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GRASSMANN–GRASSMANN CONORMAL VARIETIES, INTEGRABILITY, AND PLANE PARTITIONS

by Allen KNUTSON & Paul ZINN-JUSTIN (*)

ABSTRACT. — We give a conjectural formula for sheaves supported on (irreducible) conormal varieties inside the cotangent bundle of the Grassmannian, such that their equivariant K -class is given by the partition function of an integrable loop model, and furthermore their K -theoretic pushforward to a point is a solution of the level 1 quantum Knizhnik–Zamolodchikov equation. We prove these results in the case that the Lagrangian is smooth (hence is the conormal bundle to a subGrassmannian). To compute the pushforward to a point, or equivalently to the affinization, we simultaneously degenerate the Lagrangian and sheaf (over the affinization); the sheaf degenerates to a direct sum of cyclic modules over the geometric components, which are in bijection with plane partitions, giving a geometric interpretation to the Razumov–Stroganov correspondence satisfied by the loop model.

RÉSUMÉ. — Nous donnons une formule conjecturale pour des faisceaux cohérents de support des variétés conormales dans le fibré cotangent de la Grassmannienne, tels que leur classe de K -théorie équivariante est donnée par la fonction de partition d'un modèle de boucles intégrable, et que de plus leur image dans la K -théorie d'un point est solution de l'équation de Knizhnik–Zamolodchikov quantique de niveau 1. Nous démontrons ces résultats dans le cas où la Lagrangienne est lisse (donc le fibré conormal d'une sous-Grassmannienne). Pour pousser en avant vers un point, ou de manière équivalente vers son affinisation, nous dégenérons simultanément la Lagrangienne et son faisceau (sur l'affinisation); le faisceau dégénère en une somme directe de modules cycliques sur les composantes géométriques, qui sont en bijection avec des partitions planes, ce qui donne une interprétation géométrique à la correspondance de Razumov–Stroganov satisfaite par le modèle de boucles.

Keywords: Quantum Knizhnik–Zamolodchikov equation, equivariant K -theory, cotangent bundle of the Grassmannian, loop model.

2010 *Mathematics Subject Classification:* 14M15, 82B23, 14Q99.

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1. Introduction

In [17] solutions to the rational Yang–Baxter equation (YBE) were constructed using cohomology classes living on “symplectic resolutions”, in particular on the cotangent bundles of Grassmannians (the main symplectic resolutions considered here). Each class has a geometric origin, as the (usually reducible) singular support of a certain \mathcal{D} -module on the Grassmannian. In this paper we begin the study of a geometric origin for the corresponding trigonometric YBE, constructing sheaves on these cotangent bundles whose equivariant K -classes (once pushed to a point) we conjecture to satisfy the (trigonometric) quantum Knizhnik–Zamolodchikov equation.

For each Schubert variety $X^r \subseteq \mathrm{Gr}(n, N)$, we define a sheaf σ_r supported on its conormal variety $CX^r \subseteq T^*\mathrm{Gr}(n, N)$ (definitions appearing in Section 1.2). We give a conjectural formula for its equivariant K -theory class as a rectangular-domain partition function of a “quantum integrable” model. The sheaf cohomology groups of σ_r are modules over the affinization $\mu(CX^r)$; conjecturally, all cohomology vanishes unless the affinization map is birational, not dropping dimension. As the paper’s title indicates, in this paper we focus attention on (and prove the conjectures in) the case that X^r is a subGrassmannian (the only time X^r is smooth); the affinization $\mu(CX^r)$ of CX^r is then an A_3 quiver cycle.

The Stanley–Reisner ring of a simplicial complex has a basis given by monomials, and a “shelling” of the simplicial complex gives a partitioning of the monomials into orthants, allowing one to count those monomials without inclusion-exclusion. We do something closely analogous with a degeneration of $(\mu(CX^r), \mu_*\sigma_r)$, giving a partitioning into cones of a \mathbb{C} -basis of the module $\mu_*\sigma_r = H^0(CX^r; \sigma_r)$. Geometrically, this degenerates the base $\mu(CX^r)$ of the sheaf $\mu_*\sigma_r$ to a highly reducible scheme with very simple components, each bearing one summand of the degenerate sheaf (though our shelling statement is stronger).

One key difference between the results of [17, 21] and ours is that we work directly with sheaves on $T^*\mathrm{Gr}(n, N)$, not just K -classes thereof, which is a sort of positivity statement; in the subGrassmannian case we get a similar positivity on their pushforwards from the vanishing of their higher cohomology (Proposition 4.2). Another difference is that we work with (sheaves supported on) individual conormal varieties, whereas [17, 21] work with the stable basis (supported on unions). Their basis is positive upper triangular w.r.t. ours, the change of basis being given by maximal parabolic Kazhdan–Lusztig polynomials (the same change of basis which relates the

corresponding quantum integrable models: the six-vertex model for the stable basis and the Temperley–Lieb loop model for ours).

We now describe in more detail our main results.

1.1. Various combinatorial gadgets

We shall use interchangeably three sets in bijection:

- *Subsets* of $[N] := \{1, \dots, N\}$ of cardinality n ;
- *Young diagrams* fitting inside an $(N - n) \times n$ rectangle;
- *Link patterns* of size N with at most $\min(n, N - n)$ chords, that is, pairings of N vertices on a line (some possibly being left unpaired), in such a way that there are at most $\min(n, N - n)$ pairings, drawn as chords, and that the chords, as well as half-infinite lines coming out of unpaired vertices, may be drawn in a half-plane without any crossings.

We call the set of any of these objects $\binom{[N]}{n}$, and now describe the bijections.

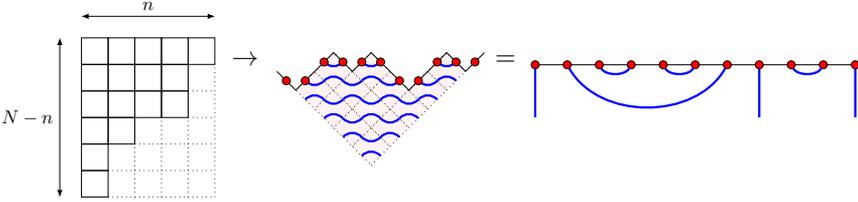
Given a Young diagram, we associate to it a subset as follows: if we number from 1 to N the boundary edges of the Young diagram from bottom left to top right, then the subset $r = \{r_1, \dots, r_n\}$ consists of all steps to the right. We always order increasingly elements of the subset, that is we always have $r_i < r_{i+1}$, $i = 1, \dots, n - 1$. We shall denote by \bar{r} its complement in $\{1, \dots, N\}$, which therefore consists of all steps up.

Given a link pattern with at most $\min(n, N - n)$ chords, then the subset consists of all “closings” of chords (vertices paired to another vertex to the left, where the numbering is from left to right), completed to cardinality n by including unpaired vertices starting from the left.

Finally, we also provide the bijection from Young diagrams to link patterns, since it will be useful later: rotate the Young diagram 45 degrees clockwise, then fill the boxes of its *complement* with the picture ; the connectivity of the lines emerging at the top reproduces the link pattern.

On an example,

$$(1.1) \quad r = \{1, 4, 6, 7, 10\} \in \binom{[11]}{5} :$$



Also, denote by $|r|$ the number of boxes of r ; equivalently, $|r| = n(N - n) - \sum_{i=1}^n (r_i - i)$.

We endow $\binom{[N]}{n}$ with the following order relation: \subseteq denotes inclusion of Young diagrams, or equivalently of the corresponding Schubert varieties. An example of the poset structure is shown in Figure 1.1.

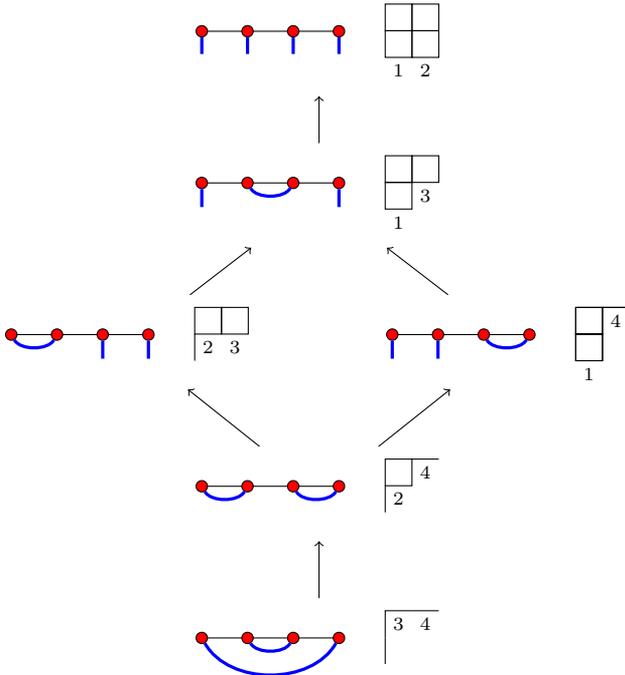
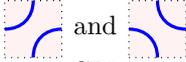
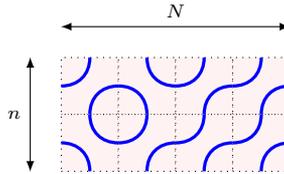


Figure 1.1. The poset structure for $\binom{[4]}{2}$.

When this order is reformulated in terms of subsets, it corresponds to pointwise greater or equal; we shall therefore denote the opposite order \leq , hoping this does not create any confusion:

$$r \subseteq s \Leftrightarrow s \leq r \Leftrightarrow s_i \leq r_i, \quad i = 1, \dots, n$$

A *Completely Packed Loop configuration* (or CPL, in short) is an assignment of the two possible plaquettes  to the faces of a $n \times N$ square grid, e.g., for $N = 4, n = 2$, one CPL is

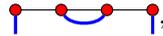


We say that a CPL has top-connectivity given by $r \in \binom{[N]}{n}$ iff when following the paths made by the (blue) lines, the connectivity of the external edge midpoints obeys the following rules:

- Denoting (l,r,b,t) for a midpoint situated on the left, right, bottom, or top sides, the allowed connectivities are $(b,l), (b,r), (b,t), (l,t), (t,t)$.
- The connectivity of the N midpoints across the top edge (ignoring connectivities outside the top side, i.e., declaring a midpoint connected to the left or bottom to be unpaired) reproduces the link pattern r .

Note that these conditions imply that if r has m chords, then a CPL with top-connectivity r has m pairings (t,t) , $n - m$ (t,l) and $N - n - m$ (b,t) .

A CPL may have paths which close onto themselves; we call them *loops*, and their number is denoted $|\text{loops}|$.

For example, the CPL above has top-connectivity , and $|\text{loops}| = 1$.

1.2. The geometric setup

Given two integers n and N such that $0 \leq n \leq N$, consider the Grassmannian $\text{Gr}(n, N) = \{V \leq \mathbb{C}^N : \dim V = n\}$ and its cotangent bundle $T^* \text{Gr}(n, N)$. $\text{GL}(N)$ acts on each of these, as do its Borel subgroups B_{\pm} and diagonal matrices $T_0 = B_+ \cap B_-$. An additional circle \mathbb{C}^{\times} acts on

$T^* \text{Gr}(n, N)$ by scaling of the cotangent spaces. Let $T := T_0 \times \mathbb{C}^\times$, with representation rings $K_{T_0} = \mathbb{Z}[z_1^\pm, \dots, z_N^\pm], K_{\mathbb{C}^\times} = \mathbb{Z}[t^\pm]$.

The T_0 -fixed points in $\text{Gr}(n, N)$ (or equivalently, T -fixed points in $T^* \text{Gr}(n, N)$, viewing $\text{Gr}(n, N)$ as its zero section) are coordinate subspaces, thereby labeled by subsets $r \in \binom{[N]}{n}$. Their B_- -orbits X_o^r are called *Schubert cells*, with conormal bundles denoted

$$CX_o^r := \{(x, \vec{v}) : x \in X_o^r, \vec{v} \in T_x^* \text{Gr}(n, N), \vec{v} \perp T_x(X_o^r)\} \subseteq T^* \text{Gr}(n, N).$$

Their closures $X^r := \overline{X_o^r}$ and $CX^r := \overline{CX_o^r}$ we call *Schubert varieties* and *conormal Schubert varieties*, respectively.

We shall consider certain T -equivariant coherent sheaves on $T^* \text{Gr}(n, N)$, and their classes in the equivariant K -theory ring $K_T(T^* \text{Gr}(n, N)) \cong K_T(\text{Gr}(n, N)) \cong K_{T_0}(\text{Gr}(n, N)) \otimes \mathbb{Z}[t^\pm]$. For $r \in \binom{[N]}{n}$, define the *restriction map*

$$\begin{aligned} |_r : \mathbb{Z}[y_1^\pm, \dots, y_n^\pm, z_1^\pm, \dots, z_N^\pm] &\rightarrow K_{T_0} \\ f &\mapsto f|_r := f(z_{r_1}, \dots, z_{r_n}, z_1, \dots, z_N) \end{aligned}$$

in which case

$$K_{T_0}(\text{Gr}(n, N)) \cong \mathbb{Z}[y_1^\pm, \dots, y_n^\pm, z_1^\pm, \dots, z_N^\pm]^{\mathcal{S}_n} \Big/ \bigcap_{r \in \binom{[N]}{n}} \ker(|_r)$$

where the \mathcal{S}_n permutes the y Laurent variables, which are the Chern roots of the tautological n -plane bundle on the Grassmannian. Thus to describe a class, it suffices to give a Laurent polynomial and check its symmetry in the y s.

CONJECTURE 1.1. — Assume N even. There exists a T -equivariant coherent rank 1 sheaf σ_r supported on CX^r , defined in Section 3.3, whose class in $K_T(T^* \text{Gr}(n, N))$ is represented by

$$(1.2) \quad [\sigma_r] = m_r \prod_{i,j=1}^n a(y_i/y_j)^{-1} \sum_{\substack{\text{CPLs with} \\ \text{top-connectivity } r}} \tau^{|\text{loops}|} \prod_{i=1}^n \prod_{j=1}^N \begin{cases} a(y_i/z_j) \\ b(y_i/z_j) \end{cases} \begin{matrix} \text{[diagram]} \\ \text{[diagram]} \end{matrix}$$

where the product is over rows i and columns j of the grid (the choice of a or b depending on the type of plaquette at (i, j)), the various plaquette

weights are given by

$$\begin{aligned} a(x) &= t^{-1/2}x^{1/2} - t^{1/2}x^{-1/2} \\ b(x) &= x^{-1/2} - x^{1/2} \\ \tau &= t^{-1/2} + t^{1/2}, \end{aligned}$$

and m_r is a monomial (with unit coefficient) in the $z_i^{1/2}$ and $t^{1/2}$.

The r.h.s. of (1.2) may not seem well-defined in $K_T(T^* \text{Gr}(n, N))$ due to the presence of a denominator, but we show in Remark 2.4 that it is. The symmetry in the y s will also be proven, in Lemma 2.2. Lastly, note that m_r can be absorbed into $[\sigma_r]$, up to issues with square roots of the z_i , by tensoring σ_r with a trivial line bundle; one could even dispense with the “ N even” hypothesis by allowing square roots of the Chern roots y_i as well.

In the sections to come we won’t be concerned with the sheaves σ_r so often as their sheaf cohomology groups, which we conjecture to all vanish unless the map from CX^r to its affinization doesn’t drop dimension. In Section 3.4 this geometric condition will be shown equivalent to a Dyck path condition on r , and the vanishing conjecture established for H^0 when X^r is Gorenstein.

1.3. Polynomial solution of the level 1 quantum Knizhnik–Zamolodchikov equation

We now restrict to the case $N = 2n$. This allows for the possibility to have full link patterns, that is, link patterns for which every vertex is paired. We denote the set of full link patterns by $LP(N)$; from the point of view of Young diagrams, it is exactly the subset of Young diagrams which are inside the “staircase” diagram $\nabla := \{2i, 1 \leq i \leq n\}$. Its cardinality is the Catalan number $c_n = \frac{(2n)!}{n!(n+1)!}$.

We first recall the following

THEOREM 1.2 ([6, 12, 19]). — *The space of polynomials in N variables z_1, \dots, z_N of degree at most $n(n - 1)$ satisfying the wheel condition*

$$\left\{ P \in \mathbb{C}(t^{1/2})[z_1, \dots, z_N] : P(\dots, z, \dots, tz, \dots, t^2z, \dots) = 0 \right\}$$

is of dimension c_n over $\mathbb{C}(t^{1/2})$. It has a basis indexed by link patterns $(\Psi_r)_{r \in LP(N)}$ given by the dual basis condition

$$\Psi_r \left(z_i = \begin{cases} t^{-1/2} & i \in \bar{s} \\ t^{1/2} & i \in s \end{cases}, i = 1, \dots, N \right) = \delta_{r,s} \tau^{|r|}, \quad r, s \in LP(N)$$

where $\tau = t^{1/2} + t^{-1/2}$.

The Ψ_r are homogeneous of degree $n(n - 1)$. They have remarkable properties, many of which follow from the (level 1) quantum Knizhnik–Zamolodchikov (qKZ) equation. Given a vector Ψ with entries Ψ_r in a basis indexed by $LP(N)$, the level 1 qKZ system is:

$$\begin{aligned} \Psi(z_1, \dots, z_{i+1}, z_i, \dots, z_N) &= \check{R}_i(z_i/z_{i+1})\Psi(z_1, \dots, z_i, z_{i+1}, \dots, z_N), \\ \Psi(z_2, \dots, z_N, t^3 z_1) &= (-t^{1/2})^{3(n-1)}\rho\Psi(z_1, \dots, z_N) \end{aligned}$$

with $i = 1, \dots, N - 1$, where $\check{R}_i(z) = \frac{z-t+t^{1/2}(1-z)e_i}{1-tz}$, e_i is the Temperley–Lieb operator  acting on link patterns by connecting i and $i + 1$ (with a weight of τ if they were already connected),⁽¹⁾ and ρ is the rotation operator that shifts cyclically to the right link patterns. We shall not make use of this system of equations in the present work and refer to [5, 31] for details.

We claim the following

CONJECTURE 1.3. — *The pushforward of σ_r to a point in localized T -equivariant K -theory is equal, up to normalization, to Ψ_r :*

$$\pi_*[\sigma_r] = \begin{cases} 0 & r \not\subseteq \nabla \\ (1 - t)^{n(n-1)}\tilde{m}_r \prod_{1 \leq i < j \leq N} (1 - tz_i/z_j)^{-1} \Psi_r & r \subseteq \nabla \end{cases}$$

Here \tilde{m}_r is another monomial in the $z_i^{1/2}$, related to the previous one by $\tilde{m}_r = \prod_{i=1}^N z_i^{n/2+1-i} m_r$.

We will prove these conjectures in a forthcoming paper.

In fact, it would not be difficult to show that Conjecture 1.1 implies Conjecture 1.3 (the first equation of the qKZ system is naturally satisfied by $\pi_*[\sigma_r]$ up to normalization of the R -matrix, and together with the initial condition of $\pi_*[\sigma_\emptyset]$, it determines them uniquely).

