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# METABELIAN GROUPS WITH LARGE RETURN PROBABILITY

by Lison JACOBONI (\*)

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ABSTRACT. — We investigate the asymptotic behaviour of the return probability of the random walk in finitely generated metabelian groups. For such groups with exponential volume growth, we obtain a characterization of metabelian groups whose return probability is the largest in purely algebraic terms, namely the Krull dimension of the group. Along the way, we give lower bounds on the return probability for metabelian groups with torsion derived subgroup, according to the dimension. We also establish a variation of the famous embedding theorem of Kaloujinine and Krasner for metabelian groups that respects the Krull dimension. Finally, we study specific sections of these groups, and use them to give upper bounds on the return probability in terms of the Krull dimension.

RÉSUMÉ. — Dans cet article, on s'intéresse au comportement asymptotique de la probabilité de retour de la marche aléatoire dans un groupe métabélien de type fini. Pour de tels groupes à croissance exponentielle, on obtient une caractérisation de ceux dont la probabilité de retour est la plus grande en des termes purement algébriques, à l'aide de la dimension de Krull du groupe. Nous obtenons d'abord des bornes inférieures dépendant de la dimension de Krull sur la probabilité de retour des groupes métabéliens dont le sous-groupe dérivé est de torsion. On prouve également une variante respectant la dimension de Krull d'un théorème de Kaloujinine et Krasner pour les groupes métabéliens. Enfin, on étudie des sections particulières de ces groupes afin de donner des bornes supérieures sur la probabilité de retour en fonction de la dimension de Krull.

## 1. Introduction

Let  $G$  be a countable group. For any probability measure  $\mu$  on  $G$ , one can consider the *random walk on  $G$  driven by  $\mu$* : this is a sequence  $(S_i)_{i \geq 0}$  of random variables valued in  $G$  such that  $S_0$  is an arbitrary point of  $G$  and, for any  $n \in \mathbb{N}$ ,  $S_{n+1} = S_n X_{n+1}$ , where  $(X_i)_{i \geq 1}$  are independent and identically distributed random variables with probability distribution  $\mu$ . If

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the current state is  $x$ , the probability of being in  $y$  at the next step is  $\mu(x^{-1}y)$ . This implicitly defines a probability measure  $\mathbb{P}_\mu$  on  $G^{\mathbb{N}}$  such that

$$\mathbb{P}_\mu(S_n = y \mid X_0 = x) = \mu^{(n)}(x^{-1}y),$$

where  $\mu^{(n)}$  denotes the  $n$ -fold convolution of  $\mu$ . Recall that if  $f$  and  $h$  are two functions on  $G$ , the convolution of  $f$  and  $h$  is  $f * h(g) = \sum_k f(k)h(k^{-1}g)$ .

An interesting class of examples arises from a probability measure whose support generates the whole group  $G$  as a semigroup. Within this class, a fundamental example is given by a finitely generated group  $G$  with  $\mu = \mu_S$ , the uniform distribution on a finite and symmetric generating set  $S$ . Such a probability measure satisfies  $\mu_S(g) = \mu_S(g^{-1})$ , for all  $g \in G$ . In general, one says that a probability measure  $\mu$  is *symmetric* whenever this latter relation is true.

Let  $p_{2n}^{\mu, G} = \mathbb{P}_\mu(X_{2n} = e \mid X_0 = e)$  denote the probability of return in  $2n$  steps to the origin  $e$  of  $G$  (even times  $2n$  are considered to avoid parity issues: namely, the simple random walk on  $\mathbb{Z}$ , with usual generating set, cannot reach 0 at odd times).

If  $\varphi, \psi$  denote two monotone functions, we use the notation  $\varphi \lesssim \psi$  if there exist positive constants  $c$  and  $C$  such that  $c\varphi(Ct) \leq \psi(t)$  (possibly for  $t$  in  $\mathbb{N}$  if  $\mathbb{N}$ -valued functions are considered). If the symmetric relation  $\varphi \gtrsim \psi$  also holds, we write  $\varphi \sim \psi$  and say that  $\varphi$  and  $\psi$  have the same asymptotic behaviour. This is an equivalence relation.

A theorem of Pittet and Saloff-Coste [22] asserts that any two symmetric and finitely supported probability measures with generating support give rise to equivalent return probabilities. Let  $p_{2n}^G$  denote this invariant, dropping the  $G$  whenever the group is clear from the context. Moreover, if  $\mu$  is a symmetric probability measure with generating support and finite second moment, that is  $\sum_g |g|^2 \mu(g) < \infty$ , then  $p_{2n}^{\mu, G}$  belongs to the class of  $p_{2n}^G$ . In addition, an important feature of the return probability is the fact that it increases when taking quotients or going to a subgroup.

Understanding how the random walk behaves allows to have insight into the large-scale geometry of the group. We give below a brief picture of what is known about  $p_{2n}^G$ , more details are to follow in Subsection 3.1.

In his thesis, Kesten proved that non-amenable groups are characterized by the fact that they behave the worst, for their return probability decays exponentially fast ([17]). On the other hand, a nilpotent group, which necessarily has polynomial growth of some definite degree  $d$  behaves like  $\mathbb{Z}^d$ : this follows from Varopoulos ([31, 33]).

What about amenable groups of exponential growth? Hebisch and Saloff-Coste ([14]) proved that if  $G$  has exponential growth, then

$$(1.1) \quad p_{2n}^G \lesssim \exp(-n^{\frac{1}{3}}).$$

Moreover, if  $G$  is a discrete subgroup of a connected Lie group, then  $G$  is amenable of exponential growth if and only if  $p_{2n}^G \sim \exp(-n^{\frac{1}{3}})$ .

However, the world of amenable groups of exponential growth can be very wild, and many other behaviours can occur. As an example, for any nontrivial finite group  $F$ , the return probability of the wreath product  $F \wr \mathbb{Z}^d$  is

$$p_{2n}^{F \wr \mathbb{Z}^d} \sim \exp\left(-n^{\frac{d}{d+2}}\right).$$

Let us also mention the return probability of  $\mathbb{Z} \wr \mathbb{Z}^d$ :

$$(1.2) \quad p_{2n}^{\mathbb{Z} \wr \mathbb{Z}^d} \sim \exp\left(-n^{\frac{d}{d+2}} (\log n)^{\frac{2}{d+2}}\right).$$

We refer to the paper [10] by Erschler for these examples (and many other).

It appears that, for every small positive  $\epsilon$ , there exist  $1 - \epsilon < \alpha < 1$ , and a solvable group whose return probability is equivalent to  $\exp(-n^\alpha)$  (see also [23]). These solvable groups can be chosen to be 2-step solvable (in other words *metabelian* or equivalently having abelian derived subgroup) and hence are relatively tame within the class of solvable groups of arbitrary derived length. Comparing with the situation for discrete subgroups of Lie groups, the following question arises naturally.

QUESTION. — *Among finitely generated solvable groups, is it possible to characterize the groups whose return probability satisfies  $p_{2n} \lesssim \exp(-n^{\frac{1}{3}})$ ?*

So far, (1.1) is known to be sharp for lamplighter group  $F \wr \mathbb{Z}$ , with  $F$  a nontrivial finite group ([30]), for polycyclic groups ([1]), for solvable Baumslag–Solitar ([8]), and for solvable groups of finite Prüfer rank ([18, 22]). A group has *finite Prüfer rank* if there exists an integer  $r$  such that any finitely generated subgroup can be generated by at most  $r$  elements and the least such  $r$  is the rank. Lastly, Tessera proved in [28] the reverse inequality for a class of groups containing all discrete subgroups of solvable algebraic groups over a local field. This class contains in particular lamplighter groups and torsion-free solvable groups of finite Prüfer rank.

Here, we answer the above question in the case of metabelian groups, showing a connection between the Krull dimension of a metabelian group and the asymptotic behaviour of its return probability.

A metabelian group  $G$  is an extension of an abelian group by another abelian group, namely

$$[G, G] \hookrightarrow G \twoheadrightarrow G_{ab},$$

where we denote by  $[G, G]$  the derived subgroup of  $G$  and by  $G_{ab}$  its abelianization  $G/[G, G]$ . The subgroup  $[G, G]$  carries a natural structure of  $\mathbb{Z}G_{ab}$ -module, coming from the action by conjugation.

The notion of Krull dimension, introduced in [25], plays an important role in the theory of rings and modules. In [29], Tushev extended this notion to groups. We recall the classical treatment of Krull dimension in non-commutative ring theory and its generalization to groups in Section 2, after which we study it for metabelian groups.

For the moment, it is enough to mention that the Krull dimension of a metabelian group  $G$ , denoted  $\text{Krull}(G)$ , is characterized by the following two properties (see Proposition 2.30):

- If the Krull dimension of the  $\mathbb{Z}G_{ab}$ -module  $[G, G]$  is positive, then they are equal:

$$\text{Krull}(G) = \text{Krull}_{\mathbb{Z}G_{ab}}([G, G]).$$

- otherwise, when the module  $[G, G]$  has Krull dimension 0 (ie is finite),  $\text{Krull}(G)$  is defined to be 1 if  $G$  is infinite, 0 if it is finite.

**THEOREM 1.1.** — *Let  $G$  be a finitely generated metabelian group of Krull dimension  $k$ . Then,*

$$k \leq 1 \iff p_{2n}^G \succsim \exp(-n^{\frac{1}{3}}).$$

Together with (1.1), this implies that the finitely generated metabelian groups with exponential growth and large return probability, that is  $p_{2n}^G \sim \exp(-n^{\frac{1}{3}})$ , are exactly those of dimension 0 or 1.

This is a consequence of studying the impact of Krull dimension on the structure of the group, which yields the more precise lower bound:

**THEOREM 1.2.** — *Let  $G$  be a metabelian group of Krull dimension  $k$ ,  $k \geq 1$ . Assume that  $[G, G]$  is torsion. Then,*

$$p_{2n}^G \succsim \exp(-n^{\frac{k}{k+2}}).$$

To obtain these estimates, we reduce to a more tractable case. A metabelian group  $G$  is *split* if it admits an exact sequence

$$M \hookrightarrow G \twoheadrightarrow Q,$$

that splits, where  $M$  and  $Q$  are abelian groups. The group  $G$  is therefore isomorphic to the semi-direct product  $M \rtimes Q$ .

The previous lower bound is actually optimal in that situation.

PROPOSITION 1.3. — *Let  $G$  be a finitely generated split metabelian group of Krull dimension  $k$ . If  $[G, G]$  is torsion, then*

$$p_{2n}^G \sim \exp(-n^{\frac{k}{k+2}}).$$

To restrict to the split case, we use the following embedding theorem. A more detailed statement can be found in Subsection 3.3.

THEOREM 1.4. — *Every finitely generated metabelian group, which is the extension of an abelian group by a finitely generated abelian group  $Q$ , can be embedded inside a finitely generated split metabelian group  $B \rtimes Q$ , of the same Krull dimension, with  $B$  abelian.*

The other direction in the proof of Theorem 1.1 is a consequence of the existence of some special subgroups. We write  $B_d^{(p)}$  for the free  $p$ -metabelian group of rank  $d$ , see part 2.3.3 for the definition.

PROPOSITION 1.5. — *Let  $G$  be a metabelian group of Krull dimension  $k$ . If  $k \geq 2$ , then  $G$  has a subgroup isomorphic to either  $\mathbb{Z} \wr \mathbb{Z}$ , or to  $B_2^{(p)}$  for some prime  $p$ .*

Saloff-Coste and Zheng ([26]) computed the return probability of the free metabelian group of rank  $d$ . We notice that, up to some adjustments, their method allows to compute that of the free  $p$ -metabelian group (see part 3.2).

PROPOSITION 1.6. — *The return probability of the free  $k$ -metabelian group of rank  $d$  is equivalent to  $\exp(-n^{\frac{d}{d+2}})$ .*

Actually, we can rewrite Theorem 1.1 in a way involving these two subgroups.

THEOREM 1.7 (Equivalent version of Theorem 1.1). — *Let  $G$  be a finitely generated metabelian group. Then,*

- *either  $G$  contains a subgroup isomorphic to  $\mathbb{Z} \wr \mathbb{Z}$  or  $B_2^{(p)}$  for some prime  $p$ ,*
- *or  $p_{2n}^G \asymp \exp(-n^{\frac{1}{3}})$ .*

Together with (1.2), we note that there is a gap in the return probabilities of finitely generated metabelian groups.

COROLLARY 1.8. — *Let  $G$  be a finitely generated metabelian group. Then, either*

$$p_{2n}^G \asymp \exp\left(-n^{\frac{1}{3}}\right),$$

or

$$p_{2n}^G \asymp \exp\left(-n^{\frac{1}{3}}(\log n)^{\frac{2}{3}}\right).$$

Finally, we provide upper bounds for the return probability in terms of the Krull dimension. We refer to part 6.1.3 for the definition of a nice section.

**THEOREM 1.9.**

- (1) *Let  $G$  be a finitely generated non-abelian metabelian torsion-free group of Krull dimension  $k \geq 2$ . Then the return probability of  $G$  satisfies*

$$p_{2n} \lesssim \exp\left(-n^{\frac{k-1}{k+1}} (\log n)^{\frac{2}{k+1}}\right).$$

- (2) *Let  $G$  be a finitely generated non-abelian metabelian group of Krull dimension  $k \geq 1$  whose derived subgroup is torsion. Assume that  $G$  admits a nice section whose torsion does not divide  $k$ . Then the return probability of  $G$  satisfies*

$$p_{2n} \lesssim \exp\left(-n^{\frac{k}{k+2}}\right).$$

### 1.1. Organization

In Section 2, we recall the definition and some useful properties of the Krull dimension of a module or a group. Then, we establish basic facts in the metabelian case. Section 3 contains preliminary results: the adaptation to the torsion case of computations by Saloff-Coste and Zheng ([26]) of the return probability of the free metabelian group, and the proof of Theorem 1.4, which is a variation of an embedding theorem of Kaloujinine and Krasner. Proposition 1.5 is proved in Section 4. In Section 5, we study sequences of Følner couples in order to apply the machinery of [8] for lower bounds on the return probability: we state in particular that such sequences go to a quotient and explain how to construct them for specific extensions. Finally, in Section 6, we highlight and study specific sections of finitely generated groups, and show in certain cases that they contain a power of a finitely generated free metabelian group. As a consequence, we give upper bounds in terms of the Krull dimension on the return probability.

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## 2. Krull dimension

Before defining the Krull dimension of a metabelian group, we recall the definition for a module. In this paper, all rings considered will be *commutative with one*.

### 2.1. Krull dimension of a module

We mention two equivalent definitions: the first one will be generalized later to define the Krull dimension of a group, while the second will be sometimes easier to handle.

#### 2.1.1. Krull dimension interpreted as the deviation of a poset

We follow [20].

Let  $A$  be a poset. If  $a \leq b$ , let  $[a, b] = \{x \in A \mid a \leq x \leq b\}$ . This is a subposet of  $A$ , called *interval*, or *factor*, from  $a$  to  $b$ . A *descending chain* is a chain  $(a_i)_i$  of elements of  $A$  such that  $a_1 \geq a_2 \geq \dots$ , and the intervals  $[a_{i+1}, a_i]$  are the *factors of the chain*. If every such descending chain is eventually constant, we say that  $A$  satisfy the *descending chain condition*. Similarly, one defines the *ascending chain condition*. A poset  $A$  is *trivial* if  $a \leq b$  implies  $a = b$ , for all  $a, b \in A$ .

