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THEORY OF BESSEL POTENTIALS. PART III [1]. POTENTIALS ON REGULAR MANIFOLDS

by R. D. ADAMS, N. ARONSZAJN and M. S. HANNA.

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Introduction.

The present part of the « Theory of Bessel Potentials » contains Chapter IV dealing with potentials on regular Riemannian manifolds. The next (and last) part to be published, Part IV, will contain Chapter V treating potentials on manifolds with singularities.

The classes of potentials $P^{\alpha}_{\mathfrak{M}}$ on a manifold $\mathfrak{M}$, or rather the classes $H^{\alpha}_{\mathfrak{M}}$ which are the classes $P^{\alpha}_{\mathfrak{M}}$ saturated relative to the class of sets of measure zero, have been used extensively in recent years (1). However, they were introduced essentially for compact C^{∞} manifolds or for compact bordered C^{∞} manifolds. In these cases the Riemannian metric on the manifold is not so essential since for different Riemannian metrics on such manifolds we obtain the same classes $P^{\alpha}_{\mathfrak{M}}$ with possibly changed (but equivalent) norm. Not so anymore is the case of a non-compact manifold, when the class $P^{\alpha}_{\mathfrak{M}}$ as well as its norm depend essentially on the metric. Therefore, the natural setting for the theory of Bessel potentials on a general manifold is the class of Riemannian manifolds, i.e., differentiable manifolds with *fixed* Riemannian metric. Our aim here is to develop the theory of the classes $P^{\alpha}_{\mathfrak{M}}$ for regular (C^{∞}) Riemannian manifolds (or bordered manifolds).

In Section 1 we consider for a manifold $\mathfrak{M}$ the classes $P^{\alpha}_{loc}(\mathfrak{M})$ which are easily defined and investigated by transfer to local coordinate patches and use of the corresponding classes in Euclidean domains as introduced in Chapter II, Part I.

(1) For instance, in the questions connected with the Atiyah-Singer index theorem.

In Section 2 we introduce the classes of potentials $P^{\alpha}_{\mathfrak{M}}$ on $\mathfrak{M}$. We take the natural and direct definition of $P^{m}_{\mathfrak{M}}$ for integral m and use quadratic interpolation to define $P^{\alpha}_{\mathfrak{M}}$ for $m < \alpha < m+1$. This kind of definition is not as direct as would be desirable but from the theoretical point of view it is the easiest to handle. Despite the fact that quadratic interpolation was already introduced and used by several authors (see for instance [2, 9]), in view of our specific needs we found it necessary to describe this interpolation method in Appendix I, especially as concerns interpolation between functional spaces. We needed also in this section the notion of equivalence of two different Riemannian metrics g and $\hat{g}$ on the same manifold, which is a sufficient (and possibly necessary) condition for equality of the two spaces $P^{\alpha}_{\mathfrak{M},g}$ and $P^{\alpha}_{\mathfrak{M},\hat{g}}$ for all $\alpha \geqslant 0$ when $\mathfrak{M}$ is provided with the two metrics. This notion of equivalent Riemannian metrics is investigated in Appendix II. Among the several propositions in this section we will mention Prop. 7 which allows us to define more directly the classes $P^{\alpha}_{\mathfrak{M}}$ in cases when the manifold can be covered by coordinate patches satisfying rather strong restrictions. In Prop. 8, under much weaker assumptions than in Prop. 7, we obtain the result that $P^{m+1}_{\mathfrak{M}}$ is dense in $P^{m}_{\mathfrak{M}}$ (in the norm of $P^{m}_{\mathfrak{M}}$). The problem is open if this result holds for all manifolds (2).

Section 3 is concerned with k-dimensional Riemannian submanifolds $\mathfrak{N}$ of the n-dimensional manifold $\mathfrak{M}$ (3). We consider restrictions u' to $\mathfrak{N}$ of functions $u \in P^{\alpha}_{\mathfrak{M}}$ and also extensions of functions u' defined on $\mathfrak{N}$. If $k = n$ it is obvious that restriction from $\mathfrak{M}$ to $\mathfrak{N}$ transforms $P^{\alpha}_{\mathfrak{M}}$ into $P^{\alpha}_{\mathfrak{N}}$ with bound $\leqslant 1$; we give sufficient conditions on $\mathfrak{N}$ that there exist a bounded linear extension mapping from $P^{\alpha}_{\mathfrak{N}}$ into $P^{\alpha}_{\mathfrak{M}}$. For the case $k < n$, we give sufficient conditions that the restriction map transform $P^{\alpha}_{\mathfrak{M}}$ boundedly into $P^{\alpha-(n-k)/2}_{\mathfrak{N}}$, $\alpha > (n-k)/2$, and sufficient conditions that there exist a bounded linear extension map from $P^{\alpha-(n-k)/2}_{\mathfrak{N}}$ into $P^{\alpha}_{\mathfrak{M}}$, $\alpha > (n-k)/2$. Since we use in this section simul-

(2) The density of $P^{m+1}_{\mathfrak{M}}$ in $P^{\alpha}_{\mathfrak{M}}$ for $m < \alpha < m+1$ results from the definition of quadratic interpolation.

(3) That is, C^{∞} submanifolds regularly imbedded in $\mathfrak{M}$ with the metric induced by the metric of $\mathfrak{M}$.

taneous extensions of functions in $P^{\alpha-(n-k)/2}(R^k)$ to functions in $P^{\alpha}(R^n)$ for all $\alpha > (n-k)/2$, we describe these extension mappings in Appendix III (4).

In Section 4 we investigate the classes of potentials on C^{∞} bordered manifolds. For a bordered manifold $\mathfrak{M}$ we define the classes $P^{\alpha}_{loc}(\mathfrak{M})$ and show that via the restriction map $P^{\alpha}_{loc}(\mathfrak{M})$ can be identified with a subspace of $P^{\alpha}_{loc}(\mathfrak{M}^i)$ where $\mathfrak{M}^i$ is the inner part of $\mathfrak{M}$. For a Riemannian bordered manifold $\mathfrak{M}$ we define the spaces $P^{\alpha}_{\mathfrak{M}}$ and show that the restriction map establishes an isometric isomorphism of $P^{\alpha}_{\mathfrak{M}}$ onto $P^{\alpha}_{\mathfrak{M}^i}$. Hence, in particular, each $u \in P^{\alpha}_{\mathfrak{M}^i}$ has « border values » $u' \in P^{\alpha-1/2}_{loc}(\partial\mathfrak{M})$, where $\partial\mathfrak{M}$ is the border of $\mathfrak{M}$. Also, in this section, for a C^{∞} bordered Riemannian manifold $\mathfrak{M}$, we introduce the notion of the regular completion of $\mathfrak{M}$ — in a sense the largest C^{∞} bordered Riemannian manifold containing $\mathfrak{M}$ as a dense subset.

In Section 5 we give a few examples answering questions connected with our considerations in the preceding sections. The first two examples show that if a domain D in Euclidean space is made into a Riemannian manifold by using the Euclidean metric, then P^{α}_{D} is in general different as well from $P^{\alpha}(D)$ as from $\check{P}^{\alpha}(D)$. Examples 3 and 4 show that, for a general k-dimensional submanifold $\mathfrak{N}$ of the n-dimensional manifold $\mathfrak{M}$, the restriction of $u \in P^{\alpha}_{\mathfrak{M}}$, $\alpha > (n-k)/2$, need not be in $P^{\alpha-(n-k)/2}_{\mathfrak{N}}$ and a function u' belonging to $P^{\alpha-(n-k)/2}_{\mathfrak{N}}$ need not have an extension $u \in P^{\alpha}_{\mathfrak{M}}$.

In the present part we did not put remarks concerning extensions of our results to potentials connected with L^p classes as we did in the preceding Part II. The treatment of the classes $P^{\alpha,p}_{\mathfrak{M}}$ is quite analogous to the treatment in the present paper of $P^{\alpha}_{\mathfrak{M}}$, which is the class $P^{\alpha,2}_{\mathfrak{M}}$. However, the formulas are much more complicated and the essential difference is that instead of using quadratic interpolation we have to use another method of interpolation (for instance the complex interpolation [5, 10]).

In the text we will refer to preceding parts of the « Theory of Bessel Potentials » without mentioning the number of the

(4) In Chapter II, Part I, we gave such extension mappings which were simultaneous extension mappings not for all $\alpha > (n-k)/2$ but for α in a fixed interval.

part, but only the number of the chapter, since in all parts the chapters are numbered in succession. We remind the reader that Part I [4] consists of Chapters I and II, Part II [1] of Chapter III, and the present Part III of Chapter IV. Thus, 3), § 9, II means Proposition 3 from § 9, Chapter II (of Part I).

CHAPTER IV

POTENTIALS ON REGULAR MANIFOLDS

1. $P^{\alpha}_{loc}(\mathfrak{M})$.

Throughout this section $\mathfrak{M}$ is an n-manifold with a C^{∞} structure. All definitions are made using a particular C^{∞} atlas $\{(U_i, h_i)\}$ for $\mathfrak{M}$. It is then shown that the concepts defined are independent of the particular atlas used. For convenience we use the notation $V_i = h_i(U_i)$.

For each $\alpha \geqslant 0$ we define $P^{\alpha}_{loc}(\mathfrak{M})$ to be the class of functions u on $\mathfrak{M}$ such that $u \circ h_i^{-1}$ belongs to $P^{\alpha}_{loc}(V_i)$ for each i. It follows from 3), § 9, II that $P^{\alpha}_{loc}(\mathfrak{M})$ is well-defined, i.e. does not depend on the atlas used in its definition. Similarly, for each $\alpha \geqslant 0$ we define $\mathfrak{A}_{2\alpha}(\mathfrak{M})$ to be the class of all subsets A of $\mathfrak{M}$ such that for each i, $h_i(A \cap U_i)$ belongs to $\mathfrak{A}_{2\alpha}$, the class of subsets R^n with 2α-capacity zero. $\mathfrak{A}_{2\alpha}(\mathfrak{M})$ is well-defined, by virtue of 20), § 6, II and the fact that a subset A of R^n belongs to $\mathfrak{A}_{2\alpha}$ iff each point of A has a neighborhood whose intersection with A belongs to $\mathfrak{A}_{2\alpha}$. The sets in $\mathfrak{A}_{2\alpha}(\mathfrak{M})$ are called the subsets of $\mathfrak{M}$ with 2α-capacity zero. $\mathfrak{A}_{2\alpha}(\mathfrak{M})$ is an exceptional class and $P^{\alpha}_{loc}(\mathfrak{M})$ is a saturated linear functional class rel. $\mathfrak{A}_{2\alpha}(\mathfrak{M})$. (For definitions of these terms see § 1, I.) Also we have the inclusion relations: $P^{\alpha}_{loc}(\mathfrak{M}) \subset P^{\beta}_{loc}(\mathfrak{M})$ and $\mathfrak{A}_{2\alpha}(\mathfrak{M}) \subset \mathfrak{A}_{2\beta}(\mathfrak{M})$ if $\beta \leqslant \alpha$. If $\alpha = 0$, we will write $L^{2}_{loc}(\mathfrak{M})$ for $P^{\alpha}_{loc}(\mathfrak{M})$ and « a.e. » for « exc. $\mathfrak{A}_{2\alpha}(\mathfrak{M})$ ». Sets in $\mathfrak{A}_0(\mathfrak{M})$ will be called sets of measure zero. By virtue of 2), § 2, II., we have the following proposition.

1) *If two functions in* $P^{\alpha}_{loc}(\mathfrak{M})$ *are equal a.e., they are equal exc.* $\mathfrak{A}_{2\alpha}(\mathfrak{M})$.

If D is an open set in R^n and k is an integer $\geqslant 0$, we say that $\varphi \in C^{(k,1)}_{loc}(D)$ iff $\varphi \in C^k(D)$ and every derivative of order $\leqslant k$ is locally Lipschitzian on D. For each integer $k \geqslant 0$, we define $C^{(k,1)}_{loc}(\mathfrak{M})$ to be the class of all functions φ on $\mathfrak{M}$ such that $\varphi \circ h_i^{-1} \in C^{(k,1)}_{loc}(V_i)$ for each i. Also for convenience we define $C^{(-1,1)}_{loc}(\mathfrak{M})$ to be the class of all functions φ on $\mathfrak{M}$ such that, for each i, $\varphi \circ h_i^{-1}$ is a locally essentially bounded measurable function on V_i. It is easily seen that $C^{(k,1)}_{loc}(\mathfrak{M})$ is well-defined. We sometimes write $L^{\infty}_{loc}(\mathfrak{M})$ for the class $C^{(-1,1)}_{loc}(\mathfrak{M})$. From 1), § 9, II. and 6), § 2, II. we obtain:

2) *If* $u \in P^{\alpha}_{loc}(\mathfrak{M})$, $\alpha \geqslant 0$, *and* $\varphi \in C^{(\alpha^*,1)}_{loc}(\mathfrak{M})$, *then* $\varphi u \in P^{\alpha}_{loc}(\mathfrak{M})$. (*Here* α^* *denotes the largest integer strictly less than* α.)

The classes $L^p_{loc}(\mathfrak{M})$, $1 \leqslant p < +\infty$, can be defined in the same way as $L^2_{loc}(\mathfrak{M})$ and $L^{\infty}_{loc}(\mathfrak{M})$. If $u \in L^1_{loc}(\mathfrak{M})$, we say that a point P on $\mathfrak{M}$ belongs to the Lebesgue set of u iff for some U_i containing P, the point $h_i(P)$ belongs to the Lebesgue set of $u \circ h_i^{-1}$. In case P is in the Lebesgue set of u, the Lebesgue correction of u at P, written $u^L(P)$, is defined to be $(u \circ h_i^{-1})^L(h_i(P))$. From 5), § 0, III it follows that the Lebesgue set of u and the Lebesgue correction of u are well-defined. Also, by classical theorems concerning Lebesgue corrections and by the results of § 0, III, we have the following four propositions:

3) *If* $u \in L^1_{loc}(\mathfrak{M})$, *then the complement of the Lebesgue set of* u *has measure zero and* $u^L(P) = u(P)$ *a.e.*

4) *If* $u, v \in L^1_{loc}(\mathfrak{M})$ *and* $u(P) = v(P)$ a.e., *then* $u^L = v^L$.

5) *If* $u \in P^{\alpha}_{loc}(\mathfrak{M})$, *then* $u^L \in P^{\alpha}_{loc}(\mathfrak{M})$ *and* $u^L(P) = u(P)$ *exc.* $\mathfrak{A}_{2\alpha}(\mathfrak{M})$.

6) *If* u *is equal a.e. to a function in* $P^{\alpha}_{loc}(\mathfrak{M})$, *then* $u^L \in P^{\alpha}_{loc}(\mathfrak{M})$.

If $\mathfrak{M}'$ is a C^{∞} p-manifold contained in $\mathfrak{M}$, we say that a coordinate patch U in $\mathfrak{M}$ agrees with $\mathfrak{M}'$ iff the points in $U \cap \mathfrak{M}'$ are characterized by the equations $x^k = x^k_0$, $k > p$, for some constants x^k_0, and $U \cap \mathfrak{M}'$ is a coordinate patch

in $\mathfrak{M}'$ with coordinates equal to the first p coordinates in U. We say that $\mathfrak{M}'$ is a submanifold of $\mathfrak{M}$ iff it can be covered by coordinate patches in $\mathfrak{M}$ which agree with it (5). If $\mathfrak{M}'$ is a p-dimensional submanifold of $\mathfrak{M}$, then 23), § 6 and § 8, 9 in Ch. II, give us the following:

7) *If* $A \in \mathfrak{A}_{2\alpha}(\mathfrak{M})$ *and* $2\alpha > n - p$, *then the set* $A' = A \cap \mathfrak{M}'$ *belongs to* $\mathfrak{A}_{2\alpha-(n-p)}(\mathfrak{M}')$.

8) *If* $u \in P^{\alpha}_{loc}(\mathfrak{M})$ *and* $2\alpha > n - p$, *then the restriction* u' *of* u *to* $\mathfrak{M}'$ *belongs to* $P^{\alpha-\frac{n-p}{2}}_{loc}(\mathfrak{M}')$.

We now suppose that $\mathfrak{M}$ has a positive definite Riemannian metric of class C^{∞}. If U is a coordinate patch in $\mathfrak{M}$ with coordinates $\{x_i\}$ we let $\{g_{ij}\}$ denote the components of the metric tensor on U with respect to these coordinates. The Riemannian metric induces on $\mathfrak{M}$ a structure of a measure space. A subset A of $\mathfrak{M}$ is measurable iff $h(A \cap U)$ is Lebesgue measurable for each coordinate patch (U, h). Also, if A is a measurable subset of $\mathfrak{M}$ which is contained in U, then the induced measure μ is given by:

$$\mu(A) = \int_{h(A)} \sqrt{g}(y)\, dy,$$

where as usual g denotes the determinant of the matrix $\{g_{ij}\}$. Since $\mathfrak{M}$ now has a measure space structure, the concepts $L^{p}_{loc}(\mathfrak{M})$, $1 \leqslant p \leqslant +\infty$, and « set of measure zero » now have a direct meaning. It is easily seen that the definitions of these concepts in terms of the measure space structure agree with the earlier definitions for manifolds not assumed to have a Riemannian metric.

In order to have a direct definition of corrections for functions on $\mathfrak{M}$, we use the concept of a normal coordinate neighborhood. For every point $P \in \mathfrak{M}$ there is an open neighborhood U such that (i) each point $Q \in U$ can be joined to P by a unique geodesic arc, and (ii) this arc is uniquely determined by its tangent vector at P. Hence, new coordinates $x_1, \ldots, x_n$ can be introduced in U such that the geodesic

(5) This definition is equivalent to the following: $\mathfrak{M}'$ is a submanifold of $\mathfrak{M}$ iff $\mathfrak{M}'$ is contained in $\mathfrak{M}$ and the injection map is of class C^{∞} and has a non-singular differential at every point.

arc through P with unit tangent vector T has the equations

$$x_i = t_i s, \qquad i = 1, \ldots, n,$$

where s is the arc length measured from P and the t_i are the components of T with respect to some fixed orthonormal basis. Such a neighborhood U together with the coordinates $x_1, \ldots, x_n$ is called a normal coordinate neighborhood of P. When U is such a coordinate neighborhood we use the notation $U(P, \rho)$ for the neighborhood of P defined by $\sum_{i=1}^{n} x_i^2 < \rho^2$.

We can now define the Lebesgue correction of a function $u \in L^1_{loc}(\mathfrak{M})$ directly. We say that a point $P \in \mathfrak{M}$ is a Lebesgue point of u iff there is a number $u^L(P)$ such that

$$\frac{1}{\mu[U(P, \rho)]} \int_{U(P, \rho)} |u(Q) - u^L(P)| \, d\mu(Q) \to 0$$

as $\rho \to 0$. In this case $u^L(P)$ is called the Lebesgue correction of u at P. It is easily checked that this direct definition of the Lebesgue correction agrees with the earlier definition for manifolds without Riemannian metrics.

We can also define an analogue on $\mathfrak{M}$ of the correction u^φ (see § 0, Ch. III). However, we first define a more general correction in Euclidean space which will be useful later (see the proof of 4), § 4). Let D be an open subset of R^n. For fixed $x \in D$ and each ρ, $0 < \rho \leqslant \rho_0(x)$, let $\varphi_\rho(x, y)$ be a measurable function of y defined for all $y \in R^n$ and such that

(i) $|\varphi_\rho(x, y)| \leqslant M_x \rho^{-n}$ for $0 < \rho \leqslant \rho_0(x)$, $y \in R^n$,

(ii) $\varphi_\rho(x, y) = 0$ for $|y - x| \geqslant C_x \rho$, $0 < \rho \leqslant \rho_0(x)$,

(iii) $\int_{R^n} \varphi_\rho(x, y) \, dy \to 1$ as $\rho \to 0$.

If $u \in L^1_{loc}(D)$, we define

$$u^{(\varphi)}(x) = \lim_{\rho \searrow 0} \int_{R^n} \varphi_\rho(x, y) u(y) \, dy,$$

provided the limit exists and is finite. If x is a Lebesgue point of u, then $u^{(\varphi)}(x)$ is defined and equals $u^L(x)$. Hence, in case $\varphi_\rho(x, y)$ is defined for all $x \in D$, $u^{(\varphi)}$ is an extension of u^L. Also, suppose that ψ is a bounded measurable function on R^n vanishing outside a compact set and having

$\int_{R^n} \psi\, dx = 1$; if we define $\varphi_\rho(x, y) = \rho^{-n}\psi\left(\frac{x-y}{\rho}\right)$ for each $x \in D$, then $u^{(\varphi)} = u^{\psi}$.

We now define an analogous correction $u^{(\varphi)}$ on $\mathfrak{M}$. For fixed $P \in \mathfrak{M}$ let $U(P, \rho_0(P))$ be a normal coordinate neighborhood of P and for each ρ, $0 < \rho \leqslant \rho_0(P)$, let $\varphi_\rho(P, Q)$ be a measurable function of Q defined for $Q \in \mathfrak{M}$ and such that:

(i) $|\varphi_\rho(P, Q)| \leqslant M_P \rho^{-n}$ for $0 < \rho \leqslant \rho_0(P)$, $Q \in \mathfrak{M}$.