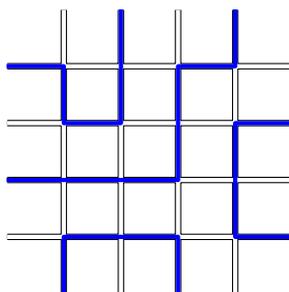
The factor $\prod_{1 \leq i < j \leq N} (1 - tz_i/z_j)$ can be naturally interpreted in terms of the weights of the space of strict upper triangular matrices $\text{Mat}_{<}(N)$; see Section 3.2 for details.

The limit $t \rightarrow 1$ corresponds on the integrable side to the “rational” limit from the quantum KZ equation to the difference KZ equation; on the geometric side, to the limit from K -theory to cohomology, where some results similar to Conjecture 1.3 are known [5, 22, 32].

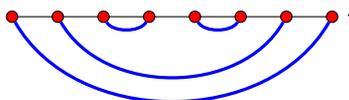
⁽¹⁾Note that if we similarly identify the identity with , then \check{R}_i is nothing but the combination of plaquettes occurring in Conjecture 1.1, up to normalization. See also Section 2.

1.4. Fully Packed Loops and the Razumov–Stroganov correspondence

A *Fully Packed Loop configuration* (in short, *FPL*) is an assignment of two possible states (empty, occupied) to the edges of a $n \times n$ square grid such that every vertex is traversed by exactly one path, and the external edges are alternatingly occupied and empty (declaring that the topmost left external edge is occupied); e.g., for $n = 4$,



Numbering the occupied external edges clockwise from the leftmost top one, we can associate to an FPL the connectivity of these external edges encoded as a (full) link pattern; in the present example, we find



(Actually the choice of the starting point for the labelling of the external edges is irrelevant, at least for enumerative purposes, since Wieland [28] constructs a bijection of FPLs which rotates cyclically the connectivity of the external edges.) Denote by FPL_r the set of FPLs with connectivity given by link pattern r .

One way to see the connection with what precedes is

CONJECTURE 1.4. — Ψ_r can be decomposed as a sum of products of the form

$$\Psi_r = \sum_{f \in \text{FPL}_r} \prod_{\alpha=1}^{n(n-1)} \frac{t^{-r_{f,\alpha}/2} z_{j_{f,\alpha}} - t^{r_{f,\alpha}/2} z_{i_{f,\alpha}}}{t^{-1/2} - t^{1/2}}$$

where $r_{f,\alpha} \in \{1, 2\}$.

This conjecture is formulated somewhat implicitly in [3, §4].

Specializing the z_i to 1 and τ to 1 (i.e., t to a nontrivial cubic root of unity), leads to the following result, which, by combining the q KZ approach [4, 5, 31] to loop models and the proof [2] by Cantini and Sportiello of the Razumov–Stroganov conjecture [20], is actually a theorem:

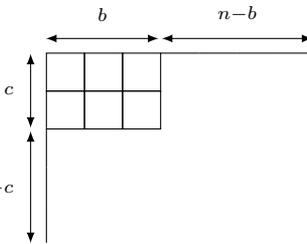
THEOREM 1.5.

$$\Psi_r(z_i = 1, i = 1, \dots, N; \tau = 1) = |\text{FPL}_r|$$

Deriving Conjecture 1.4 from the properties of the sheaf σ_r would provide a geometric justification for the Razumov–Stroganov correspondence.

1.5. The rectangular case and plane partitions

In the present work, we study σ_r in the case that r is a rectangular Young diagram:

(1.3) $r =$ 

for two nonnegative integers b, c . We also define for future use $a = N/2 - (b + c)$. We then prove Conjecture 1.1 in that case.

We further specialize to $N = 2n$, as in Section 1.3. If $a < 0$, we shall immediately conclude that $\pi_*[\sigma_r] = 0$, corresponding to the trivial case of Conjecture 1.3. If $a \geq 0$, we note that the link pattern corresponding to r is of type “ (a, b, c) ”, that is, of the form

(1.4) $r =$ 

We shall then prove Conjecture 1.3 and 1.4 in that case. Note that such a type of link pattern was already considered in [7, 30, 33] in the context of FPL enumeration and the Razumov–Stroganov conjecture, but without

any connection to geometry. In particular, the following result will play a role in what follows:

THEOREM 1.6 ([7]). — *There is a bijection between $FPL_{(a,b,c)}$ (FPLs with connectivity (a, b, c)) and $PP(a, b, c)$, which is by definition the set of plane partitions of size $c \times b$ and maximal height a .*

For the purposes of this paper, it is best to use the following definition of $PP(a, b, c)$: it is the set of a -tuplets of $\binom{[b+c]}{b}$ for which the order \leq (pointwise comparison of ordered elements of subsets) is a total order:

$$PP(a, b, c) := \{(s_1, \dots, s_a) : s_1 \leq s_2 \leq \dots \leq s_a\}$$

In turn, we can depict elements of $PP(a, b, c)$ in various equivalent ways, as demonstrated in Figure 1.2: from top left to bottom right, plane partitions, lozenge tilings, Non-Intersecting Lattice Paths (NILPs), dimer configurations. These representations will be discussed again in what follows when they are needed.

1.6. Plan of the paper

In Section 2 we study CPLs using integrability and in particular the Yang–Baxter equation, with which we show that the conjectured formula 1.1 has the right symmetry and vanishing properties. In Section 3 we give detail on conormal varieties to Grassmannian Schubert varieties, and define the sheaves referenced in Conjectures 1.1 and 1.3. Our definition of these sheaves in Section 3.3, for arbitrary G/P , is much more general than is required for the rest of the paper. In Section 4 we specialize to the case of a smooth Schubert subvariety of a Grassmannian $\text{Gr}(n, 2n)$, determine the conjectured sheaf in this case, and compute the degree $2^{bc} |PP(a, b, c)|$ of the conormal variety. In Section 5 we define the family degenerating the conormal variety and the sheaf it bears, whose special fiber we determine in Section 6. It turns out to have one component for each element of $PP(a, b, c)$, a toric complete intersection of bc many quadrics. With this in hand, we prove in Section 7 our Conjectures 1.1 and 1.3 in the Grassmann–Grassmann case.

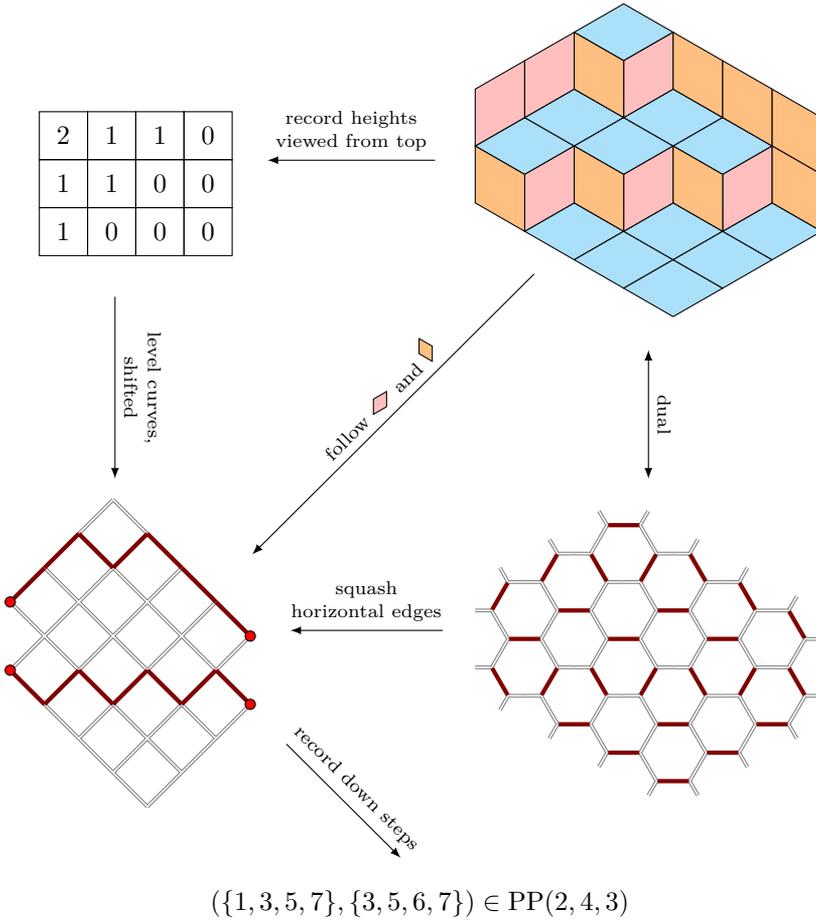


Figure 1.2. Various depictions of a particular $a = 2, b = 4, c = 3$ configuration.

2. Integrability of the CPL model

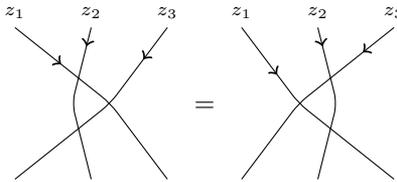
2.1. Basic properties

Introduce the following notation, borrowed from integrable models:

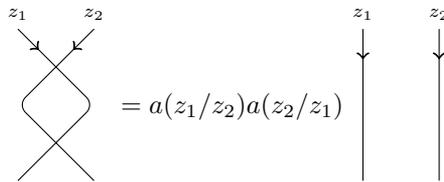
$$\begin{array}{c} z \\ \downarrow \\ \boxed{\begin{array}{c} \cdot \\ \downarrow \\ \cdot \\ \downarrow \\ \cdot \end{array}} \leftarrow y = a(y/z) \begin{array}{c} \boxed{\cdot} \\ \boxed{\cdot} \\ \boxed{\cdot} \end{array} + b(y/z) \begin{array}{c} \boxed{\cdot} \\ \boxed{\cdot} \\ \boxed{\cdot} \end{array}
 \end{array}$$

where recall from Conjecture 1.1 that $a(x) := t^{-1/2}x^{1/2} - t^{1/2}x^{-1/2}$ and $b(x) := x^{-1/2} - x^{1/2}$. The dotted lines are purely cosmetic and will be frequently omitted.

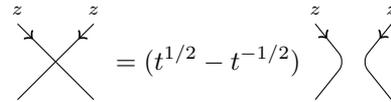
PROPOSITION 2.1. — *The following identities hold: the Yang-Baxter equation*

(2.1) 

the unitarity equation

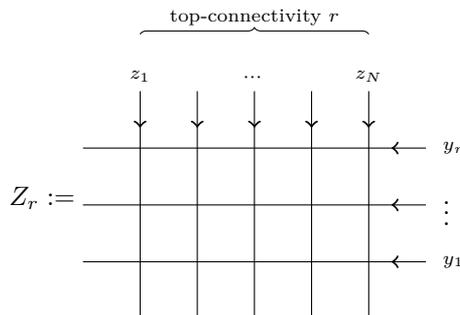
(2.2) 

and the special value

(2.3) 

All these identities should be understood as an equality of the coefficients on both sides of all the diagrams with a given connectivity of the external points, with the rule that each closed loop incurs a weight of τ . The proof of the proposition is a standard calculation.

Let us now introduce the CPL partition function with given top-connectivity $r \in \binom{[N]}{n}$:



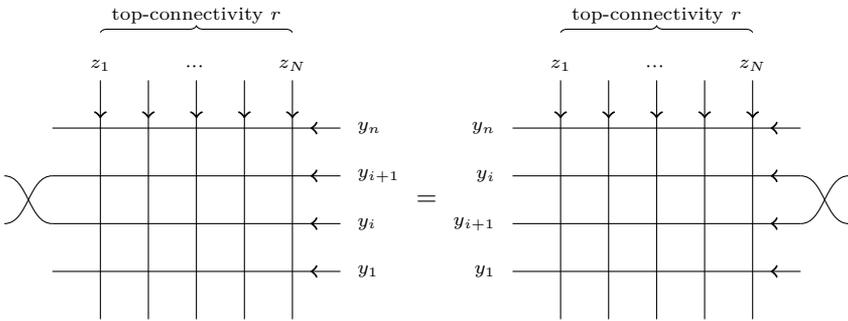
where we recall the connectivity rules: (b,l), (b,r), (b,t), (l,t), (t,t), and the connectivity of the top midpoints is given by the link pattern r .

Denote by τ_i the elementary transposition $i \leftrightarrow i+1$. By abuse of notation, also let τ_i denote the operator acting on polynomials of the variables $z_i, i = 1, \dots, N$ that permutes z_i and z_{i+1} .

LEMMA 2.2. — We have the two symmetry properties:

- Z_r is a symmetric function of the y_i .
- Assume i and $i + 1$ are not connected in r . Then $\tau_i Z_r = Z_r$.

Proof. — This is a standard Yang–Baxter-based proof. Repeated application of (2.1) leads to



Imposing top-connectivity r means in particular that left vertices are not allowed to connect between themselves, and the same for right vertices. This implies that the extra plaquette on either side of the equation above (represented by a crossing) must be of the form , so it does not affect connectivity and can be removed, and its weight $a(y_i/y_{i+1})$ compensates as well between l.h.s. and r.h.s. Once these plaquettes are removed, we get back to Z_r itself, except on the left hand side y_i and y_{i+1} are switched. This implies symmetry of Z_r by exchange of y_i and y_{i+1} for all $i = 1, \dots, n - 1$, and therefore by any permutation of the y 's.

The same argument, with the extra crossing inserted at the bottom (necessarily of the form ) and then moved upwards, leads to the formula

$$(2.4) \quad a(z_{i+1}/z_i)Z_r = a(z_{i+1}/z_i)\tau_i Z_r + b(z_{i+1}/z_i) \sum_{s: e_i s=r} \tau^{\delta_{r,s}} \tau_i Z_s$$

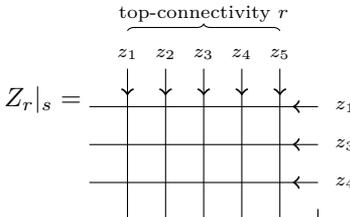
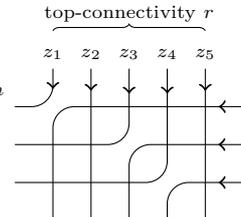
where the summation is over link patterns s which can be obtained from r by concatenating them with $e_i = \text{crossing with blue dots}$. This shows in particular that if i and $i+1$ are not connected in r , the summation is empty and $\tau_i Z_r = Z_r$. \square

2.2. Reformulation of Conjecture 1.1

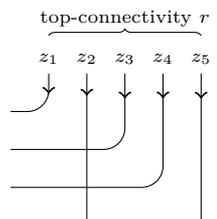
With these notations, Conjecture 1.1 is written

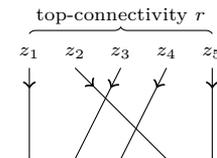
$$[\sigma_r] = m_r \prod_{i,j=1}^n a(y_i/y_j)^{-1} Z_r$$

We now propose an alternate formulation of this conjecture. A class in $K_T(T^* \text{Gr}(n, N))$ is entirely determined by its restrictions to T -fixed points, which are indexed by $\binom{[N]}{n}$. Recall from Section 1.2 that the restriction map to the coordinate subspace \mathbb{C}^s , $s \in \binom{[N]}{n}$, denoted by $|_s$, amounts to the specialization $y_i = z_{s_i}$, $i = 1, \dots, n$. We are thus naturally led to the computation of Z_r after such a substitution, which we can perform with the help of Proposition 2.1; here shown on an example:

(2.5) $Z_r|_s =$  $= a(1)^n$ 

$s = \{1, 3, 4\}$

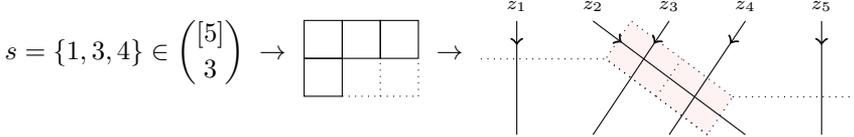
$= \prod_{i \in s, j \in s} a(z_i/z_j) \prod_{\substack{i \in s, j \in \bar{s} \\ i < j}} a(z_i/z_j)$ 

$= \prod_{i \in s, j \in s} a(z_i/z_j) \prod_{\substack{i \in s, j \in \bar{s} \\ i < j}} a(z_i/z_j)$ 

where between the first and second line we have pulled out all the south-east lines using Proposition 2.1, producing the first factor, and then removed them altogether because the constraint that bottom vertices cannot connect between themselves forces plaquettes of type a , producing the second factor. After rearranging the lines to produce the final picture, the connectivity must be understood as follows: the top vertices have connectivity given

by the link pattern r , whereas the first n bottom vertices cannot connect between themselves, and similarly for the last $N - n$.

There are two ways to understand the resulting picture. On the one hand, reintroducing the dotted lines, we recognize the complement of the Young diagram of s (cf. (1.1)):



On the other hand, we also recognize this diagram to be the graphical representation of a (or any) reduced word of the Grassmannian permutation of $\{1, \dots, N\}$ which sends $\{1, \dots, n\}$ (at the bottom) to s (at the top).⁽²⁾

Let us therefore define $Z_{r,s}$ to be the prefactor $\prod_{\substack{i \in s, j \in \bar{s} \\ i < j}} a(z_i/z_j)$ times the CPL partition function for the diagram defined in either of the two ways above, with top-connectivity given by r , in which we recall that each crossing represents as above one of the two plaquettes with their weights and that closed loops have a weight of τ .

We can thus reformulate Conjecture 1.1 as follows:

CONJECTURE 1.1'. — *The restriction of the K_T -class of the sheaf σ_r to the fixed point s satisfies*

$$(2.6) \quad [\sigma_r|_s] = m_r Z_{r,s}$$

This is the same *sort* of restriction-to-fixed-points formula as in [25] (or the AJS/Billey and Graham/Willems formulae for restricting Schubert not conormal Schubert classes), except that [25] is working with Maulik–Okounkov’s stable basis, not our basis, and (less importantly) that formula in [25] is in H_T^* not K_T .

Remark 2.4. — For $i, j \in \{1, \dots, N\}$, and $s = \{\dots, i, \dots\}$, $s' = \{\dots, j, \dots\}$ related by the transposition $i \leftrightarrow j$, because of the very definition of the $Z_{r,s}$ as a specialization $(y_1, \dots, y_n) = (z_{s_1}, \dots, z_{s_n})$ of Z_r , the following congruence holds:

$$z_i - z_j \mid (Z_r|_s - Z_r|_{s'})$$

This is exactly the K -theoretic GKM criterion for being in the image of the restriction map (see e.g. [23, A.4], [26, Cor. 5.11]). Contrary to Z_r itself,

⁽²⁾ In fact, the Yang–Baxter equation (2.1) would ensure independence of the choice of reduced word, but is not necessary in the Grassmannian case, nor more generally in the 321-avoiding case. Moreover, because of the unitarity equation (2.2), the reducedness of the word is irrelevant.

the $Z_r|_s$ are Laurent polynomials, and are therefore the point restrictions of a uniquely defined element of $K_T(\text{Gr}(n, N)) \cong K_T(T^* \text{Gr}(n, N))$.

2.3. Two lemmas

We need two more lemmas.

