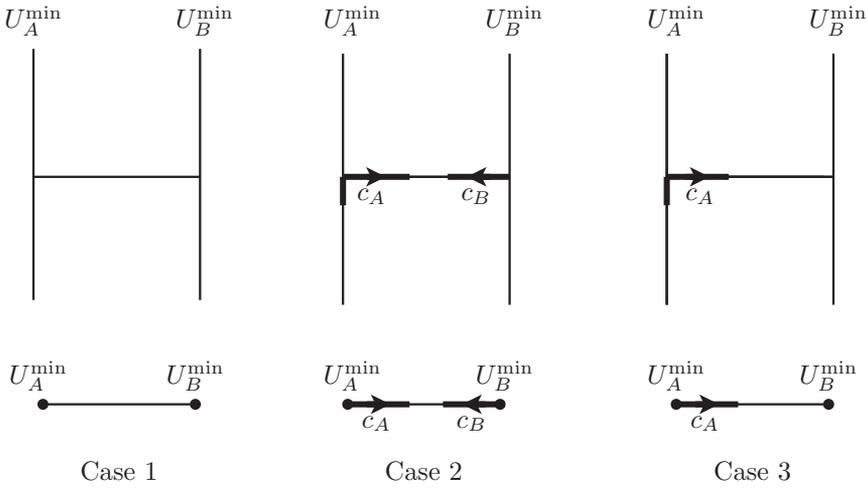


Figure 4.2. The tree U in all three cases when the splitting S is a free product.

Case of a free product. There will be several cases in the construction, depending whether the attaching point $u_A \in U$ of the vertex action $A \curvearrowright U_A$ belongs to U_A^{\min} or not (and similarly for u_B). The three possible cases are illustrated in Figure 4.2.

Case 1. — The free splitting S is the Bass–Serre tree of a decomposition of F_N as a free product $F_N = A * B$, and both $u_A \in U_A^{\min}$ and $u_B \in U_B^{\min}$.

Case 1.1. — Both A and B have rank at least 2.

Corollary 2.7 yields a continuous map $b_A : \widehat{cv}(A) \rightarrow \text{Map}(A, \widehat{cv}(A))$, such that $b_A(U_A^{\min})(*) = u_A$. We will abuse notation and write $b_A(T_A)$ for $b_A(T_A)(*)$. Similarly, we have a map b_B . We make the following definition.

DEFINITION 4.8. — Under the assumptions of Case 1.1, the set $X_{S,U} \subseteq \widehat{cv}_N$ is the set of all trees T that split as a graph of actions over S , with the condition that both vertex actions T_A and T_B are minimal, and the attaching point on the A -side (resp. on the B -side) is $b_A(T_A)$ (resp. $b_B(T_B)$).

Notice that we allow the possibility that T_A or T_B (or both) is the trivial tree.

PROPOSITION 4.9. — Let $T_0 \in \widehat{cv}_N$, let \mathcal{U} be an open neighborhood of T_0 in \widehat{cv}_N , and let S, U be chosen as above, and assume Case 1.1.

Then there exist a continuous map $\rho : \widehat{cv}_N \rightarrow X_{S,U}$ with $\rho(T_0) = U$, and a homotopy $H : \widehat{cv}_N \times [0, 1] \rightarrow \widehat{cv}_N$ from the identity to ρ , such that $H(\{T_0\} \times [0, 1]) \subseteq \mathcal{U}$ and $H(cv_N \times [0, 1]) \subseteq cv_N$.

We will start by defining $\rho(T)$ for all $T \in \widehat{cv}_N$. Let T_A be the A -minimal subtree of T , and let T_B be the B -minimal subtree of T . The tree $\rho(T)$ is defined as the tree that splits as a graph of actions over S with vertex actions T_A and T_B , and attaching points $b_A(T_A)$ and $b_B(T_B)$, where the simplicial edge from the splitting is assigned length $d_T(b_A(T_A), b_B(T_B))$.

Let now $f_T : \rho(T) \rightarrow T$ be the morphism which is the identity when restricted to T_A or T_B , and sends the simplicial edge linearly onto its image in T . Observe that by construction we have $\rho(T_0) = U$ and $f_{T_0} = f$.