(ii) $\varphi_\rho(P, Q)$ vanishes outside $U(P, C_P\rho)$, for

$$0 < \rho \leqslant \frac{1}{C_P}\rho_0(P).$$

(iii) $\int_{\mathfrak{M}} \varphi_\rho(P, Q)\, d\mu(Q) \to 1$ as $\rho \to 0$.

If $u \in L^1_{loc}(\mathfrak{M})$, we define

$$u^{(\varphi)}(P) = \lim_{\rho \searrow 0} \int_{\mathfrak{M}} \varphi_\rho(P, Q) u(Q)\, d\mu(Q),$$

provided the limit exists and is finite.

9) *If* $u \in L^1_{loc}(\mathfrak{M})$ *and* P *is a Lebesgue point of* u, *then* $u^{(\varphi)}(P)$ *exists and equals* $u^L(P)$.

Proof. — This follows from the direct definition of u^L and the fact that

$$\frac{\mu[U(P, \rho)]}{|S(0, \rho)|} \to 1 \qquad \text{as} \qquad \rho \to 0,$$

where $S(0, \rho)$ is the open ball in R^n with center 0 and radius ρ.

Thus, if $\varphi_\rho(P, Q)$ is defined for all $P \in \mathfrak{M}$, $u^{(\varphi)}$ is an extension of u^L.

10) *If* $u, v \in L^1_{loc}(\mathfrak{M})$ *and* $u(Q) = v(Q)$ *a.e., then* $u^{(\varphi)}(P)$ *exists iff* $v^{(\varphi)}(P)$ *exists, and in this case they are equal.*

There are various ways to choose the correcting function $\varphi_\rho(P, Q)$. One choice is:

$$\varphi(P, Q) = \frac{\chi_\rho(P, Q)}{\mu[U(P, \rho)]}$$

where $\chi_\rho(P, Q)$ is the characteristic function of $U(P, \rho)$. Another way to choose $\varphi_\rho(P, Q)$ is as follows: Let ψ be a

bounded measurable function on R^n vanishing outside a compact set and having $\int_{R^n} \psi \, dx = 1$; for each $P \in \mathfrak{M}$ let $(U(P, \rho_0(P)), h_P)$ be a fixed normal coordinate patch at P and define

$$\varphi_\rho(P, Q) = \rho^{-n} \psi\left(\frac{h_P(Q)}{\rho}\right).$$

An important advantage of the correction $u^{(\varphi)}$ over the Lebesgue correction u^L is that, while u^L corresponds roughly to the average of u over spheres, for suitable choice of $\varphi_\rho(P, Q)$, $u^{(\varphi)}$ can be the average of u over more general sets.

If u is any sufficiently smooth function on $\mathfrak{M}$, the Riemannian metric on $\mathfrak{M}$ enables us to define the tensor $\nabla^k u$ of the k^{th} order covariant derivatives of u. These are defined inductively starting with $\nabla^0 u = u$. If $\{A_{i_1 \dots i_k}\}$ are the components of $\nabla^k u$, then the components of $\nabla^{k+1} u$ are given by:

$$A_{i_1 \dots i_k l} = \frac{\partial A_{i_1 \dots i_k}}{\partial x^l} - \left\{ \begin{matrix} j \\ i_1 l \end{matrix} \right\} A_{j i_2 \dots i_k} - \left\{ \begin{matrix} j \\ i_2 l \end{matrix} \right\} A_{i_1 j \dots i_k} - \dots - \left\{ \begin{matrix} j \\ i_k l \end{matrix} \right\} A_{i_1 \dots i_{k-1} j}.$$

Here we use the usual summation convention; also, the $\left\{ \begin{matrix} j \\ il \end{matrix} \right\}$ are the Christoffel symbols defined by:

$$\left\{ \begin{matrix} j \\ il \end{matrix} \right\} = \frac{1}{2} g^{hj} \left(\frac{\partial g_{ih}}{\partial x^l} + \frac{\partial g_{lh}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^h} \right),$$

where $\{g^{hk}\}$ is the inverse maxtrix to $\{g_{ij}\}$. By the results of § 7, 9 in Ch. II, we have:

11) *If* $u \in P^\alpha_{loc}(\mathfrak{M})$ *and* $k \leqslant \alpha$, *then the components of* $\nabla^k u$ *with respect to the coordinates in any particular coordinate patch* (U, h), *considered as functions on* $V = h(U)$, *belong to* $P^{\alpha-k}_{loc}(V)$. *In particular, the tensor* $\nabla^k u$ *is defined exc.* $\mathfrak{A}_{2\alpha-2k}(\mathfrak{M})$.

The Riemannian metric defines a natural norm on the tensor spaces associated with $\mathfrak{M}$. In the particular case of the tensor $\nabla^k u$ the expression for the norm in terms of coordinates is:

$$|\nabla^k u|^2 = A_{i_1 \dots i_k} A_{j_1 \dots j_k} g^{i_1 j_1} \dots g^{i_k j_k},$$

where $\{A_{i_1 \dots i_k}\}$ are the components of $\nabla^k u$. It follows from the preceding proposition that if $u \in P^{\alpha}_{loc}(\mathfrak{M})$ and $k \leqslant \alpha$, then $|\nabla^k u|^2$ belongs to $L^1_{loc}(\mathfrak{M})$.

Remark. — Many of the notions discussed here extend to the case where $\mathfrak{M}$ has a C^m or $C^{(m,1)}_{loc}$ structure, $m \geqslant 0$. If $\mathfrak{M}$ has a $C^{(m,1)}_{loc}$ structure, we can define $P^{\alpha}_{loc}(\mathfrak{M})$ for $0 \leqslant \alpha \leqslant m+1$ and $C^{(k,1)}_{loc}(\mathfrak{M})$ for $0 \leqslant k \leqslant m$. In order to define $\mathfrak{A}_{2\alpha}(\mathfrak{M})$ for any $\alpha \geqslant 0$, we require only that $\mathfrak{M}$ have a $C^{(0,1)}_{loc}$ structure. Similarly, if $\mathfrak{M}$ has a $C^{(0,1)}_{loc}$ structure, we can define the classes $L^p_{loc}(\mathfrak{M})$, $1 \leqslant p \leqslant \infty$, and the Lebesgue correction of a function in $L^1_{loc}(\mathfrak{M})$. Provided $\mathfrak{M}$ has sufficient structure so that the classes involved are defined, propositions 1) through 6) remain true.

If $\mathfrak{M}'$ is a manifold contained in $\mathfrak{M}$ and if $\mathfrak{M}$ and $\mathfrak{M}'$ both have C^m or $C^{(m,1)}_{loc}$ structure, then our definition of submanifold still makes sense. Propositions 7) and 8) remain true provided the classes involved are still defined.

For $\mathfrak{M}$ to have a Riemannian metric, $\mathfrak{M}$ must have at least a C^1 structure; also, if $\mathfrak{M}$ has a C^m structure, $m \geqslant 1$, then a Riemannian metric on $\mathfrak{M}$ is at best of class C^{m-1}. If $\mathfrak{M}$ has at least a C^1 structure and a C^0 metric, then the metric induces a measure space structure on $\mathfrak{M}$ and the spaces $L^p_{loc}(\mathfrak{M})$, $1 \leqslant p \leqslant +\infty$, are the same as those defined without the use of a metric. If $\mathfrak{M}$ has at least a C^3 structure and a C^2 metric, then normal coordinate neighborhoods exist and belong to the natural C^1 structure on $\mathfrak{M}$. In this case the Lebesgue correction defined using normal coordinate neighborhoods is the same as that defined earlier without them, and propositions 9) and 10) hold.

If $\mathfrak{M}$ has a $C^{(m,1)}_{loc}$ structure and a $C^{(m-1,1)}_{loc}$ metric, $m \geqslant 1$, then proposition 11) holds for $\alpha \leqslant m+1$.

2. The space $P^{\alpha}_{\mathfrak{M}}(P^{\alpha}_{\mathfrak{M},g})$.

We shall assume that $\mathfrak{M}$ is a separable oriented Riemannian manifold with a C^{∞} metric g.

If m is an integer we definie $P^m_{\mathfrak{M}}$ as the subspace of $P^m_{loc}(\mathfrak{M})$ on which the m-norm, $|u|_{m,\mathfrak{M}}$ (defined below), is

finite; for non-integral α, $P^{\alpha}_{\mathfrak{M}}$ is defined by quadratic interpolation between $P^{m}_{\mathfrak{M}}$ and $P^{m+1}_{\mathfrak{M}}$ where $m = \alpha^*$. If we have two metrics g and $\hat{g}$ on the same manifold $\mathfrak{M}$ we shall add a suffix to prevent confusion, e.g. $P^{\alpha}_{\mathfrak{M},g}$ or $|u|_{\alpha,\mathfrak{M},g}$ [6].

For $u \in P^{m}_{loc}(\mathfrak{M})$ we define the *Dirichlet integral of order* m by:

$$d_{m,\mathfrak{M}}(u) = \int_{\mathfrak{M}} |\nabla^m_g u(x)|^2_g \sqrt{g}\, dx, \tag{2.1}$$

and the *m-norm* by:

$$|u|^2_{m,\mathfrak{M}} = \sum_{l=0}^{m} \binom{m}{l} d_{l,\mathfrak{M}}(u), \tag{2.2}$$

a hermitian quadratic norm.

If $D \subset R^n$ and e is the Euclidean metric then it is clear from the definitions in § 2, III that $P^{m}_{D,e} = \check{P}^m(D)$ and that the corresponding norms are equal.

1) *If* m *is an integer then* $P^{m}_{\mathfrak{M}}$ *is a complete functional space relative* $\mathfrak{A}_{2m}(\mathfrak{M})$ *and it is the perfect functional completion of* $C^{\infty}(\mathfrak{M}) \cap P^{m}_{\mathfrak{M}}$.

Proof. — Let $U \subset \mathfrak{M}$ be open and such that its closure is a compact subset of some coordinate patch (U_0, h) and set $V = h(U)$.

Now if $\{u_n\}$ is Cauchy in $P^{m}_{\mathfrak{M}}$, it is clear from Theorems I and II, App. II that $u_n \circ h^{-1}|_V$ is Cauchy in $P^{m}_{V,e} = \check{P}^m(V)$ and converges to a function in $\check{P}^m(V)$. From this it follows that $P^{m}_{\mathfrak{M}}$ is complete.

Let $\{U_k\}$ be a covering of $\mathfrak{M}$, each U_k having the same closure properties as U above. Then by considering $u_n|_{U_k}$ and the remarks of the previous paragraph it is easy to see that $P^{m}_{\mathfrak{M}}$ is a functional space rel. $\mathfrak{A}_{2m}(\mathfrak{M})$.

Suppose in addition that $\{U_k\}$ is locally finite. Let $\{\varphi_k\}$ be a partition of unity with $\varphi_k \in C^{\infty}_0(U_k)$. Then by Prop. 6), § 2, III, $(\varphi_k u) \circ h_k^{-1} \in \check{P}^m(h_k(U_k))$ for $u \in P^{m}_{\mathfrak{M}}$ and has compact support in $h_k(U_k)$. Therefore there is a $w_k \in C^{\infty}_0(h_k(U_k))$ such that $|(\varphi_k u) \circ h_k^{-1} - w_k|_{m,h_k(U_k),g} < \varepsilon/2^k$; $w_k \circ h_k$ extended by

[6] If $\mathfrak{M}$ is a domain in R^n we shall always add the suffix to prevent confusion with the standard α-norms, Cf. § 2, III.

0 to $\mathfrak{M}$ is clearly in $P^m_{\mathfrak{M}}$. Since $\{U_k\}$ is locally finite, $w = \Sigma w_k \circ h_k \in C^\infty(\mathfrak{M})$ and

$$|u - w|_{m,\mathfrak{M}} \leqslant \sum_k |\varphi_k u - w_k \circ h_k|_{m,\mathfrak{M}}$$
$$= \sum_k |\varphi_k u \circ h_k^{-1} - w_k|_{m,h_k(U_k),g} < \varepsilon,$$

which proves $C^\infty(\mathfrak{M}) \cap P^m_{\mathfrak{M}}$ is dense in $P^m_{\mathfrak{M}}$.

If $\{\psi_n\} \subset C_0^\infty(V)$ ($V = h(U)$ as considered in the first paragraph of this proof) are Cauchy in $\check{P}^m(V)$, and therefore in $P^m_{V,g}$, then $\psi_n \circ h$ extended by 0 to $\mathfrak{M}$ is Cauchy in $P^m_{\mathfrak{M}}$. From this we can see that there cannot be a functional completion of $C^\infty(\mathfrak{M}) \cap P^m_{\mathfrak{M}}$ relative a smaller exceptional class that $\mathfrak{A}_{2m}(\mathfrak{M})$. This completes the proof of Prop. 1).

From § 1 and (2.2) we see that $P^{m+1}_{\mathfrak{M}} \Subset P^m_{\mathfrak{M}}$ (7). Let G be the non-negative bounded operator assigned by the Lemma of Appendix I to the Hilbert subspace $P^{m+1}_{\mathfrak{M}}$ of $P^m_{\mathfrak{M}}$ ($P^{m+1}_{\mathfrak{M}}$ being saturated relative $\mathfrak{A}_{2m}(\mathfrak{M})$) and for $m < \alpha < m+1$ define $W^{(m)}_{\alpha-m}$ to be the Hilbert subspace of $P^m_{\mathfrak{M}}$ corresponding to $G^{\alpha-m}$. By Theorem III, Appendix I, $W^{(m)}_{\alpha-m}$ is the $(\alpha - m)$ — *th* interpolation space between $P^m_{\mathfrak{M}}$ and $P^{m+1}_{\mathfrak{M}}$.

THEOREM I. — $P^{m+1}_{\mathfrak{M}}$, *provided with the norm of* $W^{(m)}_{\alpha-m}$, *has a perfect functional completion relative* $\mathfrak{A}_{2\alpha}(\mathfrak{M})$. *This completion is denoted by* $P^\alpha_{\mathfrak{M}}$; *furthermore,* $P^\alpha_{\mathfrak{M}} = P^\alpha_{\mathrm{loc}}(\mathfrak{M}) \cap W^{(m)}_{\alpha-m}$.

Proof. — $P^{m+1}_{\mathfrak{M}}$ is dense in $W^{(m)}_{\alpha-m}$ and $W^{(m)}_{\alpha-m}$ is a functional space rel. $\mathfrak{A}_{2m}(\mathfrak{M})$. Hence, $W^{(m)}_{\alpha-m}$ is a functional completion of $P^{m+1}_{\mathfrak{M}}$ rel. $\mathfrak{A}_{2m}(\mathfrak{M})$. However, each equivalence class of functions in $W^{(m)}_{\alpha-m}$ rel. $\mathfrak{A}_{2m}(\mathfrak{M})$ contains a subclass of functions in $P^\alpha_{\mathrm{loc}}(\mathfrak{M})$. To see this, let u be a function in $W^{(m)}_{\alpha-m}$ and let $\{u_n\}$ be a sequence in $P^{m+1}_{\mathfrak{M}}$ which converges to u in the norm of $W^{(m)}_{\alpha-m}$. Let $\{(U_k, h_k)\}$ be a covering of $\mathfrak{M}$ by coordinate patches such that for each k the closure of U_k is a compact subset of a larger coordinate patch $(U'_k, h'_k)_k$, h is the restriction of h'_k, and

$$V_k = h_k(U_k) \in \mathfrak{E}([0, \infty))$$

(see § 7, III). It follows from Theorems I, II of Appendix II

(7) $P^{m+1}_{\mathfrak{M}} \Subset P^m_{\mathfrak{M}}$ means $P^{m+1}_{\mathfrak{M}} \subset P^m_{\mathfrak{M}}$ and $|u|_{m+1,\mathfrak{M}} \geqslant |u|_{m,\mathfrak{M}}$.

and Theorem II and Corollary 4′ of Appendix I that for each k the map $u \to u \circ h_k^{-1}$ transforms $P^l_{\mathfrak{M}}$ boundedly into $\check{P}^l(V_k)$, $l = m$, $m+1$, and $W^{(m)}_{\alpha-m}$ boundedly into $\check{P}^\alpha(V_k)$. Hence for each k the sequence $\{u_n \circ h_k^{-1}\}$ lies in $\check{P}^{m+1}(V_k)$ and is Cauchy with respect to the norm of $\check{P}^\alpha(V_k)$. By taking successive subsequences and using the diagonal process we get a subsequence $\{u_{n_i}\}$ such that for each k, $\{u_{n_i} \circ h_k^{-1}\}$ converges pointwise to a function in $\check{P}^\alpha(V_k)$ exc. $\mathfrak{A}_{2\alpha}(V_k)$. Hence the sequence $\{u_{n_i}\}$ converges pointwise to a function $u^* \in P^\alpha_{\mathrm{loc}}(\mathfrak{M})$ exc. $\mathfrak{A}_{2\alpha}(\mathfrak{M})$. It follows that $u = u^*$ exc. $\mathfrak{A}_{2m}(\mathfrak{M})$.

By 1), § 1, any two functions in $P^\alpha_{\mathrm{loc}}(\mathfrak{M}) \cap W^{(m)}_{\alpha-m}$ which are equal exc. $\mathfrak{A}_{2m}(\mathfrak{M})$ are necessarily equal exc. $\mathfrak{A}_{2\alpha}(\mathfrak{M})$. Hence the equivalence classes in $P^\alpha_{\mathrm{loc}}(\mathfrak{M}) \cap W^{(m)}_{\alpha-m}$ relative to equality exc. $\mathfrak{A}_{2\alpha}(\mathfrak{M})$ are in one-one correspondence with the equivalence classes in $W^{(m)}_{\alpha-m}$ relative to equality exc. $\mathfrak{A}_{2m}(\mathfrak{M})$. To show that $P^\alpha_{\mathrm{loc}}(\mathfrak{M}) \cap W^{(m)}_{\alpha-m}$ is a functional completion of $P^{m+1}_{\mathfrak{M}}$ rel. $\mathfrak{A}_{2\alpha}(\mathfrak{M})$, it remains only to check that it is a functional space rel. $\mathfrak{A}_{2\alpha}(\mathfrak{M})$. This is done easily by using the same procedure as in the preceding paragraph.

That there cannot be a functional completion of $P^{m+1}_{\mathfrak{M}}$ relative to a smaller exceptional class is verified in the same manner as the corresponding statement in Prop. 1).

Since $P^\alpha_{\mathrm{loc}}(\mathfrak{M}) \subset P^\beta_{\mathrm{loc}}(\mathfrak{M})$ and $W^{(m)}_{\alpha-m} \subseteq W^{(m)}_{\beta-m}$, we have $P^\alpha_{\mathfrak{M}} \subseteq P^\beta_{\mathfrak{M}}$ for $m \leqslant \beta \leqslant \alpha \leqslant m+1$. Since this also holds for α and β integers ($\beta \leqslant \alpha$), it holds for all α and β, ($0 \leqslant \beta \leqslant \alpha$).

Note that $P^\alpha_{\mathfrak{M}}$, $m < \alpha < m+1$, is the perfect functional interpolation space between $P^m_{\mathfrak{M}}$ and $P^{m+1}_{\mathfrak{M}}$ relative to the norm of $W^{(m)}_{\alpha-m}$, according to the terminology introduced in Remark 1, Appendix I. Also, note that $W^{(m)}_{\alpha-m} = P^\alpha_{\mathfrak{M}}$ saturated relative $\mathfrak{A}_{2m}(\mathfrak{M})$; therefore, we have by Prop. 6), § 1, that if $u \in W^{(m)}_{\alpha-m}$, then $u^{\mathrm{L}} \in P^\alpha_{\mathfrak{M}}$. This provides an alternative definition of $P^\alpha_{\mathfrak{M}}$ as $\{u^{\mathrm{L}} : u \in W^{(m)}_{\alpha-m}\}$ saturated rel. $\mathfrak{A}_{2\alpha}(\mathfrak{M})$.

If g and $\hat{g}$ are two C^∞ Riemannian metrics on $\mathfrak{M}$, we shall call them *equivalent* if there are constants $\Lambda_l \leqslant \Lambda_u$ and B_m, $m = 1, 2, \ldots$ such that

i) the eigenvalues of $\hat{g}$ relative g lie between Λ_l and Λ_u for all $x \in \mathfrak{M}$ and

ii) $\sup_{x\in\mathfrak{M}} \{N_m(g;\hat{g})(x)\} \leqslant B_m$ where

$$N_m(g;\hat{g})(x) = \max_{\nu=1,\ldots,m} \{(|\hat{g}^{-1}(x)|_g |\nabla_g^\nu \hat{g}(x)|_g)^{m/\nu}\}.$$

It follows from Theorem I, Appendix II that this is an equivalence relation.

2) *If* g *and* $\hat{g}$ *are equivalent then* $P^\alpha_{\mathfrak{M},g} = P^\alpha_{\mathfrak{M},\hat{g}}$ *for all* α *and*

$$C_{l,\alpha}|u|_{\alpha,\mathfrak{M},g} \leqq |u|_{\alpha,\mathfrak{M},\hat{g}} \leqslant C_{u,\alpha}|u|_{\alpha,\mathfrak{M},g}$$

where $C_{l,\alpha}$ *and* $C_{u,\alpha}$ *depend only on* $\Lambda_l, \Lambda_u, B_{\alpha^*+1}, \alpha^*$ *and* n (8).

Proof. — For α an integer, Theorem II, Appendix II supplies the proof. For non-integral α we apply Theorem II, Appendix I.