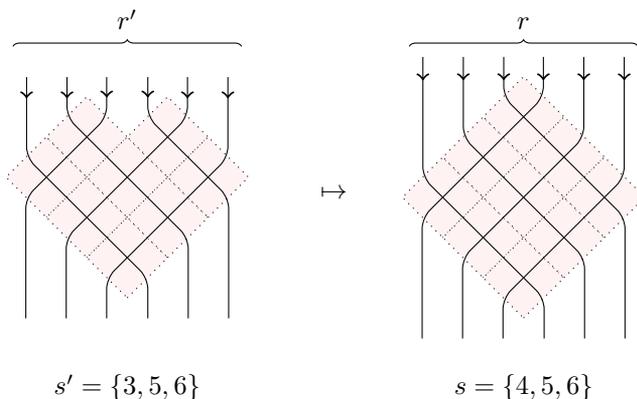
LEMMA 2.5. — $Z_{r,s}$ satisfies the triangularity property: $Z_{r,s} = 0$ unless $s \subseteq r$, and

$$(2.7) \quad Z_{r,r} = \prod_{\substack{i \in r, j \in \bar{r} \\ i < j}} a(z_i/z_j) \prod_{\substack{i \in r, j \in \bar{r} \\ i > j}} b(z_i/z_j).$$

Proof. — Induction on s . If $s = \{1, \dots, n\}$, the diagram of $Z_{r,s}$ contains no plaquette, and the resulting top-connectivity is the completely unpaired link pattern, i.e., $r = s$.

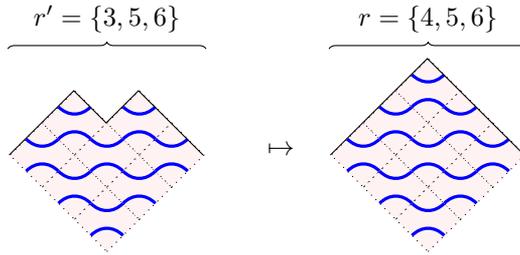
Now, pick $s \in \binom{[N]}{n}$, and assume that the property is true for any s' , $s \subsetneq s'$. Denote by s^C the complement of the Young diagram of s in the rectangle $(N - n) \times n$.

Pick any protruding box in s^C . Choose s' to be the Young diagram obtained from s by adding that box. Equivalently, choose an i such that $i \notin s$, $i + 1 \in s$, and define the subset $s' = \tau_i s$. This means that the $Z_{r,s}$ can be obtained from the $Z_{r',s'}$ by adding an extra crossing, e.g.,



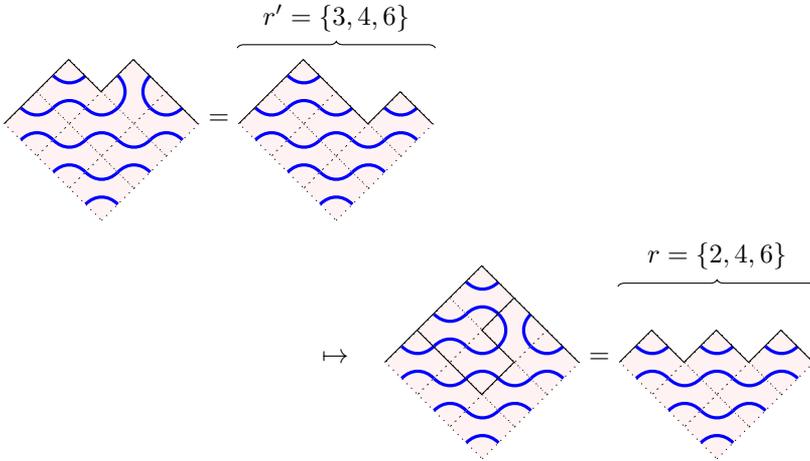
By the induction hypothesis, the only nonzero $Z_{r',s'}$ are the ones for which $s' \subseteq r'$. The extra crossing can take two forms:

- (1)  $($, which does not change the connectivity; these contribute to $Z_{r',s}$, which satisfies the upper triangularity of the lemma as $s \subsetneq s' \subseteq r'$.
- (2) , that is, $r = e_i r'$, where as before we denote $e_i r'$ the link pattern obtained from r' by pasting this extra plaquette. There are four possibilities, depending on the local configuration of r' :
 - (2a) $i \in r', i+1 \notin r'$: then $e_i r' = \tau_i r'$, i.e., the Young diagram of $(e_i r')^C$ is obtained from r'^C by adding an extra box:



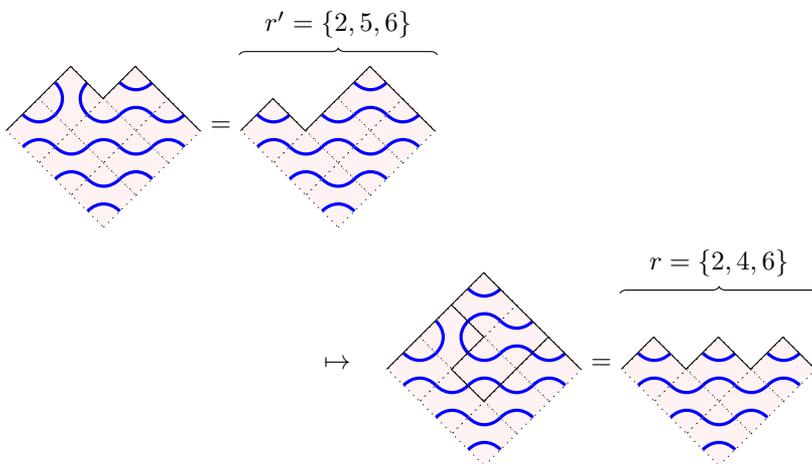
Since the same procedure goes from s'^C to s^C , we still have $(e_i r')^C \subseteq s^C$, or $s \subseteq e_i r' = r$.

- (2b) $i, i+1 \in r'$:

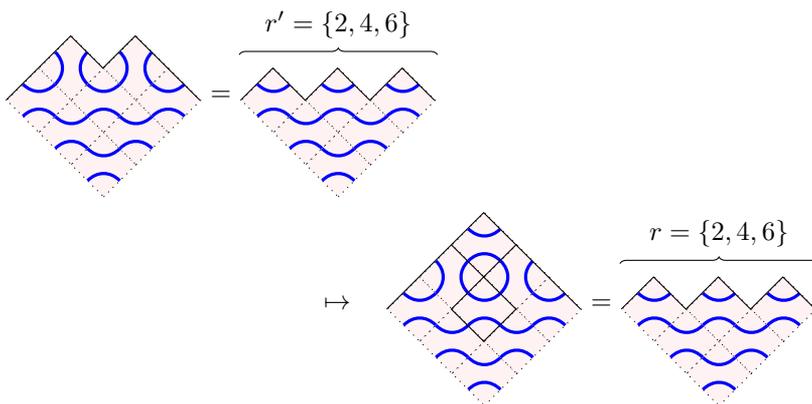


Effectively using the Temperley–Lieb relation $e_i e_{i-1} e_i = e_i$, we see that the extra e_i destroys the plaquette south-west of it, resulting in $e_i r' = \tau_{i-1} r'$, so that $s \subsetneq s' \subseteq r' \subsetneq e_i r' = r$.

(2c) $i, i + 1 \notin r'$ is treated similarly as the previous case:



(2d) $i \notin r', i \in r'$:



i.e., r' pairs i and $i + 1$, in which case effectively using $e_i^2 \propto e_i$, we find $e_i r' = r'$, and once again we note $s \subsetneq s' \subseteq r' = r$.

Only possibility (2a) can lead to $r = s$, and only in the case $r' = s'$ (as in the example); we immediately obtain inductively the formula for $Z_{r,r}$. \square

LEMMA 2.6. — Assume i and $i + 1$ are not connected in r . Then $\tau_i Z_{r,s} = Z_{r,\tau_i s}$.

Proof. — This is a direct consequence of the two parts of Lemma 2.2. \square

3. Geometry and the conjectured sheaves

3.1. Parametrization

We coordinatize the Grassmannian via its Plücker embedding:

$$(3.1) \quad \text{Gr}(n, N) = \left\{ \left[p_s, s \in \binom{[N]}{n} \right] \in \mathbb{P}^{\binom{N}{n}-1} : \forall s_{\pm} \in \binom{[N]}{n \pm 1}, \sum_{i \in s_{+} \setminus s_{-}} p_{s_{-} \cup i} p_{s_{+} \setminus i} = 0 \right\}$$

(We use here the implicit convention that when adding/subtracting indices from a subset, the index is added/subtracted at the end, but the sign of the permutation sorting the indices into increasing order must be introduced.)

In these coordinates, Hodge showed that Schubert varieties and cells are easily defined as:

$$X^r = \left\{ \left[p_s, s \in \binom{[N]}{n} \right] \in \text{Gr}(n, N) : p_s = 0 \text{ unless } s \subseteq r \right\},$$

$$X_o^r = X^r \cap \{p_r \neq 0\}$$

One has $\dim X^r = |r|$, the area of the partition.

The cotangent bundle $T^* \text{Gr}(n, N)$ can be identified with

$$T^* \text{Gr}(n, N) = \{(V, u) \in \text{Gr}(n, N) \times \text{Mat}(N) : \text{Im } u \subseteq V \subseteq \text{Ker } u\}$$

and its projection μ (the *moment map*) to the second factor has image $\{u \in \text{Mat}(N) : u^2 = 0, \text{rank}(u) \leq n\}$, the closure of a nilpotent $\text{GL}(N)$ -orbit (of which μ is the Springer resolution, which won't be especially relevant). In Plücker coordinates, and denoting by $M = (M_{i,j})$ the transposed (for convenience) matrix of u , this space is defined by the following equations:

$$(3.2) \quad T^* \text{Gr}(n, N) = \left\{ \left[p_s, s \in \binom{[N]}{n} \right] \in \text{Gr}(n, N), (M_{i,j}) \in \text{Mat}(N) : \right.$$

$$\left. \begin{aligned} & \sum_{j \in s_{+}} p_{s_{+} \setminus j} M_{i,j} = 0, \quad s_{+} \in \binom{[N]}{n+1}, 1 \leq i \leq n \\ & \sum_{i \in \bar{s}_{-}} p_{s_{-} \cup i} M_{i,j} = 0, \quad s_{-} \in \binom{[N]}{n-1}, 1 \leq j \leq n \end{aligned} \right\}$$

Let $\text{Mat}_{<}(N)$ denote the space of strictly upper triangular matrices. If we consider pairs $([p_s], M) \in T^* \text{Gr}(n, N)$ such that $M \in \text{Mat}_{<}(N)$, we get the union of the conormal bundles of Schubert cells, or equivalently, the union

of the conormal Schubert varieties CX^r ; this is because $M \in \text{Mat}_{<}(N)$ is the moment map condition for the action of the Borel subgroup B_- of invertible upper triangular matrices.

$$(3.3) \quad \bigcup_{r \in \binom{[N]}{n}} CX^r = \left\{ \left[\begin{array}{l} p_s, s \in \binom{[N]}{n} \\ \sum_{j \in s_+} p_{s_+ \setminus j} M_{i,j} = 0, s_+ \in \binom{[N]}{n+1}, 1 \leq i \leq n \\ \sum_{i \in \bar{s}_-} p_{s_- \cup i} M_{i,j} = 0, s_- \in \binom{[N]}{n-1}, 1 \leq j \leq n \end{array} \right] \in \text{Gr}(n, N), (M_{i,j}) \in \text{Mat}_{<}(N) : \right\}$$

(the only difference from (3.2) being the $<$ subscript on the Mat).

The CX^r are the irreducible components of that space; under the map μ they are sent into the *orbital scheme* $\{M \in \text{Mat}_{<}(N) : M^2 = 0, \text{rank}(M) \leq \min(n, N - n)\}$. More precisely, if the link pattern r has the maximal number of chords, then CX^r is sent to an irreducible component of that orbit closure, called an *orbital variety* (the other images $\mu(CX^r)$ have smaller dimension than the components, i.e. the general fibers have positive dimension; we work out these general fibers in Section 3.4). As the action of B_+ on the nilpotent orbit closure $\{M : M^2 = 0\}$ has finitely many orbits (it is *spherical*), each $\mu(CX^r)$ is a B_+ -orbit closure. It is easy to describe a representative of that orbit [16]: define $r_{<}$ to be the upper triangular matrix with 1s at (i, j) for each chord $i < j$ of the link pattern of r , 0s elsewhere. Then

$$\mu(CX^r) = \overline{B_+ \cdot r_{<}}$$

where \cdot means conjugation action.

Rank conditions of Southwest submatrices are preserved by B_+ -conjugation, so that we find the following inclusion and (by working a little harder [24]) even the equality of sets

$$(3.4) \quad \mu(CX^r) = \{M = (M_{i,j}) : M^2 = 0, \text{ and for each } (i, j), \text{ rank of } M \text{ Southwest of } (i, j) \leq \text{rank of } r_{<} \text{ Southwest of } (i, j)\}.$$

3.2. Reformulation of Conjectures 1.3 and 1.4

Since $\mu : \bigcup_r CX^r \rightarrow \text{Mat}_{<}(N)$ is proper, we can define $\mu_*[\sigma_r] \in K_T(\text{Mat}_{<}(N)) \cong K_T(pt) \cong \mathbb{Z}[t^\pm, z_1^\pm, \dots, z_N^\pm]$ without localization (i.e.,

without tensoring with the fraction field of $K_T(pt)$). Its relation

$$(3.5) \quad \mu_*[\sigma_r] = \pi_*[\sigma_r] \prod_{1 \leq i < j \leq N} (1 - t z_i/z_j)$$

to $\pi_*[\sigma_r]$, the (improper) pushforward to a point, derives from the weights $t z_i/z_j$, $1 \leq i < j \leq N$ of the T -action on the affine space $\text{Mat}_{<}(N)$.

Conjecture 1.3 can then be reformulated in terms of $\mu_*[\sigma_r]$ as

CONJECTURE 1.3'. — *The pushforward of σ_r to $\text{Mat}_{<}$ in T -equivariant K -theory is equal, up to normalization, to Ψ_r :*

$$\mu_*[\sigma_r] = \begin{cases} 0 & r \not\subseteq \nabla \\ (1-t)^{n(n-1)} \tilde{m}_r \Psi_r & r \subseteq \nabla \end{cases}$$

And similarly, Conjecture 1.4 can be rewritten assuming Conjecture 1.3 as

CONJECTURE 3.2. — *$\mu_*[\sigma_r]$ can be decomposed as a sum of products of the form*

$$\mu_*[\sigma_r] = \sum_{f \in \text{FPL}_r} m_f \prod_{a=1}^{n(n-1)} (1 - t^{r_{f,a}} z_{i_{f,a}}/z_{j_{f,a}})$$

where $r_{f,a} \in \{1, 2\}$, and m_f is a monomial. Explicitly,

$$m_f = \tilde{m}_r t^{-|\{a: r_{f,a}=2\}|/2} \prod_{a=1}^{n(n-1)} z_{j_{f,a}}.$$

We comment on the vanishing conclusion of Conjecture 1.3'. If we coarsen from K -homology to ordinary (Borel–Moore) homology, the K -class $[\sigma_r]$ maps to the fundamental class $[CX^r]$. As we will show in Section 3.4, the condition $r \subseteq \nabla$ is equivalent to $\mu : CX^r \rightarrow \mu(CX^r)$ being birational, so when $r \not\subseteq \nabla$ we get the homology vanishing result $\mu_*[CX^r] = 0$. In this sense, the subtle construction of σ_r in the next section is our attempt to refine this simple vanishing in homology to a much more precise vanishing in K -homology. (What one learns for free is that the sheaves $R\mu_i\sigma_r$ are supported on proper subschemes of $\mu(CX^r)$ when $r \not\subseteq \nabla$, rather than learning that their support is actually empty.)

3.3. The conjectured sheaves $\{\sigma_r\}$

We define a sheaf σ_r on any conormal Schubert variety $CX^r \subseteq T^*G/P$, although our most general conjectures concern the case G/P a Grassmannian.

We will need to twist the structure sheaf of $CX^r := \overline{CX^r_\circ}$ by a line bundle not available on T^*G/P . So let $g : G/B \rightarrow G/P$ be the G -equivariant projection, and $w \in W$ the minimum-length lift of $r \in W/W_P$, making $g : X^w_\circ \rightarrow X^r_\circ$ an isomorphism. These give us the commuting squares

$$\begin{array}{ccccc}
 G/B & \longleftarrow & g^*(T^*G/P) & \longleftrightarrow & \widetilde{CX^r} := \text{closure of} \\
 & & \downarrow & & \{(x \in X^w_\circ, \vec{v} \in T^*_{f(x)}G/P) : \vec{v} \perp T_{f(x)}X^r_\circ\} \\
 \downarrow g & & \downarrow & & \downarrow \wr \\
 G/P & \longleftarrow & T^*G/P & \longleftrightarrow & CX^r := \text{closure of} \\
 & & & & \{(y \in X^r_\circ, \vec{v} \in T^*_yG/P) : \vec{v} \perp T_yX^r_\circ\}
 \end{array}$$

where the closures taken of the fourth column, to define the third, are taken inside the second. This fourth vertical map (before taking the closures), taking $(x, \vec{v}) \mapsto (g(x), \vec{v})$, is an isomorphism. Hence its closure $\widetilde{CX^r} \rightarrow CX^r$ is birational; call this map Cg . Denote the composite $G/B \leftarrow \widetilde{CX^r}$ (with image X^w) of the left two maps on top by $f_{G/B}$, and the corresponding composite $G/P \leftarrow CX^r$ (with image X^r) on bottom by $f_{G/P}$. The first and third columns are then a commuting square

$$\begin{array}{ccc}
 X^w & \xleftarrow{f_{G/B}} & \widetilde{CX^r} \\
 \downarrow g & & \downarrow Cg \\
 X^r & \xleftarrow{f_{G/B}} & CX^r
 \end{array}$$

The space G/P comes with a list of fundamental weights ω_i orthogonal to the negative simple roots in P . Let $\omega_{G/P}$ be the sum of these, i.e. its Borel–Weil line bundle $\mathcal{O}(\omega_{G/P})$ is the smallest ample line bundle on G/P .

On some homogeneous spaces G/Q , in particular every G/B , the anti-canonical line bundle possesses a (unique) square root which we will denote $\mathcal{O}(\rho_{G/Q})$, e.g. $\rho_{G/B} = \omega_{G/B}$. (By contrast, for $G/P = \text{Gr}(n, N)$ we have $\rho_{G/P} = \frac{N}{2}\omega_{G/P}$.) Then $\mathcal{O}(-\rho_{G/Q})$ is the Borel–Weil line bundle corresponding to $-\rho_{G/Q} := -\frac{1}{2} \sum_{\beta \in \Delta^+_{\mathbb{C}} \setminus \Delta^+_{\mathbb{R}}} \beta$, and all of its sheaf cohomology groups vanish (unless $P = G$). We assume that G/P has such a square root, with which to define our sheaf:

$$\begin{aligned}
 \sigma_r &:= f_{G/P}^* \mathcal{O}(\rho_{G/P} - \omega_{G/P}) \otimes (Cg)_* f_{G/B}^* \mathcal{O}(-\rho_{G/B}) \\
 &= (Cg)_* f_{G/B}^* \mathcal{O} \left(-g^*(\omega_{G/P}) - \frac{1}{2} \sum_{\beta \in \Delta^+_{\mathbb{R}}} \beta \right)
 \end{aligned}$$

For example if $G/P = G/B$ (where $\rho_{G/P} = \omega_{G/B}$ and $Cg = Id$), this simplifies to $f_{G/B}^* \mathcal{O}(-\rho_{G/B})$.