LEMMA 4.10. — *Under the assumptions of Case 1.1, the morphism $f_T : \rho(T) \rightarrow T$ depends continuously on the tree $T \in \widehat{cv}_N$.*

Proof. — We need to establish that the tree $\rho(T)$ depends continuously on T . Once this is established, continuity of $T \mapsto f_T$ follows from the construction. Continuity of $\pi(\rho(T)) \in \widehat{cv}_N$ is obvious from the continuity of the choice of basepoints.

Now suppose that $T_n \rightarrow T$ and notice that all nontrivial arc stabilizers in $\rho(T)$ are conjugate into either A or B . By Lemma 3.5, we only need to associate to any element $\alpha \in F_N$ fixing a nondegenerate arc in $\rho(T)$, an element $\beta \in F_N$ that is hyperbolic in $\rho(T)$ and whose axis has nondegenerate intersection with $\text{Char}_T(\alpha)$, such that the relative orientations of (α, β) eventually agree in $\rho(T_n)$ and in $\rho(T)$. The element α fixes a nondegenerate arc in either the A - or the B -minimal subtree of T . As the minimal A -invariant subtree of T_n converges (as an A -tree in $\widehat{cv}(A)$) to the minimal A -invariant subtree of T , we are done by choosing β to be in A or B . \square

Proof of Proposition 4.9. — For all $T \in \widehat{cv}_N$ and all $t \in [0, 1]$, we let $H(T, t) := \widehat{H}(f_T, 1 - t)$, with the notation from Proposition 4.5. Since $\rho(T_0) = U$ belongs to \mathcal{W} , Lemma 4.6 implies that $\widehat{H}(f_{T_0}, t) \in \mathcal{U}$ for all $t \in [0, 1]$. In addition, we have $\widehat{H}(f_T, 0) = \rho(T)$ and $\widehat{H}(f_T, 1) = T$ for all $T \in \widehat{cv}_N$, and $\widehat{H}(f_T, t) \in cv_N$ for all $(T, t) \in cv_N \times [0, 1]$ in view of Remark 2.4. \square

Proof of Theorem 4.4 in Case 1.1. — Properties (1) to (4) have been established in Proposition 4.9, so there remains to establish Property (5). Assume that $(\widehat{cv}_k, \widehat{cv}_k \setminus cv_k)$ is LCC for all $k < N$. We will prove that $(X_{S,U}, X_{S,U} \setminus cv_N)$ is LCC at every point of $X_{S,U} \setminus cv_N$.

A tree T in $X_{S,U}$ is completely determined by T_A, T_B and the length of the arc connecting the basepoints $b_A(T_A)$ and $b_A(T_B)$. This gives a canonical homeomorphism

$$X_{S,U} \cong \widehat{cv(A)} \times \widehat{cv(B)} \times [0, \infty)$$

Let (T_A, T_B, d) be a point in the product space and \mathcal{U} a given neighborhood of it. After shrinking \mathcal{U} we may assume it has the form

$$\mathcal{U} = \mathcal{U}_A \times \mathcal{U}_B \times J$$

where the three factors are open in their respective spaces and J is an interval.

We claim that there exists a neighborhood $\mathcal{V}_A \subseteq \mathcal{U}_A$ of T_A in $\widehat{cv(A)}$ such that the inclusion $\mathcal{V}_A \cap cv(A) \hookrightarrow \mathcal{U}_A \cap cv(A)$ is nullhomotopic. Indeed, if $T_A \in cv(A)$, this follows from local contractibility of $cv(A)$. If $T_A \notin cv(A)$, this follows from our induction hypothesis which ensures that $(\widehat{cv(A)}, \widehat{cv(A)} \setminus cv(A))$ is LCC. Similarly, there exists a neighborhood $\mathcal{V}_B \subseteq \mathcal{U}_B$ such that the inclusion $\mathcal{V}_B \cap cv(B) \hookrightarrow \mathcal{U}_B \cap cv(B)$ is nullhomotopic. The neighborhood

$$\mathcal{V} = \mathcal{V}_A \times \mathcal{V}_B \times J$$

of T is then such that the inclusion $\mathcal{V} \cap cv_N \hookrightarrow \mathcal{U} \cap cv_N$ is nullhomotopic (notice indeed that $\mathcal{V} \cap cv_N = (\mathcal{V}_A \cap cv(A)) \times (\mathcal{V}_B \cap cv(B)) \times J$). \square

Case 1.2. — We have $\text{rk}(A) \geq 2$ and $\text{rk}(B) = 1$.

In this case, we have a map b_A as in Case 1.1.