The terminology « multiplier » was introduced in § 1, III. For the analogous definition on manifolds we call $\varphi \in C^{(m,1)}_{\text{loc}}(\mathfrak{M})$ a *multiplier of order* m if

$$|\varphi|_{m+1,\infty,\mathfrak{M}} = \max_{0\leqslant l\leqslant m+1} \left\{\operatorname*{Ess\,Sup}_{x\in\mathfrak{M}} |\nabla_g^l \varphi(x)|_g\right\} < \infty \text{ (9)}.$$

(If $m = -1$ this means that

$$\varphi \in L^\infty(\mathfrak{M}) \quad \text{and} \quad |\varphi|_{0,\infty,\mathfrak{M}} = \operatorname*{Ess\,Sup}_{\mathfrak{M}} |\varphi(x)| < \infty.)$$

If $\varphi \in C^\infty(\mathfrak{M})$ and $|\varphi|_{m,\infty,\mathfrak{M}} < \infty$ for $m = 0, 1, \ldots$ then φ is a *multiplier of order* ∞.

3) *If* $u \in P^\alpha_{\mathfrak{M}}$ *and* φ *is a multiplier of order* α^* *then* $\varphi u \in P^\alpha_{\mathfrak{M}}$ *and*

$$|\varphi u|_{\alpha,\mathfrak{M}} \leqslant 3^{\alpha/2}|\varphi|_{\alpha^*+1,\infty,\mathfrak{M}}|u|_{\alpha,\mathfrak{M}}.$$

Proof. — By Prop. 2, § 1, $\varphi u \in P^\alpha_{\text{loc}}(\mathfrak{M})$. Let $\alpha = m$, an integer, let $l \leqslant m$, and let (U, h) be a coordinate patch. Then by the usual formulas for covariant differentiation we have:

$$[\nabla_g^l(\varphi u)]_i = \sum_{j'\cup j=i} [\nabla_g^{|j'|}\varphi]_{j'}[\nabla_g^{|j|}u]_j.$$

Here the left-hand side is the i^{th} component of the tensor $\nabla_g^l(\varphi u)$, where $i = (i_1, \ldots, i_l)$ is an indicial set. The summa-

(8) See Theor. II, App. II for a more precise expression for $C_{u,\alpha}$ and $C_{l,\alpha}$.

(9) If we wish to specify a particular metric we shall write $|\varphi|_{m,\infty,\mathfrak{M},g}$.

tion is over all partitions of the indicial set i into two complementary indicial sets j and j'. The symbol $|j|$ denotes the number of elements in the indicial set j. For a fixed integer k, there are exactly $\binom{l}{k}$ distinct partitions $j' \cup j = i$ where $|j| = k$. Hence, we have:

$$|\nabla_g^l(\varphi u)(x)|_g^2 \leq \left[\sum_{k=0}^{l}\binom{l}{k}|\nabla_g^{l-k}\varphi(x)|_g|\nabla_g^k u(x)|_g\right]^2$$

$$\leq 2^l|\varphi|_{l,\infty,\mathfrak{M}}^2\sum_{k=0}^{l}\binom{l}{k}|\nabla_g^k u(x)|_g^2 \text{ a.e.}$$

Thus

$$d_{l,\mathfrak{M}}(\varphi u) \leq 2^l|\varphi|_{l,\infty,\mathfrak{M}}^2\sum_{k=0}^{l}\binom{l}{k}d_{k,\mathfrak{M}}(u)$$

and

$$|\varphi u|_{m,\mathfrak{M}}^2 \leq |\varphi|_{m,\infty,\mathfrak{M}}^2\sum_{l=0}^{m}\left[\binom{m}{l}2^l\sum_{k=0}^{l}\binom{l}{k}d_{k,\mathfrak{M}}(u)\right]$$

$$\leq |\varphi|_{m,\infty,\mathfrak{M}}^2\sum_{k=0}^{m}\left[\sum_{l=k}^{m}2^l\binom{l}{k}\binom{m}{l}\right]d_{k,\mathfrak{M}}(u)$$

$$\leq 3^m|\varphi|_{m,\infty,\mathfrak{M}}^2|u|_{m,\mathfrak{M}}^2.$$

For non-integral α, an application of Theorem II, Appendix I completes the proof.

We shall call $\{(U_k, \varphi_k)\}$ *a uniform system in* $\mathfrak{M}$ *with constants* p *and* c_m, $m = 0, 1, \ldots$ *if:*

i) $(\overline{U}_k)^0 = U_k$ and $\{U_k\}$ has reduced rank p, i.e. every $x \in \mathfrak{M}$ is contained in at most p sets U_k [10].

ii) $\{\varphi_k\} \subset C^\infty(\mathfrak{M})$, $\varphi_k \geq 0$,
$\{x : \varphi_k \neq 0\} \subset U_k$ and $|\varphi_k|_{m,\infty,\mathfrak{M}} \leq c_m$, $m = 0, 1, \ldots$

(φ_k is a multiplier of order ∞).

There will be no confusion between the « uniform systems in $\mathfrak{M}$ » and the « uniform systems of q-cells » introduced in § 10, III.

If A is a subset of $\mathfrak{M}$ and the uniform system satisfies the additional condition:

iii) $\Sigma\varphi_k = 1$ on A $\left(\text{hence } A \subset \bigcup_k U_k\right)$,

[10] « Reduced rank p » is a weaker condition than « rank p » introduced in § 5, III.

then we shall call $\{(U_k, \varphi_k)\}$ *a uniform system in* $\mathfrak{M}$ *covering* A. If the U_k in the uniform system $\{(U_k, \varphi_k)\}$ are coordinate patches, i.e. possess a corresponding homeomorphism h_k into R^n, then we shall call $\{(U_k, h_k, \varphi_k)\}$ *a uniform system of coordinate patches.*

The uniform systems are the replacement on manifolds of the loose coverings with finite rank introduced in § 5, III. The remaining propositions of this section give some of their properties and applications.

4) *Let* $\{(U_k, \varphi_k)\}$ *be a uniform system in* $\mathfrak{M}$ *with constants* p *and* c_m. *We define the mappings* I_1 *and* I_2 *by:*

$$I_1: \quad P^0_{\mathfrak{M}} \to \Sigma\perp P^0_{U_k}, \qquad (I_1 u)_k = u|_{U_k}$$
$$I_2: \quad \Sigma\perp P^0_{U_k} \to P^0_{\mathfrak{M}}, \qquad I_2(u_1, \ldots, u_k, \ldots) = \Sigma \varphi_k u_k,$$

where $\varphi_k u_k$ *is extended by* 0 *outside* U_k. *Then for all* α, $I_1(P^\alpha_{\mathfrak{M}}) \subset \Sigma\perp P^\alpha_{U_k}$ *with bound* $\leqslant p^{1/2}$ *and* $I_2(\Sigma\perp P^\alpha_{U_k}) \subset P^\alpha_{\mathfrak{M}}$ *with bound* $\leqslant c_{\alpha^*+1} 3^{\alpha/2} p^{1/2}$.

Proof. — If $u \in P^\alpha_{\mathfrak{M}}$ then clearly $(I_1 u)_k \in P^\alpha_{\mathrm{loc}}(U_k)$ and for $\alpha = m$ an integer,

$$|I_1 u|^2_{\Sigma\perp P^m_{U_k}} = \sum_k |u|^2_{m, U_k} \leqslant p |u|^2_{m, \mathfrak{M}}.$$

The conclusion about I_1 now follows by Theorems II and V of Appendix I.

To prove our statement concerning the mapping I_2, we first prove the following three facts. (We use the notation $\varphi_k u_k$ to denote the extension of $\varphi_k u_k$ by zero outside U_k; no confusion will result from this).

(2.3) *If* $u_k \in P^m_{U_k}$, *then* $\varphi_k u_k \in P^m_{\mathrm{loc}}(\mathfrak{M})$.

(2.4) *If* $u_k \in P^m_{U_k}$, *then* $\nabla^l(\varphi_k u_k) = 0$ a.e. *outside* U_k *for* $0 \leqslant l \leqslant m$.

(2.5) *If* $u_k \in P^m_{U_k}$, *then* $(\varphi_k u_k)^{\mathrm{L}} = \varphi_k u_k$ *on* $\mathfrak{M}$. (We assume here that u_k is a corrected function.)

Let u_k be a corrected function in $P^m_{U_k}$. It is obvious that (2.3) and (2.5) hold on U_k and that (2.3), (2.4) and (2.5) hold on $(\mathfrak{M} - U_k)^0$. Consider a point $x \in \partial U_k$. Let U' be a neighborhood of x such that $\overline{U}'$ is a compact subset of a coordinate patch U with homeomorphism h, and such that $h(U')$ is an open sphere about $h(x)$. Let $V = h(U)$,

$V' = h(U')$, $D_1 = h(U_k \cap U')$, $D_2 = V' - \overline{D}_1$, and

$$D = [D_1 \cup D_2 \cup (\partial D_1 \cap \partial D_2)]^0.$$

By referring to the property $(\overline{U}_k)^0 = U_k$ we obtain easily that $D \supset V'$. Define

$$\rho(\xi) = u_k \circ h^{-1}(\xi) \qquad \text{for} \qquad \xi \in D_1$$

and

$$\psi(\xi) = \varphi_k \circ h^{-1}(\xi) \qquad \text{for} \qquad \xi \in V.$$

We have $\rho \in P^m_{D_1, g} = \check{P}^m(D_1)$, $\psi \in C^\infty(V)$, $|\psi|_{l, V', e} < +\infty$ for all l, and $D_i\psi(\xi) = 0$ for $\xi \in V' - D_1$ and all i. It follows that there exists a constant M such that

$$(2.6) \quad |D_i\psi(\xi)| < M r_{D_2}(\xi)^{m+n} \text{ (11)} \qquad \text{for} \qquad |i| \leqslant m, \quad \xi \in V'.$$

Formula (2.6) implies that $J_{m, D_1, D_2}(\psi\rho) < +\infty$. By Theorem I, part (a), § 9, III (and its proof) it follows that the extension of $\psi\rho$ to V' gotten by setting it zero outside D_1 is equivalent to a function in $\check{P}^m(V')$. For convenience we denote this extension by $\psi\rho$. By applying (2.6) we see that $(\psi\rho)^L = \psi\rho$ in V', so that $\psi\rho$ actually belongs to $\check{P}^m(V')$. Also, by using (2.6) and the Lebesgue correction we see that all derivatives $D_i(\psi\rho)$, $|i| \leqslant m$, are zero on $V' - D_1$ except possibly on a set of measure zero.

Next we prove a finite version of our statement about I_2, namely:

$$(2.7) \quad \textit{If} \quad u_k \in P^\alpha_{U_k} \quad \textit{for} \quad k = 1, \ldots, N, \quad \textit{and} \quad u = \sum_{k=1}^{N} \varphi_k u_k,$$

$$\textit{then} \quad u \in P^\alpha_{\mathfrak{M}} \quad \textit{and} \quad |u|^2_{\alpha, \mathfrak{M}} \leqslant p c^2_{\alpha^*+1} 3^\alpha \sum_{k=1}^{N} |u_k|^2_{\alpha, U_k}.$$

First suppose $\alpha = m$, an integer. Then for each k we have $\varphi_k u_k \in P^m_{\text{loc}}(\mathfrak{M})$ by (2.3), and

$$|\varphi_k u_k|_{m, \mathfrak{M}} = |\varphi_k u_k|_{m, U_k}$$

by (2.4). By applying Prop. 3 to the manifold U_k we see that $\varphi_k u_k \in P^m_{\mathfrak{M}}$ and

$$|\varphi_k u_k|_{m, \mathfrak{M}} \leqslant c_m 3^{m/2} |u_k|_{m, U_k}.$$

(11) $r_{D_2}(\xi)$ is the distance from ξ to D_2. For the definition of J_{m, D_1, D_2}, see § 9, III.

Statement (2.7) for the case $\alpha = m$ follows directly once we notice that due to (2.4) and the finite reduced rank of the system $\{U_k\}$ we have:

$$\left| \sum_{k=1}^{N} \varphi_k u_k \right|^2_{m, \mathfrak{M}} \leqslant p \sum_{k=1}^{N} |\varphi_k u_k|^2_{m, \mathfrak{M}}.$$

For the case $m < \alpha < m + 1$, Theorems II and V of Appendix I give that $u = \sum_{k=1}^{N} \varphi_k u_k$ belongs to $W^{(m)}_{\alpha - m}$ and satisfies

$$\|u\|^2_{W^{(m)}_{\alpha-m}} \leqslant p c^2_{\alpha+1} 3^\alpha \sum_{k=1}^{N} |u_k|^2_{\alpha, U_k}.$$

However, by (2.5) we have $u^L = u$, so $u \in P^\alpha_{\mathfrak{M}}$.

Finally, suppose $(u_1, u_2, \ldots) \in \sum_{k=1}^{\infty}{}^{\perp} P^\alpha_{U_k}$. If we define $w_N = \sum_{k=1}^{N} \varphi_k u_k$, then by (2.7) $\{w_N\}$ is a Cauchy sequence in $P^\alpha_{\mathfrak{M}}$. On the other hand, w_N converges pointwise to $u = \sum_{k=1}^{\infty} \varphi_k u_k$ everywhere on $\mathfrak{M}$. Hence, $u \in P^\alpha_{\mathfrak{M}}$; the bound for $|u|_{\alpha, \mathfrak{M}}$ follows directly.

Let $d_{g, \mathfrak{M}}(x, y)$ be the geodesic distance from x to y in $\mathfrak{M}$ with respect to the metric g (e.g. $d_{e, R^n}(x, y) = |x - y|$). U^δ was defined in § 1, III as $\{y : d_{e, R^n}(y, R^n - U) > \delta\}$. We now introduce a more general definition which contains this one. Let U be an open subset of $\mathfrak{M}$; we define

$$U^{\delta, g, \mathfrak{M}} = \{x : d_{g, \mathfrak{M}}(x, \mathfrak{M} - U) > \delta\}.$$

If $U \subset D \subset R^n$ then $U^{\delta, e, D} \supset U^{\delta, e, R^n}$ (and this may be a strict inclusion). For convenience, and consistency with Chapter III, we shall write U^δ for U^{δ, e, R^n} when $U \subset R^n$.

It was already remarked that the uniform systems in $\mathfrak{M}$ replace for Riemannian manifolds the notion of loose coverings of finite rank which was used extensively in Chapter III. The notions introduced above allow one to speak about a loose open covering $\{U_k\}$ of a set A in $\mathfrak{M}$. Namely, it would be a covering such that, for some $\delta > 0$, $A \subset \bigcup_k U^{\delta, g, \mathfrak{M}}_k$.

The next two propositions will show that if $\{U_k, \varphi_k\}$ is a uniform system in $\mathfrak{M}$ covering A, then actually $\{U_k\}$ is a loose covering of A (Prop. 5); whereas if we have a

loose covering $\{U_k\}$ of the whole of $\mathfrak{M}$, it is only under some additional strong assumptions that we are able to prove the existence of functions φ_k such that $\{U_k, \varphi_k\}$ is a uniform system covering $\mathfrak{M}$ (see Prop. 6). Since the use of loose coverings was important because they allowed one to define corresponding partitions of unity, we have to accept, in the case of manifolds, the replacement of loose coverings by uniform covering systems which have already built in the corresponding partitions of unity.

5) *If* $\{(U_k, \varphi_k)\}$ *is a uniform system covering* $\mathfrak{M}$ *with constants* p *and* c_m, *then* $\mathfrak{M} = \bigcup_k U_k^{\delta, g, \mathfrak{M}}$ *for any* $\delta < 1/(pc_1)$.

Proof. — Suppose $x_1 \in \mathfrak{M} - \bigcup_k U_k^{\delta, g, \mathfrak{M}}$. Since $\sum_k \varphi_k(x_1) = 1$ and there are at most p non-vanishing terms, there is a k_1 such that $\varphi_{k_1}(x_1) \geqslant 1/p$ and

$$d_{g, \mathfrak{M}}(x_1, \mathfrak{M} - U_{k_1}) = \inf_{y \in \mathfrak{M} - U_{k_1}} d_{g, \mathfrak{M}}(x_1, y) \leqslant \delta.$$

Let $\{x(t)\}_{0 \leqslant t \leqslant 1}$ be a C^1 arc in $\mathfrak{M}$ such that $x(1) = x_1$ and $x(0) = x_0 \in \mathfrak{M} - U_{k_1}$, i.e. $\varphi_{k_1}(x_0) = 0$.

Then, $\dot{x}(t)$ denoting the tangent vector $\frac{dx}{dt}$,

$$1/p \leqslant \varphi_{k_1}(x_1) - \varphi_{k_1}(x_0) = \int_0^1 [\dot{x}(t) \circ \nabla_g \varphi(x(t))]\, dt$$
$$\leqslant |\varphi_{k_1}|_{1, \infty, \mathfrak{M}} \times [\text{arc length of } \{x(t)\}].$$

Since the arc length can be made smaller than any $\delta' > \delta$, we have $\delta \geqslant 1/(pc_1)$, a contradiction.

6) *Let* $\{(U_k, h_k)\}$ *be a set of coordinate patches such that:*

i) $(\overline{U}_k)^0 = U_k$ *and* $\{U_k\}$ *has reduced rank* p,

ii) $\mathfrak{M} = \bigcup_k U_k^{\delta, g, \mathfrak{M}}$,

iii) g *is uniformly equivalent to* e *on each* $h_k(U_k)$, *i.e. the constants* Λ_l, Λ_u *and* B_m *in the equivalence relation are independent of* k,

iv) *there are constants* δ *and* $b \geqslant 1$, *independent of* k, *such that for any* $x, y \in h_k(U_k)$, *if* $|x - y| < \delta$, *then* $d_{g, h_k(U_k)}(x, y) \leqslant b|x - y|$.

Then there are φ_k'*s such that* $\{(U_k, h_k, \varphi_k)\}$ *is a uniform system of coordinate patches covering* $\mathfrak{M}$ *with constants* p *and* $c_m = c_m' p^m \Lambda_l^{-m/2}\left(B_m + \frac{b}{\delta}\right)^m$ *where* c_m' *depends only on* n *and* m.

Proof. — Let $V_k = h_k(U_k^{\delta, g, \mathfrak{M}})$ and $W_k = S\left(V_k, \frac{\delta}{2b}\right)$ (the $\frac{\delta}{2b}$ neighborhood of V_k in R^n). Then $\overline{V}_k \subset W_k^{\delta/4b}$. Suppose $x \in W_k \cap h_k(U_k - U_k^{\delta/2, g, \mathfrak{M}})$. Then by iv) and since $x \in W_k$,

$$\frac{\delta}{2} \geqslant b d_{e, R^n}(x, V_k) \geqslant d_{g, h_k(U_k)}(x, V_k) \geqslant d_{g, \mathfrak{M}}(h_k^{-1}(x), U_k^{\delta, g, \mathfrak{M}}).$$

Thus

$$d_{g, \mathfrak{M}}(h_k^{-1}(x), \mathfrak{M} - U_k) \geqslant d_{g, \mathfrak{M}}(\mathfrak{M} - U_k, U_k^{\delta, g, \mathfrak{M}}) - d_{g, \mathfrak{M}}(U_k^{\delta, g, \mathfrak{M}}, h_k^{-1}(x)) \geqslant \delta - \frac{\delta}{2}$$

contradicting the fact that $x \in h_k(U_k - U_k^{\delta/2, g, \mathfrak{M}})$ so that $W_k \cap h_k(U_k - U_k^{\delta/2, g, \mathfrak{M}}) = 0$.

By Lemma 1, § 1, III there is a $\psi_k \in C^\infty(R^n)$, $0 \leqslant \psi_k \leqslant 1$, $\psi_k = 1$ on $\overline{V}_k$, $= 0$ outside W_k; hence $\psi_k = 0$ on $h_k(U_k - U_k^{\delta/2, g, \mathfrak{M}})$ and $|D_i \psi_k(x)| \leqslant \left(\frac{\delta}{4b}\right)^{|i|} C_{|i|}$, $C_{|i|}$ depending only on n and $|i|$.

Therefore if we transfer ψ_k to U_k and extend it by 0 to $\mathfrak{M} - U_k$, $\psi_k \circ h_k \in C^\infty(\mathfrak{M})$. By i) and ii), $1 \leqslant \Sigma \psi_k \circ h_k \leqslant p$ so that the desired partition of unity is given by

$$\varphi_k(x) = \psi_k \circ h_k(x) \Big/ \sum_l \psi_l \circ h_l(x).$$

The inequality follows from (AII. 4) of Appendix II and an easy calculation.

7) *Let* $\{(U_k, h_k, \varphi_k)\}$ *be a uniform system of coordinate patches covering* $\mathfrak{M}$ *with constants* p *and* c_m *such that:*

i) g *is uniformly equivalent to* e *on each* $h_k(U_k)$, i.e. *the equivalence constants* Λ_l, Λ_u *and* B_m *are independent of* k,

ii) $V_k = h_k(U_k) \in \mathcal{E}([m, m+1])$ *with extension constant* $\Gamma = \Gamma([m, m+1])$ *independent of* k.

