UNIFORM APPROXIMATION OF HARMONIC FUNCTIONS

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Introduction.

Let $\omega$ be a bounded open set in Euclidian $n$-space ($n > 1$), with closure $\overline{\omega}$ and frontier $\omega^*$. Corollary 1 below gives a necessary and sufficient condition that each continuous real-valued function on $\overline{\omega}$ harmonic in $\omega$, may be uniformly approximated on $\overline{\omega}$ by functions harmonic in a neighbourhood of $\overline{\omega}$. The purpose of this paper is to extend corollary 1 to axiomatic potential theory.

Suppose $a_p$ is a sequence of points chosen one from each domain in $\bigcap \overline{\omega}$. Let $\Phi_{a_p}^p$ be the elementary harmonic functions relative to $a_p$ [10, § 1]. Then $\Phi_{a_p}^p$ is a potential of support $a_p$, $n = 1, 2, \ldots$. If $C(\omega)$ denotes the space of continuous real-valued functions on $\omega$, then following Deny [9], [10, § 4] and de La Pradelle [16], we consider the following linear function spaces:

$M = \{f \in C(\overline{\omega}) : f$ is harmonic in $\omega\}$;

$L = \{f \in C(\overline{\omega}) : f$ extends to a function harmonic in a neighbourhood $U_f$ of $\overline{\omega}\}$;

$K = \{f \in C(\overline{\omega}) : f$ extends to the difference of two potentials with compact support contained in $\bigcap \overline{\omega}\}$;

$J = f \in C(\overline{\omega}) : f$ extends to a function in the linear span of the elementary harmonic functions $\Phi_{a_p}^p\}$. 
Theorem 1. — \( J \) is uniformly dense in \( M \) if and only if the sets \( \{ \omega \} \) and \( \{ \overline{\omega} \} \) are effilé (thin) at the same points.

The points at which \( \{ \omega \} \) is not thin [7, ch. VII, § 1] are precisely the regular points of \( \omega^* \) for the Dirichlet problem [7, ch. VIII, § 6], while the points where \( \{ \overline{\omega} \} \) is not thin are precisely the stable points of \( \omega^* \) for the Dirichlet problem.

Suppose now that \( \omega \) is a relatively compact open subset of a harmonic space \( \Omega \) which satisfies Brelot's axioms 1, 2 and 3, and on which there exists a strictly positive potential. Suppose also that the topology of \( \Omega \) has a countable base of completely determining open sets, that potentials with the same one point support are proportional, and that adjoint potentials with one point support are proportional. De La Pradelle [16, th. 5] proves the following generalisation of theorem 1.

Theorem 1'. — \( K \) is uniformly dense in \( M \) if and only if the sets \( \{ \omega \} \) and \( \{ \overline{\omega} \} \) are thin at the same points.

Deny's proof of theorem 1 consists of showing that the same measures on annihilate \( J \) and \( M \), and the same method is used to prove theorem 1'. In this paper the conditions on \( \Omega \) are relaxed, and the following corollary to theorem 1 is generalised.

Corollary 1. — \( L \) is uniformly dense in \( M \) if and only if every regular point of \( \omega^* \) is stable.

The proof of corollary 1, using elementary harmonic functions, does not adapt to axiomatic potential theory. In example 2 we give a proof which does generalise. This proof is rather satisfying, since it uses Bauer's characterisation of regular points, and the following generalisation of the Stone-Weierstrass theorem [13, th. 5].

Theorem 2. — Suppose that \( X \) is a compact Hausdorff space, that \( L \) is a linear subspace of \( C(X) \) which contains the
constant functions, separates the points of \( X \), and has the weak Riesz separation property, and that \( L \) is contained in the linear subspace \( M \) of \( C(X) \). Then \( L \) is uniformly dense in \( M \) if and only if \( \partial_L(X) = \partial_M(X) \).

\( L \) is said to have the weak Riesz separation property (R.s.p.) if whenever \( \{f_1, f_2, g_1, g_2\} \subset L \) with \( f_1 \lor f_2 < g_1 \land g_2 \), there exists \( h \in L \) with \( f_1 \lor f_2 \leq h \leq g_1 \land g_2 \). The Choquet boundary of \( M \) is denoted \( \partial_M(X) \) [15] and Bauer [1, th. 6] shows that in the classical case \( \partial_M(\omega) \) is precisely the set of regular points of \( \omega^* \). Brelot [7, ch. viii, § 1] remarks that this remains true when \( \omega \) is a relatively compact open subset of a harmonic space satisfying Brelot's axioms 1, 2 and 3', and that in this case \( \partial_L(\omega) \) is precisely the set of stable points of \( \omega^* \). Using Bauer's results, corollary 1 is an immediate consequence of Theorem 2, both in the classical case, and when \( \omega \) is a relatively compact open subset of a harmonic space satisfying Brelot's axioms 1, 2 and 3'.

If \( \omega \) is a relatively compact open subset of one of the harmonic spaces of Boboc and Cornea [4], which are more general than those of Brelot, then the set of regular points of \( \omega^* \) corresponds not to \( \partial_M(\omega) \) but to \( \omega^* \cap \partial_W(\omega) \), where \( W \subset C(\omega) \) is the min-stable wedge of continuous functions on \( \omega \) superharmonic in \( \omega \). In this case we need a strengthened form of theorem 2, which, together with this characterisation of regular points, has corollary 1 as a direct consequence. This we supply in theorem 4.