DEFINITION 2.1. — *The deviation of  $A$ , denoted  $\text{dev } A$ , if it exists, is*

- $-\infty$ , if  $A$  is empty or trivial,
- $0$ , if  $A$  is non-trivial and satisfies the descending chain condition,
- and in general by induction:  $\text{dev } A$  is defined and equal to an ordinal  $n$ , provided  $\text{dev } A$  is not equal to  $m$  for every  $m < n$ , and in any descending chain of  $A$ , all but finitely many factors have deviation defined and less than  $n$ .

Note that a poset may not have a deviation. A sufficient condition is



PROPOSITION 2.2 ([20]). — *Any poset with the ascending chain condition has a deviation.*

We may now define the Krull dimension of a module as the deviation of a natural associated poset.

DEFINITION 2.3. — *Let  $M$  be a  $R$ -module. Denote by  $\mathcal{L}_R(M)$  the poset of  $R$ -submodules of  $M$ . The Krull dimension of  $M$  as a  $R$ -module is defined as*

$$\text{Krull}_R(M) = \text{dev } \mathcal{L}_R(M),$$

*whenever it exists. Otherwise,  $M$  does not have a Krull dimension. We may forget the reference to the ring whenever it is clear from the context.*

*Remarks 2.4.*

- (1) If  $A$  is a ring, the Krull dimension of  $A$  denotes the Krull dimension of the  $A$ -module  $A$ , written  $\text{Krull}(A)$ .
- (2) A module satisfying the descending chain condition is called *artinian*. Hence, modules of dimension 0 are just artinian modules.

LEMMA 2.5 ([20]). — *If  $N$  is a submodule of  $M$ , then*

$$\text{Krull}(M) = \max\{\text{Krull}(N), \text{Krull}(M/N)\}.$$

### 2.1.2. Krull dimension interpreted as the dimension of a faithful ring

For commutative Noetherian rings, there is an equivalent way to define the Krull dimension in terms of length of chains of prime ideals. One can then derive an equivalent definition for the Krull dimension of a module over a commutative Noetherian ring. These equivalent definitions will sometimes turn out to be more tractable. References for this are [9] and [20].

PROPOSITION 2.6 ([20, Theorem 4.8]). — *Let  $A$  be a commutative Noetherian ring. The Krull dimension of  $A$ , when finite, is equal to the supremum among the  $r$  such that there exists a chain*

$$\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \cdots \subsetneq \mathcal{P}_r$$

*of prime ideals in  $A$ .*

*Remark 2.7.* — For simplicity, the equivalent definition just stated is for *finite* Krull dimension. It is possible to refine it using induction so as to obtain an ordinal and get a general equivalent definition for Krull dimension (see [20, 6.4]).

Examples 2.8.

- (1) Any field has dimension 0, and any principal ideal domain that is not a field has dimension 1.
- (2) Polynomial rings over a Noetherian ring  $A$  satisfy:

$$\text{Krull}(A[X_1, \dots, X_n]) = \text{Krull}(A) + n.$$

Considering invertible variables does not change the dimension:

$$\text{Krull}(A[X_1^{\pm 1}, \dots, X_n^{\pm 1}]) = \text{Krull}(A) + n.$$

Given fields  $K \subset L$ , the *degree of transcendence* of  $L$  over  $K$  will be denoted by  $\text{trdeg}_K L$ . For any module  $M$  and any element  $c$  in  $M$ , we denote by  $M_c$  the localization of  $M$  with respect to  $\{c^n; n \in \mathbb{N}\}$ .

The second part of example 2.8 actually generalizes: there is a connexion between Krull dimension and degree of transcendence. Before stating it, we recall Noether’s normalization theorem ([21], see [3] for this generalized version).

**THEOREM 2.9** (Noether’s normalization theorem). — *Let  $A$  be an integral domain and let  $A \subset R$  be an extension of  $A$  such that  $R$  is finitely generated as an  $A$ -algebra.*

*Then, there is a non-zero element  $c$  in  $A$  and elements  $z_1, \dots, z_k$  in  $R_c$  algebraically independent over  $A_c$  such that  $R_c$  is a finitely generated  $A_c[z_1, \dots, z_k]$ -module.*

*Remark 2.10.* — The subring  $A_c[z_1, \dots, z_k]$  is isomorphic to a polynomial ring over  $A_c$ . Note that  $k$  may be zero.

Moreover, if  $A = K$  is a field, then  $K_c = K$  for every non-zero element  $c$  and the theorem says that every finitely generated  $K$ -algebra is a finitely generated module over a polynomial ring over  $K$ . The number  $k$  of indeterminates is the Krull dimension of  $R$ .

Krull dimension behaves well with respect to extension, as stated in the next proposition.

**PROPOSITION 2.11** ([9]). — *Let  $A$  be an integral domain,  $K$  its fraction field and  $\mathbb{P}$  the prime subfield of  $K$ .*

- (1) *If  $A$  is finitely generated as a  $\mathbb{P}$ -algebra, then,*

$$\text{Krull}(A) = \text{trdeg}_{\mathbb{P}} K.$$

- (2) *If  $A$  is finitely generated as a  $\mathbb{Z}$ -algebra, then*

$$\text{Krull}(A) = \text{trdeg}_{\mathbb{Q}} K + 1.$$

This point of view gives another way to compute the Krull dimension of a module, as the dimension of a faithful ring.

PROPOSITION 2.12 ([20]). — *Let  $A$  be a commutative Noetherian ring and  $M$  a finitely generated  $A$ -module. Then,*

$$(2.1) \quad \text{Krull}_A(M) = \text{Krull}(A/\text{Ann}(M)),$$

where  $\text{Ann}(M)$  denotes the annihilator of  $M$ , that is  $\text{Ann}(M) = \{a \in A \mid aM = 0\}$ .

### 2.1.3. Refinements

It is useful to distinguish the contributions to Krull dimension that are made by torsion modules and by torsion-free modules. Let  $M$  be a module and  $T(M)$  be the torsion subgroup of the group  $M$ . The group  $T(M)$  also carries a module structure, hence has finite exponent, and is a direct factor.

LEMMA 2.13 ([13]). — *Let  $M$  be a Noetherian module and denote by  $T(M)$  the torsion subgroup of  $M$ . Then,  $T(M)$  is a submodule of  $M$  and there is a torsion-free subgroup  $N$  of  $M$  such that, as groups,  $M = T(M) \oplus N$ .*

*In particular, the group  $N$  is isomorphic to  $M/T(M)$ .*

DEFINITION 2.14. — *The torsion-free Krull dimension of  $M$  is  $\text{Krull}^0(M) = \text{Krull}(M/T(M))$  and the torsion Krull dimension of  $M$  is  $\text{Krull}^t(M) = \text{Krull}(T(M))$ .*

Remark 2.15. — Proposition 2.5 implies that

$$\text{Krull}(M) = \max\{\text{Krull}^0(M), \text{Krull}^t(M)\}.$$

### 2.1.4. Krull dimension and associated prime ideals

We end this subsection collecting some lemmas about Krull dimension which will be needed hereafter.

DEFINITION 2.16. — *Let  $M$  be a  $R$ -module. A prime ideal  $\mathcal{P}$  of  $R$  is an associated prime ideal of  $M$  if it is the annihilator of some element in  $M$ . We denote by  $\text{Ass}_R(M)$  the set of associated primes of  $M$ .*

Remark 2.17. — Equivalently, a prime  $\mathcal{P}$  is an associated prime ideal of  $M$  if  $R/\mathcal{P}$  is a submodule of  $M$ . Note that all the associated prime ideals of  $M$  contain the annihilator of  $M$ .

LEMMA 2.18 ([9]). — *Let  $R$  be a Noetherian ring. Let  $M, N$  and  $Q$  be finitely generated  $R$ -modules fitting together in a short exact sequence*

$$N \hookrightarrow M \xrightarrow{\mathcal{P}} Q.$$

*Assume  $\text{Krull}(N) < d$  and that there exists an ideal  $\mathcal{P}$  such that, for every  $x \in Q - \{0\}$ ,  $\text{Ann}(x) = \mathcal{P}$ , with  $\text{Krull}(R/\mathcal{P}) = d$ .*

*Then  $\mathcal{P}$  is an associated prime ideal of  $M$ . In particular,  $M$  has a submodule isomorphic to  $R/\mathcal{P}$ .*

*Proof.* — Denote by  $I$  be the annihilator of  $N$ :  $\text{Krull}(R/I) < d$  implies  $I \not\subseteq \mathcal{P}$ . Let  $x \in I$ ,  $x \notin \mathcal{P}$ . The module  $xM$  has nontrivial image in  $Q$  and one can check that  $N \cap xM = \{0\}$ . Hence  $xM \simeq (xM) \subset Q$  and elements of  $xM$  have annihilator  $\mathcal{P}$ . □

The next structure result will be very useful in the sequel. This shows that Noetherian modules are built up from modules isomorphic to  $R/\mathcal{P}$  where  $\mathcal{P}$  is prime.

PROPOSITION 2.19 ([9]). — *Let  $R$  be a Noetherian ring,  $M$  be a finitely generated  $R$ -module. Then, there exist  $M_0, M_1, \dots, M_n$  submodules of  $M$  such that*

$$(2.2) \quad M = M_n > M_{n-1} > \dots > M_1 > M_0 = 0$$

*and  $M_{i+1}/M_i \simeq R/\mathcal{P}_i$ , where  $\mathcal{P}_i$  is a prime ideal of  $R$ .*

As a consequence, and it is probably well-known, the Krull dimension of a Noetherian module only depends on its associated prime ideals.

PROPOSITION 2.20. — *Let  $R$  be a Noetherian ring,  $M$  be a finitely generated  $R$ -module. Then, the Krull dimension of  $M$  is attained by  $R/\mathcal{P}$  for some associated prime  $\mathcal{P}$  of  $M$ .*

*Proof.* — Proposition 2.19 gives a decomposition of  $M$  as a tower of rings of the form  $R/\mathcal{P}_i$ , and Proposition 2.5 implies that

$$\text{Krull}(M) = \max_i \{\text{Krull}(R/\mathcal{P}_i)\}.$$

Look at the minimal  $i$  such that the Krull dimension of  $M_{i+1}/M_i$  is the Krull dimension of  $M$ . We can apply Lemma 2.18 to the exact sequence

$$M_i \hookrightarrow M_{i+1} \twoheadrightarrow M_{i+1}/M_i.$$

□

## 2.2. Krull dimension of a group

### 2.2.1. Definition

Tushev defined in [29] the Krull dimension of a group analogously to that of a module. For a group  $G$ , let  $\mathcal{N}(G)$  be the poset of all normal subgroups of  $G$ .

DEFINITION 2.21 ([29]). — *Let  $G$  be a group. We say that  $G$  admits a Krull dimension whenever the poset  $\mathcal{N}(G)$  admits a deviation. In this case, we set*

$$\text{Krull}(G) = \text{dev } \mathcal{N}(G).$$

Otherwise,  $G$  does not admit a Krull dimension.

*Examples 2.22.*

- (1) A finite group has Krull dimension 0.
- (2)  $\mathbb{Z}$  has Krull dimension 1. Indeed, a decreasing sequence of subgroups of  $\mathbb{Z}$  is a sequence  $(c_n\mathbb{Z})_n$ , where  $c_n$  belongs to  $\mathbb{N}$  and  $c_n \mid c_{n+1}$ . The factors are finite, except when  $c_n$  is nonzero and  $c_{n+1}$  is zero. This can only happen once.

Remark 2.23. — If the group is abelian, its Krull dimension coincide with its Krull dimension as a  $\mathbb{Z}$ -module.

Moreover, if  $K, H$  are subgroups of an abelian group, the deviation of the factor  $[H, K]$  is exactly the dimension of the quotient group  $K/H$ .

LEMMA 2.24. — *A finitely generated abelian group has Krull dimension zero if it is finite, or one if it is infinite.*

*Proof.* — Such a group  $G$  is isomorphic to  $\mathbb{Z}^d \times F$ , for some integer  $d$  and some finite group  $F$ .

Let  $G_0 < \dots < G_m$  be a series of subgroups of  $G$  in which there are  $n$  infinite factors. For each infinite factor  $G_j/G_{j-1}$  choose an element  $x_j$  in  $G_j$  which has infinite order modulo  $G_{j-1}$ . Then these  $x_j$  taken together generate a free abelian group of rank  $n$  and hence  $n \leq d$ .

One can derive an alternative proof from Remark 2.23 above. By (2.1), the Krull dimension of  $G$  is the Krull dimension of  $\mathbb{Z}/\text{Ann}(G)$ , when  $\text{Ann}(G)$  denotes the annihilator of the  $\mathbb{Z}$ -module  $G$ . Therefore  $\text{Krull}(G)$  is 1 if the group is infinite, or 0 if it is finite.  $\square$

A reformulation of Proposition 2.2 yields:

PROPOSITION 2.25. — *If  $G$  satisfies the maximal condition on normal subgroups, then  $G$  has a Krull dimension.*

This is not a necessary condition: the group  $\mathbb{Z}[\frac{1}{p}]$  does admit a Krull dimension, equal to 1.

2.2.2. Krull dimension of a  $G$ -group

To study the Krull dimension of a metabelian group and link it with the Krull dimension of certain submodules of the group, we need the following broader notion. Let  $G, H$  be two groups.  $H$  is said to be a  $G$ -group if there is an action of  $G$  on  $H$  containing the inner automorphisms of  $H$ . If  $H$  satisfies an exact sequence  $M \hookrightarrow H \twoheadrightarrow Q$ , with abelian groups  $M$  and  $Q$ , then the induced actions on  $M$  and  $Q$  endow them with a structure of  $G$ -groups.

DEFINITION 2.26. — *Let  $H$  be a  $G$ -group for some group  $G$ . Denote by  $\mathcal{N}_G(H)$  the subposet of subgroups of  $H$  that are stable under the action of  $G$ . We say that  $H$  admits a Krull dimension as a  $G$ -group whenever the poset  $\mathcal{N}_G(H)$  admits a deviation. In this case, we set*

$$\text{Krull}_G(H) = \text{dev } \mathcal{N}_G(H),$$

Otherwise,  $H$  does not admit a Krull dimension as a  $G$ -group.

Examples 2.27.

- (1) The poset  $\mathcal{N}_G(H)$  is a subposet of  $\mathcal{N}(H)$ , therefore  $\text{Krull}_G(H) \leq \text{Krull}(H)$ .
- (2) The Krull dimension of  $G$  as a  $G$ -group for the conjugation action is indeed the Krull dimension of  $G$ .
- (3) If  $K, L$  are elements of  $\mathcal{N}_G(H)$ , with  $K \subset L$ , then

$$\text{dev}[K, L] = \text{Krull}_G(L/K),$$

where  $[K, L]$  is the segment between  $K$  and  $L$  in  $\mathcal{N}_G(H)$  on the left-hand side, and  $L/K$  denotes the  $G$ -group for the induced action of  $G$  on the right-hand side.

As a consequence, if  $N$  is a subgroup of a  $G$ -group  $H$ , stable under the action of  $G$ , then  $\text{Krull}_G(N) \leq \text{Krull}_G(H)$ .

Moreover, if  $Q$  is a quotient of a  $G$ -group  $H$  by a  $G$ -stable subgroup, then the action of  $G$  on  $H$  induces a structure of  $G$ -group on  $Q$  and again  $\text{Krull}_G(H) \geq \text{Krull}_G(Q)$ .