Then for $m \leqslant \alpha \leqslant m+1$, $u \in P_{\mathfrak{M}}^\alpha$ *if and only if*

$(u \circ h_1^{-1}, \ldots, u \circ h_k^{-1}, \ldots) \in \Sigma \perp \check{P}^\alpha(V_k)$ *and*

$$p^{-1}\Gamma^{-2}C_{u,\alpha}^{-2}|u|_{\alpha,\mathfrak{M}}^2 \leqslant \sum_k |u \circ h_k^{-1}|_{\alpha,V_k}^2 \leqslant 3^\alpha c_{\alpha^*+1}^2 p\Gamma^2 C_{l,\alpha}^{-2}|u|_{\alpha,\mathfrak{M}}^2$$

where $C_{u,\alpha}$ *and* $C_{l,\alpha}$ *are the constants given in Prop.* 2).

Proof. — By Prop. 2) and Corollary 4), Appendix I, $\check{P}^\alpha(V_k) = P^\alpha_{V_k,g}$. The proposition now follows from Prop. 4) (and the inequality follows from the inequalities in the cited propositions).

In general it is not known if $\check{P}^{m+1}(D)$ is dense in $\check{P}^m(D)$, $D \subset R^n$. In § 5, III we show that this property is a weakly localized boundary property. The next proposition gives an analogous, though weaker, result for $P^m_{\mathfrak{M}}$.

8) *If* $\{(U_k, \varphi_k)\}$ *is a uniform system covering* $\mathfrak{M}$ *such that* $P^{m+1}_{U_k}$ *is dense in* $P^m_{U_k}$ *then* $P^{m+1}_{\mathfrak{M}}$ *is dense in* $P^m_{\mathfrak{M}}$.

Proof. — If $u \in P^m_{\mathfrak{M}}$, then by Prop. 4), $\sum_k |u|^2_{m,U_k} < \infty$. Now by choosing $w_k \in P^{m+1}_{U_k}$, $|w_k - u|^2_{m,U_k} < \varepsilon/2^k$, we have by Prop. 4) $w = I_2(w_1, \ldots, w_{k_0}, 0, \ldots, 0, \ldots) \in P^{m+1}_{\mathfrak{M}}$ and by choosing k_0 sufficiently large we have the desired approximation.

Remark. — The notations and results of this section extend to the case where $\mathfrak{M}$ has a C^m or $C^{(m,1)}_{loc}$ structure by obvious modifications.

3. Restrictions and extensions.

In this section $\mathfrak{M}$ is a C^∞ Riemannian n-manifold and $\mathfrak{N}$ is a k-dimensional C^∞ submanifold of $\mathfrak{M}$ with the induced Riemannian metric. We consider restrictions to $\mathfrak{N}$ of potentials on $\mathfrak{M}$ and extensions of potentials on $\mathfrak{N}$ to potentials on $\mathfrak{M}$.

First considering the case $k = n$, we have:

THEOREM I. — *If* $k = n$, *then the restriction mapping* $u \to u_1 = u|_{\mathfrak{N}}$ *transforms* $P^\alpha_{\mathfrak{M}}$ *into* $P^\alpha_{\mathfrak{N}}$ *with bound* $\leqslant 1$, *for each* $\alpha \geqslant 0$.

Proof. — For integral α this statement follows by comparing the expressions (given by integrals) for $|u_1|_{\alpha,\mathfrak{N}}$ and

$|u|_{\alpha,\mathfrak{M}}$. For non-integral α it follows from Theorem II of Appendix I.

Clearly the restriction mapping $u \to u_1$ does not necessarily map $P^\alpha_{\mathfrak{M}}$ onto $P^\alpha_{\mathfrak{N}}$. For example, let $\mathfrak{M} = R^n$ and let $\mathfrak{N}$ be a domain in R^n which does not have the extension property (for instance, a domain with a cusp on its boundary). To get sufficient conditions for the existence of a linear extension mapping, we first state a theorem which allows us to localize the problem.

THEOREM II. — *Assume* $\alpha \geqslant 0$. *Let* $\{(U_i, \varphi_i)\}$ *be a uniform system in* $\mathfrak{M}$ *covering* $\mathfrak{N}$ *with constants* p *and* $\{c_m\}$. *If for each* i *there is a linear extension map of* $P^\alpha_{U_i \cap \mathfrak{N}}$ *into* $P^\alpha_{U_i}$ *with bound* $\leqslant M_E$, M_E *independent of* i, *then there is a linear extension map of* $P^\alpha_{\mathfrak{N}}$ *into* $P^\alpha_{\mathfrak{M}}$ *with bound* $\leqslant c_{\alpha^*+1} p 3^{\alpha/2} M_E$.

Proof. — Let $u \in P^\alpha_{\mathfrak{N}}$ and let $u_i = u|_{U_i \cap \mathfrak{N}}$. By 4), § 2, the sequence $(u_1, u_2, \ldots) \in \Sigma \perp P^\alpha_{U_i \cap \mathfrak{N}}$, so the sequence of extensions $(\tilde{u}_1, \tilde{u}_2, \ldots)$ belongs to $\Sigma \perp P^\alpha_{U_i}$. Applying 4), § 2, again, we see that the function $\tilde{u} = \Sigma \varphi_i \tilde{u}_i$ belongs to $P^\alpha_{\mathfrak{M}}$. The bound for $|\tilde{u}|_{\alpha,\mathfrak{M}}$ follows from 4), § 2.

Remark 1. — It is clear that if for each i there is a simultaneous linear extension map from $P^\alpha_{U_i \cap \mathfrak{N}}$ into $P^\alpha_{U_i}$ as α varies over an interval, then there is a simultaneous extension map from $P^\alpha_{\mathfrak{N}}$ into $P^\alpha_{\mathfrak{M}}$ as α varies over the same interval.

Theorem II can be applied, for example, in the case that the sets U_i are coordinate patches. In this case the question of existence of an extension mapping from $P^\alpha_{U_i \cap \mathfrak{N}}$ into $P^\alpha_{U_i}$ can be transferred to the image sets in R^n. The following proposition gives sufficient conditions for the existence of an extension mapping from $P^\alpha_{D_1, g}$ into $P^\alpha_{D, g}$, where D is an open subset of R^n, D_1 is an open subset of D, and $\{g_{ij}\}$ is a C^∞ Riemannian metric on D.

1) *If* $\{g_{ij}\}$ *is equivalent to the Euclidean metric on* D *with constants* Λ_l, Λ_u, *and* $\{B_m\}$, *and if* $D_1 \in \mathfrak{E}([m_1, m_2])$ *with extension constant* Γ_1, *then there is a simultaneous bounded linear extension map of* $P^\alpha_{D_1, g}$ *into* $P^\alpha_{D, g}$, $m_1 \leqslant \alpha \leqslant m_2$, *with bound* $\leqslant c\Gamma_1$, *where* c *depends only on* n, m_2, Λ_l, Λ_u, *and* B_{m_2}.

Proof. — For integral α the result is gotten, with the help

of 2), § 2, by first extending to R^n and then restricting to D. For non-integral α the result follows from Theorem II of Appendix I.

We now consider the case $k < n$. By 8), § 1, if $u \in P^{\alpha}_{\mathfrak{M}}$, $\alpha > \frac{n-k}{2}$, then $u' = u|_{\mathfrak{N}}$ belongs to $P^{\alpha-(n-k)/2}_{\mathrm{loc}}(\mathfrak{N})$; however, u' does not necessarily belong to $P^{\alpha-(n-k)/2}_{\mathfrak{N}}$ (see Ex. 3, § 5). Also, a function $u' \in P^{\alpha-(n-k)/2}_{\mathfrak{N}}$, $\alpha > \frac{n-k}{2}$, may have no extension $u \in P^{\alpha}_{\mathfrak{M}}$ (see Ex. 4, § 5). We give here sufficient conditions that the restriction mapping $u \to u'$ transform $P^{\alpha}_{\mathfrak{M}}$ into $P^{\alpha-(n-k)/2}_{\mathfrak{N}}$ and that there exist an extension mapping from $P^{\alpha-(n-k)/2}_{\mathfrak{N}}$ into $P^{\alpha}_{\mathfrak{M}}$. First we state a theorem similar to Theorem II which reduces these questions to local questions.

THEOREM III. — 1° *Let* $\{U_i\}$ *be open sets in* $\mathfrak{M}$ *and* $\{\psi_i\}$ *be* C^{∞} *functions on* $\mathfrak{N}$, *such that the collection* $\{U_i\}$ *has finite reduced rank* p *in* $\mathfrak{M}$ *and* $\{(U_i \cap \mathfrak{N}, \psi_i)\}$ *is a uniform system in* $\mathfrak{N}$ *covering* $\mathfrak{N}$ *with constants* p *and* $\{c_m\}$. *If for each* i *restriction from* U_i *to* $U_i \cap \mathfrak{N}$ *transforms* $P^{\alpha}_{U_i}$ *into* $P^{\alpha-(n-k)/2}_{U_i \cap \mathfrak{N}}$ *with bound* $\leqslant M_R$, M_R *independent of* i, *then restriction from* $\mathfrak{M}$ *to* $\mathfrak{N}$ *transforms* $P^{\alpha}_{\mathfrak{M}}$ *into* $P^{\alpha-(n-k)/2}_{\mathfrak{N}}$ *with bound* $\leqslant c_{\beta^*+1} p 3^{\beta/2} M_R$ *where* $\beta = \alpha - \frac{n-k}{2}$.

2° *Let* $\{(U_i, \varphi_i)\}$ *be a uniform system in* $\mathfrak{M}$ *covering* $\mathfrak{N}$ *with constants* p *and* $\{c_m\}$. *If for each* i *there is a linear extension map of* $P^{\alpha-(n-k)/2}_{U_i \cap \mathfrak{N}}$ *into* $P^{\alpha}_{U_i}$ *with bound* $\leqslant M_E$, M_E *independent of* i, *then there is a linear extension map of* $P^{\alpha-(n-k)/2}_{\mathfrak{N}}$ *into* $P^{\alpha}_{\mathfrak{M}}$ *with bound* $\leqslant c_{\alpha^*+1} p 3^{\alpha/2} M_E$.

Proof. — The proof of 2° is the same as the proof of Theorem II; the proof of 1° is similar.

Remark 2. — A remark similar to that after Theorem II applies here. If the common bound for the restriction maps of $P^{\alpha}_{U_i}$ into $P^{\alpha-(n-k)/2}_{U_i \cap \mathfrak{N}}$ is valid for all α in an interval, then so is the bound for the restriction map of $P^{\alpha}_{\mathfrak{M}}$ into $P^{\alpha-(n-k)/2}_{\mathfrak{N}}$. Also if for each i there is a simultaneous linear extension map from $P^{\alpha-(n-k)/2}_{U_i \cap \mathfrak{N}}$ into $P^{\alpha}_{U_i}$ as α varies over an interval, then there is a simultaneous linear extension map from $P^{\alpha-(n-k)/2}_{\mathfrak{N}}$ into $P^{\alpha}_{\mathfrak{M}}$ as α varies over the same interval.

In the case where the U_i are coordinate patches in $\mathfrak{M}$ agreeing with $\mathfrak{N}$, we can transfer questions about restrictions and extensions to the image sets in Euclidean space. We now prove two propositions dealing with these questions; however, for convenience we first prove a lemma.

LEMMA. — *Let* D *be an open set in* R^n *with a* C^∞ *Riemannian metric* $\{g_{ij}\}$ *which is equivalent to the Euclidean metric on* D *with constants* Λ_l, Λ_u, *and* $\{B_m\}$. *Then:*

(*a*) *Restriction from* R^n *to* D *transforms* $P^\alpha(R^n)$ *boundedly into* $P^\alpha_{D,g}$ *for each* $\alpha \geqslant 0$. *For* $0 \leqslant \alpha \leqslant m$ *this map has a simultaneous bound depending only on* m, n, Λ_l, Λ_u, *and* B_m.

(*b*) *If* $D \in \mathcal{E}([m_1, m_2])$ *with extension constant* Γ, *then there is a simultaneous linear extension map of* $P^\alpha_{D,g}$ *into* $P^\alpha(R^n)$, $m_1 \leqslant \alpha \leqslant m_2$, *with bound* $\leqslant c\Gamma$ *where* c *depends only on* m_2, n, Λ_l, Λ_u, *and* B_{m_2}.

Proof. — These assertions follow for integral values of α directly from 2), § 2. Theorem II of Appendix I then shows that they hold for non-integral α.

In the following two propositions D is an open subset of R^n with a C^∞ Riemannian metric $\{g_{ij}\}$ and $D' = D \cap R^k$. $\mathcal{E}'([m_1, m_2])$ is the class of domains in R^k having the extension property on the interval $[m_1, m_2]$.

2) *If* $\{g_{ij}\}$ *is equivalent to the Euclidean metric on* D *with constants* Λ_l, Λ_u, *and* $\{B_m\}$, *and if* $D \in \mathcal{E}([m_1, m_2])$ *with extension constant* Γ, *then restriction from* D *to* D′ *maps* $P^\alpha_{D,g}$ *boundedly into* $P^{\alpha-(n-k)/2}_{D',g}$ *for each* $\alpha \in [m_1, m_2]$ *such that* $\alpha > (n-k)/2$. *For* α *in any interval* $[\alpha_1, \alpha_2] \subset [m_1, m_2]$ *where* $\alpha_1 > (n-k)/2$, *this map has a simultaneous bound* $\leqslant c\Gamma$ *where* c *depends only on* n, k, m_2, α_1, Λ_l, Λ_u, *and* B_{m_2}.

Proof. — Starting with a function on D, we extend it to R^n, then restrict that function to R^k, and finally restrict the resulting function to D′. The above Lemma and Theorem 1*a*, § 8, II, justify these steps.

Remark 3. — The derivation of the bound in 2) shows that c approaches infinity as $\alpha_1 \searrow (n-k)/2$.

3) *If* $\{g_{ij}\}$ *is equivalent to the Euclidean metric on* D *with constants* Λ_l, Λ_u, *and* $\{B_m\}$, *and if* $D' \in \mathcal{E}'([m_1, m_2])$ *with*

extension constant Γ', *then there is a simultaneous bounded linear extension map of* $P^{\alpha-(n-k)/2}_{D',g}$ *into* $P^{\alpha}_{D,g}$ *for* α *in any interval* $[\alpha_1, \alpha_2] \subset [m_1 + (n-k)/2, m_2 + (n-k)/2]$ *such that* $\alpha_1 > (n-k)/2$. *The simultaneous bound is* $\leqslant c\Gamma'$ *where* c *depends on* n, k, m_2, Λ_l, Λ_u, *and* $B_{\alpha_2^*+1}$.

Proof. — The proof is similar to that of 2), except that the procedure is reversed. Starting with a function on D', we extend it to R^k, then extend that function to R^n, and finally restrict the resulting function to D. The Lemma of the present section and Theorem I of Appendix III justify the steps involved.

4. Bordered manifolds.

In this section we consider potentials defined on a C^∞ bordered n-manifold $\mathfrak{M}$. Such a manifold is defined in the same way as an unbordered one except that if (U, h) is a coordinate patch in $\mathfrak{M}$, then h maps U homeomorphically onto an open subset of $\overline{R}^n_+$ rather than onto an open subset of R^n. (We use the notation R^n_+ for the open half-space $x_n > 0$; hence $\overline{R}^n_+$ is the closed half-space $x_n \geqslant 0$. We denote by R^{n-1} the hyperplane $x_n = 0$ bounding $\overline{R}^n_+$). Points in $\mathfrak{M}$ which correspond via the coordinate homeomorphisms to points in R^n_+ are called inner points of $\mathfrak{M}$; points which correspond to points in R^{n-1} are called border points of $\mathfrak{M}$. The set of all inner points of $\mathfrak{M}$ we call the inner part of $\mathfrak{M}$ and denote by $\mathfrak{M}^i$; the set of all border points we call the border of $\mathfrak{M}$ and denote by $\partial\mathfrak{M}$. We remind the reader that $\mathfrak{M}^i$ and $\partial\mathfrak{M}$ form unbordered manifolds of dimensions n and $n-1$ respectively.

If D is a relatively open subset of $\overline{R}^n_+$, we say that $u \in P^{\alpha}_{loc}(D)$, $\alpha \geqslant 0$, iff u has an extension $\tilde{u}$ to an open subset $\tilde{D}$ of R^n containing D such that $\tilde{u} \in P^{\alpha}_{loc}(\tilde{D})$. It is easily checked (by using a partition of unity) that $u \in P^{\alpha}_{loc}(D)$ iff each point in D has a neighborhood (open in $\overline{R}^n_+$) on which u coincides with some function in $P^{\alpha}(R^n)$. Now suppose $\mathfrak{M}$ is a C^∞ bordered n-manifold (without a Riemannian metric); let $\{(U_k, h_k)\}$ be an atlas for $\mathfrak{M}$. For $\alpha \geqslant 0$ we define $P^{\alpha}_{loc}(\mathfrak{M})$ to be the class of functions u on $\mathfrak{M}$ such that, for each k,

$u \circ h_k^{-1} \in P^{\alpha}_{loc}(h_k(U_k))$. By using the second characterization of $P^{\alpha}_{loc}(D)$ given above it is easily proved that $P^{\alpha}_{loc}(\mathfrak{M})$ is well-defined. The exceptional class $\mathfrak{A}_{2\alpha}(\mathfrak{M})$ is defined as in the case of unbordered manifolds; $P^{\alpha}_{loc}(\mathfrak{M})$ is a saturated linear functional class rel. $\mathfrak{A}_{2\alpha}(\mathfrak{M})$.

We now consider restrictions of functions in $P^{\alpha}_{loc}(\mathfrak{M})$ to $\mathfrak{M}^i$ and $\partial\mathfrak{M}$.

1) *For* $\alpha \geqslant 0$ *the restriction map* $u \to u^i = u|_{\mathfrak{M}^i}$ *transforms* $P^{\alpha}_{loc}(\mathfrak{M})$ *into* $P^{\alpha}_{loc}(\mathfrak{M}^i)$. *Moreover, this map is one-one.*

Proof. — The first statement is immediate. To prove the one-one-ness, suppose $u, w \in P^{\alpha}_{loc}(\mathfrak{M})$ and $u^i = w^i$. Let (U, h) be a coordinate neighborhood of a border point and consider the functions $u \circ h^{-1}$ and $w \circ h^{-1}$. If φ is a bounded measurable function vanishing outside a compact subset of R^n, such that $\int \varphi\, dx = 1$ and $\varphi = 0$ for $x_n \leqslant 0$, then by the results of § 0, III,

$$(u \circ h^{-1})^{\varphi} = u \circ h^{-1} \quad \text{and} \quad (w \circ h^{-1})^{\varphi} = w \circ h^{-1} \text{ exc. } \mathfrak{A}_{2\alpha}(h(U)).$$

However, since

$$u \circ h^{-1} = w \circ h^{-1} \quad \text{for} \quad x_n > 0, \quad (u \circ h^{-1})^{\varphi} = (w \circ h^{-1})^{\varphi}.$$

Note that the mapping $u \to u^i$ of $P^{\alpha}_{loc}(\mathfrak{M})$ into $P^{\alpha}_{loc}(\mathfrak{M}^i)$ is not in general onto. Because of 1) we can use this mapping to identify $P^{\alpha}_{loc}(\mathfrak{M})$ with a subspace (proper, in general) of $P^{\alpha}_{loc}(\mathfrak{M}^i)$.

The proof of the following proposition is direct.

2) *For* $\alpha > 1/2$ *the restriction map* $u \to u' = u|_{\partial\mathfrak{M}}$ *transforms* $P^{\alpha}_{loc}(\mathfrak{M})$ *into* $P^{\alpha-1/2}_{loc}(\partial\mathfrak{M})$.

We call the function u' the border values of u. Hence, each function $u \in P^{\alpha}_{loc}(\mathfrak{M})$, $\alpha > 1/2$, has border values $u' \in P^{\alpha-1/2}_{loc}(\partial\mathfrak{M})$. The map $u \to u'$ is not necessarily one-one; however, the following proposition shows that it is onto.

3) *For* $\alpha > 1/2$ *there is a simultaneous linear extension map* $u' \to u$ *transforming* $P^{\alpha-1/2}_{loc}(\partial\mathfrak{M})$ *into* $P^{\alpha}_{loc}(\mathfrak{M})$.

Proof. — Let $\{(U_k, h_k)\}$ be a locally finite open covering of $\partial\mathfrak{M}$ by coordinate patches in $\mathfrak{M}$, and let $\{\varphi_k\}$ be a corresponding partition of unity such that $\Sigma\varphi_k = 1$ on $\partial\mathfrak{M}$. We may assume that for each k the closure of U_k is a compact

subset of a larger coordinate patch U_k' and h_k is the restriction of the corresponding homeomorphism h_k'. Let $\psi_k \in C_0^\infty(R^{n-1})$ be such that $\psi_k = 1$ on $h_k(U_k) \cap R^{n-1}$ and $\psi_k = 0$ outside a compact subset of $h_k'(U_k') \cap R^{n-1}$. By 1'), § 9, II, given $u' \in P_{loc}^{\alpha-1/2}(\partial\mathfrak{M})$, the function $\psi_k(u' \circ h_k'^{-1})$, extended by zero outside $h_k'(U_k') \cap R^{n-1}$, belongs to $P^{\alpha-1/2}(R^{n-1})$ and agrees with $u' \circ h_k^{-1}$ on $h_k(U_k) \cap R^{n-1}$. Extending this function to R^n by means of Theorem I, Appendix III, we obtain a function $\varphi_k \in P^\alpha(R^n)$ such that $\varphi_k = u' \circ h_k^{-1}$ on $h_k(U_k) \cap R^{n-1}$. The function $\varphi_k(\varphi_k \circ h_k)$, extended by zero outside U_k, belongs to $P_{loc}^\alpha(\mathfrak{M})$. Hence, $u = \Sigma\varphi_k(\varphi_k \circ h_k) \in P_{loc}^\alpha(\mathfrak{M})$ and $u = u'$ on $\partial\mathfrak{M}$.