In order to strengthen theorem 2 we consider min-stable wedges \( \mathcal{I} \subset W \) in \( C(X) \), and a geometric simplex \( (X, \mathcal{I}, L) \). In theorem 4 we give a sufficient condition that \( L \) be uniformly dense in the space \( M \) of continuous \( W \)-affine functions on \( X \). This condition is given in terms of the Choquet boundaries \( \partial_W(X) \) and \( \partial_{\mathcal{I}}(X) \). In lemma 5 a pair of conditions equivalent to this is given. These are of a more analytic nature. Theorem 4 is deduced from proposition 1, which is a characterisation of geometric simplexes. This is proved by repeated use of filtering arguments together with the following form of Dini's theorem.

**Theorem 3.** — If \( \{f_i : i \in I\} \) is an upward filtering family in \( C(X) \) and \( g \) is an upper bounded upper semicontinuous
function such that \( g < \sup \{ f_i : i \in I \} \), then \( g < f_{i_0} \) for some \( i_0 \in I \).

\( f > 0 \) (\( \geq 0 \)) will mean that \( f(x) > 0 \) (\( \geq 0 \)) for all \( x \in X \).

A characterisation of geometric simplexes.

Let \( X \) be a compact Hausdorff space, and let \( \mathcal{G} \in \mathcal{W} \) be min-stable wedges in \( C(X) \). If \( f \wedge g \in \mathcal{W} \) whenever \( f, g \in \mathcal{W} \) then \( \mathcal{W} \) is said to be min-stable. We shall assume that \( \mathcal{G} \) contains a function \( p = \frac{1}{2} \) and a function \( q = -1 \). The Choquet theory for min-stable wedges has been developed in [11] [5] where proofs of the following results may be found.

The wedge \( \mathcal{W} \) induces a partial order \( \preceq_w \) on the positive regular Borel measures on \( X \) given by the formula

\[ \mu \preceq_w \lambda, \quad \lambda(f) \leq \mu(f) \quad \text{whenever} \quad f \in \mathcal{W}. \]

A measure which is maximal for \( \preceq_w \) is said to be \( \mathcal{W} \)-extremal. A measure \( \mu \) is \( \mathcal{W} \)-extremal if and only if

\[ \mu(g) = \inf \{ \mu(f) : g < f \in \mathcal{W} \} \]

whenever \( g \in - \mathcal{W} \) [5, Th. 1.2]. An extended real-valued function \( g \) on \( X \) is \( \omega \)-concave if the upper integral \( \int g \, d\mu \leq g(x) \) whenever \( \varepsilon_x \preceq_w \mu \). The function \( g \) is \( \mathcal{W} \)-affine if both \( g \) and \( -g \) are \( \mathcal{W} \)-concave. The min-stable wedge of lower bounded extended real-valued \( \omega \)-concave functions on \( X \) will be denoted \( \mathcal{W} \).

**Lemma 1.** — [11, Th. 1] [5, Cor. 1.4 d)]. Each \( f \in \mathcal{W} \) is the pointwise supremum of an upward filtering family in \( \mathcal{W} \).

A closed subset \( A \) of \( X \) is a \( \mathcal{W} \)-face (\( \mathcal{W} \)-absorbent set [5, § 2], \( \mathcal{W} \)-extreme set [11, § 2]) if for each \( x \in A \)

\[ \mu(X \setminus A) = 0 \quad \text{whenever} \quad \varepsilon_x \preceq_w \mu. \]

If \( A \) is a \( \mathcal{W} \)-face and \( f \in \mathcal{W} \) then the function \( f^\varepsilon_x \), equal to \( f \) on \( A \) and to \( +\infty \) on \( X \setminus A \), belongs to \( \mathcal{W} \) [11, § 2]. The \( \mathcal{W} \)-faces are ordered by inclusion, and each \( \mathcal{W} \)-face contains a minimal \( \mathcal{W} \)-face. The measure \( \varepsilon_x \) is \( \mathcal{W} \)-extremal if and only if \( x \) belongs to a minimal \( \mathcal{W} \)-face. The Choquet boundary
of \( W \) is the union of all minimal \( W \)-faces of \( X \), and is denoted \( \partial_W(X) \) [5, § 2]. Each \( \mathfrak{F} \)-face is a \( W \)-face, so that each minimal \( \mathfrak{F} \)-face contains at least one minimal \( W \)-face.

**Lemma 2.** — [2, Satz 2] [5, Cor. 2.1] A function \( f \in \mathcal{W} \) is positive if and only if it is positive on \( \partial_W(X) \).

We say that \( W \) distinguishes the points \( x, y \in X \) if there exists \( f, g \in W \) such that

\[
f(x)g(y) \neq f(y)g(x).
\]

If \( W \) contains the constant functions, then \( W \) distinguishes \( x \) and \( y \) if and only if \( W \) separates \( x \) and \( y \). The subspace \( (W - W)/p = \{ (f - g)/p : f, g \in W \} \) is a sublattice of \( C(X) \) containing the constant functions. \( (W - W)/p \) separates points of \( X \) if and only if \( W \) distinguishes points of \( X \). By Stone’s theorem, \( W - W \) is uniformly dense in \( C(X) \) if and only if \( W \) distinguishes points of \( X \). The following lemma is an immediate consequence of [5, Th. 2.1 c)].