We now describe this construction in the $\text{Gr}(n, N)$ case, with more detail (when X^r , also, is a Grassmannian) to come in Section 4.3. The anticanonical line bundle is $\mathcal{O}(N)$, possessing a square root iff N is even (which we therefore assume), and $\mathcal{O}(\omega_{G/P}) = \mathcal{O}(1)$.

First, for each weight κ the Borel–Weil line bundle $\mathcal{O}(\kappa)$ on G/B can be identified with the sheaf of rational functions on G/B bearing poles only along the Schubert divisors X_{r_α} , of order at most $\langle \alpha, \kappa \rangle$. For $\kappa = -\rho$ these orders-of-pole are all -1 , i.e. the sections must vanish along $\bigcup_\alpha X_{r_\alpha}$.

We want the intersection of this divisor $\bigcup_\alpha X_{r_\alpha}$ with $f(\widehat{X}^r) = X^w$. By Monk’s rule, the intersection of X_{r_k} with X^w is (non-equivariantly) rationally equivalent to $\sum_{i \leq k < j // w \circ (i \leftrightarrow j) \leq w} [X^{w \circ (i \leftrightarrow j)}]$ where \leq indicates a strong Bruhat cover. Summing over k , we get $\sum_{w \circ (i \leftrightarrow j) \leq w} (j - i) [X^{w \circ (i \leftrightarrow j)}]$.

Now we want to push this to CX^r , where the term $[X^{w \circ (i \leftrightarrow j)}]$ drops dimension unless $w \circ (i \leftrightarrow j)$ is again n -Grassmannian, mapping to some $X^{r'}$ with r' being r minus an outer corner. If that removed box is in position (x, y) , then $i = n + 1 - x$ and $j = n + y$, hence $j - i$ is the diagonal ($:= x + y - 1$) of the removed box.

To restate: if we push this class on G/B down to G/P , we get $\sum_{r' = r \setminus (x, y)} (x + y - 1) [X^{r'}]$, which (because of those -1 s) is off by $\mathcal{O}(1)$ from the anticanonical class of X^r [1, Prop. 2.2.8(iv)]. In particular, the sheaf σ_r is a line bundle iff X^r is Gorenstein, which (by [29]) happens exactly when all the outer corners (x, y) of the partition r have $x + y - 1 = d$ for the same d . In this Gorenstein case we can therefore skip construction of \widehat{CX}^r and just pull $\mathcal{O}(-d)$ directly from $G/P = \text{Gr}(n, N)$ to CX^r .

Finally, we twist by $\mathcal{O}(\rho_{G/P} - \omega_{G/P}) = \mathcal{O}(N/2 - 1)$, obtaining (in this Gorenstein case) the sheaf

$$\sigma_r = f_{G/P}^* \mathcal{O}(N/2 - d - 1).$$

It will be convenient below to take $d' := d + 1$, i.e. $d' = x + y$ for each outer corner (x, y) .

3.4. Fibers of μ , and a small part of Conjecture 1.3'

The general fiber of $CX^r \rightarrow \overline{B \cdot r_<}$ can be computed most easily at the point $r_<$. That fiber consists of $V \in X^r_{\circ}$ with $\text{Im } r_< \leq V \leq \text{Ker } r_<$. To describe it, we look back at the second picture from (1.1), whose n top edges run either SW/NE or NW/SE. If we picture X^r as the row-spans

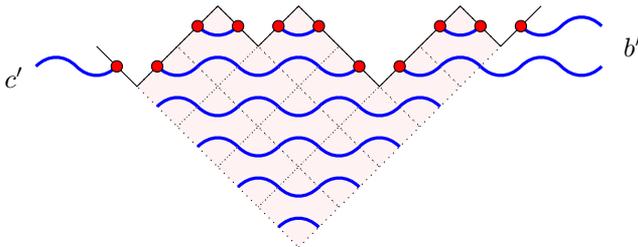
of row-echelon $k \times n$ matrices V , drawn atop a partition tilted as in that figure, then we have three conditions to impose on the matrix V :

- (1) Its row-span should lie in X^r . So we can assume that it is zero in row i to the left of r 's i th element, which lies above the i th NW/SE edge. We don't go so far as to assume that V is in reduced row-echelon form with pivots 1 in the r columns, as that would only get us X^r_0 not its closure X^r .
- (2) $\text{Im } r_{<} \leq \text{rowspan}(V)$. Equivalently, there is *indeed* a pivot above any NW/SE edge connecting to another edge (necessarily, connecting Westward to a NE/SW edge), and the rest of its row in V must be all zeroes. Of course we can use the pivotal 1 and row operations to kill the rest of the column, too.
- (3) $\text{rowspan}(V) \leq \text{Ker } r_{<}$. Equivalently, each NE/SW edge that connects to another edge (necessarily, Eastward to a NW/SE) should have an entirely 0 column.

Let R be the number of pairings, so $c' = n - R$, $b' = N - n - R$ are the number of unmatched red dots in the link pattern with rays going left and right, respectively. Then matched columns are constrained by (2), (3) above, and unmatched columns contribute to c' , b' , as indicated below the matrix in this example:

$$\begin{pmatrix} \star & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 & \star \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$c' \quad 3 \quad 3 \quad 2 \quad 3 \quad 2 \quad 2 \quad b' \quad 3 \quad 2 \quad b'$



If we remove the 0 columns from condition (3) leaving $N - R = n + b'$ columns, and remove the rows and columns of the $R = n - c'$ pivots from condition (2), we are left with a $c' \times (b' + c')$ matrix full of \star s. Therefore

our fiber is the subGrassmannian $\text{Gr}(c', c' + b')$ of n -planes inside a fixed $(n + b')$ -plane, and containing a fixed $(n - c')$ -plane.

When all the outer corners of the partition r are on the same diagonal d (the Gorenstein case), and letting $d' := d + 1$, then $c' = \min(0, d' - n)$, $b' = \min(0, d' - (N - n))$. If we assume the fiber $\text{Gr}(c', c' + b')$ is not a point, then $d' > \max(n, N - n) \geq N/2$ and $c' + b' = 2d' - N$. On a Grassmannian $\text{Gr}(\ell, m)$ which isn't a point, the line bundles $\mathcal{O}(j)$ have no⁽³⁾ sheaf cohomology if $0 > j > -m$. Our line bundle $\mathcal{O}(N/2 - d')$ on the fiber $\text{Gr}(c', c' + b')$ is exactly halfway through this cohomology desert:

$$0 > N/2 - d' > -(c' + b') = 2(N/2 - d')$$

The fact $H^0(\text{fiber}; \mathcal{O}(N/2 - d')) = 0$ alone is already enough to demonstrate the H^0 vanishing statement of Conjecture 1.3', in this Gorenstein case, since a section in $H^0(CX^r; \sigma_r)$ that vanishes on the general fiber must vanish everywhere.

4. The case of r a $c \times b$ rectangle

We now consider the Young diagram r which is a $c \times b$ rectangle, as in (1.3), and write $c \times b$ for this partition. Rectangles are precisely those diagrams for which the Schubert variety X^r is smooth, and in this case $X^{c \times b}$ is isomorphic to the Grassmannian $\text{Gr}(b, b + c)$. The embedding inside $\text{Gr}(n, N)$ is particularly simple in Plücker coordinates, namely

$$X^{c \times b} = \left\{ [p_s, s \in S] : p_s = \begin{cases} p_{\tilde{s}} & s = \{\tilde{s}_1 + \bar{n} - c, \dots, \tilde{s}_b + \bar{n} - c, \bar{n} + b + 1, \dots, N\} \\ 0 & s \cap \{1, \dots, \bar{n} - c\} \neq \emptyset \text{ or } s \not\supseteq \{\bar{n} + b + 1, \dots, N\} \end{cases} \right\}$$

where $\bar{n} = N - n$, and the $p_{\tilde{s}}$ are the Plücker coordinates of $\text{Gr}(b, b + c)$. From now on we use these “reduced” indices consisting of subsets \tilde{s} of $\{1, \dots, b + c\}$ of cardinality b , i.e., in $\binom{b+c}{b}$, as well as the corresponding order \leq (or its opposite order \subseteq).

This rectangular case is even more special than the Gorenstein case (discussed at the end of Section 3.3) where the outer corners of r lay in a single

(3) Proof: $\mathcal{O}(j)$ on $\text{Gr}(\ell, m)$ is the pushforward from $\text{GL}(m)/B$ of the $\mathcal{O}(j^\ell, 0^{m-\ell})$ Borel–Weil line bundle, so instead of pushing $\mathcal{O}(j)$ from $\text{Gr}(\ell, m)$ to a point we can push $\mathcal{O}(j^\ell, 0^{m-\ell})$ from $\text{GL}(m)/B$ to a point (through $\text{Gr}(\ell, m)$). By Borel–Weil–Bott, a line bundle $\mathcal{O}(\lambda_1 \geq \dots \geq \lambda_m)$ on $\text{GL}(m)/B$ has no cohomology iff $\lambda + (m, m - 1, \dots, 1)$ has a repeat, as $\rho_{\text{GL}(m)/B} = (m, m - 1, \dots, 1)$ up to a constant. Here, that sequence (reversed) is $[1, m - \ell], [m - \ell + 1 + j, m + j]$, and these two intervals avoid overlap only if $m - \ell + 1 + j > m - \ell$ i.e. $j > -1$, or if $m + j < 1$ i.e. $j < 1 - m$, or if one of the intervals is empty i.e., $\ell = 0, m$.

diagonal: now $r = c \times b$ has only one outer corner, in diagonal $d = c + b - 1$, and the sheaf σ_r is $\mathcal{O}(N/2 - d - 1) = \mathcal{O}(a)$. (Recall that we are assuming N even whenever we discuss the sheaf σ_r , and have defined $a := N/2 - b - c$.) Since $X^{c \times b}$ is itself a Grassmannian it has its own $\mathcal{O}(1)$ line bundle, which conveniently is the restriction of the $\mathcal{O}(1)$ from $\text{Gr}(n, N)$. Hence our space and sheaf are simply the conormal bundle $C_{\text{Gr}(n, N)} \text{Gr}(b, b + c)$ and its $\mathcal{O}(a)$ line bundle pulled up from the base $\text{Gr}(b, b + c)$.

4.1. Equations of $CX^{c \times b}$

We now wish to write the equations of the one conormal bundle $CX^{c \times b}$ (not, as in (3.3), the union of many such). Let f be the map $(V, u) \mapsto V$. Appending (3.3), rewritten in terms of the remaining Plücker coordinates, to the equations of $X^{c \times b}$ above would produce the union of all conormal Schubert varieties inside $f^{-1}(X^{c \times b})$, namely $\bigcup_{s \subseteq r} CX^s$; to exclude the others we need equations involving M as well.

We proceed as follows. Since $X^{c \times b}$ is smooth, $CX^{c \times b} \xrightarrow{f} X^{c \times b}$ is a vector bundle, and the defining equations of $CX^{c \times b}$ must be linear in the fiber $(M_{i,j})$. We therefore select among (3.4) those that are linear: they are of the form “ $M_{i,j} = 0$ if there are no nonzero entries of $r_{<}$ Southwest of (i, j) , that is, if there are no chords of r inside the interval $[i, j]$ ”. This implies the following block structure of M :

$$(4.1) \quad M = \begin{matrix} & \bar{n} - c & b + c & n - b \\ \bar{n} - c & \left(\begin{array}{ccc} 0 & B & \star \\ & 0 & C \\ n - b & & 0 \end{array} \right) \\ b + c & & & \\ n - b & & & \end{matrix}$$

where the upper-right block has not been named since its entries never occur in any equation.

We can now write (3.3) in terms of the submatrices B and C :

$$(4.2) \quad \sum_{j \in s_+} B_{i,j} p_{s_+ \setminus j} = 0, \quad s_+ \in \binom{[b+c]}{b+1}, \quad 1 \leq i \leq \bar{n} - c$$

$$(4.3) \quad \sum_{j \in \bar{s}_-} C_{j,k} p_{s_- \cup j} = 0, \quad s_- \in \binom{[b+c]}{b-1}, \quad 1 \leq k \leq n - b$$

In order to check that we have obtained *all* the equations of the vector bundle $CX^{c \times b}$, it is convenient to use the following observation. Inside $\text{GL}(N)$ acting on $\text{Gr}(n, N)$ and therefore on $T^* \text{Gr}(n, N)$, the subgroup $G := \text{GL}(\bar{n} - c) \times \text{GL}(b + c) \times \text{GL}(n - b)$ leaves $X^{c \times b}$ invariant and

the $GL(b + c)$ factor acts transitively on the base, making $CX^{c \times b}$ a G -equivariant vector bundle. Now by inspection, the block structure (4.1) as well as (4.2) and (4.3) are G -invariant, so that we only need to check that the fiber of the vector bundle $CX^{c \times b} \xrightarrow{f} X^{c \times b}$ has the correct dimension at one particular point, for example a coordinate subspace with coordinates s (where all $p_t = 0$ for $t \neq s$). For each $s_+ \in \binom{[b+c]}{b+1}$ containing s , one $B_{i,j}$ vanishes, and similarly for $s_- \in \binom{[b+c]}{b-1}$ contained in s , one $C_{j,k}$ vanishes. So we find

$$\begin{aligned} & \dim(\text{fiber}) \\ &= \dim(B) \quad + \dim(C) \quad + \dim(\star) \quad - \dim(B \text{ eqs}) - \dim(C \text{ eqs}) \\ &= (\bar{n} - c)(b + c) + (n - b)(b + c) + (\bar{n} - c)(n - b) - c(\bar{n} - c) \quad - b(n - b) \\ &= n\bar{n} - bc \end{aligned}$$

which means the total space has dimension $n\bar{n} = n(N - n)$, which is indeed the dimension of $CX^{c \times b}$ (a Lagrangian subvariety of $T^* \text{Gr}(n, N)$).

Note that the other equations of (3.4) are

$$(4.4) \quad BC = 0$$

$$(4.5) \quad \text{rank}(B) \leq b$$

$$(4.6) \quad \text{rank}(C) \leq c$$

The derivation just performed shows that they could be obtained from (4.2) and (4.3), up to saturation w.r.t. the irrelevant ideal generated by the p_s . They define an A_3 quiver locus [14] whose degree we will compute below in Proposition 4.3, to be used in Section 6.2.

Summarizing: the embedding $CX^{c \times b} \hookrightarrow \text{Gr}(b, b + c) \times \text{Mat}(N)$ gives the realization

$$(4.7) \quad CX^{c \times b} \cong \text{Proj } \mathbb{C} \left[(p_s)_{s \in \binom{[b+c]}{b}}, (B_{i,j})_{[\bar{n}-c] \times [b+c]}, (C_{j,k})_{[b+c] \times [n-b]}, \star_{[\bar{n}-c] \times [n-b]} \right] / \langle \text{Plücker relations in the } (p_s), \text{ and Equations (4.2) - (4.6)} \rangle$$

where the Plücker coordinates have degree 1, and the B, C degree 0, in this \mathbb{N} -graded ring.

4.2. Proof of Conjecture 1.1 in the rectangular case

We start with the analysis of $Z_{r,s}$.

PROPOSITION 4.1. — For r the $c \times b$ rectangle, $Z_{r,s}$ is entirely determined by Lemmas 2.5 and 2.6.

Proof. — According to Lemma 2.5, $Z_{r,s}$ is zero unless the Young diagram of s sits inside the $c \times b$ rectangle, or equivalently, the subset s is of the form $s = \{\tilde{s}_1 + \bar{n} - c, \dots, \tilde{s}_b + \bar{n} - c, \bar{n} + b + 1, \dots, N\}$, $\tilde{s} \in \binom{[b+c]}{b}$. Note that all such subsets can be obtained from $r = \{\bar{n} - c + 1, \dots, \bar{n} + b - c, \bar{n} + b + 1, \dots, N\}$ by a permutation which acts nontrivially only on $\{\bar{n} - c + 1, \dots, \bar{n} + b\}$. Now there are no chords in this subset in r , cf. (1.4). Lemma 2.6 then implies that all nonzero specializations of $Z_{r,s}$ are obtained from each other, and in particular from $Z_{r,r}$, by permutation of the $z_{\bar{n}-c+1}, \dots, z_{\bar{n}+b}$. And $Z_{r,r}$ is itself given by Lemma 2.5. \square

Next, we analyze the sheaf σ_r and show that the restrictions $[\sigma_r]_s$ of its K_T -class to fixed points satisfies the same properties as $Z_{r,s}$'s, up to normalization.

If $s \not\subseteq r$, we have $\mathbb{C}^s \not\subseteq CX^{c \times b}$, and therefore the restriction is trivially zero. At the particular fixed point $s = r$, the weight of the line bundle itself is $\prod_{i \in s} z_i^{-a}$, and the weights in the normal directions to $CX^{c \times b} \subseteq T^* \text{Gr}(n, N)$ are given by a standard calculation, resulting in

$$(4.8) \quad [\sigma_r]_r = \prod_{i \in r} z_i^{-a} \prod_{i \in r, j \notin r} \begin{cases} 1 - tz_j/z_i & i < j \\ 1 - z_i/z_j & i > j \end{cases}$$

Furthermore, the already mentioned $\text{GL}(b+c)$ -equivariance ensures that the restrictions at the fixed points indexed by $s = \{\tilde{s}_1 + \bar{n} - c, \dots, \tilde{s}_b + \bar{n} - c, \bar{n} + b + 1, \dots, N\}$, $\tilde{s} \in \binom{[b+c]}{b}$, are related to each other by the permutation of the $\{z_{\bar{n}-c+1}, \dots, z_{\bar{n}+b}\}$ which sends s to r .

Comparing (4.8) with (2.6) and (2.7), and carefully keeping track of the monomials, we conclude that Conjecture 1.1' from Section 3.2 is verified with

$$m_r = t^{bc/2} \prod_{i=1}^{\bar{n}-c} z_i^{-n/2} \prod_{i=\bar{n}-c+1}^{\bar{n}+b} z_i^{b-n/2} \prod_{i=\bar{n}+b+1}^N z_i^{b+c-n/2}$$

4.3. Vanishing of higher cohomology

Recall that $f : CX^{c \times b} \rightarrow X^{c \times b} \subseteq \text{Gr}(n, N)$ is the projection of the conormal bundle, and recall from the beginning of this section that

$$\sigma_r = f^* \mathcal{O}(a)$$

where $a = N/2 - b - c$ and is integral. Conjecture 1.3 concerns the pushforward of this line bundle to a point, which comes only from its global sections:

PROPOSITION 4.2. — *If $a \geq 0$, then the line bundle $\sigma_{c \times b}$ on $CX^{c \times b}$ has no higher sheaf cohomology, and the restriction map on sections $H^0(CX^{c \times b}; f^* \mathcal{O}(a)) \rightarrow H^0(X^{c \times b}; \mathcal{O}(a))$ is surjective. If $a < -|N/2 - n|$, then $\sigma_{c \times b}$ has no sheaf cohomology at all.*

Proof. — We are grateful to Jake Levinson for explaining Weyman’s sheaf cohomology techniques [27] to us, for the following application.