LEMMA 2.28. — *Let*

$$M \hookrightarrow H \twoheadrightarrow Q$$

*be a sequence of  $G$ -groups. Then,*

$$\text{Krull}_G(H) = \max\{\text{Krull}_G(M), \text{Krull}_G(Q)\}.$$

*Proof.* — The fact that  $\text{Krull}_G(H) \geq \max\{\text{Krull}_G(M), \text{Krull}_G(Q)\}$  follows from the remarks.

To show the reverse inequality, we use induction on the ordinal

$$q = \max\{\text{Krull}_G(M), \text{Krull}_G(Q)\}.$$

It holds if  $q$  is  $-\infty$  or  $0$ . Assume that the reverse inequality is true for any such extension where the corresponding maximum is strictly less than  $q$ .

Let  $(M_n)_n$  be a decreasing sequence of  $\mathcal{N}_G(H)$ . We need to show that all but finitely many of the factors  $[M_{n+1}, M_n]$  have deviation less than  $q$ . Equivalently, we need to show that all but finitely many of the  $G$ -groups  $M_n/M_{n+1}$  have dimension less than  $q$ . Denote by  $p$  the projection of  $H$  into  $Q$  and set  $I_n = M_n \cap M$  and  $P_n = p(M_n)$ : these are decreasing sequences of  $\mathcal{N}_G(M)$  and  $\mathcal{N}_G(Q)$  respectively. By definition of  $q$ , all but finitely many of the factors of these sequences have deviation less than  $q$ .

To apply the induction hypothesis, consider  $S_n = M_{n+1}I_n$  in  $\mathcal{N}_G(H)$ . We have

$$S_n/M_{n+1} \hookrightarrow M_n/M_{n+1} \xrightarrow{p} M_n/S_n$$

and  $M_n/S_n \simeq P_n/P_{n+1}$  and  $S_n/M_{n+1} \simeq I_n/I_{n+1}$ , hence they have dimension less than  $q$ :

$$\text{Krull}_G(M_n/M_{n+1}) \leq \max\{\text{Krull}_G(I_n/I_{n+1}), \text{Krull}_G(P_n/P_{n+1})\} < q. \quad \square$$

**COROLLARY 2.29.** — *Let  $H, Q$  be two groups so that  $Q$  is a quotient of  $H$ . Then,*

$$\text{Krull}(H) \geq \text{Krull}(Q).$$

*Proof.* — If  $G = H$ , then  $\text{Krull}(H) \geq \text{Krull}_H(Q)$ . Moreover the poset  $\mathcal{N}_H(Q)$  is isomorphic to  $\mathcal{N}(Q)$ . Hence the corollary. □

### 2.2.3. Krull dimension of a metabelian group

The derived series  $(G^{(j)})_j$  of a group  $G$  is defined inductively by  $G^{(0)} = G$  and  $G^{(j+1)} = [G^{(j)}, G^{(j)}]$ . The group  $G$  is *solvable* if  $G^{(j)} = \{e\}$  for some  $j$ , and the smallest such  $j$  is the *derived length* of  $G$ . Recall that a metabelian group  $G$  is a solvable group with derived length 2. We will denote by  $G_{ab}$  the abelianization  $G/[G, G]$  of  $G$ . The group  $G$  fits into the exact sequence

$$[G, G] \hookrightarrow G \twoheadrightarrow G_{ab}.$$

We now consider a general extension of one abelian group by another. Let  $G$  be a metabelian group such that

$$M \hookrightarrow G \twoheadrightarrow Q,$$

where  $M$  and  $Q$  are abelian groups. The group  $G$  acts on  $M$  by conjugation, and the action of an element  $g$  in  $G$  only depends on its projection in the quotient group  $Q$ . Hence, it induces an action from  $Q$  on  $M$ , endowing  $M$  with a structure of  $\mathbb{Z}Q$ -module. If  $Q$  is finitely generated then  $\mathbb{Z}Q$  is Noetherian and if  $G$  is finitely generated then both  $\mathbb{Z}Q$  and the  $\mathbb{Z}Q$ -module  $M$  are Noetherian.

If  $G$  is a finitely generated metabelian group, note that  $Q$  is virtually  $\mathbb{Z}^d$ , for some  $d$ . Hence, the group  $G$  has a finite index metabelian subgroup  $G_1$  with torsion-free quotient  $\mathbb{Z}^d$ :

$$(2.3) \quad M \hookrightarrow G_1 \twoheadrightarrow \mathbb{Z}^d,$$

and the ring  $\mathbb{Z}\mathbb{Z}^d$  identifies with  $\mathbb{Z}[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ .

PROPOSITION 2.30. — *Let  $G$  be a metabelian group, admitting an exact sequence of the form*

$$M \hookrightarrow G \twoheadrightarrow Q,$$

where  $M$  and  $Q$  are abelian groups. The Krull dimension of  $M$  as a  $G$ -group for the conjugation action coincide with the Krull dimension of  $M$  as a  $\mathbb{Z}Q$ -module.

Then, if  $\text{Krull}(M) > 0$ , we have  $\text{Krull}(G) = \text{Krull}(M)$ . Otherwise,  $\text{Krull}(G)$  is 0 if the group is finite, 1 if not.

*Proof.* — This follows from Lemmas 2.24 and 2.28, applied to the action of  $G$  on itself by conjugation. Note that  $\text{Krull}_G(Q)$  is the Krull dimension of the group  $Q$ . □

As a consequence, the Krull dimension of a metabelian group  $G$  is the Krull dimension of  $[G, G]$  as a  $\mathbb{Z}G_{ab}$ -module, except when the former is zero and the group infinite: we retain especially

$$\text{Krull}(G) = \text{Krull}_{\mathbb{Z}G_{ab}}([G, G]), \text{ if positive.}$$

In particular, if the dimension is at least 2, we may use any exact sequence expressing  $G$  as an extension of an abelian group by another one to compute the Krull dimension of  $G$  as the Krull dimension of the module involved.

Remark 2.31. — An easy consequence of this proposition is that finitely generated metabelian groups have finite Krull dimension.

In general, a metabelian group  $G$  admits a Krull dimension if and only if it has a finite series of normal subgroups each of whose factor meets the maximal or the minimal condition for  $G$ -invariant subgroups [29]. Recall that a  $G$ -group  $H$  is said to satisfy the maximal (minimal) condition for



$G$ -invariant subgroups if every descending (ascending) chain of  $G$ -invariant subgroups eventually terminates.

Up to passing to a finite index subgroup, we may consider exact sequences with torsion-free abelian quotient as (2.3).

PROPOSITION 2.32. — *Let  $G$  be a finitely generated metabelian group. Consider an exact sequence*

$$M \hookrightarrow G \twoheadrightarrow Q,$$

with  $M$  and  $Q$  abelian.

There exists a subgroup  $G'$  of finite index in  $G$  such that

- (1)  $\text{Krull}(G) = \text{Krull}(G')$ .
- (2)  $G'$  is an extension of  $M$  by a finitely generated free abelian group.

The proof of the proposition requires the following lemmas.

LEMMA 2.33. — *Let  $A \supset B$  be two rings such that  $A$  is finitely generated as a  $B$ -module. Then,*

$$\text{Krull}(A) = \text{Krull}(B) = \text{Krull}_B(A).$$

*Proof.* — Proposition 9.2 of [9] states that  $\text{Krull}(A) = \text{Krull}(B)$ . We compare this quantity with  $\text{Krull}_B(A)$ . First,  $\text{Krull}_B(A) = \text{dev } \mathcal{L}_B(A) \geq \text{dev } \mathcal{L}_A(A) = \text{Krull}(A)$ . On the other hand,  $\text{Krull}_B(A) \leq \text{Krull}(B) = \text{Krull}(A)$ .  $\square$

LEMMA 2.34. — *Let  $Q$  be a finitely generated group and  $Q'$  be a subgroup of finite index in  $Q$ . Let  $M$  be a Noetherian  $\mathbb{Z}Q$ -module. Then*

$$\text{Krull}_{\mathbb{Z}Q}(M) = \text{Krull}_{\mathbb{Z}Q'}(M).$$

*Proof.* — Take a decomposition of  $M$  as a  $\mathbb{Z}Q$ -module, as given in (2.2):  $M = M_n \supset M_{n-1} \supset \cdots \supset M_1 \supset \{0\}$ , where  $M_{i+1}/M_i = \mathbb{Z}Q/\mathcal{P}_i$ , for some prime  $\mathcal{P}_i$ . Then,

$$\text{Krull}_{\mathbb{Z}Q}(M) = \max_i \{\text{Krull}(\mathbb{Z}Q/\mathcal{P}_i)\}.$$

Lemma 2.33 above implies that

$$\text{Krull}(\mathbb{Z}Q/\mathcal{P}_i) = \text{Krull}_{\mathbb{Z}Q'/(\mathcal{P}_i \cap Q')}(\mathbb{Z}Q/\mathcal{P}_i) = \text{Krull}_{\mathbb{Z}Q'}(\mathbb{Z}Q/\mathcal{P}_i)$$

and  $\max_i \{\text{Krull}_{\mathbb{Z}Q'}(\mathbb{Z}Q/\mathcal{P}_i)\} = \text{Krull}_{\mathbb{Z}Q'}(M)$ .  $\square$

*Proof of Proposition 2.32.* — Let  $p : G \rightarrow Q$  be the projection. The group  $Q$  is a finitely generated abelian group: we may write  $Q = Q' \times T$

where  $T$  is a finite abelian group and  $Q'$  is finitely generated free abelian. Take  $G' = p^{-1}(Q')$ , it has finite index in  $G$  and

$$M \hookrightarrow G' \twoheadrightarrow Q'.$$

Hence, we are left to show that  $\text{Krull}(G) = \text{Krull}(G')$ . By Lemma 2.34 above,  $\text{Krull}_{\mathbb{Z}Q}(M) = \text{Krull}_{\mathbb{Z}Q'}(M)$ . When this dimension is nonzero, Proposition 2.30 ensures that we are done. When it is zero,  $\text{Krull}(G)$  is 0 if the group is finite, 1 if not. As  $G'$  has finite index, we have the same dichotomy.  $\square$

LEMMA 2.35. — *Let  $G$  be a metabelian group. Assume that  $G$  satisfies an exact sequence*

$$M \hookrightarrow G \twoheadrightarrow Q$$

with  $M$  and  $Q$  abelian groups.

- If  $\text{Krull}^0(M) \geq 2$ , then

$$\text{Krull}^0(M) = \text{Krull}(G/T(G))$$

where  $T(G)$  denotes the torsion subgroup of  $G$ . As a consequence, for any other exact sequence

$$\hat{M} \hookrightarrow G \twoheadrightarrow \hat{Q}$$

satisfied by  $G$ ,  $\text{Krull}^0(\hat{M}) = \text{Krull}^0(M)$  and the latter does not depend on the choice of the exact sequence satisfied by  $G$ .

- Denote by  $M_0$  the derived subgroup of  $G$ . If  $\text{Krull}^t(M) \geq 1$ , then

$$\text{Krull}^t(M) = \text{Krull}^t(M_0)$$

and the latter does not depend on the choice of the exact sequence satisfied by  $G$ .

*Proof.* — Assume first that  $\text{Krull}^0(M) \geq 2$ . As in the proof of Proposition 2.32 above, write  $Q = Q' \times T$ , where  $T$  is a finite abelian group and  $Q'$  is finitely generated free abelian. The exact sequence

$$M \hookrightarrow G \twoheadrightarrow Q$$

provides an exact sequence satisfied by the quotient group  $G/T(G)$ , where  $T(G)$  denotes the torsion subgroup of  $G$  and  $\pi : G \rightarrow Q$  is the canonical projection:

$$M/(T(G) \cap M) \hookrightarrow G/T(G) \twoheadrightarrow Q/\pi(T(G))$$

Note that  $T(G) \cap M = T(M)$  and the left-hand side is actually  $M/T(M)$ . As  $\pi(T(G)) \subset T, Q/\pi(T(G))$  is isomorphic to  $Q' \times T'$  for some finite abelian group  $T'$ . Using Lemma 2.34, we have

$$\begin{aligned} \text{Krull}^0(M) &= \text{Krull}_{\mathbb{Z}Q}(M/T(M)) = \text{Krull}_{\mathbb{Z}Q'}(M/T(M)) \\ &= \text{Krull}_{\mathbb{Z}(Q' \times T')}(M/T(M)). \end{aligned}$$

Hence the latter is greater or equal to 2, therefore equal to  $\text{Krull}(G/T(G))$  and  $\text{Krull}^0(M) = \text{Krull}(G/T(G))$  as announced.

The proof of the consequence is similar.

Assume now that  $\text{Krull}^t(M) \geq 1$  and denote by  $Q_0$  the abelianization of  $M_0$ . As  $M_0$  is the derived subgroup of  $G$ , it is a  $\mathbb{Z}Q$ -module and  $\text{Krull}_{\mathbb{Z}Q}(M_0) = \text{Krull}_{\mathbb{Z}Q_0}(M_0)$ . One have  $M_0 \subset M$  and consequently  $M_0^t = M^t \cap M_0$ . The quotient  $M^t/M_0^t$  is isomorphic to a subgroup of the torsion subgroup of  $Q_0^t$ , hence has Krull dimension  $-\infty$  or 0 as a  $\mathbb{Z}Q$ -module. Therefore

$$\begin{aligned} \text{Krull}^t(M) &= \text{Krull}_{\mathbb{Z}Q}(M^t) = \text{Krull}_{\mathbb{Z}Q}(M_0^t) = \text{Krull}_{\mathbb{Z}Q_0}(M_0^t) \\ &= \text{Krull}^t(M_0). \end{aligned} \quad \square$$

As in the case of modules, we may define the following refinements. The previous lemma ensures that this is well-defined.

DEFINITION 2.36. — *Let  $G$  be a metabelian group. Assume that  $G$  satisfies an exact sequence*

$$M \hookrightarrow G \twoheadrightarrow Q$$

with  $M$  and  $Q$  abelian groups.

- If  $\text{Krull}^0(M) \geq 2$ , the torsion-free Krull dimension of  $G$  is defined as  $\text{Krull}^0(G) = \text{Krull}^0(M) = \text{Krull}(G/T(G))$ .  
If  $\text{Krull}^0(M) \leq 1$ , set  $\text{Krull}^0(G) = 1$ .
- If  $\text{Krull}^t(M) \geq 1$ , the torsion Krull dimension of  $G$  is defined as  $\text{Krull}^t(G) = \text{Krull}^t(M)$ .  
If  $\text{Krull}^t(M) \leq 0$ , set  $\text{Krull}^t(G) = 0$ .

As a consequence, if  $G$  satisfies an exact sequence  $M \hookrightarrow G \twoheadrightarrow Q$  such that  $\text{Krull}_{\mathbb{Z}Q}(M) \geq 2$ , we have

$$\text{Krull}(G) = \max\{\text{Krull}^0(G), \text{Krull}^t(G)\}.$$

Example 2.37. — To explain the dichotomies in the previous two definitions, consider the following example, suggested by Gouëzel. Let  $G =$

$\mathbb{Z} \oplus ((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^k)$ . The group  $G$  has Krull dimension  $k$ . It satisfies the exact sequence

$$\bigoplus_{\mathbb{Z}^k} \mathbb{Z}/2\mathbb{Z} \hookrightarrow G \twoheadrightarrow \mathbb{Z} \oplus \mathbb{Z}^k.$$

Set  $M_1 = \bigoplus_{\mathbb{Z}^k} \mathbb{Z}/2\mathbb{Z}$ . Then,  $\text{Krull}^0(M_1) = \text{Krull}(0) = -\infty$  and  $\text{Krull}^t(M_1) = \text{Krull}(M_1) = k$ .