Remark 1. — We say that a function $u \in P_{loc}^\alpha(\mathfrak{M}^i)$, $\alpha > 1/2$, has border values iff there exists a $u' \in P_{loc}^{\alpha-1/2}(\partial\mathfrak{M})$ such that the function $\tilde{u}$ defined by $\tilde{u} = u$ on $\mathfrak{M}^i$, $\tilde{u} = u'$ on $\partial\mathfrak{M}$, belongs to $P_{loc}^\alpha(\mathfrak{M})$. It is clear from 1) that if a function $u \in P_{loc}^\alpha(\mathfrak{M}^i)$ has border values, they are unique. Also we see that the subspace of functions in $P_{loc}^\alpha(\mathfrak{M}^i)$ having border values is exactly the subspace which we have identified with $P_{loc}^\alpha(\mathfrak{M})$.

Now assume that the bordered manifold $\mathfrak{M}$ has a C^∞ Riemannian metric. For m an integer $\geqslant 0$ we define $P_{\mathfrak{M}}^m$ to be the subspace of $P_{loc}^m(\mathfrak{M})$ on which the norm $|u|_{m,\mathfrak{M}}$ (defined in the same way as in the unbordered case) is finite. $P_{\mathfrak{M}}^m$ (with the norm $|u|_{m,\mathfrak{M}}$) is a complete functional space rel. $\mathfrak{A}_{2m}(\mathfrak{M})$. For $m < \alpha < m+1$ we define $P_{\mathfrak{M}}^\alpha$ by quadratic interpolation between $P_{\mathfrak{M}}^{m+1}$ and $P_{\mathfrak{M}}^m$. That is, if $W_{\alpha-m}^{(m)}$ is the $(\alpha - m) - th$ interpolation space between $P_{\mathfrak{M}}^{m+1}$ (saturated rel. $\mathfrak{A}_{2m}(\mathfrak{M})$) and $P_{\mathfrak{M}}^\alpha$, then $P_{\mathfrak{M}}^{m+1}$ (provided with the norm of $W_{\alpha-m}^{(m)}$) has a perfect functional completion rel. $\mathfrak{A}_{2\alpha}(\mathfrak{M})$; denoting this completion by $P_{\mathfrak{M}}^\alpha$, we have $P_{\mathfrak{M}}^\alpha = P_{loc}^\alpha(\mathfrak{M}) \cap W_{\alpha-m}^{(m)}$. With these definitions we have the following theorem.

THEOREM I. — *For* $\alpha \geqslant 0$ *the restriction map* $u \to u^i = u|_{\mathfrak{M}^i}$ *is an isometric isomorphism of* $P_{\mathfrak{M}}^\alpha$ *onto* $P_{\mathfrak{M}^i}^\alpha$.

Proof. — Suppose $\alpha = m$, an integer. It is immediate that the mapping $u \to u^i$ transforms $P_{\mathfrak{M}}^m$ isometrically into $P_{\mathfrak{M}^i}^m$. To show that this mapping is onto $P_{\mathfrak{M}^i}^m$, it is enough to show that each $w \in P_{\mathfrak{M}^i}^m$ has an extension $\tilde{w} \in P_{loc}^m(\mathfrak{M})$. To do this,

let $w \in P^m_{\mathfrak{M}^i}$, let $\{(U_k, h_k)\}$ be a locally finite covering of $\mathfrak{M}$ by coordinate patches, and let $\{\varphi_k\}$ be a corresponding partition of unity. We may assume that, for each k, $\overline{U}_k$ is a compact subset of a larger coordinate patch U'_k and h_k is the restriction of the corresponding homeomorphism h'_k. Let $\psi_k \in C^\infty(\overline{R}^n_+)$ be such that $\psi_k = 1$ on $h_k(U_k)$ and $\psi_k = 0$ outside a compact subset of $h'_k(U'_k)$. Then $\psi_k(w \circ h'^{-1}_k)$, extended by zero outside $h'_k(U'_k) \cap R^n_+$, belongs to $\check{P}^m(R^n_+)$ and hence has an extension in $P^m(R^n)$. Thus the function $w_k = w|_{U_k \cap \mathfrak{M}^i}$ has an extension $\tilde{w}_k \in P^m_{\mathrm{loc}}(U_k)$. Setting $\tilde{w} = \Sigma \varphi_k \tilde{w}_k$, we get that $\tilde{w} \in P^m_{\mathrm{loc}}(\mathfrak{M})$ and $\tilde{w} = w$ on $\mathfrak{M}^i$.

For non-integral α the theorem follows by Theorem II of Appendix I.

By Theorem I and Prop. 2) each $u \in P^\alpha_{\mathfrak{M}^i}$, $\alpha > 1/2$, has border values $u' \in P^{\alpha-1/2}_{\mathrm{loc}}(\partial\mathfrak{M})$. Moreover, we can give a formula expressing u' in terms of u. To do this we introduce the notion of a normal coordinate neighborhood of a border point P of $\mathfrak{M}$. First, let $U'(P, \rho)$ be a normal coordinate neighborhood of P in $\partial\mathfrak{M}$, where we consider $\partial\mathfrak{M}$ as a Riemannian $(n-1)$ — manifold with the metric induced by $\mathfrak{M}$, and let $x_1, \ldots, x_{n-1}$ be the coordinates in $U'(P, \rho)$. From each $Q \in U'(P, \rho)$ there issues a unique geodesic arc in $\mathfrak{M}$ normal to $\partial\mathfrak{M}$. Also, for each $\rho \leqslant$ some fixed ρ_0 there exists a $\sigma > 0$ such that the normal geodesic arcs up to length σ are mutually disjoint and cover an open neighborhood of P. In this neighborhood we choose the coordinates $x_1, \ldots, x_n$ where $x_1, \ldots, x_{n-1}$ are as before and x_n is the arc length along geodesics normal to $\partial\mathfrak{M}$. Such a coordinate neighborhood we call a normal coordinate neighborhood of P. We use the notation $U(P, \rho)$ for the particular normal coordinate neighborhood defined by:

$$\sum_{i=1}^{n-1} x_i^2 < \rho^2, \qquad 0 \leqslant x_n < \rho.$$

4) *Let* $u \in P^\alpha_{\mathfrak{M}^i}$, $\alpha > 1/2$, *and let* u' *be the border values of* u. *Then:*

$$u'(P) = \lim_{\rho \to 0} \frac{1}{\mu[U(P, \rho)]} \int_{U(P, \rho)} u(Q)\, d\mu(Q) \tag{4.1}$$

for all $P \in \partial\mathfrak{M}$ *exc.* $\mathfrak{A}_{2\alpha-1}(\partial\mathfrak{M})$.

Proof. — Since we assume that $\mathfrak{M}$ satisfies the second axiom of countability, it suffices to prove that each point $P_0 \in \partial\mathfrak{M}$ has a neighborhood U such that (4.1) holds for all $P \in U \cap \partial\mathfrak{M}$ exc. $\mathfrak{A}_{2\alpha-1}(U \cap \partial\mathfrak{M})$. Fix a point $P_0 \in \partial\mathfrak{M}$ and let (U, h) be a coordinate neighborhood of P_0. If $u \in P^{\alpha}_{\mathfrak{M}^i}$, $\alpha > 1/2$, then we know that $\rho = u \circ h^{-1}$ has an extension $\tilde{\rho}$ to an open subset $\tilde{V}$ of R^n containing $V = h(U)$, such that $\tilde{\rho} \in P^{\alpha}_{loc}(\tilde{V})$. We will show that if x is a Lebesgue point of $\tilde{\rho}$, $x \in V \cap R^{n-1}$, and $P = h^{-1}(x)$, then the limit in (4.1) exists and equals $\tilde{\rho}^L(x)$. To do this we apply the generalized correction defined in § 1. For y near x define

$$\varphi_\rho(x, y) = \frac{\chi_{W(x,\rho)}(y)\sqrt{g(y)}}{\mu[U(P, \rho)]},$$

where $W(x, \rho)$ is the image of $U(P, \rho)$ under h. That $\varphi_\rho(x, y)$ satisfies the conditions (i)-(iii) for a correcting function stated in § 1 follows directly once we notice that there exist positive constants c_1 and c_2 (depending on x) such that

$$H(x, c_1\rho) \subset W(x, \rho) \subset H(x, c_2\rho),$$

where $H(x, \rho)$ denotes the half-sphere

$$\{y : |x - y| < \rho, 0 \leqslant y_n\}.$$

If $u \in P^{\alpha}_{\mathfrak{M}}$, $\alpha > 1/2$, the border values u' are not necessarily in $P^{\alpha-1/2}_{\partial\mathfrak{M}}$. (Example 3, § 5, can be modified to show this.) Also, if $u' \in P^{\alpha-1/2}_{\partial\mathfrak{M}}$, $\alpha > 1/2$, u' may not have an extension $u \in P^{\alpha}_{\mathfrak{M}}$. (Example 4, § 5, can be modified to give an example of this.) Sufficient conditions that the restriction map $u \to u'$ transform $P^{\alpha}_{\mathfrak{M}}$ into $P^{\alpha-1/2}_{\partial\mathfrak{M}}$ and sufficient conditions that there exist a bounded linear extension map transforming $P^{\alpha-1/2}_{\partial\mathfrak{M}}$ into $P^{\alpha}_{\mathfrak{M}}$ can be obtained by methods similar to those used in § 3.

We conclude this section with a discussion of the problem of completion of a bordered Riemannian manifold. Myers and Steenrod in their paper [11] and more recently and precisely Palais in his note [12] have proved that, given a metric space X with a distance function d, there exists at most one *unbordered* C^∞ Riemannian manifold structure on X such that the geodesic metric agrees with d. Moreover, given a metric space (X, d), there is at most one *bordered* C^∞ Riemannian

manifold structure on X such that the geodesic metric agrees locally with d [12]. From this fact it follows that, given a metric space (X, d) and given $n > 0$ there is a uniquely defined largest open subset U of X having a bordered C^∞ Riemannian n-manifold structure with geodesic metric agreeing locally with d.

Now let $\mathfrak{M}$ be a C^∞ bordered Riemannian n-manifold, let $\tilde{\mathfrak{M}}$ be the abstract completion of $\mathfrak{M}$ with respect to the geodesic metric, and let $\tilde{d}$ be the metric on $\tilde{\mathfrak{M}}$. We denote by $\mathfrak{M}^*$ the largest open subset of $\tilde{\mathfrak{M}}$ having a bordered C^∞ Riemannian n-manifold structure with geodesic metric agreeing locally with $\tilde{d}$. Since $\mathfrak{M}$ is an open subset of $\tilde{\mathfrak{M}}$, $\mathfrak{M} \subset \mathfrak{M}^*$. Moreover, $\mathfrak{M}^i$ is a submanifold of $(\mathfrak{M}^*)^i$ and $\partial\mathfrak{M}$ is a submanifold of $\partial\mathfrak{M}^*$. Also, since $\mathfrak{M}^i$ is dense in $\mathfrak{M}$, we have $(\mathfrak{M}^i)^* = \mathfrak{M}^*$. We call $\tilde{\mathfrak{M}}$ the *full completion* of $\mathfrak{M}$ and $\mathfrak{M}^*$ the *regular completion* of $\mathfrak{M}$. For example, if $\mathfrak{M}$ is an open square in the plane with the usual Riemannian structure, then $\tilde{\mathfrak{M}}$ is the closed square while $\mathfrak{M}^*$ is the closed square minus the corner-points.

As an example of the use of the of notion regular completion, suppose that $\mathfrak{M}$ is an (unbordered) C^∞ Riemannian n-manifold with a compact regular completion $\mathfrak{M}^*$ such that $(\mathfrak{M}^*)^i - \mathfrak{M}$ has $(n-1)$-dimensional measure zero (for instance, $\mathfrak{M} =$ an open disk in the plane with the usual Riemannian structure). Then each $u \in P^\alpha_{\mathfrak{M}}$, $\alpha > 1/2$, determines unique border values $u' \in P^{\alpha-1/2}_{\partial\mathfrak{M}^*}$. Also, the class $P^{m+1}_{\mathfrak{M}}$ is dense in $P^m_{\mathfrak{M}}$ for each integer $m \geqslant 0$. These statements are proved by using Theorem I′, § 3, III, and the fact that we can cover $\mathfrak{M}^*$ by a finite number of coordinate patches which are as regular as we please.

5. Examples.

Example 1.

From Corollary 4′ of Appendix I it follows that if D is an open subset of R^n belonging to $\mathcal{E}([m, m+1])$, then

[12] In addition, one can give metric properties on (X, d) which are necessary and sufficient for the existence of such a structure. These results will be proved in a later paper.

$\check{P}^{\alpha}(D) = P^{\alpha}_{D,e}$ for $m \leqslant \alpha \leqslant m+1$ (and the norms on these spaces are equivalent for each α). Here we give an example of an open set $D \subset R^1$ such that $\check{P}^{\alpha}(D) \neq P^{\alpha}_{D,e}$ for $1/2 \leqslant \alpha < 1$.

Let D be the domain obtained by removing an interior point from a finite open interval. Let D_1 and D_2 be the two components of D, and for any function u on D let u_1 and u_2 be the restrictions of u to D_1 and D_2 respectively. By Theorem V of Appendix I, the correspondence $u \longleftrightarrow \{u_1, u_2\}$ defines an isometric isomorphism between $P^{\alpha}_{D,e}$ and $P^{\alpha}_{D_1,e} \pm P^{\alpha}_{D_2,e}$ for each $\alpha \geqslant 0$. Also, if $u \in \check{P}^{\alpha}(D)$, then $\{u_1, u_2\} \in \check{P}^{\alpha}(D_1) \pm \check{P}^{\alpha}(D_2)$ and $\sum_k |u_k|^2_{\alpha, D_k} \leqslant c|u|^2_{\alpha, D}$. Since for $k = 1, 2$ and all $\alpha \geqslant 0$ the spaces $\check{P}^{\alpha}(D_k)$ and $P^{\alpha}_{D_k,e}$ coincide and have equivalent norms,

$$\check{P}^{\alpha}(D) \underset{c}{\subset} P^{\alpha}_{D,e} \quad \text{for} \quad \alpha \geqslant 0, \quad (\underset{c}{\subset} \text{ means continously imbedded}).$$

On the other hand, if $1/2 \leqslant \alpha < 1$, the function $u \equiv 0$ on D_1, $\equiv 1$ on D_2, belongs to $P^{\alpha}_{D,e}$ but not to $\check{P}^{\alpha}(D)$.

Example 2.

This is another example where $\check{P}^{\alpha}(D) \neq P^{\alpha}_{D,e}$; in this case D is connected.

Let D be an annulus (in the plane) which is slit along a radius. By introducing polar coordinates relative to the center of the annulus and the slit, we define a homeomorphism T of D onto a rectangle D^*, such that T and T^{-1} are C^{∞} with bounded derivatives. T defines a correspondence between functions u on D and v on D^* given by: $u(x) = v(Tx)$. From the properties of T and the Interpolation Theorem of Appendix I, it follows that for each $\alpha \geqslant 0$, $u \in P^{\alpha}_{D,e}$ iff $v \in P^{\alpha}_{D^*,e}$ and in this case their norms are equivalent. Also, since T^{-1} is Lipschitzian, $u \in \check{P}^{\alpha}(D)$ implies $v \in {}^{\alpha}\check{P}(D^*)$ and

$$|v|_{\alpha, D^*} \leqslant c|u|_{\alpha, D}, \qquad \alpha \geqslant 0.$$

Since the spaces $P^{\alpha}_{D^*,e}$ and $\check{P}^{\alpha}(D^*)$ are equal and have equivalent norms, $\check{P}^{\alpha}(D) \underset{c}{\subset} P^{\alpha}_{D,e}$ for $\alpha \geqslant 0$. On the other

hand, if $1/2 \leqslant \alpha < 1$, a smooth function u which $\equiv 1$ near the upper edge of the slit and $\equiv 0$ near the lower edge of the slit, belongs to $P^{\alpha}_{D,e}$ but not to $\check{P}^{\alpha}(D)$.

Example 3.

Let $\mathfrak{M}$ be an n-dimensional Riemannian manifold, let $\mathfrak{N}$ be a k-dimensional submanifold, and suppose $\alpha > (n-k)/2$. Theorem III, § 3, gives sufficient conditions that restriction from $\mathfrak{M}$ to $\mathfrak{N}$ transform $P^{\alpha}_{\mathfrak{M}}$ boundedly into $P^{\alpha-(n-k)/2}_{\mathfrak{N}}$. In the example below this restriction property does not hold.

Let $f(x)$ be a positive continuous function on $-\infty < x < +\infty$ such that $f(x) \to 0$ as $|x| \to +\infty$. Let $\mathfrak{M}$ be the open set in the plane: $|y| < f(x)$, $-\infty < x < +\infty$, with the Euclidean metric, and let $\mathfrak{N}$ be the x-axis. Assume that f is chosen so that $\mathfrak{M}$ has finite area. Then the function $u \equiv 1$ belongs to $P^{1}_{\mathfrak{M}}$, but its restriction u' to $\mathfrak{N}$ does not belong to $P^{1/2}_{\mathfrak{N}}$ (since in particular $u' \notin P^{0}_{\mathfrak{N}}$).

Example 4.

Again, let $\mathfrak{M}$ be an n-dimensional Riemannian manifold, let $\mathfrak{N}$ be a k-dimensional submanifold, and assume $\alpha > (n-k)/2$. Theorem III, § 3, gives sufficient conditions for the existence of a bounded linear extension map of $P^{\alpha-(n-k)/2}_{\mathfrak{N}}$ into $P^{\alpha}_{\mathfrak{M}}$. Here we give an example where such an extension map does not exist.

Let $\mathfrak{M} = R^3$ with coordinates x, y, z. Let $\mathfrak{N}$ be the surface gotten by revolving a curve $y = f(x)$ about the x-axis. Assume that $f(x)$ is positive and C^{∞} on $-\infty < x < +\infty$, that $f(x) \to 0$ as $|x| \to +\infty$ in such a way that $\mathfrak{N}$ has finite area, and that $f^{(k)}(x)$ is bounded on $-\infty < x < +\infty$ for each $k \geqslant 0$ (for example, take $f(x) = e^{-x^2}$). The intersection of $\mathfrak{N}$ with the (x, y) — plane consists of two symmetric generating curves. Let C denote the curve lying in the half-plane $y > 0$.

The function $u' \equiv 1$ on $\mathfrak{N}$ clearly belongs to $P^{1}_{\mathfrak{N}}$. On the other hand, u' does not have an extension $u \in P^{3/2}_{\mathfrak{M}}$, because if it did, the restriction u'' of u to C would belong to $P^{1/2}_{C}$. (This can be verified as follows. Define a transformation

$(\xi, \eta, \zeta) = T(x, y, z)$ of $\mathfrak{M}$ onto R^3 by:

$$\xi = x, \qquad \eta = y - f(x), \qquad \zeta = z.$$

T is a homeomorphism which maps C onto the ξ-axis. If $\varphi(\xi, \eta, \zeta) = u(x, y, z)$, then $\varphi \in P^{3/2}(R^3)$ and hence the restriction φ'' of φ to the ξ-axis belongs to $P^{1/2}(R^1)$. This implies $u'' \in P_C^{1/2}$.)

APPENDIX I

The quadratic interpolation.

For completeness sake we give here the definitions and proofs concerning quadratic interpolation. They were presented in the literature [2, 9] in different forms and not so completely as needed here.

We start by considering a compatible couple of Hilbert spaces V and W. The shortest definition of what such a compatible couple means is that both spaces V and W are Hilbert subspaces of a common topological Hausdorff vector space ([13]). This topological vector space plays no role whatsoever in the considerations, and it is of importance to state the intrinsic properties of the couple which make it compatible. These characteristic properties are the following [3]:

1° [V, W] form a linear couple, i.e. $V \cap W$ is a linear subspace of V as well as W and the corresponding identification (coupling) mapping π is a linear isomorphism of $V \cap W$ as subspace of V onto $V \cap W$ as subspace of W.

2° The identification mapping π is a closed mapping from V into W, i.e. if $\{x_n\} \subset V \cap W$ is a Cauchy sequence in V and in W then its limits in V and in W are equal and thus contained in $V \cap W$.

For a compatible Hilbert couple [V, W] the vector space $V \cap W$ has a direct meaning. The vector space $V + W$ has still a direct meaning if V and W are Hilbert subspaces of a common vector space, whereas if we use the intrinsic defi-

([13]) Hilbert subspace means a subspace with its own hilbertian structure such that the corresponding injection mapping (identity mapping) is continuous.

nition we identify $V + W$ with the quotient space $(V \dot{+} W/Z)$ where Z is the closed subspace of the direct sum $V \dot{+} W$ composed of couples $\{\varphi, \psi\}$ with $\varphi = -\psi \in V \cap W$.