As a vector bundle over $\text{Gr}(b, b+c)$, and ignoring the irrelevant \star variables of (4.1), the space $CX^{c \times b}$ is the vector bundle

$$\text{Hom}(\mathbb{C}^{\bar{n}-c}, S) \otimes \text{Hom}(Q, \mathbb{C}^{n-b}) \cong (Q^*)^{\oplus n-b} \oplus S^{\oplus \bar{n}-c}$$

where S, Q are the tautological sub and quotient bundles on $\text{Gr}(b, b+c)$, from the sequence

$$0 \rightarrow S \rightarrow \mathbb{C}^n \otimes \mathcal{O}_{\text{Gr}(b, b+c)} \rightarrow Q \rightarrow 0.$$

Sheaf cohomology is about the (derived) pushforward of $f^* \mathcal{O}(a)$ to a point. We push it first to $\text{Gr}(b, b+c)$ (then from there to a point):

$$\begin{aligned} f_* f^*(\mathcal{O}(a)) &\cong \mathcal{O}(a) \otimes f_*(\mathcal{O}_{CX^{c \times b}}) \\ &\cong (\text{Alt}^b S^*)^{\otimes a} \otimes \text{Sym}^\bullet(((Q^*)^{\oplus n-b} \oplus S^{\oplus \bar{n}-c})^*) \\ &\cong (\text{Alt}^b S^*)^{\otimes a} \otimes \text{Sym}^\bullet(Q)^{\otimes n-b} \otimes \text{Sym}^\bullet(S^*)^{\otimes \bar{n}-c} \end{aligned}$$

We can decompose each functor $(\text{Sym}^\bullet)^{\otimes m}$ into Schur functors Sc_λ

$$\begin{aligned} (\text{Sym}^\bullet)^{\otimes m} &\cong \left(\bigoplus_k Sc_{(k, 0, \dots, 0)} \right)^{\otimes m} \\ &\cong \bigoplus_{k_1, \dots, k_m} \bigotimes_{i=1}^m Sc_{(k_i, 0, \dots, 0)} \\ &\cong \bigoplus_{\lambda=(\lambda_1 \geq \dots \geq \lambda_m)} (Sc_\lambda)^{\oplus \dim S_\lambda(\mathbb{C}^m)} \end{aligned}$$

The third isomorphism uses the Pieri rule $m - 1$ times to assemble SSYT out of m horizontal strips of various lengths (k_i) . The number of ways to achieve a particular shape λ is the number of SSYT with values $\leq m$, which

is the dimension of the $GL(m)$ -irrep $S_\lambda(\mathbb{C}^m)$. Now

$$\begin{aligned}
 & f_* f^*(\mathcal{O}(a)) \\
 & \cong \bigoplus_{\substack{\lambda=(\lambda_1 \geq \dots \geq \lambda_{n-b}) \\ \mu=(\mu_1 \geq \dots \geq \mu_{\bar{n}-c)}}} S_{c_{(a,a,\dots,a)}}(S^*) \otimes S_{c_\lambda}(Q)^{\oplus \dim S_{c_\lambda}(\mathbb{C}^{n-b})} \\
 & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \otimes S_{c_\mu}(S^*)^{\oplus \dim S_{c_\mu}(\mathbb{C}^{\bar{n}-c})} \\
 & \cong \bigoplus_{\substack{\lambda=(\lambda_1 \geq \dots \geq \lambda_{n-b}) \\ \mu=(\mu_1 \geq \dots \geq \mu_{\bar{n}-c)}}} S_{c_\lambda}(Q)^{\oplus \dim S_{c_\lambda}(\mathbb{C}^{n-b})} \otimes S_{c_{\mu+(a,\dots,a)}}(S)^{\oplus \dim S_{c_\mu}(\mathbb{C}^{\bar{n}-c})}
 \end{aligned}$$

If $n - b > \dim Q = c$, then $S_{c_\lambda}(Q)$ will vanish if $\lambda_{c+1} > 0$. Whereas if $n - b < c$, we can pad out λ with $c - (n - b)$ zeroes. The same remarks apply to $\bar{n} - c$ vs $\dim S = b$. So hereafter we regard λ, μ as sequences of length c, b respectively. The total padding is

$$\begin{aligned}
 & \min(0, b + c - n) + \min(0, b + c - \bar{n}) \\
 & = \begin{cases} 0 & \text{if } b + c \leq \min(n, \bar{n}) \text{ hence } a \geq 0 \\ b + c - \min(n, \bar{n}) > -a & \text{if } b + c \in (\min(n, \bar{n}), \max(n, \bar{n})] \\ 2(b + c) - N = -2a > -a & \text{if } b + c > \max(n, \bar{n}) \end{cases}
 \end{aligned}$$

depending on whether we pad neither, one of, or both of λ, μ with 0s. The first and third cases will be the ones addressed in the proposition, as in the third case,

$$-a = b + c - N/2 > \max(n, \bar{n}) - N/2 = \max(n - N/2, \bar{n} - N/2) = |n - N/2|.$$

The summand $S_{c_\lambda}(Q) \otimes S_{c_{\mu+(a,\dots,a)}}(S^*)$ on $\text{Gr}(b, b+c)$ is itself the push-forward of the line bundle $\mathcal{O}(\lambda, -(a, \dots, a) - w_0\mu)$ on $GL(b+c)/B$ along the fiber bundle $GL(b+c)/B \rightarrow \text{Gr}(b, b+c)$. Now we need to study the sequence $(\lambda, -(a, \dots, a) - w_0\mu) + \rho$ (recall $\rho = (c+b, c+b-1, \dots, 3, 2, 1)$ from footnote (3)) to apply Borel-Weil-Bott.

If $a \geq 0$, then this sequence $(\lambda, -(a, \dots, a) - w_0\mu)$ is weakly decreasing, so its line bundle is dominant and has no higher cohomology. The first summand $\mathcal{O}(a) \otimes \text{Sym}^0((S^{\oplus \bar{n}-c} \oplus (Q^*)^{\oplus n-b})^*) \cong \mathcal{O}(a)$ already gives us enough sections over $CX^{c \times b}$ to restrict to the sections we want on $X^{c \times b}$.

If we're in the third case, so both λ, μ are each padded out with a positive number of zeroes (indeed, strictly more than $a' := -a$ of them total), then $(\lambda, -w_0\mu)$ looks like this:

$$\left(\overbrace{\lambda_1, \dots, \lambda_{n-b}}^c, \underbrace{0, \dots, 0}_{>a'}, \overbrace{0, \dots, 0}^b, -\mu_{\bar{n}-c}, \dots, -\mu_1 \right)$$

Choose a segment of length $a' + 1$ in that region of 0s, say in positions $k \dots k + a'$, including a 0 from each side (this is where we use $b + c > \max(n, \bar{n})$) i.e. $k < c < c + 1 < k + a'$. When we add (a', \dots, a') to $-w_0\mu$, that segment becomes $(\dots, 0, a', \dots)$. Then when we add ρ , that segment becomes $(N - k + 1, N - k - 2, \dots, N - k + 2, N - k + 1)$, and this repeat of $N - k + 1$ (and Borel–Weil–Bott) kills all the sheaf cohomology. \square

If we factor that pushforward through the affinization $\mu : CX^{c \times b} \rightarrow \text{Mat}_{<}(N), (V, M) \mapsto M$ where $M = \begin{pmatrix} 0 & B & \star \\ 0 & 0 & C \\ 0 & 0 & 0 \end{pmatrix}$ as in (4.1), then we obtain a module over the ring $\mathbb{C}[(B_{i,j}), (C_{j,k})]$, namely the degree a component of the graded ring in (4.7).

In particular, a set of $\mathbb{C}[(B_{i,j}), (C_{j,k})]$ -module generators of the global sections of σ_r is formed by all the monomials of degree a in the projective coordinates $(p_s)_{s \in S}$, and using the Plücker relations as straightening law we can already generate this module only using weakly Bruhat-decreasing a -tuplets from $\binom{[b+c]}{b}$.

4.4. The degree of the orbital variety (assuming hereafter that $N = 2n$ and $a \geq 0$)

In the rest of this paper, we assume that $N = 2n$, i.e., $\bar{n} = n$ and $a + b + c = n$. According to Proposition 4.2, σ_r has no higher sheaf cohomology, and according to Section 3.4, if $a < 0$, H^0 is also zero, which is consistent with the first case of Conjecture 1.3'. We therefore also assume that $a \geq 0$.

Note that the block structure of M of (4.1) takes the slightly more symmetric form

$$(4.9) \quad M = \begin{matrix} & a+b & b+c & c+a \\ \begin{matrix} a+b \\ b+c \\ c+a \end{matrix} & \begin{pmatrix} 0 & B & \star \\ & 0 & C \\ & & 0 \end{pmatrix} \end{matrix}$$

and (4.2) and (4.3) become

$$(4.10) \quad \sum_{j \in s_+} B_{i,j} p_{s_+ \setminus j} = 0, \quad s_+ \in \binom{[b+c]}{b+1}, \quad 1 \leq i \leq a+b$$

$$(4.11) \quad \sum_{j \in \bar{s}_-} C_{j,k} p_{s_- \cup j} = 0, \quad s_- \in \binom{[b+c]}{b-1}, \quad 1 \leq k \leq a+c$$

The affinization $\mu(CX^{c \times b})$ of this conormal bundle is the orbital variety defined by (4.4)–(4.6).

For Section 6.2 to come, we need the following result:

PROPOSITION 4.3. — For $r = c \times b$, the degree of the affine cone $\mu(CX^{c \times b})$ is $2^{bc}|\text{PP}(a, b, c)|$.

Proof. — We drop the \star variables, since as they are unconstrained they don't affect the degree. We will reduce to the $a = 0$ case, where the rank conditions (4.5)–(4.6) are automatically satisfied and the variety $\{(B, C) : BC = 0\}$ is a quadratic complete intersection of degree $2^{bc} = 2^{bc}|\text{PP}(0, b, c)|$.

For general a , (4.4)–(4.6) define a quiver cycle for the quiver + dimension vector $\mathbb{C}^{a+b} \xrightarrow{B} \mathbb{C}^{b+c} \xrightarrow{C} \mathbb{C}^{c+a}$, whose degree we compute with the “pipe formula” of [14].

The lacing diagrams [14, §3] for this quiver cycle are very simple: the bottom b dots in the $a + b$ stack are connected (noncrossingly) to some subset $s \in \binom{[b+c]}{b}$ of the dots in the $b + c$ stack, leaving the complement \bar{s} to connect to the bottom c dots in the $c + a$ stack.

We extend these partial permutations to permutations $\pi(s), \rho(s)$ of $a + b + c$, as in [14, §2.1]:

- $\pi(s) \in S_{a+b+c}$ takes
 - $[1, b]$ to $s \subseteq [1, b + c]$
 - $[b + 1, b + a]$ to $[b + c + 1, b + c + a]$, then possibly a descent,
 - $[b + a + 1, b + a + c]$ to \bar{s} , while
- $\rho(s) \in S_{a+b+c}$ takes
 - \bar{s} to $[1, c]$,
 - $[b + c + 1, b + c + a]$ to $[c + 1, c + a]$, then possibly a codescent,
 - s to $[c + a + 1, c + a + b]$.

So far the pipe formula tells us

$$\text{deg}(\mu(CX^{c \times b})) = \sum_{s \in \binom{[b+c]}{b}} |\{\text{pipe dreams for } \pi(s)\}| |\{\text{pipe dreams for } \rho(s)\}|.$$

Since $\pi(s)$ and $\rho(s)^{-1}$ are Grassmannian permutations, their pipe dreams are in natural correspondence with SSYT [15]. The shapes of the corresponding Young diagrams are given respectively by the subsets \bar{s} (viewed a c -subset of $[a + b + c]$, so that the Young diagram sits inside the $(a + b) \times c$ rectangle) and s (viewed a b -subset of $[a + b + c]$, so that the Young diagram sits inside the $(a + c) \times b$ rectangle):

$$\begin{aligned} \text{pipe dreams for } \pi(s) &\longleftrightarrow \begin{array}{l} \text{SSYT}(\bar{s}) \\ \text{with entries } \leq a + b \end{array} \\ \text{pipe dreams for } \rho(s) &\longleftrightarrow \text{pipe dreams for } \rho(s)^{-1} \longleftrightarrow \begin{array}{l} \text{SSYT}(s) \\ \text{with entries } \leq a + c \end{array} \end{aligned}$$

(Equivalently, viewing s as a Young diagram inside the $c \times b$ rectangle, and similarly \bar{s} as its transpose followed by a complementation inside $b \times c$, then we extend them by concatenating them vertically with a rectangular block of size $a \times b$ for the former, $a \times c$ for the latter.)

These numbers compute the dimensions of certain $\mathrm{GL}(a+b), \mathrm{GL}(a+c)$ representations; if we denote by $V_{r,i}$ the representation of $\mathrm{GL}(i)$ associated with the Young diagram of the subset r , then one has the Weyl dimension formula $\dim V_{r,i} = \frac{\Delta(\bar{r})}{\Delta(1, \dots, i)}$ where $\Delta(\cdot)$ denotes the Vandermonde determinant. Here,

$$\begin{aligned} & \dim V_{\bar{s}, a+b} \dim V_{s, a+c} \\ &= \frac{\Delta(s_1, \dots, s_b, b+c+1, \dots, b+c+a) \Delta(\bar{s}_1, \dots, \bar{s}_c, b+c+1, \dots, b+c+a)}{\Delta(1, \dots, a+b) \Delta(1, \dots, a+c)} \\ &= \frac{\Delta(s) \Delta(\bar{s}) \prod_{i=1}^b \prod_{j=1}^a (b+c+j-s_i) \prod_{i=1}^c \prod_{j=1}^a (b+c+j-\bar{s}_i)}{\Delta(1, \dots, b) \Delta(1, \dots, c) \prod_{i=1}^b \prod_{j=1}^a (i+j-1) \prod_{i=1}^c \prod_{j=1}^a (i+j-1)} \\ &= |\mathrm{PP}(a, b, c)| \frac{\Delta(s) \Delta(\bar{s})}{\Delta(1, \dots, b) \Delta(1, \dots, c)} \\ &= |\mathrm{PP}(a, b, c)| \dim V_{\bar{s}, b} \dim V_{s, c} \end{aligned}$$

Plugging into the previous formula, this gives

$$\begin{aligned} \deg(\mu(CX^{c \times b})) &= \sum_{s \in \binom{[b+c]}{b}} \dim V_{\bar{s}, a+b} \dim V_{s, a+c} \\ &= |\mathrm{PP}(a, b, c)| \sum_{s \in \binom{[b+c]}{b}} \dim V_{\bar{s}, b} \dim V_{s, c} \\ &= |\mathrm{PP}(a, b, c)| 2^{bc} \end{aligned}$$

the last step being the $a = 0$ case solved at the beginning (or one could use the RSK* correspondence). □

5. The degeneration

Hereafter, let A be the ring with generators $\{B_{i,j}, C_{j,k}, \star_{i,k}\}$ and relations (4.4)–(4.6), whose Spec is the orbital variety $\mu(CX^{c \times b})$, i.e., the degree 0 part of the homogeneous coordinate ring of $CX^{c \times b}$ presented in (4.7).

By Proposition 4.2, the K_T -theoretic pushforward $\mu_*(\sigma_{c \times b}) \in K^T(\mathrm{Mat}_{<}(N))$ is the class of the degree a component of that homogeneous coordinate ring. This is naturally a module over the degree 0 component A .

As such, if we take F_r to be the free module spanned by degree a monomials in the Plücker coordinates, then we have a short exact sequence

$$0 \rightarrow M_r \rightarrow F_r \rightarrow H^0(CX^{c \times b}; p^* \mathcal{O}(a)) \rightarrow 0$$

where M_r gives the relations between the degree a monomials.

Using the “straightening relations” on Plücker coordinates, we can shrink our generating set to the Plücker monomials $p_S := \prod_{i=1}^a p_{s_i}$ where $S = (s_1 \leq \dots \leq s_a)$, i.e. S lies in $\text{PP}(a, b, c)$, giving a smaller presentation

$$0 \rightarrow M'_r \rightarrow F'_r \rightarrow H^0(CX^{c \times b}; p^* \mathcal{O}(a)) \rightarrow 0$$

However, we understand the relations generating M_r much better than those generating M'_r , so we will need to work with both sequences.

In order to analyze $\mu_*(\sigma_{c \times b})$ in more detail, we define in this section a degeneration of M_r by assigning weights to the generators of F_r and the B, C variables of A , and then keeping only lowest weight terms of elements of M_r . Our principal goal in the next two sections is the following theorem:

THEOREM 5.1. — *The l.h.s. of (4.10), (4.11), (4.4) times suitable Plücker monomials form a Gröbner basis⁽⁴⁾ for the A -submodule $M'_r \leq F'_r$, i.e.*

$$\begin{aligned} & \text{in}(M'_r) \\ &= \left\langle \text{in} \left(\sum_{j \in s_+} B_{i,j} p_{s_+ \setminus j} p_{s_2} \dots p_{s_a} \right), s_+ \in \binom{[b+c]}{b+1}, 1 \leq i \leq a+b, \right. \\ & \quad \text{in} \left(\sum_{j \in \bar{s}_-} C_{j,k} p_{s_- \cup j} p_{s_2} \dots p_{s_a} \right), s_- \in \binom{[b+c]}{b-1}, 1 \leq k \leq a+c \\ & \quad \left. \text{in} \left(\sum_{j=1}^{b+c} B_{i,j} C_{j,k} p_{s_1} \dots p_{s_a} \right), 1 \leq i \leq a+b, 1 \leq k \leq a+c \right\rangle \\ & \leq \text{in}(F'_r) := \text{the free in}(A)\text{-module with basis } \{p_S : S \in \text{PP}(a, b, c)\} \end{aligned}$$

where $s_1 \leq \dots \leq s_a$ run over $\text{PP}(a, b, c)$. In particular, the $\text{in}(A)$ -module $\text{in}(F'_r)/\text{in}(M'_r)$ has the same T -equivariant Hilbert series as the A -module F'_r/M'_r .

In Theorem 5.4 in Section 5.2 we will be more precise about the actual leading forms. The first two types of equations will have single terms, and those of the third type will have two terms.

⁽⁴⁾In Proposition 5.5 we give a foundational result that defines the sense of “Gröbner basis” used here.

We prove this in three big steps. The first (Proposition 5.2) is about showing that, with the right term order on the Plücker variables, the Plücker monomials from $PP(a, b, c)$ dominate (essentially allowing us to consider F'_r instead of F_r). The second step (Sections 5.3–5.4) is about finding the leading forms of the relations in Theorem 5.1, as just described. Then in Section 6 we show that those leading terms define a module of the correct Hilbert series (and not larger), i.e. that the basis is Gröbner.