The group  $G$  also satisfies the exact sequence

$$\mathbb{Z} \oplus \left( \bigoplus_{\mathbb{Z}^k} \mathbb{Z}/2\mathbb{Z} \right) \hookrightarrow G \twoheadrightarrow \mathbb{Z}^k.$$

Set  $M_2 = \mathbb{Z} \oplus (\bigoplus_{\mathbb{Z}^k} \mathbb{Z}/2\mathbb{Z})$ . Then,  $\text{Krull}^0(M_2) = \text{Krull}(\mathbb{Z}) = 1$  and  $\text{Krull}^t(M_2) = \text{Krull}(\bigoplus_{\mathbb{Z}^k} \mathbb{Z}/2\mathbb{Z}) = k$ .

The group  $H = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z} \wr \mathbb{Z}^k)$  provides a similar example for the torsion Krull dimension.

### 2.3. Examples

We study now three classes of examples. First, we consider small dimensional metabelian groups and study the rank of the torsion-free ones. Then, in the second and third paragraphs, we give the Krull dimension of some metabelian wreath products and of the free ( $p$ -)metabelian groups (see the definitions below). These two last classes will appear to be fundamental in the sequel.

#### 2.3.1. Small dimensional metabelian groups

Finitely generated metabelian groups of Krull dimension 0 are finite, as stated in Proposition 2.30. When the dimension is 1 and the group is torsion-free, we can also say something about the structure. Recall that a group has *finite Prüfer rank* if there exists an integer  $r$  such that any finitely generated subgroup can be generated by at most  $r$  elements. The least such  $r$  is the *Prüfer rank* of the group.

**PROPOSITION 2.38.** — *Let  $G$  be a finitely generated torsion-free metabelian group of dimension one. Then, it has finite Prüfer rank.*

This Proposition is false if we do not assume that the group is torsion-free: the lamplighter is a counter-example, as it has dimension 1 but infinite Prüfer rank. Moreover, we shall see later in Section 4 that a finitely generated metabelian group of dimension 1 whose derived subgroup is torsion has a subgroup isomorphic to a lamplighter.

*Proof.* — Let  $G$  be a finitely generated torsion-free metabelian group of dimension 1. By Proposition 2.32, up to passing to a finite index subgroup, we may assume that  $G$  fits inside an exact sequence

$$M \hookrightarrow G \twoheadrightarrow \mathbb{Z}^d,$$

for some integer  $d$ , and abelian group  $M$ . The group  $M$  is a Noetherian  $\mathbb{Z}\mathbb{Z}^d$ -module and, by Proposition 2.19, it admits an increasing sequence of submodules  $M_i$  whose factors have the form  $\mathbb{Z}\mathbb{Z}^d/\mathcal{P}$ , for  $\mathcal{P}$  a prime ideal.

We first consider the case where  $M$  is a ring. It has characteristic zero, and Proposition 2.11 implies that the transcendental degree of its fraction field over  $\mathbb{Q}$  is zero, so  $M$  is an algebraic number field. Algebraic number fields have finite Prüfer rank. Hence,  $M$  has finite Prüfer rank.

In the general case, we look at a decomposition of  $M$ : the subquotients  $M_{i+1}/M_i$  have dimension either zero or one. If the dimension is one, it is enough to show that the characteristic is not positive. By contradiction, suppose  $M_{k+1}/M_k$  has positive characteristic  $p$  and dimension 1.  $M_{k+1}/M_k \simeq \mathbb{Z}\mathbb{Z}^d/\mathcal{P}$ , with  $\mathcal{P}$  prime, hence  $p$  is prime. Then, the transcendental degree of the fraction field of  $M_{k+1}/M_k$  is 1, and Proposition 4.3 will imply that it contains  $\mathbb{F}_p[X^{\pm 1}]$ . Pulling back the transcendental element  $X$  in  $M_{k+1}$ : we still get a transcendental element, which contradicts the fact that  $M$  is torsion-free of dimension 1.

The property of being of finite Prüfer rank is stable under extension, and finitely generated abelian groups have finite Prüfer rank. Hence, as  $M$  has finite Prüfer rank,  $G$  has finite Prüfer rank as well.  $\square$

### 2.3.2. Wreath products

Let  $K$  and  $H$  be countable groups. The wreath product  $K \wr H$  with base group  $H$  is the semi-direct product  $(\bigoplus_{h \in H} K_h) \rtimes H$  where  $K_h$  are copies of  $K$  indexed by  $H$ , and  $H$  acts by translation of the indices. More precisely, one may identify elements in  $\bigoplus_{h \in H} K_h$  with finitely supported functions from  $H$  to  $K$ . Thus, for any two elements  $(f, h), (f', h')$  in the group, their multiplication is given by

$$(f, h)(f', h') = (ff'(h^{-1}\bullet), hh').$$

Similarly, the unrestricted wreath product of  $K$  by  $H$ , denoted by  $K \wr \wr H$  is the semi-direct product  $(\sum_H K_h) \rtimes H$ .

The most classical example of a wreath product is the lamplighter group  $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ . This group is generated by two elements  $(0, 1)$  and  $(\delta_0, 0)$ , where  $\delta_0$  is the function from  $\mathbb{Z}$  to  $\mathbb{Z}/2\mathbb{Z}$  taking value 1 in 0, and vanishing

elsewhere. This group can be described in the following way: imagine a bi-infinite pathway of lamps, one for each integer. Each one of those lamps can be on or off, and a lamplighter is walking along the line. An element of the group corresponds to a configuration for the lamps, with only finitely many of them lit, and a position for the lamplighter. The first generator  $(0, 1)$  changes the position of the lamplighter while the second generator  $(0, \delta_0)$  has him change the state of the lamp at his current position. A configuration is a function from  $\mathbb{Z}$  to  $\mathbb{Z}/2\mathbb{Z}$  and corresponds to an element in  $\mathbb{F}_2[X, X^{-1}]$ . This subgroup has dimension 1 as a  $\mathbb{Z}[X, X^{-1}]$ -module.

In other words, the lamplighter group  $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$  has an elementary abelian subgroup of dimension 1 such that the quotient is  $\mathbb{Z}$ . Hence, its Krull dimension is 1.

Other examples are:

- (1)  $\text{Krull}(\mathbb{Z} \wr \mathbb{Z}^d) = d + 1$ .
- (2)  $\text{Krull}(F \wr \mathbb{Z}^d) = d$ , where  $F$  is a finite group.

### 2.3.3. Free ( $p$ -)metabelian groups

Denote by  $F_d$  the free group of rank  $d$ . Let  $F'_d$  be its derived subgroup and  $F''_d$  the second term of its derived series.

The free metabelian group of rank  $d$  is

$$B_d^{(p)} := F_d / F''_d$$

and the free  $p$ -metabelian group of rank  $d$  is

$$B_d^{(p)} := F_d / F''_d (F'_d)^p.$$

$B_d^{(p)}$  is the freest metabelian group whose derived subgroup has exponent  $p$  and is the quotient of the free metabelian group of rank  $d$ ,  $B_d$  by the  $p$ -powers of its commutators.

These two groups have abelianization  $\mathbb{Z}^d$  and looking at the associated exact sequence, one may compute their dimensions:

$$\text{Krull}(B_d) = d + 1, \text{ and } \text{Krull}(B_d^{(p)}) = d.$$

## 3. Preliminaries

### 3.1. Generalities on the return probability

We review some known properties and known behaviours for  $p_{2n}$  to provide a larger picture to the reader.

## 3.1.1. Stability properties ([22])

- (1) Let  $H$  be a finitely generated subgroup of  $G$ . Then  $p_{2n}^H \lesssim p_{2n}^G$ .
- (2) Let  $Q$  be a quotient of  $G$ . Then  $p_{2n}^Q \lesssim p_{2n}^G$ .

If the index of the subgroup, respectively the kernel of the quotient, is finite, then the inequality is an equivalence.

As a consequence, when studying the return probability of a finitely generated metabelian group of a given Krull dimension, we may assume that it admits an exact sequence such as (2.3), up to passing to a finite index subgroup (see Proposition 2.32).

## 3.1.2. Known behaviours: amenability and growth

The following results were mentioned in the introduction.

- (1)  $G$  has polynomial growth of degree  $d$  iff  $p_{2n}^G \sim n^{-\frac{d}{2}}$  (see [14, 32]).
- (2)  $G$  is non-amenable iff  $p_{2n}^G \sim \exp(-n)$  ([17]).
- (3) If  $G$  has exponential growth then  $p_{2n}^G \lesssim \exp(-n^{\frac{1}{3}})$  ([14]). Moreover, if  $G$  is a discrete subgroup of a connected Lie group, there is an equivalence:  $G$  is amenable with exponential growth iff  $p_{2n}^G \sim \exp(-n^{\frac{1}{3}})$ .

These three previous behaviours are the only possible ones in this case.

In the introduction, we mentioned that outside the world of discrete subgroups of connected Lie groups, far more behaviours happen. Here are some other examples.

## 3.1.3. Known behaviours: among solvable groups

- (1) Lamplighter groups:

- $\mathbb{Z} \wr \mathbb{Z}^d : p_{2n}^{\mathbb{Z} \wr \mathbb{Z}^d} \sim \exp(-n^{\frac{d}{d+2}} (\log n)^{\frac{2}{d+2}})$ ,
- $F \wr \mathbb{Z}^d : p_{2n}^{F \wr \mathbb{Z}^d} \sim \exp(-n^{\frac{d}{d+2}})$  (see [10]).

- (2) Free solvable groups on  $d$  generators:

- Free metabelian group  $B_d : p_{2n}^{B_d} \sim \exp(-n^{\frac{d}{d+2}} (\log n)^{\frac{2}{d+2}})$ ,
- Free  $r$ -solvable group ( $r > 2$ )  $B_{d,r} : p_{2n}^{B_{d,r}} \sim \exp(-n^{\frac{\log_{[r-1]} n}{\log_{[r-2]} n} \frac{2}{d}})$  ([26]).

### 3.2. Return probability of $B_d^{(p)}$

In [26], Saloff-Coste and Zheng computed the asymptotic of the return probability of the free solvable group of rank  $d$  (and, in particular, that of the free metabelian group of rank  $d$ ), using a method based on the Magnus embedding. In this part, we explain why it is possible to use their techniques to get the return probability of the free  $p$ -metabelian group, obtained by adding  $p$ -torsion to the derived subgroup. The only thing to do is to check that the Magnus embedding behaves well in this case: this was done by Bachmuth in [2].

Recall that the Magnus embedding allows to embed groups of the form  $F_d/[N, N]$ , with  $N$  a normal subgroup of  $F_d$ , into the wreath product  $\mathbb{Z}^d \wr (F_d/N)$  ([19]).

In the case of  $B_d$ , the free metabelian group of rank  $d$ , this applies to  $N = F'_d$ . The resulting wreath product is  $\mathbb{Z}^d \wr \mathbb{Z}^d$ , that is, the semi-direct product  $\bigoplus_{\mathbb{Z}^d} \mathbb{Z}^d \rtimes \mathbb{Z}^d$ , where  $\mathbb{Z}^d$  acts by shift on  $\bigoplus_{\mathbb{Z}^d} \mathbb{Z}^d$ .

The embedding takes the following form. Let  $s_1, \dots, s_d$  denote the classical generators of  $F_d$ , and  $a_1, \dots, a_d$  their images in the abelianization  $\mathbb{Z}^d$ .

Consider matrices of the form

$$\begin{pmatrix} b & m \\ 0 & 1 \end{pmatrix}$$

with  $b$  in  $\mathbb{Z}^d$  and  $m \in M$ , the free  $\mathbb{Z}(\mathbb{Z}^d)$ -module of rank  $d$  with basis  $(e_i)_{i=1, \dots, d}$ . This is a matricial representation of  $\mathbb{Z}^d \wr \mathbb{Z}^d$ . Let  $i$  be the extension to a homomorphism of

$$s_i \mapsto \begin{pmatrix} a_i & e_i \\ 0 & 1 \end{pmatrix}.$$

Magnus proved that the kernel of this homomorphism is  $[[F_d, F_d], [F_d, F_d]]$  ([19]). Therefore,  $i$  induces an embedding from  $B_d$  to  $\mathbb{Z}^d \wr \mathbb{Z}^d$ .

Bachmuth studied this representation in the case of free  $k$ -metabelian groups, for any positive integer  $k$ . He proved that moding out by  $k$  in the derived subgroup corresponds to moding out by  $k$  in  $M$ , that is

PROPOSITION 3.1 ([2, Proposition 1]). — *The Magnus embedding induces an embedding of the free  $k$ -metabelian group of rank  $d$ ,  $B_d^{(k)}$  into  $(\mathbb{Z}/k\mathbb{Z})^d \wr \mathbb{Z}^d$ .*

A lower bound for  $p_{2n}^{B_d^{(k)}}$  follows directly from the comparison with  $(\mathbb{Z}/k\mathbb{Z})^d \wr \mathbb{Z}^d$  :

$$p_{2n}^{B_d^{(k)}} \gtrsim \exp(-n^{\frac{d}{d+2}}).$$



In [26], Saloff-Coste and Zheng used the Magnus embedding to produce upper bounds for the return probability of groups of the form  $F_d/[N, N]$ , for some normal subgroup  $N$  of  $F_d$ .

They introduced the notion of exclusive pair in such a group  $F_d/[N, N]$ , made of a subgroup  $\Gamma$  together with an element  $\rho$  of the derived subgroup. The pair is designed so that the images of  $\Gamma$  and  $\rho$  in the  $\mathbb{Z}$ -module  $M$  should have minimal interaction (we refer to [26, §4] for a precise definition). From this, they derived a comparison ([26, Theorem 4.13]): the return probability of  $F_d/[N, N]$  is smaller than the return probability of the subgroup  $\mathbb{Z} \wr \bar{\Gamma}$  in  $\mathbb{Z}^d \wr (F_d/N)$ , where  $\bar{\Gamma}$  is the image of  $\Gamma$  in  $F_d/N$ .

In the case of the free metabelian group  $B_d$ ,  $\Gamma$  can be chosen so that  $\bar{\Gamma}$  has finite index in  $\mathbb{Z}^d$  (more generally, this is possible whenever  $F_d/N$  is nilpotent).

Their techniques still apply to the free  $k$ -metabelian groups, if one consider  $\mathbb{Z}/k\mathbb{Z}$ -modules instead of  $\mathbb{Z}$ -modules and yield

$$p_{2n}^{B_d^{(k)}} \lesssim p_{2n}^{\mathbb{Z}/k\mathbb{Z} \wr \mathbb{Z}^d} \lesssim \exp(-n^{\frac{d}{d+2}}).$$

As a conclusion, we get

PROPOSITION 3.2. — *The return probability of the free  $k$ -metabelian group of rank  $d$  is equivalent to  $\exp(-n^{\frac{d}{d+2}})$ .*

### 3.3. An embedding theorem for metabelian groups that preserves the Krull dimension

In this section, we consider a finitely generated metabelian group  $G$ , extension of an abelian group  $M$  by a finitely generated abelian group  $Q$ . Recall that  $M$  also carries a structure of  $\mathbb{Z}Q$ -module.