**Lemma 3.** — \( W \) distinguishes \( x, y \in \partial_W(X) \) if and only if \( x \) and \( y \) belong to different minimal \( W \)-faces of \( X \).

**Example 1.** — Let \( X = [0, 1] \times [0, 1] \), and let \( \mathcal{G} = \{ f \in C(X) : y \mapsto f(x, y) \mbox{ is convex for each } x, \mbox{ and } x \mapsto f(x, y) \mbox{ is affine with } f(1, y) = 2f(0, y) \mbox{ for each } y \} \). Then the sets \( A = \{(x, 0) : x \in [0, 1] \} \) and \( B = \{(x, 1) : x \in [0, 1] \} \) are minimal \( \mathcal{G} \)-faces. \( \mathcal{G} \) separates, yet does not distinguish the points of \( A \). The Choquet boundary

\[
\partial_{\mathcal{G}}(X) = A \cup B.
\]

The \( \mathcal{G} \)-affine functions are the \( f \in \mathcal{G} \) which are affine in \( y \) for each \( x \).

**Lemma 4.** — If \( \mathcal{G} \subset W \) are min-stable wedges in \( C(X) \), and if \( \mathcal{G} \) contains a positive function \( p \) and a negative function \( q \), then the following conditions are equivalent:

(i) For each pair of (disjoint) minimal \( \omega \)-faces \( A_1, A_2 \), there exists a pair of (disjoint) \( \mathcal{G} \)-faces \( B_1, B_2 \), such that \( A_1 \subset B_1 \) and \( A_2 \subset B_2 \);

(ii) Same statement as (i) but with \( B_1, B_2 \) minimal \( \mathcal{G} \)-faces;
(iii) \( \partial_w(X) \subset \partial_\mathcal{G}(X) \) and \( \mathcal{G} \) distinguishes points of \( \partial_w(X) \) which are distinguished by \( W \).

Proof. — (i) \( \Rightarrow \) (ii). Let \( A \) be a minimal \( W \)-face, and put \( G = \cap \{ F: F \text{ is an } \mathcal{G} \text{-face and } A \subset F \} \). Then \( G \) is an \( \mathcal{G} \)-face, and contains a minimal \( \mathcal{G} \)-face \( H \). Now \( H \) is a \( W \)-face and contains a minimal \( W \)-face \( A' \). If \( A \cap A' = \emptyset \), then there exist disjoint \( \mathcal{G} \)-faces \( B, B' \) such that \( A \in B \) and \( A' \in B' \). Then \( B \cap G \) is an \( \mathcal{G} \)-face properly contained in \( G \), which contradicts the definition of \( G \). Therefore \( A = A' \), so that \( G \subset H \) and \( G \) is a minimal \( \mathcal{G} \)-face. It follows immediately that if \( A_1, A_2 \) are disjoint minimal \( W \)-faces, then \( A_1 \subset G_1 \) and \( A_2 \subset G_2 \), where \( G_1 \) and \( G_2 \) are disjoint minimal \( \mathcal{G} \)-faces.

(ii) \( \Rightarrow \) (iii). \( \partial_w(X) = \cup \{ A: A \text{ is a minimal } W \text{-face} \} \subset \cup \{ B: B \text{ is a minimal } \mathcal{G} \text{-face} \} = \partial_\mathcal{G}(X) \). Suppose \( W \) distinguishes \( x_1 \) and \( x_2 \in \partial_w(X) \), then by lemma 3 there are disjoint minimal \( W \)-faces \( A_1 \) and \( A_2 \) with \( x_1 \in A_1 \) and \( x_2 \in A_2 \). Therefore there are disjoint minimal \( \mathcal{G} \)-faces \( B_1, B_2 \) with \( x_1 \in A_1 \subset B_1 \) and \( x_2 \in A_2 \subset B_2 \), and by lemma 3 \( \mathcal{G} \) distinguishes \( x_1 \) and \( x_2 \).

(iii) \( \Rightarrow \) (ii) \( \Rightarrow \) (i). If \( A_1 \) and \( A_2 \) are disjoint minimal \( W \)-faces, then the points \( x_1 \in A_1 \) and \( x_2 \in A_2 \) are distinguished by \( W \). Therefore \( x_1 \) and \( x_2 \) are distinguished by \( \mathcal{G} \). Since \( x_1, x_2 \in \partial_w(X) \subset \partial_\mathcal{G}(X) \) there are disjoint minimal \( \mathcal{G} \)-faces \( B_1, B_2 \) with \( x_1 \in B_1 \) and \( x_2 \in B_2 \). Since \( A_1 \) is minimal \( A_1 \subset A_1 \cap B_1 \), so that \( A_1 \subset B_1 \). Similarly \( A_2 \subset B_2 \).

If \( L \) and \( M \) are linear subspaces of \( C(X) \), then we will put 
\[ \mathcal{L} = \{ f_1 \wedge \cdots \wedge f_r: f_i \in L, \; i = 1 \ldots r \} \]
and
\[ \mathcal{M} = \{ f_1 \wedge \cdots \wedge f_r: f_i \in M, \; i = 1 \ldots r \}. \]
Then \( \mathcal{L} \) and \( \mathcal{M} \) are min-stable wedges in \( C(X) \) and if the functions in \( L \) are \( \mathcal{G} \)-affine then \( \mathcal{L} \subset \mathcal{G} \).