5.1. Plücker relations

Given a monomial $p_{s_1} \dots p_{s_a}$, we always order elements of each subset increasingly: $s_\ell = \{s_{\ell,1} < \dots < s_{\ell,b}\}$, effectively indexing monomials with $a \times b$ arrays of integers which are increasing along rows. We will often use the uppercase letter as a shorthand notation for $S := (s_1, \dots, s_a) = (s_{\ell,m})_{1 \leq \ell \leq a, 1 \leq m \leq b}$, and similarly denote $p_S := p_{s_1} \dots p_{s_a}$.

Given a two-dimensional array of integers $S = (s_{\ell,m})$, $\ell = 1, \dots, a$, $m = 1, \dots, b$, with no required monotonicity property, define

$$w(S) := \sum_{m=1}^b \sum_{\ell=1}^a \left(\frac{1}{2} s_{\ell,m} + \ell - m \right)^2$$

We define the weight of a generator of F_r , that is a monomial of degree a in the p_s , to be

$$(5.1) \quad \mathbf{wt}(p_{s_1} \dots p_{s_a}) = \max_{\sigma \in S_a} w(\sigma(S)), \quad \sigma(S) = (s_{\sigma(1)}, \dots, s_{\sigma(a)})$$

(we postpone to Section 5.2 the definition of the weights of the B, C variables of the ring, which we do not need for now). Finally, for x any linear combination of such monomials, we define $\mathbf{in}(x)$ to be the sum of monomials of x for which the function \mathbf{wt} is *minimal*.

If we let \mathcal{P} denote the space of Plücker relations, cf. (3.1), viewed as linear forms on such monomials, namely

$$\mathcal{P} = \text{span} \left(p_{s_1} \dots p_{s_{a-2}} \sum_{i \in s_+} p_{s_- \cup i} p_{s_+ \setminus i}, \right. \\ \left. (s_1, \dots, s_{a-2}) \in \binom{[N]}{n}, s_{\pm} \in \binom{[N]}{n \pm 1} \right)$$

(assuming $a \geq 2$; otherwise $\mathcal{P} = \{0\}$), then we can determine its degeneration:

PROPOSITION 5.2.

$$\text{in}(\mathcal{P}) = \text{span}(p_{s_1} \dots p_{s_a}, (s_1, \dots, s_a) \notin \text{PP}(a, b, c))$$

Compare with [18, Thm. 14.6], whose proof we adapt here.

Proof. — Introduce the Stiefel map

$$\phi(p_s) = \det_{1 \leq m, m' \leq b} (x_{m, s_{m'}})$$

where the $x_{m,j}$, $1 \leq m \leq b$, $1 \leq j \leq b + c$ are formal variables. The First Fundamental Theorem of Invariant Theory says that ϕ is an isomorphism from the homogeneous coordinate ring of the Grassmannian to the ring $\mathbb{C}[x_{m,j}]^{\text{SL}(b)}$ of $\text{SL}(b)$ -invariants. Then

$$(5.2) \quad \phi(p_S) = \phi(p_{s_1}) \dots \phi(p_{s_a}) = \sum_{J=(j_{\ell,m})} \kappa_J \prod_{m=1}^b \prod_{\ell=1}^a x_{m, j_{\ell,m}}$$

To each monomial in this expansion associate its $J = (j_{\ell,m})$, where each column is assumed to be weakly increasing: $j_{1,m} \leq \dots \leq j_{a,m}$, $m = 1, \dots, b$, and to that its weight

$$(5.3) \quad \widetilde{\text{wt}} \left(\prod_{m=1}^b \prod_{\ell=1}^a x_{m, j_{\ell,m}} \right) := w(J)$$

By a slight abuse of notation, also write $\widetilde{\text{wt}}(P)$ for the minimum of $\widetilde{\text{wt}}$ over all such monomials of P .

We now have the key lemma:

LEMMA 5.3. — *The unique term in (5.2) with lowest weight is the product of diagonal terms*

$$\prod_{\ell=1}^a x_{m, s_{\ell,m}}$$

Furthermore, $\widetilde{\text{wt}}(\phi(p_S)) \leq \text{wt}(p_S)$ with equality iff $S \in \text{PP}(a, b, c)$.

Proof of Lemma 5.3. — From the determinant structure, we see that any monomial with array J can be obtained from S by a sequence of two operations:

- Permuting each row individually, noting that the original rows are sorted: $s_{\ell,1} < \dots < s_{\ell,b}$, $\ell = 1, \dots, a$.
- Reordering each column individually so that $j_{1,m} \leq \dots \leq j_{a,m}$, $m = 1, \dots, b$.

Each of these two operations increases \widetilde{wt} . We want to show that the minimum weight is attained when the first of them is the identity permutation.

Denote the second operation $S \mapsto S^o$. We have the diagram

$$\begin{array}{ccc}
 S & \xrightarrow{\text{permute rows}} & T \\
 \downarrow \text{reorder columns} & & \downarrow \text{reorder columns} \\
 S^o & & T^o
 \end{array}$$

where $J = T^o$, and we want to compare $w(S^o)$ and $w(T^o)$.

We compute

$$\begin{aligned}
 w(T^o) - w(S^o) &= \sum_{\ell=1}^a \sum_{m=1}^b \left(\left(\frac{1}{2} s_{\ell,m}^o + \ell - m \right)^2 - \left(\frac{1}{2} t_{\ell,m}^o + \ell - m \right)^2 \right) \\
 &= \sum_{\ell=1}^a \sum_{m=1}^b (\ell - m)(t_{\ell,m}^o - s_{\ell,m}^o) \\
 &= \sum_{\ell=1}^a \sum_{m=1}^b \ell(t_{\ell,m}^o - s_{\ell,m}^o) - \sum_{m=1}^b m \sum_{\ell=1}^a (t_{\ell,m} - s_{\ell,m})
 \end{aligned}$$

where in the last line we have used the fact that for each ℓ , the $t_{\ell,m}$ are a permutation of the $s_{\ell,m}$.

We now proceed by induction on the sum of inversion numbers of the permutations of the rows taking S to T . Suppose $T \neq S$; then there exist two successive columns $r, r + 1$ where inversions occur on some row(s). We shall show that removing the inversions on these two columns decreases strictly the weight (this will be effectively equivalent to showing the property in the case $b = 2$).

Consider the contribution of these two columns to $w(T^o) - w(S^o)$; it takes the form

$$\begin{aligned}
 (5.4) \quad & \sum_{\ell=1}^a \sum_{m=r,r+1} \ell(t_{\ell,m}^o - s_{\ell,m}^o) \\
 & - \sum_{\ell=1}^a \left(r \sum_{\ell=1}^a (t_{\ell,r} - s_{\ell,r}) + (r + 1)(t_{\ell,r+1} - s_{\ell,r+1}) \right) \\
 & = \sum_{\ell=1}^a \sum_{m=r,r+1} \ell(t_{\ell,m}^o - s_{\ell,m}^o) + \sum_{\ell=1}^a (s_{\ell,r+1} - t_{\ell,r+1})
 \end{aligned}$$

The second part is easy to analyze: each inversion $s_{\ell,r+1} = t_{\ell,r} > t_{\ell,r+1} = s_{\ell,r}$ produces a strict increase of the weight by $s_{\ell,r+1} - s_{\ell,r}$. All we need to prove is that the first part is greater or equal to zero.

We proceed by induction again, this time on a . Pick among all the entries of T in the columns $r, r + 1$ the largest one; call it m . Pick a row on which this entry appears (since it may appear multiple times); call m' the other entry on this row at the same columns $r, r + 1$. We may always assume, by reordering of the rows, that this row is the last one. Apply the induction to the other rows. We have

$$\sum_{\ell=1}^{a-1} \sum_{m=r, r+1} \ell(t_{\ell, m}^{o(\ell-1)} - s_{\ell, m}^{o(\ell-1)}) \geq 0$$

where the superscript $o(\ell - 1)$ means that the reordering of the rows only affects the $\ell - 1$ first rows.

Now compare to $w(t^o) - w(s^o)$:

$$\begin{aligned} & \sum_{\ell=1}^a \sum_{m=r, r+1} \ell(t_{\ell, m}^o - s_{\ell, m}^o) \\ &= \sum_{\ell=1}^{a-1} \sum_{m=r, r+1} \ell(t_{\ell, m}^{o(\ell-1)} - s_{\ell, m}^{o(\ell-1)}) \\ & \quad + \sum_{\ell: t_{\ell, m} > m'} (t_{\ell, m} - m') - \sum_{\ell: s_{\ell, r} > m'} (s_{\ell, r} - m') \quad (m' = t_{\ell, m}) \end{aligned}$$

where the extra terms of the r.h.s. take into account the reordering of m' . Note that $t_{\ell, m} \geq s_{\ell, r}$ for $m \in \binom{r}{r+1}$. We conclude that $\sum_{\ell=1}^a \ell(t_{\ell, m}^o - s_{\ell, m}^o) \geq 0$, which is the induction hypothesis, and combining the inequalities for the two parts of (5.4), we obtain the first part of the lemma.

Therefore,

$$\begin{aligned} (5.5) \quad \widetilde{\mathbf{wt}}(\phi(p_S)) &= \widetilde{\mathbf{wt}} \left(\prod_{\ell=1}^a x_{m, s_{\ell, m}} \right) \\ &= \sum_{m=1}^b \max_{\sigma \in \mathcal{S}_a} \left(\sum_{\ell=1}^a \left(\frac{1}{2} s_{\sigma(\ell), m} + \ell - m \right)^2 \right) \end{aligned}$$

where, rather than use s^o , we have emphasized the maximum property for each column. This is to be compared with (5.1), where the maximum is outside the summation over m (one is only allowed to permute the rows globally). This immediately implies the inequality $\widetilde{\mathbf{wt}}(\phi(p_S)) \leq \mathbf{wt}(p_S)$. In case of equality, the ordering of each column agrees, or equivalently the s_a are totally ordered, which means $S \in \text{PP}(a, b, c)$. \square

(Continuing the proof of Proposition 5.2.) Using the “straightening law” [18, Thm. 14.6], the p_S for $S \notin \text{PP}(a, b, c)$, can be expressed as linear

combinations of noncrossing ones modulo the Plücker relations:

$$(5.6) \quad p_S = \sum_{T \in \text{PP}(a,b,c)} c_T p_T$$

Now pick $T \in \text{PP}(a, b, c)$ such that $\text{wt}(p_T)$ is minimal among the terms with nonzero coefficients in the r.h.s. of (5.6). Applying ϕ to (5.6) and noting that diagonal terms are all distinct for distinct T (see a similar argument in the proof of [18, Thm. 14.16]), we find that the coefficient of the diagonal term of $\phi(p_T)$ in $\phi(\text{r.h.s. of (5.6)})$ is precisely c_T (no compensation can occur). Therefore it must also appear in the l.h.s. (and be equal to ± 1), so that

$$\widetilde{\text{wt}}(\phi(p_S)) \leq \widetilde{\text{wt}}(\phi(p_T)) = \text{wt}(p_T)$$

We conclude that

$$\text{wt}(p_S) < \widetilde{\text{wt}}(\phi(p_S)) \leq \text{wt}(p_T) \quad \forall t : c_t \neq 0$$

so that the initial term of (5.6) is p_S .

It remains to show that $\text{in}(\mathcal{P})$ is no larger than $\text{span}(p_{s_1} \dots p_{s_a}, (s_1, \dots, s_a) \notin \text{PP}(a, b, c))$. For this, we use the dimension count $\dim H^0(X^{c \times b}; \mathcal{O}(a)) = |\text{PP}(a, b, c)|$ explained at the beginning of Section 5.1, and the second statement from Proposition 4.2. □

Consequently, we have a smaller presentation

$$0 \rightarrow M'_r \rightarrow F'_r \rightarrow H^0(CX^{c \times b}; p^* \mathcal{O}(a)) \rightarrow 0$$

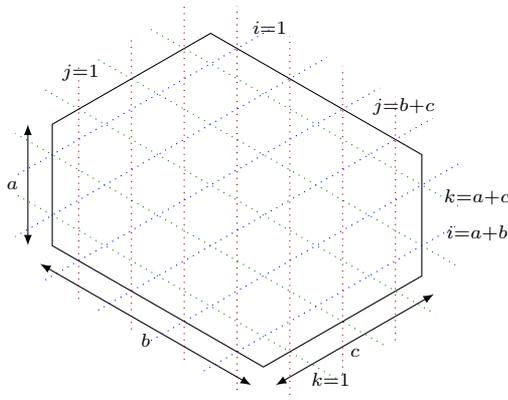
where

$$F'_r := \text{span}(p_{s_1} \dots p_{s_a}, (s_1, \dots, s_a) \in \text{PP}(a, b, c)).$$

5.2. Lozenge tilings and weights

We have seen in the previous section the appearance of the subset $\text{PP}(a, b, c)$ of increasing a -tuples from $\binom{[b+c]}{b}$. As we continue our degeneration, we shall see that it is intimately connected to the combinatorics of lozenge tilings, which are one possible graphical description of elements of $\text{PP}(a, b, c)$. As shown on the top-right picture from Figure 1.2, they are fillings of a $a \times b \times c$ hexagon with lozenges (of unit edge length) in three orientations. We shall use the following (redundant) coordinate system on

such hexagons, based on an underlying Kagome lattice:



The blue (resp. red, green) lines are constant i (resp. j, k) curves. We have the relation $i - a + k - j - 1/2 = 0$.

Let us call lozenges  “of type B ”,  “of type C ”, and  “of type BC ”. We notice that the center of a lozenge of type B has integer coordinates (i, j) , that of type C integer coordinates (j, k) , and that of type BC integer coordinates (i, k) . We shall always use such coordinates for each type of lozenge. On the example of Figure 1.2, the  lozenges of type C have coordinates $(3, 4), (5, 5), (6, 5), (7, 5), (1, 1), (3, 2), (5, 3), (7, 4)$, listed in English-reading order.

We record a few facts about the plane partitions, hexagons, dimers, and rhombi. One can act on a plane partition by adding or removing a box, which on the dimer configuration corresponds to rotating a 3-dimer hexagon by 60° . Hence the number of dimers in each orientation is constant, and can be computed from the case of the empty plane partition; there are ac dimers of type B , ab of type C , and bc of type BC . It is also easy to compute the number of hexagons: if we replace (a, b, c) by $(0, b + a, c + a)$, adding $\binom{a}{2}$ on each side, the region becomes a parallelogram with $(b + a - 1)(c + a - 1)$ hexagons. Hence there are $(b + a - 1)(c + a - 1) - 2\binom{a}{2} = ab + ac + bc - a - b - c + 1$ hexagons.

With these conventions, the bijection from lozenge tilings to subsets in $PP(a, b, c)$ is to keep track of the j coordinates of every lozenge of type C . More precisely, once we draw (as on the lower-left of Figure 1.2) paths made of lozenges of type B and C , then the subset $s_i, i = 1, \dots, a$, records the locations of down steps of path i indexed from bottom to top.

Finally we introduce one more coordinate, called y , which is the vertical coordinate of the center of any lozenge, where the lower left corner of the hexagon has coordinate $y = 1/2$. With these conventions, the conversion from (i, j, k) to y , effectively giving the y coordinate of the centers of the three types of lozenges, is

$$(5.7) \quad y_{i,j}^B = \frac{1}{2}j - (i - a) + \frac{1}{2}$$

$$(5.8) \quad y_{j,k}^C = k - \frac{1}{2}j$$

$$(5.9) \quad y_{i,k}^{BC} = k - (i - a) + \frac{1}{2}$$

We are now ready to introduce the weights of the variables $B_{i,j}$ and $C_{j,k}$. We define

$$(5.10) \quad \mathbf{wt}(B_{i,j}) = (y_{i,j}^B)^2 \quad 1 \leq i \leq a + b, 1 \leq j \leq b + c$$

$$(5.11) \quad \mathbf{wt}(C_{j,k}) = (y_{j,k}^C)^2 \quad 1 \leq j \leq b + c, 1 \leq k \leq a + c$$

With this we can define the Gröbner degeneration $\mathbf{in}(A)$ of the coordinate ring A of the orbital variety defined by (4.4)–(4.6). We will not directly determine a Gröbner basis for A ; rather, *it will be easiest to study $\mathbf{in}(A)$ by its action on the summands of $\mathbf{in}(M'_r)$.*

These weights (5.10) and (5.11), combined with the definition of $\mathbf{wt}(p_S)$ given in (5.1), also allow definition of the weight of an arbitrary monomial $\prod B_{i,j} \prod C_{j,k} p_S$ in F_r as the sum of weights of its factors. For $x \in F_r$ we then define the initial form $\mathbf{in}(x)$ to be the sum of monomials of x for which the function \mathbf{wt} is minimal.

THEOREM 5.4. — *For $S \in \text{PP}(a, b, c)$ with corresponding dimer configuration D_S , and $d \in D_S$ a dimer, we obtain a generator of $\mathbf{in}(M'_r)$ as follows.*

- *If d lies in a lozenge of type B with center (i, j) , then $B_{ij}p_S \in \mathbf{in}(M'_r)$.*
- *If d lies in a lozenge of type C with center (j, k) , then $C_{jk}p_S \in \mathbf{in}(M'_r)$.*
- *If d lies in a lozenge of type BC with center (i, k) , then $(B_{i,i+k-a-1}C_{i+k-a-1,k} + B_{i,i+k-a}C_{i+k-a,k})p_S \in \mathbf{in}(M'_r)$.*

Taken over all S and $d \in D_S$, these monomial and binomial relations generate the $\mathbf{in}(A)$ -submodule $\mathbf{in}(M'_r)$.

Showing that these generators are the leading forms of the generators from Theorem 5.1 will occupy Sections 5.3–5.4. Then showing they actually generate will come in Section 6.