A famous theorem of Kaloujnine and Krasner allows to embed  $G$  in a split metabelian group.

THEOREM 3.3 (see [16]). — *Any extension of a group  $A$  by a group  $B$  embeds in the non-restricted wreath product  $A \wr B$  of  $A$  with  $B$ .*

We first recall their construction and then explain how to modify it so as to preserve the Krull dimension.

#### 3.3.1. The embedding of Kaloujnine and Krasner

Fix a section  $s : Q \rightarrow G$  of the group. The embedding  $i : G \hookrightarrow M \wr Q = \sum_Q M \rtimes Q$  is given by

$$g \mapsto (f_g, \bar{g}).$$

where  $\bar{g}$  is the image of  $g$  in the quotient  $Q$  and

$$f_g : \begin{cases} Q \rightarrow M \\ q \mapsto s(gq)^{-1}gs(q). \end{cases}$$

Note that the embedding does depend on  $s$ . This embedding commutes with the projection on  $Q$ , and its restriction to  $M$  is  $\mathbb{Z}Q$ -equivariant. Let  $S$  be a finite generating set for  $G$  and  $\pi : M \wr Q \rightarrow \sum_Q M$  denote the canonical projection. Set  $T = \pi(i(S))$ , it is a finite subset of the base of the wreath product  $M \wr Q$ . The subgroup  $\hat{G}$ , generated by  $T$  and  $Q$  in  $M \wr Q$ , contains  $i(G)$  and is the semi-direct product  $B \rtimes Q$ , where  $B$  is the submodule of  $\sum_Q M$  generated by  $T$ .

Therefore,  $G$  embeds in the finitely generated split metabelian group  $\hat{G}$ .

### 3.3.2. Respecting the Krull dimension

We now explain how to modify this embedding so that the Krull dimension of the target group equals  $\text{Krull}(G)$ .

Let

$$\mathcal{S} = \{C \subset B \text{ submodule} \mid C \cap i(G) = \{0\}\}.$$

Zorn’s lemma provides us a maximal element  $C_0$  of  $\mathcal{S}$ . Set  $B_0 = B/C_0$ . By construction,  $G$  embeds in  $B_0 \rtimes Q$ . We still denote by  $i$  the composition of  $i$  with the projection onto  $B_0 \rtimes Q$ . Note that every submodule of  $B_0$  now intersects  $i(G)$ .

*Claim.* — The associated prime ideals of  $B_0$  and  $M$  are the same.

Indeed,  $\text{Ass}(M) \subset \text{Ass}(B_0)$ , because  $M \subset B$ . On the other hand, if  $\mathcal{P}$  is an associated prime of  $B_0$ , then  $\mathbb{Z}Q/\mathcal{P}$  is a submodule of  $B_0$ , hence does intersect  $i(G)$ : there exists  $g$  in  $G$  such that  $i(g) \in \mathbb{Z}Q/\mathcal{P}$ . More precisely,  $g$  projects trivially onto  $Q$  and thus belongs to  $M$ . It generates a submodule isomorphic to  $\mathbb{Z}Q/\mathcal{P}$  in  $M$ .

Lemma 2.20 then implies that  $M$  and  $B_0$  have the same Krull dimension. Now (see Proposition 2.30), if this dimension is positive, this is also the dimension of  $G$  and  $B_0 \rtimes Q$ , and we are done. Otherwise,  $M$  and  $B_0$  have dimension 0. If  $G$  is infinite, so is  $B_0 \rtimes Q$ , and they have dimension 1. The last case arises when  $\text{Krull}(G) = \text{Krull}(M) = 0$ , that is  $G$  is finite:  $B_0 \rtimes Q$  is finite as well, hence has dimension 0.

We just proved

THEOREM 3.4. — *Let  $G$  be a metabelian group, given as an extension*

$$M \hookrightarrow G \twoheadrightarrow Q,$$

*with  $M$  and  $Q$  abelian.*

*Then, there exists an embedding of  $G$  inside a split metabelian group  $B \rtimes Q$ , commuting with the projection on  $Q$ , such that  $M$  and  $B$  have the same associated prime ideals. In particular,  $\text{Krull}(G) = \text{Krull}(B \rtimes Q)$ .*

*Moreover, if  $G$  is finitely generated,  $B \rtimes Q$  can be chosen to be finitely generated as well.*

#### 4. Special subgroups of metabelian groups

The purpose of this section is to prove

PROPOSITION 4.1. — *Let  $G$  be a metabelian group. Assume  $G$  has Krull dimension at least 2. Then,  $G$  has a subgroup isomorphic to either  $\mathbb{Z} \wr \mathbb{Z}$ , or to  $B_2^{(p)}$  for some prime  $p$ .*

*The first option happens whenever  $\text{Krull}^0(G) \geq 2$ , and the consequence for the return probability of  $G$  is*

$$p_{2n}^G \lesssim \exp\left(-n^{\frac{1}{3}}(\log n)^{\frac{2}{3}}\right).$$

*The second option happens whenever  $\text{Krull}^t(G) \geq 2$ , and yields*

$$p_{2n}^G \lesssim \exp\left(-n^{\frac{1}{2}}\right).$$

##### 4.1. Looking for transcendental elements

We first show that a group of dimension  $k$  contains a maximal polynomial ring of the same dimension.

PROPOSITION 4.2. — *Let  $G$  be a finitely generated metabelian group, satisfying*

$$(4.1) \quad M \hookrightarrow G \twoheadrightarrow Q,$$

*with  $M$  abelian and  $Q$  free abelian. Assume that  $G$  has Krull dimension  $k \geq 2$ .*

*Then,  $M$  contains a ring isomorphic to  $\mathbb{Z}[X_1^{\pm 1}, \dots, X_{k-1}^{\pm 1}]$  or  $\mathbb{F}_p[X_1^{\pm 1}, \dots, X_k^{\pm 1}]$ , for some prime  $p$ . The first option happens whenever  $\text{Krull}^0(G) = k$ , the second whenever  $\text{Krull}^t(G) = k$ .*

To prove Proposition 4.2, it will be enough to deal with the case of  $M$  being a ring, thanks to Propositions 2.19 and 2.20 on the structure of  $M$ .

4.1.1. The case of a ring

In this paragraph, let  $Q$  be a finitely generated free abelian group and  $A$  be a ring isomorphic to  $\mathbb{Z}Q/\mathcal{P}$ , for some prime ideal  $\mathcal{P}$ . The group  $Q$  acts on  $A$  by multiplication by the corresponding monomial. The characteristic of  $A$  is either a prime  $p$ , or zero. Denote by  $K$  be the fraction field of  $A$ , and by  $\mathbb{P}$  its prime field.

PROPOSITION 4.3. — *Let  $D$  be the degree of transcendence of  $K$  over  $\mathbb{P}$ . Then  $A$  contains a family of  $D$  transcendental monomials which is algebraically free. The subring generated by this family is, according to the characteristic, either of the form*

$$\mathbb{Z}[X_1^{\pm 1}, \dots, X_D^{\pm 1}] \text{ or } \mathbb{Z}/p\mathbb{Z}[X_1^{\pm 1}, \dots, X_D^{\pm 1}],$$

for some prime  $p$ .

As a consequence, if  $A$  has Krull dimension  $d$ , we have the following dichotomy:

- when the ring  $A$  has characteristic  $p$ ,  $A$  has a subring isomorphic to  $\mathbb{Z}/p\mathbb{Z}[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ .
- when  $A$  has characteristic zero,  $A$  has a subring isomorphic to  $\mathbb{Z}[X_1^{\pm 1}, \dots, X_{d-1}^{\pm 1}]$ .

LEMMA 4.4. — *Let  $C$  be a subfield of  $K$ . If the degree of transcendence of  $K$  over  $C$  is positive then there is a transcendental monomial in  $A$ .*

*Proof.* — By hypothesis, there exists in  $K$  an element which is transcendental over  $C$ . We can write it  $\frac{t_1}{t_2}$ , with  $t_1, t_2 \in A$ . The set of all elements of  $K$  that are algebraic over  $C$  is a subfield of  $K$ , therefore one of the  $t_i$ 's has to be transcendental. This transcendental element belongs to  $A$  and is a polynomial modulo  $\mathcal{P}$ , hence a linear combination of monomials. Again, one of its monomials has to be transcendental. □

We may now proceed to the proof of the proposition.

*Proof of Proposition 4.3.* — The proof is by induction on  $D$ .

Applying the previous lemma with  $C = \mathbb{P}$  yields a monomial  $m_1 \in A$ , transcendental over  $\mathbb{P}$ . Let  $k \leq D$ . Suppose we have constructed an algebraically free family  $(m_1, \dots, m_{k-1})$  of transcendental monomials of  $A$  over  $\mathbb{P}$ . Then, the transcendence degree of  $K$  over  $\mathbb{P}(m_1, \dots, m_{k-1})$  is  $D - k + 1 > 0$ . We apply the lemma again with  $C = \mathbb{P}(m_1, \dots, m_{k-1})$  to get  $m_k$ . □

## 4.1.2. Proof of Proposition 4.2

*Proof.* — The group  $G$  fits into an exact sequence such as (4.1) with quotient  $Q \simeq \mathbb{Z}^d$ , for some  $d$ . As a consequence of Lemma 2.13, one can write the module  $M$  as  $M = T(M) \oplus M_0$ , where  $T(M)$  is the torsion subgroup of  $M$  and  $M_0$  is torsion-free. We have

$$\text{Krull}^0(G) = \text{Krull}(M/T(M)) \text{ and } \text{Krull}^t(G) = \text{Krull}(T(M)).$$

Proposition 2.20 implies that  $T(M)$ , respectively  $M/T(M)$ , contains a ring isomorphic to  $\mathbb{Z}\mathbb{Z}^d/\mathcal{P}$  of Krull dimension  $\text{Krull}(T(M))$ , respectively  $\text{Krull}(M/T(M))$ .

Conclusion then follows from Propositions 2.11 and 4.3, up to pulling back the polynomial ring in the second case.  $\square$

*Remark 4.5.* — According to its proof, this proposition rewrites as follows:

- If  $\text{Krull}^0(G) = k$ , there exists  $Q_0 \subset Q$ , with  $Q_0 \simeq \mathbb{Z}^{k-1}$ , and  $m \in A$  such that  $\mathbb{Z}Q_0.m \simeq \mathbb{Z}Q_0$  as  $\mathbb{Z}Q_0$ -modules.
- If  $\text{Krull}^t(G) = k$ , there exists  $Q_0 \subset Q$ , with  $Q_0 \simeq \mathbb{Z}^k$ , and  $m \in A$  such that  $\mathbb{Z}Q_0.m \simeq (\mathbb{Z}/p\mathbb{Z})Q_0$  as  $(\mathbb{Z}/p\mathbb{Z})Q_0$ -modules, for some prime divisor  $p$  of  $n$ .

## 4.2. Wreath products inside metabelian groups

We use the results of the previous subsection to exhibit wreath products as subgroups of some (split) metabelian groups.

## 4.2.1. In split metabelian groups

PROPOSITION 4.6. — *Let  $G$  be a split finitely generated metabelian group, that is  $G = M \rtimes Q$ , with  $M$  and  $Q$  abelian. Let  $k$  be the Krull dimension of  $G$  and assume  $k \geq 1$ . Then,*

- (1) *if  $\text{Krull}^0(G) = k$ , then  $G$  has a subgroup isomorphic to  $\mathbb{Z} \wr \mathbb{Z}^{k-1}$ . As a consequence,*

$$p_{2n}^G \lesssim p_{2n}^{\mathbb{Z} \wr \mathbb{Z}^{k-1}} \sim \exp\left(-n^{\frac{k-1}{k+1}} (\log n)^{\frac{2}{k+1}}\right).$$

- (2) *otherwise, there exists a prime  $p$  such that  $G$  has a subgroup isomorphic to  $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}^k$ . As a consequence,*

$$p_{2n}^G \lesssim p_{2n}^{(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}^k} \sim \exp\left(-n^{\frac{k}{k+2}}\right).$$

*Proof.* — Up to passing to a finite index subgroup, we may assume that  $Q$  is a finitely generated free abelian group (Proposition 2.32). This does not change the dimension nor the return probability. In both cases, Proposition 4.2 implies that  $M$  contains a ring isomorphic to either  $\mathbb{Z}[X_1^{\pm 1}, \dots, X_{k-1}^{\pm 1}]$ , or to  $\mathbb{F}_p[X_1^{\pm 1}, \dots, X_k^{\pm 1}]$  for some prime  $p$ . We denote this ring by  $B$ .

Recall that these algebraically free transcendental elements are actually monomials of  $Q \simeq \mathbb{Z}^d$ . The subgroup of  $G$  generated by  $B$  and these monomials is either  $\mathbb{Z} \wr \mathbb{Z}^{k-1}$  or  $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}^k$ . □

### 4.2.2. In general metabelian groups

In the general case, we may not be able to lift commutatively the transcendental elements obtained as we do in Proposition 4.6.

For instance, in the free metabelian group  $B_2$ , the images of the two generators of  $\mathbb{Z}^2$  by any section  $s$  from  $\mathbb{Z}^2$  to  $B_2$  will never commute.

We still have the following corollary.

**COROLLARY 4.7.** — *Let  $G$  be a metabelian group. If  $\text{Kru}ll^0(G) \geq 2$ , then  $G$  has a subgroup isomorphic to  $\mathbb{Z} \wr \mathbb{Z}$  and*

$$p_{2n}^G \lesssim p_{2n}^{\mathbb{Z} \wr \mathbb{Z}} \sim \exp\left(-n^{\frac{1}{3}}(\log n)^{\frac{2}{3}}\right).$$

This proves the first part of Proposition 4.1. The next part deals with the case  $\text{Kru}ll^t(G) \geq 2$ .

### 4.3. Extensions of torsion modules by $\mathbb{Z}^2$

Let  $p$  be a prime. The  $p$ -metabelian free group of rank 2,  $B_2^{(p)}$  fits into an exact sequence

$$(4.2) \quad [B_2^{(p)}, B_2^{(p)}] \hookrightarrow B_2^{(p)} \twoheadrightarrow \mathbb{Z}^2.$$

Denote by  $\alpha$  and  $\beta$  its two generators. As a group,  $[B_2^{(p)}, B_2^{(p)}]$  is generated by conjugates of  $[\alpha, \beta]$ . Hence, it is generated by  $[\alpha, \beta]$  as a  $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$ -module and  $[B_2^{(p)}, B_2^{(p)}]$  is a cyclic module. As it is also a torsion-free module, it is isomorphic to  $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$ .

**LEMMA 4.8.** — *Let  $H$  be an extension of  $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$  by  $\mathbb{Z}^2$ , with the usual action of  $\mathbb{Z}^2$  on  $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$ . Then  $H$  is metabelian and has a subgroup isomorphic to  $B_2^{(p)}$ .*

*Proof.* —  $H$  is metabelian because the action is not trivial.