DEFINITION 4.11. — *Under the assumptions of Case 1.2, the set $X_{S,U} \subseteq \widehat{cv_N}$ is the set of all trees T that split as a graph of actions over S , with the condition that both vertex actions T_A and T_B are minimal, and the attaching point on the A -side is $b_A(T_A)$.*

Notice that there is no condition on the attaching point on the B -side. Indeed, if T_B is reduced to a point, then there is a unique way of attaching. Otherwise it is a line, and the tree T does not depend on a choice of attaching point in T_B .

Using Corollary 2.7, we also get a continuous map $b_B : \widehat{cv_N} \rightarrow \text{Map}(F_N, \widehat{cv_N})$ such that for all $T \in \widehat{cv_N}$, the range of $b_B(T)$ is T , and $b_B(T_0)(*) = u_B$, and $b_B(T)(*)$ is contained in the characteristic set of B .

Given $T \in \widehat{cv_N}$, we define $\rho(T)$ as follows. Let T_A be the A -minimal subtree of T , and let T_B be either the trivial B -action on a point if B is elliptic in T , or else the B -minimal subtree of T . The tree $\rho(T)$ is defined

as the tree that splits as a graph of actions over S with vertex actions T_A and T_B , with attaching point $b_A(T_A)$ on the A -side, where the simplicial edge from the splitting is assigned length $d_T(b_A(T_A), b_B(T))$.

Let now $f_T : \rho(T) \rightarrow T$ be the morphism which is the identity when restricted to T_A or T_B , and sends the simplicial edge linearly onto the segment $[b_A(T_A), b_B(T)]$. Observe that by construction we have $\rho(T_0) = U$ and $f_{T_0} = f$.

Then $\rho(T)$ and f_T depend continuously on T . The rest of the proof is identical to Case 1.1.

Case 1.3. — We have $N = 2$, and the free splitting S is the Bass–Serre tree of a decomposition of F_N as a free product $F_N = A * B$ with $\text{rk}(A) = 1$ and $\text{rk}(B) = 1$.

The argument is then exactly the same as in Case 1.2 (where both sides are treated as the B -side in the argument).

Case 2. — The free splitting S is the Bass–Serre tree of a decomposition of F_N as a free product $F_N = A * B$ and $u_A \notin U_A^{\min}, u_B \notin U_B^{\min}$.

According to Proposition 4.7, u_A is an endpoint of an arc $I_A \subset U$ with nontrivial stabilizer $\langle c_A \rangle$ with $c_A \in A$. We choose c_A so that the orientation of the characteristic set of c_A is pointing away from U_A^{\min} . Note that $I_A \cap U_A^{\min}$ is a (perhaps degenerate) subarc of I_A containing the other endpoint of I_A . We define c_B similarly. Notice that in this case the minimality of U implies that both A and B have rank at least 2.

DEFINITION 4.12. — *Under the assumptions of Case 2, the set $X_{S,U} \subseteq \widehat{cv}_N$ is the set of all trees T that split as graphs of actions over the splitting S and with attaching points in the vertex trees T_A, T_B belonging to the characteristic sets of c_A and c_B respectively.*

Next, for $T \in \widehat{cv}_N$ define the basepoint $b'_A(T) \in \text{Char}_T(c_A)$ as the projection of $\text{Char}_T(c_B)$ (if the two are disjoint) or the midpoint of the overlap with $\text{Char}_T(c_B)$ (if they are not disjoint). It follows from Corollary 2.7 that this basepoint varies continuously with T . Similarly define $b'_B(T) \in \text{Char}_T(c_B)$. Let T_A be the smallest A -invariant subtree of T that contains both T_A^{\min} and $b'_A(T)$, and similarly define T_B .

We now define $\rho : \widehat{cv}_N \rightarrow X_{S,U}$. Given $T \in \widehat{cv}_N$, we let $\rho(T)$ be the tree that splits as a graph of actions over the free splitting S , with vertex actions T_A and T_B , with attaching points $b'_A(T), b'_B(T)$, and with arc length $d_T(b'_A(T), b'_B(T))$. We also have the obvious morphism $f_T : \rho(T) \rightarrow T$. Note that $\rho(T_0) = U$.