As discussed in Section 5.1, we are mostly interested in the generators p_S where $S \in \text{PP}(a, b, c)$; we note that (5.1) can be rewritten, in that case, as

$$(5.12) \quad \text{wt}(p_{s_1} \dots p_{s_a}) = \sum_{\substack{\text{lozenges of type } C \\ \text{of } S \text{ at } (j, k)}} (y_{j,k}^C)^2, \quad S = (s_1 \leq \dots \leq s_a)$$

(note that the maximum is attained for the identity permutation, then use (5.8)). This seems to break the symmetry between lozenges of type B and those of type C ; however it is not hard to show that

$$(5.13) \quad \text{wt}(p_{s_1} \dots p_{s_a}) = \text{const} + \sum_{\substack{\text{lozenges of type } B \\ \text{of } S \text{ at } (i, j)}} (y_{i,j}^B)^2, \quad S = (s_1 \leq \dots \leq s_a)$$

where const is an irrelevant constant (depending only on a and $c - b$). Equivalently, we have the two explicit expressions

$$(5.14) \quad \text{wt}(p_{s_1} \dots p_{s_a}) = \sum_{i=1}^a \sum_{j=1}^b \left(\frac{1}{2} s_{i,j} + i - j \right)^2$$

$$(5.15) \quad = \text{const} + \sum_{i=1}^a \sum_{k=1}^c \left(-\frac{1}{2} \bar{s}_{i,k} + i + k - \frac{1}{2} \right)^2$$

To state our main result of the next two sections, we need some foundational results on Gröbner bases of modules [8, §15]:

PROPOSITION 5.5. — *Let $R \geq I$ be a polynomial ring and an ideal, with quotient $A := R/I$. Fix also a partial term order on R 's monomials with which to define Gröbner degenerations such as $\text{in}(A) := R/\text{in}(I)$.*

Let P be an indexing set, and $M \leq A^P$ an A -submodule, with pullback

$$\begin{array}{ccc} \widetilde{M} & \hookrightarrow & R^P \\ \downarrow & & \downarrow \\ M & \hookrightarrow & A^P. \end{array}$$

Then the R -module $R^P/\text{in}(\widetilde{M})$ descends to an $\text{in}(A)$ -module, and any Gröbner basis for $\widetilde{M} \leq R^P$ descends to generating sets for $M \leq A^P$ and (stronger) for

$$\text{in}(M) := \text{in}(A) \otimes_R \text{in}(\widetilde{M}) \leq \text{in}(A) \otimes_R R^P \cong \text{in}(A)^P.$$

Proof. — The descent-of-basis claim is equivalent to the vanishing of $\text{in}(I) \otimes_R R^P/\text{in}(\widetilde{M})$, computable from

$$\text{in}(I \otimes_R R^P/\widetilde{M}) \cong \text{in}(I \otimes_R A^P/M) = \text{in}(0).$$

For the first claim of generation, a Gröbner basis \mathcal{B} for \widetilde{M} generates \widetilde{M} , hence descends to generate its quotient M . But to be Gröbner, it must descend to generate $\text{in}(\widetilde{M})$, of which $\text{in}(M)$ is a quotient, giving the second claim of generation. \square

With this in mind, we can speak sensibly of Gröbner bases for submodules of free A -modules, where A is presented as a quotient of a polynomial ring (as our A is, in (4.4)–(4.6)).

While the B, C relations (4.10)–(4.11) define the module F'_r/M'_r , their initial forms aren't enough to generate $\text{in}(M'_r)$; we need the BC relations from (4.4) as well.

In fact, more detailed analysis (in Section 5.4) will show that we only need a subset of these to produce a Gröbner basis, one for each dimer of each dimer configuration.

5.3. The linear equations

We first discuss the equations (4.10) and (4.11), which are linear in the variables $B_{i,j}$ or $C_{j,k}$, multiplied by “spectator” monomials $p_{s_2} \dots p_{s_a}$ as in Theorem 5.1.

It is perhaps instructive to consider first the special case of $a = 1$. The Plücker relations do not appear in that case, and the linear relations are simply (4.10) and (4.11) without spectators.

Start with (4.10). The weights of its monomials are given by

$$\begin{aligned}
 (5.16) \quad \text{wt}(B_{i,j} p_{s_+ \setminus j}) &= \left(\frac{1}{2}j - (i - 1) + \frac{1}{2}\right)^2 + \sum_{m=1}^b \left(\frac{1}{2}((s_+ \setminus j)_m + 1 - m)\right)^2 \\
 &= \left(\frac{1}{2}j - i + \frac{3}{2}\right)^2 + \sum_{m=1}^{h-1} \left(\frac{1}{2}(s_{+m} + 1 - m)\right)^2 \quad j = s_{+,h} \\
 &\quad + \sum_{m=h+1}^{b+1} \left(\frac{1}{2}s_{+,m} + 2 - m\right)^2, \\
 &= \kappa_1 + \left(\frac{1}{2}j - i + \frac{3}{2}\right)^2 + \sum_{m=1}^{h-1} (s_{+,m} - 2m + 1) - \left(\frac{1}{2}j + 2 - h\right)^2 \\
 &= \kappa_2 - (i + 1/2)j + hj - \sum_{m=1}^{h-1} s_{+,m}
 \end{aligned}$$

where s_+ is of cardinality $b+1$, and we order the elements of $s_+ \setminus j$ as usual, $(s_+ \setminus j)_1 < \dots < (s_+ \setminus j)_b$. From the third line to the fourth line, we have subtracted $\sum_m (\frac{1}{2}s_{+,m} + 2 - m)^2$, resulting in an irrelevant constant κ_1 (which is independent of h or j). κ_2 is another such constant.

We claim that (5.16) has a unique minimum at $h = i$, i.e., $j = s_{+,i}$. Indeed, compute the difference

$$\mathbf{wt}(B_{i,s_+,h+1}p_{s_+ \setminus s_{+,h+1}}) - \mathbf{wt}(B_{i,s_+,h}p_{s_+ \setminus s_{+,h}}) = (h - i + 1/2)(s_{+,h+1} - s_{+,h})$$

which is negative for $h \leq i - 1$ and positive for $h \geq i$. We conclude that

$$(5.17) \quad \mathbf{in} \left(\sum_{j \in s_+} B_{i,j}p_{s_+ \setminus j} \right) = B_{i,s_{+,i}}p_{s_+ \setminus s_{+,i}}$$

We can repeat the analysis for (4.11):

$$\begin{aligned} (5.18) \quad \mathbf{wt}(C_{j,k}p_{s_- \cup j}) &= (k - \frac{j}{2})^2 + \sum_{m=1}^b \left(\frac{1}{2}(s_- \cup j)_m + 1 - m \right)^2 \\ &= (k - \frac{j}{2})^2 + \sum_{m=1}^{j-h} \left(\frac{1}{2}s_{-,m} + 1 - m \right)^2 \\ &\quad + (h - \frac{1}{2}j)^2 + \sum_{m=j-h+1}^{b-1} \left(\frac{1}{2}s_{-,m} - m \right)^2 \quad j = \bar{s}_{-,h} \\ &= \kappa_3 + \frac{1}{2}j^2 - j(k+h) + \sum_{m=1}^{j-h} (s_{-,m} - 2m + 1) \\ &= \kappa_3 - \frac{1}{2}j^2 + j(h-k) + \sum_{m=1}^{j-h} s_{-,m} \end{aligned}$$

and once again

$$\mathbf{wt}(C_{\bar{s}_{-,h+1},k}p_{s_- \cup \bar{s}_{-,h+1}}) - \mathbf{wt}(C_{\bar{s}_{-,h},k}p_{s_- \cup \bar{s}_{-,h}}) = (h-k+1/2)(\bar{s}_{-,h+1} - \bar{s}_{-,h})$$

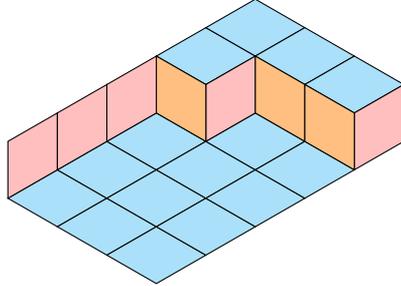
which is negative for $h \leq k - 1$ and positive for $h \geq k$, so that

$$(5.19) \quad \mathbf{in} \left(\sum_{j \in \bar{s}_-} C_{j,k}p_{s_- \cup j} \right) = C_{\bar{s}_{-,k},k}p_{s_- \cup \bar{s}_{-,k}}$$

Of course we could have made the reasoning even more similar to the previous one by using (5.15) instead of (5.14).

Results (5.17) and (5.19) both have a simple diagrammatic interpretation. Given $s \in \binom{[b+c]}{b}$, we can draw the corresponding lozenge tiling:

$$a = 1, b = 3, c = 5, s = \{4, 6, 7\} :$$



There are exactly $b + c$ lozenges of type B or C . Consider one of these lozenges of type B . Its coordinates are (i, j) where $j \notin s$ and i is one plus the number of elements of s less than j . (For example, the fifth lozenge of type B on the example has coordinates $(2, 5)$.) Therefore, defining $s_+ = s \cup j$, one has $j = s_{+,i}$ and one can naturally associate to it the initial term of an equation as in (5.17) indexed by s_+ and i . Similarly, to a lozenge of type C , with coordinates (j, k) , is naturally associated the initial term of (5.19) with $s_- = s \setminus j$ and $j = \bar{s}_{-,k}$.

We have found that to each $s \in \binom{[b+c]}{b}$ viewed as a lozenge tiling of a $1 \times b \times c$ hexagon, we can associate equations of the form $p_s B_{i,j} = 0$ and $p_s C_{j,k} = 0$ where (i, j) (resp. (j, k)) runs over the coordinates of lozenges of type B (resp. type C). We now wish to extend this conclusion to general a . The difference from the $a = 1$ case is that we shall pick a subset of equations to do so (with the implicit assumption, to be proven subsequently, that all other equations are redundant after the degeneration).

Reversing the logic we now start, for a arbitrary, with an $S \in \text{PP}(a, b, c)$ viewed as a lozenge tiling, and one lozenge of type B at (i, j) . In the NILP representation of S , it corresponds to a certain path labelled ℓ (between 1 and a), with $j \in \bar{s}_\ell$. We pick among (4.10) the one with $s_+ = s_\ell \cup j$, and multiply it by $\prod_{\ell' \neq \ell} p_{s_{\ell'}}$. We now carefully evaluate the weights of the various monomials in it.

The key observation is that from the definition (5.1) of the weight, we can bound from below the weight of each monomial $p_{s_+ \setminus j'} \prod_{\ell' \neq \ell} p_{s_{\ell'}}$ by the expression $w(s_1, \dots, s_{\ell-1}, s_+ \setminus j', s_{\ell+1}, \dots, s_a)$ (note the ordering); in the particular case $j' = j$, this bound is achieved because $S \in \text{PP}(a, b, c)$. At that stage we can do the exact same calculation as in the case $a = 1$;

skipping the details, we find

$$\begin{aligned} & w(s_1, \dots, s_{\ell-1}, s_+ \setminus s_{+,h+1}, s_{\ell+1}, \dots, s_a) + \mathbf{wt}(B_{i,s_+,h+1}) \\ & - w(s_1, \dots, s_{\ell-1}, s_+ \setminus s_{+,h}, s_{\ell+1}, \dots, s_a) - \mathbf{wt}(B_{i,s_+,h}) \\ & = (s_{+,h+1} - s_{+,h})(h - i + 1/2 + a - \ell) \end{aligned}$$

so that this function has a strict minimum at $h = i + a - \ell$, and we easily compute $s_{+,h} = j$, thus obtaining

$$(5.20) \quad \mathbf{in} \left(\sum_{j' \in s_+} B_{i,j'} p_{s_+ \setminus j'} \prod_{\ell' \neq \ell} p_{s_{\ell'}} \right) = p_S B_{i,j}$$

The exact same reasoning applies to a lozenge of type C of S at (j, k) (on the path labelled ℓ); computing

$$\begin{aligned} & w(s_1, \dots, s_{\ell-1}, s_- \cup \bar{s}_{-,h+1}, s_{\ell+1}, \dots, s_a) + \mathbf{wt}(C_{\bar{s}_{-,h+1},k}) \\ & - w(s_1, \dots, s_{\ell-1}, s_- \cup \bar{s}_{-,h}, s_{\ell+1}, \dots, s_a) - \mathbf{wt}(C_{\bar{s}_{-,h},k}) \\ & = (\bar{s}_{-,h+1} - \bar{s}_{-,h})(h - k - 1/2 + \ell) \end{aligned}$$

leads to the initial term

$$(5.21) \quad \mathbf{in} \left(\sum_{j' \in \bar{s}_-} C_{j',k} p_{s_- \cup j'} \prod_{\ell' \neq \ell} p_{s_{\ell'}} \right) = p_S C_{j,k}$$

5.4. The quadratic equations

These are (4.4), which are equations of the support of $\mu_* \sigma_{c \times b}$, i.e., which are true acting on any $p_{s_1} \cdots p_{s_a}$. Once we apply the degeneration given by weights (5.10) and (5.11), obvious minimization of the quadratic form in j results in

$$\begin{aligned} & \mathbf{in}(p_S(BC)_{i,k}) \\ & = p_S \begin{cases} B_{i,1} C_{1,k} & i+k \leq a+1 \\ B_{i,i+k-a-1} C_{i+k-a-1,k} + B_{i,i+k-a} C_{i+k-a,k} & a+1 < i+k < n+1 \\ B_{i,b+c} C_{b+c,k} & i+k \geq n+1 \end{cases} \end{aligned}$$

Rather than keeping all these initial terms, we shall show that many of them are redundant, i.e., can be derived from the initial terms (5.20) and (5.21) of the linear equations. Because of Proposition 5.2, we may always assume that $S \in \text{PP}(a, b, c)$.

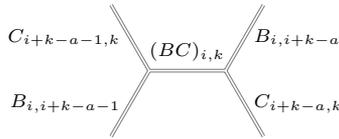
Let us start with the first type, that is $B_{i,1}C_{1,k}$, $i+k \leq a+1$. If we look at the $j = 1$ slice of a lozenge tiling (i.e., the leftmost vertical slice), we find that it always consists, from bottom to top, of a series of  lozenges of type C , then a  lozenge of type BC , then a series of  lozenges of type B . This means that among the initial terms of (5.20) and (5.21), we have

$$C_{1,1}, \dots, C_{1,i}, B_{a-i,1}, \dots, B_{1,1}$$

times p_S for some i between 0 and a . This immediately implies that if $i+k \leq a+1$, one of $B_{i,1}$ or $C_{1,k}$ is found in this list.

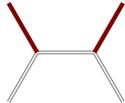
Similarly, one can show that the last case $B_{i,b+c}C_{b+c,k}$ is redundant because of the form of the rightmost slice $j = b+c$ of any lozenge tiling.

Finally, consider the middle case. In order to study it, it is convenient to go over to the dual picture of lozenge tilings, i.e., dimers, cf. the lower right picture of Figure 1.2. According to the previous section, each non-horizontal edge corresponds to a certain $B_{i,j}$ or $C_{j,k}$ depending on its orientation, and this edge is occupied by a dimer precisely when that variable times p_S is the initial term of an equation. It is natural to associate to $(BC)_{i,k}$, $a+1 < i+k < n+1$, the location (i,k) , which on this dual picture corresponds to a certain horizontal edge. Now it is easy to see that the two terms $B_{i,i+k-a-1}C_{i+k-a-1,k}$ and $B_{i,i+k-a}C_{i+k-a,k}$ are precisely the product of the variables attached to the edges adjacent to the horizontal edge at either endpoint:



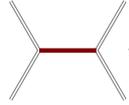
The dimer condition means that every vertex belongs to exactly one dimer. This means that there are two scenarios:

- The horizontal edge is empty, and then at either endpoint one adjacent edge must be occupied by a dimer, e.g.,



This immediately implies that the initial term associated to the horizontal edge is redundant.

- The horizontal edge is occupied:



which equivalently means that there is a lozenge of type BC at (i, k) . Then we decide to keep the corresponding initial term.

In the end, we see that a beautiful picture emerges: to each lozenge of the tiling is associated exactly one initial term, either linear in two orientations or quadratic in the last one (type BC). Note in the latter type, we get *binomials* instead of only getting monomials, as we would in a generic degeneration. This binomial behavior will lead to the appearance of certain toric varieties, as we explain now.

6. The special fiber

We recall the notion of a *shelling* of a simplicial complex and explain what modifications to it are necessary to describe the special fiber of our degeneration. All the definitions in the next paragraph are standard; our reference is [18, §1 and §13].

A collection $\Delta \subseteq 2^V$ of subsets of a “vertex set” V is a *simplicial complex* if $F \in \Delta, G \subseteq F \implies G \in \Delta$. It has a corresponding union of coordinate subspaces

$$SR(\Delta) := \bigcup_{F \in \Delta} \mathbb{C}^F \subseteq \mathbb{C}^V$$

called its (*affine*) *Stanley–Reisner scheme*, and every such union $S \subseteq \mathbb{C}^V$ comes from a unique simplicial complex $\Delta(S) := \{F \subseteq V : \mathbb{C}^V \subseteq S\}$. The maximal elements of Δ are called its *facets*; if they all have the same size $d + 1$ then Δ is called *pure of dimension d* . A *shelling* of a pure simplicial complex Δ is an ordering F_1, \dots, F_m of its facets such that $\{G \subseteq F_i : \exists j < i, F_j \supset G\}$ is again pure and of codimension 1 in F_i , for each i . (Shellings only exist for nice-enough Δ ; for example a union of two solid triangles at a point is an unshellable complex.)

Let $A := \mathbb{C}[x_v : v \in V] / \langle \prod_{g \in G} x_g \mid G \notin \Delta \rangle$ be the *Stanley–Reisner ring* of Δ , the coordinate ring of $SR(\Delta)$. Given a shelling of Δ , we can associate a list of ring elements

$$r_i := \prod \left\{ x_j \in F_i : F_i \setminus j \subseteq \bigcup_{j < i} F_j \right\}$$

and a filtration $M_i := \langle r_i, \dots, r_m \rangle \leq A$ of the regular module $M_1 = A$. Then the following is straightforward:

PROPOSITION 6.1. — *Each summand of the A -module*

$$\text{gr } M_1 := \bigoplus_{i \leq M} M_i/M_{i+1}$$

is a regular module over one component of A , i.e. $M_i/M_{i+1} = A \cdot \overline{r_i}$ and the annihilator ideals $\text{ann}(\overline{r_i})$ are the prime components of the zero ideal.

In this section we have a similar situation, but requiring three directions of generalization:

- The module M_1 is rank 1 and torsion-free, but not actually free.
- The ring A also has to degenerate; it is not just the module M_1 that degenerates (to its associated graded).
- The components of the degenerate scheme $\text{Spec in}(A)$ are still toric varieties ⁽⁵⁾ and complete intersections, but aren't quite coordinate subspaces; while some of their defining equations are coordinates (as in the Stanley–Reisner case), others are quadratic binomials (a new phenomenon).

For $S \in \text{PP}(a, b, c)$, let $F_S := \text{in}(A) \cdot p_S$ be the cyclic submodule of $\text{in}(F'_r)/\text{in}(M'_r)$ generated by the element p_S . Since $\text{in}(F'_r)$ is freely generated by the $\{p_S\}$ as an $\text{in}(A)$ -module, the quotient $\text{in}(F'_r)/\text{in}(M'_r)$ is the sum $\sum_{S \in \text{PP}(a, b, c)} F_S$. But much more is true:

THEOREM 6.2. — *The $\text{in}(A)$ -module $F'_r/\text{in}(M'_r)$ is the direct sum $\bigoplus_{S \in \text{PP}(a, b, c)} F_S$. Each F_S is supported on a single component of $\text{Spec in}(A)$, and this gives a correspondence between $\text{PP}(a, b, c)$ and the components of $\text{Spec in}(A)$.*

The rest of the section is devoted to its proof, and to determination of the individual F_S . We will then use this computation to finish the proof of Theorem 5.1, and more importantly, to pave the way to proving (in Section 7) our Conjectures 1.3 and 1.4 in the $c \times b$ rectangle case.

6.1. The individual F_S

Fix $S \in \text{PP}(a, b, c)$ for the rest of this subsection, which we will usually think of as a dimer configuration as in the Southeast picture in Figure 1.2.

⁽⁵⁾The equations defining toric varieties are binomial, but can be of high degree, and toric varieties are very rarely complete intersections.