Take  $a, b$  in  $H$  so that they do not commute and so that their projections in  $\mathbb{Z}^2$  generate  $\mathbb{Z}^2$ . We claim that  $\langle a, b \rangle$  is isomorphic to  $B_2^{(p)}$ . Indeed,  $\langle a, b \rangle$  is a metabelian group generated by two elements. Its derived subgroup is the ideal  $([a, b])$  in  $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$ , which is isomorphic to  $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$ . It satisfies

$$(4.3) \quad \mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}] \hookrightarrow \langle a, b \rangle \twoheadrightarrow \mathbb{Z}^2.$$

Moreover, it is a quotient of the free  $p$ -metabelian group of rank 2 : there exists

$$p : B_2^{(p)} \twoheadrightarrow \langle a, b \rangle.$$

This  $p$  is a morphism of extensions (4.2) to (4.3), hence an isomorphism.  $\square$

*Remark 4.9.* — If the extension splits, the group  $H$  has a subgroup isomorphic to the bigger group  $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}^2$ .

We may now complete the proof of Proposition 4.1.

*Proof of Proposition 4.1.* — Let  $G$  be a finitely generated metabelian group. Again, we can reduce by Proposition 2.32 to the case of  $G$  fitting in an exact sequence such as (4.1) with torsion-free quotient  $\mathbb{Z}^d$ , for some  $d$ . Assume that  $\text{Krull}(G) \geq 2$ . There are two cases:

- (1)  $\text{Krull}^0(G) \geq 2$ . Then, Corollary 4.7 implies that  $G$  has a subgroup isomorphic to  $\mathbb{Z} \wr \mathbb{Z}$ .
- (2)  $\text{Krull}^t(G) \geq 2$ . Let

$$M \hookrightarrow G \twoheadrightarrow Q$$

with  $M$  abelian, and  $Q$  torsion-free abelian of finite rank. Then,  $M$  has a submodule isomorphic to  $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$ , for some prime  $p$  (see Proposition 4.2). The elements  $X$  and  $Y$  come from monomials of  $Q \simeq \mathbb{Z}^d$ . The subgroup generated by the ring  $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$  and lifts of the two corresponding monomials is an extension of  $\mathbb{F}_p[X^{\pm 1}, Y^{\pm 1}]$  by  $\mathbb{Z}^2$ , with the usual action of  $\mathbb{Z}^2$ . We conclude with an application of Lemma 4.8.

Consequences in term of return probability follow from computations for  $\mathbb{Z} \wr \mathbb{Z}$  (see [10]) and  $B_2^{(p)}$  (see Proposition 3.2).  $\square$

### 5. Return probability lower bounds via the construction of sequences of Følner couples

#### 5.1. Følner couples and return probability

DEFINITION 5.1 (see [8] and [10]). — Let  $G$  be a finitely generated metabelian group. We denote by  $S$  a finite and symmetric generating set of  $G$ , and consider the associated word distance. Let  $\mathcal{V}$  be a positive continuous increasing function on  $[1, +\infty)$  whose inverse is defined on  $[\mathcal{V}(1), +\infty)$ . We say that  $G$  admits a sequence of Følner couples adapted to  $\mathcal{V}$  if there exists a sequence  $(\Omega_m, \Omega'_m)_{m \in \mathbb{N}}$  of pairs of non-empty finite sets  $\Omega'_m \subset \Omega_m$  in  $G$ , with  $\#\Omega_m \nearrow \infty$  such that

- (1)  $\#\Omega'_m$  is a positive proportion of  $\#\Omega_m : \#\Omega'_m \geq c_0 \#\Omega_m$ .
- (2)  $\Omega'_m$  lies linearly inside  $\#\Omega_m : \Omega'_m S^m \subset \Omega_m$ .
- (3)  $\mathcal{V}$  controls the size:  $\#\Omega_m \leq \mathcal{V}(m)$ .

When there exists a positive constant  $C$  such that  $\mathcal{V}(t) = Ce^{Ct}$ , we should say that  $G$  admits a sequence of Følner couples of exponential size.

Remark 5.2. — Note that if  $(\Omega_m, \Omega'_m)_m$  is a sequence of Følner couples adapted to  $\mathcal{V}$ , so is the sequence  $(\Omega'_m S^m, \Omega'_m)_m$ . It is in some sense the smallest one and we will refer to it as the *smaller version* of  $(\Omega_m, \Omega'_m)_m$ .

Given this, one can deduce a lower bound for the return probability, depending on  $\mathcal{V}$  (see Coulhon, Grigor'yan and Pittet [8], as well as Erschler [10] for more general statements). We only recall the corollary that we need.

COROLLARY 5.3 ([8]). — If a group  $G$  admits a sequence of Følner couples adapted to a function of the form  $\mathcal{V}(t) = C \exp(Ct^d)$ , then

$$p_{2n}^G \gtrsim \exp(-n^{\frac{d}{d+2}}).$$

#### 5.2. Følner couples for split metabelian groups

In this part, we prove Theorem 1.2. For any finitely generated metabelian group, Proposition 2.32 provides us with a finite index subgroup  $G$ , of the same Krull dimension, admitting an exact sequence

$$(5.1) \quad M \hookrightarrow G \twoheadrightarrow \mathbb{Z}^d,$$

with  $M$  abelian. The return probability of  $G$  is equivalent to that of the initial group. Moreover, as the return probability increases when going to



a subgroup, Theorem 3.4 imply that, when looking for lower bounds on the return probability for metabelian groups, we can reduce to the split case.

Therefore, it will be enough to prove Theorem 1.2 in the split case, namely for a finitely generated metabelian group  $G$  of the form  $G = M \rtimes \mathbb{Z}^d$ , with  $M$  abelian. Our goal is then to produce sequences of Følner couples for such a group  $G$  so as to apply Corollary 5.3. These sequences will take the following form.

DEFINITION 5.4. — *Let  $G = A \rtimes B$  be a finitely generated semi-direct product. Equip  $G$  with a generating set  $S = S_A \sqcup S_B$  where  $S_A \subset A$  generates  $A$  as a  $\mathbb{Z}B$ -module and  $S_B \subset B$  generates  $B$ . We should say that  $G$  admits a sequence of split Følner couples if it has a sequence of Følner couples of the form  $(\Omega_m \times F_m, \Omega'_m \times F'_m)_m$ , with  $\Omega_m, \Omega'_m \subset A$  and  $F_m, F'_m \subset B$ .*

Remark 5.5. — In the previous definition, the sequence  $(F_m, F'_m)_m$  is a sequence of Følner couples for  $B$ . Moreover,  $\#\Omega'_m$  is a positive proportion of  $\#\Omega_m$ .

5.2.1. When  $M$  is a ring of positive prime characteristic

We deal first with the case of  $M$  being a ring  $\mathbb{Z}\mathbb{Z}^d/\mathcal{P}$  of positive characteristic, with  $\mathcal{P}$  a prime ideal. Hence, in this part,  $M$  is  $\mathbb{F}_p[X_1^{\pm 1}, \dots, X_d^{\pm 1}]/\mathcal{P}$ , for some prime  $p$  and some prime ideal  $\mathcal{P}$ . Denote by  $\pi$  the canonical projection

$$\pi : \mathbb{F}_p[X_1^{\pm 1}, \dots, X_d^{\pm 1}] \rightarrow \mathbb{F}_p[X_1^{\pm 1}, \dots, X_d^{\pm 1}]/\mathcal{P}.$$

and by  $B_m$  the polynomials of degree bounded by  $m$ , that is

$$(5.2) \quad B_m = \{P \in \mathbb{F}_p[X_1^{\pm 1}, \dots, X_d^{\pm 1}] \mid \text{Supp}(P) \subset \llbracket -m, m \rrbracket^d\}.$$

Note that  $B_m$ , as well as  $\pi(B_m)$  are abelian groups.

The following proposition is straightforward.

PROPOSITION 5.6. — *The sequence*

$$(\Omega_m, \Omega'_m)_m = (\pi(B_{2m}) \times \llbracket -2m, 2m \rrbracket^d, \pi(B_{2m}) \times \llbracket -m, m \rrbracket^d)_m$$

is a sequence of split Følner couples adapted to

$$\mathcal{V}(m) = \#B_{2m} \leq C \exp(Cm^d).$$

Remarks 5.7. — Obviously, this  $\mathcal{V}$  is not optimal, although it gives, by Corollary 5.3, a lower bound on the return probability depending only on the rank of  $G_{ab}$ :

$$p_{2n}^G \gtrsim \exp(-n^{\frac{d}{d+2}}).$$

The following lemma aims to improve it, making use of Noether’s normalization theorem 2.9 to control the size.

LEMMA 5.8. — *There exists a constant  $C$  such that*

$$(5.3) \quad \#\pi(B_m) \leq C \exp(Cm^k),$$

where  $k$  is the Krull dimension of  $\mathbb{F}_p[X_1^{\pm 1}, \dots, X_d^{\pm 1}]/\mathcal{P}$ .

*Proof.* — By Noether’s normalization theorem 2.9, there exists elements  $z_1, \dots, z_k$  in  $\mathbb{F}_p[X_1^{\pm 1}, \dots, X_d^{\pm 1}]/\mathcal{P}$  giving it the structure of a finitely generated module over its subring  $\mathbb{F}_p[z_1, \dots, z_k]$ . Hence, each element  $X_i^{\pm 1}$  is integral over  $\mathbb{F}_p[z_1, \dots, z_k]$  and is a root of a monic polynomial with coefficients in  $\mathbb{F}_p[z_1, \dots, z_k]$ . We may assume that these polynomials have the same degree  $D$ . Let  $N$  be such that the support of all coefficients appearing in these polynomials lies in  $\llbracket -N, N \rrbracket^k$ .

Then, for  $X$  being one of the  $X_i^{\pm 1}$ , we may write each power  $X^{D+r}$  as a polynomial in  $X, \dots, X^{D-1}$ , whose coefficients have support inside  $\llbracket -rN, rN \rrbracket^k$ .

Therefore, we can write any element of  $\pi(B_m)$  as a linear combination of monomials  $X_1^{n_1}, \dots, X_d^{n_d}$  with  $(n_1, \dots, n_d) \in [-D, D]^d$  and coefficients which are elements of  $\mathbb{F}_p[z_1, \dots, z_k]$ , with support in  $\llbracket -Cm, Cm \rrbracket^k$ , for some constant  $C$ . □

### 5.2.2. Følner sequences for extensions

In this part, we recall and prove Proposition 1.3 stated in the introduction.

PROPOSITION 5.9. — *Let  $G$  be a finitely generated split metabelian group, whose derived subgroup is torsion. Assume  $G$  has Krull dimension  $k \geq 1$ . Then*

$$p_{2n} \sim \exp(-n^{\frac{k}{k+2}}).$$

*Proof.* — As mentioned at the beginning of this subsection, we may assume that  $G$  fits inside an exact sequence such as (5.1), with torsion-free quotient.

The upper bound comes from Proposition 4.6. We are left with the lower bound: it is enough (see Corollary 5.3) to show that  $G$  admits a sequence of Følner couples adapted to  $\mathcal{V}(m) = C \exp(Cm^k)$ , for some  $C$ .

The proof of this fact uses recurrence along the decomposition (2.2) of the subgroup  $M$  as an increasing sequence of submodules. Initialization is given by Proposition 5.6 and Lemma 5.8, and iteration follows from the next lemma. □

LEMMA 5.10. — *Let  $G = M \rtimes \mathbb{Z}^d$  be a split metabelian group, with  $M$  a torsion  $\mathbb{Z}\mathbb{Z}^d$ -module of Krull dimension at most  $k$ .*

*Assume that*

- (1) *there exists a submodule  $M_1$  in  $M$  such that  $M_1 \rtimes \mathbb{Z}^d$  admits a split sequence of Følner couples  $(\Omega_m \times \llbracket -2m, 2m \rrbracket^d, \Omega'_m \times \llbracket -m, m \rrbracket^d)_m$  adapted to  $\mathcal{V}(m) = C \exp(Cm^k)$ , where  $\Omega_m$  and  $\Omega'_m$  are abelian subgroups of  $M_1$ ,*
- (2)  *$M/M_1$  is a ring of the form  $\mathbb{Z}(\mathbb{Z}^d)/\mathcal{P}$ , with  $\mathcal{P}$  a prime ideal.*

*Then, the group  $G$  admits a split sequence of Følner couples adapted to a function  $\mathcal{V}'(m) = C' \exp(C'm^k)$ , of the form  $(\Gamma_m \times \llbracket -2m, 2m \rrbracket^d, \Gamma'_m \times \llbracket -m, m \rrbracket^d)_m$ . Moreover, the projections  $\Gamma_m, \Gamma'_m$  on  $M$  are abelian subgroups.*

*Proof.* — We have  $M/M_1 = \mathbb{Z}\mathbb{Z}^d/\mathcal{P}$  for some prime ideal  $\mathcal{P}$ , and this module has dimension at most  $k$ . Its characteristic is a prime  $p$ . Let  $1_Q$  be a lift of the unit of  $M/M_1$  in  $M$ . The  $\mathbb{Z}\mathbb{Z}^d$ -module  $M_2$  generated by  $1_Q$  surjects onto  $M/M_1$ , and is of the form  $\mathbb{Z}\mathbb{Z}^d/I$ , for some ideal  $I \subset \mathcal{P}$ . Because  $M$  is a torsion module,  $M_2$  is actually a ring of characteristic  $n$ , for some multiple  $n$  of  $p$ .

We cannot apply directly Noether’s normalization theorem to  $M_2$ . To get around this, note that the projection  $M_2 \rightarrow M/M_1$  have finite fibres of cardinality  $\frac{n}{p}$ . Hence, we can do as in part 5.2.1: consider the finite subgroup  $B_m$  in  $\mathbb{Z}\mathbb{Z}^d$  (defined in (5.2)). Noether’s normalization theorem provides us with an upper bound on the cardinality of its projection  $\pi(B_m)$  into  $M/M_1$ , where  $\pi : \mathbb{Z}\mathbb{Z}^d \rightarrow M/M_1$ . Let  $\Lambda_m$  be the pullback of  $\pi(B_{2m})$  in  $M_2$ . It is also the projection of  $B_{2m}$  in  $M_2 = \mathbb{Z}\mathbb{Z}^d/I$ . Because the fibres are finite, the cardinality of  $\Lambda_m$  satisfies (5.3), for some constant  $C_1$ :

$$\#\Lambda_m \leq C_1 \exp(C_1 m^k) = \mathcal{V}_1(m).$$

Note that  $\Lambda_m$  is an abelian subgroup of  $M$ .

We fix a generating set  $S$  for  $G$  as follows:

$$S = S_1 \cup \{(\pm 1_Q, 0)\},$$

such that  $S_1 = S'_1 \cup \{(0, \pm e_1), \dots, (0, \pm e_d)\}$  with  $S'_1$  a finite and symmetric generating set for the finitely generated  $\mathbb{Z}(\mathbb{Z}^d)$ -module  $M_1$  and  $(e_i)_{i=1}^d$  being the canonical basis of  $\mathbb{Z}^d$ . We choose  $S_1$  so as to be compatible with the sequence of Følner couples of  $M_1 \rtimes \mathbb{Z}^d$ .

Consider the following sequence of couples indexed by  $m \in \mathbb{N}$ :

$$F_m = \{P.1_Q + \omega \mid P \in \Lambda_m, \omega \in \Omega_m\} \times \llbracket -2m, 2m \rrbracket^d$$

$$F'_m = \{P.1_Q + \omega \mid P \in \Lambda_m, \omega \in \Omega'_m\} \times \llbracket -m, m \rrbracket^d.$$

We claim that this is a sequence of Følner couples for the group  $G$ , adapted to  $\mathcal{V}'(m) = C' \exp(C'm^k)$ , for some constant  $C'$ . Note that the projections on  $M$  are again abelian subgroups. Indeed,  $\#F_m \leq \mathcal{V}_1(m)\#\Omega_m(4m+1)^d \leq C' \exp(C'm^k)$ . The choice of a sequence of Følner couples for  $\Omega_m$  and  $\Omega'_m$  implies  $\#\Omega'_m \geq c\#\Omega_m$ , and that  $F'_m S^m \subset F_m$ .