Denoting by $\| \ \|_V$ and $\| \ \|_W$ the norms in V and W we define the norms on $V \cap W$ and $V + W$ as follows:

$$\text{(A I.1)} \qquad \|u\|^2_{V\cap W} = \|u\|^2_V + \|u\|^2_W,$$

$$\text{(A I.2)} \qquad \|u\|^2_{V+W} = \min_{\substack{v\in V,\ w\in W \\ v+w=u}} [\|\varphi\|^2_V + \|\psi\|^2_W].$$

It is immediately proved that the norm (A I.1) on $V \cap W$ is quadratic and makes $V \cap W$ into a Hilbert space. The proof that $V + W$ with the norm (A I.2) is a Hilbert space is less immediate; the shortest way to prove it is to take the definition of $V + W$ as the quotient space $(V \dot{+} W)/Z$ (as done above) when $V \dot{+} W$ is made into a Hilbert space by putting $\|\{\varphi, \psi\}\|^2 = \|\varphi\|^2_V + \|\psi\|^2_W$. Then the quotient space is identified with the orthogonal complement of Z and the norm (A I.2) on $V + W$ is just the norm of $V \dot{+} W$ restricted to this complement.

The preceding definitions assign to each compatible Hilbert couple $[V, W]$ a quadruple of well determined Hilbert spaces $[V, W, V \cap W, V + W]$. Even if V is contained in W, when $V \cap W = V$ and $V + W = W$, these equalities only mean equalities of spaces, but we still put norm (A I.1) on V considered as $V \cap W$ and norm (A I.2) on W considered as $V + W$. In this way all that follows will be valid without any exceptions.

LEMMA. — *Let $\mathcal{H}$ be a Hilbert space and V a Hilbert subspace of $\mathcal{H}$. Then there exists a linear non-negative bounded operator G on $\mathcal{H}$ such that*

1° *For $u \in \mathcal{H}$, $Gu \in V$ and $(u, \varphi)_{\mathcal{H}} = (Gu, \varphi)_V$ for all $\varphi \in V$.*

2° *The null-space of G is $\mathcal{H} \ominus \overline{V}$ ($\overline{V}$ is the closure of V in $\mathcal{H}$)*

3° *If $G^{1/2}$ is the positive square root of G then $G^{1/2}(\mathcal{H}) = V$ and for $f \in \overline{V}$, $\|f\|_{\mathcal{H}} = \|G^{1/2} f\|_V$.*

4° *The upper bound of $G = \sup_{v\in V} \dfrac{\|\varphi\|^2_{\mathcal{H}}}{\|\varphi\|^2_V}$.*

The operator G is unique. The so-established correspondence

between Hilbert subspaces V *of* $\mathcal{H}$ *and the non-negative bounded linear operators* G *is* 1 — 1 *and onto* ([14]).

Proof. — Since V is a Hilbert subspace of $\mathcal{H}$ we have $c = \sup \{\|\varphi\|^2_{\mathcal{H}}/\|\varphi\|^2_{\mathrm{V}} : \varphi \in \mathrm{V}\} < \infty$. Consequently the antilinear functional $(u, \varphi)_{\mathcal{H}}$ of φ which is bounded on $\mathcal{H}$ is, a fortiori, bounded on V with its norm and hence realized in the Hilbert space V as $(Gu, \varphi)_{\mathrm{V}}$. This defines Gu uniquely as an element of V. It is clear that G is linear. By putting $\varphi = Gu$ we get $(u, Gu)_{\mathcal{H}} = (Gu, Gu)_{\mathrm{V}} \geqslant 0$. Hence G is non-negative. To see that it is bounded by c, write

$$\|Gu\|^2_{\mathcal{H}} \leqslant c(Gu, Gu)_{\mathrm{V}} = c(u, Gu)_{\mathcal{H}} \leqslant c\|u\|_{\mathcal{H}}\|Gu\|_{\mathcal{H}}.$$

Thus, 1° is proved. 2° follows immediately from 1°.

To prove 3°, notice first that the range of G is dense in V (in the norm of V). Otherwise there would exist φ_0, $0 \neq \varphi_0 \in \mathrm{V}$, such that $(u, \varphi_0)_{\mathcal{H}} = (Gu, \varphi_0)_{\mathrm{V}} = 0$ for all $u \in \mathcal{H}$ which is impossible. On the other hand, the range of any self-adjoint operator is dense in the norm of $\mathcal{H}$ in the orthogonal complement of its null-space. Hence the elements $f = G^{1/2}u$ are dense in $\overline{\mathrm{V}}$, whereas the elements $G^{1/2}f = Gu$ are dense in V in the norm of V. By the formula $(Gu, Gu)_{\mathrm{V}} = (u, Gu)_{\mathcal{H}}$ we get $(G^{1/2}f, G^{1/2}f)_{\mathrm{V}} = (f, f)_{\mathcal{H}}$. Therefore the mapping $G^{1/2}$ is a linear isometry of the range of $G^{1/2}$ provided with the norm of $\mathcal{H}$ into V provided with the norm of V. By continuity it is extendable to an isometry of $\overline{\mathrm{V}}$ onto V. Finally, $G^{1/2}(\mathcal{H}) = G^{1/2}(\overline{\mathrm{V}}) = \mathrm{V}$.

To prove 4°, we use 3° and write

$$\sup_{u \in \mathcal{H}} \frac{(u, Gu)_{\mathcal{H}}}{(u, u)_{\mathcal{H}}} = \sup_{u \in \overline{\mathrm{V}}} \frac{(G^{1/2}u, G^{1/2}u)_{\mathcal{H}}}{(u, u)_{\mathcal{H}}} = \sup_{v \in \mathrm{V}} \frac{(\varphi, \varphi)_{\mathcal{H}}}{(\varphi, \varphi)_{\mathrm{V}}} = c.$$

To finish our proof it remains to show that if G is a linear non-negative bounded operator on $\mathcal{H}$ then there exists a unique Hilbert subspace V to which it corresponds. To this effect we take the positive square root $G^{1/2}$; put $\mathrm{V} = G^{1/2}(\mathcal{H})$. The orthogonal complement of the null-space of G is then $\overline{\mathrm{V}}$ and $G^{1/2}$ restricted to $\overline{\mathrm{V}}$ is 1 — 1 and onto. We put then

([14]) This lemma is well-known; the first version, not quite complete, was established by K. Friedrichs [7]. We give the proof for completeness sake.

for $f \in \overline{V}$, $\|G^{1/2}f\|_V = \|f\|_{\mathcal{H}}$. Then if we put for $\varphi \in V$, $\varphi = G^{1/2}f$, $f \in \overline{V}$, we have

$$(Gu,\ \varphi)_V = (G^{1/2}G^{1/2}u,\ G^{1/2}f)_V = (G^{1/2}u,\ f)_{\mathcal{H}} = (u,\ G^{1/2}f)_{\mathcal{H}} = (u,\ \varphi)_{\mathcal{H}}$$

which shows that G corresponds to V. By part 3° of the lemma there is no other possible choice of V and $\|\ \|_V$.

THEOREM I. — 1° *Let* [V, W] *be a compatible Hilbert couple and* G, H *be the operators on* V + W *which, following the Lemma, correspond to* V *and* W *respectively. Then* G + H = I *(identity) and the operators corresponding to* V ∩ W *in* V + W, V *and* W *are* GH, H *restricted to* V *and* G *restricted to* W *respectively.*

2° *If, for two Hilbert subspaces* V, W *of a common Hilbert space* $\mathcal{H}$ *the corresponding operators* G *and* H *satisfy* G + H = I *then* $\mathcal{H} \cong V + W$ [15].

Proof. — 1° We use the intrinsic definition of a compatible couple and the corresponding definition of V + W. We denote by S the direct sum $V \dotplus W$. By V_0 and W_0 we denote the subspaces of S of elements $\{\varphi, 0\}$ and $\{0, \psi\}$ respectively. By σ and τ we denote the canonical mappings $\varphi \to \{\varphi, 0\}$, $\psi \to \{0, \psi\}$. We put $Z = \{\{u, -u\} : u \in V \cap W\}$. Z is a closed subspace of S and we put $T = S \ominus Z$. π_V, π_W and π_T are the orthogonal projections in S on V_0, W_0 and T respectively. We identify now $\varphi \in V$ with $\pi_T\sigma\varphi$ and $\psi \in W$ with $\pi_T\tau\psi$. If $u \in V \cap W$ then

$$\pi_T\sigma u - \pi_T\tau u = \pi_T\{u, -u\} = 0.$$

Hence the identifications are consistent with the coupling of V and W which allows the identification of [V, W] with $[\pi_T\sigma V, \pi_T\tau W]$. With this identification $V + W \cong T$ where T is taken with the norm of S. In view of the identification, σ and τ may be considered as the inverses of π_T restricted to V_0 and W_0 respectively. Consider on T the operators G and H corresponding to $V \equiv \pi_T\sigma V$ and $W \equiv \pi_T\tau W$. We claim that $G = \pi_T\pi_V$ restricted to T and,

[15] For Banach spaces A, B, $A \cong B$ means that A and B are identical as vector spaces and, moreover, have the same norms.

similarly, $H = \pi_T\pi_W$. It is enough to show this for G. In fact, for $t \in T$ and $\rho \in V$, we have,

$$(\pi_T\pi_V t,\ \rho)_V = (\sigma\pi_T\pi_V t,\ \sigma\rho)_S = (\pi_V t,\ \sigma\rho)_S = (t,\ \sigma\rho)_S$$
$$= (t,\ \pi_T\sigma\rho)_S = (t,\ \rho)_T.$$

This equation, valid for every ρ in V, shows that $\pi_T\pi_V t = Gt$. It follows immediately that

$$Gt + Ht = \pi_T\pi_V t + \pi_T\pi_W t = \pi_T(\pi_V + \pi_W)t = \pi_T t = t.$$

To prove that GH corresponds to $V \cap W$ we notice that for $u \in V + W$, $GHu = G(I - G)u = HGu$, hence $HGu \in V \cap W$. Then, if $x \in V \cap W$, we have

$$(HGu,\ x)_{V\cap W} = (GHu,\ x)_V + (HGu,\ x)_W$$
$$= (Hu,\ x)_{V+W} + (Gu,\ x)_{V+W} = (u,\ x)_{V+W}$$

which proves our assertion. To show that H corresponds to $V \cap W$ in V, take any $\rho \in V$ and $x \in V \cap W$. Then $\rho = G^{1/2}f$, $x = G^{1/2}g = H^{1/2}h$ where f and g are in $\overline{V}$ and $h \in \overline{W}$. We have $H\rho = HG^{1/2}f = G^{1/2}Hf \in V \cap W$ and

$$(H\rho,\ x)_{V\cap W} = (HG^{1/2}f,\ G^{1/2}g)_V + (HG^{1/2}f,\ H^{1/2}h)_W$$
$$= (Hf,\ g)_{V+W} + (H^{1/2}G^{1/2}f,\ h)_{V+W}$$
$$= (f,\ g)_{V+W} - (G^{1/2}f,\ x)_{V+W} + (G^{1/2}f,\ x)_{V+W} = (\rho,\ x)_V,$$

which shows that H restricted to V corresponds to $V \cap W$ in V. The assertion about the restriction of G to W is proved similarly.

2° Since for $x \in \mathcal{H}$, $Gx \in V$ and $Hx \in W$, it follows that $x = Gx + Hx \in V + W$. Hence, $\mathcal{H}$ as vector space is the same as $V + W$. We still have to check the equality of norms. Since V and W are Hilbert subspaces of $\mathfrak{H}$ there exists a $c > 0$ such that $\|\rho\|_V^2 \geqslant c\|\rho\|_{\mathcal{H}}^2$ for $\rho \in V$ and $\|w\|_V^2 \geqslant c\|w\|_{\mathcal{H}}^2$ for $w \in W$. We have then

$$\|x\|_{V+W}^2 = \min\{\|\rho\|_V^2 + \|w\|_W^2 :\ \rho \in V,\ w \in W,\ x = \rho + w\}$$
$$\geqslant c \min\{\|\rho\|_{\mathcal{H}}^2 + \|w\|_{\mathcal{H}}^2 :\ \rho \in V,\ w \in W,\ x = \rho + w\}$$
$$\geqslant \frac{c}{2}\|x\|_{\mathcal{H}}^2\ (16).$$

(16) This follows from $\|x - w\|^2 + \|w\|^2 = \frac{1}{2}\|x\|^2 + 2\left\|w - \frac{x}{2}\right\|^2$.

The closed graph theorem then gives that there exists a constant C such that $\|x\|^2_{V+W} \leqslant C\|x\|^2_{\mathcal{H}}$. Therefore the scalar product in $V + W$ is given by a bounded, positive operator M with bounded inverse: $(x, y)_{V+W} = (Mx, y)_{\mathcal{H}}$. It follows that the operator G' corresponding to V in $V + W$ by our Lemma is given by

$$(G'x, \varphi)_V = (x, \varphi)_{V+W} = (Mx, \varphi)_{\mathcal{H}} = (GMx, \varphi)_V.$$

Hence $G' = GM$ and similarly, $H' = HM$ and

$$G' + H' = (G + H)M.$$

Since $G' + H' = I$ by 1° and $G + H = I$ by hypothesis, $M = I$, which finishes the proof.

Consider now a fixed compatible Hilbert couple $[V_0, V_1]$ and the corresponding operators G_0 and $G_1 = I - G_0$ in $V_0 + V_1$. It will be convenient to use the spectral decomposition E_λ of G_0 in the space $V_0 + V_1$. The spectrum is contained in the closed interval $0 \leqslant \lambda \leqslant 1$. The null-space N_0 of G_0 is the eigenspace corresponding to the eigenvalue 0 and the null-space N_1 of G_1 is the eigenspace for $\lambda = 1$. For any $u \in V_0 + V_1$ we have

$$\text{(AI.3)} \quad \begin{cases} u = \int_0^1 dE_\lambda u, \quad G_0^{1/2}u = \int_0^1 \lambda^{1/2}\, dE_\lambda u, \\ \qquad\qquad\qquad\qquad G_1^{1/2}u = \int_0^1 (1-\lambda)^{1/2}\, dE_\lambda u \\ G_0^{1/2}G_1^{1/2}u = \int_0^1 \lambda^{1/2}(1-\lambda)^{1/2}\, dE_\lambda u \end{cases}$$

$$\text{(AI.4}a\text{)} \quad u \in V_0 \iff \|u\|^2_{V_0} \equiv \int_0^1 \lambda^{-1} d\|E_\lambda u\|^2 < \infty$$

$$\text{(AI.4}b\text{)} \quad u \in V_1 \iff \|u\|^2_{V_1} \equiv \int_0^1 (1-\lambda)^{-1} d\|E_\lambda u\|^2 < \infty$$

$$\text{(AI.4}c\text{)} \quad u \in V_0 \cap V_1 \iff \|u\|^2_{V_0 \cap V_1} \equiv \int_0^1 \lambda^{-1}(1-\lambda)^{-1} d\|E_\lambda u\|^2 < \infty.$$

We have further

$$\text{(AI.5)} \quad \begin{cases} N_1 \subset V_0, \ \overline{V}_0 = (V_0 + V_1) \ominus N_0, \ N_0 \subset V_1, \\ \qquad\qquad\qquad\qquad \overline{V}_1 = (V_0 + V_1) \ominus N_1 \\ \overline{V_0 \cap V_1} = (V_0 + V_1) \ominus (N_0 + N_1). \end{cases}$$

For every number τ, $0 \leqslant \tau \leqslant 1$, we define the interpolation spaces V_τ between V_0 and V_1 as follows:

(AI.6) V_τ *is the Hilbert subspace of* $V_0 + V_1$ *corresponding to the operator* $G_0^{1-\tau}G_1^{\tau}$.

Clearly, for $\tau = 0$ and $\tau = 1$ the notation we started with is consistent with this definition. Using the resolution of identity E_λ we get an equivalent definition

(AI.6′) $u \in V_\tau \Leftrightarrow \|u\|_\tau^2 \equiv \int_0^1 \lambda^{\tau-1}(1-\lambda)^{-\tau} d\|E_\lambda u\|^2 < \infty.$

From (AI.6′) and (AI.4*c*) it follows immediately that

(AI.7) $V_0 \cap V_1 \subset V_\tau \subset \overline{V_0 \cap V_1}$ *for* $0 < \tau < 1.$

By approximating $u \in V_\tau$ by $(E_{1-1/n} - E_{1/n})u$ with $n \nearrow \infty$ we check that

(AI.8) $V_0 \cap V_1$ *is dense in* V_τ *(in the norm of* V_τ*).*

We remark next that for $0 < \tau < 1$,

$$V_\tau = G_0^{(1-\tau)/2} G_1^{\tau/2}(\overline{V_0 \cap V_1})$$

but for $\tau = 0$ or 1 we get

(AI.9)
$$V_0' = G_0^{1/2}(\overline{V_0 \cap V_1}) = \text{orthogonal complement in } V_0 \text{ of } N_1$$
$$V_1' = G_1^{1/2}(\overline{V_0 \cap V_1}) = \text{orthogonal complement in } V_1 \text{ of } N_0.$$

Then by checking the norms as expressed in terms of the resolution of identity E_λ one sees immediately that for any real σ_0 and σ_1 the operator $G_0^{i\sigma_0} G_1^{i\sigma_1}$ is defined on $\overline{V_0 \cap V_1}$ and is a unitary operator on all the following spaces provided with their respective norms:

$\overline{V_0 \cap V_1}$ *with norm of* $V_0 + V_1$, $V_0 \cap V_1$, V_0', V_1', V_τ *for* $0 < \tau < 1$.

If $[W_0, W_1]$ is another compatible Hilbert couple we will repeat the definitions, notations and constructions as above except that the operators will now be denoted H_0 and $H_1 = I - H_0$ and we will obtain the interpolation spaces W_τ.

We can now state the interpolation theorem.

THEOREM II (INTERPOLATION THEOREM). — *Let* T *be a linear mapping of* $V_0 + V_1$ *into* $W_0 + W_1$ *such that it transforms continuously* V_0 *into* W_0 *and* V_1 *into* W_1 *with respective bounds* M_0 *and* M_1. *Then*

1° T *transforms continuously* $V_0 + V_1$ *into* $W_0 + W_1$ *with bound* $\leqslant \max(M_0, M_1)$.

2° T *transforms continuously* $V_0 \cap V_1$ *into* $W_0 \cap W_1$ *with bound* $\leqslant \max(M_0, M_1)$.

3° T *transforms continuously* V_τ *into* W_τ, $0 < \tau < 1$, *with bound* $\leqslant M_0^{1-\tau} M_1^{\tau}$.

Proof. — 1° If $u = u_0 + u_1$, $u_0 \in V_0$ and $u_1 \in V_1$, then by (A I.2)

$$\begin{aligned}\|Tu\|^2_{W_0+W_1} &\leqslant \min(\|Tu_0\|^2_{W_0} + \|Tu_1\|^2_{W_1}) \\ &\leqslant \max(M_0^2, M_1^2) \min(\|u_0\|^2_{V_0} + \|u_1\|^2_{V_1}) \\ &= \max(M_0^2, M_1^2)\|u\|^2_{V_0+V_1}.\end{aligned}$$

2° If $u \in V_0 \cap V_1$, $Tu \in W_0 \cap W_1$ and by (A I.1)

$$\|Tu\|^2_{W_0\cap W_1} = \|Tu\|^2_{W_0} + \|Tu\|^2_{W_1} \leqslant \max(M_0^2, M_1^2)\|u\|^2_{V_0\cap V_1}$$

3° Since in this part we are considering only V_τ with $0 < \tau < 1$, we can restrict our consideration to $\overline{V_0 \cap V_1}$, a closed subspace of $V_0 + V_1$. We will retain the notation G_0, G_1 for the restrictions of G_0 and G_1 to $\overline{V_0 \cap V_1}$. With this convention for $\alpha > 0$, G_0^α and G_1^α will have well-determined inverses $G_0^{-\alpha}$ and $G_1^{-\alpha}$ (they may be unbounded). The transformation T maps $V_0 \cap V_1$ into $W_0 \cap W_1$. Hence $\overline{V_0 \cap V_1}$ is mapped into $\overline{W_0 \cap W_1}$. Thus we can restrict our considerations to $\overline{W_0 \cap W_1}$. The above remarks apply also to H_0 and H_1 restricted to $\overline{W_0 \cap W_1}$. We can now make the following remark which is checked immediately by using the spectral representation of our operators.

$G_0^{(1-\zeta)/2} G_1^{\zeta/2}$ (or $H_0^{\zeta/2} H_1^{(1-\zeta)/2}$) is an operator valued function of the complex variable ζ which is analytic in the uniform operator topology for $\zeta = \tau + i\sigma$ in the open strip $0 < \tau < 1$ and is continuous in the strong operator topology in the closed strip $0 \leqslant \tau \leqslant 1$.

Consider now $\varphi \in V_0 \cap V_1$. Then $G_0^{(1-\zeta)/2} G_1^{\zeta/2}\varphi \in V_0 \cap V_1$,

$$T(G_0^{(1-\zeta)/2} G_1^{\zeta/2}\varphi) \in W_0 \cap W_1, \quad H_0^{-1/2} H_1^{-1/2} T(G_0^{(1-\zeta)/2} G_1^{\zeta/2}\varphi) \in \overline{W_0 \cap W_1}$$

and also

$$\begin{aligned}&H_0^{(-1+\zeta)/2} H_1^{-\zeta/2} T(G_0^{(1-\zeta)/2} G_1^{\zeta/2}\varphi) \\ &\qquad = H_0^{\zeta/2} H_1^{(1-\zeta)/2} H_0^{-1/2} H_1^{-1/2} T(G_0^{(1-\zeta)/2} G_1^{\zeta/2}\varphi) \in \overline{W_0 \cap W_1}.\end{aligned}$$

Consequently the operator valued function

$$H_0^{(-1+\zeta)/2} H_1^{-\zeta/2} T(G_0^{(1-\zeta)/2} G_1^{\zeta/2})$$

is analytic in $0 < \tau < 1$ in the uniform operator topology and continuous in $0 \leqslant \tau \leqslant 1$ in the strong operator topology. Therefore

$$\varphi(\zeta) = M_0^{\zeta-1} M_1^{-\zeta} H_0^{(-1+\zeta)/2} H_1^{-\zeta/2} T(G_0^{(1-\zeta)/2} G_1^{\zeta/2} \rho)$$

is a vector valued function analytic in the open strip and continuous in the closed strip.