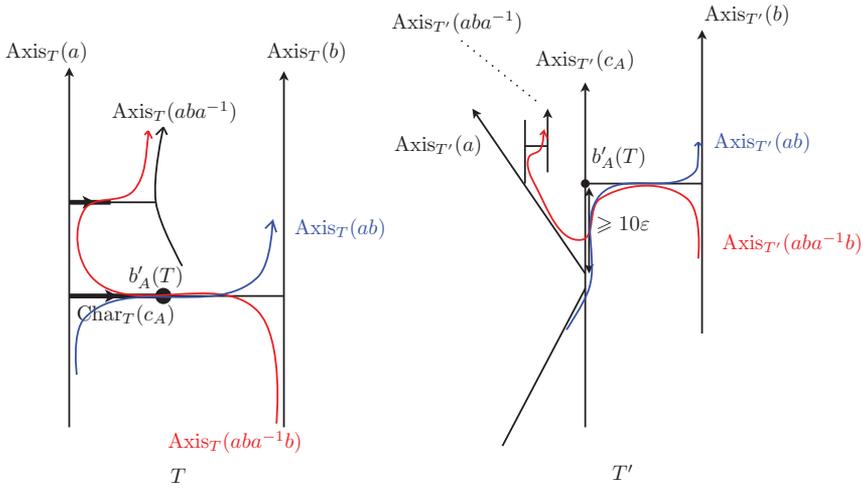


Figure 4.3. The tree T and a tree T' in the neighborhood of T in the last case of the proof of Lemma 4.13. In the picture, we have represented the situation where c_A becomes hyperbolic in T' .

LEMMA 4.13. — Under the assumptions of Case 2, the morphism $f_T : \rho(T) \rightarrow T$ depends continuously on T .

Proof. — Again we need to establish that the tree $\rho(T)$ depends continuously on T . Continuity of $\pi(\rho(T)) \in \overline{c\bar{v}N}$ follows from the construction and Guirardel’s Reduction Lemma from [16, Section 4].

Notice that all nontrivial arc stabilizers in $\rho(T)$ are conjugate into either A or B . By Lemma 3.5, in order to complete the proof of the continuity of ρ , we only need to associate to any element $\alpha \in F_N$ fixing a nondegenerate arc in $\rho(T)$, an element $\beta \in F_N$ that is hyperbolic in $\rho(T)$ and whose axis has nondegenerate intersection with $\text{Char}_T(\alpha)$, such that the relative orientations of (α, β) eventually agree in $\rho(T_n)$ and in $\rho(T)$. This is clear if α fixes a nondegenerate arc in either the A - or the B -minimal subtree of T (see the proof of Lemma 4.10).

If these minimal subtrees have empty intersection in $\rho(T)$, and α fixes a nondegenerate arc, say $\alpha = c_A$, on the segment that joins them in T , then one can choose for β an element which is a product ab , where $a \in A$ and $b \in B$ are hyperbolic in T . Indeed (see Figure 4.3), for every $\varepsilon > 0$ small enough, we can find a neighborhood of T such that for all trees T' in this neighborhood, the intersection of the characteristic sets of c_A and c_B has size at most ε , and the distance between $b'_A(T)$ and the projection of the

axis of b to $\text{Char}_{T'}(c_A)$ is at most ε , while the distance between $\text{Char}_{T'}(a)$ and $\text{Char}_{T'}(b)$ is at least 10ε . In addition, the characteristic sets of c_A and of $baba^{-1}$ have nondegenerate intersection whose orientations disagree in a neighborhood of T . This implies that for all trees T' in a neighborhood of T , the point $b'_A(T)$ is at positive distance from $\text{Char}_{T'}(a) \cap \text{Char}_{T'}(c_A)$ (or from the bridge between these two characteristic sets in case their intersection is empty). It then follows that the characteristic sets of c_A and ab have nondegenerate intersection with agreeing orientations in a neighborhood of T . □

Proof of Theorem 4.4 in Case 2. — Using Lemma 4.13, Properties (1) to (4) are proved in the same way as in Case 1 (Proposition 4.9 stays valid in Case 2 with the same proof). It remains to prove Property (5). First, using the first part of Corollary 2.7, we can choose continuous basepoints $b_A : \widehat{cv(A)} \rightarrow \text{Map}(A, \widehat{cv(A)})$ and $b_B : \widehat{cv(B)} \rightarrow \text{Map}(B, \widehat{cv(B)})$ that belong to characteristic sets of c_A and c_B respectively (Corollary 2.7 was stated for $\widehat{cv_N}$ but orientations of intervals just carry along). Then construct a map

$$\Phi : X_{S,U} \rightarrow \widehat{cv(A)} \times \widehat{cv(B)} \times (-\infty, \infty) \times (-\infty, \infty) \times [0, \infty)$$

by sending $T \in X_{S,U}$ to $(T_A^{\min}, T_B^{\min}, l_A, l_B, d)$ where l_A is the signed distance, measured along the oriented characteristic set of c_A , from $b_A(T_A)$ (viewed as a point in T) to $b'_A(T)$, and similarly for l_B , and d is the length of the attached arc. This map is continuous because the various basepoints vary continuously with T and $d = d_T(b'_A(T), b'_B(T))$.