The last condition, namely  $\frac{\#F'_m}{\#F_m} \geq c$ , follows from the following straightforward fact, applied to  $H = M, \Omega = \Omega_m, \Omega' = \Omega_m$  and  $X = \Lambda_m$ .  $\square$

*Fact 5.11.* — Let  $H$  be a group and  $\Omega' \subset \Omega$  be two subgroups of  $H$ . Assume that  $X$  is a subset of  $H$  so that  $\Omega X$  and  $\Omega' X$  are subgroups. Then

$$[\Omega X : \Omega' X] \leq [\Omega : \Omega'].$$

We may now complete the proof of Theorem 1.2.

*Proof of Theorem 1.2.* — As the return probability increases when taking subgroups, the theorem follows from Corollary 5.9 and Theorem 3.4.  $\square$

### 5.3. Proof of Theorem 1.1

The proof of Theorem 1.1 requires to manage with the torsion submodule and torsion-free quotient of  $M$  so as to deal with both cases. Therefore, Lemma 5.10, that provides Følner couples for extension in the torsion case, will not suffice.

We first prove a proposition stating that admitting Følner couples descends to the quotient in finitely generated groups. The fact that it goes to a subgroup is due to Erschler [10].

**PROPOSITION 5.12.** — *Let  $G$  be a finitely generated group and  $Q$  be a quotient of  $G$ .*

*Assume that  $G$  admits a sequence of Følner couples adapted to a function  $\mathcal{V}$ . Then, so does  $Q$ .*

*Proof.* — Equip  $G$  with a finite and symmetric generating set  $S$  and  $H$  with the projection  $T$  of  $S$  to the quotient.

Denote by  $(\Omega_n, \Omega'_n)_n$  the given sequence of Følner couples. If  $h$  is a function from a discrete group  $K$  to  $\mathbb{R}$ ,  $\|h\|_1 = \sum_{k \in K} |h(k)|$  designates its  $L^1$ -norm.

Set  $f_n = 1_{\widehat{\Omega}'_n}$  and  $|\nabla_n f_n|(g) = \sup_{x \in S^n} |f_n(g) - f_n(gx)|$ . The latter gradient measures the variations of  $f_n$  along  $S^n$ . We have, for some constant  $c$ :

$$\|\nabla_n f_n\|_1 \leq c \|f_n\|_1.$$

Let  $\widehat{f}_n$  be the function from  $Q$  to  $\mathbb{R}$  defined by

$$\widehat{f}_n(q) = \sum_{g \in \pi^{-1}(q)} f(g).$$

The norm is preserved:  $\|\widehat{f}_n\|_1 = \|f_n\|_1 = \#\Omega'_n$ . Similarly, consider the gradient  $|\widehat{\nabla}_n \widehat{f}_n|(q) = \sup_{x \in T^n} |\widehat{f}_n(q) - \widehat{f}_n(qx)|$ . Then, we have:  $\|\widehat{\nabla}_n \widehat{f}_n\|_1 \leq \|\nabla_n f_n\|_1 \leq c \|\widehat{f}_n\|_1$ .

For a subset  $A$  of  $G$  we denote by  $\partial_n A$  the  $n$ -boundary of  $A$ , that is:  $\partial_n A = AS^n \cap A^C S^n$ . The following version of the co-area formula (see for instance [27] (3.1) for a proof):

$$\frac{1}{2} \int_{\mathbb{R}^+} \#\partial_n \{f > t\} dt \leq \|\nabla_n f\|_1 \leq \int_{\mathbb{R}^+} \#\partial_n \{f > t\} dt,$$

implies that there exist  $t \geq 0$  and  $C > 0$ , such that

$$\#(\partial_n \{\widehat{f}_n > t\}) \leq C \#\{\widehat{f}_n > t\}.$$

Set  $\widehat{\Omega}'_n = \{\widehat{f}_n > t\}$  and  $\widehat{\Omega}_n = \widehat{\Omega}'_n \cup \partial_n \widehat{\Omega}'_n$ . We claim that  $(\widehat{\Omega}_n, \widehat{\Omega}'_n)$  is a sequence of Følner couples for  $Q$  adapted to  $\mathcal{V}$ . □

*Remark 5.13.* — It is possible to elaborate on the technique appearing in this proof and to generalize this lemma to the context of locally compact compactly generated groups. The good setting for this seems to be the theory of isoperimetric profiles of groups through an approach similar to [27, 28]. This will be the content of a forthcoming paper, see [15] for more details.

**COROLLARY 5.14.** — *Let  $G = M \rtimes Q$  be a finitely generated semi-direct product and  $H = M' \rtimes Q$ , with  $M'$  a quotient of  $M$  as a  $\mathbb{Z}Q$ -module.*

*Assume  $G$  admits a sequence of split Følner couples of exponential size whose projection onto  $Q$  is  $(F_n, F'_n)_n$ , then  $H$  admits a sequence of split Følner couples of exponential size that projects onto  $(F'_n S_Q^n, F'_n)_n$ .*

*Proof.* — Write  $(\Omega_n, \Omega'_n) = (A_n \times F_n, A'_n \times F'_n)_n$  for the sequence of split Følner couples of  $G$  and denote by  $\pi$  the projection of the  $\mathbb{Z}Q$ -module  $M$  onto  $M'$ . Coming back to the proof of the proposition above, a computation gives: for any  $(m', q) \in H$ ,

$$\widehat{f}_n(m', q) = \begin{cases} \#(A'_n \cap \pi^{-1}(m')) & \text{if } q \in F'_n \\ 0 & \text{otherwise.} \end{cases}$$

Hence,  $\widehat{\Omega}'_n = \widehat{A}'_n \times F'_n$  for some  $\widehat{A}'_n \subset \pi(A'_n)$  and by construction the projection onto  $Q$  of  $\widehat{\Omega}'_n$  is  $F'_n S^n_Q$ .  $\square$

The next lemma will allow to combine Følner couples along the decomposition of a finitely generated module.

LEMMA 5.15. — *Let  $Q$  be a finitely generated abelian group and  $M$  be a finitely generated  $\mathbb{Z}Q$ -module. Let  $S_Q$  be a finite generating set for  $Q$  and  $S_M$  be a finite generating set for the  $\mathbb{Z}Q$ -module  $M$ . Assume that*

- (1)  *$M$  has submodules  $M_1$  and  $M_2$  so that  $M_2$  is a cyclic  $\mathbb{Z}Q$ -module and its projection onto  $M/M_1$  is the whole of  $M/M_1$ .*
- (2) *The group  $M_1 \rtimes Q$  admits a sequence of split Følner couples of exponential size that projects onto a sequence  $(F_n, F'_n)_n$  in  $Q$ .*
- (3) *The group  $M_2 \rtimes Q$  admits a sequence of split Følner couples of exponential size that projects onto a sequence  $(F_n, F'_n)_n$  in  $Q$ .*

*Then, the group  $M \rtimes Q$  admits a sequence of Følner couples of exponential size that projects onto the sequence  $(F'_n S^n_Q, F'_n)_n$  in  $Q$ .*

*Proof.* — The module  $M_1 \times M_2$  surjects onto  $M$ , hence the group  $H = (M_1 \times M_2) \rtimes Q$ , where  $Q$  acts diagonally, surjects onto  $M \rtimes Q$ . By Corollary 5.14, it is enough to construct a sequence of split Følner couples for  $H$ , whose projection onto  $Q$  is  $(F_n, F'_n)_n$ .

Write  $(\Omega_n \times F_n, \Omega'_n \times F'_n)_n$ , resp.  $(\Lambda_n \times G_n, \Lambda'_n \times F'_n)_n$  the sequence of Følner couples of  $M_1 \rtimes Q$ , resp.  $M_2 \rtimes Q$ , and set for all  $n$

$$\Delta_n = (\Omega_n \times \Lambda_n) \times F_n, \quad \Delta'_n = (\Omega'_n \times \Lambda'_n) \times F'_n.$$

The sequence  $(\Delta_n, \Delta'_n)_n$  is a sequence of Følner couples of exponential size, that projects onto the sequence  $(F_n, F'_n)_n$  in  $Q$ .  $\square$

The proof of Theorem 1.1 makes use the following ingredient of the proof of the main theorem in Pittet and Saloff-Coste [24].

PROPOSITION 5.16 ([24, Proposition 7.8]). — *Let  $\Gamma$  be a finitely generated torsion-free soluble group of finite Prüfer rank satisfying*

$$N \hookrightarrow \Gamma \twoheadrightarrow A,$$

*with  $N$  nilpotent and torsion-free and  $A$  free abelian of finite rank  $d$ .*

*Then,  $\Gamma$  admits a sequence of Følner couples of exponential size. If the extension splits, so does the Følner sequence. Moreover, the projection onto  $A$  of this sequence is the classical Følner sequence  $(\llbracket -2n, 2n \rrbracket^d, \llbracket -n, n \rrbracket^d)_n$  of Følner couples of  $A$ .*

*Proof of Theorem 1.1.* — The reverse implication follows from Proposition 4.1.

To prove the direct implication, let  $G$  be a finitely generated metabelian group of Krull dimension at most 1. We wish to prove a lower bound on its return probability, therefore, by Proposition 2.32 and Theorem 3.4, we may assume that  $G$  is a semi-direct product  $M \rtimes Q$  where  $M$  is abelian and  $Q$  is free abelian. Equip  $G$  with a generating set  $S = S_M \sqcup S_Q$  where  $S_M \subset M$  generates  $M$  as a  $\mathbb{Z}Q$ -module and  $S_Q \subset Q$  generates the group  $Q$ . Denote by  $d$  the rank of  $Q$  and set  $F = (F_n, F'_n)_n = (\llbracket -2n, 2n \rrbracket^d, \llbracket -n, n \rrbracket^d)_n$  the classical sequence of Følner couples for  $Q$ .

Proposition 2.19 provides us with submodules of  $M$

$$\{0\} = M_0 \leq M_1 \leq \dots \leq M_n = M$$

such that  $M_{i+1}/M_i$  is isomorphic to  $\mathbb{Z}Q/\mathcal{P}_i$  with  $\mathcal{P}_i$  a prime ideal of  $\mathbb{Z}Q$ . Each of these cyclic modules  $\mathbb{Z}Q/\mathcal{P}_i$  has dimension at most 1 and by Proposition 5.6 and Lemma 5.8 for the torsion case, and Propositions 2.38 and 5.16 for the torsion-free case, the groups  $\mathbb{Z}Q/\mathcal{P}_i \rtimes Q$  admits Følner couples of exponential size whose projection onto  $Q$  is  $F$ .

We combine them, or possibly their smaller version as defined in Remark 5.2, by iterative applications of Lemma 5.15 to obtain a sequence of Følner couples of exponential size in  $G$ . The conclusion then follows from 5.3. □

### 5.4. Application to the $L^2$ -isoperimetric profile

(see for instance [7]) The  $L^2$ -isoperimetric profile of a finitely generated group  $G$ , equipped with a finite and symmetric generating set  $S$ , is the non-decreasing function

$$j_G(v) = \sup_{\#A \leq v} \sup_{f \in L^2(A)} \frac{\|f\|_2}{\sup_{s \in S} \|f - f(s^{-1} \cdot)\|_2},$$

where  $L^2(A)$  denotes the set of functions  $f : G \rightarrow \mathbb{R}$  supported in  $A$ .

We recall the following theorem due to Coulhon and Grigor'yan [6, 11]. Tessera then generalized it to the setting of metric measure spaces in [27].

**THEOREM 5.17.** — *Let  $G$  be a finitely generated group. Then the isoperimetric profile of  $G$  satisfies  $j_G(v) \sim \ln v$  if and only if  $p_{2n}^G \sim \exp(-n^{\frac{1}{3}})$ .*

Hence, our main theorem may be stated as follows:

**COROLLARY 5.18.** — *Let  $G$  be a finitely generated metabelian group of exponential growth. Let  $d$  be the Krull dimension of  $G$ . Then,*

$$j_G(v) \sim \ln v \iff d \leq 1.$$

## 6. Section of metabelian groups and upper bounds for the return probability

In this section, we describe a method to obtain upper bounds on the return probability of a metabelian group, depending only on its Krull dimension.

When passing to a subgroup, or going to a quotient, the return probability increases. Hence, to establish an upper bound for this quantity, we may perform either of these operations. We do so in the first part. Below, in the second part, we will produce upper bounds for the return probability of the section obtained. This rests on studying some quotients of the free  $(p)$ -metabelian group whose derived subgroup has rank 1.

### 6.1. Sections

We describe a way to obtain a section of a metabelian group respecting the Krull dimension. The section obtained is an extension of a torsion-free module of rank 1 over a polynomial ring by a finitely generated free abelian group (see Lemma 6.1 below). We note in the last paragraph that it embeds inside a wreath product of the same Krull dimension.

#### 6.1.1. When the derived subgroup is torsion

Let  $G$  be a finitely generated metabelian non-abelian group of Krull dimension  $k \geq 2$ , whose derived subgroup is torsion. Up to passing to a finite index subgroup (see Proposition 2.32), we may assume that  $G$  fits in

$$A \hookrightarrow G \twoheadrightarrow Q$$

with  $Q$  free abelian of rank  $d \geq k$  and  $A$  a finitely generated  $\mathbb{Z}Q$ -module, whose underlying additive group is torsion. Because  $G$  is finitely generated and  $\mathbb{Z}Q$  is a Noetherian ring,  $A$  is a finitely generated Noetherian module and has uniform torsion, that is there exists  $n \in \mathbb{N}$  such that  $nA = \{0\}$ .

A reformulation of Proposition 4.2 (see Remark 4.5) is that there exists  $Q_0 \subset Q$ , with  $Q_0 \simeq \mathbb{Z}^k$ , and  $m \in A$  such that  $\mathbb{Z}Q_0 \cdot m \simeq (\mathbb{Z}/p\mathbb{Z})Q_0$  as  $(\mathbb{Z}/p\mathbb{Z})Q_0$ -modules, for some prime divisor  $p$  of  $n$ . Denote by  $L_0$  this  $\mathbb{Z}Q_0$ -module generated by  $m$ .

Consider the subgroup  $G_1$  of  $G$  generated by  $m$  and pull-back  $g_1, \dots, g_k$  of generators of  $Q_0$  in  $G$ . This subgroup satisfies

$$M' \hookrightarrow G_1 \twoheadrightarrow Q_0$$



where  $M'$  is a  $\mathbb{Z}Q_0$ -module containing  $L_0$ . Note that  $M = M'/pM'$  is now a  $(\mathbb{Z}/p\mathbb{Z})Q_0$ -module containing  $L_0$ . We denote by  $G_2$  the corresponding quotient of  $G_1$ .