For $\zeta = i\sigma (\tau = 0)$ this vector valued function becomes

$$\varphi(i\sigma) = M_0^{i\sigma-1} M_1^{-i\sigma} H_0^{i\sigma/2} H_1^{-i\sigma/2} H_0^{-1/2} T(G_0^{-i\sigma/2} G_1^{i\sigma/2} G_0^{1/2} \rho).$$

Hence

$$\|\varphi(i\sigma)\|_{\overline{W_0 \cap W_1}} = \|H_0^{1/2} \varphi(i\sigma)\|_{W_0} = M_0^{-1} \|T(G_0^{-i\sigma/2} G_1^{i\sigma/2} G_0^{1/2} \rho)\|_{W_0}$$
$$\leqslant \|G_0^{1/2} \rho\|_{V_0} = \|\rho\|_{\overline{V_0 \cap V_1}}.$$

Similarly we prove that $\|\varphi(1 + i\sigma)\|_{\overline{W_0 \cap W_1}} \leqslant \|\rho\|_{\overline{V_0 \cap V_1}}$. By the maximum modulus theorem [6] for vector valued analytic functions we get $\|\varphi(\zeta)\|_{\overline{W_0 \cap W_1}} \leqslant \|\rho\|_{\overline{V_0 \cap V_1}}$ for all ζ in the strip. In particular for $0 < \tau < 1$,

$$\|\varphi(\tau)\|_{\overline{W_0 \cap W_1}} = M_0^{-1+\tau} M_1^{-\tau} \|T(G_0^{(1-\tau)/2} G_1^{\tau/2} \rho)\|_{W_\tau}$$
$$\leqslant \|\rho\|_{\overline{V_0 \cap V_1}} = \|G_0^{(1-\tau)/2} G_1^{\tau/2} \rho\|_{V_\tau}.$$

When ρ varies over $V_0 \cap V_1$, $G_0^{(1-\tau)/2} G_1^{\tau/2} \rho$ varies over a dense subspace of V_τ. Hence the last inequality shows that T is a bounded transformation of V_τ into W_τ with bound $\leqslant M_0^{1-\tau} M_1^{\tau}$.

THEOREM III. — *Let* H *be a linear positive bounded operator on a Hilbert space* $\mathcal{H}$. *Let* W_α *be the Hilbert subspace of* $\mathcal{H}$ *corresponding to the operator* $H^{2\alpha}$, $\alpha \geqslant 0$. *Then for any* τ, $0 \leqslant \tau \leqslant 1$, *and* $0 \leqslant \alpha < \beta$, $W_{\alpha(1-\tau)+\beta\tau}$ *is the* τ*-th interpolation space by quadratic interpolation between* W_α *and* W_β.

Proof. — Consider the subspace W′ corresponding to the operator $H^{2\alpha} + H^{2\beta}$. By using the spectral decomposition E_λ of H we see that the elements $u \in W_\alpha$ are characterized by $\|u\|^2_{W_\alpha} = \int \lambda^{-2\alpha} \, d\|E_\lambda u\|^2 < \infty$, and the elements $u \in W'$ are

characterized by $\|u\|^2_{W'} = \int (\lambda^{2\alpha} + \lambda^{2\beta})^{-1}\, d\|E_\lambda u\|^2 < \infty$. It follows that $W_\alpha \subset W'$ and similarly, $W_\beta \subset W'$. W_α and W_β are then necessarily Hilbert subspaces of W'. Using the expression of norms and scalar products in W_α and W' in terms of the resolution of identity E_λ we check immediately that the operator H'_α which corresponds to W_α in W' corresponds to multiplication by $\dfrac{\lambda^{2\alpha}}{(\lambda^{2\alpha} + \lambda^{2\beta})}$, i.e. it is the operator $H^{2\alpha}(H^{2\alpha} + H^{2\beta})^{-1}$. Similarly the operator $H^{2\beta}(H^{2\alpha} + H^{2\beta})^{-1}$ corresponds to W_β in W'. Since the sum of these two operators is $= I$ it follows, by Theorem I, 2°, that $W' = W_\alpha + W_\beta$. Hence the τ-th interpolation space between W_α and W_β corresponds to the operator on W' which is $H^{2\alpha(1-\tau)+2\beta\tau}(H^{2\alpha} + H^{2\beta})^{-1}$. Therefore this subspace corresponds in $\mathcal{H}$ to the operator $H^{2\alpha(1-\tau)+2\beta\tau}$ as is stated in our theorem.

As application we give the following corollary.

COROLLARY 3'. — *For the space of potentials* $P^\alpha(R^n)$ *we have that* $P^{\alpha(1-\tau)+\beta\tau}(R^n)$ *is the* τ*-th interpolation space by quadratic interpolation between* $P^\alpha(R^n)$ *and* $P^\beta(R^n)$.

Proof. — We have only to remind the reader that $P^\alpha(R^n) = G_\alpha(L^2(R^n))$ where G_α are positive bounded integral operators on $L^2(R^n)$ which, in view of the convolution formula $G_\alpha * G_\beta = G_{\alpha+\beta}$, are necessarily the α-powers of G_1, i.e. $G_\alpha = G_1^\alpha$. Since for $f \in L^2(R^n)$, $\|G_\alpha f\|_{P^\alpha} = \|f\|_{L^2}$ we recognize that the spaces $P^\alpha(R^n)$ are Hilbert subspaces of $L^2(R^n)$ corresponding to the operators $G_{2\alpha} = G_1^{2\alpha}$. Hence the preceding theorem applies.

Remark 1. — Some clarification should be given concerning the last Corollary. The spaces $P^\alpha(R^n)$ as introduced in Ch. II are not exactly subspaces of $L^2(R^n)$. The elements of $L^2(R^n)$ are classes of equivalence rel. $\mathfrak{A}_0$ (i.e. relative to the sets of Lebesgue measure 0), whereas the elements of $P^\alpha(R^n)$ are classes of equivalence relative to the corresponding class of exceptional sets, the class $\mathfrak{A}_{2\alpha} \subset \mathfrak{A}_0$. However, each class of equivalence in $L^2(R^n)$ has elements in common with at most one class of equivalence in $P^\alpha(R^n)$ (and then the former

contains the latter) and therefore we may identify the elements of $P^{\alpha}(R^n)$ with elements of $L^2(R^n)$. By virtue of this identification we can proceed as in the proof of the Corollary, but then we obtain the interpolation space as a class of elements in $L^2(R^n)$. In other words we obtain, by Corollary 3', the class $P^{\alpha(1-\tau)+\beta\tau}(R^n)$ saturated rel. $\mathfrak{A}_0$. To recapture the proper class $P^{\alpha(1-\tau)+\beta\tau}(R^n)$ we have to apply the above identification in reverse which can be obtained by correcting the function in the class saturated rel. $\mathfrak{A}_0$.

This situation may present itself in a more general case when we deal with two saturated functional Hilbert spaces ([17]) $\mathcal{F}^1$ and $\mathcal{F}^2$ on a common basic set $\mathcal{E}$ relative to two exceptional classes $\mathfrak{A}^1$ and $\mathfrak{A}^2$ respectively.

We first remind the reader that a saturated functional Hilbert space $\mathcal{F}$ rel. $\mathfrak{A}$ is, in the first place, a linear class of functions defined on $\mathcal{E}$ with a quadratic pseudo-norm having the property that $\|u\| = 0 \iff u(x) = 0$ exc. $\mathfrak{A}$. Since each such class is a subspace of the linear class of all functions defined on $\mathcal{E}$, we have a natural meaning for the classes $\mathcal{F}^1 \cap \mathcal{F}^2$ and $\mathcal{F}^1 + \mathcal{F}^2$. We define then, by formulas (A I.1) and (A I.2) the pseudo-norms on $\mathcal{F}^1 \cap \mathcal{F}^2$ and $\mathcal{F}^1 + \mathcal{F}^2$. The compatibility of the two functional spaces means that for no function $f^k \in \mathcal{F}^k$ with $\|f^k\|_{\mathcal{F}^k} \neq 0$ one gets $\|f^k\|_{\mathcal{F}^1+\mathcal{F}^2} = 0$ for $k = 1, 2$. This is equivalent to the fact that if $f \in \mathcal{F}^1 \cap \mathcal{F}^2$, then $\|f\|_{\mathcal{F}^1} = 0 \iff \|f\|_{\mathcal{F}^2} = 0$. Assuming that the compatibility holds, we notice immediately that the class $\mathcal{F}^1 + \mathcal{F}^2$ is a functional Hilbert space rel.

$$\mathfrak{A}^1 \cup \mathfrak{A}^2 = \{(A^1 \cup A^2) : A^1 \in \mathfrak{A}^1,\ A^2 \in \mathfrak{A}^2\}$$

and that $\mathcal{F}^1 \cap \mathcal{F}^2$ is a functional Hilbert space rel. $\mathfrak{A}^1 \cap \mathfrak{A}^2$. However, to apply the abstract interpolation here, we must identify the elements of the Hilbert spaces $\mathcal{F}^1$, $\mathcal{F}^2$ and $\mathcal{F}^1 \cap \mathcal{F}^2$, i.e. the equivalence classes rel. $\mathfrak{A}^1$, $\mathfrak{A}^2$ and $\mathfrak{A}^1 \cap \mathfrak{A}^2$ respectively, with the equivalence classes rel. $\mathfrak{A}^1 \cup \mathfrak{A}^2$ containing them, which are elements of the Hilbert space $\mathcal{F}^1 + \mathcal{F}^2$. This leads to interpolation spaces which are functional spaces rel. $\mathfrak{A}^1 \cup \mathfrak{A}^2$. However, we can give a more precise definition of the interpolation spaces as functional spaces, proceeding

([17]) Similar developments can be made in case of functional Banach spaces.

as follows. The functional space $\mathfrak{F}^1 \cap \mathfrak{F}^2$ rel. $\mathfrak{A}^1 \cap \mathfrak{A}^2$ is dense in the interpolation space with its norm. Therefore the interpolation space is a functional completion of $\mathfrak{F}^1 \cap \mathfrak{F}^2$ relative to the interpolation norm. Hence we can ask for the best possible completion, i.e. with the smallest possible exceptional class. In particular, if there exists a perfect functional completion of $\mathfrak{F}^1 \cap \mathfrak{F}^2$ rel. $\mathfrak{A}^1 \cap \mathfrak{A}^2$ with the interpolation norm on it, we will call this *the perfect functional interpolation space.* Its exceptional class will always be contained between $\mathfrak{A}^1 \cap \mathfrak{A}^2$ and $\mathfrak{A}^1 \cup \mathfrak{A}^2$.

In the case of the Corollary 3′, we recapture in this way the interpolation space $P^{\alpha(1-\tau)+\beta\tau}(R^n)$ with its proper exceptional class $\mathfrak{A}_{2\alpha(1-\tau)+2\beta\tau}$.

In order to state the next theorem in its strongest form we recall a few facts about general interpolation methods between Banach spaces (see [3]).

The notion of a compatible couple of Banach spaces $[V, W]$ and the definitions of $V \cap W$ and $V + W$ as Banach spaces are similar to those given above for Hilbert spaces (except for some differences in the definitions of norms of $V \cap W$ and $V + W$). For a compatible couple $[V, W]$ we say that the Banach space A is an *intermediate space* between V and W if it is a Banach subspace of $V + W$ and $V \cap W \subset A \subset V + W$. An intermediate space A is an *interpolation space* between V and W if any linear operator T transforming $V + W$ into $V + W$ which transforms continuously V into V and W into W, also transforms continuously A into A.

An interpolation method F defined on a class $\mathfrak{K}$ of Banach couples is a function which assigns to each couple $[V, W] \in \mathfrak{K}$ an intermediate Banach space $F[V, W]$ between V and W ([18]) with the following property: if T is a linear mapping of $V_0 + W_0$ into $V_1 + W_1$ for $[V_i, W_i] \in \mathfrak{K}$, $i = 0, 1$, and if T transforms continuously V_0 into V_1 and W_0 into W_1 then T transforms $F[V_0, W_0]$ into $F[V_1, W_1]$ continuously.

One of the main results of [3] can be briefly stated as follows:

1° If F is an interpolation method on some class $\mathfrak{K}$ of

([18]) If F' assigns the same spaces as F but with different (necessarily equivalent) norms, then F' is considered different from F.

Banach couples, then $F[V, W]$ is an interpolation space between V and W for any $[V, W] \in \mathfrak{K}$.

2° If A is an interpolation space between V and W for some compatible Banach couple $[V, W]$, then there exist general interpolation methods ([19]) which assign A to $[V, W]$.

It is obvious now, in view of Theorem II, that for the class $\mathfrak{K}$ of compatible Hilbert couples, the assignment of V_τ to $[V_0, V_1]$ forms an interpolation method and hence V_τ is an interpolation space between V_0 and V_1.

THEOREM IV. — *Let* E *and* E' *be two Hausdorff topological vector spaces and let* T *be a linear mapping of* E *onto* E' *and* S *a linear mapping of* E' *into* E *such that* $TS = I$ *(identity on* E'*). Suppose now that* V *and* W *are Banach subspaces of* E *and that* $V' = T(V)$ *and* $W' = T(W)$ *are Banach subspaces of* E' *such that* T *transforms continuously* V *onto* V' *and* W *onto* W', *and* S *transforms continuously* V' *into* V *and* W' *into* W. *Then if* F *is any interpolation method defined for both couples* $[V, W]$ *and* $[V', W']$ *we have* $F[V', W'] = T(F[V, W])$ *and* $S(F[V', W']) \subset F[V, W]$.

Proof. — By our hypothesis we have $T(F[V, W]) \subset F[V', W']$ and $S(F[V', W']) \subset F[V, W]$. Since $TS = I$ we get $F[V', W'] \subset T(F[V, W])$. Hence the statement of our theorem.

Remark 2. — In view of Theorem IV, if we consider any general interpolation method which assigns to the couple $[V, W]$ a fixed interpolation space A between V and W, this interpolation method will assign to the couple $[T(V), T(W)]$ the same space $T(A)$ which, in particular, must be an interpolation space between $T(V)$ and $T(W)$. However, the norm on $T(A)$ will depend on the interpolation method.

We can apply Theorem IV to the following case. We take a closed interval $[\tau_0, \tau_1]$, $0 \leqslant \tau_0 < \tau_1$, and consider an open subset $D \subset R^n$ with the simultaneous extension property over this closed interval, i.e. $D \in \mathfrak{E}([\tau_0, \tau_1])$ (see § 7, III). We take as E the class $P^{\tau_0}(R^n)$, and as E', $\check{P}^{\tau_0}(D)$. The simultaneous extension property means that there exists a

([19]) General in the sense that they are defined for all compatible Banach couples.

linear extension mapping S, $S(E') \subset E$ such that it transforms every $\check{P}^{\alpha}(D)$, $\tau_0 \leqslant \alpha \leqslant \tau_1$, continuously into $P^{\alpha}(R^n)$ with a uniform bound for all such $\alpha's$. By T we denote the restriction mapping assigning to any function defined in R^n its restriction to D. Even if D is not in a simultaneous extension class we have always that T transforms $P^{\alpha}(R^n)$ continuously into $\check{P}^{\alpha}(D)$ (with a bound $\leqslant 1$). If $D \in \mathcal{E}([\tau_0, \tau_1])$, and since S is an extension mapping, it follows that $TS = I$. Hence $T(P^{\alpha}(R^n)) = \check{P}^{\alpha}(D)$ for $\tau_0 \leqslant \alpha \leqslant \tau_1$. By taking in Theorem IV $V = P^{\alpha}(R^n)$ and $W = P^{\beta}(R^n)$ with $\tau_0 \leqslant \alpha < \beta \leqslant \tau_1$, and applying Corollary 3′ we obtain the following corollary:

COROLLARY 4′. — *The τ-th interpolation space by quadratic interpolation between $\check{P}^{\alpha}(D)$ and $\check{P}^{\beta}(D)$ is equal to $\check{P}^{\alpha(1-\tau)+\beta\tau}(D)$. However, the interpolation norm is in general different from the usual norm on $\check{P}^{\alpha(1-\tau)+\beta\tau}(D)$ (but the two norms are equivalent).*

Remark 3. — We can obtain the same space $\check{P}^{\gamma}(D)$ by quadratic interpolation between two different couples $[\check{P}^{\alpha}(D), \check{P}^{\beta}(D)]$ and $[\check{P}^{\alpha'}(D), \check{P}^{\beta'}(D)]$ with $\alpha < \gamma < \beta$, $\alpha' < \gamma < \beta'$. For each couple we have to choose different τ, namely $(\gamma - \alpha)/(\beta - \alpha)$ and $(\gamma - \alpha')/(\beta' - \alpha')$ respectively. The two so obtained interpolation spaces will have equivalent but different norms.

The next theorem is concerned with direct orthogonal sums. If V^i are Hilbert spaces for i belonging to a set of indices $\mathfrak{J}$ we denote by $\sum_{i \in \mathfrak{J}} \perp V^i$ the vector space formed by all systems $\{v^i\}_{i \in \mathfrak{J}}$, $v^i \in V^i$, such that $\|\{v^i\}\|^2 = \sum_{i \in \mathfrak{J}} \|v^i\|^2_{V^i} < \infty$. As is well known this is a Hilbert space. Suppose now that for each $i \in \mathfrak{J}$ we have a compatible couple $[V^i, W^i]$. We consider then $V = \sum_{i \in \mathfrak{J}} \perp V^i$ and $W = \sum_{i \in \mathfrak{J}} \perp W^i$. The natural definition of $V \cap W$ is the set of all systems $\{u^i\}_{i \in \mathfrak{J}}$ with $u^i \in V^i \cap W^i$ and the two norms $\|\{u^i\}\|_V$ and $\|\{u^i\}\|_W$ finite; one checks immediately that $[V, W]$ is a compatible Hilbert couple and that $V \cap W = \Sigma\perp(V^i \cap W^i)$ and $V + W = \Sigma\perp(V^i + W^i)$.

If G_{i0} and G_{i1} are operators on $V^i + W^i$ which correspond to V^i and W^i, then one verifies immediately that the operators G_0 and G_1 on $V + W$ corresponding to V

and W respectively, are given by

$$G_0\{u^i\}_{i\in\mathfrak{J}} = \{G_{i0}u^i\}_{i\in\mathfrak{J}}, \qquad G_1\{u^i\}_{i\in\mathfrak{J}} = \{G_{i1}u^i\}_{i\in\mathfrak{J}}.$$

By using the definition (A I.6) we obtain then the following theorem.

THEOREM V. — *If* $[V_0^i, V_1^i]$, $i \in \mathfrak{J}$, *are compatible Hilbert couples and* V_τ^i, $0 \leqslant \tau \leqslant 1$, *are the corresponding interpolation spaces by quadratic interpolation, and if we write* $V_0 = \sum_{i\in\mathfrak{J}} \perp V_0^i$, $V_1 = \sum_{i\in\mathfrak{J}} \perp V_1^i$ *then the corresponding interpolation space* V_τ *is obtained as* $V_\tau \cong \sum_{i\in\mathfrak{J}} \perp V_\tau^i$.

APPENDIX II

Equivalence of metrics.

Let $\mathfrak{M}$ be an n-dimensional C^∞ manifold, g and $\hat{g}$ be two C^∞ metrics on $\mathfrak{M}$. If T is a k-covariant (or contravariant) tensor on $\mathfrak{M}$ we shall write $\nabla_g^m T(x)$ for the tensor of the m-covariant derivatives of T and $|T(x)|_g$ for the norm of T at x, both computed with respect to g, and we shall write g^{-1} for the contravariant tensor associated with g. A similar notation will be used for $\hat{g}$. The eigenvalues of $\hat{g}$ relative g will be designated by $\lambda_1(x) \leqslant \cdots \leqslant \lambda_n(x)$.

In this appendix we will show that if the eigenvalues of $\hat{g}$ relative g are bounded (above and below) and $|\nabla_g^\nu \hat{g}(x)|_g$, $\nu = 1, \ldots, m$, is bounded uniformly on $\mathfrak{M}$, then the potential norms $|u|_{m,\mathfrak{M},g}$ and $|u|_{m,\mathfrak{M},\hat{g}}$ are equivalent. Furthermore we shall explicitly display the dependence of the constants of equivalence on these bounds.

We define for integers r:

$$\text{(AII.1)} \quad N_r(g;\ \hat{g})(x) = \begin{cases} 1 \quad \textit{for} \quad r = 0, \\ \underset{\nu=1,\ldots,r}{\text{Max}} \{(|\hat{g}^{-1}(x)|_g |\nabla_g^\nu \hat{g}(x)|_g)^{r/\nu}\} \\ \qquad\qquad \textit{for} \quad r \geqslant 1. \end{cases}$$

For positive constants c and $\hat{c}$ we note (by using the fact that $\nabla_{cg}^\nu = \nabla_g^\nu$ and the homogeneity of the norms) that $N_r(cg;\ \hat{c}\hat{g}) = c^{-r/2} N_r(g;\ \hat{g})$.