We claim that the map Φ is a homeomorphism. Once this claim is established, the argument that $(X_{S,U}, X_{S,U} \setminus cv_N)$ is LCC is similar to the proof of Case 1. To prove the claim, we construct the inverse Ψ . Given (T_A, T_B, l_A, l_B, d) , where T_A (resp. T_B) is a minimal A -tree (resp. B -tree), construct $b'_A(T)$ by taking the point in $\text{Char}_{T_A}(c_A)$ (which is oriented) at the correct signed distance of $b_A(T_A)$. More precisely, if c_A is hyperbolic, then the point $b'_A(T)$ is uniquely defined. If c_A is elliptic, it may happen that $|l_A|$ is larger than the distance between $b_A(T_A)$ and the suitable endpoint of $\text{Fix}_{T_A}(c_A)$. In that case attach an arc to this endpoint and add it to the fixed set of c_A (and equivariantly attach an orbit of arcs). Its orientation is determined by the sign of l_A . This yields a tree T'_A . Similarly construct T'_B (in the case where c_A is hyperbolic, we just let $T'_A = T_A$, and likewise on the B -side). Then $\Psi(T_A, T_B, l_A, l_B, d) := T$ is the graph of actions obtained by gluing an arc of length d between $b'_A(T_A)$ and $b'_B(T_B)$. By construction Ψ is an inverse of Φ .

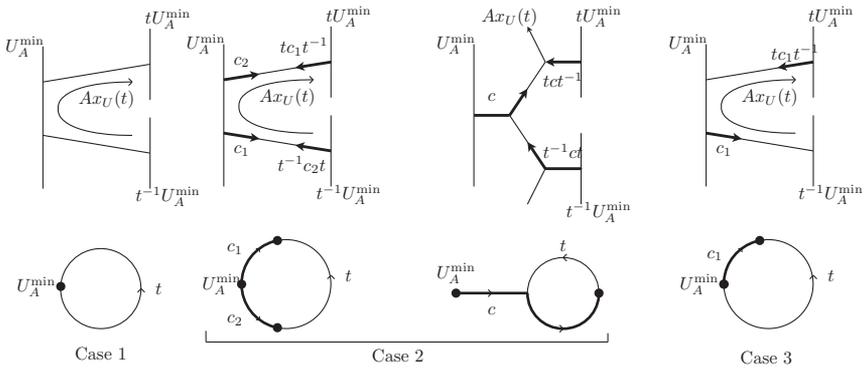


Figure 4.4. The tree U in Case 2 of the proof of Proposition 4.9, and the quotient of groups.

We claim that the map $(T_A, l_A) \mapsto (T'_A, b'_A(T_A))$ is continuous; continuity of Ψ will then follow from Guirardel’s Reduction Lemma [16, Section 4]. To prove the claim, it is enough to observe that if (T_A^n, l_A^n) converges to (T_A, l_A) , then for all $g \in F_N$, the distance $d_{(T'_A)^n}(b'_A(T_A^n), gb'_A(T_A^n))$ converges to $d_{T'_A}(b'_A(T_A), gb'_A(T_A))$. This is proved by using the analogous convergence for the basepoints b_A instead of b'_A , and the fact that the basepoints b_A and b'_A always lie at the same distance l_A on the characteristic set of c_A . \square

Remark 4.14. — Notice that the above proof of Property (5) does not work in $\overline{cv_N}$ where we do not have a well-defined notion of signed distance along the characteristic set of an element of F_N .

Case 3. — The free splitting S is the Bass–Serre tree of a decomposition of F_N as a free product $F_N = A * B$ and $u_A \in U_A^{\min}$, $u_B \notin U_B^{\min}$.