Let H denote the set of hexagons in that picture, whose non-horizontal edges we corresponded (in Section 5.2) with some of the B, C ring generators. We computed $|H| = ab + ac + bc - a - b - c + 1$ in Section 5.2.

(5.20)–(5.21) and Section 5.4 show F_S is a cyclic module over the ring

$$A_S := \mathbb{C}[B_{i,j}, C_{j,k}, \star_{i,k}]$$

$$\left\langle \begin{array}{l} B_{i,j} \\ C_{j,k} \\ B_{i,i+k-a-1}C_{i+k-a-1,k} + B_{i,i+k-a}C_{i+k-a,k} \end{array} \right\rangle \begin{array}{l} \text{of type } B \text{ in } S \\ \text{of type } C \text{ in } S \\ \text{of type } BC \text{ in } S \end{array}$$

who itself has $(a + b)(b + c) + (b + c)(c + a) + (c + a)(a + b)$ generators, and one relation for each dimer in S (a total of $ab + ac + bc$, as recorded in Section 5.2). More specifically, we have a natural map $A_S \rightarrow \text{in}(A)/\text{ann}(F_S)$, which we will soon show is an isomorphism. A key tool will be the following:

PROPOSITION 6.3. — *Call a generator of A_S relevant if it appears on a (non-horizontal) edge in the diagram of H (including the half-edges around the boundary), but is not in S . There are $|H| + (a + c - 1) + 2b - ac = ab + bc + b$ relevant B generators, and similarly $ac + bc + c$ relevant C generators. Let A'_S be the subring generated by those, suffering the bc many quadratic binomial equations from the third group above.*

Let H_+ be H plus the partial hexagons around the outside, and $L = \mathbb{Z}^{H_+}$, the space of \mathbb{Z} -valued functions on H_+ . Then A'_S has an L -grading, where the B or C generator corresponding to a non-horizontal edge γ is given weight $f_\gamma \in L$, defined by

$$f_\gamma(h) = \begin{cases} +1 & \text{if } h \text{ is the (partial) hexagon above } \gamma \\ -1 & \text{if } h \text{ is the (partial) hexagon below } \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, this grading is fine, meaning that its homogeneous components are 1-dimensional.

Proof. — Each hexagon in H has a type B edge on its Northwest side, which gives all the B edges except for the $a + c - 1$ of them on the East and Southeast, and $2b$ one-ended edges coming off the Northeast and Southwest sides. Of those, there are ac in S we must remove. Flip left/right for the type C edges statement.

We have to check that the quadratic binomial relation coming from a horizontal edge is L -homogeneous:

$$\begin{aligned}
 \begin{array}{c} \diagup \quad + \\ \text{---} \\ \diagdown \quad - \end{array} &= \begin{array}{c} \diagup \quad + \\ \text{---} \\ \diagdown \quad - \end{array} - \begin{array}{c} \diagdown \quad - \\ \text{---} \\ \diagup \quad + \end{array} + \begin{array}{c} \diagdown \quad - \\ \text{---} \\ \diagup \quad + \end{array} \\
 &= - \begin{array}{c} \diagdown \quad - \\ \text{---} \\ \diagup \quad + \end{array} + \begin{array}{c} \diagup \quad + \\ \text{---} \\ \diagdown \quad - \end{array}
 \end{aligned}$$

The exponent vector of a Laurent monomial m is a \mathbb{Z} -valued function g on H_+ 's nonhorizontal edges, vanishing on S (and on the half-edges around the boundary). Each horizontal dimer $\gamma \in S$ gives a relation of the form $B_{i,i+k-a-1}C_{i+k-a-1,k}/B_{i,i+k-a}C_{i+k-a,k} = -1$, whose exponent vector we call r_γ . To show the grading is fine (as could have been predicted from [9]), we need to show that

$$f := \sum_{\text{edges } \gamma} g(\gamma)f_\gamma = 0$$

implies that g is in the span of the $(r_\gamma : \gamma \in S \text{ horizontal})$.

We can use the (r_γ) to modify g as follows, from left to right: for each horizontal $\gamma \in S$, connected to some B -edge γ' on its West side (necessarily not in S), add $g(\gamma')r_\gamma$ to g thereby making the new value at γ' be 0. Now we can assume that g vanishes not only on S , but on each B -edge to the Southwest of a horizontal S -edge. In short, if a B -edge is has nonzero g -value, then the C -edge below it has zero g -value (since it's in S), unless both are on the right boundary of H (so don't have a full horizontal edge giving an r_γ to use).

The displayed equation above says that in each partial hexagon $h \in H_+$, the sum of the NW and NE g -values equals the sum of the SW and SE. Each partial hexagon h on the West side of H_+ (there are initially $a + 1$ of them) has only NE and SE contributors to $f(h)$, and we've just shown that one of them vanishes. Hence the other one does too. Rip off this vertical line of (initially a many) hexagons from the West side and repeat the argument. The new shape will again be a hexagon of hexagons, albeit not with edge-lengths (a, b, c, a, b, c) , but that doesn't affect the argument. \square

In particular,

$$\begin{aligned}
 \dim A'_S &\geq \# \text{generators} - \# \text{relations} \\
 &= (ab + bc + b) + (ac + bc + c) - bc = ab + ac + bc + b + c
 \end{aligned}$$

with equality iff it is a complete intersection of those bc quadric hypersurfaces. So now we compute its dimension (to show that this inequality is indeed strict).

Let $L_S \leq L$ be the set of actual L -gradings of monomials occurring in A'_S . To analyze L_S , we first consider its perp (with respect to the dot product on L where the hexagon vectors $\{\vec{v}_h : h \in H_+\}$ are orthonormal).

PROPOSITION 6.4.

- (1) $\dim L = ab + ac + bc + a + b + c + 1$.
- (2) L_S is a cone (i.e. closed under $+$), and A'_S is its monoid algebra, a domain of dimension $\dim L_S$.
- (3) $f \in L_S^\perp$ iff it is constant on the regions of $S \cup \{\text{all horizontal edges}\}$.
- (4) Hence $\dim L_S^\perp = a + 1$ and $\dim L_S = ab + ac + bc + b + c$.
- (5) A'_S and A_S are complete intersections, of degree 2^{bc} .

Proof.

- (1) The number of partial hexagons in the H_+ for (a, b, c) is the number of hexagons that H would have for $(a + 1, b + 1, c + 1)$, namely $(a+1)(b+1) + (a+1)(c+1) + (b+1)(c+1) - (a+1) - (b+1) - (c+1) + 1$ or $ab + ac + bc + a + b + c + 1$.
- (2) The binomial relations defining A'_S don't involve any of the variables being killed in the linear relations. So any monomial m in the non-killed variables must be nonzero in A'_S , since each binomial relation just lets us rewrite m as another monomial (up to sign). Hence if $f_1, f_2 \in L$ come from nonzero monomials m_1, m_2 , then $m_1 m_2$ is nonzero also and has L -grading $f_1 + f_2$.

By [9], if A'_S were not a domain then it would satisfy a relation $m(p - q) = 0$ where m, p, q are monomials. But then mp, mq would be monomials with the same L -grading, as was just now forbidden in Proposition 6.3. The dimension statement is standard in toric geometry [10].

- (3) Each non-horizontal dimer γ not in S gives a nonvanishing generator of A_S , hence a vector $f_\gamma \in L$. To be perpendicular to that, $f \in L_S^\perp$ needs to take on the same value on the two hexagons that would have been separated by γ .
- (4) Looking at the bottom row of Figure 1.2, we notice that including all horizontal edges has the same topological effect as contracting them all to produce a NILP configuration out of a dimer configuration. In particular, since there are a NILPs, the number of regions separated by occupied edges is obviously $a + 1$. Then subtract this from $\dim L$ computed in (1).

- (5) The dimension of A'_S is at least $ab + ac + bc + b + c$, as we checked just before the proposition. Since by parts (1,4) that is the actual dimension of A'_S , it is a complete intersection, so its degree is the product of the degrees of its defining equations. There is one degree 2 equation for each horizontal dimer, of which there are bc (as seen in Section 5.2). Since A_S is A'_S tensor a polynomial ring, it too is a complete intersection of this degree 2^{bc} . □

6.2. The total initial module

So far we have a surjection

$$\bigoplus_{S \in \text{PP}(a,b,c)} A_S \twoheadrightarrow \text{in}(F'_r)/\text{in}(M'_r)$$

$$(a_S \in A_S : S \in \text{PP}(a,b,c)) \mapsto \sum_{S \in \text{PP}(a,b,c)} a_S p_S$$

that we need to show is an isomorphism. We will use a module-theoretic extension of the argument in [13, Lem. 1.7.5]:

LEMMA 6.5. — *Both sides of the surjection above are graded modules over the polynomial ring $\mathbb{C}[B_{i,j}, C_{j,k}, \star_{i,k}]$. Hence we can speak of their degrees, and since their supports are of the same dimension (namely $ab + ac + bc$), the surjection gives an inequality on degrees. If the map is not an isomorphism, then this inequality on degrees is strict.*

Proof. — As with (quotients by) ideals, the degree of a graded module M is defined as the leading coefficient of its Hilbert polynomial (times $\dim(\text{supp}(M))!$). There is a minor annoyance that the map defined above only becomes a graded map if we shift the grading on each A_S summand by $\deg(p_S)$, but shifting the argument of the Hilbert polynomial doesn't change the leading term, so we can ignore this subtlety.

If this map has an element $(k_S \in A_S)_{S \in \text{PP}(a,b,c)} \neq 0$ in its kernel K , then the kernel contains the module $\bigoplus A_S k_S$. Since each A_S is a domain by Proposition 6.4, this submodule of the kernel again has the dimension $ab + ac + bc$, so $\deg(RHS) = \deg(LHS) - \deg(K) < \deg(LHS)$. □

Proofs of Theorems 6.2, 5.1, and 5.4. — We already knew that $F'_r/\text{in}(M'_r)$ is generated by (F_S) , i.e. is a quotient of $\bigoplus F_S$, and that each F_S is supported only on the component $\text{Spec } A_S$ of $\text{Spec } A$.

By Proposition 4.3, the degree of A_S , hence also of its free rank 1 module F_S , is 2^{bc} . So $\bigoplus F_S$ has degree $2^{bc}|\text{PP}(a,b,c)|$.

Meanwhile its quotient $F'_r/\text{in}(M'_r)$ is a degeneration of a rank 1 sheaf over $\text{Spec } A$, which we calculated to be $2^{bc}|\text{PP}(a, b, c)|$ in Proposition 4.3. Now we use Lemma 6.5 to know that the map $\bigoplus F_S \rightarrow F'_r/\text{in}(M'_r)$ is an isomorphism. This completes the proof of Theorem 6.2.

To establish the Gröbnerness of Theorems 5.1 and 5.4, we need to know that the family doesn't have any components supported only on the special fiber. But we've determined the components (F_S) of the special fiber, and if we cut any of them down the degree would decrease below that of the general fiber. □

7. Conclusion

We now use the results of the last two sections, in particular Theorems 5.1, 6.2 and Proposition 6.4, to prove the remaining two Conjectures 1.3 and 1.4 (or more precisely, the equivalent Conjecture 1.3' and 3.2), in the (a, b, c) case.

First, we consider Conjecture 3.2.

7.1. Polynomiality and Conjecture 3.2

According to Theorem 5.1, if we wish to compute the Hilbert series of $\mu_*\sigma_{c \times b}$, we can instead use that of the degenerated A -module F'_r/M'_r . The latter, according to Theorem 6.2, is a direct sum $\bigoplus_{S \in \text{PP}(a,b,c)} F_S$ where each F_S is free of rank 1 over a complete intersection A_S .

As a result, we have the following formula:

$$\begin{aligned}
 (7.1) \quad \mu_*[\sigma_{c \times b}] = & \prod_{\substack{1 \leq i < j \leq a+b \\ \text{or} \\ a+b+1 \leq i < j \leq a+2b+c \\ \text{or} \\ a+2b+c+1 \leq i < j \leq 2n}} (1 - t z_i/z_j) \sum_{S \in \text{PP}(a,b,c)} \prod_{\substack{i \in s_\ell, \\ 1 \leq \ell \leq a}} z_i^{-1} \\
 & \prod_{\substack{\text{lozenges } (i, k) \\ \text{of type } BC \text{ of } S}} (1 - t^2 z_i/z_{k+a+2b+c}) \prod_{\substack{\text{lozenges } (i, j) \\ \text{of type } B \text{ of } S}} (1 - t z_i/z_{j+a+b}) \\
 & \prod_{\substack{\text{lozenges } (j, k) \\ \text{of type } C \text{ of } S}} (1 - t z_{j+a+b}/z_{k+a+2b+c})
 \end{aligned}$$

where the prefactors correspond to the 0s of M , cf. (4.9), the monomial is the weight of the generator p_S of F_S , and the other factors come from the various equations of A_S .

Now we recall Theorem 1.6, namely that lozenge tilings of a $a \times b \times c$ hexagon are in bijection with FPLs with connectivity (a, b, c) . Therefore, (7.1) is of the form conjectured in Conjecture 3.2.

In order for the monomials m_f to have the right form, we need to have

$$(7.2) \quad \tilde{m}_r^{-1} = t^{-bc/2} \prod_{i=1}^{a+b} z_i^{i-1} \prod_{j=1}^{b+c} z_{j+a+b}^{a+j-1} \prod_{k=1}^{a+c} z_{k+a+2b+c}^{b+k-1}$$

and carefully rearranging the monomials, we find

$$(7.3) \quad \tilde{m}_r^{-1} \mu_*[\sigma_{c \times b}] = \prod_{\substack{1 \leq i < j \leq a+b \\ \text{or } a+b+1 \leq i < j \leq a+2b+c \\ \text{or } a+2b+c+1 \leq i < j \leq 2n}} (z_j - t z_i) \\ \sum_{S \in \text{PP}(a,b,c)} \prod_{\substack{\text{lozenges } (i,k) \\ \text{of type } BC \text{ of } S}} t^{-1/2} (z_{k+a+2b+c} - t^2 z_i) \\ \prod_{\substack{\text{lozenges } (i,j) \\ \text{of type } B \text{ of } S}} (z_{j+a+b} - t z_i) \prod_{\substack{\text{lozenges } (j,k) \\ \text{of type } C \text{ of } S}} (z_{k+a+2b+c} - t z_{j+a+b})$$

which fits precisely with the form of the monomials in Conjecture 1.4.

We also find

COROLLARY 7.1.

$$\tilde{m}_r^{-1} \mu_*[\sigma_{c \times b}] = \prod_{\substack{1 \leq i < j \leq a+b \\ \text{or } a+b+1 \leq i < j \leq a+2b+c \\ \text{or } a+2b+c+1 \leq i < j \leq 2n}} (t^{-1/2} z_j - t^{1/2} z_i) \Phi_r$$

where Φ_r is a symmetric polynomial in the $\{z_1, \dots, z_{a+b}\}$, the $\{z_{a+b+1}, \dots, z_{a+2b+c}\}$, and the $\{z_{a+2b+c+1}, \dots, z_{2n}\}$.

Proof. — (compare with [4, Thm. 1]). The polynomiality (as opposed to Laurent polynomiality) is explicit in (7.3). The symmetry stems from the fact that $\pi_*[\sigma_{c \times b}]$, the pushforward to a point of a $\text{GL}(a+b) \times \text{GL}(b+c) \times \text{GL}(c+a)$ -equivariant sheaf, possesses the required symmetry, the prefactor coming from (3.5) and (7.2). \square

Now we move on to Conjecture 1.3. It should be noted that general equivariant localization arguments allow to show that Conjecture 1.1 implies Conjecture 1.3 without any need to use Gröbner degenerations. However, in the case (a, b, c) , is simpler to prove Conjecture 1.3' directly using the degeneration, as we show now.

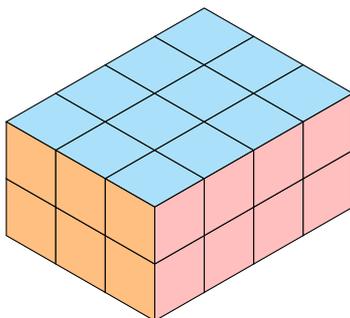
7.2. Wheel condition, specialization and Conjecture 1.3'

Given $1 \leq i_1 < i_2 < i_3 \leq N$, we investigate the specialization $z_{i_\ell} = t^{\ell-1}z$, $\ell = 1, 2, 3$, of $\tilde{m}_r^{-1}\mu_*[\sigma_r]$. If two of the indices i_1, i_2, i_3 fall in the same interval $\{1, \dots, a+b\}$, $\{a+b+1, \dots, a+2b+c\}$, $\{a+2b+c+1, \dots, 2n\}$, then according to Corollary 7.1, the prefactor of $\tilde{m}_r^{-1}\mu_*[\sigma_r]$ vanishes. Now assume $i_1 = i$, $i_2 = j+a+b$, $i_3 = a+2b+c+k$, $1 \leq i \leq a+b$, $1 \leq j \leq b+c$, $1 \leq k \leq a+c$. With the notation of Corollary 7.1, due to the symmetry of Φ_r , we can choose i, j, k as we wish and we may as well assume $j = i+k-a-1$. Now in each plane partition appearing in (7.3), there must be a lozenge of type B at (i, j) , of type C at (j, k) or of type BC at (i, k) (corresponding to the fact that one of the equations $B_{i,j} = 0$, $C_{j,k} = 0$ or $(BC)_{i,k} = 0$ must be satisfied). Therefore Φ_r vanishes, and we conclude that $\tilde{m}_r^{-1}\mu_*[\sigma_r]$ satisfies the wheel condition of Theorem 1.2.

Next we investigate the “dual basis” specializations of the same theorem. Pick $s \in LP(N)$ and set $z_i = t^{\pm 1/2}$ depending on whether $i \in s$ or not. Because of the same prefactors in Corollary 7.1, if a $t^{-1/2}$ occurs before a $t^{1/2}$ in each of the three same intervals mentioned above, the specialization is zero. Furthermore, $s \in LP(N)$ implies that all these specializations satisfy the “Dyck path” condition that in any sequence (z_1, \dots, z_ℓ) , $\ell = 1, \dots, N$, there must be more $t^{-1/2}$ than $t^{1/2}$. This leaves the unique possibility

$$(z_1, \dots, z_N) = (\underbrace{t^{-1/2}, \dots, t^{-1/2}}_{a+b}, \underbrace{t^{1/2}, \dots, t^{1/2}}_b, \underbrace{t^{-1/2}, \dots, t^{-1/2}}_c, \underbrace{t^{1/2}, \dots, t^{1/2}}_{a+c})$$

which is exactly the case $s = r$. In the sum of (7.3), a single term survives, corresponding to the “full” lozenge tiling of the type



and we compute, after cancellations of various powers of $t^{1/2}$:

$$\tilde{m}_r^{-1}\mu_*[\sigma_r]|_{\text{specialization above}} = (1-t)^{n(n-1)}(t^{1/2} + t^{-1/2})^{bc}$$

which means $\mu_*[\sigma_r] = (1-t)^{n(n-1)}\tilde{m}_r\Psi_r$, thus proving Conjecture 1.3'.

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