Suppose \( L \) is a linear subspace of continuous \( \mathcal{G} \)-affine functions on \( X \). The triple \( (X, \mathcal{G}, L) \) is a geometric simplex if given \( f \in - \mathcal{G} \) and \( g \in \mathcal{G} \) with \( f < g \), then there exists
Proposition 1. — \((X, \mathcal{I}, L)\) is a geometric simplex if and only if \(L\) has the weak R.s.p., \(\partial_\mathcal{I}(X) \subset \partial_\mathcal{J}(X)\) and \(L\) distinguishes points of \(\partial_\mathcal{J}(X)\) which are distinguished by \(\mathcal{I}\).

Proof. — Let \((X, \mathcal{I}, L)\) be a geometric simplex and suppose that \(\{f_1, f_2, g_1, g_2\} \subset L\) with \(f_1 \vee f_2 < g_1 \wedge g_2\). Since \(g_1 \wedge g_2 \in \mathcal{J}\) there exists a family \(\Lambda = \{h_i \in \mathcal{I}: h_i < g_1 \wedge g_2, i \in I\}\) filtering up to \(g_1 \wedge g_2\). By Dini's theorem there exists \(h_i \in \Lambda\) such that \(f_1 \vee f_2 < h_i < g_1 \wedge g_2\). Similarly, there exists \(h_i \in -\mathcal{I}\) such that \(f_1 \vee f_2 < h_i < h_i < g_1 \wedge g_2\). Since \((X, \mathcal{I}, L)\) is a geometric simplex there exists \(h \in L\) such that

\[
\begin{align*}
f_1 \vee f_2 &< h \leq h_i \leq g_1 \wedge g_2
\end{align*}
\]

and \(L\) has the weak R.s.p.

Suppose \(x_i \in \partial_\mathcal{I}(X), i = 1, 2,\) and \(f_j \in -\mathcal{I}, j = 1, 2\). Then \(f_j \in -\mathcal{I}\) and by (1)

\[
\begin{align*}
f_j(x_i) &= \inf \{h(x_i): f_j < h \in \mathcal{I}\}, \\
&= \inf \{g(x_i): g \in L, f_j < g < h \in \mathcal{I}\},
\end{align*}
\]

since \((X, \mathcal{I}, L)\) is a geometric simplex. Therefore \(x_i \in \partial_\mathcal{J}(X),\) and \(\partial_\mathcal{I}(X) \subset \partial_\mathcal{J}(X)\). If \(\varepsilon > 0\) then by (2) there exists \(g_1, g_2 \in L\) such that

\[
|g_j(x_i) - f_j(x_i)| < \varepsilon, \quad i, j = 1, 2.
\]

If \(f_1\) and \(f_2\) distinguish \(x_1\) and \(x_2\), and \(\varepsilon\) is small enough, then \(g_1\) and \(g_2\) distinguish \(x_1\) and \(x_2\), and the conditions of the proposition are necessary.

Suppose that \((X, \mathcal{I}, L)\) satisfies the given conditions, and that \(f \in -\mathcal{I}, g \in \mathcal{I}\) with \(f < g\). If \(A\) is a minimal \(\mathcal{I}\)-face, then by lemma 4 \(A\) is contained in a minimal \(\mathcal{J}\)-face \(B\). If \(\alpha\) is the smallest real number such that \(\alpha f \geq f\) on \(B\), then

\[
D = \{x \in B: (\alpha f - f)(x) = 0\} = \{x \in X: (\alpha f - f)(x) = 0\}
\]
is a \( \mathcal{F} \)-face [5, prop. 2.2]. \( \mathcal{D} \) contains a minimal \( \mathcal{G} \)-face \( A' \), and by lemma 4, \( A = A' \). Similarly

\[ A \subset \{ x \in B : (g - \beta l)(x) = 0 \}, \]

where \( \beta \) is the greatest real number such that \( \beta l \leq g \) on \( B \). Since \( l \) is strictly positive, \( \alpha < \beta \), and if \( \alpha < \gamma < \beta \), then \( f < \gamma l < g \) on \( B \). By lemma 1, the function \( (\gamma l)^p \) is the supremum of an increasing filtering family \( \{ f_i \in \mathcal{D} : i \in I \} \).

Since \( f < (\gamma l)^p \), it follows from Dini's theorem that \( f < f_{i_0} (= h_1 \wedge \cdots \wedge h_n : h_i \in \mathcal{L}, r = 1, \ldots, n) \) for some \( i_0 \in I \).

Therefore there exists \( h \in \mathcal{L} \) with \( f < h \) on \( X \) and \( h < g \) on \( B \).