Fix a nontrivial element $g_A \in A$. In this hybrid case we define basepoints $b_A(T)$ as in Case 1, using Lemma 2.7, and $b'_B(T)$ as in Case 2, projecting the characteristic set of g_A to $\text{Char}_T(c_B)$ (or taking the midpoint of the overlap). The argument in this case follows Cases 1 and 2 and is left to the reader.

Case of an HNN extension. We now briefly explain how to adapt the above construction in the case of an HNN extension. As the argument is the same as in the free product case (details differ only in notation), we leave the proof of Proposition 4.9 to the reader in this case.

We first assume that $N \geq 3$. The tree U has one of the shapes represented in Figure 4.4, where t denotes a stable letter for the splitting; this leads to three cases as in the case of a free product. There are two attaching points in U_A , which we denote by u_1 and u_2 .

The first case is the case where both u_1 and u_2 belong to U_A^{\min} . As $N \geq 3$, the rank of A is at least 2. By Corollary 2.7, there exist continuous maps $b_1 : \widehat{cv}(A) \rightarrow \text{Map}(A, \widehat{cv}(A))$ and $b_2 : \widehat{cv}(A) \rightarrow \text{Map}(A, \widehat{cv}(A))$ such that $b_1(U_A^{\min})(*) = u_1$ and $b_2(U_A^{\min})(*) = u_2$. We then define $X_{S,U} \subseteq \widehat{cv}_N$ to be the set of all trees T that split as a graph of actions over S , with the condition that both vertex actions T_A and T_B are minimal, and the attaching points are $b_1(T_A)$ and $b_2(T_B)$ respectively. We then argue as in Case 1 of the free product case.

The second case is when neither u_1 nor u_2 belongs to U_A^{\min} . Then u_1 and u_2 belong to arcs stabilized by elements $c_1, c_2 \in F_N$ (in the case where $c_1 = c_2$, the points u_1 and u_2 may be the two extremities of the segment fixed by $c_1 = c_2$, but it might also happen that one of them is not an extremity, see Case 2 in Figure 4.4). We then define a basepoint $b'_1(T)$ to be the projection of $\text{Char}_T(t^{-1}c_2t)$ to $\text{Char}_T(c_1)$ if these sets have nondegenerate intersection, or else the midpoint of their intersection. Similarly, we let $b'_2(T)$ be the projection of $\text{Char}_T(tc_1t^{-1})$ to $\text{Char}_T(c_2)$ if these sets have nondegenerate intersection, or else as the midpoint of their intersection. We then define $X_{S,U} \subseteq \widehat{cv}_N$ to be the set of all trees T that split as graphs of actions over the splitting S and with attaching points belonging to the characteristic sets of c_1 and c_2 respectively. We argue as in Case 2 of the free product case.

Finally, in the hybrid case where exactly one of the points u_1, u_2 belongs to U_A^{\min} , we define one basepoint as in Case 1, and one basepoint as in Case 2.

We now assume that $N = 2$, so $A = \langle a \rangle$ is isomorphic to \mathbb{Z} . In this case, we define $X_{S,U}$ as the set of all trees T that split as a graph of actions over S , where the vertex action is either a tree in \widehat{cv}_1 or an oriented arc stabilized by a . There is a homeomorphism between $X_{S,U}$ and $\widehat{cv}_1 \times \mathbb{R} \times \mathbb{R}_+$, a point in $X_{S,U}$ being completely determined by the data of a tree in \widehat{cv}_1 , together with the signed distance between the two attaching points (when the tree in \widehat{cv}_1 is trivial, this is the signed length of the arc fixed by a), and the length of the arc coming from S .

Appendix A. A Čech homology lemma

We regard the n -sphere S^n as the boundary of the unit ball B^{n+1} in \mathbb{R}^{n+1} and $S^{n-1} \subset S^n$ as the equator $S^n \cap (\mathbb{R}^n \times \{0\})$. The northern (resp. southern) hemisphere B_+^n (resp. B_-^n) is the set of points in S^n whose last coordinate is nonnegative (resp. nonpositive). We will say that a map $h : S^n \rightarrow \mathbb{R}$ is *standard* if $h^{-1}(0) = S^{n-1}$, $h^{-1}([0, \infty)) = B_+^n$ and $h^{-1}((-\infty, 0]) = B_-^n$. The reader is referred to [13, Chapters IX, X] for basic facts about Čech homology.