Let  $C$  be a  $(\mathbb{Z}/p\mathbb{Z})Q_0$ -submodule of  $M$  maximal among the submodules of  $M$  having trivial intersection with  $L_0$  and look at the quotient group  $H := G_2/C$ . It satisfies

$$L \hookrightarrow H \twoheadrightarrow Q_0$$

with  $L$  an *essential extension* for  $L_0$ , that is: for every submodules  $B$  of  $L$ , if  $B \cap L_0 = \{0\}$  then  $B = \{0\}$ . In particular,  $L$  is a torsion-free  $(\mathbb{Z}/p\mathbb{Z})Q_0$ -module of rank 1.

### 6.1.2. When the group is torsion-free

Let  $G$  be a finitely generated metabelian non-abelian group of Krull dimension  $k \geq 2$ . Assume  $G$  is torsion-free. We can argue similarly as above, without the considerations about the torsion. One difference relates to the reformulation of Proposition 4.2 (see Remark 4.5). If  $G$  satisfies  $A \hookrightarrow G \twoheadrightarrow Q$  with  $Q$  free abelian of rank  $d \geq k-1$  and  $A$  a finitely generated  $\mathbb{Z}Q$ -module, it reformulates as: there exists  $Q_0 \subset Q$ , with  $Q_0 \simeq \mathbb{Z}^{k-1}$ , and  $m \in A$  such that  $\mathbb{Z}Q_0.m \simeq \mathbb{Z}Q_0$  as  $\mathbb{Z}Q_0$ -modules.

Following the argument yields a finitely generated non-abelian metabelian section  $H$  of  $G$ , with Krull dimension  $k$  satisfying

$$L \hookrightarrow H \twoheadrightarrow Q_0$$

where  $Q_0 \simeq \mathbb{Z}^{k-1}$  and  $L$  is a torsion-free  $\mathbb{Z}Q_0$ -module of rank 1.

### 6.1.3. Conclusion

We summarize in the next lemma the content of the last two paragraphs.

LEMMA 6.1. — *Let  $G$  be a finitely generated non-abelian metabelian group of Krull dimension  $k \geq 2$ . Then there exists a finitely generated non-abelian metabelian section  $H$  of  $G$  with Krull dimension  $k$  satisfying*

$$L \hookrightarrow H \twoheadrightarrow Q$$

where

- (1) *If the derived subgroup of  $G$  is torsion:  $Q \simeq \mathbb{Z}^k$  and  $L$  is a torsion-free  $(\mathbb{Z}/p\mathbb{Z})Q$ -module of rank 1.*
- (2) *If  $G$  is torsion-free and  $k \geq 2$ :  $Q \simeq \mathbb{Z}^{k-1}$  and  $L$  is a torsion-free  $\mathbb{Z}Q$ -module of rank 1.*

*Remark 6.2.* — In general, a finitely generated non-abelian metabelian group has sections of each of the forms listed above, where  $k$  is respectively the torsion or torsion-free Krull dimension of the group.

Such a section is, in some sense, minimal among the sections of maximal Krull dimension of  $G$ . In the following, we will refer to it as a *nice section*.

6.1.4. Inside wreath products

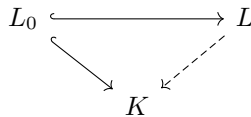
We prove in this paragraph that the above sections embed inside wreath products of the same Krull dimension.

LEMMA 6.3. — *Let  $G$  be a finitely generated non-abelian metabelian group of Krull dimension  $k \geq 2$ , whose derived subgroup is torsion. Then,  $G$  has a section which embeds in  $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}^k$ , for some prime  $p$ .*

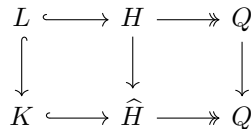
*Similarly, let  $H$  be a finitely generated non-abelian metabelian torsion-free group of Krull dimension  $k \geq 3$ . Then,  $H$  has a section which embeds in  $\mathbb{Z} \wr \mathbb{Z}^{k-1}$ .*

*Proof.* — We prove the first statement. The second one can be proved similarly.

By Lemma 6.1,  $G$  has a finitely generated non-abelian metabelian section  $H$  of Krull dimension  $k$  satisfying  $L \hookrightarrow H \twoheadrightarrow Q$  where  $Q \simeq \mathbb{Z}^k$  and  $L$  is a torsion-free  $(\mathbb{Z}/p\mathbb{Z})Q$ -module of rank 1. Moreover,  $L$  has an essential cyclic submodule  $L_0$ , with  $L_0 \simeq (\mathbb{Z}/p\mathbb{Z})Q$ , as  $(\mathbb{Z}/p\mathbb{Z})Q$ -modules. Denote by  $K$  the fraction field of  $(\mathbb{Z}/p\mathbb{Z})Q$ . By injectivity of  $K$ , there exists a morphism  $L \rightarrow K$  such that the following diagram commute



Because  $L_0$  is an essential submodule, this map is injective and  $L$  embeds in  $K$ . Hence, we can push forward the extension and get



The latter extension is split: indeed, the cohomology group  $H^2(Q, K)$  is reduced to  $\{0\}$ .

We now exhibit the wreath product. Fix a sequence  $(f_n)_n$  of elements of  $(\mathbb{Z}/p\mathbb{Z})Q$  so that the following holds

- (1) for every  $i$  in  $\mathbb{N}$ ,  $f_i \mid f_{i+1}$ .
- (2) for every  $f$  in  $(\mathbb{Z}/p\mathbb{Z})Q$ , there exists an integer  $i$  so that  $f \mid f_i$ .

We have

$$\frac{1}{f_i}(\mathbb{Z}/p\mathbb{Z})Q \subset \frac{1}{f_{i+1}}(\mathbb{Z}/p\mathbb{Z})Q$$

and  $K$  is the nondecreasing union of the  $\frac{1}{f_i}(\mathbb{Z}/p\mathbb{Z})Q$ . As  $L$  is a finitely generated  $(\mathbb{Z}/p\mathbb{Z})Q$ -module, there exists a  $f$  such that  $L \subset \frac{1}{f}(\mathbb{Z}/p\mathbb{Z})Q$ . As a consequence,  $H$  actually sits in the following subgroup of  $\widehat{H}$ :

$$\frac{1}{f}(\mathbb{Z}/p\mathbb{Z})Q \hookrightarrow \tilde{H} \twoheadrightarrow Q,$$

isomorphic to  $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}^k$ . □

### 6.2. Upper bound

We now study the sections given by Lemma 6.1 in order to obtain upper bounds for the return probability of the initial group.

Let  $G$  be a finitely generated metabelian non-abelian group of Krull dimension  $k \geq 2$  satisfying

$$L \hookrightarrow G \twoheadrightarrow Q$$

where either

- (1)  $Q \simeq \mathbb{Z}^k$  and  $L$  is a torsion-free  $(\mathbb{Z}/p\mathbb{Z})Q$ -module of rank 1,
- (2) or  $k \geq 2$ ,  $Q \simeq \mathbb{Z}^{k-1}$  and  $L$  is a torsion-free  $\mathbb{Z}Q$ -module of rank 1.

If this extension splits, then it contains either  $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}^k$  or  $\mathbb{Z} \wr \mathbb{Z}^{k-1}$  and we can derive the corresponding upper bound for the return probability of  $G$ . If it does not split, then the subgroup generated by any pull-back of generators of  $Q$  still satisfies the exact sequence above and is either  $k$ -generated or  $(k-1)$ -generated. We still denote it  $G$ . Hence,  $G$  is a quotient of either  $B_k^{(p)}$  or  $B_{k-1}$ . Denote by  $H$  this latter group. We have the commuting diagram

$$\begin{array}{ccccc} [H, H] & \hookrightarrow & H & \twoheadrightarrow & Q \\ \downarrow & & \downarrow & & \downarrow \\ L & \hookrightarrow & G & \twoheadrightarrow & Q \end{array}$$

Let  $N$  be the kernel of this morphism, we have  $N \subset [H, H]$  and we are left to study the extensions of  $Q$  with a rank 1 quotient of the derived subgroup  $[H, H]$ .

We start with the torsion-free case, that is  $H = B_k$ .

PROPOSITION 6.4. — *The free metabelian  $B_k$  group embeds as a subgroup of the power  $(B_k/N)^{k!}$ .*

*Proof.* — Consider the action of the symmetric group  $\mathbf{S}_k$  on  $B_k$  permuting the generators. The derived subgroup  $M = [B_k, B_k]$  has rank  $k - 1$  ([5]) and is invariant under this action. Consider the diagonal homomorphism

$$B_k \rightarrow \prod_{\sigma \in \mathbf{S}_k} B_k / \sigma(N).$$

Lemma 6.5 below precisely says that this homomorphism is injective. Since  $B_k / \sigma(N)$  is isomorphic to  $B_k / N$ , this proves the proposition.  $\square$

LEMMA 6.5. — *The intersection  $\bigcap_{\sigma \in \mathbf{S}_k} \sigma(N)$  is reduced to  $\{0\}$ .*

The proof of the lemma requires the following fact.

*Fact.* — For every  $k \geq 0$ , the group  $\mathbf{S}_k$  has no faithful complex representation of dimension  $\leq k - 2$ . In particular, every faithful complex representation of dimension  $k - 1$  is irreducible.

*Proof.* — This is trivial for  $k \leq 1$ . For  $k \geq 2$  distinct of 4, tables (see [12] and [4] Note C) shows that the only irreducible representations of dimension  $< k - 1$  are the trivial and signature 1-dimensional representations, and are thus not faithful. For  $k = 4$ , there is an additional one, 2-dimensional, factoring through a quotient of order 6, completing the proof. The second statement immediately follows.  $\square$

*Proof of Lemma 6.5.* — Let  $K$  be the fraction field of  $\mathbb{Z}Q$ . By flatness of  $K$ , we have an exact sequence

$$0 \rightarrow N \otimes K \rightarrow M \otimes K \rightarrow M/N \otimes K \rightarrow 0.$$

Since  $M/N$  is torsion-free of rank 1, we deduce that  $N \otimes K$  has positive codimension in  $M \otimes K$ . The intersection  $\bigcap_{\sigma \in \mathbf{S}_k} \sigma(N \otimes K)$  is a  $\mathbf{S}_k$ -invariant subspace of positive codimension in  $M \otimes K$ . By the fact, it is therefore reduced to  $\{0\}$ . Since  $M$  is embedded into  $M \otimes K$ , it follows that  $\bigcap_{\sigma \in \mathbf{S}_k} \sigma(N) = \{0\}$ .  $\square$

Combined with Lemma 6.1(2), we get the following estimate.

COROLLARY 6.6. — *Let  $G$  be a finitely generated non-abelian metabelian torsion-free group of Krull dimension  $k \geq 2$ . Then the return probability of  $G$  satisfies*

$$p_{2n} \lesssim \exp\left(-n^{\frac{k-1}{k+1}}(\log n)^{\frac{2}{k+1}}\right).$$

Remark 6.7. — The representation above is the standard representation for the symmetric group. Indeed, by Magnus’ embedding,  $B_d$  embeds in the matrix group

$$\left\{ \begin{pmatrix} b & m \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{Z}^d, m \in M \right\}$$

where  $M$  denotes the free  $\mathbb{Z}(\mathbb{Z}^k)$ -module of rank  $d$  with basis  $(e_i)_{i=1,\dots,k}$ , via

$$s_i \mapsto \begin{pmatrix} a_i & e_i \\ 0 & 1 \end{pmatrix}.$$

Here,  $s_1, \dots, s_k$  denotes the generators of  $B_k$  and  $a_i$  is the image of  $s_i$  in the abelianization  $\mathbb{Z}^k$ . The symmetric group acts on  $M$  by permuting the elements of the basis.

The derived subgroup of  $B_k$  embeds in the submodule  $N_0 = \{\sum b_i e_i \mid \sum b_i(1 - X_i) = 0\}$  and they both have rank  $d - 1$  ([5]). Therefore, their tensor product by  $K$  are equal and we denote by  $V$  this subspace of  $M \otimes K$ . Up to multiplying the  $e_i$ ’s by the scalars  $(1 - X_i)^{-1}$ ,  $V$  is the subspace of vectors  $\sum b_i e_i$  satisfying  $\sum b_i = 0$ .

Remark 6.8. — Proposition 6.4 remains true in positive characteristic  $p$ , as long as  $p$  does not divide  $k$ . Indeed, in this setting the standard representation is irreducible: any  $v \in V \setminus \{0\}$  has coordinates  $x_i$  and  $x_j$  such that  $x_i \neq x_j$ . Set  $a = x_i - x_j \neq 0$ . Then

$$a^{-1}(x - (i, j).x) = (0, \dots, \underset{i}{\uparrow} -1, 0, \dots, \underset{j}{\uparrow} 1, \dots, 0)$$

generates a basis of  $V$ .

COROLLARY 6.9. — *Let  $G$  be a finitely generated non-abelian metabelian group of Krull dimension  $k \geq 2$  whose derived subgroup is torsion. Assume that  $G$  admits a nice section whose torsion does not divide  $k$ .*

*Then the return probability of  $G$  satisfies*

$$p_{2n} \lesssim \exp\left(-n^{\frac{k}{k+2}}\right).$$

Remark 6.10. — Recall that, if  $G$  is a finitely generated metabelian group such that  $[G, G]$  is torsion, then the torsion is uniform. Let  $n \geq 1$  be the smallest integer so that  $n[G, G] = \{1\}$ . A sufficient condition for  $G$  to

admit a nice section whose torsion does not divide its Krull dimension  $k$  is therefore to assume that  $n$  and  $k$  are coprime.

### 6.3. Free metabelian groups as subgroups of wreath products

Let  $G$  be a finitely generated non-abelian metabelian group. In the last two subsections, we exhibited a section of  $G$  that embeds in a wreath product and so that a power of it contains a free ( $p$ -)metabelian group as a subgroup. All groups mentionned here have the same Krull dimension. To complete the picture, we note, as a corollary of the Magnus' embedding, that the free ( $p$ -)metabelian group embeds in a wreath product of the same Krull dimension.

**COROLLARY 6.11.** — *Let  $d \geq 2$ . The free metabelian group  $B_d$  embeds in the wreath product  $\mathbb{Z} \wr \mathbb{Z}^d$ . Similarly, for any prime  $p$ , the free  $p$ -metabelian group  $B_d^{(p)}$  embeds in the wreath product  $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}^d$ .*

*Proof.* — By Magnus' embedding, the free metabelian group embeds in the wreath product  $\mathbb{Z}^d \wr \mathbb{Z}^d$  and the free  $p$ -metabelian group embeds in  $(\mathbb{Z}/p\mathbb{Z})^d \wr \mathbb{Z}^d$ . Hence, it is enough to prove that  $\mathbb{Z}^d \wr \mathbb{Z}^d$  embeds in  $\mathbb{Z} \wr \mathbb{Z}^d$ , and respectively that  $(\mathbb{Z}/p\mathbb{Z})^d \wr \mathbb{Z}^d$  embeds in  $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}^d$ . We indicate the proof in the torsion-free case, the proof for the torsion case is similar.

Denote by  $B$  the base of the wreath product  $G = \mathbb{Z} \wr \mathbb{Z}^d$ . The group satisfies the split extension

$$B \hookrightarrow G \twoheadrightarrow \mathbb{Z}^d.$$

Consider the finite index subgroup  $H$  of  $G$  mapping to  $Q = \mathbb{Z}^{d-1} \times d\mathbb{Z}$  in the quotient. Under the action of  $Q$ , the base  $B$  becomes a free  $\mathbb{Z}Q$ -module of rank  $d$ . Therefore  $H$  is isomorphic to the wreath product  $\mathbb{Z}^d \wr \mathbb{Z}^d$ .  $\square$

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