Suppose that \( f < h_1 \wedge h_2 \) with \( h_1, h_2 \in \mathcal{L} \). Since \( \mathcal{L} \) has the weak R.s.p. and contains a positive function, the family \( \{ k \in \mathcal{L} : k < h_1 \wedge h_2 \} \) filters up. Therefore

\[ \bar{k} = \sup \{ k' \in \mathcal{L} : k' < h_1 \wedge h_2 \} = \sup \{ k \in \mathcal{L} : k < h_1 \wedge h_2 \}. \]

Thus \( \bar{k} \) is the supremum of a filtering family of continuous \( \mathcal{L} \)-affine functions and is therefore \( \mathcal{L} \)-affine and lower semicontinuous. Therefore \( \bar{k} \in \mathcal{H} \). It follows from (1) that \( \bar{k} = h_1 \wedge h_2 \) on \( \delta_{\mathcal{H}}(x) \). Since \( \delta_{\mathcal{H}}(X) \subset \delta_{\mathcal{F}}(X) \), the function \( \bar{k} - f \) is strictly positive on \( \delta_{\mathcal{F}}(X) \). By lemma 2, \( \bar{k} > f \). By Dini's theorem there exists \( h \in \mathcal{L} \) such that \( f < h < h_1 \wedge h_2 \), and the family \( \mathcal{F} = \{ h \in \mathcal{L} : f < h \} \) is filtering down.

Therefore the function \( h = \inf \{ h \in \mathcal{L} : f < h \} \) is upper semicontinuous \( \mathcal{L} \)-affine and \( \mathcal{F} \)-affine. If \( A \) is a minimal \( \mathcal{F} \)-face, then there exists \( h \in \mathcal{F} \) with \( h < g \) on \( A \). Therefore \( h < g \) on \( \delta_{\mathcal{F}}(X) \), and by lemma 2, \( h < g \). By Dini's theorem there exists \( h \in \mathcal{L} \) such that \( f < h < g \). Therefore \( (X, \mathcal{F}, \mathcal{L}) \) is a geometric simplex.

We may now extend the density theorem in [13].

**Theorem 4.** — Suppose that \( \mathcal{F} \subset \mathcal{W} \) are min-stable wedges in \( C(X) \), and that \( \mathcal{F} \) contains a positive function \( p \) and a negative function \( q \). Let \( M = \{ f \in C(X) : f \text{ is } \mathcal{W} \text{-affine} \} \) and let \( \mathcal{L} \subset C(X) \) be a linear subspace of \( \mathcal{F} \)-affine functions. If \( (X, \mathcal{F}, \mathcal{L}) \) is a geometric simplex and if \( \delta_{\mathcal{W}}(X) \subset \delta_{\mathcal{F}}(X) \) and if \( \mathcal{F} \) distinguishes points of \( \delta_{\mathcal{W}}(X) \) which are distinguished by \( \mathcal{W} \), then \( \mathcal{L} \) is uniformly dense in \( M \).
Proof. — It follows from proposition 1 that \( \partial_W(X) \subset \partial_{2b}(X) \) and that \( 2 \) distinguishes points of \( \partial_W(X) \) distinguished by \( W \). Therefore \((X, W, L)\) is a geometric simplex. If \( f \in M \) and \( \varepsilon > 0 \), then by lemma 1 and by Dini's theorem there exist \( h \in -W, k \in W \) such that

\[
f + \varepsilon q < h < k < f + \varepsilon p.
\]

Since \((X, \omega, L)\) is a geometric simplex, there exists \( g \in L \) such that \( f + \varepsilon q < h \leq g \leq k < f + \varepsilon p \), and \( L \) is uniformly dense in \( M \).

Suppose that \( L \subset M \) are linear subspaces of \( C(X) \) containing the constant functions, and that \( L \) has the weak R.s.p. Then \( \mathcal{L} \) and \( \mathfrak{M} \) are min-stable wedges, \( \partial_{2b}(X) = \partial_{L}(X) \) the Choquet boundary of \( L \), and \( \partial_{\mathcal{M}}(X) = \partial_{M}(X) \), the Choquet boundary of \( M \) [15], and \((X, \mathcal{L}, L)\) is a geometric simplex. Since \( L \) contains the constant functions, points are distinguished by \( \mathcal{L} \) (resp. \( \mathfrak{M} \)) if and only if they are separated by \( L \) (resp. \( M \)). We have therefore the following corollary to theorem 4.

**Corollary 1.** — [13, cor. to th. 5]. If \( \partial_L(X) = \partial_M(X) \) and \( L \) separates the points of \( \partial_M(X) \) which are separated by \( M \), then \( L \) is uniformly dense in \( M \).

We may replace the conditions in proposition 1 and theorem by a pair of conditions very similar to those used by D. A. Edwards [12].

Suppose we are given wedges \( W_0 \) and \( \mathfrak{g}_0 \) such that the min-stable wedges \( \{f_1 \land \cdots \land f_r : f_i \in \omega_0, i = 1, \ldots, r\} \) and \( \{f_1 \land \cdots \land f_r : f_i \in \mathfrak{g}_0, i = 1, \ldots, r\} \) are uniformly dense in \( W \) and \( \mathfrak{g} \) respectively. For example, in corollary 1 we could take \( M = W_0 \) and \( L = \mathfrak{g}_0 \). Since \( \mathfrak{g} \) contains a positive element it follows that \( \mathfrak{g}_0 \) contains a positive element which we may take as \( p \). We consider the following conditions:

(a) If \( x \in \partial_W(X), \varepsilon > 0 \) and \( f_1, f_2 \in \mathfrak{g}_0 \), then there exists \( g \in -\mathfrak{g} \) such that \( g < f_1 \land f_2 \) and \( f_1 \land f_2(x) < g(x) + \varepsilon \).