LEMMA A.1. — *Let $n \geq 1$, and let $\tilde{h} : B^{n+1} \rightarrow \mathbb{R}$ be a continuous map whose restriction to $S^n = \partial B^{n+1}$ is standard. Then the inclusion*

$$S^{n-1} \hookrightarrow Y := \tilde{h}^{-1}(0)$$

is trivial in Čech homology \check{H}_{n-1} with $\mathbb{Z}/2$ -coefficients (one has to consider reduced homology if $n = 1$).

Proof. — The proof is illustrated in Figure A.1. Represent Y as a nested intersection $\cap_i K_i$ of compact neighborhoods. By the continuity of Čech homology [13, Theorem X.3.1], we have

$$\check{H}_{n-1}(Y) = \lim_{\leftarrow} H_{n-1}(K_i),$$

so it suffices to show that $S^{n-1} \hookrightarrow K_i$ is trivial in (singular) H_{n-1} for every i .

Fix one such K_i and choose $\varepsilon > 0$ so that $\tilde{h}^{-1}(-2\varepsilon, 2\varepsilon) \subset K_i$. Let $g : B^{n+1} \rightarrow \mathbb{R}$ be a smooth map such that $|\tilde{h}(x) - g(x)| < \varepsilon/2$ for every x and so that g and $g_0 = g|_{S^n}$ have ε as a regular value. Thus $V = g_0^{-1}(\varepsilon) \subset B_+^n$ (in blue on the figure) is an $(n - 1)$ -manifold and it bounds the n -manifold $g^{-1}(\varepsilon) \subset \tilde{h}^{-1}(\varepsilon/2, 3\varepsilon/2) \subset K_i$ (in red on the figure). On the other hand, V is disjoint from S^{n-1} and $V \sqcup S^{n-1}$ bounds the n -manifold $g_0^{-1}[0, \varepsilon] \subset \tilde{h}^{-1}(-\varepsilon/2, 3\varepsilon/2) \subset K_i$ (in green on the figure). This shows that the cycle $[S^{n-1}]$ is null-homologous in K_i . □

Appendix B. Proving that a space is an AR: a topological criterion

In this appendix, we establish the ANR criterion that suits our purpose. Recall from the introduction that a metric space X is *locally n -connected* (LC^n) if for every $x \in X$ and every open neighborhood \mathcal{U} of x , there exists an open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of x such that the inclusion $\mathcal{V} \hookrightarrow \mathcal{U}$ is trivial

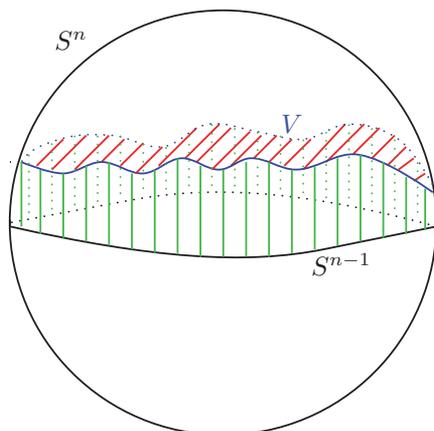


Figure A.1. The situation in the proof of Lemma A.1 (illustrated on the figure in the case where $n = 1$).

in π_i for all $0 \leq i \leq n$. A nowhere dense closed subset $Z \subseteq X$ is *locally complementarily n -connected* (LCC^n) in X if for every $z \in Z$ and every open neighborhood \mathcal{U} of z in X , there exists a smaller neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of z in X such that the inclusion $\mathcal{V} \setminus Z \hookrightarrow \mathcal{U} \setminus Z$ is trivial in π_i for all $0 \leq i \leq n$.

It is a classical fact that a compact n -dimensional LC^n metrizable space is an ANR [23, Theorem V.7.1], and we need the following extension. The methods we use are classical, but we could not find this statement in the literature, so we include a proof. We use [23] as a reference, but the techniques were established earlier, see [7, 12, 25, 26].

THEOREM B.1. — *Let X be a compact metrizable space with $\dim X \leq n$, and let $Z \subset X$ be a nowhere dense closed subset which is LCC^n in X and such that $X \setminus Z$ is LC^n . Then X is an ANR and Z is a Z -set in X .*

If in addition $X \setminus Z$ is assumed to be contractible, then X is an AR.

Proof. — The key is to prove the following statement.