Our main theorems here are the following.

THEOREM I. — *For any* $x \in \mathfrak{M}$ *and* $r \geqslant 1$

$$\text{(AII.2)} \quad |\nabla_{\hat{g}}^r g(x)|_{\hat{g}} \leqslant C_r n^{1/2} \lambda_1(x)^{-\frac{1}{2}(r+2)} N_r(g;\ \hat{g})(x)$$

and

$$\text{(AII.2')}\quad N_r(\hat{g};\, g)(x) \leqslant C_r'\lambda_1(x)^{-r/2}\left(n\frac{\lambda_n(x)}{\lambda_1(x)}\right)^r N_r(g;\, \hat{g})(x)$$

where C_r *depends only on* r *and* $C_r' = \underset{\nu=1,\ldots,r}{\text{Max}}\{C_\nu^{r/\nu}\}$.

THEOREM II. — *If there are positive constants* Λ_l, Λ_u *and* B_m, $m = 1, 2, \ldots$, *with* $1 \leqslant B_m \leqslant B_{m+1}$ *(and setting* $B_{-1} = B_0 = 1$*) such that* $\Lambda_l \leqslant \lambda_1(x) \leqslant \cdots \leqslant \lambda_n(x) \leqslant \Lambda_u$ *and* $N_m(g, \hat{g})(x) \leqslant B_m$ *for all* $x \in \mathfrak{M}$ *then for* $u \in P^m_{\text{loc}}(\mathfrak{M})$, m *a non-negative integer,*

$$\text{(AII.3)}\quad |u|^2_{m,\mathfrak{M},\hat{g}} \leqslant C_m^{(1)}\Lambda_u^{n/2}(1 + \Lambda_l^{-m})B_{m-1}^2|u|^2_{m,\mathfrak{M},g}$$

and

$$\text{(AII.3')}\quad |u|^2_{m,\mathfrak{M},g} \leqslant C_m^{(2)}\Lambda_l^{-n/2}(1+\Lambda_l^{-m})^2\left(n\frac{\Lambda_u}{\Lambda_l}\right)^{2(m-1)}B_{m-1}^2|u|^2_{m,\mathfrak{M},\hat{g}}$$

where $C_m^{(1)}$ *and* $C_m^{(2)}$ *depend only on* m.

The proofs of Theorems I and II are based on the following lemma.

LEMMA 1. — *Let* T *be a* k*-covariant tensor on* $\mathfrak{M}$ *and* (U, h) *a coordinate patch on* $\mathfrak{M}$. *If* $\nabla_g^r T(x)$ *exists at* $x \in h(U)$ *then*

$$\text{(AII.4)}\quad |\nabla_{\hat{g}}^r T(x)|_{\hat{g}} \leqslant \lambda_1(x)^{-\frac{1}{2}(k+r)}\sum_{s=0}^{r} c_s N_s(g;\, \hat{g})(x)|\nabla_g^{r-s}T(x)|_g,$$

where $c_0 = 1$ *and* c_s, $s \geqslant 1$, *is a constant depending only on* r, k *and* s.

Proof of Theorem I. — In (AII.4) we set $T = g$ and note that $|\nabla_g^r g(x)|_g = n^{1/2}$ for $r = 0$, and $= 0$ for $r > 0$. By an obvious computation (AII.2) now follows (cf. (AII.5), ii) and iii)).

From (AII.2) (and (AII.5), ii) and iii)) we have

$$(|g^{-1}(x)|_{\hat{g}}|\nabla_{\hat{g}}^\nu g(x)|_{\hat{g}})^{r/\nu} \leqslant \lambda_1(x)^{-r/2}\left(C_\nu n\frac{\lambda_n(x)}{\lambda_1(x)}N_\nu(g;\, \hat{g})(x)\right)^{r/\nu},$$

and (AII.2') is now clear.

Proof of Theorem II. — Let (U, h) be a coordinate patch on $\mathfrak{M}$. Since the metric densities satisfy $\sqrt{\hat{g}(x)} \leqslant \Lambda_u^{n/2}\sqrt{g(x)}$, (AII.3) is clear for $m = 0$.

Now suppose $m \geqslant 1$ and $\nabla_g^m u(x)$ exists at $x \in h(U)$ it exists exc. $\mathfrak{A}_{2m}(h(U))$ on $h(U)$).

Since $\nabla_{\hat{g}} u(x) = \nabla_g u(x)$ we apply (AII.4) with $T = \nabla_g u$; note that $\lambda_1(x)^{-s} \leqslant 1 + \Lambda_l^{-m}$, $0 \leqslant s \leqslant m$, and apply the Cauchy-Schwarz inequality. Thus for $0 \leqslant r \leqslant m-1$ we have

$$|\nabla_{\hat{g}}^{r+1} u(x)|_{\hat{g}}^2 \leqslant \lambda_1(x)^{-(r+1)} \left(\sum_{s=0}^{r} c_s^2 \binom{m}{r-s+1}^{-1} N_s^2(g; \hat{g})(x) \right) \times \left(\sum_{s=0}^{r} \binom{m}{r-s+1} |\nabla_g^{r-s+1} u(x)|_g^2 \right) \leqslant c'_m (1 + \Lambda_l^{-m}) B_{m-1}^2 \sum_{s=0}^{m} \binom{m}{s} |\nabla_g^s u(x)|_g^2$$

where c'_m depends only on m. From this and the inequality between the metric densities (AII.3) follows for $m \geqslant 1$.

(AII.3′) follows directly from (AII.3) and (AII.2′) by interchanging g and $\hat{g}$ and noting that

$$\lambda_1(x)^{-r} \leqslant 1 + \Lambda_l^{-(m-1)} \leqslant 1 + \Lambda_l^{-m} \quad \text{for} \quad 0 \leqslant r \leqslant m-1,$$

$\dfrac{\lambda_n(x)}{\lambda_1(x)} \geqslant 1$, and that the eigenvalues of g relative $\hat{g}$ are bounded above and below by Λ_l^{-1} and Λ_u^{-1} respectively.

Proof of Lemma 1. — We fix a coordinate patch (U, h) and a point $x \in V = h(U)$. All tensors are transferred to V and in what follows we do not need to mention x and V. Let A and B be a- and b-covariant tensors and C a c-contravariant tensor. Then AB will be the $(a+b)$-covariant tensor formed by taking the product of A and B. If $a \geqslant c$ we shall use the symbol $C \circ A$ for any of the covariant tensors obtained by contraction of all the indices of C with certain indices of A followed by some permutation of the free indices. We remind the reader of a few well known facts:

(AII.5) i) $|AB|_g = |A|_g |B|_g$, $|C \circ A|_g \leqslant |C|_g |A|_g$,
$\nabla(C \circ B) = (\nabla C) \circ B + C \circ (\nabla B)$,

ii) $\lambda_n^{-a} |A|_g^2 \leqslant |A|_{\hat{g}}^2 \leqslant \lambda_1^{-a} |A|_g^2$, $\lambda_1^c |C|_g^2 \leqslant |C|_{\hat{g}}^2 \leqslant \lambda_n^c |C|_g^2$,

iii) $|g|_g^2 = |g^{-1}|_g^2 = n$.

Let $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ and $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}^{\wedge}$ be the Riemann-Christoffel symbols

of g and $\hat{g}$, and $\Phi = \left\{\Phi^i_{jk} : \Phi^i_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}\hat{} - \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}\right\}$. From the usual formulae for covariant differentiation we have for a k-covariant tensor T

$$\text{(AII.6)} \quad \nabla_{\hat{g}}\mathrm{T} - \nabla_g \mathrm{T} = \mathrm{R} \quad \text{where} \quad \mathrm{R} = \left\{ \sum_{\nu=1}^{k} \Phi^{\alpha}_{i_\nu i_{k+1}} \mathrm{T}_{i_1 \dots i_{\nu-1} \alpha i_{\nu+1} \dots i_k} \right\}$$

with i_{k+1} the index of differentiation. We can also deduce from (AII.6) that Φ is a tensor (cf. (AII.7)).

To determine an explicit expression for Φ, let $\hat{g}_{ij|k}$ be a component of $\nabla_g \hat{g}$, k the index of differentiation. Then $\hat{g}_{ij|k} = \dfrac{\partial \hat{g}_{ij}}{\partial x_k} - \hat{g}_{\alpha j}\{^{\alpha}_{ik}\} - \hat{g}_{i\alpha}\{^{\alpha}_{ik}\}$, and since $\nabla_{\hat{g}}\hat{g} = 0$, $0 = \dfrac{\partial \hat{g}_{ij}}{\partial x_k} - \hat{g}_{\alpha j}\{^{\alpha}_{ik}\}\hat{} - \hat{g}_{i\alpha}\{^{\alpha}_{jk}\}\hat{}$. Thus

$$\hat{g}_{ij|k} = \hat{g}_{\alpha j}\Phi^{\alpha}_{ik} + \hat{g}_{i\alpha}\Phi^{\alpha}_{jk}.$$

Noting that Φ^k_{ij} is symmetric in i and j we have by a direct calculation

$$\text{(AII.7)} \qquad \Phi^k_{ij} = \frac{1}{2}\hat{g}^{k\alpha}(\hat{g}_{i\alpha|j} + \hat{g}_{\alpha j|i} - \hat{g}_{ij|\alpha})$$

and from the obvious symmetry in g and $\hat{g}$

$$\text{(AII.7')} \qquad -\Phi^k_{ij} = \frac{1}{2} g^{k\alpha}(g_{i\alpha|\hat{j}} + g_{\alpha j|\hat{i}} - g_{ij|\hat{\alpha}})$$

where $g_{ij|\hat{k}}$ is a component of $\nabla_{\hat{g}} g$.

Then from (AII.6) we have

$$\text{(AII.8)} \qquad \nabla_{\hat{g}}\mathrm{T} = \nabla_g \mathrm{T} + \frac{1}{2}\Sigma \pm g^{-1} \circ [(\nabla_g \hat{g})\mathrm{T}]$$

where the summation consists of $2k$ positive terms of the form

$$\{\hat{g}^{\alpha\beta}\hat{g}_{i_l\alpha|i_{k+1}}\mathrm{T}_{i_1 \dots i_{l-1}\beta i_{l+1} \dots i_k}\}$$

or

$$\{\hat{g}^{\alpha\beta}\hat{g}_{\alpha i_{k+1}|i_l}\mathrm{T}_{i_1 \dots i_{l-1}\beta i_{l+1} \dots i_k}\}$$

and k negative terms of the form

$$\{\hat{g}^{\alpha\beta}\hat{g}_{i_l i_{k+1}|\alpha}\mathrm{T}_{i_1 \dots i_{l-1}\beta i_{l+1} \dots i_k}\}.$$

By noting that $\nabla_{\hat{g}}\hat{g}^{-1} = 0$, we have from (AII.8) by an easy induction on r,

$$\text{(AII.9)}\quad \nabla^r_{\hat{g}}T = \nabla^r_g T + \sum_{\nu}\sum \pm \left(\frac{1}{2}\right)^l (\hat{g}^{-1})^l \circ [(\nabla^{\nu_1}_g \hat{g}) \cdots (\nabla^{\nu_l}_g \hat{g})(\nabla^{r-\Sigma\nu_j}T)]$$

where the outer summation is taken over all systems $\nu = (\nu_1, \ldots, \nu_l)$, with $\nu_j \geqslant 1$, $\sum_{j=1}^{l} \nu_j \leqslant r$, $l = 1, \ldots, r$ and the inner summation is taken over certain contractions of all the indices of the $2l$-contravariant tensor $(\hat{g}^{-1})^l = \underbrace{\hat{g}^{-1} \cdots \hat{g}^{-1}}_{l\text{-times}}$ with the $(2l + r + k)$-covariant tensor $(\nabla^{\nu_1}_g \hat{g}) \cdots (\nabla^{\nu_l}_g \hat{g})(\nabla^{r-\Sigma\nu_j}T)$ followed by a permutation of the free indices. For each fixed system $\{\nu_j\}$ the number of terms in the inner summation depends only on the system $\{\nu_j\}$, r and k. Thus

(AII.10)

$$|\nabla^r_{\hat{g}}T|_g \leqslant |\nabla^r_g T|_g + \sum_{\nu} c'_\nu |\hat{g}^{-1}|^l_g |\nabla^{\nu_1}_g \hat{g}|_g \cdots |\nabla^{\nu_l}_g \hat{g}|_g |\nabla^{r-\Sigma\nu_j}_g T|_g$$
$$\leqslant |\nabla^r_g T|_g + \sum_{s=1}^{r} \Big[\sum_{\Sigma\nu_j = s} c'_\nu |\hat{g}^{-1}|^l_g |\nabla^{\nu_1}_g \hat{g}|_g \cdots |\Delta^{\nu_l}_g \hat{g}|_g\Big] |\Delta^{r-s}_g T|_g,$$

where c'_ν depends only on ν, k and r and the inner summation is taken over all $\nu = (\nu_1, \ldots, \nu_l)$, $\nu_j \geqslant 1$, $l = 1, \ldots, r$, $\nu_1 + \cdots + \nu_l = s$.

If we consider any term in the inner summation then by (AII.1)

$$c'_\nu |\hat{g}^{-1}|^l_g |\nabla^{\nu_1}_g \hat{g}|_g \cdots |\nabla^{\nu_l}_g \hat{g}|_g$$
$$= c'_\nu [(|\hat{g}^{-1}|_g |\nabla^{\nu_1}_g \hat{g}|_g)^{1/\nu_1}]^{\nu_1} \cdots [(|\hat{g}^{-1}|_g |\nabla^{\nu_l}_g \hat{g}|_g)^{1/\nu_l}]^{\nu_l}$$
$$\leqslant c'_\nu \Big[\operatorname*{Max}_{j=1,\ldots,l} \{(|\hat{g}^{-1}|_g |\nabla^{\nu_j}_g \hat{g}|_g)^{1/\nu_j}\}\Big]^s$$
$$\leqslant c'_\nu N_s(g; \hat{g}).$$

Hence from (AII.10), we have

$$\text{(AII.11)}\quad |\nabla_{\hat{g}}T|^r_{\hat{g}} \leqslant \lambda_1^{-(k+r)/2} |\nabla^r_{\hat{g}}T|_g$$
$$\leqslant \lambda_1^{-(k+r)/2} \sum_{s=0}^{r} c_s N_s(g; \hat{g}) |\nabla^{r-s}_g T|_g$$

where $c_0 = 1$ and $c_s = \sum_{\Sigma\nu_j = s} c'_\nu$ which depends only on k, r and s. This proves Lemma 1.

APPENDIX III

Simultaneous extensions from subspaces of R^n.

Theorem 1*b*, § 8, II, gives a formula assigning to each function $u' \in P^{\alpha-(n-k)/2}(R^k)$ an extension $u \in P^\alpha(R^n)$. Theorem 1*c*, § 8, II, gives a formula which assigns to a system of functions $\{\varphi_0, \varphi_1, \ldots, \varphi_r\}$ defined on R^{n-1} a function $u \in P^\alpha(R^n)$ which has the functions φ_p as successive normal derivatives on the hyperplane $x_n = 0$. In this appendix we state similar formulas which give maps which are simultaneous, i.e., such that the function u assigned to u' (or to $\{\varphi_0, \varphi_1, \ldots, \varphi_r\}$) does not depend on the potential class of which u' (or each φ_p) is considered a member. The proofs are omitted since they are slight variations of those in § 8, II.

Remark. — The maps defined by Theorems 1*b* and 1*c* are simultaneous to some extent. Consider formula (8.3) of Theorem 1*b* for some fixed $\alpha = \alpha_0 > (n-k)/2$. If $u' \in P^{\alpha-(n-k)/2}(R^k)$, $(n-k)/2 < \alpha \leqslant \alpha_0$, then the function u defined by (8.3) belongs to $P^\alpha(R^n)$,

$$|u|_{\alpha, R^n} \leqslant c|u'|_{\alpha-(n-k)/2, R^k},$$

and $u|_{R^k} = u'$. The simultaneity properties of formula (8.5) of Theorem 1*c* were noted in Remark 2, § 8, II. However, the maps defined by the theorems of this appendix are simultaneous for larger ranges of α.

THEOREM I. — *Let* Φ *be an integrable function on* R^{n-k} *such that*

1° $\int_{R^{n-k}} |\Phi(\eta'')|^2(1 + |\eta''|^2)^\alpha \, d\eta'' < +\infty$ *for all* $\alpha \geqslant 0$.

2° $\int_{R^{n-k}} \Phi(\eta'') \, d\eta'' = 1$.

If $u' \in P^{\alpha-(n-k)/2}(R^k)$, $(n-k)/2 < \alpha$, *then the formula*

(AIII.1)

$$\hat{u}(\xi) = (2\pi)^{(n-k)/2}\Phi\left(\frac{\xi''}{(1+|\xi'|^2)^{1/2}}\right)(1+|\xi'|^2)^{-(n-k)/2}\widehat{u'}(\xi')$$

defines a function $u \in P^\alpha(R^n)$ *such that*

$$|u|^2_{\alpha, R^n} = (2\pi)^{n-k}\int_{R^{n-k}}(1+|\eta''|^2)^\alpha|\Phi(\eta'')|^2\,d\eta''\,|u'|^2_{\alpha-(n-k)/2, R^k}$$

and $u|_{R^k} = u'$.

Remark. — Formula (AIII.1) defines a bounded linear map of $H^{\alpha-(n-k)/2}(R^k)$ into $H^\alpha(R^n)$ ([20]) for each real α, with the property

$$\text{(AIII.2)}\quad \widehat{u'}(\xi') = (2\pi)^{(k-n)/2}\int_{R^{n-k}}\hat{u}(\xi', \xi'')\,d\xi'' \quad \text{a.e.}$$

For $\alpha > (n-k)/2$, (AIII.2) implies that u' is the restriction of u. For $\alpha \leqslant (n-k)/2$, R^k is an exceptional set for a function $u \in P^\alpha(R^n)$ and hence in general the restriction of such a function is not defined.

An example of a function Φ having the properties required in Theorem I is:

$$\Phi(\eta'') = (2\pi)^{(k-n)/2}e^{-(1/2)|\eta''|^2}.$$

For this Φ we can evaluate the bound of the map defined by Theorem I explicitly. We have

$$(2\pi)^{n-k}\int_{R^{n-k}}(1+|\eta''|^2)^\alpha|\Phi(\eta'')|^2\,d\eta'' = (\pi)^{(n-k)/2}\Psi((n-k)/2,\quad (n-k)/2+\alpha+1;\ 1),$$

where $\Psi(a, b; z)$ is the confluent hypergeometric function of the second kind. Also, in the case $\alpha = (n-k)/2$ (with this Φ) it can be shown that the corrected function u defined by (AIII.1) is continuous for $x'' \neq 0$ and that, if $u_{x''}(x') = u(x', x'')$, $u_{x''}$ converges to u' in $L^2(R^k)$ as x'' approaches 0.

([20]) For the definition of the spaces $H^\alpha(R^n)$ see, for instance, Hörmander [8, p. 45]. For $\alpha > 0$, $H^\alpha(R^n)$ is the space $P^\alpha(R^n)$ saturated relative to the class of sets of measure zero.

THEOREM II. — *Let* r *be an integer* $\geqslant 0$ *and assume* $0 \leqslant p \leqslant r$. *Let* $\psi_{p,r}(t)$ *be an integrable function on* R^1 *such that:*

1^o $\int_{-\infty}^{\infty}(1+t^2)^{\alpha}|\psi_{p,r}(t)|^2\,dt < +\infty$ *for all* $\alpha \geqslant 0$,

2^o $\int_{-\infty}^{\infty}(it)^q\psi_{p,r}(t)\,dt = \delta_{pq}$ *for* $0 \leqslant q \leqslant r$.

If $\varphi \in \mathrm{P}^{\alpha-p-1/2}(\mathrm{R}^{n-1})$, $p + 1/2 \leqslant \alpha$, *then the formula*

(AIII.3)

$$\hat{u}(\xi) = (2\pi)^{1/2}\psi_{p,r}\left(\frac{\xi_n}{(1+|\xi'|^2)^{1/2}}\right)(1+|\xi'|^2)^{-(p+1)/2}\hat{\varphi}(\xi')$$

defines a function $u \in \mathrm{P}^{\alpha}(\mathrm{R}^n)$ *such that*

$$|u|^2_{\alpha,\mathrm{R}^n} = (2\pi)\int_{-\infty}^{\infty}(1+t^2)^{\alpha}|\psi_{p,r}(t)|^2\,dt\,|\varphi|^2_{\alpha-p-1/2,\mathrm{R}^{n-1}}$$

and

$$\left.\frac{\partial^q u}{\partial x_n^q}\right|_{\mathrm{R}^{n-1}} = \delta_{pq}\varphi \qquad \text{for} \qquad 0 \leqslant q \leqslant r, \quad q < \alpha - 1/2.$$

The functions $\psi_{p,r}$ can be chosen in various ways. For example, let $\psi_{p,r}(t) = i^{-p}e^{-(1/2)t^2}\varphi_{p,r}(t)$ where $\varphi_{p,r}(t)$ is the polynomial of degree $\leqslant r$ such that 2^o holds.

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