(a') Same as (a), but with \( g \in -\mathfrak{g}_0 \).

(b) If \( x_1 \) and \( x_2 \in \partial_W(X), \varepsilon > 0 \) and \( 0 < f \in W_0 \), then there exists \( g \in \mathfrak{g}_0 \) such that \( |f(x_i) - g(x_i)| < \varepsilon, i = 1, 2 \).

Suppose that \( \mathfrak{g}_0 \) satisfies condition (a). Then there exists \( \{h_1, \ldots, h_n\} \subset -\mathfrak{g}_0 \) such that \( g \leq h_1 \lor \cdots \lor h_n < f_1 \land f_2 \).
Then $h_i < f_1 \land f_2$ and $f_1 \land f_2(x) < h_i(x) + \varepsilon$ for some $i$ with $1 \leq i \leq n$. Therefore $(a)$ implies $(a')$ and since $(a')$ implies $(a)$, the two conditions are equivalent.

**Lemma 5.** — $\delta_{w}(X) \subset \delta_{\mathcal{G}}(X)$ if and only if $\mathcal{G}_0$ satisfies condition $(a)$.

**Proof.** — It follows from (1) that $x \in \delta_{\mathcal{G}}(X)$ if and only if whenever $f \in \mathcal{G}$ there exists $g \in - \mathcal{G}$ with $g < f$ and $f(x) < g(x) + \varepsilon$. Therefore the condition is necessary.

If $\mathcal{G}_0$ satisfies condition $(a)$ then it satisfies $(a')$. Consider $x \in \delta_{w}(X), \varepsilon > 0$ and $f \in \mathcal{G}$. If $\delta > 0$ choose $\{f_i, \ldots, f_n\} \subset \mathcal{G}$ such that $|f - f_1 \land \cdots \land f_n| < \delta$. Let

$$c = \min \{f_i(x) : i = 1, \ldots, n\}.$$ 

By condition $(a')$ there exists $k \in \mathcal{G}_0$ such that $k(x) = -c$ and $\{g_1, \ldots, g_n\} \subset - \mathcal{G}_0$ such that

$$g_i < (f_i + k) \land 0, \quad g_i(x) > -\varepsilon/n, \quad i = 1, \ldots, n.$$ 

Then

$$g_0 = \sum\{g_i : i = 1, \ldots, n\}$$

$$< (f_1 + k) \land \cdots \land (f_n + k) = f_1 \land \cdots \land f_n + k,$$

and $g_0(x) > -\varepsilon$. Therefore $g_0 - k = h \in - \mathcal{G}_0$ and $h < f_1 \land \cdots \land f_n < f + \delta$ with $h(x) > c - \varepsilon > f(x) - \delta - \varepsilon$.

Choosing $\delta$ such that $\delta(1 + p(x)) < \varepsilon$ and then putting $g = h - \delta p$ it follows that $g < f$ and $g(x) > f(x) - 2\varepsilon$. It follows from (1) that $x \in \delta_{\mathcal{G}}(X)$ and that $\delta_{w}(X) \subset \delta_{\mathcal{G}}(X)$.

**Lemma 6.** — $\delta_{w}(X) \subset \delta_{\mathcal{G}}(X)$ and $\mathcal{G}$ distinguishes points of $\delta_{w}(X)$ which are distinguished by $W$ if and only if $\mathcal{G}_0$ and $W_0$ satisfy conditions $(a)$ and $(b)$.

**Proof.** — If $W$ distinguishes the points $x_1$ and $x_2$ of $\delta_{w}(X)$, then there exists $f \in W$ such that

$$f(x_1)p(x_2) \neq f(x_2)p(x_1).$$

Since $p \in W$, we may assume that $f > 0$. If $\mathcal{G}_0$ satisfies condition $(b)$ and $\varepsilon < 0$, then there exists $g \in \mathcal{G}_0$ such that $|g(x_i) - f(x_i)| < \varepsilon$, $i = 1, 2$. If $\varepsilon$ is small enough, then $g(x_1)p(x_2) \neq g(x_2)p(x_1)$, and $\mathcal{G}$ distinguishes $x_1$ and $x_2$. 
If $\mathcal{S}_0$ also satisfies condition (a) then $\delta_w(X) \subset \delta_S(X)$, by lemma 4.

Conversely, suppose that $x_1, x_2 \subset \delta_w(X)$, $\varepsilon > 0$ and $0 < f \in W_0$. We consider the following cases:

(i) $f(x_1)p(x_2) = f(x_2)p(x_1)$. Choose real $c$ such that $cp(x_1) = f(x_1)$ and $cp(x_2) = f(x_2)$. Then $cp = g \in \mathcal{S}_0$ and $|f(x_i) - g(x_i)| = 0 < \varepsilon$, $i = 1, 2$.