Claim. — Let Y be a compact metrizable space, with $\dim Y \leq n + 1$, let $A \subset Y$ be a closed subset, and let $f : A \rightarrow X$ be a map. Then there exists an open neighborhood \mathcal{O} of A in Y and an extension $\tilde{f} : \mathcal{O} \rightarrow X$ of f such that $\tilde{f}(\mathcal{O} \setminus A) \subseteq X \setminus Z$.

We start by explaining how to derive Theorem B.1 from the claim. To prove that Z is a Z -set, take $Y = X \times [0, 1]$ and $A = X \times \{0\}$, and let

$f : A \rightarrow X$ be the identity. Then an extension to a neighborhood produces an instantaneous homotopy of X off of Z , so Z is a Z -set (indeed, since X is compact, a neighborhood of $X \times \{0\}$ includes $X \times [0, \varepsilon]$ for some $\varepsilon > 0$, and we can reparametrize to $X \times [0, 1]$).

We now argue that X is LC^n , and this will establish that X is an ANR by [23, Theorem V.7.1]. We only need to show that X is LC^n at every point $z \in Z$. Let $z \in Z$ and let \mathcal{U} be a neighborhood of z in X . Choose a neighborhood \mathcal{V} of z in X such that $\mathcal{V} \setminus Z \hookrightarrow \mathcal{U} \setminus Z$ is trivial in π_i for all $i \leq n$. Let $f : S^i \rightarrow \mathcal{V}$ be a given map. Using the instantaneous deformation, we can homotope f to $f' : S^i \rightarrow \mathcal{V} \setminus Z$ within \mathcal{V} . But now f' is nullhomotopic within \mathcal{U} by assumption.

It remains to prove the claim. Choose an open cover \mathcal{W} of $Y \setminus A$ whose multiplicity is at most $n + 2$ and so that the size of the open sets gets small close to A . For example, one could arrange that if $y \in W \in \mathcal{W}$ then $\text{diam}W < \frac{1}{2}d(y, A)$ with respect to a fixed metric d on Y (such coverings are called *canonical*, see e.g. [23, II.11]). Next, let N be the nerve of \mathcal{W} , thus $\dim N \leq n + 1$. There is a natural topology on $A \cup N$ where A is closed, N is open, and a neighborhood of $a \in A$ induced by an open set $\mathcal{O} \subset Y$ with $a \in \mathcal{O}$ is $\mathcal{O} \cap A$ together with the interior of the subcomplex of N spanned by those $W \in \mathcal{W}$ contained in \mathcal{O} . For more details see e.g. [23, II.12].

Since there is a natural map $\pi : Y \rightarrow A \cup N$ that takes A to A and $Y \setminus A$ to N (given by a partition of unity) it suffices to prove the claim after replacing Y by $A \cup N$.

The extension is constructed by induction on the skeleta of N , and uses the method of e.g. [23, Theorem V.2.1]. The only difference is that we want in addition $\tilde{f}(\mathcal{O} \setminus A) \subseteq X \setminus Z$.

To extend $f : A \rightarrow X$ to the vertices of N , use the assumption that Z is nowhere dense in X to send a vertex v close to some $a \in A$ to a point in $X \setminus Z$ close to $f(a)$. Inductively, suppose $0 \leq i \leq n$, and f has been extended to the i -skeleton of some subcomplex N_i of N , in such a way that

- $f(N_i) \subseteq X \setminus Z$,
- $A \cup N_i$ contains a neighborhood of A in $A \cup N$, and
- for every $\delta > 0$, there exists a neighborhood $\mathcal{U}_{i,\delta}$ of A in $A \cup N_i$ such that the f -image of every i -simplex of N_i contained in $\mathcal{U}_{i,\delta}$ has diameter at most δ .

Since X is compact and Z is LCC^n in X , there exists $\delta_{i+1} > 0$, and a function $r : (0, \delta_{i+1}) \rightarrow \mathbb{R}_+$, with $r(t) \rightarrow 0$ as t decreases to 0, such that any map $\phi : S^i \rightarrow X \setminus Z$ whose image has diameter at most $d < \delta_{i+1}$,

extends to a map $\tilde{\phi} : B^{i+1} \rightarrow X \setminus Z$ whose image has diameter at most $r(d)$. Now apply this to every $(i+1)$ -simplex in N_i such that the image of its boundary has diameter strictly smaller than δ_{i+1} . When extending to the simplex, always arrange that the diameter of the image is controlled by the function r . This completes the inductive step. \square

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