(ii) $f(x_1)p(x_2) < f(x_2)p(x_1)$. If $\delta_w(X) \subset \delta_S(X)$ and $\mathcal{S}$ distinguishes points of $\delta_w(X)$ distinguished by $W$, then $\mathcal{S}$ distinguishes $x_1$ and $x_2$, and $x_1$ belongs to a minimal $\mathcal{S}$-face $A$. Then the function $0^* \in \mathcal{S}$. It follows from lemma 1 that there exists $k \in \mathcal{S}$ such that $k(x_1) < 0$ and $k(x_2) > 0$. Since $\mathcal{S}_0$ is a wedge containing $p$, there exists $h \in \mathcal{S}_0$ such that $h(x_1) = 0$ and $h(x_2) > 0$. Define $g \in \mathcal{S}_0$ by the formula

$$g = \frac{f(x_1)}{p(x_1)} p + \frac{f(x_2)p(x_1) - f(x_1)p(x_2)}{f(x_1)h(x_2)} h.$$

Then $|f(x_i) - g(x_i)| = 0 < \varepsilon$, $i = 1, 2$, and $W_0$ and $\mathcal{S}_0$ satisfy the conditions (a) and (b).

**Application to axiomatic potential theory.**

Let $\omega$ be an open relatively compact MP subset [4, § 2] of a harmonic space which satisfies one of the axiomatic systems [4, $H_0$, ..., $H_4$] [3, $A_1$, ..., $A_3$]. Let

$$W = \{ f \in C(\omega): f \text{ is superharmonic in } \omega \},$$

$$\mathcal{S} = \{ f \in C(\omega): f \text{ extends to a function superharmonic in an open neighbourhood } U_f \text{ of } \omega \},$$

and define $L$ and $M$ as in the introduction. Then $\mathcal{S} \subset W$ are min-stable wedges in $C(\omega)$, $M$ is the space of continuous $W$-affine functions, and $L$ is the space of continuous $\mathcal{S}$-affine functions on $\omega$. We suppose that $\mathcal{S}$ contains a positive function $p$ and a negative function $q$, and distinguishes points of $\omega^*$. We have

**Lemma 7.** If $A$ is a minimal $W$-face of $\omega$, then $A \cap \omega^* \neq \emptyset$.

**Proof.** The function $O_A$ belongs to $\hat{W}$ and is therefore hyperharmonic [4, § 1]. Suppose $A \cap \omega^* = \emptyset$, then $O_{\lambda}^* - p$
is non-negative on $\omega \setminus A$, and for any point $x_0 \in \omega^*$, \[
\liminf \{(0^\omega - p)(x) : x \to x_0\} = \infty.
\] Since $\omega$ is an MP set, $0^\omega - p > 0$ and therefore $A = \emptyset$. Therefore $A \cap \omega^* \neq \emptyset$.

We now recall the definitions and some properties of regular and stable points of $\omega^*$. If $f \in C(\omega^*)$ put $\Phi^o_f = \{\nu : \nu$ is hyperharmonic in $\omega\}$ and

\[
\liminf \{\nu(x) : x \in \omega, \ x \to x_0\} \geq f(x_0), \ x \in \omega^*\},
\]
put $\bar{H}^\omega_f = \inf \{\nu : \nu \in \Phi^o_f\}$, and put $\bar{H}^\omega_f = - \bar{H}^\omega_{-f}$. Since $(\mathcal{A} - \mathcal{B})|_{\omega^*}$ is uniformly dense in $C(\omega^*)$ it may be shown as in [7, ch. VIII, § 3] [14] [3, Satz 24], that $\bar{H}^\omega_f = \bar{H}^\omega_{-f} = H^\omega_f$ whenever $f \in C(\omega^*)$. Moreover $f \mapsto H_f$ is a linear map from $C(\omega^*)$ to the bounded continuous functions on $\omega$, which is continuous for the supremum norms. A point $x_0 \in \omega^*$ is regular if $\lim \{H_f(x) : x \in \omega, \ x \to x_0\} = f(x_0)$ whenever $f \in C(\omega^*)$. Since $(\mathcal{A} - \mathcal{B})|_{\omega^*}$ is dense in $C(\omega^*)$ and the map $f \mapsto H_f$ is continuous, $x_0$ is regular if and only if $\lim \{H_f(x) : x \in \omega, \ x \to x_0\} = f(x_0)$ whenever $f \in - \mathcal{B}|_{\omega^*}$.

If $f \in C(\omega^*)$ then put $\Psi^o_f = \{\nu : \nu$ is hyperharmonic in a neighbourhood of $\omega\}$ and

\[
\liminf \{\nu(x) : x \in \omega, \ x \to x_0\} \geq f(x_0)\},
\]
put $K^o_f = \inf \{\nu : \nu \in \Psi^o_f\}$ and put $K^o_f = - K^o_{-f}$. As in [6, § 2] it may be shown that $K^o_f = K^o_{-f} = K^o_f$, a continuous function on $\overline{\omega}$, harmonic in $\omega$, whenever $f \in C(\omega^*)$. The map $f \mapsto K^o_f$ is a linear map from $C(\omega^*)$ to $C(\overline{\omega})$ continuous for the supremum norms. If $f(x) = K^o_f(x)$ whenever $f \in C(\omega^*)$ then $x$ is a stable point of $\omega^*$. As with regular points, $x$ is stable if and only if $f(x) = K^o_f(x)$ whenever $f \in - \mathcal{B}|_{\omega^*}$.

Suppose that $F \in - \mathcal{B}$, and let $F$ be a continuous subharmonic function defined on an open neighbourhood $U_F$ of $\overline{\omega}$, which equals $F$ on $\overline{\omega}$. If $\overline{\omega} = \bigcap \{\omega_i : i \in I\}$ the intersection of a decreasing filtering family of open subsets of $U_F$, then (by an abuse of language) $\{H^\omega_{\mathcal{B}} : i \in I\}$ is a decreasing filtering family in $L$, and $K_F = \inf \{H^\omega_{\mathcal{B}} : i \in I\}$ [6, § 2]. If $x_0 \in \omega^*$ is stable, then

$$F(x_0) = \inf \{H^\omega_{\mathcal{B}}(x_0) : i \in I\} \geq \inf \{h(x_0) : F < h \in \mathcal{B}\}.$$
so that \( x_0 \in \partial g(\omega) \) by (1). Conversely, if \( x_0 \in \partial g(\omega) \cap \omega^* \) and \( F \in -\mathcal{F}, \ G \in \mathcal{F} \) with \( F < G \), then \( F|_{\omega_i} < G|_{\omega_i} \) for some \( i \in I \). Therefore \( F < H^\omega_i < G \) on. Therefore \((\omega, \mathcal{F}, L)\) is a geometric simplex [11, prop. 5] [5, p. 521]. It follows that \( F(\omega) = \inf g(x_0) : F < g \in \mathcal{F} \geq \inf \{ H^\omega_i(x_0) : i \in I \} \geq F(x_0) \). Therefore \( x_0 \) is stable and the following lemma holds.

**Lemma 7.** — *The set of stable points of \( \omega^* \) is precisely \( \partial g(\omega) \cap \omega^* \).*

**Example 2.** — *The classical case.* Let \( \omega \) be a bounded open subset of \( \mathbb{R}^n, n > 1 \). The affine functions on \( \mathbb{R}^n \) are harmonic, \( \partial_M(\omega) \) is precisely the set of regular points of \( \omega^* \), while \( \partial_L(\omega) \) is precisely the set of stable points of \( \omega^* \). Since \( L \) contains the constant functions, separates the points of \( \overline{\omega} \), and has the weak R.s.p., the following theorem is an immediate consequence of theorem 2.

**Theorem 5.** — *L is uniformly dense in \( M \) if and only if every regular point of \( X \) is stable.*

We now return to the general case.

**Theorem 6.** — *If every regular point of \( \omega^* \) is stable, then \( L \) is uniformly dense in \( M \).*

**Proof.** — Suppose \( x_i \) belongs to the minimal W-face \( A_i, i = 1, 2 \). Since \( \mathcal{F} \) distinguishes points of \( \omega^* \) it follows from lemma 3, that \( A_i \cap \omega^* \) is a one point set \( \{y_i\} \). If \( F \in -\mathcal{F} \) and \( f = F|_{\omega^*} \) then \( \inf \{ G : G \in \omega, F < G \} \geq H^\omega_f \geq F \) on \( \omega \). Since \( y_i \in \partial_W(\overline{\omega}) \), \( F(y_i) = \inf \{ G(y_i) : G \in W, F < G \} \). Therefore \( \lim \{ H_f(x) : x \in \omega, x \to y_i \} = f(y_i) \), and \( y_i \) is regular. Therefore \( y_i \) is stable. By lemma 7 there exist minimal \( \mathcal{F} \)-faces \( B_i, \ y_i \in B_i, \ i = 1, 2 \). Since \( A_i \cap B_i \neq \emptyset \) and \( A_i \) is minimal, \( A_i \subset B_i \). Therefore \( \partial_W(\overline{\omega}) \subset \partial_{\mathcal{F}}(\overline{\omega}) \). If \( \omega \) distinguishes \( x_1 \) and \( x_2 \) then by lemma 3 \( \omega \) distinguishes \( y_1 \) and \( y_2 \), and \( y_1 \neq y_2 \). Therefore \( \mathcal{F} \) distinguishes \( y_1 \) and \( y_2 \) so that \( B_1 \neq B_2 \), and \( \mathcal{F} \) distinguishes \( x_1 \) and \( x_2 \). It follows from theorem 4 that \( L \) is uniformly dense in \( M \).

Boboc and Cornea [5, th. 4.3], with the additional hypothesis that \( \omega \) is weakly determining, show that \( (\overline{\omega}, W, M) \) is a
geometric simplex, and that the set of regular points of $\omega^*$ is precisely $\delta_w(\overline{\omega}) \cap \omega^*$. In this case we have a complete generalisation of theorem 5 to axiomatic potential theory.

**Corollary 2.** — If $\omega$ is weakly determining, then $L$ is uniformly dense in $M$ if and only if every regular point of $\omega^*$ is stable.

**Proof.** — If $x$ is a regular point of $\overline{\omega}$ then $x \in \delta_\omega(\overline{\omega})$ [5, th. 4.3]. $(\omega, W, M)$ is a geometric simplex so by proposition 1, $x \in \delta_M(\overline{\omega})$. If $L$ is dense in $M$, then $L$-faces are $M$-faces, and $x$ belongs to a minimal $L$-face $A$. Since $(\overline{\omega}, \emptyset, L)$ is a geometric simplex, it follows from proposition 1 and lemma 4 that $A$ contains a unique minimal $\emptyset$-face $B$ and a unique minimal $W$-face $C$. Therefore $x \in C \subseteq B$, so that $x \in \delta_\emptyset(\overline{\omega})$ and $x$ is stable by lemma 7. The corollary is now an immediate consequence of theorem 6.

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